EDGEWORTH-TYPE EXPANSION IN THE ENTROPIC FREE CLT

G. P. CHISTYAKOV[1,2] AND F. GÖTZE[1,2]

Abstract. We prove an expansion for densities in the free CLT and apply this result to an expansion in the entropic free central limit theorem assuming a moment condition of order four for the free summands.

1. Introduction

Free convolutions were introduced by D. Voiculescu [32], [33] and have been studied intensively in context of non commutative probability. The key concept here is the notion of freeness, which can be interpreted as a kind of independence for non commutative random variables. As in classical probability theory where the concept of independence gives rise to the classical convolution, the concept of freeness leads to a binary operation on the probability measures, the free convolution. Many classical results in the theory of addition of independent random variables have their counterparts in Free Probability, such as the Law of Large Numbers, the Central Limit Theorem, the Lévy-Khintchine formula and others. We refer to Voiculescu, Dykema and Nica [34], Hiai and Petz [20], and Nica and Speicher [26] for an introduction to these topics.

In this paper we obtain an analogue of Esseen's expansion for a density of normalized sums of free identically distributed random variables under a fourth moment assumption on the free summands. Using this expansion we establish the rate of convergence of the free entropy of normalized sums of free identically distributed random variables.

The paper is organized as follows. In Section 2 we formulate and discuss the main results of the paper. In Section 3 and 4 we state auxiliary results. In Section 5 we discuss the passage to probability measures with bounded supports. In Section 6 we obtain a local asymptotic expansion for a density in the CLT for free identically distributed random variables. In Section 7 we study the behaviour of subordination functions in the free CLT for truncated free summands. In Section 8 we discuss the closeness of subordination functions in the free CLT for bounded and unbounded free random variables. In Section 9 we investigate the rate of convergence for densities in the free CLT in $L_1(-\infty,+\infty)$ and Section 10 is devoted to study the rate of convergence for the free entropy of normalized

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1) Faculty of Mathematics , University of Bielefeld, Germany.

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sums of free identically distributed random variables. In Section 11 we derive rates of convergence for the free Fisher information of normalized sums of free identically distributed random variables.

2. Results

Denote by $\mathcal{M}$ the family of all Borel probability measures defined on the real line $\mathbb{R}$. Let $\mu \boxplus \nu$ be the free (additive) convolution of μ and ν introduced by Voiculescu [32] for compactly supported measures. Free convolution was extended by Maassen [24] to measures with finite variance and by Bercovici and Voiculescu [8] to the class $\mathcal{M}$. Thus, $\mu \boxplus \nu = \mathcal{L}(X+Y)$, where X and Y are free random variables such that $\mu = \mathcal{L}(X)$ and $\nu = \mathcal{L}(Y)$.

Henceforth $X, X_1, X_2, \dots$ stands for a sequence of identically distributed random variables with distribution $\mu = \mathcal{L}(X)$. Define $m_k(\mu) := \int_{\mathbb{R}} u^k \,\mu(du)$, where $k = 0, 1, \dots$.

The classical CLT says that if $X_1, X_2, \dots$ are independent and identically distributed random variables with a probability distribution μ such that $m_1(\mu) = 0$ and $m_2(\mu) = 1$, then the distribution function $F_n(x)$ of

$$Y_n := \frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}} \tag{2.1}$$

tends to the standard Gaussian law $\Phi(x)$ as $n \to \infty$ uniformly in x.

A free analogue of this classical result was proved by Voiculescu [31] for bounded free random variables and later generalized by Maassen [24] to unbounded random variables. Other generalizations can be found in [9], [10], [16], [21]–[23], [27], [37], [38].

For $t > 0$, the centered semicircle distribution of variance t is the probability measure with density $p_{w_t}(x) := \frac{1}{2\pi t}\sqrt{(4t - x^2)_+}$, $x \in \mathbb{R}$, where $a_+ := \max\{a, 0\}$ for $a \in \mathbb{R}$. Denote by μ_{w_t} the probability measure with the distribution function $w_t(x)$. In the sequel we use the notations $w_1(x) = w(x)$.

When the assumption of independence is replaced by the freeness of the non commutative random variables $X_1, X_2, \dots, X_n$, the limit distribution function of (2.1) is the semicircle law $w(x)$. We denote as well by μ_n the probability measure with the distribution function $F_n(x)$.

It was proved in [5] that if the distribution μ of X is not a Dirac measure, then in the free case $F_n(x)$ is Lebesgue absolutely continuous when $n \geq n_1 = n_1(\mu)$ is sufficiently large. Denote by $p_n(x)$ the density of $F_n(x)$.

In the sequel we denote by $c(\mu), c_1(\mu), c_2(\mu), \dots$ positive constants depending on μ only. By $c(\mu)$ we denote generic constants in different (or even in the same) formula. The symbols $c_1(\mu), c_2(\mu), \dots$ will denote explicit constants. By $\{\varepsilon_{nk}\}$ denote positive numbers such that $\varepsilon_{nk} \to 0$ as $n \to \infty$.

Wang [38] proved that under the condition $m_2(\mu) < \infty$ the density $p_n(x)$ of $F_n(x)$ is continuous for sufficiently large n and

$$p_n(x) \leq c(\mu), \quad x \in \mathbb{R}. \tag{2.2}$$

Assume that $m_4(\mu) < \infty, m_1(\mu) = 0, m_2(\mu) = 1$ and denote

$$a_n := \frac{m_3(\mu)}{\sqrt{n}}, \quad b_n := \frac{m_4(\mu) - m_3^2(\mu) - 1}{n}, \quad d_n := \frac{m_4(\mu) - m_3^2(\mu)}{n}, \quad n \in \mathbb{N}. \tag{2.3}$$

Furthermore, let $e_n := (1 - b_n)/\sqrt{1 - d_n}$ and let I_n and I_n^* denote intervals of the form

$$I_n := \Big\{x \in \mathbb{R} : |x - a_n| \le \frac{2}{e_n} - \frac{\varepsilon_{n1}}{n}\Big\}, \qquad I_n^* := \Big\{x \in \mathbb{R} : |x - a_n| \le \frac{2}{e_n} - \sqrt{\frac{\varepsilon_{n1}}{n}}\Big\}. \tag{2.4}$$

In the sequel we denote by θ a real-valued quantity such that $|\theta| \le 1$.

We have derived an asymptotic expansion of $p_n(x)$ for bounded free random variables $X_1, X_2, \dots$ in the paper [17]. Improving the methods of this paper and [18] we obtain an asymptotic expansion of $p_n(x)$ for the case $m_4(\mu) < \infty$. Denote by $v_n(x)$ the function

$$v_n(x) = \Big(1 + \frac{1}{2}d_n - a_n^2 - \frac{1}{n} - a_n x - \Big(b_n - a_n^2 - \frac{1}{n}\Big)x^2\Big)p_w(e_n x), \quad x \in \mathbb{R}. \tag{2.5}$$

Theorem 2.1. *Let $m_4(\mu) < \infty$ and $m_1(\mu) = 0$, $m_2(\mu) = 1$. Then there exist sequences $\{\varepsilon_{n1}\}$ and $\{\varepsilon_{n2}\}$ such that*

$$p_n(x + a_n) = v_n(x) + \rho_{n1}(x) + \rho_{n2}(x), \quad x \in I_n^* - a_n, \tag{2.6}$$

where, for $x \in I_n^* - a_n$,

$$|\rho_{n1}(x)| \le \frac{\varepsilon_{n1}}{n} \frac{c(\mu)}{(4 - (e_n x)^2)^{3/2}}, \tag{2.7}$$

and for $x \in (I_n - a_n) \setminus (I_n^* - a_n)$,

$$|\rho_{n1}(x)| \le \sqrt{\frac{\varepsilon_{n1}}{n}} \frac{c(\mu)}{(4 - (e_n x)^2)^{1/2}}. \tag{2.8}$$

In (2.6) $\rho_{n2}(x)$ is a continuous function such that

$$0 \le \rho_{n2}(x) \le c(\mu) \quad \text{and} \quad \int_{I_n - a_n} \rho_{n2}(x)\,dx = o(1/n^2). \tag{2.9}$$

Moreover,

$$\int_{\mathbb{R}\setminus I_n} p_n(x)\,dx \le \frac{\varepsilon_{n2}}{n}. \tag{2.10}$$

Corollary 2.2. *Let $m_4(\mu) < \infty$ and $m_1(\mu) = 0$, $m_2(\mu) = 1$, then*

$$\int_{\mathbb{R}} |p_n(x) - p_w(x)|\,dx = \frac{2|m_3(\mu)|}{\pi\sqrt{n}} + c(\mu)\theta\Big(\Big(\frac{\varepsilon_{n1}}{n}\Big)^{3/4} + \frac{1}{n}\Big). \tag{2.11}$$

In [17] we proved analogous results for *bounded* free random variables and in [18] assuming a finite moment of order eight.

Recall that, if the random variable X has density f, then the classical entropy of a distribution of X is defined as $h(X) = -\int_{\mathbb{R}} f(x)\log f(x)\,dx$, provided the positive part of the integral is finite. Thus we have $h(X) \in [-\infty, \infty)$. A much stronger statement than the classical CLT – the entropic central limit theorem – indicates that, if for some

n_0, or equivalently, for all $n \ge n_0$, Y_n from (2.1) have absolutely continuous distributions with finite entropies $h(Y_n)$, then there is convergence of the entropies, $h(Y_n) \to h(Y)$, as $n \to \infty$, where Y is a standard Gaussian random variable. This theorem is due to Barron [3]. Artstein, Bally, Barthez, and Naor [2] have solved an old question raised by Shannon about the monotonicity of entropy under convolution. The relative entropy

$$D(X) = D(X||Z) = h(Z) - h(X),$$

where the normal random variable Z have the same mean and the same variance as X, is nonnegative and serves as kind of a distance to the class of normal laws. Thus, the entropic central limit theorem may be reformulated as $D(Y_n) \downarrow 0$, as long as $D(Y_{n_0}) < +\infty$ for some n_0.

Recently Bobkov, Chistyakov and Götze [12] found the rate of convergence to zero of $D(Y_n)$ and for the random variables X with $\mathbb{E}|X|^k < \infty$, $k \ge 4$, have obtained an Edgeworth-type expansion of $D(Y_n)$ as $n \to \infty$.

Let ν be a probability measure on $\mathbb{R}$. We assume below that $m_1(\nu) = 0$ and $m_2(\nu) = 1$. The quantity

$$\chi(\nu) = \int\int_{\mathbb{R}\times\mathbb{R}} \log|x-y|\,\nu(dx)\nu(dy) + \frac{3}{4} + \frac{1}{2}\log 2\pi,$$

called free entropy, was introduced by Voiculescu in [35]. Free entropy χ behaves like the classical entropy h. In particular, the free entropy is maximized by the standard semicircular law w with the value $\chi(w) = \frac{1}{2}\log 2\pi e$ among all probability measures with variance one [20], [36]. Shlyakhtenko [29] has proved that $\chi(\mu_n)$ decreases monotonically, i.e., the Shannon hypothesis holds in the free case as well.

Wang [38] has proved a free analogue of Barron's result: the free entropy $\chi(\mu_n)$ converges to the semicircular entropy. As in the classical case a relative free entropy

$$D(\nu||\mu_w) = \chi(\mu_w) - \chi(\nu)$$

is nonnegative and serves as kind of a distance to the class of semicircular laws.

We derive an optimal rate of convergence in the free CLT for free random variables with a finite moment of order four. In previous results [17] we showed an analogous result for bounded free random variables and in [18] for free random variables with a finite moment of order eight.

Corollary 2.3. *Let $m_4(\mu) < \infty$ and $m_1(\mu) = 0$, $m_2(\mu) = 1$. Then, for every fixed $1 < q \le 1.01$,*

$$D(\mu_n||\mu_w) = \frac{m_3^2(\mu)}{6n} + \theta\Big(c(\mu,q)\Big(\frac{\varepsilon_{n1}}{n}\Big)^{1+\frac{1}{2q}} + c(\mu)\frac{\varepsilon_{n2}}{n}\Big), \tag{2.12}$$

where $c(\mu,q) > 0$ is a constant depended on μ and q only.

Hence the remainder term in (2.12) is of order $o(n^{-1})$ provided that $m_4(\mu) < \infty$. In Sections 5 and 8 we explicitly describe the sequences $\{\varepsilon_{n1}\}$ and $\{\varepsilon_{n2}\}$. If we assume that $m_6(\mu) < \infty$, then it follows from Remarks 8.13 and 8.14 (see the end of Section 8) that the remainder term in (2.12) is of order $O(n^{-3/2})$.

Given a random variable X with an absolutely continuous density f, the Fisher information of X is defined by $I(X) = \int_{-\infty}^{+\infty} \frac{f'(x)^2}{f(x)}\, dx$, where f' denotes the Radon-Nikodym derivative of f. In all other cases, let $I(X) = +\infty$. With the first two moments of X being fixed, $I(X)$ is minimized for the normal random variable Z with the same mean and the same variance as X, i.e. $I(X) \geq I(Z)$ (which is a variant of Cramér-Rao's inequality).

Baron and Johnson have proved in [4] that $I(Y_n) \to I(Z)$, as $n \to \infty$, if and only if $I(Y_{n_0}) < \infty$. In classical probability and statistics the relative Fisher information

$$I(X||Z) = I(X) - I(Z)$$

is used as a strong measure of the probability distribution of X being near to the Gaussian distribution. The result of Baron and Johnson is equivalent to the fact that $I(Y_n||Z) \to 0$ as $n \to \infty$, if and only if $I(Y_{n_0}||Z) < \infty$.

Bobkov, Chistyakov and Götze [13] found the rate of convergence to zero of $I(Y_n||Z)$ and for the random variables X with $\mathbb{E}|X|^k < \infty$, $k \geq 4$, have obtained an Edgeworth-type expansion of $I(Y_n||Z)$ as $n \to \infty$.

Suppose that the measure ν has a density p in $L^3(\mathbb{R})$. Then, following Voiculescu [36], the free Fisher information is

$$\Phi(\nu) = \frac{4\pi^2}{3} \int_{\mathbb{R}} p(x)^3\, dx.$$

It is well-known that $\Phi(\mu_w) = 1$. The free Fisher information has many properties analogous to those of classical Fisher information. These include the free analog of the Cramér-Rao inequality.

