ESTIMATES FOR THE CLOSENESS OF CONVOLUTIONS OF PROBABILITY DISTRIBUTIONS ON CONVEX POLYHEDRA

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ABSTRACT. The aim of the present work is to show that the results obtained earlier on the approximation of distributions of sums of independent summands by the accompanying compound Poisson laws and the estimates of the proximity of sequential convolutions of multidimensional distributions may be transferred to the estimation of the closeness of convolutions of probability distributions on convex polyhedra.

Let us first introduce some notation. Let $\mathfrak{F}_d$ denote the set of probability distributions defined on the Borel σ-field of subsets of the Euclidean space $\mathbf{R}^d$ and let $\mathcal{L}(\xi) \in \mathfrak{F}_d$ be the distribution of a d-dimensional random vector ξ. Let $\mathfrak{F}_d^s \subset \mathfrak{F}_d$ be the set of symmetric distributions. For $F \in \mathfrak{F}_d$, we denote the corresponding characteristic functions by $\widehat{F}(t)$, $t \in \mathbf{R}^d$, and distribution functions by $F(x) = F\{(-\infty, x_1] \times \cdots \times (-\infty, x_d]\}$, $x = (x_1, \ldots, x_d) \in \mathbf{R}^d$. The uniform Kolmogorov distance is defined by

$$\rho(G, H) = \sup_{x \in \mathbf{R}^d} \big|G(x) - H(x)\big|, \quad G, H \in \mathfrak{F}_d.$$

By the symbols c and $c(\,\cdot\,)$ we denote (generally speaking various) positive absolute constants and quantities depending only on the arguments in brackets. For $0 \leq \alpha \leq 2$, we denote

$$\mathfrak{F}_d^{(\alpha)} = \Big\{F \in \mathfrak{F}_d^s : \widehat{F}(t) \geq -1 + \alpha, \text{ for all } t \in \mathbf{R}^d\Big\}, \quad \mathfrak{F}_d^+ = \mathfrak{F}_d^{(1)}.$$

Products and powers of measures are understood in the convolution sense: $GH = G * H$, $H^m = H^{m*}$, $H^0 = E = E_0$, where E_x is the distribution concentrated at a point $x \in \mathbf{R}^d$. A natural approximating infinitely divisible distribution for $\prod_{i=1}^n F_i$ is the accompanying compound Poisson distribution $\prod_{i=1}^n e(F_i)$, where

$$e(H) = e^{-1} \sum_{k=0}^{\infty} \frac{H^k}{k!}, \quad H \in \mathfrak{F}_d,$$

and, more generally,

$$e(\alpha H) = e^{-\alpha} \sum_{k=0}^{\infty} \frac{\alpha^k H^k}{k!}, \quad \alpha > 0. \tag{1}$$

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It is well-known that the distribution $e(\alpha H)$ is infinitely divisible.

Arak [1] showed that, if F is a symmetric one-dimensional distribution with a nonnegative characteristic function for all $t \in \mathbf{R}$, then

$$\rho(F^n, e(nF)) \le c\, n^{-1}, \tag{2}$$

He introduced and used the so-called method of triangular functions (see [2, Chapter 3, Sections 2–4]).

Zaitsev [6] applied the methods which were used by Arak while proving inequality (2) (see [2, Chapter 5, Sections 2, 5–7]). Later, he managed to modify these methods, adapting them to the multidimensional case (see [7]–[11]). In particular, in [10], a multidimensional analogue of inequality (2) was obtained.

Using the method of triangular functions and its generalizations, several bounds of the type

$$\rho(G, H) \le c(d)\, \varepsilon \tag{3}$$

were obtained, where $0 < \varepsilon < 1$ is small, $G, H \in \mathfrak{F}_d$, and the inequalities

$$\sup_{t \in \mathbf{R^d}} \big|\widehat{G}(t) - \widehat{H}(t)\big| \le c\, \varepsilon \tag{4}$$

are valid (see a discussion for $d = 1$ in [2, Chapter 3, Section 3]). Note that, in the general case, (4) does not imply (3).

