# Pathwise vs. Path-by-Path Uniqueness

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**Abstract:** We construct a series of stochastic differential equations of the form $dX_t = b(t, X_t)dt + dB_t$ which exhibit *nonuniqueness* in the *path-by-path* sense while having a unique adapted solution in the sense of stochastic processes, i.e. *pathwise* uniqueness holds.

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## 1. Introduction

In this paper we consider the stochastic differential equation

$$\mathrm{d}X_t = b(t, X_t)\,\mathrm{d}t + \mathrm{d}B_t,\ X_0 = x_0, \tag{1.1}$$

where $b\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a Borel measurable mapping, $(B_t)_{t\geq 0}$ is a standard $d$–dimensional Brownian motion, $x_0 \in \mathbb{R}^d$. Let us recall that a solution (*weak* solution) to SDE (1.1) is a pair of a Brownian motion $(B_t)_{t\geq 0}$ defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$ and a stochastic process $(X_t)_{t\in[0,T]}$ adapted to the filtration $(\mathcal{F}_t)_{t\in[0,\infty)}$ such that $\mathbb{P}$-a.s.

$$X_t = x_0 + \int_{[0,t]} b(s, X_s)\ \mathrm{d}t + B_t,\ t \in [0,T]. \tag{1.2}$$

The solution $(B, X)$ is called a *strong* solution if the process $(X_t)_{t\in[0,T]}$ is adapted to the augmented filtration (i.e. the completed filtration) $(\overline{F^B})_{t\geq 0}$ generated by the Brownian motion. The classical example of an SDE which has a *weak* solution but no *strong* solutions is Tanaka's equation

$$\mathrm{d}X_t = \mathrm{sgn}(X_t)\,\mathrm{d}B_t,\ X_0 = 0.$$

It is also worth mentioning the celebrated example due to B. Tsirelson (see [Ts75]) of an SDE of the form

$$\mathrm{d}X_t = b(X_{\leq t}, t)\,\mathrm{d}t + \mathrm{d}B_t,\ X_0 = 0,$$

where $b$ is a bounded Borel measurable function of $t$ and the "past" of $X$ up to the time $t$, which admits a *weak* solution but no *strong* solutions.

**Definition 1.** *For SDE (1.1) pathwise uniqueness holds if for any two solutions $(B, X)$, $(B, Y)$ defined on the same filtered probability space with the same Brownian motion $(B_t)_{t\geq 0}$ there exists a measurable set $\Omega' \subseteq \Omega$ with $\mathbb{P}[\Omega'] = 1$ such that*

$$X_t(\omega) = Y_t(\omega), \qquad \omega \in \Omega', t \in [0, T].$$

At the same time one can consider random ordinary differential equation (1.2) and ask whether the uniqueness holds in the pure ODE setting.

**Definition 2.** *For SDE (1.1) path-by-path uniqueness holds if there exists a measurable set $\Omega' \subset C([0, T], \mathbb{R}^d)$ of full Wiener measure such that for any Brownian trajectory from $\Omega'$ integral equation (1.2) has a unique solution.*

Let us point out that in Definition 1 of *pathwise uniqueness* the set $\Omega' \subset \Omega$ of full measure a priori is allowed to depend on the both processes $X$ and $Y$. This is in stark contrast to *path-by-path* uniqueness, where there is a set of full measure $\Omega'' := B^{-1}(\Omega')$, where $\Omega' \subset C([0, T], \mathbb{R}^d)$ is the set of "good" Brownian trajectories from Definition 2, such that the functions $t \to X_t(\omega)$ and $t \to Y_t(\omega)$ have to coincide for all $\omega \in \Omega''$. Furthermore, in the same way as in [Fla11], we call a map $\Omega \supseteq \Omega' \to X(\omega) \in C([0, T], \mathbb{R}^d)$ a *path-by-path solution* if $\mathbb{P}[\Omega'] = 1$ and $X(\omega)$ solves ODE (1.2) for all $\omega \in \Omega'$. Note that *path-by-path* solutions are not required to be adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$ but every solution of the corresponding SDE yields a *path-by-path* solution. However, the converse whether every *path-by-path* solution can be obtained from a solution to the SDE was posed as an open question in [AL17] and also was mentioned in the book [Fla15] (see the discussion on p. 12). In this paper we show that, in general, this is not true. Moreover, we construct SDEs such that a *strong* solution exists, *pathwise* uniqueness holds, but *path-by-path* uniqueness fails to hold.

