Heat Kernels and Besov Spaces Associated with Second Order Divergence Form Elliptic Operators

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Abstract Let $\mathcal{L} = -\mathrm{div}(A\nabla)$ be a uniformly elliptic operator in $\mathbb{R}^n$ with real, symmetric, measurable coefficients. We study the identity of two families of Besov spaces $B^{s,\mathcal{L}}_{p,q}$ and $B^s_{p,q}$, where the former one is defined using the heat semigroup of $\mathcal{L}$, while the latter one is defined in a classical way, using the metric structure of $\mathbb{R}^n$. A sharp range of parameters p, q, s ensuring the identity $B^{s,\mathcal{L}}_{p,q} = B^s_{p,q}$ is given by a Hardy-Littlewood-Sobolev-Kato diagram.

Contents

1 Introduction

Let $A := (a_{i,j})$ be an $n \times n$ real-valued matrix function defined on the Euclidean space $\mathbb{R}^n$, which is symmetric (namely, $a_{i,j} \equiv a_{j,i}$) and satisfies the *uniform ellipticity condition* that there exists

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$\Lambda > 0$ such that for any $\xi \in \mathbb{R}^n$ and a.e. $x \in \mathbb{R}^n$,

$$\Lambda^{-1}|\xi|^2 \le \sum_{i,j=1}^{n} a_{i,j}(x)\,\xi_i\xi_j \le \Lambda|\xi|^2. \tag{1.1}$$

Define the *bilinear form* $\mathcal{E}$ associated with A by setting for any $f,\, g \in \mathrm{dom}\,[\mathcal{E}]$,

$$\mathcal{E}(f,\, g) := \int_{\mathbb{R}^n} \sum_{i,j=1}^{n} a_{i,j}(x)\frac{\partial}{\partial x_j} f(x)\frac{\partial}{\partial x_i} g(x)\,dx,$$

where the domain $\mathrm{dom}\,[\mathcal{E}] := W^{1,2}(\mathbb{R}^n)$ denotes the usual first order Sobolev space on $\mathbb{R}^n$.

It is well-known that the bilinear form $\mathcal{E}$ is symmetric, closed and Markovian in $L^2(\mathbb{R}^n)$, namely, $\mathcal{E}$ is a Dirichlet form. Thus, by the classical theory of Dirichlet forms [21], there exists a correspondence between $\mathcal{E}$ and its associated operator $\mathcal{L}$, which is nonngegative and self-adjoint in $L^2(\mathbb{R}^n)$. We call the operator $\mathcal{L}$, written formally $\mathcal{L} := -\mathrm{div}(A\nabla)$, the *second order divergence form elliptic operator* with the domain $\mathrm{dom}\,[\mathcal{L}]$ consisting of all $f \in \mathrm{dom}\,[\mathcal{E}]$ satisfying the following condition: there exists $g := \mathcal{L}f \in L^2(\mathbb{R}^n)$ such that for any $h \in \mathrm{dom}\,[\mathcal{E}]$,

$$\mathcal{E}(f,\, h) = (g,\, h)_{L^2(\mathbb{R}^n)},$$

where $(\cdot,\, \cdot)_{L^2(\mathbb{R}^n)}$ denotes the $L^2(\mathbb{R}^n)$ inner product.

The second order divergence form elliptic operator $\mathcal{L}$ generates a strongly continuous contractive Markovian semigroup $\{e^{-t\mathcal{L}}\}_{t>0}$ on $L^2(\mathbb{R}^n)$. The regularity theory of the associated parabolic problems (see [2, 3, 14, 27, 40, 41] and their references) asserts that $e^{-t\mathcal{L}}$ has the integral kernel $P_t(x,\, y)$ (called also *heat kernel*) that is defined on $\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n$, is symmetric in x, y, and satisfies the following properties:

(a) **Gaussian upper and lower bounds**: for any $t \in (0,\, \infty)$ and $x,\, y \in \mathbb{R}^n$,

$$\frac{c_0}{t^{n/2}} \exp\left\{-\frac{\alpha_0|x-y|^2}{t}\right\} \le P_t(x,\, y) \le \frac{c_1}{t^{n/2}} \exp\left\{-\frac{\alpha_1|x-y|^2}{t}\right\}. \tag{1.2}$$

(b) **Hölder continuity**: $\forall\ t \in (0,\, \infty)$, $P_t(x,\, y)$ is jointly continuous in $(x,\, y)$ and, for any x, $y,\, y' \in \mathbb{R}^n$ satisfying $|y - y'| \le \sqrt{t}$,

$$\left|P_t(x,\, y) - P_t(x,\, y')\right| \le c_2 \left(\frac{|y-y'|}{\sqrt{t}}\right)^{\Theta} \frac{1}{t^{n/2}} \exp\left\{-\frac{\alpha_2|x-y|^2}{t}\right\}, \tag{1.3}$$

where all constants Θ, c_0, c_1, c_2, α_0, α_1, α_2 are positive and depend only on n and Λ; besides $\Theta \in (0,\, 1)$.

The properties (a) and (b) will be denoted shortly by $\mathbf{G}(\Theta)$. Let us emphasize that the value of Θ can be arbitrary small (see [55, 46]).

It is also known that the semigroup $\{e^{-t\mathcal{L}}\}_{t>0}$ is conservative (see, for example, [4, Chapter 2.5]), that is, $e^{-t\mathcal{L}}1 = 1$, which implies that for any $t \in (0,\, \infty)$ and $x \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} P_t(x,\, y)\,dy \equiv 1. \tag{1.4}$$

Let us also mention the following further properties of operator $\mathcal{L}$ in the Lebesgue space $L^p(\mathbb{R}^n)$, $p \in (1,\, \infty)$, that follow from $\mathbf{G}(\Theta)$ (except for (i) that follows from (1.4)).

(i) **Extrapolation of semigroup to** $L^p(\mathbb{R}^n)$: the semigroup $\{T_t\}_{t>0} = \{e^{-t\mathcal{L}}\}_{t>0}$ on $L^2(\mathbb{R}^n)$ can be extrapolated to a strongly continuous contractive semigroup on $L^p(\mathbb{R}^n)$ for any $p \in (1, \infty)$, which is consistent with the original semigroup $\{e^{-t\mathcal{L}}\}_{t>0}$ in $L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ (see [13, 48] and their references). We use the same notation $\{T_t\}_{t>0}$ for all the semigroups on $L^p(\mathbb{R}^n)$.

(ii) **Independence of spectrum in** $L^p(\mathbb{R}^n)$: let $p \in (1, \infty)$ and $\mathcal{L}_p$ be the generator of the semigroup $\{T_t\}_{t>0}$ on $L^p(\mathbb{R}^n)$. The Gaussian estimates (1.2) imply that the spectrum $\sigma(\mathcal{L}_p) \subset [0, \infty)$ is independent of $p \in (1, \infty)$ (see [1, 15]). We use the same notation $\mathcal{L}$ to denote all the generators $\mathcal{L}_p$ in $L^p(\mathbb{R}^n)$, and $\{e^{-t\mathcal{L}}\}_{t>0}$ to denote the associated semigroup on $L^p(\mathbb{R}^n)$.

(iii) **Functional calculus of** $\mathcal{L}$ **in** $L^p(\mathbb{R}^n)$: for any $p \in (1, \infty)$, the operator $\mathcal{L}$ satisfies the *bounded* H^∞ *functional calculus in* $L^p(\mathbb{R}^n)$ (see [5, 16]). More precisely, for any $\mu \in (0, \pi)$, let $\Sigma_\mu := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \mu\}$ be an open sector in the complex plane $\mathbb{C}$ and $H^\infty(\Sigma_\mu)$ the space of all bounded holomorphic functions in Σ_μ. Then there exists a positive constant C such that for any $\varphi \in H^\infty(\Sigma_\mu)$ and $f \in L^p(\mathbb{R}^n)$,

$$\|\varphi(\mathcal{L})f\|_{L^p(\mathbb{R}^n)} \leq C\|\varphi\|_{L^\infty(\Sigma_\mu)}\|f\|_{L^p(\mathbb{R}^n)}, \tag{1.5}$$

where $\varphi \to \varphi(\mathcal{L})$ is a bounded homomorphism from the Banach algebra $H^\infty(\Sigma_\mu)$ into the Banach algebra $\mathscr{L}(L^p(\mathbb{R}^n))$ of all bounded linear operators on $L^p(\mathbb{R}^n)$. The H^∞ functional calculus is consistent with the functional calculus of L^2-spectral theory and can be extended from $H^\infty(\Sigma_\mu)$ to the following *extended Dunford-Riesz class* $\mathscr{E}(\Sigma_\mu)$, which is defined for some fixed $c, s, s' > 0$ as follows:

$$\mathscr{E}(\Sigma_\mu) := \left\{f \text{ is holomorphic in } \Sigma_\mu : \text{ for any } \xi \in \Sigma_\mu, |f(\xi)| \leq c \max\left\{|\xi|^s, |\xi|^{-s'}\right\}\right\}. \tag{1.6}$$

As the functions in $\mathscr{E}(\Sigma_\mu)$ may be unbounded, this extension may allow us to define some unbounded operators such as the fractional power $\mathcal{L}^s$ of $\mathcal{L}$ for any $s \in \mathbb{C}_+$ (namely, $s \in \mathbb{C}$ with $\operatorname{Re} s > 0$). See [39, 12, 28] for more systematic descriptions on the H^∞ functional calculus.

(iv) **Holomorphic semigroup of** $\mathcal{L}$ **on** $L^p(\mathbb{R}^n)$: By the uniform ellipticity condition (1.1), we know that $\mathcal{L}$ is a 0-sectorial operator in $L^2(\mathbb{R}^n)$ (see, for example, [28, Chapter 2] for the definition of the sectorial operator). This immediately implies that $\mathcal{L}$ generates a bounded holomorphic semigroup $\{e^{-z\mathcal{L}}\}$ on $L^2(\mathbb{R}^n)$ for any $z \in \mathbb{C}_+$ (see [28]). This bounded holomorphic semigroup $\{e^{-z\mathcal{L}}\}$ can be extrapolated from $L^2(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for any $p \in (1, \infty)$, due to the property $\mathbf{G}(\Theta)$ (see [42, 29]). Moreover, it was proved in [13, 43] that $e^{-z\mathcal{L}}$ has an integral kernel P_z satisfying the following estimate:

$$|P_z(x, y)| \leq \frac{c_3}{(\operatorname{Re} z)^{n/2}} \exp\left\{-\frac{\alpha_3|x-y|^2}{|z|}\right\}, \tag{1.7}$$

for any $z \in \mathbb{C}_+$ and $x, y \in \mathbb{R}^n$, where c_3 and α_3 are positive constants.

The aforementioned analytical properties of $\mathcal{L}$ form basis for us to construct function spaces related to smoothness properties of $\mathcal{L}$. For example, let us introduce the following *heat Besov space* $B^{s,\mathcal{L}}_{p,q}$ for any $p, q \in (1, \infty)$ and $s \in [0, \infty)$ by

$$B^{s,\mathcal{L}}_{p,q} := \left\{f \in L^p(\mathbb{R}^n) : \|f\|_{B^{s,\mathcal{L}}_{p,q}} := \|f\|_{L^p(\mathbb{R}^n)} + \|f\|_{\dot{B}^{s,\mathcal{L}}_{p,q}} < \infty\right\}, \tag{1.8}$$

where

$$\|f\|_{\dot{B}^{s,\mathcal{L}}_{p,q}} := \left\{ \int_0^\infty \left[t^{-s/2} \left\| (t\mathcal{L})^k e^{-t\mathcal{L}} f \right\|_{L^p(\mathbb{R}^n)} \right]^q \frac{dt}{t} \right\}^{1/q} \tag{1.8a}$$

with $k \in \mathbb{Z}_+ \cap (s/2, \infty)$. Clearly, $B^{s,\mathcal{L}}_{p,q}$ is a Banach space. Moreover, as pointed out in [35, Theorem 6.1] or [26, Proposition 2.9], the norms $\|\cdot\|_{B^{s,\mathcal{L}}_{p,q}}$ in (1.8) are equivalent for different choices of $k \in \mathbb{Z}_+ \cap (s/2, \infty)$, so that $B^{s,\mathcal{L}}_{p,q}$ does not depend on k.

In the case $\mathcal{L} = -\Delta$, where Δ is the Laplacian, this family of Besov spaces was first introduced by Taibleson [49] to characterize the classical Besov spaces via temperatures (Gaussian kernel) and harmonic functions (Poisson kernel) by using the Littlewood-Paley method (see [49, 50, 51, 20] and their references). Triebel [54] introduced a certain family of Besov spaces in a more general set up, using instead of $\{e^{-t\mathcal{L}}\}_{t>0}$ an abstract semigroup $\{T_t\}_{t>0}$. Haase [28] proved that if $\{T_t\}_{t>0}$ is generated by a sectorial operator $\mathcal{L}$, the Besov spaces of Triebel coincide with those defined as in (1.8)-(1.8a). In the setting of metric measure space, the heat Besov spaces were introduced by Hu and Zähle [33]. See also [8, 35, 26, 38, 7] for recent developments of this topic.

On the other hand, it is known that one of the main motivations for O. V. Besov to introduce his notion of Besov spaces was to provide a unified scale of function spaces that contain both the Sobolev and Hölder spaces (see [52] for an excellent historical review). As both of those spaces are defined via the difference, it is natural to define the Besov spaces in the same way. To be precise, for any $p, q \in (1, \infty)$ and $s \in (0, \infty)$, define the *Lipschitz Besov space* $B^s_{p,q}$ by

$$B^s_{p,q} := \left\{ f \in L^p(\mathbb{R}^n) : \|f\|_{B^s_{p,q}} := \|f\|_{L^p(\mathbb{R}^n)} + \|f\|_{\dot{B}^s_{p,q}} < \infty \right\}, \tag{1.9}$$

where

$$\|f\|_{\dot{B}^s_{p,q}} := \left\{ \int_0^\infty \left[\iint_{|x-y|<r} \frac{|f(x) - f(y)|^p}{r^{n+sp}} \, dy \, dx \right]^{q/p} \frac{dr}{r} \right\}^{1/q}. \tag{1.9a}$$

Let us emphasize that this definition does not depend on the operator $\mathcal{L}$.

It is known that for any $s \in (0, 1)$ and $p \in (1, \infty)$, the space $B^s_{p,p}$ equals to the fractional order Sobolev space $W^{s,p}(\mathbb{R}^n)$ defined via the Gagliardo norm. Also, for any $s \in (0, 1)$, with the usual modification in (1.9)-(1.9a) when $p = q = \infty$, the space $B^s_{\infty,\infty}$ coincides with the Hölder space $C^s(\mathbb{R}^n)$ (see [51, 19, 20, 52]). In the general setting of metric measure space, the Lipschitz Besov space also plays an important role in the geometric analysis of the underlying space (see [34, 44, 24, 56, 25, 45, 27]).

Returning to our setting, we have now two families of Besov spaces $B^{s,\mathcal{L}}_{p,q}$ and $B^s_{p,q}$. In 2010, Pietruska-Pałuba [44] raised a question on the relationships between these two families of Besov spaces. The following is the main problem considered in this paper.

Main problem: when the spaces $B^{s,\mathcal{L}}_{p,q}$ and $B^s_{p,q}$ are identical?

If $\mathcal{L} = -\Delta$, then the identity

$$B^{s,\mathcal{L}}_{p,q} = B^s_{p,q} \tag{1.10}$$

is known to be true for the full range of all $p, q \in (1, \infty)$ and $s \in (0, 1)$ (see [51, 52]). For general $\mathcal{L}$ as above, the situation is more complicated. Hu and Zähle [33] first showed that (1.10) holds for $p = q = 2$ and any $s \in (0, 1)$. Later, Grigor'yan and Liu [26] proved that (1.10) holds for any $p, q \in (1, \infty)$ and any $s \in (0, \Theta)$, where Θ denotes the Hölder exponent as in (1.3) (see also [56] for some similar results in the setting of RD-spaces). Figure 1 below shows the range of parameters p and s in the $(1/p, s)$-plane, where (1.10) was known before.

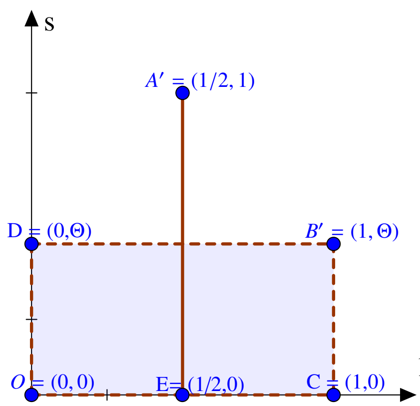

Figure 1: previously known range for p and s

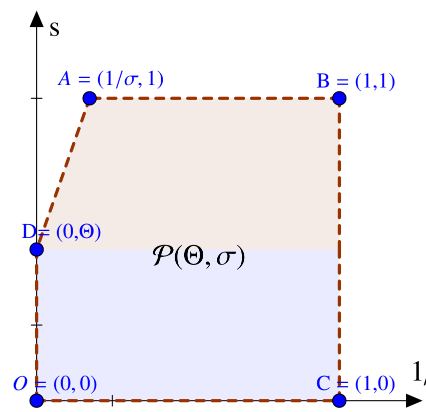

Figure 2: new range for p and s

Our main result says that this range can be extended as on Figure 2. More precisely, let

$$\sigma \in (2, \infty) \tag{1.11}$$

be *the upper limit of the numbers* $p \in (1, \infty)$ *satisfying that the Riesz transform* $\nabla \mathcal{L}^{-1/2}$ *is bounded in* $L^p(\mathbb{R}^n)$. For Θ and σ respectively as in (1.3) and (1.11), let $\mathcal{P}(\Theta, \sigma)$ be the *Hardy-Littlewood-Sobolev-Kato diagram* in the $(1/p, s)$-plane defined by

$$\mathcal{P}(\Theta, \sigma) := \left\{ \left(\frac{1}{p}, s \right) \in (0, 1) \times (0, 1) : \frac{1}{p} \in \begin{cases} (0, 1), & s \in (0, \Theta), \\ (\frac{s-\Theta}{(1-\Theta)\sigma}, 1), & s \in [\Theta, 1) \end{cases} \right\}. \tag{1.12}$$

Our result is stated in the next theorem.

Theorem 1.1. *Let* $\mathcal{P}(\Theta, \sigma)$ *be the open range of* $(1/p, s)$ *as on Figure 2 (see also* (1.12)*). Then for any* $(1/p, s) \in \mathcal{P}(\Theta, \sigma)$ *and* $q \in (1, \infty)$*,*

$$B^{s,\mathcal{L}}_{p,q} = B^s_{p,q}. \tag{1.13}$$

The range $\mathcal{P}(\Theta, \sigma)$ of parameter $(1/p, s)$ is optimal in the entire class $\mathscr{E}(\mathbb{R}^n)$ of all operators $\mathcal{L}$ as above. Indeed, denote by $\mathcal{P}$ the intersection of all $\mathcal{P}(\Theta, \sigma)$ for any $\Theta \in (0, 1)$ and $\sigma > 2$ (see Figure 5 and (4.24)).

Theorem 1.2. *For any* $(1/p, s) \in (0, 1) \times (0, 1) \setminus \mathcal{P}$ *with* $\mathcal{P}$ *as on Figure 5 (see also* (4.24)*), there exists* $\mathcal{L} \in \mathscr{E}(\mathbb{R}^n)$ *such that for any* $q \in (1, \infty)$*,*

$$B^{s,\mathcal{L}}_{p,q} \neq B^s_{p,q}.$$

From Theorems 1.1 and 1.2, it seems that the parameter q is irrelevant to the identity (1.13). Also, as it follows from the proof of Theorem 1.1, the inclusion $B^s_{p,q} \subset B^{s,\mathcal{L}}_{p,q}$ is in fact true for the entire range $(1/p, s) \in (0,1) \times (0,1)$. Let us emphasize that under some additional assumptions on $\mathcal{L}$, the parameters Θ and σ may reach their boundary values. For example, if the coefficients of the matrix A belong to the Hölder space $C^\gamma(\mathbb{R}^n)$ for some $\gamma \in (0, 1)$, then by [17, Theorem 1.3], we have that $\sigma = \infty$, which implies that $\mathcal{P}(\Theta, \sigma) = (0, 1) \times (0, 1)$ so that the identity (1.13) holds for the full range of all $p, q \in (1, \infty)$ and $s \in (0, 1)$ (see Remark 5.11).

Both Theorems 1.1 and 1.2 are proved in Section 4.2. The new idea of the proof of Theorem 1.1 consists of using two versions of Triebel-Lizorkin spaces (heat and Lipschitz) that are denoted by $F^{s,\mathcal{L}}_{p,q}$ and $F^s_{p,q}$, and defined similarly to the above two versions of Besov spaces (see (2.4)-(2.4a) and (2.6)-(2.6a)). We first consider the endpoint case $s = 1$ when it is well-known that

$$F^1_{p,2}(\mathbb{R}^n) = W^{1,p}(\mathbb{R}^n) \text{ and } F^{1,\mathcal{L}}_{p,2} = \mathrm{dom}_p[\mathcal{L}^{1/2}],$$

where $F^1_{p,2}(\mathbb{R}^n)$ is the classical Triebel-Lizorkin space on $\mathbb{R}^n$, which is a suitable substitute of $F^1_{p,2}$ when the smoothness parameter $s = 1$ (see Remark 2.3). By the result of [4, Chapter 4] (see also [5]) solving the L^p-Kato square root problem, we have

$$W^{1,p}(\mathbb{R}^n) = \mathrm{dom}_p[\mathcal{L}^{1/2}]$$

for all $p \in (1, \sigma)$, where σ is as above, which yields the identity

$$F^1_{p,2}(\mathbb{R}^n) = F^{1,\mathcal{L}}_{p,2}$$

for all $p \in (1, \sigma)$. Then we prove the identity

$$F^s_{p,q} = F^{s,\mathcal{L}}_{p,q}$$

for all $s \in (0, \Theta)$ and arbitrary p, q, which is the most technical part of the proof. By using Hardy-Littlewood-Sobolev-Kato estimates similar to [4], we extend the range of parameters p and s to $\mathcal{P}(\Theta, \sigma)$ as on Figure 2. In the last step in the proof we transfer the above identity of Triebel-Lizorkin spaces to Besov spaces, by using real interpolation.