Inequality (3) is equivalent to the validity of the inequality

$$\big|G\{X\} - H\{X\}\big| \le c(d)\, \varepsilon \tag{5}$$

for all sets X of the form

$$X = \big\{x \in \mathbf{R}^d : a_j \le \langle x, e_j\rangle \le b_j,\ j = 1, \dots, d\big\}, \tag{6}$$

where $e_j \in \mathbf{R}^d$ are the vectors of the standard Euclidean basis, $-\infty \le a_j \le b_j \le \infty$, $j = 1, \dots, d$.

For $m \in \mathbf{N}$ we denote by $\mathfrak{X}_m$ the collection of convex polyhedra $X \subset \mathbf{R}^d$ representable in the form

$$X = \big\{x \in \mathbf{R}^d : a_j \le \langle x, t_j\rangle \le b_j,\ j = 1, \dots, m\big\},$$

where $t_j \in \mathbf{R}^d$, $-\infty \le a_j \le b_j \le \infty$, $j = 1, \dots, m$, and, for $H = \mathcal{L}(\xi) \in \mathfrak{F}_d$, $X \in \mathfrak{X}_m$,

$$q(H, X) = \inf_{t \in \mathbf{R}^d, \|t\|=1} Q\big(\mathcal{L}\big(\langle \xi, t\rangle\big), \lambda\big\{\{\langle x, t\rangle : x \in X\}\big\}\big),$$

where $\lambda\{\,\cdot\,\}$ is the Lebesgue measure and $Q(F, b) = \sup_x F\big\{[x,\, x + b]\big\}$ is the concentration function of $F \in \mathfrak{F}_1$. Define a distance over all polyhedra $X \in \mathfrak{X}_m$ as

$$\rho_m(G, H) = \sup_{X \in \mathfrak{X}_m} \big|G\{X\} - H\{X\}\big|.$$

In [11], Zaitsev has proved several non-uniform inequalities of the form $|G\{X\} - H\{X\}| \le c(m)\, \varepsilon\, \beta(G, H, X) + o(\varepsilon)$ containing a factor $\beta(G, H, X)$, depending on the compared distributions and on the set $X \in \mathfrak{X}_m$, which satisfies the inequality $\beta(G, H, X) \le c(m)$ and may turn out to be small if the polyhedron X is sufficiently small in a certain sense. Theorems 1 and 2 have been obtained in [11] as consequences of the corresponding results of [8]–[10] which were proved for the case when X is a parallelepiped (6) with faces parallel to the coordinate axes. The aim of the present paper is to formulate and to discuss similar bounds for the quantities $\rho_m(G,H)$ and $\big|G\{X\}-H\{X\}\big|$, $X\in\mathfrak{X}_m$, where $G,H\in\mathfrak{F}_d$ are certain convolutions of probability distributions which have not been studied in the literature before.

Theorem 1. *Let* $F\in\mathfrak{F}_d^{(\alpha)}$, $0\le\alpha\le2$, $m,n\in\mathbf{N}$, $X\in\mathfrak{X}_m$, $D=e(nF)$, $q_1=q(D,X)$. *Then*

$$\big|(F^n)\{X\}-D\{X\}\big|\le c(m)\,\Big(n^{-1}q_1^{1/5}\big(\big|\log q_1\big|+1\big)^{(17m+24)/5}+\exp(-n\alpha+c\,m\,\log^3 n)\Big). \quad (7)$$

and, moreover,

$$\big|(F^n)\{X\}-(F^{n+1})\{X\}\big|\le c(m)\,\Big(n^{-1}q_1^{1/3}\big(\big|\log q_1\big|+1\big)^{3m+2}+\exp(-n\alpha+c\,m\,\log^3 n)\Big). \quad (8)$$

Hence,

$$\max\big\{\rho_m(F^n,e(nF)),\rho_m(F^n,F^{n+1})\big\}\le c(m)\,\Big(n^{-1}+\exp(-n\alpha+c\,m\,\log^3 n)\Big). \quad (9)$$