Assume now that $m_1(\nu) = 0$ and $m_2(\nu) = 1$. Consider the free relative Fisher information

$$\Phi(\nu||\mu_w) = \Phi(\nu) - \Phi(\mu_w) \geq 0$$

as a strong measure of closeness of ν to Wigner's semicircle law. Here we obtain an Edgeworth-type expansion for free random variables with a finite moment of order four.

Corollary 2.4. *Let $m_4(\mu) < \infty$ and $m_1(\mu) = 0$, $m_2(\mu) = 1$. Then*

$$\Phi(\mu_n||\mu_w) = \int_{\mathbb{R}} p_n(x)^3\, dx - \Phi(\mu_w) = \frac{m_3^2(\mu)}{n} + c(\mu)\theta\, \frac{\varepsilon_{n1} + \varepsilon_{n2}}{n}. \tag{2.13}$$

As in the formula (2.12) the remainder term here is of order $o(n^{-1})$ if $m_4(\mu) < \infty$ and of order $O(n^{-3/2})$ provided that $m_6(\mu) < \infty$.

In contrast to the classical case (see [12] and [13]) we expect that the asymptotic expansion in (2.12) and (2.13) holds with an error of order n^{-1} only.

3. Auxiliary results

We need results about some classes of analytic functions (see [1], Section 3.

The class $\mathcal{N}$ (Nevanlinna, R.) is the class of analytic functions $f(z) : \mathbb{C}^+ \to \{z : \Im z \geq 0\}$. For such functions there is an integral representation

$$f(z) = a + bz + \int\limits_{\mathbb{R}} \frac{1+uz}{u-z}\,\tau(du) = a + bz + \int\limits_{\mathbb{R}} \Big(\frac{1}{u-z} - \frac{u}{1+u^2}\Big)(1+u^2)\,\tau(du), \quad z \in \mathbb{C}^+, \tag{3.1}$$

where $b \geq 0$, $a \in \mathbb{R}$, and τ is a non-negative finite measure. Moreover, $a = \Re f(i)$ and $\tau(\mathbb{R}) = \Im f(i) - b$. From this formula it follows that $f(z) = (b + o(1))z$ for $z \in \mathbb{C}^+$ such that $|\Re z|/\Im z$ stays bounded as $|z|$ tends to infinity (in other words $z \to \infty$ non tangentially to $\mathbb{R}$). Hence if $b \neq 0$, then f has a right inverse $f^{(-1)}$ defined on the region $\Gamma_{\alpha,\beta} := \{z \in \mathbb{C}^+ : |\Re z| < \alpha \Im z,\ \Im z > \beta\}$ for any $\alpha > 0$ and some positive $\beta = \beta(f, \alpha)$.

A function $f \in \mathcal{N}$ admits the representation

$$f(z) = \int\limits_{\mathbb{R}} \frac{\sigma(du)}{u-z}, \quad z \in \mathbb{C}^+, \tag{3.2}$$

where σ is a finite non-negative measure, if and only if $\sup_{y \geq 1} |yf(iy)| < \infty$. Moreover $\sigma(\mathbb{R}) = -\lim_{y \to +\infty} iyf(iy)$.

For $\mu \in \mathcal{M}$, consider its Cauchy transform $G_\mu(z)$

$$G_\mu(z) = \int_{\mathbb{R}} \frac{\mu(du)}{z-u}, \quad z \in \mathbb{C}^+. \tag{3.3}$$

The measure μ can be recovered from $G_\mu(z)$ as the weak limit of the measures

$$\mu_y(dx) = -\frac{1}{\pi}\Im G_\mu(x+iy)\,dx, \quad x \in \mathbb{R},\ y > 0,$$

as $y \downarrow 0$. If the function $\Im G_\mu(z)$ is continuous at $x \in \mathbb{R}$, then the probability distribution function $D_\mu(t) = \mu((-\infty, t))$ is differentiable at x and its derivative is given by

$$D'_\mu(x) = -\Im G_\mu(x)/\pi. \tag{3.4}$$

This inversion formula allows to extract the density function of the measure μ from its Cauchy transform.

Following Maassen [24] and Bercovici and Voiculescu [8], we shall consider in the following the *reciprocal Cauchy transform*

$$F_\mu(z) = \frac{1}{G_\mu(z)}. \tag{3.5}$$

The corresponding class of reciprocal Cauchy transforms of all $\mu \in \mathcal{M}$ will be denoted by $\mathcal{F}$. This class coincides with the subclass of Nevanlinna functions f for which $f(z)/z \to 1$ as $z \to \infty$ non tangentially to $\mathbb{R}$.

The following lemma is well-known, see [1], Th. 3.2.1, p. 95.

Lemma 3.1. *Let μ be a probability measure such that*

$$m_k = m_k(\mu) := \int\limits_{\mathbb{R}} u^k \, \mu(du) < \infty, \qquad k = 0, 1, \dots, 2n, \quad n \geq 1. \tag{3.6}$$

Then the following relation holds

$$\lim_{z\to\infty} z^{2n+1}\Big(G_\mu(z) - \frac{1}{z} - \frac{m_1}{z^2} - \dots - \frac{m_{2n-1}}{z^{2n}}\Big) = m_{2n} \tag{3.7}$$

uniformly in the angle $\delta \leq \arg z \leq \pi - \delta$, where $0 < \delta < \pi/2$.

Conversely, if for some function $G(z) \in \mathcal{N}$ the relation (3.7) *holds with real numbers m_k for $z = iy, y \to \infty$, then $G(z)$ admits the representation* (3.3)*, where μ is a probability measure with moments* (3.6)*.*

As shown before, $F_\mu(z)$ admits the representation (3.1) with $b = 1$. From Lemma 3.1 the following proposition is immediate.

Proposition 3.2. *In order that a probability measure μ satisfies the assumption* (3.6)*, where $m_1(\mu) = 0$, it is necessary and sufficient that*

$$F_\mu(z) = z + \int_{\mathbb{R}} \frac{\tau(du)}{u - z}, \quad z \in \mathbb{C}^+, \tag{3.8}$$

where τ is a nonnegative measure such that $m_{2n-2}(\tau) < \infty$. Moreover

$$m_k(\mu) = \sum_{l=1}^{[k/2]} \sum_{s_1+\dots+s_l=k-2,\, s_j\geq 0} m_{s_1}(\tau)\dots m_{s_l}(\tau), \quad k = 2, \dots, 2n. \tag{3.9}$$

Voiculescu [35] showed for compactly supported probability measures that there exist unique functions $Z_1, Z_2 \in \mathcal{F}$ such that $G_{\mu_1\boxplus\mu_2}(z) = G_{\mu_1}(Z_1(z)) = G_{\mu_2}(Z_2(z))$ for all $z \in \mathbb{C}^+$. Using Speicher's combinatorial approach [30] to freeness, Biane [11] proved this result in the general case.

Bercovici and Belinschi [6], Belinschi [7], Chistyakov and Götze [15], proved, using complex analytic methods, that there exist unique functions $Z_1(z)$ and $Z_2(z)$ in the class $\mathcal{F}$ such that, for $z \in \mathbb{C}^+$,

$$z = Z_1(z) + Z_2(z) - F_{\mu_1}(Z_1(z)) \quad \text{and} \quad F_{\mu_1}(Z_1(z)) = F_{\mu_2}(Z_2(z)). \tag{3.10}$$

The function $F_{\mu_1}(Z_1(z))$ belongs again to the class $\mathcal{F}$ and there exists $\mu \in \mathcal{M}$ such that $F_{\mu_1}(Z_1(z)) = F_\mu(z)$, where $F_\mu(z) = 1/G_\mu(z)$ and $G_\mu(z)$ is the Cauchy transform as in (3.3). The measure μ depends on μ_1 and μ_2 only and $\mu = \mu_1 \boxplus \mu_2$.

Specializing to $\mu_1 = \mu_2 = \dots = \mu_n = \mu$ write $\mu_1 \boxplus \dots \boxplus \mu_n = \mu^{n\boxplus}$. The relation (3.10) admits the following consequence (see for example [15], Section 2, Corollary 2.3).

Proposition 3.3. *Let $\mu \in \mathcal{M}$. There exists a unique function $Z \in \mathcal{F}$ such that*

$$z = nZ(z) - (n-1)F_\mu(Z(z)), \quad z \in \mathbb{C}^+, \tag{3.11}$$

and $F_{\mu^{n\boxplus}}(z) = F_\mu(Z(z))$.

Using the representation (3.1) for $F_\mu(z)$ we obtain

$$F_\mu(z) = z + \Re F_\mu(i) + \int\limits_{\mathbb{R}} \frac{(1+uz)\,\tau(du)}{u-z}, \quad z \in \mathbb{C}^+, \tag{3.12}$$

where τ is a nonnegative measure such that $\tau(\mathbb{R}) = \Im F_\mu(i) - 1$. Denote $z = x + iy$, where $x, y \in \mathbb{R}$. We see that, for $\Im z > 0$,

$$\Im\Big(nz - (n-1)F_\mu(z)\Big) = y\Big(1 - (n-1)I_\mu(x,y)\Big), \quad \text{where} \quad I_\mu(x,y) := \int\limits_{\mathbb{R}} \frac{(1+u^2)\,\tau(du)}{(u-x)^2+y^2}.$$

For every real fixed x, consider the equation

$$y\Big(1 - (n-1)I_\mu(x,y)\Big) = 0, \quad y > 0. \tag{3.13}$$

Since $y \mapsto I_\mu(x,y)$, $y > 0$, is positive and monotone, and decreases to 0 as $y \to \infty$, it is clear that the equation (3.13) has at most one positive solution. If such a solution exists, denote it by $y_n(x)$. Note that (3.13) does not have a solution $y > 0$ for any given $x \in \mathbb{R}$ if and only if $I_\mu(x,0) \le 1/(n-1)$. Consider the set $S := \{x \in \mathbb{R} : I_\mu(x,0) \le 1/(n-1)\}$. We put $y_n(x) = 0$ for $x \in S$. We proved in [17], Section 3, p.13, that the curve γ_n given by the equation $z = x + iy_n(x)$, $x \in \mathbb{R}$, is continuous and simple.

Consider the open domain $\tilde{D}_n := \{z = x + iy,\ x, y \in \mathbb{R} : y > y_n(x)\}$.

Lemma 3.4. *Let $Z \in \mathcal{F}$ be the solution of the equation* (3.11). *The function $Z(z)$ maps $\mathbb{C}^+$ conformally onto $\tilde{D}_n$. Moreover the function $Z(z)$, $z \in \mathbb{C}^+$, is continuous up to the real axis and it establishes a homeomorphism between the real axis and the curve γ_n.*

This lemma was proved in [17] (see Lemma 3.4). The following lemma was proved as well in [17] (see Lemma 3.5).

Lemma 3.5. *Let μ be a probability measure such that $m_1(\mu) = 0, m_2(\mu) = 1$. Assume that $\int_{|u|>\sqrt{(n-1)/8}} u^2\,\mu(du) \le 1/10$ for some positive integer $n \ge 10^3$. Then the following inequality holds*

$$|Z(z)| \ge \sqrt{(n-1)/8}, \qquad z \in \mathbb{C}^+, \tag{3.14}$$

where $Z \in \mathcal{F}$ is the solution of the equation (3.11).

The next lemma was proved in [24] and [38].

Lemma 3.6. *There exists a unique probability measure ν such that such that $F_\mu(z) = z - G_\nu(z)$, $z \in \mathbb{C}^+$, and, for every $n \ge 1$, $F_{\mu_n}(z) = z - G_{\nu_{n-1}\boxplus w_t}(z)$, $z \in \mathbb{C}^+ \cup \mathbb{R}$, where the measure ν_{n-1} is given by $d\nu_{n-1}(x) = d\nu(\sqrt{n}x)$ and $t = t(n) = (n-1)/n$.*

Biane [11] gave the following bound.

Lemma 3.7. *Fix $t > 0$ and the probability measure ν. Then $|G_{\nu\boxplus w_t}(z)| \le t^{-1/2}$, $z \in \mathbb{C}^+ \cup \mathbb{R}$.*

4. Free Meixner measures

Consider the three-parameter family of probability measures $\{\mu_{a,b,d} : a \in \mathbb{R}, b < 1, d < 1\}$ with the reciprocal Cauchy transform

$$\frac{1}{G_{\mu_{a,b,d}}(z)} = a + \frac{1}{2}\Big((1+b)(z-a) + \sqrt{(1-b)^2(z-a)^2 - 4(1-d)}\Big), \quad z \in \mathbb{C}, \tag{4.1}$$

which we will call the free centered (i.e. with mean zero) Meixner measures. In this formula we choose the branch of the square root determined by the condition $\Im z > 0$ implies $\Im(1/G_{\mu_{a,b,d}}(z)) \geq 0$. These measures are counterparts of the classical measures discovered by Meixner [25]. The free Meixner type measures occurred in many places in the literature, see for example [14], [28].

Saitoh and Yoshida [28] have proved that the absolutely continuous part of the free Meixner measure $\mu_{a,b,d}, a \in \mathbb{R}, b < 1, d < 1$, is given by

$$\frac{\sqrt{4(1-d) - (1-b)^2(x-a)^2}}{2\pi f(x)}, \tag{4.2}$$

when $a - 2\sqrt{1-d}/(1-b) \leq x \leq a + 2\sqrt{1-d}/(1-b)$, where

$$f(x) := bx^2 + a(1-b)x + 1 - d;$$

Saitoh and Yoshida proved as well that for $0 \leq b < 1$ the (centered) free Meixner measure $\mu_{a,b,d}$ is $\boxplus$-infinitely divisible.

As we have shown in [17], Section 4, it follows from Saitoh and Yoshida's results that the probability measure μ_{a_n,b_n,d_n} with the parameters a_n, b_n, d_n from (2.3) is $\boxplus$-infinitely divisible and it is absolutely continuous with a density of the form (4.2) where $a = a_n, b = b_n, d = d_n$ for sufficiently large $n \geq n_1(\mu)$.