Concerning the historical development of *path-by-path* uniqueness to our knowledge the first result was obtained by A. M. Davie in [Dav07] for the case when $b$ is Borel measurable and bounded. Later, Davie extended his result and proved that *path-by-path* uniqueness holds in the non-degenerate multiplicative noise case (see [Dav11]). The original result of Davie was established with a different method by the first author in [Sh16] (see also some corrections in [Sh17]), which enabled him to present a simpler proof of the main theorem from [Dav07] and strengthen it in multiple directions. In particular, in [Sh16] *path-by-path* uniqueness was obtained for some unbounded drift coefficients $b$ and by carefully examining the arguments one can show that the set $\Omega' \subseteq \Omega$ where *path-by-path* uniqueness holds can be constructed independently of the initial condition. R. Catellier and M. Gubinelli in [CG16] showed that *path-by-path* uniqueness can be established if the Wiener process is replaced by a fractional Brownian motion in $\mathbb{R}^d$ with Hurst parameter $H$. Furthermore, the drift $b$ was allowed to be merely a distribution as long as $H$ was sufficiently small. In the work [BFGM14] L. Beck, F. Flandoli, M. Gubinelli and M. Maurelli proved that *path-by-path* uniqueness does not only hold for SDEs, but also for SPDEs. In [Pri18] E. Priola considered equations driven by a Lévy process such that the

Lévy measure fulfills some integrability condition, see also [Zh18] for related results. In 2016 O. Butkovsky and L. Mytnik showed in [BM16] that *path-by-path* uniqueness holds for the stochastic heat equation with space-time white noise for bounded Borel measurable drifts. In the works [Wre16, Wre17] the second author established *path-by-path* uniqueness for the case where $\mathbb{R}^d$ is replaced by a Hilbert space $H$ and $W$ is a cylindrical Wiener process as long as the linear negative operator is added to the SDE and the nonlinear part is bounded with respect to a specific norm, a condition which is trivial if $\dim H < \infty$. In the recent paper [Pri19] E. Priola improved the aforementioned result by allowing a time–dependent coefficient in front of the Lévy noise which can in essence be as degenerate as in the condition for *pathwise* uniqueness.

In conclusion, *path-by-path* uniqueness can be established when the drift is singular and also in the case when the noise term is degenerate. However, in general, the conditions to establish *path-by-path* uniqueness are stricter than those for *pathwise* uniqueness so in some cases there is a "gap" between the available *pathwise* and the *path-by-path* results. In this paper we would like to add a new point of focus. By carefully constructing an SDE we can determine that any global "solution" must know something about its own future and cannot be adapted to a filtration with respect to which the "driving" stochastic process remains a Brownian motion. Next, we can construct an SDE having a unique adapted solution and with probability one having some other non–adapted solutions to the corresponding ODE.

## 2. Bessel processes

The constructions in the next sections are based on the properties of the stochastic differential equations governing Bessel processes. For the sake of completeness below we recall the known results which will be used in the subsequent considerations.

**Definition 3.** *For $\delta > 0$, $Z_0 \geq 0$ the unique strong solution of the SDE*

$$Z_t = Z_0 + \delta t + 2\int_{[0,t]} \sqrt{|Z_t|}\,\mathrm{d}B_s \tag{2.1}$$

*is called the square of the $\delta$–dimensional Bessel process started at $Z_0$.*

We refer to Chapter 11 in [RY99] for the basic properties of this equation. In particular, it is well–known that although the diffusion coefficient is non–Lipschitz for equation (2.1) *pathwise* uniqueness holds and, moreover, with probability one the solution is non-negative. The process $\sqrt{Z_t}$ is called the $\delta$–dimensional Bessel process started at $\sqrt{Z_0}$.

Now let us introduce for $\delta > 1$ the Bessel SDE

$$X_t = X_0 + \int_{[0,t]} \mathbb{1}_{X_s \neq 0} \frac{\delta - 1}{2X_s}\,\mathrm{d}s + \mathrm{B}_t. \tag{2.2}$$

One can verify that for $\delta > 1$ the process $\sqrt{Z_t}$ satisfies (2.2) with $X_0 = \sqrt{Z_0}$.

The next theorem was obtained by A. Cherny in [Ch00].