In order to prove the sharpness of the range of p and s (Theorem 1.2), we use a counterexample from [5, p. 120] (see Lemma 4.5 below), where the authors proved that $\inf \sigma = 2$ over all $\mathcal{L} \in \mathscr{E}(\mathbb{R}^n)$. Let us emphasize that under higher regularity of the coefficients one can extend the identity (1.13) also to $s \geq 1$ (see Section 5).

As illustrated above, the idea used to prove Theorems 1.1 and 1.2 depends heavily on the Euclidean structure of the underlying space. It is natural to ask if similar results can be extended to the setting of a general metric measure space. We consider this question in a forthcoming paper [10], where it is proved that (1.10) holds for parameters lie in a proper subset of (1.12) by applying some new methods that are different to the present paper.

This paper is organized as follows. Sections 2 and 3 are preparations. In Section 2 we review some basic notions and properties of the function spaces associated with $\mathcal{L}$ which include three versions of Besov and Triebel-Lizorkin spaces: heat, Lipschitz and spectral. Then in Section 3, we prove the identity of the spectral and Lipschitz versions of Triebel-Lizorkin spaces for any $s \in (0, \Theta)$. In Section 4 we prove Theorems 1.1 and 1.2. Finally, in Section 5, we extend the above considerations to the case $s \in (0,\mu)$ for some $\mu > 1$, by adding higher order regularity assumption on the heat kernel.

Notation. Let $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. For any subset $E \subset \mathbb{R}^n$, $\mathbf{1}_E$ denotes its *characteristic function*. We use C to denote a *positive constant* that is independent of the main parameters involved, whose value may differ from line to line. Constants with subscripts, such as C_1, do not change in different occurrences. For any function f on $\mathbb{R}^n$, let $\mathcal{M}(f)$ be its *Hardy-Littlewood maximal function* defined by setting for any $x \in \mathbb{R}^n$,

$$\mathcal{M}(f)(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)|\, dy, \tag{1.14}$$

where the supremum is taken over all balls in $\mathbb{R}^n$ containing x. For any $\xi \in \mathbb{R}^n$, let

$$\widehat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\, dx \tag{1.15}$$

be the *Fourier transform* of f. For any qualities f, g and h, if $f \le Cg$, we write $f \lesssim g$ and, if $f \lesssim g \lesssim f$, we then write $f \simeq g$. We also use the following convention: if $f \le Cg$ and $g = h$ or $g \le h$, we then write $f \lesssim g \simeq h$ or $f \lesssim g \lesssim h$, *rather than* $f \lesssim g = h$ or $f \lesssim g \le h$. Finally, for any $s \in \mathbb{R}$, we use $[s]$ to denote the largest integer not greater than s and $\{s\}$ to denote the number $s - [s] \in [0, 1)$.

2 Function spaces associated with $\mathcal{L}$

In this section, we provide the preliminaries on the function spaces associated with the second order divergence form elliptic operator $\mathcal{L}$. We review some basic notions and properties of the Besov and Triebel-Lizorkin spaces from three different point of views: heat, Lipschitz and spectral, which are introduced respectively in Sections 2.1, 2.2 and 2.3. The reader may skip this section and go directly to Section 3 if she or he is familiar with these notions and properties. We also refer the reader to [56, 8, 28, 35, 26, 32, 37, 38, 22, 9, 7] and their references for a complete description of this topic.

2.1 The heat Besov and Triebel-Lizorkin spaces

Let p, $q \in (1, \infty)$ and $s \in [0, \infty)$. Recall that the *heat Besov space* $B^{s,\mathcal{L}}_{p,q}$ is defined by setting

$$B^{s,\mathcal{L}}_{p,q} := \left\{ f \in L^p(\mathbb{R}^n) : \ \|f\|_{B^{s,\mathcal{L}}_{p,q}} := \|f\|_{L^p(\mathbb{R}^n)} + \|f\|_{\dot{B}^{s,\mathcal{L}}_{p,q}} < \infty \right\}, \tag{2.1}$$

where

$$\|f\|_{\dot{B}^{s,\mathcal{L}}_{p,q}} := \left\{ \int_0^\infty \left[t^{-s/2} \left\| (t\mathcal{L})^k e^{-t\mathcal{L}} f \right\|_{L^p(\mathbb{R}^n)} \right]^q \frac{dt}{t} \right\}^{1/q} \tag{2.1a}$$

with $k \in \mathbb{Z}_+ \cap (s/2, \infty)$.

Remark 2.1. (i) The norms $\|\cdot\|_{B^{s,\mathcal{L}}_{p,q}}$ in (2.1) are equivalent for different choices of $k \in \mathbb{Z}_+ \cap (s/2, \infty)$ (see [35, Theorem 6.1] or [26, Proposition 2.9]).

(ii) By the bounded functional calculus in $L^p(\mathbb{R}^n)$, we known that for any $t \in (0, \infty)$ and $f \in L^p(\mathbb{R}^n)$, $\|(t\mathcal{L})^k e^{-t\mathcal{L}} f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$, which then implies that for any $k \in \mathbb{Z}_+ \cap (s/2, \infty)$ and $f \in L^p(\mathbb{R}^n)$,

$$\|f\|_{B^{s,\mathcal{L}}_{p,q}} \simeq \|f\|_{L^p(\mathbb{R}^n)} + \left(\int_0^1 \left[t^{-s/2} \left\| (t\mathcal{L})^k e^{-t\mathcal{L}} f \right\|_{L^p(\mathbb{R}^n)} \right]^q \frac{dt}{t} \right)^{1/q}. \tag{2.2}$$

(iii) The spaces $B^{s,\mathcal{L}}_{p,q}$ belong to a more general family of Komatsu type spaces (or McIntosh-Yagi spaces) $\mathcal{X}^s_{p,q}$, which arise to characterize the immediate spaces of real interpolation of the domains of the fractional powers of sectorial operators in Banach spaces (see [36, 54, 28]).

The following real interpolation result on the heat Besov spaces $B^{s,\mathcal{L}}_{p,q}$ is essentially established in [28, Chapter 6].

Proposition 2.2 ([28])**.** *Let $p, q \in (1, \infty)$ and $s \in (0, \infty)$. Then*

(i) *for any $\theta \in (0, 1)$,*

$$\left(L^p(\mathbb{R}^n), \operatorname{dom}_p[\mathcal{L}^{s/2}] \right)_{\theta,q} = B^{s\theta,\mathcal{L}}_{p,q},$$

where for any $p \in (1, \infty)$ and $s \in (0, \infty)$, $\operatorname{dom}_p[\mathcal{L}^{s/2}]$ denotes the domain of the fractional power $\mathcal{L}^{s/2}$ of $\mathcal{L}$ in $L^p(\mathbb{R}^n)$, endowed with the norm

$$\|f\|_{\operatorname{dom}_p[\mathcal{L}^{s/2}]} := \|f\|_{L^p(\mathbb{R}^n)} + \left\| \mathcal{L}^{s/2} f \right\|_{L^p(\mathbb{R}^n)} \simeq \left\| (1+\mathcal{L})^{s/2} f \right\|_{L^p(\mathbb{R}^n)}; \tag{2.3}$$

(ii) *for any $\theta \in (0, 1)$, $q_0, q_1 \in (1, \infty)$ and $s_0, s_1 \in (0, \infty)$,*

$$\left(B^{s_0,\mathcal{L}}_{p,q_0}, B^{s_1,\mathcal{L}}_{p,q_1} \right)_{\theta,q} = B^{s,\mathcal{L}}_{p,q},$$

where $s \in (0, \infty)$ satisfies $s = s_0(1-\theta) + s_1\theta$.

Similar to the heat Besov space, one can also define the *heat Triebel-Lizorkin space* $F^{s,\mathcal{L}}_{p,q}$ with $p, q \in (1, \infty)$ and $s \in [0, \infty)$ by setting

$$F^{s,\mathcal{L}}_{p,q} := \left\{ f \in L^p(\mathbb{R}^n) : \ \|f\|_{F^{s,\mathcal{L}}_{p,q}} := \|f\|_{L^p(\mathbb{R}^n)} + \|f\|_{\dot{F}^{s,\mathcal{L}}_{p,q}} < \infty \right\}, \tag{2.4}$$

where

$$\|f\|_{\dot{F}^{s,\mathcal{L}}_{p,q}} := \left\| \left(\int_0^\infty \left| t^{-s/2} (t\mathcal{L})^k e^{-t\mathcal{L}} f \right|^q \frac{dt}{t} \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \tag{2.4a}$$

with $k \in \mathbb{Z}_+ \cap (s/2, \infty)$. The heat Triebel-Lizorkin space $F^{s,\mathcal{L}}_{p,q}$ satisfies properties similar to those of $B^{s,\mathcal{L}}_{p,q}$ (see [37, 35] and their references for more related properties). Here, we only note the following identity on the space $F^{s,\mathcal{L}}_{p,q}$ that for any $p \in (1, \infty)$,

$$F^{0,\mathcal{L}}_{p,2} = L^p(\mathbb{R}^n),$$

which can be derived from the $\mathcal{L}$-adapted square function characterization of the space $L^p(\mathbb{R}^n)$ (see [4, Theorem 6.1]).

2.2 The Lipschitz Besov and Triebel-Lizorkin spaces

Let $p, q \in (1, \infty)$ and $s \in (0, \infty)$. Recall that the *Lipschitz Besov space* $B^s_{p,q}$ is defined by

$$B^s_{p,q} := \left\{ f \in L^p(\mathbb{R}^n) : \ \|f\|_{B^s_{p,q}} := \|f\|_{L^p(\mathbb{R}^n)} + \|f\|_{\dot{B}^s_{p,q}} < \infty \right\}, \tag{2.5}$$

where

$$\|f\|_{\dot{B}^s_{p,q}} := \left\{ \int_0^\infty \left[\iint_{|x-y|<r} \frac{|f(x) - f(y)|^p}{r^{n+sp}} \, dy \, dx \right]^{q/p} \frac{dr}{r} \right\}^{1/q}. \tag{2.5a}$$

Similarly, the *Lipschitz Triebel-Lizorkin space* $F^s_{p,q}$ with $p, q \in (1, \infty)$ and $s \in (0, \infty)$ is defined by

$$F^s_{p,q} := \left\{ f \in L^p(\mathbb{R}^n) : \ \|f\|_{F^s_{p,q}} := \|f\|_{L^p(\mathbb{R}^n)} + \|f\|_{\dot{F}^s_{p,q}} < \infty \right\}, \tag{2.6}$$

where

$$\|f\|_{\dot{F}^s_{p,q}} := \left\| \left[\int_0^\infty \left(\int_{B(\cdot,r)} \frac{|f(\cdot) - f(y)|}{r^{n+s}} \, dy \right)^q \frac{dr}{r} \right]^{1/q} \right\|_{L^p(\mathbb{R}^n)}. \tag{2.6a}$$

Remark 2.3. (i) Observe in (2.5)-(2.5a) and (2.6)-(2.6a), the spaces $B^s_{p,q}$ and $F^s_{p,q}$ don't depend on the operator $\mathcal{L}$. Moreover, from their definitions, it is easy to see $B^s_{p,q}$ and $F^s_{p,q}$ are consistent with the classical Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ and Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$ on $\mathbb{R}^n$ for any p, $q \in (1, \infty)$ and $s \in (0, 1)$ (see [51, 52]).

(ii) For $s \geq 1$, as the walk dimension of $\mathbb{R}^n$ is 2, the above Lipschitz Besov and Triebel-Lizorkin spaces may degenerate to $\{0\}$ (see, for example, [34, 24, 26]). To obtain non-trivial and more suitable function spaces, we need to replace the first order difference in (2.5)-(2.5a) and (2.6)-(2.6a) by higher order differences (see [52, Chapter 1] for a more detailed explanation and Section 5.2 below).

We end this subsection with the following characterizations of the norm of the space $F^s_{p,q}$.

Lemma 2.4. *Let $p, q \in (1, \infty)$ and $s \in (0, \infty)$. Assume that $f \in L^p(\mathbb{R}^n)$. Then the following two assertions hold.*

(i) $\|f\|_{F^s_{p,q}} \simeq \|f\|_{\widetilde{F^s_{p,q}}} := \|f\|_{L^p(\mathbb{R}^n)} + \|f\|_{\widetilde{\dot{F}^s_{p,q}}}$, *where*

$$\|f\|_{\widetilde{\dot{F}^s_{p,q}}} := \left\| \left[\int_0^1 \left(\int_{B(\cdot,r)} \frac{|f(\cdot) - f(y)|}{r^{n+s}} \, dy \right)^q \frac{dr}{r} \right]^{1/q} \right\|_{L^p(\mathbb{R}^n)} \tag{2.7}$$

and the implicit constants are independent of f.

(ii) $\|f\|_{F^s_{p,q}} \simeq \|f\|_{\widetilde{\widetilde{F^s_{p,q}}}} := \|f\|_{L^p(\mathbb{R}^n)} + \|f\|_{\widetilde{\widetilde{\dot{F}^s_{p,q}}}}$, *where*

$$\|f\|_{\widetilde{\widetilde{\dot{F}^s_{p,q}}}} := \left\| \left[\sum_{j=0}^\infty \left(2^{j(s+n)} \int_{B(\cdot,2^{-j})} |f(\cdot) - f(y)| \, dy \right)^q \right]^{1/q} \right\|_{L^p(\mathbb{R}^n)} \tag{2.8}$$

and the implicit constants are independent of f.

Proof. We first prove (i). By (2.6), (2.6a) and (2.7), we only need to show that

$$\mathrm{A} := \left\| \left[\int_1^\infty \left(\int_{B(\cdot,r)} \frac{|f(\cdot)-f(y)|}{r^{n+s}}\,dy \right)^q \frac{dr}{r} \right]^{1/q} \right\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}. \tag{2.9}$$

To this end, we write

$$\mathrm{A} \leq \left\{ \int_{\mathbb{R}^n} \left[\int_1^\infty \left(\int_{B(x,r)} \frac{|f(x)|+|f(y)|}{r^{n+s}}\,dy \right)^q \frac{dr}{r} \right]^{p/q} dx \right\}^{1/p} =: \mathrm{A}_1 + \mathrm{A}_2. \tag{2.10}$$

As $sq > 0$, we have

$$\mathrm{A}_1 \lesssim \|f\|_{L^p(\mathbb{R}^n)} \left[\int_1^\infty r^{-sq} \frac{dr}{r} \right] \lesssim \|f\|_{L^p(\mathbb{R}^n)}. \tag{2.11}$$

For A_2, by the boundedness of the Hardy-Littlewood maximal function $\mathcal{M}$ defined as in (1.14) on $L^p(\mathbb{R}^n)$ and an argument similar to (2.11), we find

$$\mathrm{A}_2 \lesssim \left\| \left[\int_1^\infty r^{-sq} \left(\mathcal{M}(f)(\cdot)\right)^q \frac{dr}{r} \right]^{1/q} \right\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)},$$

which combined with (2.10) and (2.11) shows (2.9) holds true. This proves (i).

We now turn to the proof of (ii). By (2.7) and (2.8), we have that

$$\begin{aligned} \|f\|_{\widetilde{\dot{F}^s_{p,q}}} &\simeq \left\| \left[\sum_{j=0}^\infty \int_{2^{-j-1}}^{2^{-j}} \left(2^{j(s+n)} \int_{B(\cdot,r)} |f(\cdot)-f(y)|\,dy \right)^q \frac{dr}{r} \right]^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\ &\lesssim \left\| \left[\sum_{j=0}^\infty \left(2^{j(s+n)} \int_{B(\cdot,2^{-j})} |f(\cdot)-f(y)|\,dy \right)^q \right]^{1/q} \right\|_{L^p(\mathbb{R}^n)} \simeq \|f\|_{\widetilde{\widetilde{\dot{F}^s_{p,q}}}}. \end{aligned} \tag{2.12}$$

On the other hand, by using an argument similar to (2.12), we obtain

$$\begin{aligned} \|f\|_{\widetilde{\dot{F}^s_{p,q}}} &\gtrsim \left\| \left[\sum_{j=0}^\infty \left(2^{j(s+n)} \int_{B(\cdot,2^{-j-1})} |f(\cdot)-f(y)|\,dy \right)^q \right]^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\ &\simeq \left\| \left[\sum_{\widetilde{j}=1}^\infty \left(2^{\widetilde{j}(s+n)} \int_{B(\cdot,2^{-\widetilde{j}})} |f(\cdot)-f(y)|\,dy \right)^q \right]^{1/q} \right\|_{L^p(\mathbb{R}^n)}, \end{aligned}$$

which together with (2.8) and the fact that $\| \int_{B(\cdot,1)} |f(\cdot)-f(y)|\,dy\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$ immediately implies

$$\|f\|_{\widetilde{F^s_{p,q}}} \gtrsim \|f\|_{\widetilde{\widetilde{F^s_{p,q}}}}. \tag{2.13}$$

Altogether (2.12), (2.13) and (i), we conclude that (ii) holds true. This finishes the proof of Lemma 2.4.

□

2.3 The spectral Besov and Triebel-Lizorkin spaces

The spectral Besov and Triebel-Lizorkin spaces are defined via the Littlewood-Paley decomposition adapted to $\mathcal{L}$. To this end, we need introduce a resolution of unity. Recall that a couple of functions (Φ_0, Φ) in $C_c^\infty([0, \infty))$ is said to satisfy the *unity resolution condition* (**UR**), if

(i) $\operatorname{supp}\Phi_0 \subset [0, 2]$, $\Phi_0^{(2\nu+1)}(0) \equiv 0$ for any $\nu \in \mathbb{Z}_+$, $|\Phi_0(\lambda)| \geq c$ for any $\lambda \in [0, 2^{-3/4}]$ and some $c > 0$;

(ii) $\operatorname{supp}\Phi \subset [1/2, 2]$ and $|\Phi(\lambda)| \geq c$ for any $\lambda \in [2^{-3/4}, 2^{3/4}]$ and some $c > 0$.

We refer the reader to [11, p. 1043] for a construction of such couple of functions.

Now, let (Φ_0, Φ) be a couple of functions satisfies the condition (**UR**). For any $p, q \in (1, \infty)$ and $s \in [0, \infty)$, the *spectral Besov space* $\widetilde{B}^{s,\mathcal{L}}_{p,q}$ is defined to be

$$\widetilde{B}^{s,\mathcal{L}}_{p,q} := \left\{ f \in L^p(\mathbb{R}^n) : \|f\|_{\widetilde{B}^{s,\mathcal{L}}_{p,q}} < \infty \right\}, \tag{2.14}$$

where

$$\|f\|_{\widetilde{B}^{s,\mathcal{L}}_{p,q}} := \left\{ \sum_{j=0}^{\infty} \left[2^{js} \left\| \Phi_j(\sqrt{\mathcal{L}}) f \right\|_{L^p(\mathbb{R}^n)} \right]^q \right\}^{1/q} \tag{2.14a}$$

with Φ_j defined by

$$\Phi_j(\lambda) := \Phi\left(2^{-j}\lambda\right) \tag{2.15}$$

for any $j \in \mathbb{N}$ and $\lambda \in [0, \infty)$.

Similarly, the *spectral Triebel-Lizorkin space* $\widetilde{F}^{s,\mathcal{L}}_{p,q}$ with $p, q \in (1, \infty)$ and $s \in [0, \infty)$ is defined to be

$$\widetilde{F}^{s,\mathcal{L}}_{p,q} := \left\{ f \in L^p(\mathbb{R}^n) : \|f\|_{\widetilde{F}^{s,\mathcal{L}}_{p,q}} < \infty \right\}, \tag{2.16}$$

where

$$\|f\|_{\widetilde{F}^{s,\mathcal{L}}_{p,q}} := \left\| \left\{ \sum_{j=0}^{\infty} \left[2^{js} \left| \Phi_j(\sqrt{\mathcal{L}}) f \right| \right]^q \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)}. \tag{2.16a}$$

Remark 2.5. (i) The spectral spaces in (2.14)-(2.14a) and (2.16)-(2.16a) are identical for different choices of couples (Φ_0, Φ) satisfying (**UR**) (see [35, 32, 38, 7]). A classical example of functions satisfying (**UR**) is as follows. Let $\varphi_0 \in C_c^\infty([0, \infty))$ satisfy $\operatorname{supp}\varphi_0 \subset [0, 2]$, $0 \leq \varphi_0 \leq 1$ and $\varphi_0 \equiv 1$ on $[0, 1]$. Let $\varphi(\cdot) := \varphi_0(\cdot) - \varphi_0(2\cdot)$ and for any $j \in \mathbb{N}$ and $\lambda \in [0, \infty)$, $\varphi_j(\lambda) := \varphi\left(2^{-j}\lambda\right)$. It is easy to see that the couple (φ_0, φ) of functions satisfies the condition (**UR**). Moreover, we have that for any $\lambda \in [0, \infty)$, $\sum_{j=0}^{\infty} \varphi_j(\lambda) \equiv 1$, which further implies the following Calderón reproducing formula that for any $p \in (1, \infty)$ and $f \in L^p(\mathbb{R}^n)$,

$$f = \sum_{j=0}^{\infty} \varphi_j(\sqrt{\mathcal{L}}) f$$

in $L^p(\mathbb{R}^n)$ (see, for example, [35, Proposition 5.5]). In the general case of (Φ_0, Φ) in $C_c^\infty([0,\infty))$ satisfying (**UR**). By applying a corresponding scalar identity (see [20, Lemma 6.9] or [6, (3.20)]), we still have the following Calderón reproducing formula that for any $p \in (1,\infty)$ and $f \in L^p(\mathbb{R}^n)$,

$$f = \sum_{j=0}^{\infty} \Psi_j(\sqrt{\mathcal{L}})\Phi_j(\sqrt{\mathcal{L}})f = \sum_{j=0}^{\infty} \Phi_j(\sqrt{\mathcal{L}})\Psi_j(\sqrt{\mathcal{L}})f \tag{2.17}$$

in $L^p(\mathbb{R}^n)$ (see also [38, (2.17)]), where for any $j \in \mathbb{N}$, Φ_j and Ψ_j are defined as in (2.15). Moreover, (Ψ_0, Ψ) are in $C_c^\infty([0,\infty))$ and also satisfy the condition (**UR**).