5. Passage to measures with bounded supports

Let us assume that $\mu \in \mathcal{M}$ and $m_4(\mu) < \infty$. In addition let $m_1(\mu) = 0$ and $m_2(\mu) = 1$. By Proposition 3.3, there exists $Z(z) \in \mathcal{F}$ such that (3.11) holds, and $F_{\mu^{n\boxplus}}(z) = F_\mu(Z(z))$. Hence $F_{\mu_n}(z) = F_\mu(\sqrt{n}S_n(z))/\sqrt{n}$, $z \in \mathbb{C}^+$, where $S_n(z) := Z(\sqrt{n}z)/\sqrt{n}$. Since $m_1(\mu) = 0$, $m_2(\mu) = 1$ and $m_4(\mu) < \infty$, by Proposition 3.2, we have the representation

$$F_\mu(z) = z + \int_{\mathbb{R}} \frac{\tau(du)}{u-z}, \quad z \in \mathbb{C}^+, \tag{5.1}$$

where τ is a nonnegative measure such that $\tau(\mathbb{R}) = 1$ and $m_2(\tau) < \infty$.

Denote, for $n \in \mathbb{N}$,

$$\eta(n;\tau) := inf_{0<\varepsilon\leq 10^{-1/2}} g_n(\varepsilon;\tau), \quad \text{where} \quad g_n(\varepsilon;\tau) = \varepsilon + \frac{1}{m_2(\tau)\varepsilon^2}\int_{|u|>\varepsilon\sqrt{n-1}} u^2\,\tau(du).$$

It is easy to see that $0 < \eta(n;\tau) \leq 11$ and $\eta(n;\tau) \to 0$ monotonically as $n \to \infty$. Let $\delta_n \in (0, 10^{-1/2}]$ be a point at which the infimum of the function $g_n(\varepsilon;\tau)$ is attained. This means that

$$\eta(n;\tau) = \delta_n + \frac{1}{m_2(\tau)\delta_n^2}\int_{|u|>\delta_n\sqrt{n-1}} u^2\,\tau(du). \tag{5.2}$$

Consider a function

$$F(z) = z + \int_{\mathbb{R}} \frac{\tau^*(du)}{u-z} := \int_{|u|\le\delta_n\sqrt{n-1}} \frac{\tau(du)}{u-z}, \quad z\in\mathbb{C}^+. \tag{5.3}$$

This function belongs to the class $\mathcal{F}$ and therefore there exists the probability measure μ^* such that $F_{\mu^*}(z) = F(z)$, $z\in\mathbb{R}$. The probability measure μ^* of course depends on n. Moreover we conclude from the inversion formula that $\mu^*([-\frac{\sqrt{10}}{3}\delta_n\sqrt{n-1}, \frac{\sqrt{10}}{3}\delta_n\sqrt{n-1}]) = 1$ for $n\ge n_1(\mu)$. Hence it follows that the support of μ^* is contained in the interval $[-\frac{1}{3}\sqrt{n-1}, \frac{1}{3}\sqrt{n-1}]$. By Proposition 3.2, we see as well that $m_1(\mu^*)=0$ and

$$\begin{aligned} m_2(\mu) - m_2(\mu^*) &= \tau(\mathbb{R}\setminus[-\delta_n\sqrt{n-1}, \delta_n\sqrt{n-1}]) \\ &\le \frac{1}{\delta_n^2(n-1)}\int_{|u|>\delta_n\sqrt{n-1}} u^2\,\tau(du) \le c(\mu)\frac{\eta(n;\tau)}{n-1}. \end{aligned} \tag{5.4}$$

Moreover

$$\begin{aligned} |m_3(\mu) - m_3(\mu^*)| &= |m_1(\tau)m_0(\tau) - m_1(\tau^*)m_0(\tau^*)| \\ &\le |m_1(\tau) - m_1(\tau^*)| + |m_2(\mu) - m_2(\mu^*)||m_1(\tau^*)| \\ &\le \int_{|u|>\delta_n\sqrt{n-1}} |u|\,\tau(du) + c(\mu)|m_1(\tau^*)|\frac{\eta(n;\tau)}{n-1} \le c(\mu)\frac{\eta(n;\tau)}{\sqrt{n-1}}; \end{aligned} \tag{5.5}$$

In the same way

$$\begin{aligned} |m_4(\mu) - m_4(\mu^*)| &\le |m_2(\tau) - m_2(\tau^*)| + |m_2(\mu) - m_2(\mu^*)||m_2(\tau^*)| \\ &\quad + |m_1(\tau) - m_1(\tau^*)||m_1(\tau) + m_1(\tau^*)| \le c(\mu)\eta(n;\tau). \end{aligned} \tag{5.6}$$

Here $m_k(\tau^*)$, $k=0,1,2$, denote moments of the measure τ^*.

Let $X^*, X_1^*, X_2^*, \dots$ be free identically distributed random variables such that $\mathcal{L}(X^*) = \mu^*$. Denote $\mu_n^* := \mathcal{L}((X_1^*+\dots+X_n^*)/\sqrt{n})$. As before, by Proposition 3.3, there exists $W(z)\in\mathcal{F}$ such that (3.11) holds with $Z=W$ and $\mu=\mu^*$, and $F_{(\mu^*)^{n\boxplus}}(z) = F_{\mu^*}(W(z))$. Hence $F_{\mu_n^*}(z) = F_{\mu^*}(\sqrt{n}T_n(z))/\sqrt{n}$, $z\in\mathbb{C}^+$, where $T_n(z) := W(\sqrt{n}z)/\sqrt{n}$. In the sequel we shall need more detailed information about the behaviour of the functions $T_n(z)$ and $S_n(z)$. By Lemma 3.4, these functions are continuous up to the real axis for $n\ge n_1(\mu)$. Their values for $z=x\in\mathbb{R}$ we denote by $T_n(x)$ and $S_n(x)$, respectively. In order to formulate the following results for $T_n(z)$ we introduce some notations. Denote by $M_n(z)$ the reciprocal Cauchy transform of the free Meixner measure μ_{a_n,b_n,d_n} with the parameters a_n, b_n and d_n from (2.3), i.e.,

$$M_n(z) := a_n + \frac{1}{2}\Big(\big(1+b_n\big)(z-a_n) + \sqrt{\big(1-b_n\big)^2(z-a_n)^2 - 4\big(1-d_n\big)}\Big), \quad z\in\mathbb{C}^+.$$

Denote by D_n the rectangle

$$D_n = \Big\{z \in \mathbb{C} : 0 < \Im z \le 3, |\Re z - a_n| \le \frac{2}{e_n} - \frac{\varepsilon_{n1}}{n}\Big\},$$

where $\varepsilon_{n1} := c_1(\mu)(\eta(n;\tau) + 1/\sqrt{n})$ and $c_1(\mu) > 0$ is sufficiently large. In the sequel we assume that ε_{n1} is always of this form.

Repeating step by the step the arguments of Section 7 (see Subsections 7.2–7.7) of our paper [19] we establish the following result.

Theorem 5.1. *Let $\mu \in \mathcal{M}$ such that $m_4(\mu) < \infty$ and $m_1(\mu) = 0$, $m_2(\mu) = 1$. Then there exists a constant $c(\mu)$ such that the following relation holds, for $z \in D_n$ and $n \ge n_1(\mu)$,*

$$T_n(z) = M_n(z) + \frac{\varepsilon_{n1}}{n}\frac{c(\mu)\,\theta}{\sqrt{(e_n(z - a_n))^2 - 4}}, \tag{5.7}$$

$$G_{\mu_n^*}(z) = \frac{1}{T_n(z)} + \frac{m_2(\mu^*)}{nT_n(z)^3} + \frac{c(\mu)\,\theta}{n^{3/2}}. \tag{5.8}$$

In addition

$$0 \le \Im T_n(x) \le c(\mu)\sqrt{\frac{\varepsilon_{n1}}{n}}, \quad \frac{2}{e_n} - \frac{\varepsilon_{n1}}{n} < |x - a_n| \le 3. \tag{5.9}$$

In (5.7) and (5.8) θ is a complex-valued quantities such that $|\theta| \le 1$.

Here and in the sequel constants $c(\mu) > 0$ do not depend on the constant $c_1(\mu)$.

In Section 7 of this paper we shall give a more detailed exposition of the proof of this theorem.

By Lemmas 3.4, 3.5, $|T_n(z)| \ge 1.03/3$ for $z \in \mathbb{C}^+ \cup \mathbb{R}$ and for $n \ge n_1(\mu)$. It is obvious that the same estimate holds for $M_n(z)$. Since

$$G_{\mu_n^*}(z) = \sqrt{n}G_{\mu^*}(\sqrt{n}T_n(z)) = \int\limits_{[-\sqrt{n-1}/3,\sqrt{n-1}/3]} \frac{\mu^*(du)}{T_n(z) - u/\sqrt{n}}, \qquad z \in \mathbb{C}^+, \tag{5.10}$$

we conclude that $G_{\mu_n^*}(z)$ is a continuous function up to the real axis. Denote its value for real x by $G_{\mu_n^*}(x)$. Denote $G_{\hat\mu_n}(z) := 1/T_n(z)$, $z \in \mathbb{C}^+$. This function is continuous up to the real axis as well. Therefore $\hat\mu_n$ and μ_n^* are absolutely continuous measures with continuous densities $\hat p_n(x)$ and $p_n^*(x)$, respectively,

$$\hat p_n(x) = -\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\Im\frac{1}{T_n(x + i\varepsilon)} = -\frac{1}{\pi}\Im\frac{1}{T_n(x)},$$
$$p_n^*(x) = -\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\Im G_{\mu_n^*}(x + i\varepsilon) = -\frac{1}{\pi}\Im G_{\mu_n^*}(x).$$

In addition, $\hat p_n(x) \le 1$ and $p_n^*(x) \le 50$ for all $x \in \mathbb{R}$ and $n \ge n_1(\mu)$.

Theorem 5.2. *Let $\mu \in \mathcal{M}$ such that $m_4(\mu) < \infty$ and $m_1(\mu) = 0$, $m_2(\mu) = 1$. Then, for $x \in I_n := \{x \in \mathbb{R} : |x - a_n| \le \frac{2}{e_n} - \frac{\varepsilon_{n1}}{n}\}$ and $n \ge n_1(\mu)$, the following relation holds*

$$p_n^*(x) = v_n(x - a_n) + \frac{\varepsilon_{n1}}{n} \frac{c(\mu)\,\theta}{\sqrt{4 - (e_n(x - a_n))^2}}, \tag{5.11}$$

where $v_n(x)$ is defined in (2.5).

Proof. We shall use the following estimate, for $x \in \mathbb{R}$,

$$|p_n^*(x) - p_{\mu_{a_n,b_n,d_n}}(x) - \frac{1}{n} q_n(x)| \le |\hat{p}_n(x) - p_{\mu_{a_n,b_n,d_n}} x)| + |p_n^*(x) - \hat{p}_n(x) - \frac{1}{n}\hat{q}_n(x)| \\ + \frac{1}{n}|q_n(x) - \hat{q}_n(x)|, \tag{5.12}$$

where

$$q_n(x) := -\frac{1}{\pi}\Im\frac{1}{M_n(x)^3}, \quad \text{and} \quad \hat{q}_n(x) := -\frac{1}{\pi}\Im\frac{1}{T_n(x)^3}.$$

By (5.7) and by the lower bounds

$$|T_n(x)| \ge 1.03/3, \qquad |M_n(x)| \ge 1.03, \quad x \in \mathbb{R}, \tag{5.13}$$

we easily obtain the upper bound, for $x \in I_n$ and $n \ge n_1(\mu)$,

$$|\hat{p}_n(x) - p_{\mu_{a_n,b_n,d_n}} x)| \le \frac{\varepsilon_{n1}}{n} \frac{c(\mu)}{\sqrt{4 - (e_n(x - a_n))^2}} \tag{5.14}$$

and, by (5.8), we have

$$|p_n^*(x) - \hat{p}_n(x) - \frac{1}{n}\hat{q}_n(x)| \le \frac{c(\mu)}{n^{3/2}}, \quad x \in I_n, \quad n \ge n_1(\mu). \tag{5.15}$$

Since, by (4.2),

$$p_{\mu_{a_n,b_n,d_n}}(x) := \frac{\sqrt{4(1 - d_n) - (1 - b_n)^2(x - a_n)^2}}{2\pi(b_n x^2 + a_n(1 - b_n)x + 1 - d_n)}, \quad x \in \tilde{I}_n := [a_n - 2/e_n, a_n + 2/e_n], \tag{5.16}$$

we easily conclude that

$$p_{\mu_{a_n,b_n,d_n}}(x) = \Big(1 + \frac{d_n}{2} - a_n^2 - a_n(x - a_n) - (b_n - a_n^2)(x - a_n)^2\Big) p_w(e_n(x - a_n)) + \frac{c(\mu)\theta}{n^{3/2}} \tag{5.17}$$

for $x \in \tilde{I}_n$. Using again (5.7) and (5.13), we obtain

$$|q_n(x) - \hat{q}_n(x)| \le \frac{1}{\pi}|T_n(x) - M_n(x)|\Big(\frac{1}{|M_n(x)T_n(x)^3|} + \frac{1}{|M_n(x)^2 T_n(x)^2|} + \frac{1}{|M_n(x)^3 T_n(x)|}\Big) \\ \le \frac{\varepsilon_{n1}}{n} \frac{c(\mu)}{\sqrt{4 - (e_n(x - a_n))^2}}, \quad x \in I_n. \tag{5.18}$$

On the other hand it is not difficult to show that

$$q_n(x) := \frac{1}{8\pi}\sqrt{(4(1-d_n)-(1-b_n)^2(x-a_n)^2)_+}$$
$$\times \frac{3((1+b_n)x+(1-b_n)a_n)^2+(1-b_n)^2(x-a_n)^2-4(1-d_n)}{(b_nx^2+(1-b_n)a_nx+1-d_n)^3}, \quad x\in\mathbb{R},$$

which leads to the relation

$$q_n(x) = ((x-a_n)^2-1)p_w(e_n(x-a_n)) + c(\mu)\theta(|a_n|+n^{-1}) \tag{5.19}$$

for $x \in \tilde{I}_n$.