**Theorem 1.** *For SDE* (2.2)

1. *if $\delta > 1$, $X_0 \geq 0$ then the $\delta$–dimensional Bessel process is the unique non–negative solution, moreover, it is a strong solution,*
2. *if $\delta > 2$, $X_0 \neq 0$ then pathwise uniqueness holds,*
3. *if $1 < \delta < 2$ or $X_0 = 0$ then there exist other strong solutions with the same $X_0$ and $B$, there exist weak solutions, the uniqueness in law does not hold.*

Now let us recall the SDE for the 3–Bessel bridge with the terminal value 1:

$$X_t = X_0 + \int_{[0,t]} \Big( \frac{1 - X_t}{1 - t} + \frac{1}{X_t} \Big) \,\mathrm{d}t + B_t, \;\; X_0 \geq 0, \;\; t \in [0, 1] \tag{2.3}$$

see e.g. equation (29) on p. 274 in [Pit99]. In fact, we will be interested in the SDE

$$dX_t = f(t, X_t)\, dt + dB_t, \;\; X_0 = x_0 \in \mathbb{R}, t \in [0, 1], \tag{2.4}$$

where

$$f(t, x) := \mathbb{1}_{\{x>0\}} \Big( \frac{1 - x}{1 - t} + \frac{1}{x} \Big) - \mathbb{1}_{\{x<0\}} \Big( \frac{1 - (-x)}{1 - t} + \frac{1}{-x} \Big).$$

**Proposition 1.** *For SDE* (2.4)

1. *For $x_0 = 0$ there is a unique nonnegative solution (unique in the pathwise sense among all nonnegative solutions), and analogously there is a unique nonpositive solution. Moreover, these solutions are strong solutions.*
2. *For $x_0 = 0$ any weak solution with probability one preserves its sign for $t \in (0, 1]$, in particular, never reaches $0$ for $t \in (0, 1]$ and equals $1$ or $-1$ when $t = 1$.*

*Proof.* Let us notice that there is a nonnegative *weak* solution to SDE (2.4) given by the 3-Bessel bridge with the terminal value 1 and as it is well-known this solution with probability 1 never reaches 0 for $t \in (0, 1]$. The corresponding nonpositive solution is given by the $(-1) \times 3$–Bessel bridge. Now let us establish *pathwise* uniqueness in the class of nonnegative solutions, the case of nonpositive solutions is handled completely analogously. One can see that it is sufficient to establish *pathwise* uniqueness on every interval $[0, T]$, $T \in (0, 1)$. Let us assume that $(X_1, B)$, $(X_2, B)$ are two *weak* solutions on $[0, T]$ to SDE (2.4) defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$ with the same Brownian motion $B$ and $X_1, X_2 \geq 0$ a.s. Let us set

$$\varrho_T := \exp\Bigg( - \int_{[0,T]} \frac{1}{1 - t} \,\mathrm{d}t - \frac{1}{2} \int_{[0,T]} \frac{1}{(1 - t)^2} \,\mathrm{d}B_t \Bigg).$$

Then a.s. $\varrho_T > 0$ and $\mathbb{E}\varrho_T = 1$. By Girsanov's theorem under the new measure

$$\mathrm{d}Q := \varrho_T \,\mathrm{d}P$$

the process

$$\widetilde{B}_t := \int_{[0,t]} \frac{1}{1-s}\,\mathrm{d}s + B_t,\ t \in [0,T].$$

is a Brownian motion with respect to the same filtration $(\mathcal{F}_t)_{t\geq 0}$. It is also clear that the filtrations $(\overline{\mathcal{F}^B}_t)_{t\in[0,T]}$ and $(\overline{\mathcal{F}^{\widetilde{B}}}_t)_{t\in[0,T]}$ coincide. Now one can see that on the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, Q)$ the processes $(X_1, \widetilde{B})$, $(X_2, \widetilde{B})$ are *weak* nonnegative solutions to the SDE

$$X_t = \int_{[0,t]} \mathbb{1}_{\{X_t>0\}}\Big(\frac{-X_t}{1-t} + \frac{1}{X_t}\Big)\,\mathrm{d}t + \widetilde{B}_t,\ X_0 = 0,\ t \in [0,T]. \tag{2.5}$$

Let us define $Y_{1,t} := X_{1,t}^2$, $Y_{2,t} := X_{2,t}^2$. Applying Ito's formula one can show that $(Y_{1,t}, \widetilde{B})$ and $(Y_{2,t}, \widetilde{B})$ are *weak* nonnegative solutions to the SDE

$$Y_t = \int_{[0,t]} \frac{-2Y_t}{1-t}\,\mathrm{d}t + 2t + \int_{[0,t]} \sqrt{|Y_t|}\,\mathrm{d}\widetilde{B}_t,\ t \in [0,T]. \tag{2.6}$$

where we have also used the occupation time formula for semimartingales (see e.g. [RY99], Ch. 7) applied to $X = X_1, X_2$:

$$\int_{[0,t]} \mathbb{1}_{\{X_s=0\}}\,\mathrm{d}s = \int_{\mathbb{R}} \mathbb{1}_{\{x=0\}} L_t^x(X)\,\mathrm{d}x = 0,\ t \in [0,T].$$

