On Evans' and Choquet's theorems for polar sets

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1 Introduction and main results

By classical results of G.C. Evans and G. Choquet on "good kernels G in potential theory", for every polar K_σ-set P, there exists a finite measure μ on P such that $G\mu = \infty$ on P, and a set P admits a finite measure μ on P such that $\{G\mu = \infty\} = P$ if and only if P is a polar G_δ-set.

We recall that Evans' theorem yields the solutions of the generalized Dirichlet problem for open sets by the Perron-Wiener-Brelot method using only *harmonic* upper and lower functions (see [7] and Corollary 4.3).

In this paper we intend to show that such results can be obtained, by elementary "metric" considerations and without using any potential theory, for general kernels G locally satisfying

$$G(x,z) \wedge G(y,z) \le CG(x,y).^1$$

The particular case $G(x,y) = |x-y|^{\alpha-N}$ on $\mathbb{R}^N$, $2 < \alpha < N$, solves an open problem (see [16, p. 407, III.1.1]).

ASSUMPTION. *Let X be a locally compact space with countable base and let $G\colon X \times X \to [0,\infty]$ be Borel measurable, $G > 0$, such that the following holds:*

(1) *For every $x \in X$, $\lim_{y\to x} G(x,y) = G(x,x) = \infty$ and $G(x,\cdot)$ is bounded outside every neighborhood of x.*

(2) *G has the local triangle property.*

We recall that G is said to have the *triangle property* if, for some $C > 0$,

$$G(x,z) \wedge G(y,z) \le CG(x,y) \qquad \text{for all } x,y,z \in X, \tag{1.1}$$

and that G has the *local triangle property* if every point in X admits an open neighborhood U such that $G|_{U\times U}$ has the triangle property.

It is well known that G has the triangle property if and only if there exist a metric d on X and $\gamma, C \in (0,\infty)$ such that, for all $x, y \in X$,

$$C^{-1} d(x,y)^{-\gamma} \le G(x,y) \le C d(x,y)^{-\gamma}, \tag{1.2}$$

where, by assumption (1), every such metric yields the initial topology on X.

*Supported by CRC 1283 of the German Research Council (DFG). We thank our colleague A. Grigor'yan for helping us with useful illustrations.

[1] We write $a \wedge b$ for the minimum and $a \vee b$ for the maximum of a and b.

To be more explicit we observe that (1.1) means that $\tilde{G}(x,y) := G(y,x) \le CG(x,y)$ (take $z = x$) and that $\rho := G^{-1} + \tilde{G}^{-1}$ is a *quasi-metric* on X, that is, for some $C > 0$, $\rho(x,y) \le C(\rho(x,z) + \rho(y,z))$ for all $x, y, z \in X$. A metric d satisfying (1.2) is then obtained fixing $\gamma \ge 2\log_2 C$ and defining

$$d(x,y) := \inf\{\sum\nolimits_{0<j<n} \rho(z_j, z_{j+1})^{1/\gamma} \colon n \ge 2,\, z_1 = x, z_n = y,\, z_j \in X\}$$

(see [15, Proposition 14.5]). Conversely, every G satisfying (1.2) has the triangle property, since $d(x,z) \vee d(y,z) \ge d(x,y)/2$. More generally, (1.1) holds if

$$g \circ d_0 \le G \le cg \circ d_0$$

for some metric d_0 for X and a decreasing function g on $[0, \sup\{d_0(x,y) \colon x, y \in X\}]$ satisfying $0 < g(r/2) \le cg(r) < \infty$ for $r > 0$ and $\lim_{r \to 0} g(r) = g(0) = \infty$.

In particular, the local triangle property holds for the classical Green function not only for domains in $\mathbb{R}^N$, $N \ge 3$, but also on domains X in $\mathbb{R}^2$ such that $\mathbb{R}^2 \setminus X$ is not polar ($\log(2/r) \le 2\log(1/r)$ for $0 < r \le 1/2$) and as well for Green functions associated with very general Lévy processes (see [12]).

For every Borel set B in X, let $\mathcal{M}(B)$ denote the set of all *finite* positive Radon measures μ on X such that $\mu(X \setminus B) = 0$, and let $\mathcal{M}_\eta(B)$, $\eta > 0$, be the set of all measures $\mu \in \mathcal{M}(B)$ such that the total mass $\|\mu\| := \mu(B)$ is at most η. Let

$$G\mu(x) := \int G(x,y)\, d\mu(y), \qquad \mu \in \mathcal{M}(X),\ x \in X.$$

By assumption (1), for every $\mu \in \mathcal{M}(X)$,

$$G\mu < \infty \quad \text{on } X \setminus \operatorname{supp}(\mu). \tag{1.3}$$

For every $A \subset X$, let

$$c^*(A) := \inf\{\|\mu\| \colon \mu \in \mathcal{M}(X),\ G\mu \ge 1 \text{ on } A\}.$$

We observe that $c^*(A) = 0$ if and only if there exists $\mu \in \mathcal{M}(X)$ such that $G\mu = \infty$ on A. Indeed, if there are $\mu_n \in \mathcal{M}(X)$, $n \in \mathbb{N}$, such that $\|\mu_n\| < 2^{-n}$ and $G\mu_n \ge 1$ on A, then obviously $\mu := \sum_{n \in \mathbb{N}} \mu_n \in \mathcal{M}_1(X)$ and $G\mu = \infty$ on A. Conversely, if $\mu \in \mathcal{M}(X)$ with $G\mu = \infty$ on A, then $\nu_n := (1/n)\mu$ satisfies $G\nu_n = \infty \ge 1$ on A and $\lim_{n \to \infty} \|\nu_n\| = 0$, hence $c^*(A) = 0$.

As already indicated, the main results of this paper are the next two theorems obtained by G.C. Evans [9] in the classical case (where $c^*(P)$ is the outer capacity of P, and P is polar if and only if $c^*(P) = 0$, cf. [1, Corollary 5.5.7]) and G. Choquet for "good kernels in potential theory" [4, 5].