Applying (5.14), (5.15), (5.18) and (5.17), (5.19) to (5.12) we arrive at the statement of the theorem. □

6. Local asymptotic expansion

First we prove the auxiliary result.

Theorem 6.1. *Let $\mu \in \mathcal{M}$ such that $m_4(\mu) < \infty$ and $m_1(\mu) = 0$, $m_2(\mu) = 1$. Then the following relation holds*

$$p_n(x) = p_n^*(x) + \tilde{\rho}_{n1}(x) + \tilde{\rho}_{n2}(x), \quad x\in\mathbb{R},\ n\ge n_1(\mu), \tag{6.1}$$

where

$$|\tilde{\rho}_{n1}(x)| \le c(\mu)\big(|\Im(S_n(x)-T_n(x))| + \Im T_n(x)|S_n(x)-T_n(x)| + n^{-2}\big)$$

and $\tilde{\rho}_{n2}(x)$ is a continuous function such that

$$0 \le \tilde{\rho}_{n2}(x) \le c(\mu) \quad \text{and} \quad \int_{\mathbb{R}} \tilde{\rho}_{n2}(x)\,dx = o\Big(\frac{1}{n^2}\Big).$$

Proof. Represent the density $p_n(x)$ of the measure μ_n in the form

$$p_n(x) = p_{n1}(x) + p_{n2}(x), \quad x\in\mathbb{R}, \tag{6.2}$$

where $p_{nj}(x) \ge 0$, $x\in\mathbb{R}$, $j=1,2$, and, for $z\in\mathbb{C}^+$,

$$I_1(z) := \int_{|u|\le\sqrt{n-1}/3} \frac{\mu(du)}{S_n(z)-u/\sqrt{n}} = \int_{\mathbb{R}} \frac{p_{n1}(u)\,du}{z-u},$$
$$I_2(z) := \int_{|u|>\sqrt{n-1}/3} \frac{\mu(du)}{S_n(z)-u/\sqrt{n}} = \int_{\mathbb{R}} \frac{p_{n2}(u)\,du}{z-u}.$$

Since $\lim_{y\to+\infty} iyI_2(iy) = \int_{|u|>\sqrt{n-1}/3}\mu(du) = \int_{\mathbb{R}} p_{n2}(u)\,du$, we note that

$$\int_{\mathbb{R}} p_{n2}(u)\,du = o(n^{-2}). \tag{6.3}$$

Since $S_n(x)$, $x \in \mathbb{R}$, is a continuous function and $|S_n(x)| \ge 1.03/3$ for all $x \in \mathbb{R}$, we easily see that, for $x \in \mathbb{R}$,

$$p_{n1}(x) = -\frac{1}{\pi}\Im \int_{|u|\le\sqrt{n-1}/3} \frac{\mu(du)}{S_n(x) - u/\sqrt{n}}$$

and that $p_{n1}(x)$ is a continuous function on the real line. In view of (2.2), $p_n(x)$ is a continuous function on the real line and $p_n(x) \le c(\mu)$, $x \in \mathbb{R}$, for $n \ge n_1(\mu)$. Therefore we conclude from (6.2) that $p_{n2}(x)$ is a continuous function on the real line and $p_{n2}(x) \le c(\mu)$, $x \in \mathbb{R}$, for the same n.

Now we may write

$$\begin{aligned} \pi(p_n^*(x) - p_{n1}(x)) &= I_{3,1}(x) + I_{3,2}(x) \\ &:= \Im\Big(\int_{|u|\le\sqrt{n-1}/3} \frac{\mu(du)}{S_n(x) - u/\sqrt{n}} - \int_{|u|\le\sqrt{n-1}/3} \frac{\mu(du)}{T_n(x) - u/\sqrt{n}}\Big) \\ &+ \Im\int_{|u|\le\sqrt{n-1}/3} \frac{(\mu-\mu^*)(du)}{T_n(x) - u/\sqrt{n}}. \end{aligned} \tag{6.4}$$

Since

$$\begin{aligned} \frac{\Im S_n(x)}{|S_n(x) - u/\sqrt{n}|^2} - \frac{\Im T_n(x)}{|T_n(x) - u/\sqrt{n}|^2} &= \frac{\Im S_n(x) - \Im T_n(x)}{|S_n(x) - u/\sqrt{n}|^2} \\ &+ \Im T_n(x)\Big(\frac{1}{|S_n(x) - u/\sqrt{n}|^2} - \frac{1}{|T_n(x) - u/\sqrt{n}|^2}\Big), \end{aligned}$$

we have

$$\begin{aligned} I_{3,1}(x) &= (\Im S_n(x) - \Im T_n(x)) \int\limits_{|u|\le\sqrt{n-1}/3} \frac{\mu(du)}{|S_n(x) - u/\sqrt{n}|^2} \\ &+ \Im T_n(x)\Re(T_n(x) - S_n(x)) \int\limits_{|u|\le\sqrt{n-1}/3} \frac{(\Re S_n(x) + \Re T_n(x) - 2u/\sqrt{n})\,\mu(du)}{|S_n(x) - u/\sqrt{n}|^2|T_n(x) - u/\sqrt{n}|^2} \\ &+ \Im T_n(x)\Im(T_n(x) - S_n(x)) \int\limits_{|u|\le\sqrt{n-1}/3} \frac{(\Im S_n(x) + \Im T_n(x)))\,\mu(du)}{|S_n(x) - u/\sqrt{n}|^2|T_n(x) - u/\sqrt{n}|^2}. \end{aligned}$$

By the inequalities $|S_n(x) - u/\sqrt{n}| \ge 0.01$, $|T_n(x) - u/\sqrt{n}| \ge 0.01$ for $x \in \mathbb{R}$ and $|u| \le \sqrt{n-1}/3$, we conclude that, for $x \in \mathbb{R}$,

$$|I_{3,1}(x)| \le c(\mu)|\Im S_n(x) - \Im T_n(x)| + c(\mu)\Im T_n(x)|S_n(x) - T_n(x)|. \tag{6.5}$$

On the other hand we note that, for $x \in \mathbb{R}$,

$$I_{3,2}(x) = \Im\Big(\frac{m_2(\mu)-m_2(\mu^*)}{T_n^3(x)n} + \frac{m_3(\mu)-m_3(\mu^*)}{T_n^4(x)n^{3/2}} - \sum_{j=0}^{3}\frac{1}{T_n(x)^{j+1}n^{j/2}}\int_{|u|>\sqrt{n-1}/3} u^j\,\mu(du) + \frac{1}{T_n^4(x)n^2}\int\limits_{|u|\le\sqrt{n-1}/3}\frac{u^4(\mu-\mu^*)(du)}{T_n(x)-u/\sqrt{n}}\Big).$$

Using (5.4)–(5.6) and the inequality

$$\int_{|u|>\sqrt{n-1}/3}|u|^j\,\mu(du) \le c(\mu)n^{-(4-j)/2}, \quad j=0,1,2,3,4,$$

we obtain the estimate

$$|I_{3,2}(x)| \le \frac{c(\mu)}{n^2}, \quad x\in\mathbb{R}. \tag{6.6}$$

Applying (6.5) and (6.6) to (6.4), we have, for $x \in \mathbb{R}$,

$$|p_{n1}(x)-p_n^*(x)| \le c(\mu)\big(|\Im S_n(x)-\Im T_n(x)| + \Im T_n(x)|S_n(x)-T_n(x)| + n^{-2}\big). \tag{6.7}$$

The representation (6.1) follows immediately from (6.2) if to define $\tilde\rho_{n1}(x) = p_{n1}(x)-p_n^*(x)$ and $\tilde\rho_{n2}(x) = p_{n2}(x)$ and from the bounds (6.7) and (6.3). □

7. Proof of Theorem 5.1

In this section we show how the arguments of Section 7 in [19] lead to a proof of Theorem 5.1.

Repeating the arguments of Subsection 7.2 we deduce that $T_n(z)$ satisfies the functional equation, for $z \in \mathbb{C}^+$,

$$T_n^5(z) - zT_n^4(z) + m_2(\mu^*)T_n^3(z) + \frac{\zeta_{n2}(z)}{\sqrt{n}}T_n^2(z) + \frac{\zeta_{n3}(z)}{n}T_n(z) - \frac{\zeta_{n4}(z)z}{n^2} = 0, \tag{7.1}$$

where $\zeta_{n1}(z) := \int_{\mathbb{R}}\frac{u^5\,\mu^*(du)}{W(\sqrt{n}z)-u}$, $\zeta_{n2}(z) := m_3(\mu^*) - z/\sqrt{n}$, $\zeta_{n3}(z)(z) := m_4(\mu^*)+\zeta_{n1}(z) - zm_3(\mu^*)/\sqrt{n}$ and $\zeta_{n4}(z)(z) := m_4(\mu^*)+\zeta_{n1}(z)$. As in Subsection 7.3 from [19] we obtain estimates for the functions $\zeta_{nj}(z)$, $j=1,2,3,4$, in the domain $D^* := \{|\Re z| \le 4,\ 0 < \Im z \le 3\}$

$$|\zeta_{n1}(z)| \le c(\mu)\delta_n \le c(\mu)\eta(n;\tau), \quad \sum_{j=2}^{4}|\zeta_{nj}(z)| \le c(\mu). \tag{7.2}$$

For every fixed $z\in\mathbb{C}^+$ consider the equation

$$Q(z,w) := w^5 - zw^4 + m_2(\mu^*)w^3 + \frac{\zeta_{n2}(z)}{\sqrt{n}}w^2 + \frac{\zeta_{n3}(z)}{n}w - \frac{\zeta_{n4}(z)z}{n^2} = 0. \tag{7.3}$$

Denote the roots of the equation (7.3) by $w_j = w_j(z)$, $j=1,\dots,5$.

As in Subsection 7.4 [19] we can show that for every fixed $z \in D^*$ the equation $Q(z,w)=0$ has three roots, say $w_j = w_j(z)$, $j=1,2,3$, such that

$$|w_j| < r' := c_2(\mu) n^{-1/2}, \quad j=1,2,3, \tag{7.4}$$

and two roots, say w_j, $j=4,5$, such that $|w_j| \ge r'$ for $j=4,5$.

Represent $Q(z,w)$ in the form

$$Q(z,w) = (w^2 + s_1 w + s_2)(w^3 + g_1 w^2 + g_2 w + g_3),$$

where $w^3 + g_1 w^2 + g_2 w + g_3 = (w-w_1)(w-w_2)(w-w_3)$. From this formula we derive the relations

$$s_1 + g_1 = -z, \quad s_2 + s_1 g_1 + g_2 = m_2(\mu^*), \quad s_2 g_1 + s_1 g_2 + g_3 = \frac{\zeta_{n2}(z)}{\sqrt{n}},$$

$$s_2 g_2 + s_1 g_3 = \frac{\zeta_{n3}(z)}{n}, \quad s_2 g_3 = -\frac{\zeta_{n4}(z) z}{n^2}. \tag{7.5}$$

By Vieta's formulae and (7.4), note that

$$|g_1| \le 3r', \quad |g_2| \le 3(r')^2, \quad |g_3| \le (r')^3. \tag{7.6}$$

Now we obtain from (7.5) and (7.6) the following bounds, for $z \in D^*$,

$$|s_1| \le 5 + 3r', \quad |m_2(\mu^*) - s_2| \le 3r'(4r'+5) \le 16r' \le \frac{1}{2}. \tag{7.7}$$

Then we conclude from (5.4), (7.2), (7.5)–(7.7) that, for the same z,

$$\Big| g_2 - \frac{\zeta_{n4}(z)}{n} \Big| \le \Big| g_2 - \frac{\zeta_{n3}(z)}{n} \Big| + \frac{|m_3(\mu^*)||z|}{n^{3/2}} \le \frac{|s_1|}{|s_2|}|g_3| + \frac{|s_2 - m_2(\mu^*)|}{|s_2|} \frac{|\zeta_{n3}(z)|}{n} + (r')^3$$
$$\le 11(r')^3 + 8(r')^3 + (r')^3 = 20(r')^3 \le c(\mu) n^{-3/2}. \tag{7.8}$$

Now repeating the arguments of Subsection 7.4 we deduce the inequality

$$|g_1 - a_n - b_n z| \le c(\mu) \frac{\eta(n;\tau)}{n}, \quad z \in D^*. \tag{7.9}$$

To find the roots w_4 and w_5, we need to solve the equation $w^2 + s_1 w + s_2 = 0$. Using (7.5), we have, for $j=4,5$,

$$\begin{aligned} w_j &= \frac{1}{2}\Big(-s_1 + (-1)^j \sqrt{s_1^2 - 4s_2} \Big) \\ &= \frac{1}{2}\Big(z + g_1 + (-1)^j \sqrt{(z+g_1)^2 - 4(m_2(\mu^*) + (z+g_1)g_1 - g_2)} \Big) \\ &= \frac{1}{2}\Big(z + g_1 + (-1)^j \sqrt{(z-g_1)^2 - 4m_2(\mu^*) - 4(g_1^2 - g_2)} \Big) = \frac{1}{2} r_{n1}(z) + a_n + \\ &+ \frac{1}{2}\Big(\big(1+b_n\big)(z-a_n) + (-1)^j \sqrt{\big(1-b_n\big)^2 (z-a_n)^2 - 4\big(1-d_n\big) + r_{n2}(z)} \Big), \end{aligned} \tag{7.10}$$

where

$$\begin{aligned} r_{n1}(z) &:= g_1 - a_n - b_n(z-a_n), \\ r_{n2}(z) &:= -3r_{n1}^2(z) - 2r_{n1}(z)(4a_n + (1+3b_n)(z-a_n)) + 4(1-m_2(\mu^*)) \\ &\quad + 4(g_2 - m_4(\mu^*)/n) - 4b_n(z-a_n)(2a_n + b_n(z-a_n)). \end{aligned}$$

The quantities $r_{n1}(z)$ and $r_{n2}(z)$ admit the bound (see Subsection 7.4 and 7.5 from [19])

$$|r_{n1}(z)| + |r_{n2}(z)| \le c(\mu)\frac{\eta(n;\tau)}{n}, \quad z \in D^*. \tag{7.11}$$

We choose the branch of the analytic square root according to the condition $\Im w_4(i) \ge 0$.