For SDE (2.6) the classic Yamada–Watanabe condition is statisfied and *pathwise* uniqueness follows, see e.g. [RY99], Ch. 9, Theorem 3.5. Consequently, by the Yamada–Watanabe theorem there is a unique solution to (2.6) and the solution is *strong*. Since $X_{1,t} = \sqrt{|Y_{1,t}|}$ and $X_{2,t} = \sqrt{|Y_{2,t}|}$ this easily gives the required a.s. equality

$$X_{1,t} = X_{2,t},\ t \in [0,T],$$

and applying the Yamada–Watanabe theorem again we obtain that the solution $(X_1, \widetilde{B})$ is *strong*. Taking into account the equality

$$(\overline{\mathcal{F}^B}_t)_{t\in[0,T]} = (\overline{\mathcal{F}^{\widetilde{B}}}_t)_{t\in[0,T]}$$

we prove the first claim of Proposition 1. Let $(X, B)$ be a *weak* solution to SDE (2.4). One can observe that since $f(t,0) \equiv 0$ any solution $X$ to SDE (2.4) is not identically zero on every interval $(0,t')$, $t' > 0$ with probability one. Let us define the sequence of Markov moments $\{\tau_n\}$

$$\tau_n := \inf\big\{t > 0 : |X_{\tau_n}| = 1/n\big\}.$$

Then $\lim_{n\to\infty} \tau_n = 0$ a.s. One can notice that the arguments presented above in the case $x_0 = 0$ similarly yield *pathwise* uniqueness for the SDE

$$Z_t = Z_\tau + \int_{[\tau,t]} f(t, Z_t)\,\mathrm{d}t + \mathrm{d}B_t,\ t \in [\tau,T],$$

as soon as $Z_\tau \neq 0$ with probability 1 on the set $\{\tau < T\}$. Applying this observation to $\tau := \tau_n \wedge T$ for each $n$ and taking into account the properties of the 3–Bessel bridge we prove that with probability 1 the trajectories of the process $X$ never reach 0 for $t \in (0, 1)$. □

We will also need the following classic example of a singular SDE which has no solutions at all.

**Proposition 2.** *The SDE*

$$X_t = \int_{[0,t]} \mathbb{1}_{\{X_s \neq 0\}} \frac{-1}{2X_s} \,\mathrm{d}s + B_t,\ \ X_0 = 0. \tag{2.7}$$

*has no weak solutions.*

*Proof.* For the proof see Example 2.1 in [Ch01]. □

## 3. Equation without adapted solutions

**Theorem 2.** *There exists a Borel mapping $b = (b_1, b_2) : [0, 2] \times \mathbb{R}^2 \to \mathbb{R}^2$,*

$$X_t = \int_{[0,t]} b(s, X_s) \,\mathrm{d}s + B_t,\ \ X_0 = 0,\ \ t \in [0, 2] \tag{3.1}$$

*such that*

1. *there exists a set of Brownian trajectories $\Omega$ of full measure, such that for every $B \in \Omega$ integral equation (3.1) has at least one solution defined on the whole interval $[0, 2]$,*
2. *equation (3.1) considered as an SDE has no solutions $(X, B)$ defined on the whole interval $[0, 2]$.*

*Proof.* **Step 1.** For $t \in [0, 1]$ let us set

$$b_1 := 0,$$

$$b_2 := \mathbb{1}_{\{x_2 > 0\}} \Big( \frac{1 - x_2}{1 - t} + \frac{1}{x_2} \Big) - \mathbb{1}_{\{x_2 < 0\}} \Big( \frac{1 - (-x_2)}{1 - t} + \frac{1}{-x_2} \Big).$$

In this case any solution $(X, B)$ to (3.1) for $t \in [0, 1]$ is of the following form:

$$X_{1,t} = B_{1,t},\ \ t \in [0, 1] \tag{3.2}$$

$$X_{2,t} = \int_{[0,t]} b_2(s, X_{2,s}) \,ds + B_{2,t},\ \ t \in [0, 1] \tag{3.3}$$

Now let us remind that by Proposition 1 for SDE (3.3)

1. there exists a unique nonnegative solution, i.e. the 3–Bessel bridge from 0 to 1,
2. there exists a unique nonpositive solution, i.e. the $(-1) \times$ 3–Bessel bridge from 0 to $-1$,

3. any solution does not change its sign.