(ii) For any $p \in (1,\infty)$ and $s \in [0,\infty)$, let $H_p^{s,\mathcal{L}}(\mathbb{R}^n)$ be the *Bessel potential space associated with* $\mathcal{L}$ with the norm defined by

$$\|f\|_{H_p^{s,\mathcal{L}}(\mathbb{R}^n)} := \|(I+\mathcal{L})^{s/2} f\|_{L^p(\mathbb{R}^n)}.$$

By (2.3), we know that for any $p \in (1,\infty)$ and $s \in [0,\infty)$,

$$H_p^{s,\mathcal{L}}(\mathbb{R}^n) = \mathrm{dom}_p[\mathcal{L}^{s/2}].$$

In particular, if $s = 0$ the space $H_p^{0,\mathcal{L}}(\mathbb{R}^n)$ coincides with the Lebesgue space $L^p(\mathbb{R}^n)$. Moreover, by [35, Theorem 7.8], we know that for any $p \in (1,\infty)$ and $s \in [0,\infty)$,

$$\widetilde{F}_{p,2}^{s,\mathcal{L}} = H_p^{s,\mathcal{L}}(\mathbb{R}^n). \tag{2.18}$$

(iii) The spectral spaces $\widetilde{B}_{p,q}^{s,\mathcal{L}}$ and $\widetilde{F}_{p,q}^{s,\mathcal{L}}$ satisfy the following lifting property for Bessel potential. Let $p, q \in (1,\infty)$, $0 \le \delta \le s < \infty$ and $f \in \widetilde{A}_{p,q}^{s,\mathcal{L}}$ with A being the B-space or F-space. Then $(1+\mathcal{L})^{\delta/2} f \in \widetilde{A}_{p,q}^{s-\delta,\mathcal{L}}$ and

$$\left\|(I+\mathcal{L})^{\delta/2} f\right\|_{\widetilde{A}_{p,q}^{s-\delta,\mathcal{L}}} \simeq \|f\|_{\widetilde{A}_{p,q}^{s,\mathcal{L}}},$$

where the implicit constants are independent of f (see [7, Theorem 7.1] for a detailed argument in the homogeneous case).

Remark 2.6. Let $\mathcal{S}([0,\infty))$ be the Schwartz class of all functions $f \in C^\infty((0,\infty)) \cap C([0,\infty))$ such that for any $k \in \mathbb{N}$, $f^{(k)}$ decays rapidly at infinity and the limit $\lim_{\lambda\to 0^+} f^{(k)}(\lambda)$ exists. A couple (ϕ_0, ϕ) of functions in $\mathcal{S}([0,\infty))$ is said to satisfy the *admission condition* if there exist $R > 0$ and $M \in \mathbb{Z}_+$ such that: i) for any $\lambda \in [0, 2R)$, $|\varphi_0(\lambda)| > 0$; ii) for any $\lambda \in (R/2, 2R)$, $|\varphi(\lambda)| > 0$; iii) $\lambda^{-M}\varphi(\lambda) \in \mathcal{S}([0,\infty))$. In [32], Hu proved that the spectral function spaces in (2.14)-(2.14a) and (2.16)-(2.16a) are invariant if we use the couple (ϕ_0, ϕ) satisfying the admission condition to replace the couple (Φ_0, Φ) satisfying (**UR**) in their definitions.

One example of functions satisfying the admission condition is as follows. Let $\theta \in C_c^\infty(\mathbb{R})$ be even and satisfy $\operatorname{supp}\theta \subset (-1,1)$, $\int_{\mathbb{R}} \theta(\lambda)\,d\lambda = 1$. Let $\eta(\xi) := \widehat{\theta}(\xi)$ be the Fourier transform of θ defined as in (1.15) which is also even in $\mathbb{R}$. For any $M \in \mathbb{Z}_+$ and $\lambda \in [0,\infty)$, let

$$\begin{cases} \phi_0(\lambda) := \eta(\lambda), \\ \phi(\lambda) := \lambda^{2M}\eta(\lambda). \end{cases}$$

Then the couple (ϕ_0, ϕ) of functions satisfies the admission condition (see [32, Lemma 4.7]).

The advantage of the couple (ϕ_0, ϕ) of functions satisfying the admission condition is that it admits the operator $\phi(\sqrt{\mathcal{L}})$ (also for $\phi_0(\sqrt{\mathcal{L}})$) has compact supported integral kernel. Indeed, by using the finite speed propagation property (see, for example, [30, Lemma 3.5] or [35, Proposition 2.8]), we know that for any $j \in \mathbb{Z}_+$ and $l \in \mathbb{Z}_+$,

$$\operatorname{supp}\left(2^{-j}\sqrt{\mathcal{L}}\right)^{l}\phi\left(2^{-j}\sqrt{\mathcal{L}}\right)(\cdot,\,\cdot)\subset\left\{(x,\,y)\in\mathbb{R}^n\times\mathbb{R}^n:\ |x-y|<2^{-j}\right\}, \tag{2.19}$$

where $(2^{-j}\sqrt{\mathcal{L}})^l\phi(2^{-j}\sqrt{\mathcal{L}})(\cdot,\,\cdot)$ denotes the integral kernel of the operator $(2^{-j}\sqrt{\mathcal{L}})^l\phi(2^{-j}\sqrt{\mathcal{L}})$. Similarly, the same estimates holds for the operator $(2^{-j}\sqrt{\mathcal{L}})^l\phi_0(2^{-j}\sqrt{\mathcal{L}})$. Note that the property (2.19) plays an important role in the atomic characterization of spectral function spaces (see [30, 32, 7] and their references).

We end this subsection with the following proposition from [35, 38], which establishes the identification of the function spaces defined via heat semigroup and spectrum.

Proposition 2.7 ([35, 38])**.** *Let $p,\,q \in (1,\,\infty)$ and $s \in [0,\,\infty)$. Then*

(i) $B^{s,\mathcal{L}}_{p,q} = \widetilde{B}^{s,\mathcal{L}}_{p,q}$;

(ii) $F^{s,\mathcal{L}}_{p,q} = \widetilde{F}^{s,\mathcal{L}}_{p,q}$.

3 Identification of spectral and Lipschitz Triebel-Lizorkin spaces for $s \in (0,\,\Theta)$

In this section, we establish the identity of the spectral and Lipschitz versions of Triebel-Lizorkin spaces for any $p,\,q \in (1,\,\infty)$ and $s \in (0,\,\Theta)$ with Θ as in (1.3). The main result of this section is the following Theorem 3.1.

Theorem 3.1. *Let $p,\,q \in (1,\,\infty)$ and $s \in (0,\,\Theta(\mathcal{L}))$. Then*

$$\widetilde{F}^{s,\mathcal{L}}_{p,q} = F^{s}_{p,q}$$

with equivalent norms.

The proof of Theorem 3.1 will be split into two steps: i) $F^s_{p,q} \subset \widetilde{F}^{s,\mathcal{L}}_{p,q}$ and ii) $\widetilde{F}^{s,\mathcal{L}}_{p,q} \subset F^s_{p,q}$. We first prove i), namely, $F^s_{p,q} \subset \widetilde{F}^{s,\mathcal{L}}_{p,q}$. To this end, we need the following result from [35, Theorem 3.1], which is of fundamental importance in the smooth functional calculus associated with $\mathcal{L}$.

Lemma 3.2 ([35])**.** *Let $k \in \mathbb{N}$ satisfy $k \geq n+1$. Assume that $\varphi \in C^k([0,\,\infty))$ satisfies $\varphi^{(2\nu+1)}(0) = 0$ for any $0 \leq \nu \leq (k-1)/2$ and $\operatorname{supp}\varphi \subset [0,\,R]$ for some $R \geq 1$. Then for any $\delta \in (0,\,1]$, the operator $\varphi(\delta\sqrt{\mathcal{L}})$ has an integral kernel $\varphi(\delta\sqrt{\mathcal{L}})(\cdot,\,\cdot)$ on $\mathbb{R}^n\times\mathbb{R}^n$. Moreover, for any $x,\,y,\,y' \in \mathbb{R}^n$,*

(i) $\left|\varphi(\delta\sqrt{\mathcal{L}})(x,\,y)\right| \leq C(k)D_{\delta,k}(x,y)$, *where*

$$D_{\delta,k}(x,y) := \frac{1}{\delta^n}\left(1+\frac{|x-y|}{\delta}\right)^{-k} \tag{3.1}$$

and

$$C(k) := R^n \left[(C_1 k)^k \|\varphi\|_{L^\infty([0,\,\infty))} + (C_2 R)^k \left\|\varphi^{(k)}\right\|_{L^\infty([0,\,\infty))}\right]$$

with constants C_1, C_2 being positive and depend only on n and Λ as in (1.1)*;*

(ii) *if furhter $|y - y|' \le \delta$, then*

$$\left|\varphi(\delta\sqrt{\mathcal{L}})(x,\ y) - \varphi(\delta\sqrt{\mathcal{L}})(x,\ y')\right| \le C'(k)\left(\frac{|y-y|}{\delta}\right)^{\Theta} D_{\delta,k}(x,y),$$

where $\Theta \in (0,\ 1)$ denotes the Hölder exponent as in (1.3) *and*

$$C'(k) := C_3 C(k) R^{\Theta}$$

with constant C_3 being positive and depend only on n and k;

(iii) $\int_{\mathbb{R}^n} \varphi(\delta\sqrt{\mathcal{L}})(x,\ y)\, dy \equiv \varphi(0)$.

With the help of Lemma 3.2, we now turn to the proof of the inclusion $F^s_{p,q} \subset \widetilde{F}^{s,\mathcal{L}}_{p,q}$.

Proof of the inclusion $F^s_{p,q} \subset \widetilde{F}^{s,\mathcal{L}}_{p,q}$. To begin with, note that in this part of proof, we only need $s \in (0,\ 1)$ and are not restricted by the Hölder exponent Θ in (1.3). Let $f \in F^s_{p,q}$. By (2.16a), we have that

$$\|f\|_{\widetilde{F}^{s,\mathcal{L}}_{p,q}} \simeq \left\| \left\{ \sum_{j=0}^{\infty} \left[2^{js} \left|\varphi_j(\sqrt{\mathcal{L}})f\right|\right]^q \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)}, \tag{3.2}$$

where (φ_0, φ) in $C_c^\infty([0,\ \infty))$ is a particular choice of functions satisfying the unity resolution condition **(UR)** as in Remark 2.5(i). By the bounded functional calculus in $L^p(\mathbb{R}^n)$ for any $p \in (1,\ \infty)$ and the fact $\|\varphi_0\|_{L^\infty([0,\,\infty))} = 1$, we find that for any $p \in (1,\ \infty)$,

$$\left\|\varphi_0(\sqrt{\mathcal{L}})f\right\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}. \tag{3.3}$$

For $j \ge 1$, by (2.15) and the fact $\varphi_j(\lambda) = \varphi_0(2^{-j}\lambda) - \varphi_0(2^{1-j}\lambda)$, we derive from Lemma 3.2(iii) that for any $j \in \mathbb{N}$ and $x \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} \varphi_j(\sqrt{\mathcal{L}})(x,\ y)\, dy = \int_{\mathbb{R}^n} \varphi_0(2^{-j}\sqrt{\mathcal{L}})(x,\ y)\, dy - \int_{\mathbb{R}^n} \varphi_0(2^{1-j}\sqrt{\mathcal{L}})(x,\ y)\, dy = 0,$$

which combined with Lemma 3.2(i) (with $\delta = 2^{-j}$ and $k = \tau$ for some $\tau \in ((s+n)q,\ \infty)$) further implies that for any $x \in \mathbb{R}^n$,

$$\left|\varphi_j(\sqrt{\mathcal{L}})f(x)\right| \le \int_{\mathbb{R}^n} \left|\varphi_j(\sqrt{\mathcal{L}})(x,\ y)\right| |f(x) - f(y)|\ dy \tag{3.4}$$

$$\lesssim \int_{\mathbb{R}^n} 2^{jn}\left(1+2^j|x-y|\right)^{-\tau}|f(x)-f(y)|\,dy$$
$$\lesssim \sum_{k=0}^{\infty}\int_{S_{k-j}(x)} 2^{jn-k\tau}|f(x)-f(y)|\,dy,$$

where for any $k\in\mathbb{N}$, $S_{k-j}(x) := B(x,2^{k-j})\setminus B(x,2^{k-j-1})$ and $S_{-j}(x) := B(x,2^{-j})$.

Thus, by (3.4) and Hölder's inequality, we write

$$\left\|\left[\sum_{j=1}^{\infty}\left(2^{js}\left|\varphi_j(\sqrt{\mathcal{L}})f\right|\right)^q\right]^{1/q}\right\|_{L^p(\mathbb{R}^n)} \tag{3.5}$$
$$\lesssim \left\{\int_{\mathbb{R}^n}\left[\sum_{j=1}^{\infty}\left(2^{j(s+n)}\sum_{k=0}^{\infty}2^{-k\tau}\int_{B(x,2^{k-j})}|f(y)-f(x)|\,dy\right)^q\right]^{p/q}dx\right\}^{1/p}$$
$$\lesssim \left\{\int_{\mathbb{R}^n}\left[\sum_{k=0}^{\infty}2^{-k\tau}\sum_{j=1}^{\infty}2^{j(s+n)q}\left(\int_{B(x,2^{k-j})}|f(y)-f(x)|\,dy\right)^q\right]^{p/q}dx\right\}^{1/p}$$
$$\simeq \left\{\int_{\mathbb{R}^n}\left[\sum_{k=0}^{\infty}2^{-k(\tau-sq-nq)}\sum_{\widetilde{j}=1-k}^{\infty}2^{\widetilde{j}(s+n)q}\left(\int_{B(x,2^{-\widetilde{j}})}|f(y)-f(x)|\,dy\right)^q\right]^{p/q}dx\right\}^{1/p}$$
$$\simeq \left\{\int_{\mathbb{R}^n}\left[\sum_{k=0}^{\infty}2^{-k(\tau-sq-nq)}\left(\sum_{\widetilde{j}=1-k}^{0}+\sum_{\widetilde{j}=1}^{\infty}\right)\cdots\right]^{p/q}dx\right\}^{1/p} =: \mathrm{A}_1+\mathrm{A}_2.$$

We first estimate A_1, which is trivial when $k=0$. As $\widetilde{j}\in\{1-k,\ldots,0\}$, we have $B(x,2^{-\widetilde{j}})\subset B(x,1)$. By this and the fact $\tau > sq+nq$, we obtain

$$\mathrm{A}_1 \lesssim \left\{\int_{\mathbb{R}^n}\left[\sum_{k=0}^{\infty}k2^{-k(\tau-sq-nq)}\left(\int_{B(x,1)}|f(y)-f(x)|\,dy\right)^q\right]^{p/q}dx\right\}^{1/p} \tag{3.6}$$
$$\lesssim \|\mathcal{M}(f)\|_{L^p(\mathbb{R}^n)}+\|f\|_{L^p(\mathbb{R}^n)}\lesssim \|f\|_{L^p(\mathbb{R}^n)},$$

where $\mathcal{M}(f)$ denotes of Hardy-Littlewood maximal function of f as in (1.14).

For A_2, by Hölder's inequality, the fact $\tau > (s+n)q$ again and Lemma 2.4(ii), we find

$$\mathrm{A}_2 \lesssim \left\{\int_{\mathbb{R}^n}\left[\sum_{k=0}^{\infty}2^{-k(\tau-sq-nq)}\sum_{\widetilde{j}=1}^{\infty}\left(2^{\widetilde{j}(s+n)}\int_{B(x,2^{-\widetilde{j}})}|f(y)-f(x)|\,dy\right)^q\right]^{p/q}dx\right\}^{1/p}\lesssim \|f\|_{F^s_{p,q}}. \tag{3.7}$$

Altogether (3.2) through (3.7), we conclude that for any $f\in F^s_{p,q}$, $\|f\|_{\widetilde{F}^{s,\mathcal{L}}_{p,q}}\lesssim\|f\|_{F^s_{p,q}}$, which completes the proof of the inclusion $F^s_{p,q}\subset\widetilde{F}^{s,\mathcal{L}}_{p,q}$.

□

The inclusion $F^s_{p,q} \subset \widetilde{F}^{s,\mathcal{L}}_{p,q}$ for any $s \in (0,\ 1)$ implies the following reverse inequality of the fractional order local Riesz transform $(I+\Delta)^{s/2}(I+\mathcal{L})^{-s/2}$ on $L^p(\mathbb{R}^n)$, which extends the corresponding result for $s = 1$ in [5, p. 115, Theorem 3].

Corollary 3.3. *Let $p \in (1,\ \infty)$ and $s \in (0,\ 1]$. Then there exists a positive constant C such that for any $f \in H^s_p(\mathbb{R}^n)$ in the classical Bessel potential space,*

$$\left\|(I+\mathcal{L})^{s/2} f\right\|_{L^p(\mathbb{R}^n)} \leq C \left\|(I+\Delta)^{s/2} f\right\|_{L^p(\mathbb{R}^n)}.$$

Proof. As the case $s = 1$ was already proved in [5, p. 115, Theorem 3], we only need to consider the case $s \in (0,\ 1)$. By (2.18), we have that for any $p \in (1,\ \infty)$ and $s \in [0,\ \infty)$,

$$\widetilde{F}^{s,\mathcal{L}}_{p,2} = H^{s,\mathcal{L}}_p(\mathbb{R}^n). \tag{3.8}$$

On the other hand, as pointed out in Remark 2.3(i), we know that for any $p,\ q \in (1,\ \infty)$ and $s \in (0,\ 1)$, $F^s_{p,q} = F^s_{p,q}(\mathbb{R}^n)$ is the classical Triebel-Lizorkin space. This then implies that for any $p \in (1,\ \infty)$ and $s \in (0,\ 1)$,

$$F^s_{p,2} = H^s_p(\mathbb{R}^n),$$

where the later denotes the classical Bessel potential space. By this, (3.8) and the inclusion $F^s_{p,2} \subset \widetilde{F}^{s,\mathcal{L}}_{p,2}$ for any $s \in (0,\ 1)$, we immediately conclude the inclusion that $H^s_p(\mathbb{R}^n) \subset H^{s,\mathcal{L}}_p(\mathbb{R}^n)$ and for any $f \in H^s_p(\mathbb{R}^n)$,

$$\|f\|_{H^{s,\mathcal{L}}_p(\mathbb{R}^n)} \lesssim \|f\|_{H^s_p(\mathbb{R}^n)},$$

which completes the proof of Corollary 3.3.

□

We now turn to the proof of the converse inclusion $\widetilde{F}^{s,\mathcal{L}}_{p,q} \subset F^s_{p,q}$. To this end, we need some technical lemmata.

Lemma 3.4. *Let $p,\ q \in (1,\ \infty)$ and $s \in (0,\ \infty)$. Then the following embedding holds*

$$\widetilde{F}^{s,\mathcal{L}}_{p,q} \subset L^p(\mathbb{R}^n).$$

Proof. To prove this lemma, we first claim that for any $p,\ q \in (1,\ \infty)$ and $s \in (0,\ \infty)$,

$$\widetilde{F}^{s,\mathcal{L}}_{p,q} \subset \widetilde{F}^{0,\mathcal{L}}_{p,2}. \tag{3.9}$$

Indeed, for any $f \in \widetilde{F}^{s,\mathcal{L}}_{p,q}$, by (2.16a), we have

$$\|f\|_{\widetilde{F}^{s,\mathcal{L}}_{p,q}} := \left\|\left\{\sum_{j=0}^{\infty}\left[2^{js}\left|\Phi_j(\sqrt{\mathcal{L}})f\right|\right]^q\right\}^{1/q}\right\|_{L^p(\mathbb{R}^n)}. \tag{3.10}$$

If $q \geq 2$, then by Hölder's inequality and the fact $s > 0$, we see

$$\left\{\sum_{j=0}^{\infty}\left[\left|\Phi_j(\sqrt{\mathcal{L}})f\right|\right]^2\right\}^{1/2} = \left\{\sum_{j=0}^{\infty}\left[2^{js}\left|\Phi_j(\sqrt{\mathcal{L}})f\right|\right]^2 2^{-2js}\right\}^{1/2} \lesssim \left\{\sum_{j=0}^{\infty}\left[2^{js}\left|\Phi_j(\sqrt{\mathcal{L}})f\right|\right]^q\right\}^{1/q},$$

which together with (3.10) shows that (3.9) holds true.

If $q < 2$, then using the decreasing property of the ℓ^q norm in q and the fact $s > 0$, we find

$$\left\{\sum_{j=0}^{\infty}\left[\left|\Phi_j(\sqrt{\mathcal{L}})f\right|\right]^2\right\}^{1/2} \le \left\{\sum_{j=0}^{\infty}\left[\left|\Phi_j(\sqrt{\mathcal{L}})f\right|\right]^q\right\}^{1/q} \le \left\{\sum_{j=0}^{\infty}\left[2^{js}\left|\Phi_j(\sqrt{\mathcal{L}})f\right|\right]^q\right\}^{1/q}.$$

By this and (3.10), we also see (3.9) is valid in the case $q < 2$. Altogether the above two cases, we conclude that the claim (3.9) holds true.

On the other hand, by Remark 2.5(ii), we know that for any $p \in (1, \infty)$,

$$\widetilde{F}^{0,\mathcal{L}}_{p,2} = L^p(\mathbb{R}^n),$$

which together with (3.9) implies the inclusion $\widetilde{F}^{s,\mathcal{L}}_{p,q} \subset L^p(\mathbb{R}^n)$. This finishes the proof of Lemma 3.4.