As in Subsection 7.6 from [19] we prove that $w_4(z) = T_n(z)$ for $z \in D_n$, where the constant $c(\mu)$ in (7.11) does not depend on the constant $c_1(\mu)$. Since the constant $c_1(\mu)$ is sufficiently large, we have, by (7.11),

$$|r_{n2}(z)|/|((1-b_n)^2(z-a_n)^2 - 4(1-d_n)| \le 10^{-2}, \quad z \in D_n. \tag{7.12}$$

For $z \in D_n$, using formula (7.10) with $j=4$ for $T_n(z)$, we write

$$\begin{aligned} M_n(z) - T_n(z) &= -\frac{1}{2}r_{n1}(z) \\ &- \frac{1}{2}\frac{r_{n2}(z)}{\sqrt{(1-b_n)^2(z-a_n)^2 - 4(1-d_n)} + \sqrt{(1-b_n)^2(z-a_n)^2 - 4(1-d_n) + r_{n2}(z)}}. \end{aligned} \tag{7.13}$$

Using (7.12) and the power expansion for the function $(1+z)^{1/2}$, $|z|<1$, we easily rewrite (7.13) in the form

$$M_n(z) - T_n(z) = \frac{r_{n3}(z)}{\sqrt{(1-b_n)^2(z-a_n)^2 - 4(1-d_n)}}, \quad z \in D_n, \tag{7.14}$$

where $|r_{n3}(z)| \le c(\mu)\eta(n;\tau)/n$. The relation (5.7) immediately follows from (7.14).

Using (5.10), we conclude that

$$G_{\mu_n^*}(z) = \frac{1}{T_n(z)} + \frac{m_2(\mu^*)}{nT_n^3(z)} + \frac{1}{n^{3/2}T_n^3(z)}\int_{[-\sqrt{n-1}/3,\sqrt{n-1}/3]}\frac{u^3\,\mu^*(du)}{T_n(z)-u/\sqrt{n}}, \quad z \in \mathbb{C}^+. \tag{7.15}$$

Since $|T_n(z)| \ge 1.03/3$ for $z \in \mathbb{C}^+$ and $n \ge n_1(\mu)$, we arrive at (5.8).

The function $T_n(x)$ for real x such that $\frac{2}{e_n} - \frac{\varepsilon_{n1}}{n} \le |x-a_n| \le 3$ coincide with $w_3(x)$ or $w_4(x)$ from (7.10). Here we understand $w_j(x)$ as limit values of $w_j(z)$ where $z \in D^*$ and $z \to x$. It is not difficult to conclude from the formula (7.10) that $0 \le \Im T_n(x) \le c(\mu)\sqrt{\frac{\varepsilon_{n1}}{n}}$ for $\frac{2}{e_n} - \frac{\varepsilon_{n1}}{n} \le |x-a_n| \le 3$.

8. Proof of Theorem 2.1

Recalling the definition of the function $Z(z)$, we see, by Lemma 3.4, that the function $S_n(z)$ maps $\mathbb{C}^+$ conformally onto $\hat{D}_n$, where $\hat{D}_n := \{z = x+iy, x,y \in \mathbb{R} : y > y_n(x\sqrt{n})/\sqrt{n}\}$. Denote by $\hat{\gamma}_n$ a curve given by the equation $z = x + i\hat{y}_n(x)$, where $\hat{y}_n(x) = y_n(x\sqrt{n})/\sqrt{n}$. The function $S_n(z)$ is continuous up to the real axis and it establishes a homeomorphism between the real axis and the curve $\hat{\gamma}_n$.

Note as well that the function $T_n(z)$ maps $\mathbb{C}^+$ conformally onto $\hat{D}_n^*$, where $\hat{D}_n^* := \{z = x+iy, x,y \in \mathbb{R} : y > y_n^*(x\sqrt{n})/\sqrt{n}\}$. Here $y_n^*(x)$ is defined in the same way as $y_n(x)$ if we change the measure μ by μ^*. By definition of the measure μ^*, we see that $y_n^*(x) \le y_n(x)$. Hence $S_n(\mathbb{C}^+) \subseteq T_n(\mathbb{C}^+)$. Denote by $\hat{\gamma}_n^*$ a curve given by the equation $z = x + i\hat{y}_n^*(x)$, $x \in \mathbb{R}$, where $\hat{y}_n^*(x) = y_n^*(x\sqrt{n})/\sqrt{n}$. The function $T_n(z)$ is continuous up to the real axis and it establishes a homeomorphism between the real axis and the curve $\hat{\gamma}_n^*$.

Let $x \in \mathbb{R}$. Since $S_n(z)$ and $T_n(z)$ are the conformal maps $\mathbb{C}^+$ on $\hat{D}_n$ and $\hat{D}_n^*$, respectively, which are continuous up to the real axis, we note that the functions $\Re S_n(x)$ and $\Re T_n(x)$ are monotonically increasing. Hence for every $x \in \mathbb{R}$ there exists unique $\tilde{x} \in \mathbb{R}$ such that $\Re S_n(x) = \Re T_n(\tilde{x})$. Denote $h(x) := \Im S_n(x) - \Im T_n(\tilde{x}) \ge 0$.

In order to prove Theorem 2.1 we need the following auxiliary results. In the sequel we assume that $n \ge n_1(\mu) \ge 10$.

Proposition 8.1. *For $z_1, z_2 \in T_n(\mathbb{C}^+)$,*

$$|T_n^{(-1)}(z_1) - T_n^{(-1)}(z_2)| \le c(\mu)|z_1 - z_2|.$$

Proof. Using the formula

$$T_n^{(-1)}(z) = nz - \frac{n-1}{\sqrt{n}} F_{\mu^*}(z\sqrt{n}) = z - \frac{n-1}{\sqrt{n}} \int_{-\delta_n\sqrt{n-1}}^{\delta_n\sqrt{n-1}} \frac{\tau(du)}{u - z\sqrt{n}}, \quad z \in \mathbb{C}^+, \tag{8.1}$$

we have the relation, for $z_1, z_2 \in \mathbb{C}^+$,

$$T_n^{(-1)}(z_1) - T_n^{(-1)}(z_2) = (z_1 - z_2)\Big(1 - (n-1)\int_{-\delta_n\sqrt{n-1}}^{\delta_n\sqrt{n-1}} \frac{\tau(du)}{(u - z_1\sqrt{n})(u - z_2\sqrt{n})}\Big).$$

Since, by Lemma 3.5, $|u - z\sqrt{n}| \ge 10^{-2}\sqrt{n}$ for $|u| \le \sqrt{n-1}/\pi$ and $z \in T_n(\mathbb{C}^+)$, we immediately arrive at the assertion of the proposition. □

Proposition 8.2. *For $z \in S_n(\mathbb{C}^+)$,*

$$|T_n^{(-1)}(z) - S_n^{(-1)}(z)| \le \frac{\tau(\{|u| > \delta_n\sqrt{n-1}\})}{\Im z}.$$

Proof. Using (8.1) and the formula

$$S_n^{(-1)}(z) = nz - \frac{n-1}{\sqrt{n}} F_{\mu}(z\sqrt{n}) = z - \frac{n-1}{\sqrt{n}} \int_{\mathbb{R}} \frac{\tau(du)}{u - z\sqrt{n}}, \quad z \in \mathbb{C}^+, \tag{8.2}$$

we have, taking into account that $S_n(\mathbb{C}^+) \subseteq T_n(\mathbb{C}^+)$,

$$|S_n^{(-1)}(z) - T_n^{(-1)}(z)| = \frac{n-1}{\sqrt{n}} \Big| \int_{|u|>\delta_n\sqrt{n-1}} \frac{\tau(du)}{u - z\sqrt{n}} \Big| \le \frac{\tau(\{|u| > \delta_n\sqrt{n-1}\})}{\Im z}$$

for $z \in S_n(\mathbb{C}^+)$, proving the proposition. □

Proposition 8.3. *For $x_1, x_2 \in I_n$, we have the estimate*

$$|T_n(x_1) - T_n(x_2)| \le c(\mu) \frac{|x_1 - x_2| + \varepsilon_{n1}/n}{\min_{j=1,2}\{\sqrt{4 - (e_n(x_j - a_n))^2}\}}.$$

Proof. By Theorem 5.1, we have the following relation

$$T_n(x_1) - T_n(x_2) = M_n(x_1) - M_n(x_2) + \frac{\varepsilon_{n1}}{n} \frac{c(\mu)\,\theta}{\sqrt{(e_n(x_1 - a_n))^2 - 4}} - \frac{\varepsilon_{n1}}{n} \frac{c(\mu)\,\theta}{\sqrt{(e_n(x_2 - a_n))^2 - 4}}, \tag{8.3}$$

where $x_1, x_2 \in I_n$. On the other hand it is easy to see that

$$M_n(x_1) - M_n(x_2) = \frac{(1+b_n)(x_1 - x_2)}{2} + \frac{1}{2} \frac{(1-b_n)^2(x_1 - x_2)(x_1 + x_2 - 2a_n)}{\sqrt{(1-b_n)^2(x_1 - a_n)^2 - 4(1-d_n)} + \sqrt{(1-b_n)^2(x_2 - a_n)^2 - 4(1-d_n)}}. \tag{8.4}$$

Moreover, we have, for $x_1, x_2 \in I_n$,

$$\Big|\sqrt{(1-b_n)^2(x_1 - a_n)^2 - 4(1-d_n)} + \sqrt{(1-b_n)^2(x_2 - a_n)^2 - 4(1-d_n)}\Big| = (|\sqrt{4(1-d_n) - (1-b_n)^2(x_1 - a_n)^2}| + |\sqrt{4(1-d_n) - (1-b_n)^2(x_2 - a_n)^2}|).$$

In view of this relation and (8.3), (8.4), we easily obtain the assertion of the proposition. □

Proposition 8.4. *For $x \in I_n$, the following formula holds*

$$T_n(x) = a_n + \frac{1}{2}\Big((1+b_n)(x - a_n) + (1-b_n)\sqrt{(x-a_n)^2 - 4/e_n^2}\Big) + \frac{\varepsilon_{n1}}{n} \frac{c(\mu)\,\theta}{\sqrt{(e_n(x - a_n))^2 - 4}}.$$

Proof. The proof immediately follows from Theorem 5.1. □

Proposition 8.5. *For $x \in \mathbb{R}$, the following estimates hold*

$$|S_n(x)| \le |x| + \sqrt{\frac{n}{n-1}}, \tag{8.5}$$

$$|x - \tilde{x}| \le 2\sqrt{\frac{n}{n-1}}. \tag{8.6}$$

Proof. We note from (3.11) that $S_n(z) = \frac{z}{n} + \frac{n-1}{n} F_{\mu_n}(z)$, where μ_n is the distribution of Y_n. By Proposition 3.6, we see that $S_n(z) = z - G_{\nu_{n-1}\boxplus w_t}(z)$, $z \in \mathbb{C}^+ \cup \mathbb{R}$, where the measure ν_{n-1} is given by $d\nu_{n-1}(x) = d\nu(\sqrt{n}x)$, where, by (5.1), $\nu = \tau$, and $t = t(n) = (n-1)/n$. Since, by Proposition 3.7, $|G_{\nu_{n-1}\boxplus w_t}(z)| \le \sqrt{\frac{n}{n-1}}$, $z \in \mathbb{C}^+ \cup \mathbb{R}$, we obtain the upper bound (8.5).

Now we note that in the same way as above $T_n(z) = z - G_{\zeta_{n-1}\boxplus w_t}(z)$, $z \in \mathbb{C}^+ \cup \mathbb{R}$, where the measure ζ_{n-1} is given by $d\zeta_{n-1}(x) = d\zeta(\sqrt{n}x)$, where, by (5.3), ζ a narrowing of the measure τ on the interval $[-\delta\sqrt{n-1}, \delta\sqrt{n-1}]$. Moreover $|G_{\zeta_{n-1}\boxplus w_t}(z)| \le \sqrt{\frac{n}{n-1}}$, $z \in \mathbb{C}^+ \cup \mathbb{R}$.

