Journal of Optimization Theory and Applications
https://doi.org/10.1007/s10957-019-01546-3

Stochastic Optimal Control Problem with Obstacle Constraints in Sublinear Expectation Framework

Hanwu Li[1,2] · Falei Wang[1]

Received: 3 January 2018 / Accepted: 16 May 2019

Abstract
In this paper, we consider a stochastic optimal control problem, in which the cost function is defined through a reflected backward stochastic differential equation in sublinear expectation framework. Besides, we study the regularity of the value function and establish the dynamic programming principle. Moreover, we prove that the value function is the unique viscosity solution of the related Hamilton–Jacobi–Bellman–Isaac equation.

Keywords Sublinear expectation · Reflected backward stochastic differential equations · Dynamic programming principle

Mathematics Subject Classification 93E20 · 60H10 · 35J60

1 Introduction

The present paper introduces a new class of stochastic optimal control problem with obstacle constraints under model uncertainty, which involves a class of non-dominated probability measures. Specifically, we shall consider a minimum cost problem of an agent, where the cost is defined by the solution of reflected backward stochastic differential equation (BSDE) in sublinear expectation framework.

The sublinear expectation theory, formulated by Peng [1], is a useful tool for the study of model uncertainty, which is also called G-expectation. For example, Epstein

---

Communicated by Hélène Frankowska.

✉ Falei Wang
flwang2011@gmail.com

Hanwu Li
lihanwu11@163.com

1 Shandong University, Jinan, People's Republic of China

2 Bielefeld University, Bielefeld, Germany

Published online: 25 June 2019

and Ji [2,3] applied G-expectation to tackle a recursive utility problem under volatility uncertainty. Actually, the G-expectation can be represented as an upper expectation over a non-dominated set of probability measures.

Under G-expectation framework, a new kind of Brownian motion, called G-Brownian motion, was constructed and the associated stochastic calculus was also established. Moreover, Hu et al. [4,5] studied the well-posedness of BSDE driven by G-Brownian motion (G-BSDE). In a different setting, Soner et al. [6] established the so-called 2BSDEs theory, which shares many similarities with G-BSDEs.

Thanks to the development of G-BSDE theory, Hu and Ji [7] discussed a stochastic recursive optimal control problem under volatility uncertainty, in which the cost function is defined by the solution of G-BSDE. Note that the stochastic control problem in G-framework is essentially an "inf sup problem," which can be seen as a robust optimal control problem. For more results concerning this topic, we refer to [8–10].

It is well known that the solution of reflected BSDE can be regarded as the payoff process of American option; see [11]. The reflection means that the solution is forced to be above a prescribed stochastic process, which is called obstacle. Then, Li et al. [12] study the reflected BSDE in the G-expectation framework, called reflected G-BSDE. For the completeness of the G-stochastic control theory, this paper is devoted to extending the results in [7] to the obstacle constraint case.

The contribution of this paper is threefold: The stochastic optimal control problem with model uncertainty is formulated; the cost function can be required to be bigger than a typical function; the G-stochastic representation for a class of HJBI equations is obtained. Indeed, compared with [13], our problem is essentially an "inf sup problem" involving a family of non-dominated probability measures, which makes it more delicate and challenging. The cost function is measured by the solution of a reflected G-BSDE. Moreover, our result provides a stochastic control approach for the study of a class of HJBI equations with obstacle constraints, which is easier than the game problem.

The paper is organized as follows. In Sect. 2, we introduce the stochastic recursive optimal control problem. We then establish the dynamic programming principle in Sect. 3. In Sect. 4, it is shown that the value function is the unique viscosity solution of the corresponding HJBI equations.

2 Formulation of the Problem

Let $\Omega_T = C_0^d([0, T])$ be the space of all $\mathbb{R}^d$-valued continuous paths $(\omega_t)_{0\leq t\leq T}$ starting from the origin and $B_t(\omega) = \omega_t$ be the canonical mapping equipped with the supremum norm.