**Step 2.** Now we would like to make use of equation (2.7), which will play the role of a "randomized filter". The idea is to force the equation to "select" solutions in an non–adapted way.

For $t \in (1, 2]$ let us set

$$b_1 := \mathbb{1}_{\{x_1 \neq 0\}} \frac{1}{x_1}$$

$$b_2 := \mathbb{1}_{\{x_1>0\}} \mathbb{1}_{\{x_2>0\}} \frac{-1}{2(x_2-1)} + \mathbb{1}_{\{x_1<0\}} \mathbb{1}_{\{x_2<0\}} \frac{-1}{2(x_2+1)} + \mathbb{1}_{\{x_1>0\}} \mathbb{1}_{\{x_2<0\}} \frac{1}{x_2} + \mathbb{1}_{\{x_1<0\}} \mathbb{1}_{\{x_2>0\}} \frac{1}{x_2}.$$

Any solution $(X, B)$ to (3.1) for $t \in [1, 2]$ has the form

$$X_{1,t} = X_{1,1} + \int_{[1,t]} b_1(s, X_{1,s}) \, \mathrm{d}s + B_{1,t} - B_{1,1} \tag{3.4}$$

$$X_{2,t} = X_{2,1} + \int_{[1,t]} b_2(s, X_{1,s}, X_{2,s}) \, \mathrm{d}s + B_{2,t} - B_{2,1}. \tag{3.5}$$

Let us remind that the drift $x \mapsto \mathbb{1}_{\{x \neq 0\}} \frac{1}{x}$ corresponds to the 3–Bessel SDE, therefore equation (3.4) has a unique solution and this solution preserves its sign on the interval $[1, 2]$. In turn, equation (3.5) has no solutions if $X_{1,1} \cdot X_{2,1} > 0$ and has a unique solution otherwise.

Let us consider a "global" solution $X$ to SDE (3.1) on a fixed probability space equipped with a two–dimensional Brownian motion $B$ with respect to some filtration $\{\mathcal{F}_t\}_{t \geq 0}$. We claim that if $B_{1,1} > 0$ then $X_{2,t}, t \in [0, 1)$ must stay non–positive, while if $B_{1,1} < 0$ then $X_{2,t},\ t \in [0, 1)$ must stay non–negative. Indeed, it is easy to see that if any of these conditions are violated then we have $X_{1,1} \cdot X_{2,1} > 0$ and this contradicts to the fact that SDE (2.7) has no solutions. Now it is easy to see that the random variable $X_{2,1/2}$ cannot be measurable with respect to $\mathcal{F}_{1/2}$ because the random variable $\operatorname{sgn}(B_{1,1}) = -\operatorname{sgn}(X_{2,1/2})$ is not measurable with respect to $\mathcal{F}_{1/2}$. However, it easy to construct a set of Brownian trajectories $\Omega$ of full measure such that for every $B \in \Omega$ there exists a function $t \mapsto X_t$ which solves integral equation (3.1). □

**Remark 1.** *Since the process $X_{2,t}$ never changes its sign for $t \in (0, 2]$ the same arguments show that for any solution $(B, X)$ to integral equation (3.1) the following equality holds:*

$$\operatorname{sgn}(B_{1,1}) = \lim_{n \to \infty} -\operatorname{sgn}(X_{2,1/n}) \tag{3.6}$$

**Theorem 3.** *Let $B$ be a cylindrical Brownian motion. There exists a Borel mapping $b : [0, 1] \times \mathbb{R}^\infty \to \mathbb{R}^\infty$*

$$X_t = \int_{[0,t]} b(s, X_s) \, \mathrm{d}s + B_t,\ \ X_0 = 0,\ \ t \in [0, 1], \tag{3.7}$$

*such that*

1. *there exists a set of Brownian trajectories $\Omega \subset C([0,1],\mathbb{R}^\infty)$ of full measure, such that for every $B \in \Omega$ integral equation (3.7) has at least one solution defined on the whole interval $[0,1]$,*
2. *equation (3.7) considered as an SDE has no solutions $(X,B)$ at all, even defined up to some Markov moment $\tau$ if $\tau > 0$ on the set of positive probability.*