Theorem 2. *Assume that the distributions* $G_i\in\mathfrak{F}_d$ *are represented as*

$$G_i=(1-p_i)\,E+p_i\,V_i, \quad (10)$$

where $V_i\in\mathfrak{F}_d$ *are arbitrary distributions,* $0\le p_i\le p=\max_j p_j$,

$$m\in\mathbf{N},\quad X\in\mathfrak{X}_m,\quad G=\prod_{i=1}^n G_i,\quad D=\prod_{i=1}^n e(G_i),\quad q_2=q(D_0,X),$$

where D_0 *are the* d*-variate infinitely divisible distribution with characteristic function*

$$\widehat{D}_0(t)=\prod_{i=1}^n\exp\big(-p_i(1-p_i)\big(1-\operatorname{Re}\widehat{V}_i(t)\big)\big),\quad t\in\mathbf{R}^d.$$

Then

$$\big|G\{X\}-D\{X\}\big|\le c(m)\,q_2^{1/3}\big(\big|\log q_2\big|+1\big)^{3m+2}p \quad (11)$$

and, hence,

$$\rho_m(G,D)\le c(m)\,p. \quad (12)$$

It is easy to see that, under the conditions of Theorems 1 and 2, we have $0\le q_j\le1$, $j=1,2$, and, moreover, the quantities q_j may be small. For example, for a fixed bounded set $X\in\mathfrak{X}_m$, the quantity q_1 decreases for $n\to\infty$ not slower than $O(n^{-1/2})$. Thus, inequalities (7), (8) and (11) significantly strengthen inequalities (9) and (12). At the same time, there is no reason to expect that inequalities (7), (8) and (11) are optimal with respect to the dependence of the right-hand sides on the parameters q_1 and q_2. In particular, already from Arak's results (see [2, Theorem 7.1, Chap. V]) it follows that, for $m=1$, $\alpha=1$, inequality (7) may be replaced by

$$\big|(F^n)\{X\}-D\{X\}\big|\le c\,n^{-1}q_1^{1/3}\big(\big|\log q_1\big|+1\big)^{13/3}. \quad (13)$$

In inequalities (7) and (8) we have $\varepsilon = n^{-1}$ and

$$\beta(F^n, D, X) = q_1^{1/5}\big(\big|\log q_1\big| + 1\big)^{(17m+24)/5},$$

$$\beta(F^n, F^{n+1}, X) = q_1^{1/3}\big(\big|\log q_1\big| + 1\big)^{3m+2}$$

respectively.

The proofs of Theorems 1 and 2 are based on applications of m-variate versions of bounds for the closeness of distributions on the sets of the form (6) with $d = m$. It is important that the m-variate vectors with coordinates $\langle \xi, t_j\rangle$, $\langle \eta, t_j\rangle$, $t_j \in \mathbf{R}^d$, $j = 1, \dots, m$, satisfy the same m-dimensional conditions as the random vectors $\xi, \eta \in \mathbf{R}^d$ with compared d-dimensional distributions. For example, if $F = \mathcal{L}(\xi) \in \mathfrak{F}_d^{(\alpha)}$, for some α satisfying $0 \le \alpha \le 2$, then $\mathcal{L}\big(\langle \xi, t_1\rangle, \dots, \langle \xi, t_m\rangle\big) \in \mathfrak{F}_m^{(\alpha)}$. Similarly, if $F = \mathcal{L}(\xi) \in \mathfrak{F}_d^s$, then $\mathcal{L}\big(\langle \xi, t_1\rangle, \dots, \langle \xi, t_m\rangle\big) \in \mathfrak{F}_m^s$. Analogous statements hold for n and $(n+1)$-fold convolutions of such distributions and about other distributions involved in the assertions of Theorems 1 and 2. Thus, roughly speaking, from the known estimates of the distance ρ in space $\mathbf{R}^m$ we derive estimates of the distance ρ_m in space $\mathbf{R}^d$.

The situation considered in Theorem 2 can be interpreted as a comparison of the sample containing independent observations of rare events with a Poisson point process which is obtained after a Poissonization of the initial sample (see [3], [12]).

Indeed, let $X_1, X_2, \dots, X_n$ be independent not identically distributed elements of a measurable space $(\mathfrak{X}, \mathcal{S})$ and $f : \mathfrak{X} \to \mathbf{R}^m$ be a Borel mapping. Assume that the set $\mathfrak{X}$ is represented as the union of two disjoint measurable sets: $\mathfrak{X} = \mathfrak{X}_1 \cup \mathfrak{X}_2$, with $\mathfrak{X}_1$, $\mathfrak{X}_2 \in \mathcal{S}$, $\mathfrak{X}_1 \cap \mathfrak{X}_2 = \varnothing$. We say that the i-th rare event occurs if $X_i \in \mathfrak{X}_2$. Respectively, it does not occur if $X_i \in \mathfrak{X}_1$.