It remains to write the following relation

$$\tilde{x} - \Re G_{\zeta_{n-1}\boxplus w_t}(\tilde{x}) = \Re T_n(\tilde{x}) = \Re S_n(x) = x - G_{\nu_{n-1}\boxplus w_t}(x)$$

and obtain from here the upper bound (8.6). The proposition is proved. □

Proposition 8.6. *Let $x \in \mathbb{R}$. Then*

$$h(x) \le c(\mu)\tau(\{|u| > \delta_n\sqrt{n-1}\})\frac{1 + |S_n(x)|^2}{(\Im S_n(x))^3}. \tag{8.7}$$

Proof. Without loss of generality we assume that $\Im T_n(\tilde{x}) > 0$. The case $\Im T_n(\tilde{x}) = 0$ considers in the same way. It follows from (8.1) that

$$1 = \frac{n-1}{n}\int_{-\delta_n\sqrt{n-1}}^{\delta_n\sqrt{n-1}} \frac{\tau(du)}{(\Re T_n(\tilde{x}) - u/\sqrt{n})^2 + (\Im T_n(\tilde{x}))^2}. \tag{8.8}$$

The formula (8.2) gives us

$$1 = \frac{n-1}{n}\Big(\int_{-\delta_n\sqrt{n-1}}^{\delta_n\sqrt{n-1}} + \int_{|u|>\delta_n\sqrt{n-1}}\Big)\frac{\tau(du)}{(\Re T_n(\tilde{x}) - u/\sqrt{n})^2 + (\Im T_n(\tilde{x}) + h(x))^2}. \tag{8.9}$$

For $x \in \mathbb{R}$ we have the following lower bound

$$\begin{aligned} \tilde{I}_1 :=& \int_{|u|\le\delta_n\sqrt{n-1}} \frac{\tau(du)}{(u/\sqrt{n} - \Re T_n(\tilde{x}))^2 + (\Im T_n(\tilde{x}))^2} \\ &- \int_{|u|\le\delta_n\sqrt{n-1}} \frac{\tau(du)}{(u/\sqrt{n} - \Re T_n(\tilde{x}))^2 + (\Im T_n(\tilde{x}) + h(x))^2} \\ &\ge \int_{|u|\le\delta_n\sqrt{n-1}} \frac{nh(x)(2\Im T_n(\tilde{x}) + h(x))\,\tau(du)}{(u/\sqrt{n} - \Re S_n(x))^2 + (\Im S_n(x))^2} \ge \frac{c(\mu)h(x)\Im S_n(x)}{1 + |S_n(x)|^2} \end{aligned} \tag{8.10}$$

and the upper bound

$$\tilde{I}_2 := \int_{|u|>\delta_n\sqrt{n-1}} \frac{\tau(du)}{(\Re T_n(\tilde{x}) - u/\sqrt{n})^2 + (\Im T_n(\tilde{x}) + h(x))^2} \le \frac{\tau(\{|u| > \delta_n\sqrt{n-1}\})}{(\Im S_n(x))^2}. \tag{8.11}$$

It follows from (8.8) and (8.9) that $\tilde{I}_1 = \tilde{I}_2$, therefore we obtain the assertion of the proposition from (8.10) and (8.11). □

Proposition 8.7. *Let $x \in \mathbb{R}$. Then*

$$|x-\tilde{x}| \le c(\mu)\tau(\{|u|>\delta_n\sqrt{n-1}\})\frac{1+|S_n(x)|^2}{(\Im S_n(x))^2}. \tag{8.12}$$

Proof. Without loss of generality we assume that $\Im S_n(x) > 0$. By the formula (8.1), we have

$$\tilde{x} = \Re T_n^{(-1)}(T_n(\tilde{x})) = \Re T_n(\tilde{x}) + \frac{n-1}{n}\int_{-\delta_n\sqrt{n-1}}^{\delta_n\sqrt{n-1}} \frac{(\Re T_n(\tilde{x}) - u/\sqrt{n})\,\tau(du)}{(\Re T_n(\tilde{x}) - u/\sqrt{n})^2 + (\Im T_n(\tilde{x}))^2}. \tag{8.13}$$

On the other hand, by (8.2), we obtain

$$x = \Re S_n^{(-1)}(T_n(\tilde{x}) + ih(x)) = \Re T_n(\tilde{x}) + \frac{n-1}{n}\int_{\mathbb{R}} \frac{(\Re T_n(\tilde{x}) - u/\sqrt{n})\,\tau(du)}{(\Re T_n(\tilde{x}) - u/\sqrt{n})^2 + (\Im S_n(x))^2}. \tag{8.14}$$

It follows from these formulae that

$$x = \tilde{x} + \frac{n-1}{n}(J_1(x) + J_2(x)), \tag{8.15}$$

where

$$J_1(x) := -\int_{-\delta_n\sqrt{n-1}}^{\delta_n\sqrt{n-1}} \frac{(\Re T_n(\tilde{x}) - u/\sqrt{n})h(x)(\Im S_n(x) + \Im T_n(\tilde{x}))\,\tau(du)}{((\Re T_n(\tilde{x}) - u/\sqrt{n})^2 + (\Im T_n(\tilde{x}))^2)((\Re T_n(\tilde{x}) - u/\sqrt{n})^2 + (\Im S_n(x))^2}$$

and

$$J_2(x) := \int_{|u|>\delta_n\sqrt{n-1}} \frac{(\Re T_n(\tilde{x}) - u/\sqrt{n})\,\tau(du)}{(\Re T_n(\tilde{x}) - u/\sqrt{n})^2 + (\Im S_n(x))^2}.$$

It is easy to see that

$$|J_1(x)| \le c(\mu)h(x)\Im S_n(x) \quad \text{and} \quad |J_2(x)| \le c(\mu)\frac{\tau(\{|u|>\delta_n\sqrt{n-1}\})}{\Im S_n(x)}.$$

Therefore, using (8.7), we have

$$\begin{aligned} |x-\tilde{x}| &\le c(\mu)\tau(\{|u|>\delta_n\sqrt{n-1}\})\Big(\frac{1+|S_n(x)|^2}{(\Im S_n(x))^2} + \frac{1}{\Im S_n(x)}\Big) \\ &\le c(\mu)\tau(\{|u|>\delta_n\sqrt{n-1}\})\frac{1+|S_n(x)|^2}{(\Im S_n(x))^2} \end{aligned} \tag{8.16}$$

and (8.12) is proved. □

Proposition 8.8. *For $x \in I_n^*$, the following inequalities hold*

$$\frac{1}{2}((2/e_n)^2 - (x-a_n)^2) \le (2/e_n)^2 - (\tilde{x}-a_n)^2 \le \frac{3}{2}((2/e_n)^2 - (x-a_n)^2).$$

Proof. Consider x such that $|\tilde{x}-a_n| \le \frac{2}{e_n} - \sqrt{\frac{\varepsilon_{n1}}{c_1(\mu)n}}$. By (8.5) and (8.6) we have $|x-\tilde{x}| \le 3$ and $|S_n(x)| \le 7$. In view of Proposition 8.4 and (5.4), it is easy to see that

$$\Im T_n(\tilde{x}) \ge \frac{1}{4}\Big(\frac{\varepsilon_{n1}}{c_1(\mu)n}\Big)^{1/4} \ge c(\mu)(\tau(\{|u| > \delta_n\sqrt{n-1}\}))^{1/4}.$$

By (8.12), we see that

$$|x(\tilde{x}) - \tilde{x}| \le c(\mu)\tau(\{|u| > \delta_n\sqrt{n-1}\})\frac{1+|S_n(x)|^2}{(\Im T_n(\tilde{x}))^2} \le c(\mu)\sqrt{\tau(\{|u| > \delta_n\sqrt{n-1}\})}.$$

Since $c(\mu)$ does not depend on $c_1(\mu)$, we conclude finally

$$|x(\tilde{x}) - \tilde{x}| \le \frac{1}{100}\sqrt{\frac{\varepsilon_{n1}}{n}}. \tag{8.17}$$

The function $x(\tilde{x})$ is monotone, continuous and the assertion of the proposition follows at once from (8.17). □

Proposition 8.9. *We have the bounds*

$$|\Im S_n(x) - \Im T_n(x)| \le \frac{\varepsilon_{n1}}{n}\,\frac{c(\mu)}{((2/e_n)^2 - (x-a_n)^2)^{3/2}}, \quad x \in I_n^*. \tag{8.18}$$

Proof. Let $x \in I_n^*$. We have the formula

$$\Im S_n(x) - \Im T_n(x) = \Im S_n(x) - \Im T_n(\tilde{x}) + \Im T_n(\tilde{x}) - \Im T_n(x) = h(x) + \Im T_n(\tilde{x}) - \Im T_n(x). \tag{8.19}$$

By Propostion 8.8, if $x \in I_n^*$, then $\tilde{x} \in I_n$. We see, by Propostion 8.4, that, for such x,

$$\begin{aligned}\Im T_n(x) &= \frac{1}{2}\sqrt{(1-b_n)^2(x-a_n)^2 - 4(1-d_n)} + \frac{\varepsilon_{n1}}{n}\,\frac{c(\mu)\,\theta}{\sqrt{(e_n(x-a_n))^2-4}},\\ \Im T_n(\tilde{x}) &= \frac{1}{2}\sqrt{(1-b_n)^2(\tilde{x}-a_n)^2 - 4(1-d_n)} + \frac{\varepsilon_{n1}}{n}\,\frac{c(\mu)\,\theta}{\sqrt{(e_n(\tilde{x}-a_n))^2-4}}.\end{aligned} \tag{8.20}$$

Using again Proposition 8.8 we obtain

$$\begin{aligned}(\Im T_n(\tilde{x}))^2 - (\Im T_n(x))^2 &= \frac{1}{4}(1-b_n)^2(x-\tilde{x})(x+\tilde{x}-2a_n) + \frac{\varepsilon_{n1}}{n}\sqrt{1-d_n}\\ &\quad + \Big(\frac{\varepsilon_{n1}}{n}\Big)^2\frac{c(\mu)\,\theta}{(e_n(x-a_n))^2-4)}.\end{aligned}$$

By Propositions 8.6– 8.8, and the formula (8.20), we conclude that

$$\begin{aligned}h(x) &\le c(\mu)\tau(\{|u| > \delta_n\sqrt{n-1}\})/(\Im T_n(\tilde{x}))^3 \le c(\mu)\tau(\{|u| > \delta_n\sqrt{n-1}\})/(\Im T_n(x))^3\\ &\le c(\mu)\tau(\{|u| > \delta_n\sqrt{n-1}\})/((2/e_n)^2 - (x-a_n)^2)^{3/2},\end{aligned} \tag{8.21}$$

$$|x-\tilde{x}| \le c(\mu)\tau(\{|u| > \delta_n\sqrt{n-1}/3\})/((2/e_n)^2 - (x-a_n)^2). \tag{8.22}$$

Therefore, using (8.22), we get

$$|\Im T_n(\tilde{x}) - \Im T_n(x)| = \frac{|(\Im T_n(\tilde{x}))^2 - (\Im T_n(x))^2|}{\Im T_n(\tilde{x}) + \Im T_n(x)} \leq \frac{|(\Im T_n(\tilde{x}))^2 - (\Im T_n(x))^2|}{\Im T_n(x)}$$
$$\leq \frac{c(\mu)|\tilde{x}-x|}{\Im T_n(x)} + \frac{\varepsilon_{n1}}{n}\frac{c(\mu)}{\Im T_n(x)}$$
$$\leq \frac{c(\mu)\tau(\{|u| > \delta_n\sqrt{n-1}\})}{(\Im T_n(x))^3} + \frac{\varepsilon_{n1}}{n}\frac{c(\mu)}{\Im T_n(x)} \leq \frac{\varepsilon_{n1}}{n}\frac{c(\mu)}{(\Im T_n(x))^3}. \quad (8.23)$$

Applying (8.21) and (8.23) to (8.19) we arrive at the assertion of the proposition. □

Proposition 8.10. *We have the bounds*

$$|\Re S_n(x) - \Re T_n(x)| \leq \frac{\varepsilon_{n1}}{n}\frac{c(\mu)}{(2/e_n)^2 - (x-a_n)^2}, \quad x \in I_n^*. \quad (8.24)$$

Proof. Since $\Re S_n(x) - \Re T_n(x) = \Re T_n(\tilde{x}) - \Re T_n(x)$, we conclude, using Proposition 8.4 and (8.22),

$$|\Re T_n(\tilde{x}) - \Re T_n(x)| \leq c(\mu)|\tilde{x}-x| + \frac{\varepsilon_{n1}}{n}\frac{c(\mu)}{(e_n(\tilde{x}-a_n))^2 - 4} + \frac{\varepsilon_{n1}}{n}\frac{c(\mu)}{(e_n(x-a_n))^2-4}$$
$$\leq \frac{\varepsilon_{n1}}{n}\frac{c(\mu)}{(e_n(x-a_n))^2-4} \quad (8.25)$$

for $x \in I_n^*$. The proposition is proved. □

Proposition 8.11. *For $x \in I_n \setminus I_n^*$ and $\alpha \in (0,1]$, we have*

$$|\Im S_n(x) - \Im T_n(x)| \leq \frac{\varepsilon_{n1}}{n}\frac{c(\mu)}{((2/e_n)^2 - (x-a_n)^2)^{3\alpha/2}} + \sqrt{\frac{\varepsilon_{n1}}{n}}\frac{c(\mu)}{((2/e_n)^2-(x-a_n)^2)^{1/2}}$$
$$+ c(\mu)\,((2/e_n)^2 - (x-a_n)^2)^{\alpha/2}.$$

Proof. Let $x, \tilde{x} \in I_n \setminus I_n^*$ and $\alpha \in (0,1]$. We have the two possibilities

$$a)\ \Im S_n(x) \geq (\Im T_n(x))^\alpha \quad \text{or} \quad b)\ \Im S_n(x) < (\Im T_n(x))^\alpha.$$

Consider the case a). Then, by (8.7), we have

$$h(x) \leq c(\mu)\frac{\tau(\{|u|>\delta_n\sqrt{n-1}\})}{(\Im T_n(x))^{3\alpha}} \leq c(\mu)\frac{\tau(\{|u|>\delta_n\sqrt{n-1}\})}{((2/e_n)^2-(x-a_n)^2)^{3\alpha/2}}. \quad (8.26)$$

In addition, repeating the argument of the proof of Proposition 8.9 and using the inequality $|\tilde{x}-x| \leq \sqrt{\frac{\varepsilon_{n1}}{n}}$, we obtain

$$|\Im T_n(\tilde{x}) - \Im T_n(x)| \leq c(\mu)\frac{|\tilde{x}-x|}{\Im T_n(x)} + \frac{\varepsilon_{n1}}{n}\frac{c(\mu)}{\Im T_n(x)} \leq \sqrt{\frac{\varepsilon_{n1}}{n}}\frac{c(\mu)}{((2/e_n)^2-(x-a_n)^2)^{1/2}}.$$

Hence in the case a)

$$|\Im S_n(x)-\Im T_n(x)|\le \frac{\varepsilon_{n1}}{n}\,\frac{c(\mu)}{((2/e_n)^2-(x-a_n)^2)^{3\alpha/2}}+\sqrt{\frac{\varepsilon_{n1}}{n}}\,\frac{c(\mu)}{((2/e_n)^2-(x-a_n)^2)^{1/2}}. \tag{8.27}$$

Now consider the case b). In this case we have the following simple estimate

$$|\Im S_n(x)-\Im T_n(x)|\le (\Im T_n(x))^{\alpha}+\Im T_n(x))\le c(\mu)((2/e_n)^2-(x-a_n)^2)^{\alpha/2}. \tag{8.28}$$

It remains to consider the case when $x\in I_n\setminus I_n^*$ and $\tilde{x}\notin I_n\setminus I_n^*$.