□

We also need the following discrete Calderón reproducing formula from [38, Theorem 6.1]. To this end, for any $j \in \mathbb{Z}_+$ let $\mathcal{D}_j$ be the set of all dyadic cubes in $\mathbb{R}^n$ with side length 2^{-j} and $\{Q_\tau\}_{\tau\in\mathcal{I}}$ be the set of all dyadic cubes in $\mathcal{D}_{j+j_0}$, where $j_0 \in \mathbb{N}$ is a sufficiently large number depending only on n that is fixed from now on (see [38, (8.1), (8.2) and (8.20)] for a precise requirement of j_0).

Lemma 3.5 ([38])**.** *Let* (Φ_0, Φ) *in* $C_c^\infty([0, \infty))$ *satisfy the unity resolution condition* **(UR)**. *There exist a family* $\{\Psi_j(\sqrt{\mathcal{L}})\}_{j\in\mathbb{Z}_+}$ *of operators with integral kernels such that the followings hold.*

(i) *For any* $p \in (1, \infty)$, $f \in L^p(\mathbb{R}^n)$ *and a.e.* $x \in \mathbb{R}^n$,

$$f(x) = \sum_{j=0}^{\infty}\sum_{\tau\in\mathcal{I}} 2^{-jn}\left(\Phi_j(\sqrt{\mathcal{L}})f\right)(\xi_\tau)\Psi_j(\sqrt{\mathcal{L}})(\xi_\tau, x),$$

where for any $\tau \in \mathcal{I}$, ξ_τ *can be chosen as any point in* $Q_\tau \in \mathcal{D}_{j+j_0}$.

(ii) *For any* $\gamma \in (2n, \infty)$, *there exists a positive constant* $C(\gamma)$ *such that for any* $j \in \mathbb{Z}_+$ *and* x, $y \in \mathbb{R}^n$,

$$\left|\Psi_j(\sqrt{\mathcal{L}})(x, y)\right| \le C(\gamma)D_{2^{-j},\gamma}(x, y),$$

where $D_{2^{-j},\gamma}(x, y)$ *is as in* (3.1) *with* $\delta = 2^{-j}$ *and* $k = \gamma$.

(iii) *For any* $\gamma \in (2n, \infty)$, *there exists a positive constant* $C'(\gamma)$ *such that for any* $j \in \mathbb{Z}_+$ *and* x, y, $y' \in \mathbb{R}^n$ *satisfying* $|y - y'| \le 2^{-j}$,

$$\left|\Psi_j(\sqrt{\mathcal{L}})(x, y) - \Psi_j(\sqrt{\mathcal{L}})(x, y')\right| \le C'(\gamma)\left(2^j|y - y'|\right)^\Theta D_{2^{-j},\gamma}(x, y),$$

where $\Theta \in (0, 1)$ *denotes the Hölder exponent as in* (1.3) *and* $D_{2^{-j},\gamma}(x, y)$ *is as in* (3.1) *with* $\delta = 2^{-j}$ *and* $k = \gamma$.

The following technical lemma is on the estimates for Hardy-Littlewood maximal function $\mathcal{M}$ in (1.14), whose proof can be fund in [18, 23] and [19, Theorems A.1 and A.2].

Lemma 3.6 ([18, 19, 23]). (i) *Let $p,\ q \in (1,\infty)$. Then for any sequence of functions $\{f_j\}_{j\in\mathbb{Z}_+}$ on $\mathbb{R}^n$, it holds*

$$\left\|\left[\sum_{j=0}^{\infty}\Big(\mathcal{M}(f_j)\Big)^q\right]^{1/q}\right\|_{L^p(\mathbb{R}^n)} \le C(p,q)\left\|\left[\sum_{j=0}^{\infty}\left|f_j\right|^q\right]^{1/q}\right\|_{L^p(\mathbb{R}^n)},$$

where the constant $C(p,q)$ depends only on p and q.

(ii) *Let $k\in\mathbb{Z}_+$ and $\{Q_\tau\}_{\tau\in\mathcal{I}}\subset\mathcal{D}_k$ be a sequence of dyadic cubes of the same level. Then for any $\xi_\tau\in Q_\tau$, $\{c_\tau\}_{\tau\in\mathcal{I}}\subset\mathbb{R}$ and $x\in\mathbb{R}^n$, it holds*

$$\sum_{\tau\in\mathcal{I}}|c_\tau|\Big(1+2^k|x-\xi_\tau|\Big)^{-\gamma} \le C(n,\gamma)\mathcal{M}\left(\sum_{\tau\in\mathcal{I}}|c_\tau|\mathbf{1}_{Q_\tau}\right)(x),$$

where $\gamma>n$ and the constant $C(n,\gamma)$ depends only on n and γ.

With the helps of Lemmas 3.4, 3.5 and 3.6, we now turn to the proof of the inclusion $\widetilde{F}^{s,\mathcal{L}}_{p,q}\subset F^s_{p,q}$, which combined with the inclusion $F^s_{p,q}\subset\widetilde{F}^{s,\mathcal{L}}_{p,q}$ completes the proof of Theorem 3.1.

Proof of the inclusion $\widetilde{F}^{s,\mathcal{L}}_{p,q}\subset F^s_{p,q}$. In this part of proof, we need the restriction $s\in(0,\Theta)$. For any $p,\ q\in(1,\ \infty)$ and $s\in(0,\Theta)$, let $f\in\widetilde{F}^{s,\mathcal{L}}_{p,q}$. Using Lemma 2.4(ii), we find

$$\|f\|_{F^s_{p,q}} \simeq \|f\|_{L^p(\mathbb{R}^n)} + \left\|\left[\sum_{k=0}^{\infty}\left(2^{k(s+n)}\int_{B(\cdot,2^{-k})}|f(\cdot)-f(y)|\,dy\right)^q\right]^{1/q}\right\|_{L^p(\mathbb{R}^n)}. \tag{3.11}$$

By Lemma 3.4, we immediately obtain

$$\|f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{\widetilde{F}^{s,\mathcal{L}}_{p,q}}. \tag{3.12}$$

We now deal with the second term in (3.11). For any $k\in\mathbb{Z}_+$, using Lemma 3.5, we can write for a.e. $x,\ y\in\mathbb{R}^n$ satisfying $|x-y|<2^{-k}$,

$$\begin{aligned}
&2^{k(s+n)}\,|f(y)-f(x)| \qquad (3.13)\\
&\le \sum_{j=0}^{\infty}\sum_{\tau\in\mathcal{I}}2^{k(s+n)}2^{-jn}\left|\Big(\Phi_j(\sqrt{\mathcal{L}})f\Big)(\xi_\tau)\right|\left|\Psi_j(\sqrt{\mathcal{L}})(\xi_\tau,\ y)-\Psi_j(\sqrt{\mathcal{L}})(\xi_\tau,\ x)\right|\\
&=\left[\sum_{j=0}^{k}+\sum_{j=k+1}^{\infty}\right]\sum_{\tau\in\mathcal{I}}2^{k(s+n)}2^{-jn}\left|\Big(\Phi_j(\sqrt{\mathcal{L}})f\Big)(\xi_\tau)\right|\left|\Psi_j(\sqrt{\mathcal{L}})(\xi_\tau,\ y)-\Psi_j(\sqrt{\mathcal{L}})(\xi_\tau,\ x)\right|\\
&=:\mathrm{I}_1(x,y)+\mathrm{I}_2(x,y).
\end{aligned}$$

For $\mathrm{I}_1(x,y)$, since $j \le k$, we have $2^{-k} \le 2^{-j}$. Thus, for any $x, y \in \mathbb{R}^n$ satisfying $|x-y| < 2^{-k} \le 2^{-j}$, by Lemma 3.5(iii), we know

$$\begin{aligned}
\mathrm{I}_1(x,y) &\lesssim \sum_{j=0}^{k} \sum_{\tau\in\mathcal{I}} 2^{k(s+n)} 2^{-jn} \left|\left(\Phi_j(\sqrt{\mathcal{L}})f\right)(\xi_\tau)\right| \left(2^j|x-y|\right)^{\Theta} D_{2^{-j},\gamma}(\xi_\tau,\, x) \\
&\simeq \sum_{j=0}^{k} \sum_{\tau\in\mathcal{I}} 2^{k(s+n-\Theta)} 2^{j\Theta} \left(1+2^j|x-\xi_\tau|\right)^{-\gamma} \left|\left(\Phi_j(\sqrt{\mathcal{L}})f\right)(\xi_\tau)\right| \\
&\simeq \sum_{j=0}^{k} 2^{k(s+n-\Theta)} 2^{j(\Theta-s)} \left[\sum_{\tau\in\mathcal{I}} \left(1+2^j|x-\xi_\tau|\right)^{-\gamma} 2^{js} \left|\left(\Phi_j(\sqrt{\mathcal{L}})f\right)(\xi_\tau)\right|\right] \\
&\lesssim \sum_{j=0}^{k} 2^{j(\Theta-s)} 2^{k(s+n-\Theta)} \mathcal{M}\left(\sum_{\tau\in\mathcal{I}} 2^{js} \left|\left(\Phi_j(\sqrt{\mathcal{L}})f\right)(\xi_\tau)\right| \mathbf{1}_{Q_\tau}\right)(x),
\end{aligned} \tag{3.14}$$

where $\mathcal{M}$ denotes the Hardy-Littlewood maximal function as in (1.14), and in the last inequality, we have used Lemma 3.6(ii).

Now, let

$$F_j := \sum_{\tau\in\mathcal{I}} 2^{js} \left|\left(\Phi_j(\sqrt{\mathcal{L}})f\right)(\xi_\tau)\right| \mathbf{1}_{Q_\tau}. \tag{3.15}$$

By Hölder's inequality, (3.14) and Lemma 3.6(i), we derive from $s < \Theta$ that

$$\begin{aligned}
\mathrm{J}_1 &:= \left\{\int_{\mathbb{R}^n} \left[\sum_{k=0}^{\infty} \left(\int_{B(x,2^{-k})} |\mathrm{I}_1(x,\,y)|\,dy\right)^q\right]^{p/q} dx\right\}^{1/p} \\
&\lesssim \left\{\int_{\mathbb{R}^n} \left[\sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} 2^{(j-k)(\Theta-s)} 2^{kn} \int_{B(x,2^{-k})} \mathcal{M}\left(F_j\right)(x)\,dy\right)^q\right]^{p/q} dx\right\}^{1/p} \\
&\simeq \left\{\int_{\mathbb{R}^n} \left[\sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} 2^{(j-k)(\Theta-s)} \mathcal{M}\left(F_j\right)(x)\right)^q\right]^{p/q} dx\right\}^{1/p} \\
&\lesssim \left\{\int_{\mathbb{R}^n} \left[\sum_{k=0}^{\infty} \sum_{j=0}^{k} 2^{(j-k)(\Theta-s)} \left(\mathcal{M}\left(F_j\right)(x)\right)^q\right]^{p/q} dx\right\}^{1/p} \\
&\lesssim \left\{\int_{\mathbb{R}^n} \left[\sum_{j=0}^{\infty} \left(\mathcal{M}\left(F_j\right)(x)\right)^q\right]^{p/q} dx\right\}^{1/p} \lesssim \left\{\int_{\mathbb{R}^n} \left[\sum_{j=0}^{\infty} \left|F_j(x)\right|^q\right]^{p/q} dx\right\}^{1/p}.
\end{aligned} \tag{3.16}$$

As the dyadic cubes $\{Q_\tau\}_\tau$ are disjoint and $\cup_{\tau\in\mathcal{I}} Q_\tau = \mathbb{R}^n$, we derive from (3.15) and the arbitrariness of ξ_τ in Q_τ that

$$\sum_{j=0}^{\infty} \left|F_j(x)\right|^q = \sum_{j=0}^{\infty} \left|\sum_{\tau\in\mathcal{I}} 2^{js} \left|\Phi_j(\sqrt{\mathcal{L}})f(\xi_\tau)\right| \mathbf{1}_{Q_\tau}(x)\right|^q \tag{3.17}$$

$$= \sum_{j=0}^{\infty} \sum_{\tau \in \mathcal{I}} 2^{jsq} \left| \Phi_j(\sqrt{\mathcal{L}}) f(\xi_\tau) \right|^q \mathbf{1}_{Q_\tau}(x) \le \sum_{j=0}^{\infty} 2^{jsq} \left| \Phi_j(\sqrt{\mathcal{L}}) f(x) \right|^q .$$

Combined (3.16), (3.17) and (2.16a), we conclude that

$$\mathrm{J}_1 \lesssim \left\| \left\{ \sum_{j=0}^{\infty} \left[2^{js} \left| \Phi_j(\sqrt{\mathcal{L}}) f \right| \right]^q \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)} \simeq \|f\|_{\widetilde{F}^{s,\mathcal{L}}_{p,q}} . \tag{3.18}$$

We now estimate $\mathrm{I}_2(x, y)$. To this end, we first write

$$\begin{aligned} \mathrm{I}_2(x, y) &\lesssim \sum_{j=k+1}^{\infty} \sum_{\tau \in \mathcal{I}} 2^{k(s+n)} 2^{-jn} \left| \left(\Phi_j(\sqrt{\mathcal{L}}) f\right)(\xi_\tau) \right| \left[\left| \Psi_j(\sqrt{\mathcal{L}})(\xi_\tau, y) \right| + \left| \Psi_j(\sqrt{\mathcal{L}})(\xi_\tau, x) \right| \right] \\ &=: \mathrm{I}_{2,1}(y) + \mathrm{I}_{2,2}(x). \end{aligned} \tag{3.19}$$

Without loss of generality, we may only estimate $\mathrm{I}_{2,1}(y)$. The estimates for $\mathrm{I}_{2,2}(x)$ are similar and even easier. To estimate $\mathrm{I}_{2,1}(y)$, using Lemma 3.5(ii) and an argument similar to (3.16), we have

$$\begin{aligned} \mathrm{I}_{2,1}(y) &\lesssim \sum_{j=k+1}^{\infty} \sum_{\tau \in \mathcal{I}} 2^{k(s+n)} \left(1 + 2^j |y - \xi_\tau|\right)^{-\gamma} \left| \left(\Phi_j(\sqrt{\mathcal{L}}) f\right)(\xi_\tau) \right| \\ &\simeq \sum_{j=k+1}^{\infty} 2^{k(s+n)} 2^{-js} \left[\sum_{\tau \in \mathcal{I}} \left(1 + 2^j |y - \xi_\tau|\right)^{-\gamma} 2^{js} \left| \left(\Phi_j(\sqrt{\mathcal{L}}) f\right)(\xi_\tau) \right| \right] \\ &\lesssim \sum_{j=k+1}^{\infty} 2^{-js} 2^{k(s+n)} \mathcal{M}\left(F_j\right)(y), \end{aligned} \tag{3.20}$$

where F_j is the same as in (3.15). Thus from (3.20), the fact $j > k$, Hölder's inequality and using the Lemma 3.6(i) twice, it follows

$$\begin{aligned} \mathrm{J}_2 &:= \left\{ \int_{\mathbb{R}^n} \left[\sum_{k=0}^{\infty} \left(\int_{B(x, 2^{-k})} \left| \mathrm{I}_{2,1}(y) \right| \, dy \right)^q \right]^{p/q} dx \right\}^{1/p} \\ &\lesssim \left\{ \int_{\mathbb{R}^n} \left[\sum_{k=0}^{\infty} \left(\sum_{j=k+1}^{\infty} 2^{(k-j)s} 2^{kn} \int_{B(x, 2^{-k})} \mathcal{M}\left(F_j\right)(y) \, dy \right)^q \right]^{p/q} dx \right\}^{1/p} \\ &\lesssim \left\{ \int_{\mathbb{R}^n} \left[\sum_{k=0}^{\infty} \left(\sum_{j=k+1}^{\infty} 2^{(k-j)s} \mathcal{M} \circ \mathcal{M}\left(F_j\right)(x) \right)^q \right]^{p/q} dx \right\}^{1/p} \\ &\lesssim \left\{ \int_{\mathbb{R}^n} \left[\sum_{k=0}^{\infty} \sum_{j=k+1}^{\infty} 2^{(k-j)s} \left(\mathcal{M} \circ \mathcal{M}\left(F_j\right)(x) \right)^q \right]^{p/q} dx \right\}^{1/p} \\ &\lesssim \left\{ \int_{\mathbb{R}^n} \left[\sum_{j=0}^{\infty} \left(\mathcal{M} \circ \mathcal{M}\left(F_j\right)(x) \right)^q \right]^{p/q} dx \right\}^{1/p} \lesssim \left\{ \int_{\mathbb{R}^n} \left[\sum_{j=0}^{\infty} \left| F_j(x) \right|^q \right]^{p/q} dx \right\}^{1/p}, \end{aligned} \tag{3.21}$$

which combined with (3.17) and (2.16a) implies

$$\mathrm{J}_2 \lesssim \left\| \left\{ \sum_{j=0}^{\infty} \left[2^{js} \left| \Phi_j(\sqrt{\mathcal{L}}) f \right| \right]^q \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)} \simeq \|f\|_{\widetilde{F}^{s,\mathcal{L}}_{p,q}}. \tag{3.22}$$

Altogether the estimates (3.11) through (3.13), (3.16) and (3.18) through (3.22), we conclude that for any $f \in \widetilde{F}^{s,\mathcal{L}}_{p,q}$ with $p, q \in (1, \infty)$ and $s \in (0, \Theta)$,

$$\|f\|_{F^s_{p,q}} \lesssim \|f\|_{\widetilde{F}^{s,\mathcal{L}}_{p,q}}.$$

This finishes the proof of the inclusion $\widetilde{F}^{s,\mathcal{L}}_{p,q} \subset F^s_{p,q}$.

□

The inclusion $\widetilde{F}^{s,\mathcal{L}}_{p,q} \subset F^s_{p,q}$ for any $s \in (0, \Theta)$ implies the following Corollary 3.7 on the boundedness of the fractional order local Riesz transform $(I + \Delta)^{s/2}(I + \mathcal{L})^{-s/2}$ on $L^p(\mathbb{R}^n)$ for any $p \in (1, \infty)$ and $s \in (0, \Theta)$. As its proof is similar to that of Corollary 3.3, we omit the details.

Corollary 3.7. *Let $p \in (1, \infty)$ and $s \in (0, \Theta)$. Then there exists a positive constant C such that for any $f \in H^{s,\mathcal{L}}_p(\mathbb{R}^n)$ in the Bessel potential space associated with $\mathcal{L}$,*

$$\left\|(I + \Delta)^{s/2} f\right\|_{L^p(\mathbb{R}^n)} \leq C \left\|(I + \mathcal{L})^{s/2} f\right\|_{L^p(\mathbb{R}^n)}.$$

4 Identification of heat and Lipschitz Besov spaces for $s \in (0, 1)$

In this section, we prove the main results of this paper. To be precise, we show the identity of heat and Lipschitz versions of Besov spaces with parameters in a Hardy-Littlewood-Sobolev-Kato diagram $\mathcal{P}(\Theta, \sigma)$ (Theorem 1.1), and then show that this diagram is sharp (Theorem 1.2). To begin with, we need to establish the Hardy-Littlewood-Sobolev-Kato estimates in $\mathcal{P}(\Theta, \sigma)$.

4.1 The Hardy-Littlewood-Sobolev-Kato estimates

Let $\sigma \in (2, \infty)$ be as in (1.11). Recall [5] that $\sigma = 2 + \epsilon$ for some $\epsilon \in (0, \infty)$, which may be as small as possible.

Now, let $p \in (1, \infty)$ and $s \in [0, 1]$. Consider the *Hardy-Littlewood-Sobolev-Kato diagram* $\widetilde{\mathcal{P}}(\sigma)$ involving the parameters p and s in the $\{(1/p, s)\}$-plane defined by

$$\widetilde{\mathcal{P}}(\sigma) := \left\{ \left(\frac{1}{p}, s\right) \in (0, 1) \times [0, 1] : \frac{1}{p} \in \left(\frac{s}{\sigma}, 1\right) \right\}, \tag{4.1}$$

as illustrated in the following Figure 3.

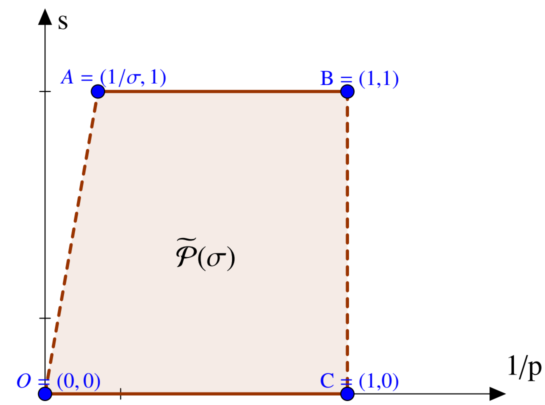

Figure 3: the Hardy-Littlewood-Sobolev-Kato diagram $\widetilde{\mathcal{P}}(\sigma)$

The following theorem was first established in [4, Theorem 5.4]. See also [31, Theorem 8.54] for a similar result.

Proposition 4.1 ([4])**.** *Let* $M := (1/p, s)$*,* $N := (1/q, r) \in \widetilde{\mathcal{P}}(\sigma)$ *as in* (4.1) *with* $p \le q$*. Assume* $\mu \in (0, \pi)$ *and* $\varphi \in \mathscr{E}(\Sigma_\mu)$ *is in the extended Dunford-Riesz class as in* (1.6) *and satisfies the following estimate*

$$\left\| z^{\alpha(M,N)} \varphi \right\|_{L^\infty(\Sigma_\mu)} < \infty,$$

where

$$\alpha(M, N) := \frac{r-s}{2} + \frac{n}{2}\left(\frac{1}{p} - \frac{1}{q}\right). \tag{4.2}$$

Then there exists a positive constant C such that for any $f \in \dot{H}^s_p(\mathbb{R}^n)$ *in the classical Riesz potential space,*

$$\|\varphi(\mathcal{L})(f)\|_{\dot{H}^r_q(\mathbb{R}^n)} \le C \left\| z^{\alpha(M,N)} \varphi \right\|_{L^\infty(\Sigma_\mu)} \|f\|_{\dot{H}^s_p(\mathbb{R}^n)}.$$

Proposition 4.1 indicates the reason why we call $\widetilde{\mathcal{P}}(\sigma)$ the Hardy-Littlewood-Sobolev-Kato diagram, as we can do the Hardy-Littlewood-Sobolev-Kato estimates associated with $\mathcal{L}$ with parameters p and s in $\widetilde{\mathcal{P}}(\sigma)$. More precisely, Proposition 4.1 implies that the domain $\mathrm{dom}_p[\mathcal{L}^{1/2}]$ of the square root of $\mathcal{L}$ in $L^p(\mathbb{R}^n)$ satisfies

$$\mathrm{dom}_p[\mathcal{L}^{1/2}] = H^1_p(\mathbb{R}^n), \tag{4.3}$$

for any $p \in (1, \sigma)$. Moreover, we can obtain the following Hardy-Littlewood-Sobolev inequality by letting $M := (1/p, 0)$, $N := (1/q, 0)$ and $\varphi(z) := (1 + z)^{-\alpha/2}$ with $\alpha = n(\frac{1}{p} - \frac{1}{q})$.