Assume that $f(x) = 0$, for all $x \in \mathfrak{X}_1$, and

$$0 \le p_i = \mathbf{P}\big\{X_i \in \mathfrak{X}_2\big\} = 1 - \mathbf{P}\big\{X_i \in \mathfrak{X}_1\big\} \le p = \max_{1\le i\le n} p_i.$$

Then $\mathcal{L}\big(f(X_i)\big) = (1 - p_i)\, E + p_i\, V_i$, where $E, V_i \in \mathfrak{F}_m$, and V_i is conditional distributions of $f(X_i)$ given $X_i \in \mathfrak{X}_2$. In [3], it was shown that

$$\rho\bigg(\mathcal{L}\Big(\sum_i f(X_i)\Big), \mathcal{L}\Big(\sum_k f(Y_k)\Big)\bigg) \le c(m)\, p, \tag{14}$$

where Y_k are the points of the corresponding Poisson point process. In particular, in the case where $\mathcal{X} = \mathbf{R}^d$, $\mathcal{X}_1 = \{0\}$, and $f(x) = \big(\langle x, t_1\rangle, \dots, \langle x, t_m\rangle\big)$, for $x \in \mathbf{R}^d$, inequality (14) turns into inequality (12).

In the rest of the paper we will study how small is the difference between F^{n+k} and F^n. A particular case of this problem is considered in inequality (7) of Theorem 1.

In the papers of Zaitsev [4], [5], [8] it was shown that one can obtain sharp bounds for the closeness of F^{n+k} and F^n without any moment conditions. Moreover, if the distribution F is centered so that all its marginal distributions have zero medians, then

$$\rho(F^n, F^{n+1}) \le c\, d\, n^{-1/2}, \tag{15}$$

where c is an absolute constant. The proof of this inequality is relatively simple and is based on classical bounds for concentration functions of convolutions. Much more complicated methods are needed to investigate the case of symmetric distributions $F \in \mathfrak{F}_d^s$. In this case inequality (15) is valid and it is optimal with respect to order in n. But it may be essentially improved in the case when the characteristic function $\widehat{F}(t)$ is uniformly separated from -1. In particular,

$$\rho(F^n, F^{n+1}) \le c(d)\, n^{-1}. \tag{16}$$

if $\widehat{F}(t) \ge 0$ for all $t \in \mathbf{R}^d$. Notice that inequality (8) is much more general compared to (16). Using this fact for the distribution F^2 with symmetric F, we obtain the paradoxical statement that for all natural numbers n and for any symmetric distribution F the inequalities

$$\rho(F^n, F^{n+1}) \le c\, d\, n^{-1/2} \quad \text{and} \quad \rho(F^n, F^{n+2}) \le c(d)\, n^{-1} \tag{17}$$

are valid and they are both optimal with respect to the order in n. Inequalities (17) imply the following Theorem 3.

Theorem 3. *Let $F \in \mathfrak{F}_d^s$, $k, n \in \mathbf{N}$. Then*

$$\rho(F^n, F^{n+2k}) \le c(d)\, k\, n^{-1}. \tag{18}$$

$$\rho(F^n, F^{n+2k+1}) \le c\, d\, n^{-1/2} + c(d)\, k\, n^{-1}. \tag{19}$$

In particular,

$$\sup_{k \le \sqrt{n}} \rho(F^n, F^{n+k}) \le c(d)\, n^{-1/2}. \tag{20}$$

It is evident that knowledge about the closeness of F^{n+k} and F^n is useful for studying distributions of the form

$$G = \sum_{s=0}^{\infty} p_s\, F^s, \quad 0 \le p_s \le 1, \quad \sum_{s=0}^{\infty} p_s = 1.$$

A result in this direction is given in our Theorem 7. In particular, using (1) and the bounds for the closeness of F^{n+k} and F^n, Zaitsev [8] proved the following Theorem 4.