Let $x_{n2}^*=a_n+2/e_n-\sqrt{\frac{\varepsilon_{n1}}{n}}$ and $x_{n2}=a_n+2/e_n-\frac{\varepsilon_{n1}}{n}$. Assume that $x_{n2}^*<x\le x_{n2}$ and $\tilde{x}\in I_n^*$. Assume as well that a) holds. By (8.12), we see that $|\tilde{x}(x_{n2}^*)-x_{n2}^*|\le c(\mu)\sqrt{\frac{\varepsilon_{n1}}{n}}$. In addition $\tilde{x}(x)$ is a monotone increasing function, therefore $|\tilde{x}(x)-x|\le c(\mu)\sqrt{\frac{\varepsilon_{n1}}{n}}$. Repeating the previous estimates we obtain the bounds (8.27) and (8.28) in the considered case. We prove the bounds (8.27) and (8.28) for $x_{n1}<x\le x_{n1}^*$, where $x_{n1}^*=a_n-2/e_n+\sqrt{\frac{\varepsilon_{n1}}{n}}$, $x_{n1}=a_n-2/e_n+\frac{\varepsilon_{n1}}{n}$ and $\tilde{x}\in I_n^*$ in the same way.

Without loss of generality let us assume now that $\tilde{x}>x_{n2}$. Then, by (8.12) and (8.20), we have

$$|x-\tilde{x}|\le c(\mu)\frac{\tau(\{|u|>\delta_n\sqrt{n-1}\})}{(\Im T_n(x))^{2\alpha}}\le c(\mu)\frac{\tau(\{|u|>\delta_n\sqrt{n-1}\})}{((2/e_n)^2-(x-a_n)^2)^{\alpha/2}}\le 1.$$

Then, by (5.9), we conclude $\Im T_n(\tilde{x})\le c(\mu)\sqrt{\frac{\varepsilon_{n1}}{n}}$ and hence

$$|\Im T_n(x)-\Im T_n(\tilde{x})|\le T_n(x)+c(\mu)\sqrt{\frac{\varepsilon_{n1}}{n}}\le c(\mu)((2/e_n)^2-(x-a_n)^2)^{1/2}+c(\mu)\sqrt{\frac{\varepsilon_{n1}}{n}}.$$

Since in our case (8.26) holds, we arrive at the upper bound

$$\begin{aligned}|\Im S_n(x)-\Im T_n(x)|\le{}&\frac{\varepsilon_{n1}}{n}\,\frac{c(\mu)}{((2/e_n)^2-(x-a_n)^2)^{3\alpha/2}}\\&+c(\mu)((2/e_n)^2-(x-a_n)^2)^{1/2}+c(\mu)\sqrt{\frac{\varepsilon_{n1}}{n}}.\end{aligned}$$

In the case b) we have obviously the estimate (8.28). The proposition is proved. □

Proposition 8.12. *For $x\in I_n\setminus I_n^*$,*

$$|\Im S_n(x)-\Im T_n(x)|\le\sqrt{\frac{\varepsilon_{n1}}{n}}\frac{c(\mu)}{\sqrt{(2/e_n)^2-(x-a_n)^2}}.$$

Proof. Note that, for every fixed $x\in I_n\setminus I_n^*$,

$$\frac{\varepsilon_{n1}}{n}\,\frac{1}{(2/e_n-|x-a_n|)^{3\alpha/2}}=(2/e_n-|x-a_n|)^{\alpha/2}$$

for

$$\alpha=\frac{1}{2}\,\frac{\log(\varepsilon_{n1}/n)}{\log(2/e_n-|x-a_n|)}\le 1.$$

Moreover, for this α,

$$(2/e_n - |x-a_n|)^{\alpha/2} = \Big(\frac{\varepsilon_{n1}}{n}\Big)^{1/4}.$$

Therefore the assertion of the proposition follows immediately from Proposition 8.11. □

Now we finish the proof of Theorem 2.1. Return to the formulations of Theorem 5.2 and 6.1. Denote

$$\rho_{n1}(x) := \tilde\rho_{n1}(x) + \frac{\varepsilon_{n1}}{n}\,\frac{c(\mu)\,\theta}{((2/e_n)^2-(x-a_n)^2)^{1/2}} \quad \text{and} \quad \rho_{n2}(x) = \tilde\rho_{n2}(x)$$

for $x \in I_n$. The statement of Theorem 2.1 for $x \in I_n$ follows immediately from Propositions 8.9– 8.10 and 8.12.

It remains to prove (2.10). From Theorem 2.6 [19] and the formula (5.17) it follows immediately that

$$\sup_{x\in\mathbb{R}} |F_n(x+a_n) - \int_{-\infty}^{x} v_n(u)\,du| \le c(\mu)\,\frac{\varepsilon_{n2}}{n}, \tag{8.29}$$

where $\varepsilon_{n2} := \eta_q(n)\frac{\beta_q(\mu)}{n^{(q-4)/2}}$ if $\beta_q(\mu) := \int_{\mathbb{R}} |u|^q\,\mu(du) < \infty$, $4 \le q < 5$. Here

$$\eta_q(n) = \inf_{0<\varepsilon\le 10^{-1/2}} g_q(\varepsilon), \quad \text{where} \quad g_q(\varepsilon) = \varepsilon^{q_*} + \frac{\varepsilon^{q_*-5}}{\beta_{5-q_*}(\mu)}\int_{|u|>\varepsilon\sqrt{n}} |u|^{5-q_*}\,\mu(du)$$

with $q_* = 5 - \min\{q,5\}$. It is easy to see that $\eta_q(n)$ are the functions such that $\eta_q(n) \le 10^{1+3/2}$ and $\eta_q(n) \to 0$ monotonically as $n\to\infty$. If $\beta_q(\mu)<\infty$, $q\ge 5$, then $\varepsilon_{n2} := \frac{\beta_5(\mu)}{n^{1/2}}$. Therefore (2.10) holds with the indicated sequence $\{\varepsilon_{n2}\}$.

Theorem 2.1 is completely proved.

Remark 8.13. It is obvious that for the considered above ε_{n2} we have the properties: $\frac{\varepsilon_{n2}}{n} = o(1/n^{(q-2)/2})$ if $\beta_q(\mu)<\infty$, $4\le q<5$, and $\frac{\varepsilon_{n2}}{n} = O(1/n^{3/2})$ if $\beta_5(\mu)<\infty$.

Remark 8.14. If $m_{2k}(\mu)<\infty$ for some $k=3,\dots$, then we choose in (5.2) $\delta_n = 1/\pi$ and define $\varepsilon_{n1} := c_1(\mu)(n\tau(\mathbb{R}\setminus[-\sqrt{n-1}/\pi, \sqrt{n-1}/\pi]) + 1/\sqrt{n})$. In this case $\varepsilon_{n1} = O(1/\sqrt{n})$. Repeating the argument of Sections 5–8 we obtain the statement of Theorem 2.1 with such ε_{n1} and ε_{n2} described before.

9. Asymptotic expansion of $\int_{\mathbb{R}} |p_n(x) - p_w(x)|\,dx$

In this section we shall prove Corollary 2.2. Indeed, by the estimate (2.10), we have, for $n \ge n_1$,

$$\int_{\mathbb{R}} |p_n(x)-p_w(x)|\,dx = \int_{I_n} |p_n(x)-p_w(x)|\,dx + c(\mu)\,\theta\,\frac{\varepsilon_{n1}+\varepsilon_{n2}}{n}. \tag{9.1}$$

Using Theorems 2.1, we easily conclude that

$$\int_{I_n} |p_n(x)-p_w(x)|\,dx=\int_{I_n} |v_n(x-a_n)-p_w(x)|\,dx$$
$$+\theta\int_{I_n-a_n}|\rho_{n1}(x)|\,dx+\theta\int_{I_n-a_n}|\rho_{n2}(x)|\,dx$$
$$=\int_{[-2,2]}|(1-a_nx)p_w(x)-p_w(x+a_n)|\,dx+c(\mu)\theta\,n^{-1}+\theta\int_{I_n-a_n}|\rho_{n1}(x)|\,dx$$
$$=\frac{|a_n|}{2\pi}\int_{[-2,2]}|x|\frac{|3-x^2|}{\sqrt{4-x^2}}\,dx+c(\mu)\theta\,n^{-1}+\theta\int_{I_n-a_n}|\rho_{n1}(x)|\,dx$$
$$=\frac{2|a_n|}{\pi}+c(\mu)\theta\,n^{-1}+\theta\int_{I_n-a_n}|\rho_{n1}(x)|\,dx.$$

By (2.7), we see that

$$\int_{I_n^*-a_n}|\rho_{n1}(x)|\,dx\le c(\mu)\Big(\frac{\varepsilon_{n1}}{n}\Big)^{3/4}.$$

and, by (2.8),

$$\int_{(I_n-a_n)\setminus(I_n^*-a_n)}|\rho_{n1}(x)|\,dx\le c(\mu)\Big(\frac{\varepsilon_{n1}}{n}\Big)^{3/4}.$$

Applying these relations to (9.1) we get the expansion (2.11).

10. Asymptotic expansion of the free entropy

In this section we prove Corollaries 2.3. First we find an asymptotic expansion of the logarithmic energy $E(\mu_n)$ of the measure μ_n. Recall that (see [20])

$$-E(\mu_n)=\int\int_{\mathbb{R}^2}\log|x-y|\,\mu_n(dx)\mu_n(dy)=I_1(\mu_n)+I_2(\mu_n)$$
$$:=\int\int_{I_n\times I_n}\log|x-y|\,\mu_n(dx)\mu_n(dy)+\int\int_{\mathbb{R}^2\setminus(I_n\times I_n)}\log|x-y|\,\mu_n(dx)\mu_n(dy). \tag{10.1}$$

Using (2.2) and the equality $\int_{\mathbb{R}}u^2\,p_n(u)\,du=1$, we get the inequality

$$\int_{\mathbb{R}}(1+|y|)^2p_n(y)^2\,dy\le c(\mu)\int_{\mathbb{R}}(1+|y|)^2p_n(y)\,dy\le c(\mu).$$

Therefore we conclude with the help of the Cauchy-Bunyakovsky inequality that

$$\int_{\mathbb{R}}|\log|x-y||\,p_n(y)dy\le\Big(\int_{\mathbb{R}}\frac{(\log|x-y|)^2}{(1+|y|)^2}\,dy\Big)^{1/2}\Big(\int_{\mathbb{R}}(1+|y|)^2p_n(y)^2\,dy\Big)^{1/2}\le c(\mu).$$

Recalling (2.10), we obtain

$$|I_2(\mu_n)| \le 4 \int_{\mathbb{R}\setminus I_n} p_n(x) \int_{\mathbb{R}} |\log|x-y||\, p_n(y)\, dy\, dx \le c(\mu) \int_{\mathbb{R}\setminus I_n} p_n(x)\, dx \le c(\mu)\, \frac{\varepsilon_{n2}}{n}. \tag{10.2}$$

Now we note that

$$\begin{aligned} I_1(\mu_n) &= \int\int_{I_n\times I_n} \log|x-y|\, p_n(x)p_n(y)\, dxdy = I_{11}(\mu_n) + 2I_{12}(\mu_n) + I_{13}(\mu_n) \\ &:= \int_{I_n\times I_n} \log|x-y|\, v_n(x-a_n)v_n(y-a_n)\, dxdy \\ &+ 2\int\int_{I_n\times I_n} \log|x-y|\, (p_n(x)-v_n(x-a_n))v_n(y-a_n)\, dxdy \\ &+ \int\int_{I_n\times I_n} \log|x-y|\, (p_n(x)-v_n(x-a_n))(p_n(y)-v_n(y-a_n))\, dxdy. \end{aligned} \tag{10.3}$$

Using the form of $v_n(x)$ we easily conclude that

$$I_{11}(\mu_n) = \int\int_{\mathbb{R}^2} \log|x-y|\, v_n(x)v_n(y)\, dx\, dy + c\theta\Big(\frac{\varepsilon_{n1}}{n}\Big)^{3/2}. \tag{10.4}$$

Recalling the definition of $v_n(x)$ we see that

$$\begin{aligned} &\int\int_{\mathbb{R}^2} \log|x-y|\, v_n(x)v_n(y)\, dx\, dy = \tilde I_1(v_n) + \tilde I_2(v_n) + \tilde I_3(v_n) + \tilde I_4(v_n) + c(\mu)\theta n^{-2} \\ &:= \Big(1+\frac{1}{2}d_n - a_n^2 - \frac{1}{n}\Big)^2 \int\int_{\mathbb{R}^2} \log|x-y|\, p_w(e_n x)\, p_w(e_n y)\, dx\, dy \\ &- 2\Big(1+\frac{1}{2}d_n - a_n^2 - \frac{1}{n}\Big)a_n \int\int_{\mathbb{R}^2} x\log|x-y|\, p_w(e_n x)\, p_w(e_n y)\, dx\, dy \\ &+ a_n^2 \int\int_{\mathbb{R}^2} xy\log|x-y|\, p_w(e_n x)\, p_w(e_n y)\, dx\, dy \\ &- 2\Big(b_n - a_n^2 - \frac{1}{n}\Big)\int\int_{\mathbb{R}^2} x^2\log|x-y|\, p_w(e_n x)\, p_w(e_n y)\, dx\, dy + c(\mu)\theta n^{-2}. \end{aligned} \tag{10.5}$$

In view of $E(\mu_w) = 1/4$ and $e_n^{-2} = 1 - d_n + 2b_n + c(\mu)\theta n^{-2}$, $\log e_n = \frac{d_n}{2} - b_n + c(\mu)\theta n^{-2}$, note that

$$\begin{aligned} \tilde I_1(v_n) &= \Big(1+\frac{1}{2}d_n - a_n^2 - \frac{1}{n}\Big)^2 e_n^{-2}\Big(\int\int_{\mathbb{R}^2} \log|x-y| p_w(x)p_w(y)\, dx\, dy - \log e_n\Big) \\ &= -E(\mu_w) + \frac{a_n^2}{2} + c(\mu)\theta n^{-2}. \end{aligned} \tag{10.6}$$

Since the function $\int_{\mathbb{R}} \log|x-y|p_w(y)\,dy$ is even, we see that $\tilde{I}_2(v_n)=0$. In order to calculate $\tilde{I}_3(v_n)$ we easily deduce that

$$\int_{-2}^{2} up_w(u)\log|x-u|\,du = -x + x^3/6, \quad x\in[-2,2]. \tag{10.7}$$