Corollary 4.2. *Let* $1 < p \le q < \infty$ *and* $\alpha = n(\frac{1}{p} - \frac{1}{q})$*. Then there exists a positive constant C such that for any* $f \in L^p(\mathbb{R}^n)$*,*

$$\left\| (I + \mathcal{L})^{-\alpha/2}(f) \right\|_{L^q(\mathbb{R}^n)} \le C \|f\|_{L^p(\mathbb{R}^n)}.$$

Motivated by Proposition 4.1 and based on Theorem 3.1, we introduce the following *Hardy-Littlewood-Sobolev-Kato diagram* $\overline{\mathcal{P}}(\Theta, \sigma)$ in the $(1/p,\ s)$-plane defined by

$$\overline{\mathcal{P}}(\Theta, \sigma) := \left\{ \left(\frac{1}{p},\ s \right) \in (0,\ 1) \times [0,\ 1] :\ \frac{1}{p} \in \begin{cases} (0,\ 1), & s \in [0,\ \Theta), \\ (\frac{s-\Theta}{(1-\Theta)\sigma},\ 1), & s \in [\Theta,\ 1] \end{cases} \right\} \tag{4.4}$$

as illustrated in the following Figure 4.

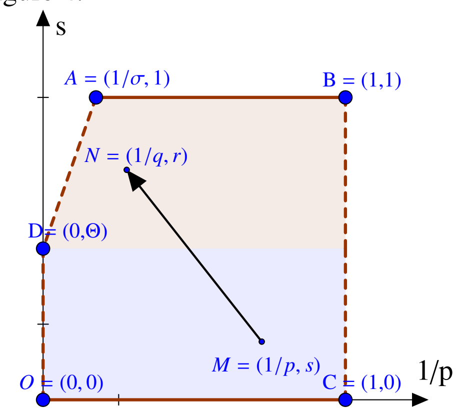

Figure 4: the Hardy-Littlewood-Sobolev-Kato diagram $\overline{\mathcal{P}}(\Theta, \sigma)$

The following theorem is an inhomogeneous version of Proposition 4.1, with parameters in the new Hardy-Littlewood-Sobolev-Kato diagram $\overline{\mathcal{P}}(\Theta, \sigma)$ as in Figure 4, which plays an essential role in the proof of Theorem 1.1.

Theorem 4.3. *Let* $M := (1/p, s)$, $N := (1/q, r) \in \overline{\mathcal{P}}(\Theta, \sigma)$ *be as in* (4.4) *with* $p \leq q$. *Assume* $\mu \in (0, \pi)$ *and* $\varphi \in \mathscr{E}(\Sigma_\mu)$ *is in the extended Dunford-Riesz class as in* (1.6) *and satisfies the following estimate*

$$\left\| (1+z)^{\alpha(M,N)} \varphi \right\|_{L^\infty(\Sigma_\mu)} < \infty,$$

where $\alpha(M, N)$ *is as in* (4.2)*. Then there exists a positive constant* C *such that for any* $f \in H^s_p(\mathbb{R}^n)$ *in the Bessel potential space,*

$$\|\varphi(\mathcal{L})(f)\|_{H^r_q(\mathbb{R}^n)} \leq C \left\| (1+z)^{\alpha(M,N)} \varphi \right\|_{L^\infty(\Sigma_\mu)} \|f\|_{H^s_p(\mathbb{R}^n)}. \tag{4.5}$$

Proof. Let $m(\overrightarrow{MN})$ be the slope of the vector $\overrightarrow{MN}$. We consider the following three cases based on the size of $|m(\overrightarrow{MN})|$: i) $|m(\overrightarrow{MN})| = 0$; ii) $|m(\overrightarrow{MN})| = \infty$; iii) $|m(\overrightarrow{MN})| \in (0,\ \infty)$. See Figure 4 for the general position of the points M and N in $\overline{\mathcal{P}}(\Theta, \sigma)$.

Case i): $|m(\overrightarrow{MN})| = 0$**.** In this case, $\overline{MN}$ is a horizontal segment. We split this case into three subcases: a) M, N are in the segment $\overline{OC}$ (see Figure 3); b) M, N are in the segment $\overline{AB}$; c) M, $N \notin \overline{AB}$ or $M, N \notin \overline{OC}$.

In case a), (4.5) is equivalent to the assertion that for any $1 < p \leq q < \infty$ and $f \in L^p(\mathbb{R}^n)$,

$$\|\varphi(\mathcal{L})(f)\|_{L^q(\mathbb{R}^n)} \lesssim \left\| (1+z)^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \varphi \right\|_{L^\infty(\Sigma_\mu)} \|f\|_{L^p(\mathbb{R}^n)}. \tag{4.6}$$

Indeed, by the bounded functional calculus in $L^q(\mathbb{R}^n)$ and Corollary 4.2, we have

$$\begin{aligned}\|\varphi(\mathcal{L})(f)\|_{L^q(\mathbb{R}^n)} &= \left\|\varphi(\mathcal{L})\,(I+\mathcal{L})^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\,(I+\mathcal{L})^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\,f\right\|_{L^q(\mathbb{R}^n)}\\ &\lesssim \left\|(1+z)^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\varphi\right\|_{L^\infty(\Sigma_\mu)}\left\|(I+\mathcal{L})^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\,f\right\|_{L^q(\mathbb{R}^n)}\\ &\lesssim \left\|(1+z)^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\varphi\right\|_{L^\infty(\Sigma_\mu)}\|f\|_{L^p(\mathbb{R}^n)},\end{aligned}$$

which implies (4.6).

In case b), (4.5) is equivalent to the assertion that for any $1<p\le q<\sigma$ and $f\in L^p(\mathbb{R}^n)$,

$$\|\varphi(\mathcal{L})(f)\|_{H^1_q(\mathbb{R}^n)} \lesssim \left\|(1+z)^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\varphi\right\|_{L^\infty(\Sigma_\mu)}\|f\|_{H^1_p(\mathbb{R}^n)}. \tag{4.7}$$

Using again the bounded functional calculus in $L^q(\mathbb{R}^n)$, Corollaries 4.2 and 3.3 and the fact that $1<p\le q<\sigma$, we know

$$\begin{aligned}\|\varphi(\mathcal{L})(f)\|_{H^1_q(\mathbb{R}^n)} &\simeq \left\|(I+\Delta)^{1/2}\,\varphi(\mathcal{L})(f)\right\|_{L^q(\mathbb{R}^n)} \lesssim \left\|(I+\mathcal{L})^{1/2}\,\varphi(\mathcal{L})(f)\right\|_{L^q(\mathbb{R}^n)}\\ &\lesssim \left\|(1+z)^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\varphi\right\|_{L^\infty(\Sigma_\mu)}\left\|(I+\mathcal{L})^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})+\frac{1}{2}}\,f\right\|_{L^q(\mathbb{R}^n)}\\ &\lesssim \left\|(1+z)^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\varphi\right\|_{L^\infty(\Sigma_\mu)}\left\|(I+\mathcal{L})^{1/2}\,f\right\|_{L^p(\mathbb{R}^n)}\\ &\lesssim \left\|(1+z)^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\varphi\right\|_{L^\infty(\Sigma_\mu)}\left\|(I+\Delta)^{1/2}\,f\right\|_{L^p(\mathbb{R}^n)},\end{aligned}$$

which shows that (4.7) holds true.

For the case c), to prove (4.5), we need to show that for any $0<s<1$, $1<p\le q<p_s(\mathcal{L})$ with

$$p_s(\mathcal{L}) := \begin{cases}\infty, & s\in(0,\,\Theta),\\ \frac{s-\Theta}{(1-\Theta)\sigma}, & s\in[\Theta,\,1)\end{cases} \tag{4.8}$$

and any $f\in H^s_p(\mathbb{R}^n)$,

$$\|\varphi(\mathcal{L})(f)\|_{H^s_q(\mathbb{R}^n)} \lesssim \left\|(1+z)^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\varphi\right\|_{L^\infty(\Sigma_\mu)}\|f\|_{H^s_p(\mathbb{R}^n)}. \tag{4.9}$$

If $s\in(0,\,\Theta)$, by Corollaries 3.3, 3.7 and Theorem 3.1, we find that

$$\begin{aligned}\|\varphi(\mathcal{L})(f)\|_{H^s_q(\mathbb{R}^n)} &\lesssim \left\|(1+z)^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\varphi\right\|_{L^\infty(\Sigma_\mu)}\left\|(I+\mathcal{L})^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})+\frac{s}{2}}\,f\right\|_{L^q(\mathbb{R}^n)}\\ &\lesssim \left\|(1+z)^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\varphi\right\|_{L^\infty(\Sigma_\mu)}\left\|(I+\mathcal{L})^{s/2}\,f\right\|_{L^p(\mathbb{R}^n)}\\ &\lesssim \left\|(1+z)^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\varphi\right\|_{L^\infty(\Sigma_\mu)}\|f\|_{H^s_p(\mathbb{R}^n)},\end{aligned} \tag{4.10}$$

which yields that (4.9) holds true.

If $s \in [\Theta, 1)$ and $\frac{1}{q} \geq \frac{1}{2}$, or $s \in [\Theta, 1)$ and $\frac{1}{p} \leq \frac{1}{2}$, then by (4.8), we have $p_s(\mathcal{L}) = \frac{s-\Theta}{(1-\Theta)\sigma}$. Mereover, there exist $s_0 \in (0, \Theta)$, $p_0, q_0 \in (1, \infty)$, $p_1, q_1 \in (1, \sigma)$ and $\theta \in (0, 1)$ satisfying

$$\begin{cases} s = (1-\theta)s_0 + \theta \\ \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \\ \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \end{cases} \quad \text{and} \quad \frac{1}{p_1} - \frac{1}{q_1} = \frac{1}{p} - \frac{1}{q} = \frac{1}{p_0} - \frac{1}{q_0}. \tag{4.11}$$

Let $E := (\frac{1}{p_0}, s_0)$, $F := (\frac{1}{q_0}, s_0)$, $G := (\frac{1}{p_1}, 1)$ and $H := (\frac{1}{q_1}, 1)$. Note that (4.11) is possible due to the convexity of the diagram $\overline{\mathcal{P}}(\Theta, \sigma)$ in Figure 4. Indeed, in the case $s \in [\Theta, 1)$ and $\frac{1}{q} \geq \frac{1}{2}$, we may let $p_0 = p_1 = p$ and $q_0 = q_1 = q$. In the case $s \in [\Theta, 1)$ and $\frac{1}{p} \leq \frac{1}{2}$, we may let $\overline{EFHG}$ be the parallelogram satisfying that the side $\overline{FH}$ is paralle to $\overline{DA}$ and the side $\overline{EF}$ is paralle to $\overline{MN}$.

As $\overrightarrow{EF}$ belongs to the case c) with $s \in (0, \Theta)$ and $\overrightarrow{GH}$ belongs to the case b), we know

$$\|\varphi(\mathcal{L})(f)\|_{H^{s_0}_{q_0}(\mathbb{R}^n)} \lesssim \left\| (1+z)^{\frac{n}{2}(\frac{1}{p_0} - \frac{1}{q_0})} \varphi \right\|_{L^\infty(\Sigma_\mu)} \|f\|_{H^{s_0}_{p_0}(\mathbb{R}^n)} \tag{4.12}$$

and

$$\|\varphi(\mathcal{L})(f)\|_{H^{1}_{q_1}(\mathbb{R}^n)} \lesssim \left\| (1+z)^{\frac{n}{2}(\frac{1}{p_1} - \frac{1}{q_1})} \varphi \right\|_{L^\infty(\Sigma_\mu)} \|f\|_{H^{1}_{p_1}(\mathbb{R}^n)}. \tag{4.13}$$

Moreover, by (4.11) and the complex interpolation of classical Bessel potential space $H^s_p(\mathbb{R}^n)$ (see [51, 54]), we have

$$\left[H^{s_0}_{p_0}(\mathbb{R}^n),\, H^1_{p_1}(\mathbb{R}^n)\right]_\theta = H^s_p(\mathbb{R}^n) \quad \text{and} \quad \left[H^{s_0}_{q_0}(\mathbb{R}^n),\, H^1_{q_1}(\mathbb{R}^n)\right]_\theta = H^s_q(\mathbb{R}^n),$$

which combined with (4.12), (4.13) and (4.11) again implies that for any such $s \in [\Theta, 1)$,

$$\|\varphi(\mathcal{L})(f)\|_{H^{s}_{q}(\mathbb{R}^n)} \lesssim \left\| (1+z)^{\frac{n}{2}(\frac{1}{p} - \frac{1}{q})} \varphi \right\|_{L^\infty(\Sigma_\mu)} \|f\|_{H^{s}_{p}(\mathbb{R}^n)}. \tag{4.14}$$

By this and (4.10), we conclude that (4.9) holds in the cases $s \in [\Theta, 1)$ and $\frac{1}{q} \geq \frac{1}{2}$, or $s \in [\Theta, 1)$ and $\frac{1}{p} \leq \frac{1}{2}$.

If $s \in [\Theta, 1)$ and $\frac{1}{q} < \frac{1}{2} < \frac{1}{p}$, then we split the vector $\overrightarrow{MN}$ into two parts as

$$\overrightarrow{MN} = \overrightarrow{ME} \cup \overrightarrow{EN}$$

with $E := (\frac{1}{2}, s)$. As the cases $\overrightarrow{ME}$ and $\overrightarrow{EN}$ respectively belongs to the cases $s \in [\Theta, 1)$, $\frac{1}{q} \geq \frac{1}{2}$ and $s \in [\Theta, 1)$, $\frac{1}{p} \leq \frac{1}{2}$, we have

$$\begin{aligned} \|\varphi(\mathcal{L})(f)\|_{H^{s}_{q}(\mathbb{R}^n)} &\lesssim \left\| (1+z)^{\theta_0 \frac{n}{2}(\frac{1}{p} - \frac{1}{q})} \varphi^{\theta_0} \right\|_{L^\infty(\Sigma_\mu)} \left\| \varphi^{1-\theta_0} f \right\|_{H^s_2(\mathbb{R}^n)} \\ &\lesssim \left\| (1+z)^{\frac{n}{2}(\frac{1}{p} - \frac{1}{q})} \varphi \right\|^{\theta_0}_{L^\infty(\Sigma_\mu)} \left\| (1+z)^{(1-\theta_0)\frac{n}{2}(\frac{1}{p} - \frac{1}{q})} \varphi^{1-\theta_0} \right\|_{L^\infty(\Sigma_\mu)} \|f\|_{H^s_p(\mathbb{R}^n)}, \end{aligned}$$

where $\theta_0 := (\frac{1}{2} - \frac{1}{q})/(\frac{1}{p} - \frac{1}{q}) \in (0, 1)$. By this and (4.14), we see (4.9) holds true in case c).

Combined the above three subcases a), b) and c), we conclude that (4.6) holds true in Case i).

Case ii): $|m(\overrightarrow{MN})| = \infty$**.** In this case, we need to show that for any $f \in H^s_p(\mathbb{R}^n)$,

$$\|\varphi(\mathcal{L})(f)\|_{H^r_p(\mathbb{R}^n)} \lesssim \left\|(1+z)^{\frac{1}{2}(r-s)}\varphi\right\|_{L^\infty(\Sigma_\mu)} \|f\|_{H^s_p(\mathbb{R}^n)}. \tag{4.15}$$

To this end, we also consider three subcases: case a) $\frac{1}{p} \in (\frac{1}{\sigma}, 1)$; case b) $\frac{1}{p} \in (0, \frac{1}{\sigma}]$ and $\max\{s, r\} < \Theta$; case c) $\frac{1}{p} \in (0, \frac{1}{\sigma}]$ and $\max\{s, r\} \geq \Theta$.

For case a), recall the following complex interpolation of the domain $\mathrm{dom}_p[\mathcal{L}^{s/2}]$ of the fractional power $\mathcal{L}^{s/2}$ of $\mathcal{L}$ with $s \in (0, 1]$ that for any $\theta \in (0, 1)$ and $p \in (1, \infty)$ (see, for example, [36] or [28, Theorem 6.6.8]),

$$\left[L^p(\mathbb{R}^n), \mathrm{dom}_p[\mathcal{L}^{s/2}]\right]_\theta = \mathrm{dom}_p[\mathcal{L}^{s\theta/2}],$$

which combined with (4.3) and Corollary 3.3 and $p \in (1, \sigma)$ shows that

$$\begin{aligned}\|\varphi(\mathcal{L})(f)\|_{H^r_p(\mathbb{R}^n)} &\simeq \left\|(I+\mathcal{L})^{r/2}\varphi(\mathcal{L})(f)\right\|_{L^p(\mathbb{R}^n)} \lesssim \left\|(1+z)^{\frac{1}{2}(r-s)}\varphi\right\|_{L^\infty(\Sigma_\mu)} \left\|(I+\mathcal{L})^{s/2} f\right\|_{L^p(\mathbb{R}^n)} \\ &\lesssim \left\|(1+z)^{\frac{1}{2}(r-s)}\varphi\right\|_{L^\infty(\Sigma_\mu)} \|f\|_{H^s_p(\mathbb{R}^n)}.\end{aligned}$$

This implies that (4.15) holds true.

For case b), by Theorem 3.1, we have

$$\|\varphi(\mathcal{L})(f)\|_{H^r_p(\mathbb{R}^n)} \simeq \left\|(I+\mathcal{L})^{r/2}\varphi(\mathcal{L})(f)\right\|_{L^p(\mathbb{R}^n)} \lesssim \left\|(1+z)^{\frac{1}{2}(r-s)}\varphi(z)\right\|_{L^\infty(\Sigma_\mu)} \|f\|_{H^s_p(\mathbb{R}^n)}, \tag{4.16}$$

which implies that (4.15) also holds true.

For case c), we split the vector $\overrightarrow{MN}$ equally into n_0 parts as

$$\overrightarrow{MN} = \overrightarrow{M_0M_1} \cup \cdots \cup \overrightarrow{M_{n_0-1}M_{n_0}} \tag{4.17}$$

with $M_0 := (\frac{1}{p}, s_0) = M$, $M_{n_0} := (\frac{1}{p}, s_{n_0}) = N$ and $M_i := (\frac{1}{p}, s_i)$ for any $i \in \{1, \ldots, n_0 - 1\}$ and satisfying that

$$s_i - s_{i-1} = \frac{1}{n_0}(r-s) < \Theta. \tag{4.18}$$

Here we can choose $n_0 \in \mathbb{N}$ as large as possible to make $s_i - s_{i-1}$ sufficiently small.

If $\overrightarrow{M_{i-1}M_i}$ belongs to case b), then by (4.16) and (4.18), we know that

$$\begin{aligned}\left\|(\varphi(\mathcal{L}))^{1/n_0}(f)\right\|_{H^{s_i}_p(\mathbb{R}^n)} &\lesssim \left\|(1+z)^{\frac{1}{2}(s_i-s_{i-1})}|\varphi(z)|^{1/n_0}\right\|_{L^\infty(\Sigma_\mu)} \|f\|_{H^{s_{i-1}}_p(\mathbb{R}^n)} \\ &\simeq \left\|(1+z)^{\frac{1}{2}(r-s)}\varphi(z)\right\|^{1/n_0}_{L^\infty(\Sigma_\mu)} \|f\|_{H^{s_{i-1}}_p(\mathbb{R}^n)}.\end{aligned} \tag{4.19}$$

If $\overrightarrow{M_{i-1}M_i}$ doesn't belong to case b), then by taking n_0 sufficiently large, there exist $E_i := (\frac{1}{\widetilde{p}_i}, \widetilde{s}_i)$, $F_i := (\frac{1}{\widetilde{p}_i}, \widetilde{r}_i)$, $G_i := (\frac{1}{\widetilde{\widetilde{p}}_i}, \widetilde{\widetilde{s}}_i)$, $H_i := (\frac{1}{\widetilde{\widetilde{p}}_i}, \widetilde{\widetilde{r}}_i) \in \overline{\mathcal{P}}(\Theta, \sigma)$ with $\overrightarrow{E_iF_i}$ belong to the case a), $\overrightarrow{G_iH_i}$

belong to the case b) and satisfying

$$\begin{cases} s_i = (1-\theta)\widetilde{r_i} + \theta\widetilde{\widetilde{r_i}}, \\ s_{i-1} = (1-\theta)\widetilde{s_i} + \theta\widetilde{\widetilde{s_i}}, \\ \frac{1}{p_i} = \frac{1-\theta}{\widetilde{p_i}} + \frac{\theta}{\widetilde{\widetilde{p_i}}}, \end{cases}$$

which together with the cases a), b) and a complex interpolation argument implies also

$$\left\|(\varphi(\mathcal{L}))^{1/n_0}(f)\right\|_{H_p^{s_i}(\mathbb{R}^n)} \lesssim \left\|(1+z)^{\frac{1}{2}(r-s)}\varphi\right\|_{L^\infty(\Sigma_\mu)}^{1/n_0} \|f\|_{H_p^{s_{i-1}}(\mathbb{R}^n)}. \tag{4.20}$$

Combined (4.17), (4.19) and (4.20), we conclude that

$$\begin{aligned} \|\varphi(\mathcal{L})(f)\|_{H_p^r(\mathbb{R}^n)} &\lesssim \left\|(1+z)^{\frac{1}{2}(r-s)}\varphi\right\|_{L^\infty(\Sigma_\mu)}^{1/n_0} \left\|\varphi(\mathcal{L})^{(n_0-1)/n_0} f\right\|_{H_p^{s_{n_0-1}}(\mathbb{R}^n)} \\ &\lesssim \left\|(1+z)^{\frac{1}{2}(r-s)}\varphi\right\|_{L^\infty(\Sigma_\mu)} \|f\|_{H_p^s(\mathbb{R}^n)}, \end{aligned}$$

which implies that (4.15) holds true.