*Proof.* One can modify the construction from Theorem 2 to obtain an equation for which the existence of adapted solutions does not hold on the interval $[0,1/n]$. Considering a system of such equations with independent 2-dimensional Brownian motions yields the required example. Indeed, the existence of a *path-by-path* solution is obvious, let us prove that no adapted solutions to (3.7) exist. Let us assume that there is a solution to SDE (3.7) defined up to the Markov moment $\tau$ and $\mathbb{P}[\tau > 0] \neq 0$. Set $\mathcal{F}_{0+} := \bigcap\limits_{t>0} F_t$, it is well–known that the Brownian motion remains independent of the $\sigma$–field $\mathcal{F}_{0+}$. Since

$$\lim_{k\to\infty} \mathbb{P}[\tau > 2/k] = \mathbb{P}[\tau > 0] > 0,$$

we can find $K \in \mathbb{N}$ such that

$$\mathbb{P}[A_K] \geq \frac{3}{4}\mathbb{P}[A] > 0,$$

where

$$A_K := \{\tau > 2/K\},\ A := \{\tau > 0\}.$$

Let us consider the $K$th 2-dimensional equation and since the number $K$ is fixed for the sake of brevity we will sill denote its solution by $(B,X)$. Taking into account Remark 1 we have the following equality:

$$\mathbb{1}_{A_K} \operatorname{sgn}(B_{1,1/K}) = \mathbb{1}_{A_K}\xi,$$

where

$$\xi := \lim_{n\to\infty} -\operatorname{sgn}(X_{2,1/n}).$$

One can notice that $A \in \mathcal{F}_{0+}$, $\xi$ is measurable with respect to $\mathcal{F}_{0+}$ and $|\xi| = 1$ a.s. Then we have the following chain of equalities:

$$\begin{aligned}
\mathbb{1}_A\xi &= \mathbb{E}\big[\mathbb{1}_A\xi|\mathcal{F}_{0+}\big] = \mathbb{E}\big[\mathbb{1}_{A_K}\xi|\mathcal{F}_{0+}\big] + \mathbb{E}\big[(\mathbb{1}_A - \mathbb{1}_{A_K})\xi|\mathcal{F}_{0+}\big] \\
&= \mathbb{E}\big[\mathbb{1}_{A_K} \operatorname{sgn}(B_{1,1/K})|\mathcal{F}_{0+}\big] + \mathbb{E}\big[(\mathbb{1}_A - \mathbb{1}_{A_K})\xi|\mathcal{F}_{0+}\big] \\
&= \mathbb{E}\big[\mathbb{1}_A \operatorname{sgn}(B_{1,1/K})|\mathcal{F}_{0+}\big] + \mathbb{E}\big[(\mathbb{1}_{A_K} - \mathbb{1}_A)\operatorname{sgn}(B_{1,1/K})|\mathcal{F}_{0+}\big] + \mathbb{E}\big[(\mathbb{1}_A - \mathbb{1}_{A_K})\xi|\mathcal{F}_{0+}\big] \\
&= \mathbb{1}_A\mathbb{E}\big[\operatorname{sgn}(B_{1,1/K})|\mathcal{F}_{0+}\big] + \mathbb{E}\big[(\mathbb{1}_{A_K} - \mathbb{1}_A)\operatorname{sgn}(B_{1,1/K})|\mathcal{F}_{0+}\big] + \mathbb{E}\big[(\mathbb{1}_A - \mathbb{1}_{A_K})\xi|\mathcal{F}_{0+}\big] \\
&= \mathbb{E}\big[(\mathbb{1}_{A_K} - \mathbb{1}_A)\operatorname{sgn}(B_{1,1/K})|\mathcal{F}_{0+}\big] + \mathbb{E}\big[(\mathbb{1}_A - \mathbb{1}_{A_K})\xi|\mathcal{F}_{0+}\big],
\end{aligned}$$

where we have used the fact that

$$\mathbb{E}\big[\operatorname{sgn}(B_{1,1/K})|\mathcal{F}_{0+}\big] = 0.$$

Then:

$$\begin{aligned}\mathbb{P}[A] &= \mathbb{E}\mathbb{1}_A|\xi| \\ &\leq \mathbb{E}\Big|\mathbb{E}\big[(\mathbb{1}_{A_K}-\mathbb{1}_A)\,\mathrm{sgn}(B_{1,1/K})|\mathcal{F}_{0+}\big]\Big| + \mathbb{E}\Big|\mathbb{E}\big[(\mathbb{1}_A-\mathbb{1}_{A_K})\xi|\mathcal{F}_{0+}\big]\Big| \\ &\qquad\qquad\qquad\qquad \leq 2\mathbb{P}[A\setminus A_K] \leq \frac{1}{2}\mathbb{P}[A].\end{aligned}$$

The established ineqaulity contradicts to the assumption $\mathbb{P}[A] = \mathbb{P}[\tau > 0] > 0$. Now it is easy to complete the proof. □