Therefore we obtain

$$\tilde{I}_3(v_n) = -a_n^2\int_{-2}^{2} xp_w(x)(x-x^3/6)\,dx + c(\mu)\theta n^{-2} = -\frac{2}{3}a_n^2 + c(\mu)\theta n^{-2}. \tag{10.8}$$

Using the following well-known formula (see [20], p. 197)

$$\int_{-2}^{2} p_w(u)\log|x-u|\,du = \frac{x^2}{4}-\frac{1}{2}, \quad x\in[-2,2], \tag{10.9}$$

we deduce

$$\int\int_{\mathbb{R}^2} x^2\log|x-y|\,p_w(x)\,p_w(y)\,dx\,dy = \int_{-2}^{2} x^2\Big(\frac{x^2}{4}-\frac{1}{2}\Big)p_w(x)\,dx = 0. \tag{10.10}$$

Therefore

$$\tilde{I}_4(v_n) = c(\mu)\theta\, n^{-2}. \tag{10.11}$$

By (10.4)–(10.11), we arrive at the formula

$$I_{11}(\mu_n) = -E(\mu_w) - \frac{1}{6}a_n^2 + c\theta\Big(\Big(\frac{\varepsilon_{n1}}{n}\Big)^{3/2} + \frac{1}{n^2}\Big). \tag{10.12}$$

Now we note, using (10.7), (10.9), (10.10), and the Hölder inequality that, for $x\in I_n - a_n$ and for any fixed positive δ,

$$\begin{aligned}&\int_{I_n-a_n} \log|x-y|\,v_n(y)\,dy\\ &= \int_{\mathbb{R}} \log|x-y|\big(1+\frac{1}{2}d_n - a_n^2 - \frac{1}{n} - a_ny - (b_n - a_n^2 - \frac{1}{n})y^2\big)\,p_w(e_ny)\,dy\\ &+ c(\mu,\delta)\theta\left(\frac{\varepsilon_{n1}}{n}\right)^{\frac{3}{2}-\delta} = \frac{x^2}{4} - \frac{1}{2} - \log e_n + a_n\big(x - \frac{x^3}{6}\big) + c(\mu,\delta)\theta\left(\frac{\varepsilon_{n1}}{n}\right)^{\frac{3}{2}-\delta},\end{aligned}$$

where $c(\mu,\delta)$ are positive constants depending on μ and δ only.

On the other hand, by (8.29), we see that

$$\Big|\int_{I_n} x^k(p_n(x+a_n) - v_n(x))\,dx\Big| \le c(\mu)\,\frac{\varepsilon_{n2}}{n}, \quad k=0,1,2,3.$$

The last two relations give us finally

$$|I_{12}(\mu_n)| \le c(\mu,\delta)\left(\frac{\varepsilon_{n1}}{n}\right)^{\frac{3}{2}-\delta} + c(\mu)\,\frac{\varepsilon_{n2}}{n}. \tag{10.13}$$

It remains to estimate $I_{13}(\mu_n)$. Write

$$I_{13}(\mu_n) = I_{13,1}(\mu_n) + 2I_{13,2}(\mu_n) + I_{13,3}(\mu_n)$$
$$:= \Big(\int\int_{I_n^*\times I_n^*} +2\int\int_{I_n^*\times(I_n\setminus I_n^*)} + \int\int_{(I_n\setminus I_n^*)\times(I_n\setminus I_n^*)} \Big)$$
$$\log|x-y|\,(p_n(x)-v_n(x-a_n))(p_n(y)-v_n(y-a_n))\,dxdy. \tag{10.14}$$

In order to estimate $I_{13,1}(\mu_n)$ we use the upper bound

$$\tilde I_{13,1} := \int_{I_n^*} |\log|x-y||\Big(|\rho_{n1}(y-a_n)| + \rho_{n2}(y-a_n)\Big)\,dy$$
$$\le c(\mu)\frac{\varepsilon_{n1}}{n}\int_{I_n^*} |\log|x-y||\frac{dy}{(4-(e_n(y-a_n))^2)^{3/2}} + \int_{I_n^*}|\log|x-y||\rho_{n2}(y)\,dy.$$

Let $p,q>1$ and $\frac{1}{p}+\frac{1}{q}=1$. We assume that q is closed to 1, i.e., $1<q\le 1.01$. By Hölder inequality, for $x\in I_n^*$,

$$\int_{I_n^*}|\log|x-y||\frac{dy}{(4-(e_n(y-a_n))^2)^{3/2}}$$
$$\le \Big(\int_{I_n^*}|\log|x-y||^p\,dy\Big)^{1/p}\Big(\int_{I_n^*}\frac{dy}{(4-(e_n(y-a_n))^2)^{3q/2}}\,dy\Big)^{1/q} \le c(\mu,q)\Big(\sqrt{\frac{n}{\varepsilon_{n1}}}\Big)^{\frac{3}{2}-\frac{1}{q}}$$

and, by (2.9),

$$\int_{I_n^*}|\log|x-y||\rho_{n2}(y-a_n)\,dy \le \Big(\int_{I_n^*}|\log|x-y||^p\,dy\Big)^{1/p}\Big(\int_{I_n^*}\rho_{n2}(y-a_n)^q\,dy\Big)^{1/q} \le \frac{c(\mu,q)}{n^{2/q}},$$

where $c(\mu,q)>0$ is a constant depended on μ and q only. From the two last upper bounds we get

$$\tilde I_{13,1} \le c(\mu,q)\Big(\Big(\frac{\varepsilon_{n1}}{n}\Big)^{\frac{1}{4}+\frac{1}{2q}} + \frac{1}{n^{2/q}}\Big) \le c(\mu,q)\Big(\frac{\varepsilon_{n1}}{n}\Big)^{\frac{1}{4}+\frac{1}{2q}}. \tag{10.15}$$

On the other hand it is easy to see that

$$\int_{I_n^*}\Big(|\rho_{n1}(y-a_n)| + \rho_{n2}(y-a_n)\Big)\,dy \le c(\mu)\Big(\Big(\frac{\varepsilon_{n1}}{n}\Big)^{\frac{3}{4}} + \frac{1}{n^2}\Big) \le c(\mu)\Big(\frac{\varepsilon_{n1}}{n}\Big)^{\frac{3}{4}}. \tag{10.16}$$

We conclude from (10.15) and (10.16) that

$$|I_{13,1}(\mu_n)| \le c(\mu,q)\Big(\frac{\varepsilon_{n1}}{n}\Big)^{1+\frac{1}{2q}}. \tag{10.17}$$

Now introduce the quantity

$$\tilde I_{13,2} := \int_{I_n\setminus I_n^*}|\log|x-y||\Big(|\rho_{n1}(y-a_n)| + \rho_{n2}(y-a_n)\Big)\,dy.$$

As above we obtain, for $x \in I_n$,

$$\int_{I_n \setminus I_n^*} |\log|x-y|| \frac{dy}{(4-(e_n(y-a_n))^2)^{1/2}}$$
$$\le \Big(\int_{I_n \setminus I_n^*} |\log|x-y||^p \, dy \Big)^{1/p} \Big(\int_{I_n \setminus I_n^*} \frac{dy}{(4-(e_n(y-a_n))^2)^{q/2}} \, dy \Big)^{1/q} \le c(\mu,q) \Big(\frac{\varepsilon_{n1}}{n} \Big)^{\frac{1}{2q}-\frac{1}{4}}.$$

Therefore we have

$$\tilde{I}_{13,2} \le c(\mu,q) \Big(\frac{\varepsilon_{n1}}{n} \Big)^{\frac{1}{2q}+\frac{1}{4}}, \quad x \in I_n. \tag{10.18}$$

We deduce from (10.16) and (10.18)

$$|I_{13,2}(\mu_n)| \le c(\mu,q) \Big(\frac{\varepsilon_{n1}}{n} \Big)^{\frac{1}{2q}+1}. \tag{10.19}$$

It remains to estimate $I_{13,3}(\mu_n)$. It is easy to verify that

$$\int_{I_n \setminus I_n^*} \Big(|\rho_{n1}(y-a_n)| + \rho_{n2}(y-a_n) \Big) \, dy \le c(\mu,q) \Big(\frac{\varepsilon_{n1}}{n} \Big)^{\frac{3}{4}}. \tag{10.20}$$

We obtain from (10.18) and (10.20) that

$$|I_{13,3}(\mu_n)| \le c(\mu,q) \Big(\frac{\varepsilon_{n1}}{n} \Big)^{\frac{1}{2q}+1}. \tag{10.21}$$

It remains to note that the upper bound

$$|I_{13}(\mu_n)| \le c(\mu,q) \Big(\frac{\varepsilon_{n1}}{n} \Big)^{\frac{1}{2q}+1}. \tag{10.22}$$

follows immediately from (10.17), (10.19), and (10.21).

In view of (10.1), (10.2), (10.12), (10.13) and (10.22) we get

$$-E(\mu_n) = -E(\mu_w) - \frac{1}{6} a_n^2 + c(\mu,q)\theta \Big(\Big(\frac{\varepsilon_{n1}}{n} \Big)^{\frac{1}{2q}+1} + \frac{\varepsilon_{n2}}{n} \Big). \tag{10.23}$$

The assertion of Corollary 2.3 follows from this relation.

11. Asymptotic expansion of the free Fisher information

Now let us prove Corollary 2.4. We shall show that the free Fisher information of the measure μ_n has the form (2.13). Denote

$$\Phi(\mu_n) = \Phi_1(\mu_n) + \Phi_2(\mu_n) := \frac{4\pi^2}{3} \int_{I_n} p_n(x)^3 \, dx + \frac{4\pi^2}{3} \int_{\mathbb{R} \setminus I_n} p_n(x)^3 \, dx. \tag{11.1}$$

As before we see that, by (2.10),

$$\Phi_2(\mu_n) \le c(\mu) \int_{\mathbb{R} \setminus I_n} p_n(x) \, dx \le c(\mu) \frac{\varepsilon_{n2}}{n}. \tag{11.2}$$

On the other hand, by (2.6), we have

$$\Phi_1(\mu_n) = \frac{4\pi^2}{3}\int_{I_n-a_n} v_n(x)^3\,dx + \frac{4\pi^2}{3}\int_{I_n-a_n} \sum_{\substack{k+l=3\\ 0\le k\le 2,\, l\ge 1}} v_n(x)^k(\rho_{n1}(x)+\rho_{n2}(x))^l\,dx. \tag{11.3}$$

In the sequal we consider nonnegative entire numbers $0 \le k \le 2$, $l \ge 1$ and $k+l=3$ only.

We see, by (2.9), that

$$\int_{I_n-a_n} |v_n(x)|^k|\rho_{n2}(x)|^l\,dx \le c(\mu)\frac{1}{n^2}. \tag{11.4}$$

Now we note, by (2.7), that

$$\begin{aligned}\int_{I_n^*-a_n} |v_n(x)|^k|\rho_{n1}(x)|^l\,dx &\le c(\mu)\Big(\frac{\varepsilon_{n1}}{n}\Big)^l\int_{I_n^*-a_n}\frac{dx}{(4-(e_nx)^2)^{(3l-k)/2}}\\ &\le c(\mu)\Big(\frac{\varepsilon_{n1}}{n}\Big)^l\Big(\frac{n}{\varepsilon_{n1}}\Big)^{\frac{3l-k-2}{4}} = c(\mu)\Big(\frac{\varepsilon_{n1}}{n}\Big)^{\frac{5}{4}}.\end{aligned} \tag{11.5}$$

Furthermore, we have, using the bound (2.8),

$$\begin{aligned}&\int_{(I_n-a_n)\setminus(I_n^*-a_n)} |v_n(x)|^k|\rho_{n1}(x)|^l\,dx\\ &\le c(\mu)\int_{(I_n-a_n)\setminus(I_n^*-a_n)} |v_n(x)|^k\Big(\frac{\varepsilon_{n1}}{n(4-(e_nx)^2)}\Big)^{l/2}\,dx \le c(\mu)\frac{\varepsilon_{n1}}{n}.\end{aligned} \tag{11.6}$$

From (11.5) and (11.6) it follows

$$\int_{I_n-a_n} |v_n(x)|^k|\rho_{n1}(x)|^l\,dx \le c(\mu)\,\frac{\varepsilon_{n1}}{n}. \tag{11.7}$$

Applying (11.4) and (11.7) to (11.3) we obtain

$$\Phi_1(\mu_n) = \frac{4\pi^2}{3}\int_{I_n-a_n} v_n(x)^3\,dx + c(\mu)\theta\,\frac{\varepsilon_{n1}}{n}. \tag{11.8}$$

It is easy to see that the integral on the right hand-side of (11.8) is equal to

$$\begin{aligned}&\Big(1+3\Big(\frac{1}{2}d_n - a_n^2 - \frac{1}{n}\Big)\Big)e_n^{-1}\int_{\mathbb{R}} p_w(x)^3\,dx - 3\Big(b_n - a_n^2 - \frac{1}{n}\Big)e_n^{-3}\int_{\mathbb{R}} x^2p_w(x)^3\,dx\\ &+ 3a_n^2e_n^{-3}\int_{\mathbb{R}} x^2p_w(x)^3\,dx + c(\mu)\theta n^{-5/2} = \frac{3}{4\pi^2}(1+a_n^2) + c(\mu)\theta\,n^{-2}.\end{aligned}$$

Therefore we finally conclude by (11.1)–(11.3) that

$$\Phi(\mu_n) = 1 + a_n^2 + \theta c(\mu)\frac{\varepsilon_{n1}+\varepsilon_{n2}}{n} = \Phi(\mu_w) + a_n^2 + c(\mu)\theta\,\frac{\varepsilon_{n1}+\varepsilon_{n2}}{n}.$$

Thus, Corollary 2.4 is proved.

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Gennadii Chistyakov
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
33501 Bielefeld
Germany
Email address: `chistyak@math.uni-bielefeld.de`

Friedrich Götze
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
33501 Bielefeld
Germany
Email address: `goetze@math.uni-bielefeld.de`