Case iii): $|m(\overrightarrow{MN})| \in (0,\infty)$**.** In this case, we let $E := (\frac{1}{q}, s)$. Then $\overrightarrow{ME}$ belongs to Case i) and $\overrightarrow{EN}$ belongs to Case ii). This implies that

$$\begin{aligned} \|\varphi(\mathcal{L})(f)\|_{H_q^r(\mathbb{R}^n)} &= \left\|(I+\mathcal{L})^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}(I+\mathcal{L})^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\varphi(\mathcal{L})(f)\right\|_{H_q^r(\mathbb{R}^n)} \\ &\lesssim \left\|(1+z)^{\frac{1}{2}(r-s)+\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\varphi\right\|_{L^\infty(\Sigma_\mu)} \left\|(I+\mathcal{L})^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}(f)\right\|_{H_q^s(\mathbb{R}^n)} \\ &\lesssim \left\|(1+z)^{\frac{1}{2}(r-s)+\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\varphi\right\|_{L^\infty(\Sigma_\mu)} \|f\|_{H_p^s(\mathbb{R}^n)}, \end{aligned}$$

which immediately shows (4.5) and hence completes the proof of Theorem 4.3.

□

Theorem 4.3 implies the following fractional Kato estimates, which when $s=1$ reduces to the Kato square root estimates established in [5, 4].

Corollary 4.4. *Let* $(\frac{1}{p}, s) \in \overline{\mathcal{P}}(\Theta,\sigma)$ *be as in* (4.4)*. Then there exist positive constants* C_4 *and* C_5 *such that for any* $f \in L^p(\mathbb{R}^n)$*,*

$$C_4\left\|(I+\Delta)^{s/2} f\right\|_{L^p(\mathbb{R}^n)} \le \left\|(I+\mathcal{L})^{s/2} f\right\|_{L^p(\mathbb{R}^n)} \le C_5\left\|(I+\Delta)^{s/2} f\right\|_{L^p(\mathbb{R}^n)}$$

Proof. Corollary 4.4 follows immediate from Theorem 4.3 by taking respectively $M := (\frac{1}{p}, 0)$, $N := (\frac{1}{p}, s)$, $\varphi(z) := (1+z)^{-s/2}$ and $M := (\frac{1}{p}, s)$, $N := (\frac{1}{p}, 0)$, $\varphi(z) := (1+z)^{s/2}$. □

4.2 Proofs of Theorems 1.1 and 1.2

In this subsection, we prove Theorems 1.1 and 1.2, which are the main results of this paper. We first turn to the proof of Theorem 1.1, based on the Hardy-Littlewood-Sobolev-Kato estimates established in Section 4.1.

Proof of Theorem 1.1. By Corollary 4.4, we see for any $(1/p,\ s) \in \overline{\mathcal{P}}(\Theta, \sigma)$ as in Figure 4 and $f \in L^p(\mathbb{R}^n)$,

$$\left\|(I+\Delta)^{s/2}\right\|_{L^p(\mathbb{R}^n)} \simeq \left\|(I+\mathcal{L})^{s/2}\right\|_{L^p(\mathbb{R}^n)},$$

which immediately implies that for any $(\frac{1}{p},\ s) \in \overline{\mathcal{P}}(\Theta, \sigma)$ in Figure 4,

$$H^s_p(\mathbb{R}^n) = \mathrm{dom}_p[\mathcal{L}^{s/2}]. \tag{4.21}$$

On the other hand, recall the following real interpolation of the classical Bessel potential space $H^s_p(\mathbb{R}^n)$ that for any $p, q \in (1,\ \infty)$, $s \in (0,\ \infty)$ and $\theta \in (0,\ 1)$,

$$\left(L^p(\mathbb{R}^n),\ H^s_p(\mathbb{R}^n)\right)_{\theta,q} = B^{s\theta}_{p,q}(\mathbb{R}^n) \tag{4.22}$$

(see for example, [52, Theorem 1.6.7]), which combined with Remark 2.3(ii) implies that for any $p, q \in (1,\ \infty)$, $s \in (0,\ 1]$ and $\theta \in (0,\ 1)$,

$$\left(L^p(\mathbb{R}^n),\ H^s_p(\mathbb{R}^n)\right)_{\theta,q} = B^{s\theta}_{p,q}. \tag{4.23}$$

On the other hand, by Proposition 2.2, we know that for any $p,\ q \in (1,\ \infty)$, $s \in (0,\ \infty)$ and $\theta \in (0,\ 1)$,

$$\left(L^p(\mathbb{R}^n),\ \mathrm{dom}_p[\mathcal{L}^{s/2}]\right)_{\theta,q} = B^{s\theta,\mathcal{L}}_{p,q},$$

which together with (4.21) and (4.23) immediately implies that (1.13) holds true for any $(\frac{1}{p},\ s) \in \mathcal{P}(\Theta, \sigma)$ as in Figure 2. This finishes the proof of Theorem 1.1. □

We now consider the sharpness of the parameters $p \in (1,\ \infty)$ and $s \in (0,\ 1)$ in the above identity. The sharpness we considered here is in the sense of the whole *class* $\mathscr{E}(\mathbb{R}^n)$, which consists of all the second order divergence form elliptic operators $\mathcal{L}$ with real symmetric coefficients satisfying (1.1). To be precise, we want to find a range of p and s such that for any $p \in (1,\ \infty)$ and $s \in (0,\ 1)$ not belong to this range, there exists a second order divergence form elliptic operator $\mathcal{L} \in \mathscr{E}(\mathbb{R}^n)$ and $q \in (1,\ \infty)$, $B^{s,\mathcal{L}}_{p,q} \neq B^s_{p,q}$.

Note the following two facts on the sharpness of the parameters s and p.

(i) For any $s \in (0,\ 1)$, there exists $\mathcal{L} \in \mathscr{E}(\mathbb{R}^n)$ such that $\Theta < s$ (see [55, 46]). This implies that $\inf \Theta = 0$ over all $\mathcal{L}$ in $\mathscr{E}(\mathbb{R}^n)$.

(ii) For any $p \in (2,\ \infty)$, there exists $\mathcal{L} \in \mathscr{E}(\mathbb{R}^n)$ such that $\sigma < p$ (see [5]). This implies that $\inf \sigma = 2$ over all $\mathcal{L}$ in $\mathscr{E}(\mathbb{R}^n)$.

Based on the above two observations, we introduce the following *sharp Hardy-Littlewood-Sobolev-Kato diagram* $\mathcal{P}$

$$\mathcal{P} := \bigcap_{\substack{\Theta \in (0,\ 1) \\ \sigma \in (2,\ \infty)}} \mathcal{P}(\Theta, \sigma) = \left\{\left(\frac{1}{p},\ s\right) \in (0,\ 1) \times (0,\ 1) :\ \frac{1}{p} \in \left[\frac{s}{2},\ 1\right)\right\}, \tag{4.24}$$

as illustrated in the following Figure 5.

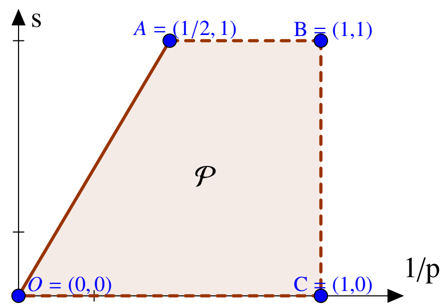

Figure 5: the sharp Hardy-Littlewood-Sobolev-Kato diagram $\mathcal{P}$

We now prove Theorem 1.2 for parameters in $\mathcal{P}$. To this end, we need the following key lemma, which provides the desired counterexample to consider the sharpness. Recall that this counterexample is also used in [5, p. 120] to show that $\inf \sigma = 2$ over all $\mathcal{L}$ in $\mathscr{E}(\mathbb{R}^n)$.

Lemma 4.5. *There exist two families $\{\mathcal{L}_\beta\}_{\beta\in(-1,\infty)} \subset \mathscr{E}(\mathbb{R}^2)$ and $\{u_\beta\}_{\beta\in(-1,\infty)}$ of functions on $\mathbb{R}^2 \setminus \{0\}$ such that*

(i) *for any $\beta \in (-1, \infty)$ and $p \in (1, \infty)$, $u_\beta \in L^p(\mathbb{R}^2)$;*

(ii) *for any $s \in (0, 1)$ and $p \in (\frac{2}{s}, \infty)$, there exists $\beta \in (-1, \frac{s}{2-s} - 1)$ such that $\Delta^{s/2} u_\beta \notin L^p(\mathbb{R}^2)$;*

(iii) *for any $s \in (0, 1)$, $\beta \in (-1, \infty)$ and $p \in (\frac{2}{s}, \infty)$, $\mathcal{L}_\beta^{s/2} u_\beta \in L^p(\mathbb{R}^2)$.*

Proof. Be begin with, we first recall the following example of divergence form elliptic operators in $\mathscr{E}(\mathbb{R}^2)$ from [5]. For any $s \in (0, 1)$, $\beta \in (-1, \infty)$ and $x \in \mathbb{R}^2 \setminus \{0\}$, let

$$A_\beta(x) := I + \frac{\beta(\beta+2)}{|x|^2} \begin{pmatrix} x_2^2, & -x_1x_2 \\ -x_1x_2, & x_1^2 \end{pmatrix} \tag{4.25}$$

be a 2×2 nonnegative symmetric real matrix on $\mathbb{R}^2$. It is easy to see that A_β satisfies the uniform ellipticity condition (1.1), due to the fact that for any $x \in \mathbb{R}^2 \setminus \{0\}$,

$$\min\left\{1, (1+\beta)^2\right\} I \le A_\beta(x) \le \max\left\{1, (1+\beta)^2\right\} I.$$

Let $\mathcal{L}_\beta := -\mathrm{div}(A_\beta \nabla) \in \mathscr{E}(\mathbb{R}^2)$ be the second order divergence form elliptic operator associated with A_β as in (4.25). Let $u_\beta := v_\beta \phi$ with $\phi \in C_c^\infty(\mathbb{R}^2)$ satisfying $\operatorname{supp} \phi \subset B(0, 1)$, $0 \le \phi \le 1$, $\phi \equiv 1$ on $B(0, 1/2)$ and

$$v_\beta(x) := x_1 |x|^\beta \tag{4.26}$$

for any $x := (x_1, x_2) \in \mathbb{R}^2 \setminus \{0\}$. Then, by an elementary calculation, we know that the function v_β defined as in (4.26) is a classical strong solution to the equation

$$\mathcal{L}_\beta v = 0 \tag{4.27}$$

on $\mathbb{R}^2 \setminus \{0\}$ and also a weak solution to (4.27) on $\mathbb{R}^2$ (see also [5, p. 120]).

We now first prove (i). Indeed, by the properties of ϕ, the fact that $\beta > -1$ and using the polar coordinate, we have

$$\|u_\beta\|_{L^p(\mathbb{R}^n)}^p \simeq \int_0^1 r^{(1+\beta)p+1}\,dr < \infty,$$

which immediately implies that (i) holds true.

To prove (ii), we first observe from (4.26) that for any $\beta \in (-1, \infty)$ and $x \in \mathbb{R}^2 \setminus \{0\}$,

$$\nabla v_\beta(x) = \left(|x|^\beta + \beta x_1^2 |x|^{\beta-2},\ \beta x_1 x_2 |x|^{\beta-2}\right) \tag{4.28}$$

and

$$\Delta v_\beta(x) = 4\beta x_1 |x|^{\beta-2} + \beta(\beta-2)(x_1^3 + x_1 x_2^2)|x|^{\beta-4}. \tag{4.29}$$

If further $\beta \in (-1, \frac{s}{2-s} - 1)$, we know that there exist $\epsilon \in (0,\ 1)$ and $\sigma \in (0,\ s)$ such that

$$-1 + \epsilon < \beta < \frac{\sigma}{2-s} - 1. \tag{4.30}$$

Now let $\delta := \frac{2}{2-\epsilon} - 1 \in (0,\ 1)$ and $p_1 := 1 + \delta$. By (4.30), it is easy to see $(\beta - 1)p_1 + 1 > -1$, which together with (4.29) implies that

$$\left\|\phi \Delta v_\beta\right\|_{L^{p_1}(\mathbb{R}^n)}^{p_1} \simeq \int_0^1 r^{(\beta-1)p_1+1}\,dr < \infty.$$

This together with the properties of ϕ and (4.28) implies that

$$\left\|\Delta u_\beta\right\|_{L^{p_1}(\mathbb{R}^n)} \lesssim \left\|\phi \Delta v_\beta\right\|_{L^{p_1}(\mathbb{R}^n)} + \left\|v_\beta \Delta \phi\right\|_{L^{p_1}(\mathbb{R}^n)} + \left\||\nabla \phi|\,|\nabla v_\beta|\right\|_{L^{p_1}(\mathbb{R}^n)} < \infty. \tag{4.31}$$

Moreover, let

$$p_0 := \frac{2(2-s)(1+\delta)}{(s-\sigma)(1+\delta) + 2(1-s)} \tag{4.32}$$

with σ as in (4.30). From (4.30), it follows $\beta < \frac{\sigma}{2-s} - 1 + \frac{2(1-s)\delta}{2-s}$, which is equivalent to $\beta p_0 + 1 < -1$. By this and (4.28), we know

$$\left\|\phi \nabla v_\beta\right\|_{L^{p_0}(\mathbb{R}^n)}^{p_0} \simeq \int_0^1 r^{\beta p_0 + 1}\,dr = \infty.$$

Using the properties of ϕ again, we find

$$\left\|\nabla u_\beta\right\|_{L^{p_0}(\mathbb{R}^n)} \gtrsim \left\|\phi \nabla v_\beta\right\|_{L^{p_0}(\mathbb{R}^n)} - \left\|v_\beta \nabla \phi\right\|_{L^{p_0}(\mathbb{R}^n)} = \infty. \tag{4.33}$$

We now continue the proof of (ii). For any $p \in (\frac{2}{s}, \infty)$, let $\widetilde{\sigma} \in (0,\ s)$ such that $p = \frac{2}{s-\widetilde{\sigma}}$. Moreover, it is easy to see that there exists $\beta \in (-1, \frac{s}{2-s} - 1)$ satisfies (4.30) with $\sigma = \widetilde{\sigma}$. By this and (4.32), we have

$$\begin{cases} 1 = s(1-\theta) + 2\theta \\ \frac{1}{p_0} = \frac{1-\theta}{p} + \frac{\theta}{p_1}, \end{cases}$$

where $\theta := \frac{1-s}{2-s} \in (0,\ 1)$.

For β fixed as above, if we assume that $\Delta^{s/2} u_\beta \in L^p(\mathbb{R}^2)$, then by (4.31) and the following interpolation inequality for functions in Riesz potential space

$$\left\|\nabla u_\beta\right\|_{L^{p_0}(\mathbb{R}^n)} \le \left\|\Delta^{s/2} u_\beta\right\|_{L^p(\mathbb{R}^n)}^{1-\theta} \left\|\Delta u_\beta\right\|_{L^{p_1}(\mathbb{R}^n)}^{\theta},$$

we obtain $\nabla u_\beta \in L^{p_0}(\mathbb{R}^n)$, which is contract with (4.33). Thus the assumption $\Delta^{s/2} u_\beta \in L^p(\mathbb{R}^2)$ doesn't hold. This proves (ii).

To prove (iii), we first claim that for any $\beta \in (-1,\ \infty)$ and $p \in (1,\ \infty)$,

$$\mathcal{L}_\beta^{s/2} u_\beta = \mathcal{L}_\beta^{-(2-s)/2} \circ \mathcal{L}_\beta u_\beta \tag{4.34}$$

in $L^p(\mathbb{R}^n)$. Indeed, let $f := \mathcal{L} u_\beta$. Using the fact that v_β is a solution to (4.27) and $u_\beta = v_\beta \phi$, we find

$$f = \mathcal{L}_\beta u_\beta = 2A_\beta \nabla\phi \cdot \nabla v_\beta + v_\beta \mathcal{L}_\beta \phi,$$

which together with the properties of ϕ shows that $\mathcal{L}_\beta u_\beta = f \in C_c^\infty(B(0,\ 1) \setminus B(0,\ 1/2)) \subset L^p(\mathbb{R}^n)$. By this and (i), we conclude that $u_\beta \in \mathrm{dom}_p[\mathcal{L}_\beta]$ for any $p \in (1,\ \infty)$, which implies (4.34) holds true.

Now by (4.34) and the bounded functional calculus in $L^p(\mathbb{R}^n)$, we write for any $p \in (1,\ \infty)$,

$$\mathcal{L}_\beta^{s/2} u_\beta = \mathcal{L}_\beta^{-(2-s)/2} f = \frac{1}{\Gamma((2-s)/2)} \int_0^\infty t^{-s/2} e^{-t\mathcal{L}_\beta} f\, dt \tag{4.35}$$

in $L^p(\mathbb{R}^n)$. As $e^{-t\mathcal{L}_\beta}$ has heat kernel satisfying (1.2), we know that the $\mathcal{L}_\beta$-adapted Riesz potential $\mathcal{L}_\beta^{-(2-s)/2}$ also has an integral kernel $\mathcal{L}_\beta^{-(2-s)/2}(x,\ y)$, which satisfies that for any $(x, y) \in (\mathbb{R}^2 \times \mathbb{R}^2) \setminus \{(0,\ 0)\}$,

$$\left|\mathcal{L}_\beta^{-(2-s)/2}(x,\ y)\right| \lesssim \int_0^\infty t^{-s/2-1} \exp\left\{-c_1 \frac{|x-y|^2}{t}\right\} dt \lesssim \frac{1}{|x-y|^s} \simeq \frac{1}{|x-y|^{2-(2-s)}},$$

where the last term is exactly the integral kernel of the classical Riesz potential $\Delta^{-(2-s)/2}$ on $\mathbb{R}^2$. Thus by (4.35), the fact $f \in L^q(\mathbb{R}^n)$ for any $q \in (1,\ \infty)$ and the boundedness of Riesz potential $\Delta^{-(2-s)/2}$ from $L^q(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for any $1 < q \le p < \infty$ satisfying

$$\frac{1}{q} - \frac{1}{p} = \frac{2-s}{2},$$

we know that $\mathcal{L}_\beta^{s/2} u_\beta \in L^p(\mathbb{R}^n)$ for any $p \in (\frac{2}{s},\ \infty)$. This implies (iii) and hence finishes the proof of Lemma 4.5.

□

Remark 4.6. (i) Lemma 4.5 shows the sharpness of the boundedness of the fractional Riesz transform $\Delta^{s/2}\mathcal{L}^{-s/2}$ on $L^p(\mathbb{R}^n)$ for any $s \in (0,\ 1)$ and $p \in (1,\ \frac{2}{s}]$. Here the sharpness is also understand in the sense of the whole class $\mathscr{E}(\mathbb{R}^n)$.

(ii) Recall that in our counterexample in (4.26), the function v_β belongs to the Hölder class $C^{1+\beta}(\mathbb{R}^2)$. As $\beta < \frac{s}{2-s} - 1$, we have $1 + \beta < \frac{s}{2-s} < s$. This shows that $s \notin (0,\ \Theta(\mathcal{L}_\beta))$ for $\mathcal{L}_\beta$ associated with the coefficient matrix A_β as in (4.25). Thus we cannot use Theorem 3.1 in this case.

With the help of Lemma 4.5, we now turn to the proof of Theorem 1.2.

Proof of Theorem 1.2. We first claim that if there exists $E := (\frac{1}{p_0}, s_0) \in (0, 1) \times (0, 1) \setminus \mathcal{P}$, $q_0 \in (1, \infty)$ and $\mathcal{L} \in \mathscr{E}(\mathbb{R}^n)$ satisfying

$$B^{s_0,\mathcal{L}}_{p_0,q_0} = B^{s_0}_{p_0,q_0}, \tag{4.36}$$

then this identity can pass from E to another point, which is very close to the boundary of $\mathcal{P}$. To be precise, we will show that for any $\epsilon \in (0, 1)$ sufficiently small, there exists $F := (\frac{1}{p}, s) \in (0, 1) \times (0, 1) \setminus \mathcal{P}$ satisfying

$$\left|\frac{1}{p} - \frac{s}{2}\right| < \epsilon \tag{4.37}$$

such that for any $q \in (1, \infty)$,

$$B^{s,\mathcal{L}}_{p,q} = B^{s}_{p,q}. \tag{4.38}$$

Indeed, let $p := p_0$ and $s < s_0$ such that the point $F = (\frac{1}{p_0}, s) \in (0, 1)\times(0, 1)\setminus\mathcal{P}$ sufficiently close to the segment $\overline{OA}$ (see Figure 5 for the existence of the point F), then we know $(\frac{1}{p_0}, s)$ satisfying (4.37). Moreover, let $G := (\frac{1}{p_0}, s_1) \in \mathcal{P}$ with $s_1 < s$. Then by Theorem 1.1, we have for any $q \in (1, \infty)$,

$$B^{s_1,\mathcal{L}}_{p_0,q} = B^{s_1}_{p_0,q}.$$

Note that there exists $\theta \in (0, 1)$ such that $s = s_0(1 - \theta) + s_1\theta$. Thus, by Proposition 2.2(ii), the real interpolation of the classical Besov space $B^s_{p,q}(\mathbb{R}^n)$ and Remark 2.3(i), we conclude that for any $q \in (1, \infty)$,

$$B^{s,\mathcal{L}}_{p_0,q} = B^{s}_{p_0,q}.$$

This verifies the claim (4.38). Note that the above claim enables us only need to consider points $(\frac{1}{p}, s) \in (0, 1)\times(0, 1)\setminus\mathcal{P}$ satisfying (4.37) for any $q \in (1, \infty)$, as other cases can be passed to this case by this claim.