Theorem 4. *Let $F \in \mathfrak{F}_d^s$, $n \in \mathbf{N}$. Then*

$$\rho(F^n, e(nF)) \le c(d)\, n^{-1/2}. \tag{21}$$

A one-dimensional version of Theorem 4 was proved somewhat earlier in [5].

It is evident that if a distribution $F \in \mathfrak{F}_d$ is concentrated on a hyperplane which does not contain zero and is orthogonal to one of coordinate axes then $\rho(F^n, F^{n+k}) = 1$ for any $n, k \in \mathbf{N}$. In particular, this is true in the case where $F = E_a$, $a \in \mathbf{R}^d$, $a \ne 0$. On the other hand, if all distributions $F^{(j)} \in \mathfrak{F}_1$, $j = 1, \dots, d$, of coordinates of the vector ξ with $\mathcal{L}(\xi) = F$ are either non-degenerate or equal to $E \in \mathfrak{F}_1$, then, as is shown in Zaitsev [4], $\rho(F^n, F^{n+1}) \to 0$ as $n \to \infty$ and, moreover,

$$\rho(F^n, F^{n+1}) \le \frac{c(F)}{\sqrt{n}}, \quad \text{for all } n \in \mathbf{N}. \tag{22}$$

Let $F \in \mathfrak{F}_1$ be a one-dimensional lattice symmetric distribution concentrated on the set of odd numbers. Then the distributions F^n, $n = 1, 2, \dots$, are concentrated either on the set of odd numbers or on the set of even ones according to the parity of the number n. Therefore, $\rho(F^n, F^{n+1}) \geq Q(F^n, 0)/2$. For many distributions, e.g. for $F = E_{-1}/2 + E_1/2$, the concentration function $Q(F^n, 0)$ behaves as $c(F)\, n^{-1/2}$ as $n \to \infty$. This indicates that the rate of decrease with respect to n of the right-hand side of (22) cannot be increased without additional assumptions.

It is easy to show that the distribution $F \in \mathfrak{F}_1^s$ is concentrated on the set of odd numbers if and only if its characteristic function $\widehat{F}(t)$ is equal to -1 at the points $t = (2k+1)\pi$ for all $k \in \mathbf{Z}$. For example, let $\widehat{F}(t) = \cos t$, for $F = E_{-1}/2 + E_1/2$. Inequality (9) of Theorem 1 says that the separation from -1 of the characteristic function of a distribution $F \in \mathfrak{F}_d^s$ leads to more quick decay of $\rho(F^n, F^{n+1})$ than the inequality (22) is able to provide.

Similarly to the proof of Theorems 1 and 2, we can use inequalities (18)–(22) in order to obtain the corresponding analogues of (18)–(22) for the closeness of convolutions of d-dimensional distributions on the convex polyhedra $X \in \mathfrak{X}_m$. The following Theorems 5–7 are the main results of the present paper.

Theorem 5. *Let $F \in \mathfrak{F}_d^s$, $k, m, n \in \mathbf{N}$. Then*

$$\rho_m(F^n, e(nF)) \leq c(m)\, n^{-1/2}, \tag{23}$$

$$\rho_m(F^n, F^{n+2k}) \leq c(m)\, k\, n^{-1}, \tag{24}$$

$$\rho_m(F^n, F^{n+2k+1}) \leq c\, m\, n^{-1/2} + c(m)\, k\, n^{-1}. \tag{25}$$

In particular,

$$\sup_{k \leq \sqrt{n}} \rho_m(F^n, F^{n+k}) \leq c(m)\, n^{-1/2}. \tag{26}$$

For $m \in \mathbf{N}$, $t_1, \dots, t_m \in \mathbf{R}^d$, we denote by $\mathfrak{X}(t_1, \dots, t_m)$ the collection of convex polyhedra $X \subset \mathbf{R}^d$ representable in the form

$$X = \big\{x \in \mathbf{R}^d : a_j \leq \langle x, t_j \rangle \leq b_j,\ j = 1, \dots, m\big\}.$$

Clearly,

$$\mathfrak{X}_m = \bigcup_{t_1,\dots,t_m} \mathfrak{X}(t_1, \dots, t_m).$$

The following Theorem 6 is a consequence of inequality (22).