## 4. Pathwise uniqueness without path-by-path uniqueness

**Theorem 4.** *There exists a Borel mapping $b = (b_1, b_2, b_3) : [0,2]\times\mathbb{R}^3 \to \mathbb{R}^3$,*

$$X_t = \int_{[0,t]} b(s, X_s)\,\mathrm{d}s + B_t,\ \ X_0 = 0,\ \ t\in[0,2] \tag{4.1}$$

*such that*

1. *there exists a set of Brownian trajectories $\Omega \subset C([0,2],\mathbb{R}^3)$ of full measure, such that for every $B\in\Omega$ integral equation (4.1) has at least two different solutions defined the whole interval $[0,2]$.*
2. *pathwise uniqueness holds in the sense that there exists a unique solution to the corresponding SDE defined on the whole interval $[0,2]$ and this solution is strong.*

*Proof.* For $t \in [0,1]$ let us set

$$b_1 := 0,$$

$$b_2 := \mathbb{1}_{\{x_2>0\}}\mathbb{1}_{\{x_3>0\}}\Big(\frac{1-x_2}{1-t}+\frac{1}{x_2}\Big) - \mathbb{1}_{\{x_2<0\}}\mathbb{1}_{\{x_3>0\}}\Big(\frac{1-(-x_2)}{1-t}+\frac{1}{-x_2}\Big),$$

$$b_3 := \mathbb{1}_{\{x_3>0\}}\Big(\frac{1-x_3}{1-t}+\frac{1}{x_3}\Big) - \mathbb{1}_{\{x_3<0\}}\Big(\frac{1-(-x_3)}{1-t}+\frac{1}{-x_3}\Big).$$

For $t\in(1,2]$ let us set

$$b_1 := \mathbb{1}_{\{x_1\neq 0\}}\mathbb{1}_{\{x_3>0\}}\frac{1}{x_1},$$

$$\begin{aligned}b_2 := {}& \mathbb{1}_{\{x_1>0\}}\mathbb{1}_{\{x_2>0\}}\mathbb{1}_{\{x_3>0\}}\frac{-1}{2(x_2-1)} + \mathbb{1}_{\{x_1<0\}}\mathbb{1}_{\{x_2<0\}}\mathbb{1}_{\{x_3>0\}}\frac{-1}{2(x_2+1)} \\ &+ \mathbb{1}_{\{x_1>0\}}\mathbb{1}_{\{x_2<0\}}\mathbb{1}_{\{x_3>0\}}\frac{1}{x_2} + \mathbb{1}_{\{x_1<0\}}\mathbb{1}_{\{x_2>0\}}\mathbb{1}_{\{x_3>0\}}\frac{1}{x_2},\end{aligned}$$

$$b_3 := \mathbb{1}_{\{x_3\neq 0\}}\frac{1}{x_3}.$$

First, let us present a *strong* solution to equation (4.1). On the interval $[0,1]$ let $X_3$ be the $(-1)\times 3$–Bessel bridge from $0$ to $-1$ and on the interval $[1,2]$ let $X_3$ be the $(-1)\times 3$–Bessel process with the initial condition $X_{3,1}=-1$. Applying Theorem 1 and Proposition 1 one can see that $X_3$ is indeed adapted to the filtration generated by the Brownian motion $B$ and negative for all $t\in(0,2]$. Let us set $X_1 := B_1$ and $X_2 = B_2$ for $t\in[0,2]$. It is clear that since $X_{3,t}$ is nonpositive for $t\in[0,2]$ then the drifts $b_1, b_2$ are identically equal to zero, therefore the constructed process $(X_1,X_2,X_3)$ is indeed a *strong* solution to SDE 4.1.

Second, one can notice, that if we take the positive "version" of $X_3$, e.g. on the interval $[0,1]$ we can define $X_3$ as the 3–Bessel bridge from $0$ to $1$ and on the interval $[1,2]$ let $X_3$ be the 3–Bessel process with the initial condition $X_{3,1}=1$, then the drifts $b_1,b_2$ coincide with the mappings constructed in Theorem 2, thus in this case the equation for $(X_1,X_2)$ has only non–adapted solutions, but the set of solutions is nonempty. We have shown that with probability one there are at least two different solutions to integral equation (4.1) (corresponding to the positive and negative "versions" of $X_3$) and *path-by-path* uniqueness does not hold.