THEOREM 1.1. *Let P be an F_σ-set in X, $P = \bigcup_{m \in \mathbb{N}} A_m$ with closed sets A_m. Then $c^*(P) = 0$ if and only if there is a measure $\mu \in \mathcal{M}(P)$ with $G\mu = \infty$ on P.*

COROLLARY 1.2. *Let P be an F_σ-set in X with $c^*(P) = 0$. Let $P_0 \subset P$ be countable and A_m, $m \in \mathbb{N}$, be closed sets such that $\bigcup_{m \in \mathbb{N}} A_m = P$ and every intersection $P_0 \cap A_m$ is dense in A_m.*

Then there is a measure $\mu \in \mathcal{M}(P_0)$ such that $G\mu = \infty$ on P.

THEOREM 1.3. *Let $P \subset X$ and let P_0 be a countable dense set in P. The following are equivalent:*

(i) *P is a G_δ-set and $c^*(P) = 0$.*

(ii) *There exists $\mu \in \mathcal{M}(P)$ such that $\{G\mu = \infty\} = P$.*

(iii) *There exists $\mu \in \mathcal{M}(P_0)$ such that $\{G\mu = \infty\} = P$.*

REMARK 1.4. Let us note that J. Deny [8] had made a step in the direction of Choquet's result in proving that, for every G_δ-set P in $\mathbb{R}^N$ which is polar (with respect to classical potential theory), there exists a measure μ on $\mathbb{R}^N$ such that $\{G\mu = \infty\} = P$.

2 Case, where G has the triangle property on X

Let us consider first the case, where G has the triangle property on X, and therefore $C^{-1}d^{-\gamma} \le G \le Cd^{-\gamma}$ for some metric d for X and $\gamma, C \in (0,\infty)$. Defining $\tilde{G} := d^{-\gamma}$ we then have $C^{-1}\tilde{G}\mu \le G\mu \le C\tilde{G}\mu$, and hence $\{\tilde{G}\mu = \infty\} = \{G\mu = \infty\}$ for every $\mu \in \mathcal{M}(X)$. So the implications of Theorems 1.1 and Theorem 1.3 follow immediately if they hold for $\tilde{G}$. Thus we may and shall assume in this section without loss of generality that

$$G(x,y) = d(x,y)^{-\gamma}, \qquad x, y \in X.$$

Then, for every $\mu \in \mathcal{M}(X)$, the "potential" $G\mu$ is lower semicontinuous on X and continuous outside the support of μ. Moreover, if A and B are Borel sets in X such that $d(A,B) := \inf\{d(x,y) \colon x \in A, y \in B\} > 0$, then

$$G\mu(x) \le d(A,B)^{-\gamma}\|\mu\| \qquad \text{for all } x \in A \text{ and } \mu \in \mathcal{M}(B).$$

The key for Theorems 1.1 and 1.3 in our setting will be Lemmas 2.1, 2.5 and 2.4.

2.1 Proof of Theorem 1.1

LEMMA 2.1. *Let $\emptyset \ne A \subset X$ be closed, let $A_0 \subset A$ be a Borel set which is dense in A, and let $\mu \in \mathcal{M}(X)$.*

(a) *If $\mu(A) = 0$, there exists $\nu \in \mathcal{M}(A_0)$ such that*

$$\|\nu\| = \|\mu\| \quad \text{and} \quad G\nu \ge 3^{-\gamma}G\mu \text{ on } A. \tag{2.1}$$

(b) *There exists $\nu \in \mathcal{M}(A)$ such that (2.1) holds.*

Proof. (a) Assuming $\mu(A) = 0$, we consider "shells" $S(A,r)$ around A defined by

$$S(A,r) := \{y \in X \colon 3r \le d(A,\{y\}) < 4r\}, \qquad r > 0.$$

Since $X \setminus A$ is obviously covered by the "shells" $S(A,(4/3)^k)$, $k \in \mathbb{Z}$, we may suppose without loss of generality that μ is supported by some $S(A,r)$, $r > 0$.

For $x \in X$ and $R > 0$, let $B(x,R) := \{y \in X : d(y,x) < R\}$. There is a sequence (x_n) in A_0 such that A is covered by the open balls $B(x_n, r)$, $n \in \mathbb{N}$, and hence $S(A,r)$ is the union of the sets

$$A_n := S(A,r) \cap B(x_n, 5r).$$

So there exist $\mu_n \in \mathcal{M}(A_n)$, $n \in \mathbb{N}$, such that $\sum_{n\in\mathbb{N}} \mu_n = \mu$.

For the moment, let us fix $n \in \mathbb{N}$ and $x \in A$. If $y \in A_n$, then $d(y,x_n) < 5r$ and $3r \le d(x,y)$, hence $d(x,x_n) \le d(x,y) + d(y,x_n) < 3d(x,y)$.

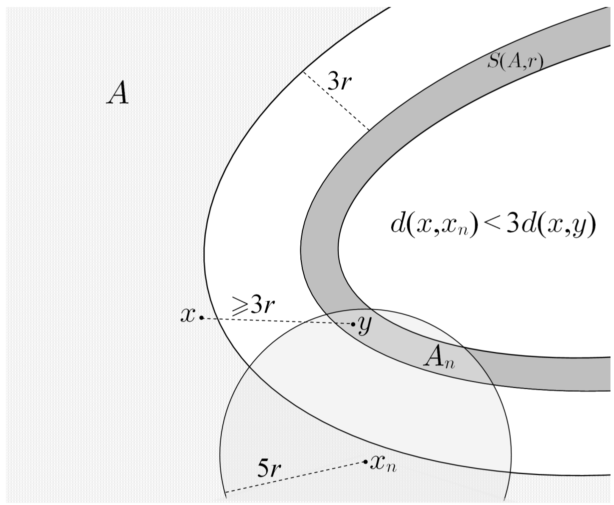

Figure 1. "Sweeping" of μ_n from A_n to x_n

So $d(x,x_n)^{-\gamma} > 3^{-\gamma} d(x,y)^{-\gamma}$. Integrating with respect to $\mu_n \in \mathcal{M}(A_n)$ we see that

$$\|\mu_n\| G(x,x_n) \ge 3^{-\gamma} G\mu_n(x).$$

Clearly, $\nu := \sum_{n\in\mathbb{N}} \|\mu_n\|\, \delta_{x_n} \in \mathcal{M}(A_0)$ (where δ_{x_n} denotes Dirac measure at x_n) and the measure ν satisfies (2.1).