Now let $F := (\frac{1}{p}, s) \in (0, 1) \times (0, 1) \setminus \mathcal{P}$ satisfying (4.37). If for any such p, s, any $q \in (1, \infty)$ and any $\mathcal{L} \in \mathscr{E}(\mathbb{R}^n)$,

$$B^{s,\mathcal{L}}_{p,q} = B^{s}_{p,q}, \tag{4.39}$$

then by the lifting properties of the Bessel potentials in the corresponding Besov spaces (see Remark 2.5(iii)), Proposition 2.7 and Remark 2.3(i), we have that for any $\delta \in (0, s)$ small enough (see (4.43) below for the precise requirement) and $f \in L^p(\mathbb{R}^n)$,

$$\left\|(I + \Delta)^{(s-\delta)/2} f\right\|_{B^{\delta}_{p,p}(\mathbb{R}^n)} \simeq \left\|(I + \mathcal{L})^{s/2} f\right\|_{B^{0,\mathcal{L}}_{p,p}} \simeq \left\|(I + \mathcal{L})^{s/2} f\right\|_{\widetilde{B}^{0,\mathcal{L}}_{p,p}}. \tag{4.40}$$

As $F = (\frac{1}{p}, s) \in (0, 1) \times (0, 1) \setminus \mathcal{P}$, we know that $p > 2$, thus by the embedding properties of the spectral Besov and Triebel-Lizorkin spaces (see Remark 2.5(i), (2.14a), (2.16a) and an elementary calculation), we know

$$L^p(\mathbb{R}^n) = \widetilde{F}^{0,\mathcal{L}}_{p,2} \subset \widetilde{F}^{0,\mathcal{L}}_{p,p} = \widetilde{B}^{0,\mathcal{L}}_{p,p},$$

which immediately shows that when $n = 2$

$$\left\|(I+\mathcal{L})^{s/2}f\right\|_{\widetilde{B}^{0,\mathcal{L}}_{p,p}} \lesssim \left\|(I+\mathcal{L})^{s/2}f\right\|_{L^p(\mathbb{R}^2)}. \tag{4.41}$$

On the other hand, since $B^{\delta}_{p,p}(\mathbb{R}^n) = F^{\delta}_{p,p}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ for any $n \in \mathbb{N}$ (see [51, 52] and also Lemma 3.4), we have that for any $f \in L^p(\mathbb{R}^2)$,

$$\left\|(I+\Delta)^{(s-\delta)/2}f\right\|_{L^p(\mathbb{R}^2)} \lesssim \left\|(I+\Delta)^{(s-\delta)/2}f\right\|_{B^{\delta}_{p,p}(\mathbb{R}^2)}. \tag{4.42}$$

Now, using (4.37), we know that there exist σ, $\delta \in (0,\ s)$ sufficiently small such that

$$p = \frac{2}{s-\delta-\sigma} > \frac{2}{s-\delta} > \frac{2}{s}. \tag{4.43}$$

By Lemma 4.5(i) and (ii), there exist $\beta \in (-1,\ \frac{s-\delta}{2-s+\delta} - 1)$, $\mathcal{L}_\beta \in \mathscr{E}(\mathbb{R}^2)$ and $u_\beta \in L^p(\mathbb{R}^2)$ such that

$$\left\|(I+\Delta)^{(s-\delta)/2}u_\beta\right\|_{L^p(\mathbb{R}^2)} = \infty, \tag{4.44}$$

which combined with (4.40) (4.41) and (4.42) implies that

$$\left\|(I+\mathcal{L}_\beta)^{s/2}u_\beta\right\|_{L^p(\mathbb{R}^2)} = \infty.$$

However, by Lemma 4.5(iii), we know that the same function u_β in (4.44) satisfies

$$\left\|(I+\mathcal{L}_\beta)^{s/2}u_\beta\right\|_{L^p(\mathbb{R}^2)} < \infty.$$

This is a contradiction. Thus, the assumption (4.39) doesn't hold true. This finishes the proof of Theorem 1.2.

□

5 Extension to the case $s \geq 1$

In this section, we extend the above considerations from $s \in (0,\ 1)$ to the case $s \geq 1$ by adding higher regularity assumption on the heat kernel. This higher regularity is introduced in Section 5.1, where the associated smooth functional calculus is also established. In Section 5.2, we introduce the function spaces of higher smoothness defined via the higher order difference and then derive the embedding relations between this version of function spaces and the function spaces defined via the heat semigroup.

5.1 Heat kernels with higher order regularity

Let $\mu := [\mu] + \{\mu\} \in [1,\ \infty)$ with $[\mu] \in \mathbb{Z}_+$ and $\{\mu\} \in [0,\ 1)$. For simplicity, *we may assume that* $\{\mu\} \in (0,\ 1)$ *throughout this paper*, that is, $\mu \in (1,\ \infty) \setminus \mathbb{N}$. Assume that the heat kernel $\{P_t\}_{t>0}$ satisfies the following *local higher order regularity estimates* $\mathbf{G}_{\mathrm{loc}}(\mu)$ that

(i) for any $k \in \{1, \ldots, [\mu]\}$ and $\alpha \in \mathbb{Z}_+^n$ satisfying $|\alpha| = k$, there exist positive constants c_4 and α_4 such that for any $t \in (0, 1]$ and $x, y \in \mathbb{R}^n$,

$$\left|D_x^\alpha P_t(x, y)\right| \le \frac{c_4}{t^{(n+k)/2}} \exp\left\{-\frac{\alpha_4|x-y|^2}{t}\right\}; \tag{5.1}$$

(ii) for any $\alpha \in \mathbb{Z}_+^n$ satisfying $|\alpha| = [\mu]$, there exist positive constants c_5 and α_5 such that for any $t \in (0, 1]$ and $x, x', y \in \mathbb{R}^n$ satisfying $|x - x'| \le \sqrt{t}$,

$$\left|D_x^{[\mu]} P_t(x, y) - D_x^{[\mu]} P_t(x', y)\right| \le c_5 \left(\frac{|x-x'|}{\sqrt{t}}\right)^{\{\mu\}} \frac{1}{t^{(n+[\mu])/2}} \exp\left\{-\frac{\alpha_5|x-y|^2}{t}\right\}. \tag{5.2}$$

Remark 5.1. Let $\mathcal{L} = -\mathrm{div}(A\nabla)$ be the second order divergence form elliptic operator on $\mathbb{R}^n$ with real symmetry coefficient entries $\{a_{i,j}\} \subset C^{\mu+\epsilon-1}(\mathbb{R}^n)$ in the Hölder space for any $\epsilon > 0$ (see (5.16) below). Then the property $\mathbf{G}_{\mathrm{loc}}(\mu)$ holds true (see [17, Theorems 1.1 and 1.5 III]).

The property $\mathbf{G}_{\mathrm{loc}}(\mu)$ can be extended to the half complex plane $\mathbb{C}_+ := \{z \in \mathbb{C} : \ \mathrm{Re}\, z > 0\}$ as follows.

Lemma 5.2. *Let $\mu \in (1, \infty) \setminus \mathbb{N}$. Assume that the heat kernel $\{P_t\}_{t>0}$ satisfies $\mathbf{G}_{\mathrm{loc}}(\mu)$. Then*

(i) *for any $k \in \{1, \ldots, [\mu]\}$ and $\alpha \in \mathbb{Z}_+^n$ satisfying $|\alpha| = k$, there exist positive constants c_6 and α_6 such that for any $z \in \mathbb{C}_+$ satisfying $\mathrm{Re}\, z \in (0, 1]$ and $x, y \in \mathbb{R}^n$,*

$$\left|D_x^\alpha P_z(x, y)\right| \le \frac{c_6}{|\mathrm{Re}\, z|^{(n+k)/2}} \exp\left\{-\frac{\alpha_6|x-y|^2}{|z|}\right\}; \tag{5.3}$$

(ii) *for any $\alpha \in \mathbb{Z}_+^n$ satisfying $|\alpha| = [\mu]$, there exist positive constants c_7 and α_7 such that for any $z \in \mathbb{C}_+$ satisfying $\mathrm{Re}\, z \in (0, 1]$ and $x, x', y \in \mathbb{R}^n$ satisfying $|x - x'| \le \sqrt{\mathrm{Re}\, z}$,*

$$\left|D_x^\alpha P_z(x, y) - D_x^\alpha P_z(x', y)\right| \le c_7 \left(\frac{|x-x'|}{\sqrt{|\mathrm{Re}\, z|}}\right)^{\{\mu\}} \frac{1}{|\mathrm{Re}\, z|^{(n+[\mu])/2}} \exp\left\{-\frac{\alpha_7|x-y|^2}{|z|}\right\}. \tag{5.4}$$

Proof. As the semigroup $\{e^{-t\mathcal{L}}\}_{t>0}$ extends to a bounded holomorphic semigroup $\{e^{-z\mathcal{L}}\}_{z\in\mathbb{C}_+}$ on $L^p(\mathbb{R}^n)$, we write for any $z := t + is \in \mathbb{C}_+$ with $t \in (0, 1]$,

$$e^{-z\mathcal{L}} = e^{-(t/2)\mathcal{L}} e^{-(t/2+is)\mathcal{L}}.$$

This immediately implies that for any such z and $x, y \in \mathbb{R}^n$,

$$P_z(x, y) = \int_{\mathbb{R}^n} P_{\frac{t}{2}}(x, u) P_{\frac{t}{2}+is}(u, y)\, du,$$

which combined with (5.1), (5.2) and (1.7) implies that (5.3) and (5.4) hold true. This finishes the proof of Lemma 5.2. □

Based on $\mathbf{G}_{\mathrm{loc}}(\mu)$, we have the following higher order smooth functional calculus, which is an extension of Lemma 3.2.

Proposition 5.3. *Assume that the heat kernel $\{P_t\}_{t>0}$ satisfies $\mathbf{G}_{\rm loc}(\mu)$ with $\mu \in (1, \infty) \setminus \mathbb{N}$. Let $\varphi \in C^{2l+4}([0, \infty))$ with $l > 2n$ and $\varphi^{(2\nu+1)}(0) = 0$ for any $\nu \in \{0, \ldots, l+1\}$.*

(i) *If $\operatorname{supp}\varphi \subset [0, R]$ for some $R \ge 1$, then for any $k \in \{1, \ldots, [\mu]\}$ and $\alpha \in \mathbb{Z}_+^n$ satisfying $|\alpha| = k$, $\delta \in (0, 1]$ and $x, y \in \mathbb{R}^n$,*

$$\left|D_x^\alpha \varphi(\delta\sqrt{\mathcal{L}})(x, y)\right| \le C(l, R) D_{\delta,l,k}(x, y), \tag{5.5}$$

where

$$D_{\delta,l,k}(x, y) := \frac{1}{\delta^{n+k}}\left(1 + \frac{|x-y|}{\delta}\right)^{-l} \tag{5.6}$$

and the positive constant $C(l, R)$ depends only on l and R.

(ii) *If $\operatorname{supp}\varphi \subset [0, R]$ for some $R \ge 1$, then for any $\alpha \in \mathbb{Z}_+^n$ satisfying $|\alpha| = [\mu]$, there exists a positive constant $C'(l, R)$, depending on l and R, such that for any $\delta \in (0, 1]$ and x, x', $y \in \mathbb{R}^n$ satisfying $|x' - x| \le \delta$,*

$$\left|D_x^\alpha \varphi(\delta\sqrt{\mathcal{L}})(x, y) - D_x^\alpha \varphi(\delta\sqrt{\mathcal{L}})(x', y)\right| \le C'(l, R)\left(\frac{|x-x'|}{\delta}\right)^{\{\mu\}} D_{\delta,l,k}(x, y). \tag{5.7}$$

(iii) *If there exist positive constant $C(l) > 0$ and $r \in (n + [\mu] + 2l + 5, \infty)$ such that for any $k \in \{0, \ldots, 2l+4\}$ and $\lambda \in (0, \infty)$,*

$$\left|\varphi^{(k)}(\lambda)\right| \le C(l)\,(1+\lambda)^{-r}, \tag{5.8}$$

then (5.5) and (5.7) still hold true. Moreover, for any $\delta \in (0, 1]$ and $x \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} \varphi(\delta\sqrt{\mathcal{L}})(x, y)\,dy \equiv \varphi(0).$$

To prove Proposition 5.3, we need the following technical lemma from [11, Theorems 3.1 and 3.4] and [35, Theorem 3.4].

Lemma 5.4 ([11, 35])**.** (i) *Let $\varphi \in C^{2l+4}([0, \infty))$ be as in the statement of Proposition 5.3 with $l > 2n$ and $\operatorname{supp}\varphi \subset [0, R]$ for some $R \ge 1$. Then there exist g_0, $g_1 : \mathbb{R} \to \mathbb{C}$ satisfying for $i \in \{0, 1\}$,*

$$\|g_i\|_{*,l} := \int_{\mathbb{R}^n} \left|\widehat{g_i}(\xi)\right| (1 + |\xi|)^l\, d\xi < \infty$$

such that for any $\lambda \in [0, \infty)$,

$$\varphi(\lambda) = g_0(\lambda^2) e^{-\lambda^2} + g_1(\lambda^2) e^{-\lambda^2},$$

where for $i \in \{0, 1\}$, $\widehat{g_i}$ denotes the Fourier transform of g_i as in (1.15). Moreover, there exists a positive constant $C(l)$ such that for $i \in \{0, 1\}$,

$$\|g_i\|_{*,l} \le C(l) R^{2l+n+4}\left(\|\varphi\|_{L^\infty} + \left\|\varphi^{(2l+4)}\right\|_{L^\infty} + \max_{0\le\nu\le 2l+4}\left\{\left|\varphi^{(\nu)}(0)\right|\right\}\right). \tag{5.9}$$

(ii) *Let* $g: \mathbb{R} \to \mathbb{C}$ *satisfy* $\|g\|_{*,l} < \infty$ *with* $l > 2n$. *Then for any* $\delta \in (0, 1]$ *and* $x, y \in \mathbb{R}^n$, *the integral kernel* $g(\delta^2\mathcal{L})e^{-\delta^2\mathcal{L}}(x, y)$ *of the operator* $g(\delta^2\mathcal{L})e^{-\delta^2\mathcal{L}}$ *satisfies*

$$g\left(\delta^2\mathcal{L}\right)e^{-\delta^2\mathcal{L}}(x, y) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{g}(\xi)P_{\delta^2(1-i\xi)}(x, y)\,d\xi.$$

(iii) *Let* $\varphi \in C^{2l+4}([0, \infty))$ *be as in the statement of Proposition 5.3 with* $l > 2n$ *and satisfy* (5.8) *with* $r \in (n + [\mu] + 2l + 5, \infty)$ *for some* $[\mu] > 0$. *Then there exists a family* $\{h_j\}_{j\in\mathbb{Z}_+}$ *of functions on* $[0, \infty)$ *with* h_j *satisfying the assumptions in (i) with* $R \equiv 2$ *such that for any* $\lambda \in [0, \infty)$,

$$\varphi(\lambda) = \sum_{j=0}^{\infty} h_j(2^{-j}\lambda).$$

Moreover, $h_0(0) = f(0)$, $h_j(0) \equiv 0$ *for any* $j \in \mathbb{N}$ *and for any* $j \in \mathbb{Z}_+$,

$$\|h_j\|_{L^\infty} + \left\|h_j^{(2l+4)}\right\|_{L^\infty} + \max_{0\le\nu\le 2l+4}\left\{\left|h_j^{(\nu)}(0)\right|\right\} \lesssim 2^{-j(n+[\mu]+1)}, \tag{5.10}$$

where the implicit constant is independent of j.

Based on Lemma 5.4, we now turn to the proof of Proposition 5.3.

Proof of Proposition 5.3. We first prove (i) and (ii). By Lemma 5.4(i), we know that there exist $g_0, g_1: \mathbb{R} \to \mathbb{C}$ satisfying for $i \in \{0, 1\}$, $\|g_i\|_{*,l} < \infty$ such that for any $\lambda \in [0, \infty)$,

$$\varphi(\lambda) = g_0(\lambda^2)e^{-\lambda^2} + g_1(\lambda^2)e^{-\lambda^2}.$$

Thus, to finish the proofs of (i) and (ii), we only need to verify that for $i \in \{0, 1\}$, the integral kernel $G_i(x, y) := g_i(\delta^2\mathcal{L})e^{-\delta^2\mathcal{L}}(x, y)$ of the operator $g_i(\delta^2\mathcal{L})e^{-\delta^2\mathcal{L}}$ satisfies (5.5) and (5.7).

Using Lemma 5.4(ii), we have for $i \in \{0, 1\}$,

$$G_i(x, y) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{g_i}(\xi)P_{\delta^2-\delta^2\xi i}(x, y)\,d\xi.$$

This combined with Lemma 5.2(i) shows that for any $k \in \{0, \ldots, [\mu]\}$, $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| = k$ and x, $y \in \mathbb{R}^n$,

$$\begin{aligned}\left|D_x^\alpha G_i(x, y)\right| &\le \frac{1}{2\pi}\int_{\mathbb{R}} \left|\widehat{g_i}(\xi)\right|\left|D_x^\alpha P_{\delta^2-\delta^2\xi i}(x, y)\right|\,d\xi \\ &\lesssim \frac{1}{2\pi}\int_{\mathbb{R}} \left|\widehat{g_i}(\xi)\right|\frac{1}{\delta^{k+n}}\exp\left\{-\frac{c|x-y|^2}{\delta^2(1+\xi^2)}\right\}\,d\xi \\ &\lesssim \frac{1}{2\pi}\int_{\mathbb{R}} \left|\widehat{g_i}(\xi)\right|\frac{1}{\delta^{k+n}}(1+|\xi|)^l\left(1+\frac{|x-y|}{\delta}\right)^{-l}\,d\xi \lesssim \frac{\|g\|_{*,l}}{\delta^{k+n}}\left(1+\frac{|x-y|}{\delta}\right)^{-l},\end{aligned} \tag{5.11}$$

which combined with (5.9) implies that $G_i(x, y)$ satisfies (5.5).

On the other hand, using Lemma 5.2(ii), we have that for any $\alpha \in \mathbb{Z}_+^n$ satisfying $|\alpha| = [\mu]$ and x, x', $y \in \mathbb{R}^n$ satisfying $|x' - x| \le \delta$,

$$\left|D_x^\alpha G_i(x, y) - D_x^\alpha G_i(x', y)\right| \le \frac{1}{2\pi}\int_{\mathbb{R}} \left|\widehat{g_i}(\xi)\right|\left|D_x^\alpha P_{\delta^2-\delta^2\xi i}(x, y) - D_x^\alpha P_{\delta^2-\delta^2\xi i}(x', y)\right|\,d\xi \tag{5.12}$$

$$\lesssim \frac{1}{2\pi}\int_{\mathbb{R}}\left|\widehat{g_i}(\xi)\right|\left(\frac{|x-x'|}{\delta}\right)^{\{\mu\}}\frac{1}{\delta^{[\mu]+n}}\exp\left\{-\frac{c|x-y|^2}{\delta^2(1+\xi^2)}\right\}d\xi$$
$$\lesssim \frac{\|g\|_{*,l}}{\delta^{k+n}}\left(\frac{|x-x'|}{\delta}\right)^{\{\mu\}}\left(1+\frac{|x-y|}{\delta}\right)^{-l},$$

which combined with (5.9) implies that $G_i(x, y)$ satisfies (5.7). Combined (5.11) and (5.12), we finish the proofs of (i) and (ii).

To prove (iii), by Lemma 5.4(iii), we know that for any $\delta \in (0, 1]$, $p \in (1, \infty)$ and $g \in L^p(\mathbb{R}^n)$,

$$\varphi(\delta\sqrt{\mathcal{L}})g = \sum_{j=0}^{\infty} h_j(2^{-j}\delta\sqrt{\mathcal{L}})g \tag{5.13}$$

in $L^p(\mathbb{R}^n)$. This together with (5.5), (5.6), (5.9) and (5.10) implies the integral kernel $\varphi(\delta\sqrt{\mathcal{L}})(\cdot,\cdot)$ of $\varphi(\delta\sqrt{\mathcal{L}})$ satisfies that for any $k \in \{1, \dots, [\mu]\}$, $\alpha \in \mathbb{Z}_+^n$ satisfying $|\alpha| = k$ and $x, y \in \mathbb{R}^n$,

$$\left|D_x^\alpha\varphi(\delta\sqrt{\mathcal{L}})(x, y)\right| \lesssim \sum_{j=0}^{\infty}\left|D_x^\alpha h_j(\delta 2^{-j}\sqrt{\mathcal{L}})(x, y)\right|$$
$$\lesssim \sum_{j=0}^{\infty}\frac{2^{-j}}{\delta^{n+k}}\left(1+\frac{|x-y|}{2^{-j}\delta}\right)^{-l} \lesssim D_{\delta,l,k}(x, y),$$

which implies that (5.5) holds true in this case. The proof of (5.7) is similar, the details being omitted.

Finally, by (5.13), Lemma 3.2(iii) and the facts $h_0(0) = \varphi(0)$, $h_j(0) \equiv 0$ for any $j \in \mathbb{N}$, we conclude that

$$\int_{\mathbb{R}^n}\varphi(\delta\sqrt{\mathcal{L}})(x, y)\,dy = \sum_{j=0}^{\infty}\int_{\mathbb{R}^n}h_j(\delta 2^{-j}\sqrt{\mathcal{L}})(x, y)\,dy = \int_{\mathbb{R}^n}h_0(\delta\sqrt{\mathcal{L}})(x, y)\,dy = h_0(0) = \varphi(0),$$

which shows that (iii) holds true and hence completes the proof of Proposition 5.3. □

5.2 Function spaces with higher order smoothness

We first recall the definitions of function spaces defined via the higher order difference. Let $M \in \mathbb{N}$, $h \in \mathbb{R}^n$ and f be a function on $\mathbb{R}^n$. Recall the following definition of the *M-order difference* $\Delta_h^M(f)$ *of* f *with step* h by setting for any $x \in \mathbb{R}^n$,

$$\Delta_h^M f(x) := \sum_{j=0}^{M}(-1)^{M-j}f(x+jh). \tag{5.14}$$

Note that if $M = 1$, then $\Delta_h^1 f(x) = \Delta_h f(x) = f(x+h) - f(x)$ is the usual first order difference, and for any $M \geq 2$, $\Delta_h^M f(x) = \Delta_h^{M-1}(\Delta_h^1 f)(x)$.