Finally, let us establish *pathwise* uniqueness for equation (4.1). Let

$$X=(X_1,X_2,X_3)$$

be a solution to stochastic differential equation 4.1 on the interval $[0,2]$ with some Brownian motion $B$. We would like to show that a.s. $X$ coincides with the strong solution presented above. Applying Theorem 1 and Proposition 1 one may notice that a.s. $X_3$ does not change its sign for $t\in(0,2]$. The arguments from the proof of Theorem 2 show that on the set

$$U := \{X_{3,1/2}>0\}\in\mathcal{F}_{1/2}$$

a.s. we have the equality

$$\operatorname{sgn}(B_{1,1}) = -\operatorname{sgn}(X_{2,1/2}).$$

Then:

$$\begin{aligned}\mathbb{E}\big[\operatorname{sgn}(B_{1,1})|\mathcal{F}_{1/2}\big] &= \mathbb{E}\big[\mathbb{1}_U\operatorname{sgn}(B_{1,1})|\mathcal{F}_{1/2}\big]+\mathbb{E}\big[\mathbb{1}_{\Omega\setminus U}\operatorname{sgn}(B_{1,1})|\mathcal{F}_{1/2}\big]\\ &= \mathbb{1}_U\mathbb{E}\big[-\mathbb{1}_U\operatorname{sgn}(X_{2,1/2})|\mathcal{F}_{1/2}\big]+\mathbb{1}_{\Omega\setminus U}\mathbb{E}\big[\operatorname{sgn}(B_{1,1})|\mathcal{F}_{1/2}\big]\\ &= -\mathbb{1}_U\operatorname{sgn}(X_{2,1/2})+\mathbb{1}_{\Omega\setminus U}\mathbb{E}\big[\operatorname{sgn}(B_{1,1})|\mathcal{F}_{1/2}\big].\end{aligned}\tag{4.2}$$

But at the same time

$$\mathbb{E}\big[\operatorname{sgn}(B_{1,1})|\mathcal{F}_{1/2}\big] = [P_{1/2}\operatorname{sgn}](B_{1,1/2}),$$

where $(P_t)_{t\geq 0}$ is the standard heat semigroup defined by the formula

$$P_tf(x)=\int_{\mathbb{R}} f(y)\frac{1}{\sqrt{2\pi}}e^{-\frac{(x-y)^2}{2t}}\,\mathrm{d}y.$$

It easy to see that for almost all $x \in \mathbb{R}$ we have the strict inequalities

$$0 < |P_{1/2}\,\mathrm{sgn}(x)| < 1,$$

consequently, the equality

$$[P_{1/2}\,\mathrm{sgn}](B_{1,1/2}) = -\mathbb{1}_U\,\mathrm{sgn}(X_{2,1/2}) + \mathbb{1}_{\Omega\setminus U}\mathbb{E}\big[\mathrm{sgn}(B_{1,1})|\mathcal{F}_{1/2}\big]$$

can hold a.s. only if $\mathbb{P}[U] = 0$. This means that a.s. $X_3$ is negative for $t \in (0, 2]$ and now it is trivial to complete the proof. □

**Theorem 5.** *Let $B$ be a cylindrical Brownian motion. There exists a Borel mapping $b : [0,1] \times \mathbb{R}^\infty \to \mathbb{R}^\infty$,*

$$X_t = \int_{[0,t]} b(s, X_s)\,\mathrm{d}s + B_t,\ \ X_0 = 0,\ \ t \in [0,1] \tag{4.3}$$

*such that*

1. *pathwise uniqueness holds in the sense that for any Markov moment $\tau$ there exists a unique solution to SDE (4.3) defined up to the moment $\tau$.*
2. *there exists a set of Brownian trajectories $\Omega \subset C([0,2], \mathbb{R}^\infty)$ of full measure such that for every $B \in \Omega$ integral equation (4.3) has at least two different solutions defined on the whole interval $[0,1]$.*

*Proof.* Let $\widetilde{b}$ be the drift constructed in the proof of Theorem 3. Let us define the new drift $b = (b_1, b_2)$ as follows:

$$b_1 : [0,1] \times \mathbb{R}^\infty \to \mathbb{R}^\infty,$$

$$b_1 := \mathbb{1}_{\{x_2>0\}}\widetilde{b}(x_1),\ \ x_1 \in \mathbb{R}^\infty.$$

$$b_2 : [0,1] \times \mathbb{R} \to \mathbb{R},$$

$$b_2 := \mathbb{1}_{\{x_2>0\}}\Big(\frac{1-x_2}{1-t} + \frac{1}{x_2}\Big) - \mathbb{1}_{\{x_2<0\}}\Big(\frac{1-(-x_2)}{1-t} + \frac{1}{-x_2}\Big),\ \ x_2 \in \mathbb{R}.$$

Then the same arguments as in the proof of Theorem 3, Theorem 4 show that the drift $b$ has the desired properties. □

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