(b) Let $\mu' := 1_{A^c}\mu$ and $\mu'' := 1_A\mu$. By (a), there exists $\nu' \in \mathcal{M}(A)$ such that $\|\nu'\| = \|\mu'\|$ and $G\nu' \ge 3^{-\gamma} G\mu'$ on A. Since $\mu = \mu' + \mu''$, we obtain that $\nu := \nu' + \mu'' \in \mathcal{M}(A)$ and (2.1) holds. □

REMARK 2.2. It is easily seen that, for every $\varepsilon > 0$, we can get (a) in Lemma 2.1 with $(2+\varepsilon)^{-\gamma}$ in place of $3^{-\gamma}$ replacing $3r$ and $4r$ in the definition of $S(A,r)$ by Mr and $(M+1)r$, M sufficiently large.

Proof of Theorem 1.1. Let $\mu \in \mathcal{M}_1(X)$, $G\mu = \infty$ on P. By Lemma 2.1, there are $\nu_m \in \mathcal{M}_1(A_m)$ with $G\nu_m = \infty$ on A_m, $m \in \mathbb{N}$. Then $\nu := \sum_{m\in\mathbb{N}} 2^{-m}\nu_m \in \mathcal{M}_1(P)$ and $G\nu = \infty$ on P. □

We might note that until now we did not use local compactness of X.

2.2 Proof of Corollary 1.2

We shall need a lemma which follows from the weak*-lower semicontinuity of the mappings $\nu \mapsto G\nu(x) = \sup_{m\in\mathbb{N}} (G \wedge m)\nu(x)$, $x \in X$, and the lower semicontinuity of the functions $G\nu$ (see [3, p. 26] or [5, Lemme 1]).

LEMMA 2.3. *Let K, L be compacts in X, $L \subset K$. Let $\mu \in \mathcal{M}(K)$ and φ be a continuous function on L such that $G\mu > \varphi$ on L. Then there exists a weak*-neighborhood $\mathcal{N}$ of μ in $\mathcal{M}(K)$ such that, for every $\nu \in \mathcal{N}$, $G\nu > \varphi$ on L.*

Given a compact $K \neq \emptyset$ in X, a measure $\mu \in \mathcal{M}(K)$ and a dense set A_0 in K, we construct *approximating measures*

$$\mu^{(n)} = \sum\nolimits_{1 \le j \le M_n} \alpha_{nj} \delta_{x_{nj}}, \quad \alpha_{nj} \ge 0,\ x_{nj} \in A_0, \quad n \in \mathbb{N}, \tag{2.2}$$

in the following way: Having fixed $n \in \mathbb{N}$, we take $x_1, \dots, x_M \in A_0$ such that the balls $B_j := B(x_j, 1/n)$ cover K, choose $\mu_j \in \mathcal{M}(B_j)$ with $\mu = \sum_{1 \le j \le M} \mu_j$, and define $\mu^{(n)} := \sum_{1 \le j \le M} \|\mu_j\| \delta_{x_j}$. Of course, $\|\mu^{(n)}\| = \|\mu\|$.

Clearly, the sequence $(\mu^{(n)})$ is weak*-convergent to μ, and, for every open neighborhood W of K, the sequence $(G\mu^{(n)})$ converges to $G\mu$ uniformly on W^c, since the functions $y \mapsto d(x,y)^{-\gamma}$, $x \in W^c$, are equicontinuous on K.

Proof of Corollary 1.2. We may assume without loss of generality that (A_m) is an increasing sequence of compacts. Given $m \in \mathbb{N}$, there exists $\mu_m \in \mathcal{M}(A_m)$ such that $G\mu_m = \infty$ on A_m, by Theorem 1.1. By Lemma 2.3 and using approximating discrete measures, we obtain $\nu_m \in \mathcal{M}_1(P_0 \cap A_m)$ such that $G\nu_m > 2^m$ on A_m. Then $\nu := \sum_{m \in \mathbb{N}} 2^{-m} \nu_m \in \mathcal{M}_1(P_0)$ and $G\nu = \infty$ on P. □

2.3 Proof of Choquet's theorem

The implication (iii) ⇒ (ii) holds trivially. Since, for every $\mu \in \mathcal{M}(X)$, the function $G\mu$ is lower semicontinuous, and therefore $\{G\mu = \infty\} = \bigcap_{n \in \mathbb{N}} \{G\mu > n\}$ is a G_δ-set, we also know that (ii) implies (i).

Based on Lemma 2.1, the next two lemmas are the additional ingredients for the proof of the implication (i) ⇒ (iii) in our setting. They replace the potential-theoretic Lemme 3 in [5]. The other steps can, more or less, be taken as in [5].

A sequence (U_n) of open sets in X with $\bigcup_{n \in \mathbb{N}} U_n = U$ will be called *exhaustion of U* provided, for every $n \in \mathbb{N}$, the closure $\overline{U}_n$ is a compact subset of U_{n+1}.