Let $p, q \in (1, \infty)$ and $s \in (0, \infty)$, define the *Lipschitz Besov space* $B_{p,q}^{s,D}$ by

$$B_{p,q}^{s,D} := \left\{f \in L^p(\mathbb{R}^n) :\ \|f\|_{B_{p,q}^{s,D}} := \|f\|_{L^p(\mathbb{R}^n)} + \|f\|_{\dot{B}_{p,q}^{s,D}} < \infty\right\}, \tag{5.15}$$

where

$$\|f\|_{\dot{B}^{s,D}_{p,q}} := \left\{ \int_0^\infty \left[\int_{\mathbb{R}^n} \int_{B(x,r)} \frac{\left|\Delta_h^M f(x)\right|^p}{r^{n+sp}} \, dh \, dx \right]^{q/p} \frac{dr}{r} \right\}^{1/q},$$

where $\Delta_h^M f(x)$ is defined as in (5.14) with $M > s$.

Similarly, the *Lipschitz Triebel-Lizorkin space* $F^{s,D}_{p,q}$ with $p, q \in (1, \infty)$ and $s \in (0, \infty)$ is defined to be

$$F^{s,D}_{p,q} := \left\{ f \in L^p(\mathbb{R}^n) : \|f\|_{F^{s,D}_{p,q}} := \|f\|_{L^p(\mathbb{R}^n)} + \|f\|_{\dot{F}^{s,D}_{p,q}} < \infty \right\},$$

where

$$\|f\|_{\dot{F}^{s,D}_{p,q}} := \left\| \left[\int_0^\infty \left(\int_{B(\cdot,r)} \frac{\left|\Delta_h^M f(\cdot)\right|}{r^{n+s}} \, dh \right)^q \frac{dr}{r} \right]^{1/q} \right\|_{L^p(\mathbb{R}^n)},$$

where $\Delta_h^M f(x)$ is defined as in (5.14) with $M > s$.

Remark 5.5. From [52, Chapter 3.5.3], we deduce that for any $p, q \in (1, \infty)$, $s \in (0, \infty)$ and $M \in \mathbb{N}$ satisfying $M > s$, the spaces $B^{s,D}_{p,q}$ and $F^{s,D}_{p,q}$ are consistent with the classical Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ and Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$. In particular, $B^{s,D}_{p,q}$ and $F^{s,D}_{p,q}$ are invariant for different choices of $M \in \mathbb{N}$ satisfying $M > s$. Thus, we don't write the parameter M in the notation of $B^{s,D}_{p,q}$ and $F^{s,D}_{p,q}$.

The following result gives the embedding relation between the spectral and Lipschitz Triebel-Lizorkin spaces with higher order smoothness.

Theorem 5.6. *Assume that the heat kernel $\{P_t\}_{t>0}$ satisfies the property* $\mathbf{G}_{\rm loc}(\mu)$ *for some $\mu \in (1, \infty) \setminus \mathbb{N}$. Then for any $p, q \in (1, \infty)$ and $s \in (0, \mu)$,*

$$F^{s,\mathcal{L}}_{p,q} \subset F^{s,D}_{p,q}.$$

We prove Theorem 5.6 by making use of the non-smooth atomic characterization of the classical Triebel-Lizorkin spaces from [53, Chapter 1.5.2]. Recall that for any $j \in \mathbb{Z}_+$, $\mathcal{D}_j$ denotes the set of all dyadic cubes in $\mathbb{R}^n$ with side length 2^{-j}.

Definition 5.7. (a) Let $\alpha \in (0, \infty) \setminus \mathbb{N}$. A function $a : \mathbb{R}^n \to \mathbb{C}$ is called a 1_α*-atom associated with the dyadic cube* $Q \in \mathcal{D}_0$ if there exists a positive constant $c > 1$ such that

(i) $\operatorname{supp} a \subset cQ$;

(ii) $\|a\|_{C^\alpha(\mathbb{R}^n)} \le 1$, where for $\alpha := [\alpha] + \{\alpha\} \in (0, \infty)$ with $[\alpha] \in \mathbb{Z}_+$ and $\{\alpha\} \in (0, 1)$,

$$\|a\|_{C^\alpha(\mathbb{R}^n)} := \sup_{0 \le |\beta| \le [\alpha]} \left\| D^\beta a \right\|_{L^\infty(\mathbb{R}^n)} + \sum_{|\beta| = [\sigma]} \sup_{x \ne y} \frac{|D^\beta a(x) - D^\beta a(y)|}{|x-y|^{\{\alpha\}}}. \tag{5.16}$$

(b) Let $s \in (0, \infty)$, $\alpha \in (0, \infty) \setminus \mathbb{N}$ and $p \in (1, \infty)$. A function $a : \mathbb{R}^n \to \mathbb{C}$ is called a $(s, p)_\alpha$-*atom associated with the dyadic cube* $Q \in \mathcal{D}_j$ for some $j \in \mathbb{N}$ if there exists a positive constant $c > 1$ such that

(i) $\operatorname{supp} a \subset cQ$;

(ii) $\|a(2^{-j}\cdot)\|_{C^\alpha(\mathbb{R}^n)} \le 2^{-j(s-n/p)}$, where the norm $\|\cdot\|_{C^\alpha(\mathbb{R}^n)}$ is as in (5.16);

(iii) $\int_{\mathbb{R}^n} a(x)\,dx = 0$.

(c) For any $p, q \in (1, \infty)$ and $\vec{\lambda} := \left\{\lambda_{j,m} \in \mathbb{C} : j \in \mathbb{Z}_+, m \in \mathbb{Z}^n\right\}$, let $f_{p,q} := \{\vec{\lambda} : \|\vec{\lambda}\|_{f_{p,q}} < \infty\}$ with

$$\|\vec{\lambda}\|_{f_{p,q}} := \left\|\left(\sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\left|\lambda_{j,m}2^{jn/p}\mathbf{1}_{Q_{j,m}}(\cdot)\right|^q\right)^{1/q}\right\|_{L^p(\mathbb{R}^n)}.$$

The following non-smooth atomic characterization of $F^s_{p,q}(\mathbb{R}^n)$ was established by Triebel (see, for example, [53, Corollary 1.23]).

Lemma 5.8 ([53]). *Let $p, q \in (1, \infty)$, $s \in (0, \infty)$ and $\alpha \in (s, \infty) \setminus \mathbb{N}$. Then for any $f \in L^p(\mathbb{R}^n)$, $f \in F^s_{p,q}(\mathbb{R}^n)$ if and only if it can be represented as*

$$f = \sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\lambda_{j,m}a_{j,m} \tag{5.17}$$

in sense of Schwartz distribution $\mathcal{S}'(\mathbb{R}^n)$, where $a_{0,m}$ are 1_α-atoms, $a_{j,m}$ for $j \ge 1$ are $(s, p)_\alpha$-atoms and $\vec{\lambda} := \{\lambda_{j,m}\} \in f_{p,q}$. Furthermore,

$$\|f\|_{F^s_{p,q}(\mathbb{R}^n)} \simeq \inf\left\{|f\|_{f_{p,q}}\right\},$$

where the infimum is taken over all admissible representation (5.17) *and the implicit constants are independent of f.*

We also need the following discrete Calderón reproducing formula from [32, Lemmas 3.6 and 3.7], which is a generalization of (2.17).

Lemma 5.9 ([32]). *Let (ϕ_0, ϕ) in $\mathcal{S}([0, \infty))$ satisfy the admission condition as in Remark 2.6. Then there exist another couple (ψ_0, ψ) of functions in $\mathcal{S}([0, \infty))$ satisfying the admission condition such that for any $p \in (1, \infty)$ and $f \in L^p(\mathbb{R}^n)$,*

$$f = \sum_{j=0}^{\infty}\phi_j(\sqrt{\mathcal{L}})\psi_j(\sqrt{\mathcal{L}})f$$

in $L^p(\mathbb{R}^n)$.

With the helps of Lemmas 5.8 and 5.9, we now turn to the proof of Theorem 5.6.

Proof of Theorem 5.6. Let (ϕ_0, ϕ) in $\mathcal{S}([0, \infty))$ satisfy the admission condition and (2.19) (see Remark 2.6 for the existence of such couple of functions). Then by Lemma 5.9, we know that there exist (ψ_0, ψ) in $\mathcal{S}([0, \infty))$ such that for any $f \in L^p(\mathbb{R}^n)$,

$$f = \sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \left(2^{-2j}\mathcal{L}\right)^M \phi_j(\sqrt{\mathcal{L}}) \left(\mathbf{1}_{Q_{j,m}} \left(2^{-2j}\mathcal{L}\right)^{-M} \psi_j(\sqrt{\mathcal{L}}) f\right) \tag{5.18}$$
$$= \sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \int_{Q_{j,m}} \left(2^{-2j}\mathcal{L}\right)^M \phi_j(\sqrt{\mathcal{L}})(\cdot, y) \left(\psi_{j,M}(\sqrt{\mathcal{L}}) f\right)(y)\, dy,$$

where for any $j \in \mathbb{Z}_+$, $m \in \mathbb{Z}^n$, $Q_{j,m}$ denotes the dyadic cube in $\mathcal{D}_j$ and for any $j \in \mathbb{Z}_+$, $M \in \mathbb{N}$,

$$\psi_{j,M}(\sqrt{\mathcal{L}}) := \left(\left(2^{-2j}\mathcal{L}\right)^{-M} \psi_j(\sqrt{\mathcal{L}})\right)$$

with the modification $M = 0$ when $j = 0$. Let us emphasize that the equalities in (5.18) hold in $L^p(\mathbb{R}^n)$.

Now, for any $j \in \mathbb{Z}_+$ and $m \in \mathbb{Z}^n$, let

$$\lambda_{j,m} := c2^{j(s-n/p)} \sup_{y\in Q_{j,m}} \left|\left(\psi_{j,M}(\sqrt{\mathcal{L}}) f\right)(y)\right| \tag{5.19}$$

and

$$a_{j,m} := \frac{1}{\lambda_{j,m}} \int_{Q_{j,m}} \left(2^{-2j}\mathcal{L}\right)^M \phi_j(\sqrt{\mathcal{L}})(\cdot, y) \left(\psi_{j,M}(\sqrt{\mathcal{L}}) f\right)(y)\, dy, \tag{5.20}$$

where the positive constant c will be determined later. Thus, by (5.18) through (5.20), we obtain the following decomposition

$$f = \sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{j,m} a_{j,m}$$

in $L^p(\mathbb{R}^n)$ and hence in the sense of $\mathcal{S}'(\mathbb{R}^n)$.

We now show that $a_{j,m}$ are the atoms in Definition 5.7. Indeed, by (2.19), we find that for any $j \in \mathbb{Z}_+$, $m \in \mathbb{Z}^n$, $x \in \mathbb{R}^n$ satisfying $\operatorname{dist}(x, Q_{j,m}) > 2^{-j}$ and any $y \in Q_{j,m}$,

$$\left(2^{-2j}\mathcal{L}\right)^M \phi_j(\sqrt{\mathcal{L}})(x, y) \equiv 0,$$

which together with the fact that $y \in Q_{j,m}$ implies that

$$\operatorname{supp} a_{j,m} \subset 4Q_{j,m}. \tag{5.21}$$

On the other hand, for any $\lambda \in [0, \infty)$, let

$$\widetilde{\phi}(\lambda) := \lambda^{2M}\phi(\lambda).$$

Since $\phi \in \mathcal{S}([0, \infty))$, we know that $\widetilde{\phi}(\lambda) \in \mathcal{S}([0, \infty))$. Moreover, as ϕ_0 and ϕ can be extended to even functions in $\mathcal{S}(\mathbb{R})$ (see Remark 2.6), we further conclude that for any $\nu \in \mathbb{N}$, $\phi_0^{(2\nu+1)}(0) = 0$

and $\widetilde{\phi}^{(2\nu+1)}(0) = 0$. Thus, $\phi_0, \widetilde{\phi} \in \mathcal{S}([0, \infty))$ and satisfy the assumptions of Proposition 5.3(iii). Using (5.20) and Proposition 5.3(iii), we know that for any $j \in \mathbb{N}$,

$$\begin{aligned}\int_{\mathbb{R}^n} a_{j,m}(x)\,dx &= \frac{1}{\lambda_{j,m}} \int_{\mathbb{R}^n} \left[\int_{Q_{j,m}} \widetilde{\phi}_j(\sqrt{\mathcal{L}})(x, y)\psi_{j,M}(\sqrt{\mathcal{L}})(y)\,dy\right] dx \\ &= \frac{1}{\lambda_{j,m}} \int_{Q_{j,m}} \psi_{j,M}(\sqrt{\mathcal{L}})(y) \left[\int_{\mathbb{R}^n} \widetilde{\phi}_j(\sqrt{\mathcal{L}})(x, y)\,dx\right] dy = 0.\end{aligned} \tag{5.22}$$

Moreover, for any $k \in \{0, \ldots, [\mu]\}$ and $\beta \in \mathbb{Z}_+^n$ satisfying $|\beta| = k$, by (5.20) and Proposition 5.3(iii), we know for any $j \in \mathbb{Z}_+$ and $l \in \mathbb{N} \cap (2n, \infty)$,

$$\begin{aligned}\left|D_x^\beta a_{j,m}(x)\right| &\lesssim 2^{-j(s-n/p)} \left(\sup_{y\in Q_{j,m}} \left|\left(\psi_{j,M}(\sqrt{\mathcal{L}})\right)(y)\right|\right)^{-1} \int_{Q_{j,m}} \left|D_x^\beta \left[\widetilde{\phi}_j\left(\sqrt{\mathcal{L}}\right)\right](x, y)\right| \\ &\quad \times \left|\left(\psi_{j,M}(\sqrt{\mathcal{L}})f\right)(y)\right| dy \\ &\lesssim 2^{-j(s-n/p)} \int_{Q_{j,m}} 2^{j(n+k)} \left(1 + 2^j|x - y|\right)^{-l} dy \lesssim 2^{-j(s-n/p)} 2^{jk},\end{aligned}$$

which together with (5.16) implies that for any $j \in \mathbb{Z}_+$, $k \in \{0, \ldots, [\mu]\}$ and $\beta \in \mathbb{Z}_+^n$ satisfying $|\beta| = k$,

$$\left\|D^\beta a_{j,m}(2^{-j}\cdot)\right\|_{L^\infty(\mathbb{R}^n)} \lesssim 2^{-j(s-n/p)}. \tag{5.23}$$

Moreover, using Proposition 5.3 again, we find that for any $\beta \in \mathbb{Z}_+^n$ satisfying $|\beta| = [\mu]$, $l \in \mathbb{N} \cap (2n, \infty)$ and $x, x' \in \operatorname{supp} a_{j,m}$, namely, $|x - x'| \le 4\sqrt{n}2^{-j}$,

$$\begin{aligned}\left|D_x^\beta a_{j,m}(x) - D_x^\beta a_{j,m}(x')\right| &\lesssim 2^{-j(s-n/p)} \left(2^j|x - x'|\right)^{\{\mu\}} \int_{Q_{j,m}} 2^{j(n+[\mu])} \left(1 + 2^j|x - y|\right)^{-l} dy \\ &\lesssim 2^{-j(s-n/p)} 2^{j\mu} |x - x'|^{\{\mu\}},\end{aligned}$$

which together with (5.16) again implies that for any $j \in \mathbb{Z}_+$,

$$\left\|D_x^\beta a_{j,m}(2^{-j}\cdot)\right\|_{C^{\{\mu\}}(\mathbb{R}^n)} \lesssim 2^{-j(s-n/p)}.$$

By this and (5.23), we see for any $j \in \mathbb{Z}_+$,

$$\left\|a_{j,m}(2^{-j}\cdot)\right\|_{C^\mu(\mathbb{R}^n)} \lesssim 2^{-j(s-n/p)},$$

which combined with (5.21) and (5.22) indicates that $a_{j,m}$ is a 1_μ-atom when $j = 0$ and a $(s, p)_\mu$-atom when $j \in \mathbb{N}$, by normalizing the constant c in (5.19) properly.

We now consider the $\|\cdot\|_{f_{p,q}}$ norm generated by the coefficients $\{\lambda_{j,m}\}_{j,m}$. By (5.19), we have that for any $j \in \mathbb{Z}_+$ and $m \in \mathbb{Z}^n$,

$$\lambda_{j,m} 2^{jn/p} \mathbf{1}_{Q_{j,m}} = 2^{js} \sup_{y\in Q_{j,m}} \left|\psi_{j,M}(\sqrt{\mathcal{L}})f(y)\right| \mathbf{1}_{Q_{j,m}}$$

with the modifications that $M = 0$ when $j = 0$. This implies that for any $j \in \mathbb{Z}_+$,

$$\sum_{m\in\mathbb{Z}^n} \left|\lambda_{j,n} 2^{jn/p} \mathbf{1}_{Q_{j,m}}\right|^q \lesssim \sum_{m\in\mathbb{Z}^n} \left(2^{js} \sup_{y\in Q_{j,m}} \left|\psi_{j,M}(\sqrt{\mathcal{L}})f(y)\right| \mathbf{1}_{Q_{j,m}} \right)^q.$$

Note that for any $\phi \in \mathcal{S}([0, \infty))$, there exists $\gamma \in (2n, \infty)$ such that for any $x \in \mathbb{R}^n$,

$$\sup_{y\in Q_{j,m}} \left|\phi(\sqrt{\mathcal{L}})f(y)\right| \mathbf{1}_{Q_{j,m}}(x) \lesssim \sup_{y\in\mathbb{R}^n} \frac{\left|\phi(\sqrt{\mathcal{L}})f(y)\right|}{(1 + 2^j|x-y|)^\gamma},$$

where the later term belongs to the system $\{(\phi_j^* f)_\gamma(x)\}_{j\in\mathbb{Z}_+}$ of *Peetre maximal function* of f associated with ϕ. Then by the Peetre maximal function characterization of the space $\widetilde{F}^{s,\mathcal{L}}_{p,q}$ (see [32, Theorem 3.4]), we conclude that

$$\begin{aligned}\left\|\{\lambda_{j,m}\}_{j,m}\right\|_{f_{p,q}} &\sim \left\| \left(\sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \left|\lambda_{j,m} 2^{(j-n)/p} \mathbf{1}_{Q_{j,m}}\right|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \\ &\lesssim \left\| \left\{ \sum_{j=0}^{\infty} \left[2^{js} (\phi_j^* f)_\gamma \right]^q \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{\widetilde{F}^{s,\mathcal{L}}_{p,q}}.\end{aligned}$$

By this and Proposition 5.8, we conclude that $f \in F^s_{p,q}(\mathbb{R}^n)$ with $\|f\|_{F^s_{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{\widetilde{F}^{s,\mathcal{L}}_{p,q}}$. This combined with Remark 5.5 finishes the proof of Theorem 5.6.

□

Based on Theorem 5.6, we have the following corollary on the embedding relation between the Besov spaces defined via the heat semigroup and the higher order difference, which extends Theorem 1.1 to the case $s \in (0, \mu)$ for $\mu \in (1, \infty) \setminus \mathbb{N}$.

Corollary 5.10. *Assume that the heat kernel $\{P_t\}_{t>0}$ satisfies the property $\mathbf{G}_{\rm loc}(\mu)$ for some $\mu \in (1, \infty) \setminus \mathbb{N}$. Then for any $p, q \in (1, \infty)$ and $s \in (0, \mu)$,*

$$B^{s,\mathcal{L}}_{p,q} \subset B^{s,D}_{p,q},$$

where $B^{s,D}_{p,q}$ is defined as in (5.15).

Proof. By Theorem 5.6 with $p \in (1, \infty)$, $q = 2$ and $s \in (0, \mu)$, we have that

$$\widetilde{F}^{s,\mathcal{L}}_{p,2} \subset F^{s,D}_{p,2},$$

which combined with (2.18) and Remark 5.5 implies that for any $p \in (1, \infty)$ and $s \in (0, \mu)$

$$\mathrm{dom}_p[\mathcal{L}^{s/2}] \subset H^s_p(\mathbb{R}^n).$$

By this, Propositions 2.2, 2.7 and (4.22) implies that $B^{s,\mathcal{L}}_{p,q} \subset B^{s,D}_{p,q}$, which completes the proof of Corollary 5.10.

□

Remark 5.11. (i) Let $s_0 \in (\Theta, \infty)$. If for any $p \in (1, \widetilde{\sigma})$ with some $\widetilde{\sigma} > 1$, the identity

$$\mathrm{dom}_p[\mathcal{L}^{s_0/2}] = H^{s_0}_p(\mathbb{R}^n) \tag{5.24}$$

holds true, then by using the same argument as in the proof of Corollary 5.10 (see also the proof of Theorem 1.1), we conclude that for any $(\frac{1}{p}, s)$ in some Hardy-Littlewood-Sobolev-Kato diagram defined in a way similar to Figure 2 (with 1 and σ therein replaced respectively by s_0 and $\widetilde{\sigma}$) and for any $q \in (1, \infty)$,

$$B^{s,\mathcal{L}}_{p,q} = B^{s,D}_{p,q}.$$

(ii) If the coefficient matrix A satisfies some regularity conditions, then the identity (5.24) is true with s_0 and $\widetilde{\sigma}$ taking different values. For example, if the entries $a_{i,j} \in \mathrm{VMO}(\mathbb{R}^n)$, Shen [47, Theorem C] proved that (5.24) holds with $s_0 = 1$ and $\widetilde{\sigma} > 3$. Moreover, ter Elst et al. [17, Theorems 1.3] proved that if $a_{i,j} \in C^\mu(\mathbb{R}^n)$ for some $\mu \in (0, 1)$, then (5.24) is true with $s_0 = 1$ and $\widetilde{\sigma} = \infty$. See also [17, Theorems 1.5 II] for a further discuss in the case $\mu \geq 1$ and $s_0 \in \mathbb{N}$.

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