Theorem 6. *Let $F = \mathcal{L}(\xi) \in \mathfrak{F}_d$, $m \in \mathbf{N}$, $t_1, \dots, t_m \in \mathbf{R}^d$, and all distributions of the random variables $\langle \xi, t_j \rangle$, $j = 1, \dots, m$, are either non-degenerate or equal to $E \in \mathfrak{F}_1$. Then, for all $X \in \mathfrak{X}(t_1, \dots, t_m)$,*

$$\big|(F^n)\{X\} - (F^{n+1})\{X\}\big| \leq c(F, t_1, \dots, t_m)\, n^{-1/2}. \tag{27}$$

Thus, we have the alternative: the left-hand side of (27) is equal to one or decreases at least as $O(n^{-1/2})$.

The quantity $c(F, t_1, \dots, t_m)$ can be larger than any absolute constant. For example, if $F = F_n \in \mathfrak{F}_1$ depends on n and $F_n\{[n, n+1)\} = 1$, then $\rho_1(F_n^n, F_n^{n+1}) = 1$.

Note, however, that there is a difference between Theorems 1–2 and Theorems 5–6. In Theorems 1–2, the bounds are non-uniform. They include factors $\beta(\,\cdot\,, \,\cdot\,, X)$ which depend on q_j, $j = 1, 2$, and may be small for small sets $X \in \mathfrak{X}_m$. The bounds of Theorems 5–6 cannot be improved even if we compare the probabilities to hit the set containing the unique point 0 only, where $d = 1$ and $F = E_{-1}/2 + E_1/2$. An exception is inequality (24). Applying inequality (8) to the distribution $F^2 \in \mathfrak{F}_d^+$, it is easy to show that

$$\big|(F^n)\{X\} - (F^{n+2k})\{X\}\big| \le c(m)\,k\,\Big(n^{-1} q_3^{1/3} \big(\big|\log q_3\big| + 1\big)^{3m+2} + \exp(-n + c\,m\,\log^3 n)\Big), \quad (28)$$

for any $X \in \mathfrak{X}_m$, where $q_3 = q(e(n_0 F^2), X)$ and n_0 is the maximal integer which is less or equal to $n/2$.

In conclusion, we formulate a result on the closeness of the distributions of sums of random number of independent identically distributed random vectors, which follows from Theorem 5. For the distance $\rho(\,\cdot\,, \,\cdot\,)$ this result is contained in [8, Theorem 1.3].

Let $\xi_1, \xi_2, \dots$ be independent identically distributed random vectors with common distribution $F \in \mathfrak{F}_d$ and let $(\mu, \nu) \in \mathbf{Z}^2$ be a two-dimensional random vector with integer non-negative coordinates, independent of the sequence $\{\xi_j\}_{j=1}^\infty$. Denote

$$U = \mathcal{L}(\mu), \;\; V = \mathcal{L}(\nu), \;\; G = \mathcal{L}(\xi_1 + \dots + \xi_\mu), \;\; H = \mathcal{L}(\xi_1 + \dots + \xi_\nu). \quad (29)$$

Then it is well known that then

$$G = \sum_{k=0}^{\infty} \mathbf{P}\{\mu = k\}\, F^k, \quad H = \sum_{k=0}^{\infty} \mathbf{P}\{\nu = k\}\, F^k. \quad (30)$$

Theorem 7. *If $F \in \mathfrak{F}_d^s$, then*

$$\rho_m(G, H) \le \inf \mathbf{E}\, \min\Big\{\frac{cm}{\sqrt{\nu + 1}} + c(m)\,\frac{|\mu - \nu|}{\nu + 1}, 1\Big\}, \quad (31)$$

and if $F \in \mathfrak{F}_d^+$, then

$$\rho_m(G, H) \le \inf \mathbf{E}\, \min\Big\{c(m)\,\frac{|\mu - \nu|}{\nu + 1}, 1\Big\}. \quad (32)$$

Here, the infimum is taken over all possible two-dimensional distributions $\mathcal{L}((\mu, \nu)) \in \mathfrak{F}_2$ such that $\mathcal{L}(\mu) = U$, $\mathcal{L}(\nu) = V$.

An interesting problem is to expand our inequalities to arbitrary convex sets X. This does not follow from Theorems 1, 2, and 5–7, since the constants $c(m)$ depend on m.

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