LEMMA 2.4. *Let $V \subset X$ be open, $\nu_0 \in \mathcal{M}(V)$, and $M > 0$. There exists $\nu \in \mathcal{M}(V)$ such that $\nu \le \nu_0$, $G\nu < \infty$ on V^c and $G\nu > M$ on $\{G\nu_0 > M+1\} \cap V$.*

Proof. Let us choose exhaustions (V_n) and (W_n) of V and $W := \{G\nu_0 > M+1\}$, respectively. We claim that there are measures $\nu_0 \ge \nu_1 \ge \nu_2 \ge \dots$ such that

$$G\nu_n > M + 2^{-n} \text{ on } W, \qquad G(\nu_{n-1} - \nu_n) < 2^{-n} \text{ on } V_n. \tag{2.3}$$

Let us fix $n \in \mathbb{N}$ and suppose that we have $\nu_{n-1} \le \nu_0$ such that $G\nu_{n-1} > M + 2^{-(n-1)}$ on W (which holds if $n = 1$). Since $V_m \uparrow V$ and therefore $G(1_{V_m}\nu_{n-1}) \uparrow G\nu_{n-1}$ as $m \to \infty$, there exists $m > n$ such that

$$B_n := W_n \cap (V \setminus V_m) \subset W_n \setminus V_{n+1} \quad \text{and} \quad \nu_n := 1_{V \setminus B_n} \nu_{n-1}$$

(see Figure 2) satisfy

$$\nu_0(B_n) < 2^{-n} d(W_n \setminus V_{n+1}, W_{n+1}^c \cup V_n)^\gamma \ \text{ and } \ G\nu_n > M + 2^{-n} \text{ on } \overline{W}_{n+1}. \tag{2.4}$$

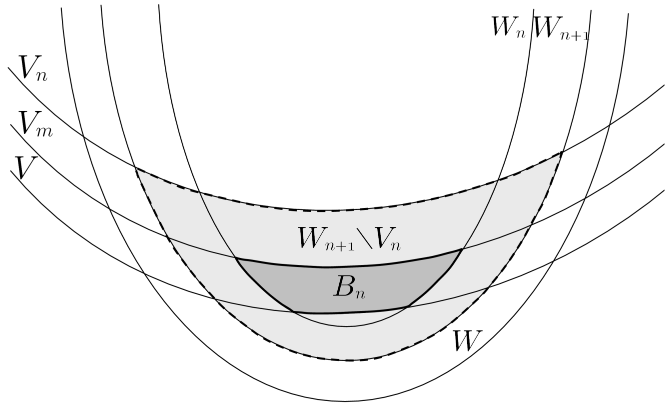

Figure 2. The choice of B_n

Then $G(\nu_{n-1}-\nu_n) \le G(1_{B_n}\nu_0) < 2^{-n}$ on $W_{n+1}^c \cup V_n$. In particular,

$$G\nu_n \ge G\nu_{n-1} - 2^{-n} > M + 2^{-(n-1)} - 2^{-n} = M + 2^{-n} \text{ on } W \setminus W_{n+1}.$$

Having the second inequality in (2.4) we conclude that (2.3) holds.

The sequence (ν_n) is decreasing to a measure ν such that, for every $n \in \mathbb{N}$,

$$G(\nu_n - \nu) = \sum\nolimits_{j=n}^{\infty} G(\nu_j - \nu_{j+1}) < \sum\nolimits_{j=n}^{\infty} 2^{-(j+1)} = 2^{-n} \text{ on } V_n,$$

and hence $G\nu > M$ on $W \cap V_n$, by (2.3). So $G\nu > M$ on $W \cap V$.

Of course, $G\nu < \infty$ on $\overline{V}^c \cup W^c$. By our construction, for every $n \in \mathbb{N}$, the support of ν_n does not intersect $W_n \cap \partial V$, and hence $G\nu \le G\nu_n < \infty$ on $W_n \cap \partial V$. Therefore $G\nu < \infty$ on $W \cap \partial V$, and we finally obtain that $G\nu < \infty$ on V^c. □

LEMMA 2.5. *Let U be a relatively compact open set in X, let P be a subset of U with $c^*(P) = 0$, and let $\varepsilon > 0$. Then there exist an open neighborhood V of P in U and $\mu \in \mathcal{M}_\varepsilon(\overline{P} \cap V)$ such that $G\mu < \infty$ on V^c and $G\mu > 2$ on $\overline{P} \cap V$.*

Proof. Let $\nu_0 \in \mathcal{M}_\varepsilon(X)$ with $G\nu_0 = \infty$ on P. Since $G(1_{U^c}\nu_0) < \infty$ on U, by (1.3), we may assume that ν_0 is supported by U. Of course,

$$P \subset V := \{x \in U : G\nu_0(x) > 9^{\gamma+1} + 1\} \subset U.$$

By Lemma 2.4, there exists $\nu \in \mathcal{M}_\varepsilon(U)$ with $G\nu < \infty$ on U^c and $G\nu > 9^{\gamma+1}$ on V. Let $\nu_1 := 1_V\nu$, $\nu_2 := 1_{U\setminus\overline{V}}\nu$ and $\sigma := 1_{U\cap\partial V}\nu$ so that $\nu = \nu_1 + \nu_2 + \sigma$.

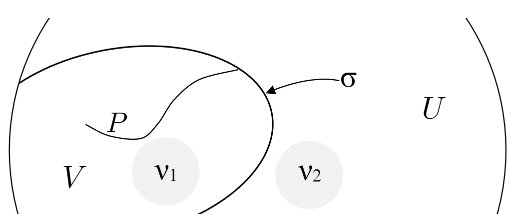

Figure 3. Decomposition of ν

By Lemma 2.1,(a), applied to U, $A := \overline{V} \cap U$, $A_0 := V$, there exists $\tilde{\nu}_2 \in \mathcal{M}(V)$ such that

$$\|\tilde{\nu}_2\| = \|\nu_2\| \quad \text{and} \quad G\tilde{\nu}_2 \ge 3^{-\gamma} G\nu_2 \text{ on } \overline{V} \cap U.$$

Of course, $G\sigma < \infty$ outside the boundary ∂V supporting σ. Further, $G\sigma \le G\nu < \infty$ on U^c and, by definition of V, on $U \setminus V$. So $G\sigma < \infty$ on X. By Lusin's theorem and a version of the continuity principle of Evans-Vasilesco (see [14, pp. 97–98]), there are $\sigma_n \in \mathcal{M}(X)$ such that each $G\sigma_n$ is continuous on X and $\sigma = \sum_{n\in\mathbb{N}} \sigma_n$.

Using (2.2) and Lemma 2.3 with $K = L = \overline{V}$, we get $\tilde{\sigma}_n \in \mathcal{M}(V)$ with

$$\|\tilde{\sigma}_n\| = \|\sigma_n\| \quad \text{and} \quad G\tilde{\sigma}_n > G\sigma_n - 2^{-n} \text{ on } \overline{V}.$$

We define $\tilde{\sigma} = \sum_{n\in\mathbb{N}} \tilde{\sigma}_n$ and $\tilde{\nu} := \nu_1 + \tilde{\nu}_2 + \tilde{\sigma}$. Then

$$\tilde{\nu} \in \mathcal{M}_\varepsilon(V) \quad \text{and} \quad G\tilde{\nu} \ge 3^{-\gamma} G\nu - 1 > 3^{\gamma+2} - 1 > 3^{\gamma+1} \text{ on } V.$$

Applying Lemma 2.1,(b) to V, we get $\mu_0 \in \mathcal{M}(\overline{P} \cap V)$ with $\|\mu_0\| = \|\tilde{\nu}\| \le \varepsilon$ and $G\mu_0 \ge 3^{-\gamma} G\tilde{\nu} > 3$ on $\overline{P} \cap V$. Finally, by Lemma 2.4, we obtain a measure $\mu \le \mu_0$ such that $G\mu < \infty$ on V^c and $G\mu > 2$ on $\overline{P} \cap V$. □

LEMMA 2.6. *Let $P \subset X$ such that $c^*(P) = 0$, U be an open neighborhood of P and $0 < \varepsilon \le 1$. There are an open neighborhood V of P in U and $\mu \in \mathcal{M}_\varepsilon(\overline{P} \cap V)$ such that $G\mu > 2$ on $\overline{P} \cap V$, $G\mu < \infty$ on V^c, and $G\mu < \varepsilon$ on U^c.*

Proof. Let (W_n) be an exhaustion of U. Let $n \in \mathbb{N}$,

$$U_n := W_{n+1} \setminus \overline{W}_{n-1}, \quad P_n := P \cap U_n, \quad \varepsilon_n := 2^{-n}\varepsilon\,(1 \wedge d(U_n, W_{n-2} \cup W_{n+2}^c)^\gamma)$$

(with $W_{-1} = W_0 = \emptyset$). By Lemma 2.5, there exist an open neighborhood V_n of P_n in U_n and $\mu_n \in \mathcal{M}_{\varepsilon_n}(\overline{P} \cap V_n)$ such that $G\mu_n > 2$ on $\overline{P} \cap V_n$ and $G\mu_n < \infty$ on V_n^c.

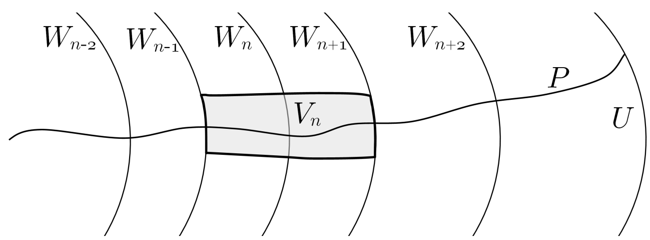

Figure 4. Choice of V_n

By our choice of ε_n, $G\mu_n \le 2^{-n}\varepsilon$ on $W_{n-2} \cup W_{n+2}^c$. It is immediately verified that $\mu := \sum_{n\in\mathbb{N}} \mu_n$ and $V := \bigcup_{n\in\mathbb{N}} V_n$ have the desired properties. □

We may now continue similarly as in [5], but in a slightly simpler way using approximating sequences instead of weak*-neighborhoods.

LEMMA 2.7. *Let $P \subset X$ with $c^*(P) = 0$ and P_0 be a countable, dense set in P. Let U be an open neighborhood of P and $\varepsilon > 0$. There exists $\nu \in \mathcal{M}_\varepsilon(P_0)$ such that*

$$G\nu > 1 \text{ on } P, \quad G\nu < \infty \text{ on } X \setminus P_0 \quad \text{and} \quad G\nu < \varepsilon \text{ on } U^c.$$

Proof (cf. the proof of [5, Lemme 2]). Let $\delta := 1 \wedge (\varepsilon/2)$. By Lemma 2.6, there exist an open neighborhood V of P in U and $\mu \in \mathcal{M}_\delta(\overline{P} \cap V)$ such that

$$G\mu > 2 \text{ on } \overline{P} \cap V, \quad G\mu < \infty \text{ on } V^c \quad \text{and} \quad G\mu < \delta \text{ on } U^c. \tag{2.5}$$

Let (V_k) be an exhaustion of V. For $k \in \mathbb{N}$, we define (taking $V_{-1} = V_0 := \emptyset$)

$$P_k := P_0 \cap (V_k \setminus V_{k-1}) \quad \text{and} \quad W_k := V_{k+1} \setminus \overline{V}_{k-2}$$

so that W_k is an open neighborhood of $\overline{P}_k$, see Figure 5.

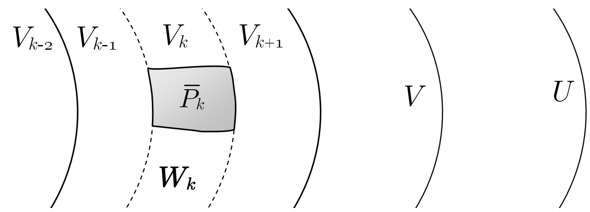

Figure 5. The open neighborhood W_k of $\overline{P}_k$

Clearly, every point in $\overline{P} \cap (V_k \setminus V_{k-1})$ is contained in the closure of $P_{k-1} \cup P_k$. Hence we have $\bigcup_{k\in\mathbb{N}} \overline{P}_k = \overline{P} \cap V$.

We choose $\mu_k \in \mathcal{M}(\overline{P}_k)$ with $\mu = \sum_{k\in\mathbb{N}} \mu_k$ and approximating sequences $(\mu_k^{(n)})_{n\in\mathbb{N}}$ for μ_k in $\mathcal{M}(P_k)$, see (2.2). So, for every $k \in \mathbb{N}$, the sequence $(\mu_k^{(n)})$ is weak*-convergent to μ_k and the sequence $(G\mu_k^{(n)})$ converges to $G\mu_k$ uniformly on W_k^c.

For the moment, we fix $k \in \mathbb{N}$. There exists $l_k \in \mathbb{N}$ such that, for all $n \geq l_k$,

$$|G\mu_k^{(n)} - G\mu_k| < 2^{-k}\delta \quad \text{on } W_k^c. \tag{2.6}$$

Let $\tau_k := \sum_{|m-k|>1} \mu_m$. Then $G\tau_k$ is continuous on $\overline{P}_k$ and, by (2.5),

$$G(\mu_{k-1} + \mu_k + \mu_{k+1}) > 2 - G\tau_k \quad \text{on } \overline{P}_k.^2$$

By Lemma 2.3 (applied with $K := \overline{P}_{k-1} \cup \overline{P}_k \cup \overline{P}_{k+1}$ and $L := \overline{P}_k$), there exists $m_k \geq l_k$ such that, for all $n \geq m_k$,

$$G(\mu_{k-1}^{(n)} + \mu_k^{(n)} + \mu_{k+1}^{(n)}) > 2 - G\tau_k \quad \text{on } \overline{P}_k. \tag{2.7}$$

We now define

$$n_k := m_{k-1} \vee m_k \vee m_{k+1}, \quad \nu_k := \mu_k^{(n_k)} \quad \text{and} \quad \nu := \sum_{k\in\mathbb{N}} \nu_k.$$

Then $\nu \in \mathcal{M}(P_0)$, $\|\nu\| = \|\mu\| \leq \delta$, and $G\nu < \infty$ on $V \setminus P_0$, since ν is supported by a subset of P_0 having no accumulation points in V. By (2.6),

$$G\nu \leq G\mu + \sum_{k\in\mathbb{N}} |G\nu_k - G\mu_k| < G\mu + \delta \quad \text{on } V^c.$$

So, by (2.5), $G\nu < \infty$ on V^c and $G\nu < 2\delta \leq \varepsilon$ on U^c. By (2.7), for every $k \in \mathbb{N}$,

$$G\nu \geq G(\nu_{k-1} + \nu_k + \nu_{k+1} + \tau_k) - \sum_{|m-k|>1} |G\nu_m - G\mu_m| > 2 - \delta \geq 1$$

on $\overline{P}_k$. Thus $G\nu > 1$ on the set $\overline{P} \cap V$ containing P. □

Proof of (i) ⇒ (iii) *in Theorem 1.3.* Let (U_m) be a decreasing sequence of open sets in X such that $P = \bigcap_{m\in\mathbb{N}} U_m$. By Lemma 2.7, there are $\nu_m \in \mathcal{M}(P_0)$, $m \in \mathbb{N}$, such that $\|\nu_m\| \leq 2^{-m}$, $G\nu_m > 1$ on P, $G\nu_m < \infty$ on $X \setminus P_0$ and $G\nu_m < 2^{-m}$ on $X \setminus U_m$. Obviously, $\nu := \sum_{m\in\mathbb{N}} \nu_m \in \mathcal{M}(P_0)$ and $\{G\nu = \infty\} = P$. □

[2] To be formally correct we have to omit, here and below, the term with subscript $k-1$ if $k = 1$.

3 The general case

Assuming that G has the local triangle property there is a locally finite covering of X by relatively compact open sets U_n such that, for each $n \in \mathbb{N}$, the restriction of G on $U_n \times U_n$ has the triangle property.

Let us consider $P \subset X$, a countable dense set P_0 in P, and $\mu \in \mathcal{M}_1(X)$ such that $G\mu = \infty$ on P. For $n \in \mathbb{N}$, we introduce

$$P_n := P \cap U_n \quad \text{and} \quad \mu_n := 1_{U_n}\mu \in \mathcal{M}_1(U_n).$$

By (1.3), $G(1_{U_n^c}\mu) < \infty$ on U_n, and hence

$$G\mu_n = \infty \text{ on } P_n. \tag{3.1}$$

a) If P is an F_σ-set, then every P_n is an F_σ-set, and applying Theorem 1.1 to U_n and $G|_{U_n \times U_n}$, we obtain $\nu_n \in \mathcal{M}_1(P_n)$ such that $G\nu_n = \infty$ on P_n. Then clearly $\nu := \sum_{n\in\mathbb{N}} 2^{-n}\nu_n \in \mathcal{M}_1(P)$ and $G\nu = \infty$ on P completing the proof of Theorem 1.1. A straightforward modification yields Corollary 1.2.

b) Let us next suppose that P is a G_δ-set and let $n \in \mathbb{N}$. Then P_n is a G_δ-set and an application of Theorem 1.3 to U_n and $G|_{U_n\times U_n}$ yields $\nu_n \in \mathcal{M}_1(P_0 \cap U_n)$ such that $\{x \in U_n : G\nu_n(x) = \infty\} = P_n$. By (1.3), $G\nu_n < \infty$ on U_n^c, and therefore

$$\{x \in X \colon G\nu_n(x) = \infty\} = P_n. \tag{3.2}$$

Obviously,

$$\nu := \sum\nolimits_{n\in\mathbb{N}} 2^{-n}\nu_n \in \mathcal{M}_1(P_0) \quad \text{and} \quad G\nu = \infty \text{ on } P.$$

To show that $G\nu < \infty$ outside P, we fix $n \in \mathbb{N}$ and note that the set I_n of all $k \in \mathbb{N}$ such that $U_k \cap U_n \neq \emptyset$ is finite. Defining $\rho_n := \sum_{k\in I_n^c} 2^{-k}\nu_k$ we know that $G\rho_n < \infty$ on U_n, by (1.3). Hence, using (3.2),

$$G\nu = G\rho_n + \sum\nolimits_{k\in I_n} 2^{-n} G\nu_k < \infty \quad \text{on } U_n \setminus P.$$

Thus $\{G\nu = \infty\} = P$.

c) To complete the proof of Theorem 1.3 we suppose that $\{G\mu = \infty\} = P$ and have to show that P is a G_δ-set. Let $n \in \mathbb{N}$. By (3.1), $\{x \in U_n \colon G\mu_n(x) = \infty\} = P_n$. By Theorem 1.3, applied to U_n and $G|_{U_n\times U_n}$, we obtain that P_n is a G_δ-set. So there exist open neighborhoods W_{nm}, $m \in \mathbb{N}$, of P_n in U_n such that $W_{nm} \downarrow P_n$ as $m \to \infty$. Since I_n is finite, we then easily see that $W_m := \bigcup_{n\in\mathbb{N}} W_{nm}$ is decreasing to $\bigcup_{n\in\mathbb{N}} P_n = P$ as $m \to \infty$. Thus P is a G_δ-set, and we are done.

4 Appendix: Application to the PWB-method

In this section we shall recall the solution to the generalized Dirichlet problem for balayage spaces by the Perron-Wiener-Brelot method and how (a generalization of) Evans theorem enables us to use only *harmonic* upper and lower functions.

So let $(X, \mathcal{W})$ be a balayage space (where the assumption in Section 1 may be satisfied or not). Let $\mathcal{B}(X)$, $\mathcal{C}(X)$ denote the set of all Borel measurable numerical functions on X, continuous real functions on X, respectively. Let $\mathcal{P}$ be the set of all *continuous real potentials* for $(X, \mathcal{W})$, that is

$$\mathcal{P} := \{p \in \mathcal{W} \cap \mathcal{C}(X) \colon \exists\, q \in \mathcal{W} \cap \mathcal{C}(X),\, q > 0,\, p/q \to 0 \text{ at infinity}\},$$

see [2, 11] for a thorough treatment and, for example, [13, 10] for an introduction to balayage spaces.

We recall that, for all open sets V in X and $x \in X$, we have positive Radon measures $\varepsilon_x^{V^c}$ on X, supported by V^c and characterized by

$$\int p \, d\varepsilon_x^{V^c} = R_p^{V^c}(x) := \inf\{w(x) \colon w \in \mathcal{W},\, w \geq p \text{ on } V^c\}, \qquad p \in \mathcal{P},$$

so that, obviously, $\varepsilon_x^{V^c} = \delta_x$ if $x \in V^c$. They lead to *harmonic kernels* H_V on X:

$$H_V f(x) := \int f \, d\varepsilon_x^{V^c}, \qquad f \in \mathcal{B}^+(X),\, x \in X.$$

Let us now fix an open set U in X for which we shall consider the generalized Dirichlet problem (see [2, Chapter VII]). Let $\mathcal{V}(U)$ denote the set of all open sets V such that $\overline{V}$ is compact in U, and let ${}^*\mathcal{H}(U)$ be the set of all functions $u \in \mathcal{B}(X)$ which are *hyperharmonic on* U, that is, are lower semicontinuous on U and satisfy

$$-\infty < H_V u(x) \leq u(x) \text{ for all } x \in V \in \mathcal{V}(U).$$

Then $\mathcal{H}(U) := {}^*\mathcal{H}(U) \cap (-{}^*\mathcal{H}(U))$ is the set of functions which are *harmonic on* U,

$$\mathcal{H}(U) = \{h \in \mathcal{B}(X) \colon h|_U \in \mathcal{C}(U),\ H_V h(x) = h(x) \text{ for all } x \in V \in \mathcal{V}(U)\}.$$

A function $f \colon X \to \overline{\mathbb{R}}$ is called *lower $\mathcal{P}$-bounded*, *$\mathcal{P}$-bounded* if there is some $p \in \mathcal{P}$ such that $f \geq -p$, $|f| \leq p$, respectively. For every numerical function f on X, we have the set of all *upper functions*

$$\mathcal{U}_f^U := \{u \in {}^*\mathcal{H}(U) \colon u \geq f \text{ on } U^c,\ u \text{ lower } \mathcal{P}\text{-bounded and l.s.c. on } X\},$$

the set $\mathcal{L}_f^U := -\mathcal{U}_{-f}^U$ of all *lower functions for f with respect to* U, and the definitions

$$\overline{H}_f^U := \inf \mathcal{U}_f, \qquad \underline{H}_f^U := \sup \mathcal{L}_f^U.$$

For every $p \in \mathcal{P}$, there exists $q \in \mathcal{P}$, $q > 0$, such that $p/q \to 0$ at infinity. Hence we may replace $\mathcal{U}_f^U$ by the smaller set of upper functions, which are positive outside a compact in X, without changing the infimum (if $f \geq -p$ consider $f + \varepsilon q$, $\varepsilon > 0$).

To avoid technicalities we state the *resolutivity result* (see [2, VIII.2.12]) only for $\mathcal{P}$-bounded functions:

THEOREM 4.1. *For every $\mathcal{P}$-bounded $f \in \mathcal{B}(X)$,*

$$H_U f = \overline{H}_f^U = \underline{H}_f^U \in \mathcal{H}(U).$$

REMARK 4.2. Let us indicate how the general approach above yields the solution to the generalized Dirichlet problem for harmonic spaces in the way the reader may be more familiar with.

So let us assume for a moment that the harmonic measures $\varepsilon_x^{V^c}$, $x \in V$, for our balayage space are supported by ∂V so that (hyper)harmonicity on U does not depend on values on U^c, and let us identify functions on U with functions on X vanishing outside U.

Let f be a Borel measurable function on ∂U which is $\mathcal{P}$-bounded (amounting to boundedness if U is relatively compact) and let $\tilde{\mathcal{U}}_f^U$ be the set of all functions u on U which are hyperharmonic on U and satisfy

$$\liminf_{x\in U,\, x\to z} u(x) \ge f(z) \quad \text{for every } z \in \partial U. \tag{4.1}$$

If $u \in \mathcal{U}_f^U$, then $\tilde{u} := 1_U u$ is hyperharmonic on U and $\liminf_{x\to z} \tilde{u}(x) \ge u(z) \ge f(z)$ for every $z \in \partial U$, hence $\tilde{u} \in \tilde{\mathcal{U}}_f^U$. If, conversely, $\tilde{u}$ is a function in $\tilde{\mathcal{U}}_f^U$ then, extending it to X by $\liminf_{x\to z} u(z)$ for $z \in \partial U$ and ∞ on $X \setminus \overline{U}$, we get a function $u \in \mathcal{U}_f^U$. Therefore Theorem 4.1 yields that $h\colon x \mapsto \varepsilon_x^{U^c}(f)$, $x \in U$, is harmonic on U and

$$h(x) = \inf \tilde{\mathcal{U}}_f^U(x) = \sup \tilde{\mathcal{L}}_f^U(x) \quad \text{for every } x \in U.$$

Let $\partial_{\mathrm{reg}} U$ denote the set of *regular* boundary points z of U, that is, $z \in \partial U$ such that $\lim_{x\to z} H_U f(x) = f(z)$ for all $\mathcal{P}$-bounded $f \in \mathcal{C}(X)$, and let $\partial_{\mathrm{irr}} U$ be the set of *irregular* boundary points of U, $\partial_{\mathrm{irr}} U := \partial U \setminus \partial_{\mathrm{reg}} U$.

COROLLARY 4.3. *Suppose that there is a lower semicontinuous function $h_0 \ge 0$ on X which is harmonic on U and satisfies $h = \infty$ on $\partial_{\mathrm{irr}} U$. Then*

$$H_U f = \inf \mathcal{U}_f^U \cap \mathcal{H}(U) = \sup \mathcal{L}_f^U \cap \mathcal{H}(U) \quad \textit{for every } \mathcal{P}\textit{-bounded } f \in \mathcal{B}(X).$$

Proof. a) Let g be $\mathcal{P}$-bounded and lower semicontinuous on X. Then there exist $\mathcal{P}$-bounded φ_n in $\mathcal{C}(X)$, $n \in \mathbb{N}$, such that $\varphi_n \uparrow g$. For all $z \in \partial_{\mathrm{reg}} U$ and $n \in \mathbb{N}$,

$$\liminf_{x\to z} H_U g(x) \ge \liminf_{x\to z} H_U \varphi_n(x) = \varphi_n(z),$$

and hence $\liminf_{x\to z} H_U g(x) \ge g(z)$. Clearly,

$$h_n := H_U g + (1/n) h_0 \in \mathcal{H}(U)$$

satisfies $\lim_{x\to z} h_n(x) = \infty$ for all $z \in \partial_{\mathrm{irr}} U$, and h_n is lower semicontinuous on X. Thus $h_n \in \mathcal{U}_f^U \cap \mathcal{H}(U)$.

b) Let $f \in \mathcal{B}(X)$ be $\mathcal{P}$-bounded, $x \in X$. There exists a decreasing sequence (g_n) of $\mathcal{P}$-bounded lower semicontinuous functions on X such that $g_n \ge f$ for every $n \in \mathbb{N}$ and

$$\int f \, d\varepsilon_x^{U^c} = \inf_{n\in\mathbb{N}} \int g_n \, d\varepsilon_x^{U^c},$$

that is, $H_U f(x) = \inf_{n\in\mathbb{N}} H_U g_n(x)$. Hence

$$H_U f(x) = \inf_{n\in\mathbb{N}} (H_U g_n + (1/n) h_0)(x),$$

where $H_U g_n + (1/n) h_0 \in \mathcal{U}_f^U \cap \mathcal{H}(U)$, by (a). Thus $H_U f = \inf \mathcal{U}_f^U \cap \mathcal{H}(U)$.

c) Further, $H_U f = -H_U(-f) = -\inf \mathcal{U}_{-f}^U \cap \mathcal{H}(U) = \sup \mathcal{L}_f^U \cap \mathcal{H}(U)$. □

REMARKS 4.4. 1. For harmonic spaces, the result in Corollary 4.3 has been proven in [7], where the solution to the generalized Dirichlet problem is obtained using *controlled convergence.*

2. In general, the set $\partial_{\mathrm{irr}} U$ is a semipolar F_σ-set. Of course, if $(X, \mathcal{W})$ satisfies Hunt's hypothesis (H), that is, if every semipolar set is polar, then $\partial_{\mathrm{irr}} U$ is polar for every U. Let us note that (H) holds if X is an abelian group such that $\mathcal{W}$ is invariant under translations and $(X, \mathcal{W})$ admits a Green function having the local triangle property (see [14]).

By Theorem 1.1, we obtain that in this situation (which covers the classical case, many translation-invariant second order PDO's as well as Riesz potentials, that is, α-stable processes, and many more general Lévy processes) the assumption of Corollary 4.3 holds.

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