2.1 The Probabilistic Setup

Given a fixed monotonic and sublinear function $G : \mathbb{S}(d) \to \mathbb{R}$, where $\mathbb{S}(d)$ denotes the space of all $d \times d$ symmetric matrices. For each $0 \le t \le T$, set

$$L_{ip}(\Omega_t) := \{\varphi(B_{t_1}, \ldots, B_{t_k}) : k \in \mathbb{N}, t_1, \ldots, t_k \in [0, t], \varphi \in C_{b.Lip}(\mathbb{R}^{k\times d})\},$$

where $C_{b.Lip}(\mathbb{R}^{k\times d})$ denotes the space of bounded and Lipschitz functions on $\mathbb{R}^{k\times d}$. Then, Peng [14] established the G-expectation $\hat{\mathbb{E}}[\cdot]$ on $(\Omega_T, L_{ip}(\Omega_T))$. The canonical process B, called G-Brownian motion, has stationary and independent increment.

Theorem 2.1 *([15]) There exists a weakly compact set $\mathcal{P}$ of non-dominated probability measures on $(\Omega_T, \mathcal{B}(\Omega_T))$, such that*

$$\hat{\mathbb{E}}[\xi] = \sup_{P\in\mathcal{P}} E_P[\xi] \textit{ for all } \xi \in L_{ip}(\Omega_T).$$

Based on the above set $\mathcal{P}$, it is natural to introduce the following capacity

$$c(A) := \sup_{P\in\mathcal{P}} P(A),\ A \in \mathcal{B}(\Omega_T).$$

A set $A \in \mathcal{B}(\Omega_T)$ is polar, if $c(A) = 0$. A property holds quasi-surely (q.s.), if it holds outside a polar set. From now on, we do not distinguish between two random variables X and Y, if $X = Y$ q.s.

In the rest of this paper, we shall make use of the following spaces: for each $0 \le t \le s \le T$, $p \ge 1$ and a compact subset U of $\mathbb{R}^m$,

- $L_{ip}(\Omega_s^t) := \{\varphi(B_{t_1}^t, \ldots, B_{t_n}^t) : n \ge 1, t_1, \ldots t_n \in [t, s], \varphi \in C_{b.Lip}(\mathbb{R}^{n\times d})\}$,
- $L_G^p(\Omega_s^t) :=$ {the completion of $L_{ip}(\Omega_s^t)$ under $\|\xi\|_{L_G^p} := \hat{\mathbb{E}}[|\xi|^p]^{\frac{1}{p}}$},
- $M_G^{0,t}(t, T) := \{\eta_s = \sum_{i=0}^{N-1} \xi_i I_{[t_i, t_{i+1})}(s) : t < t_1 < \cdots < T, \xi_i \in L_{ip}(\Omega_{t_i}^t)\}$,
- $M_G^{p,t}(t, T) :=$ {the completion of $M_G^{0,t}(t, T)$ under the norm $\|\cdot\|_{M_G^p}$},
- $H_G^{p,t}(t, T) :=$ {the completion of $M_G^{0,t}(t, T)$ under the norm $\|\cdot\|_{H_G^p}$},
- $S_G^{0,t}(t, T) = \{h(s, B_{t_1\wedge s}, \cdots, B_{t_n\wedge s}) : t_1, \ldots, t_n \in [t, T], h \in C_{b,Lip}(\mathbb{R}^{1+n\times d})\}$,
- $S_G^{p,t}(t, T) :=$ {the completion of $S_G^{0,t}(t, T)$ under the norm $\|\cdot\|_{S_G^p}$},
- $\mathcal{U}^t[t, T] := \{u : u \in M_G^{2,t}(t, T)$ with values in $U\}$,
- $\mathbb{U}[t, T] := \{u = \sum_{i=1}^n I_{A_i} u^i : n \ge 1, u^i \in \mathcal{U}^t[t, T], I_{A_i} \in L_G^2(\Omega_t^0), (A_i)_{i=1}^n$ is a partition of $\Omega_T\}$,

where $B_s^t = B_s - B_t$ and the definition of the above norms can be found in [4]. For convenience, we set $L_G^p(\Omega_s) := L_G^p(\Omega_s^0)$ and $\Xi_G^p(0, T) := \Xi_G^{p,0}(0, T)$, for $\Xi = M, H, S$.

For each $1 \le i, j \le d$, denote by $\langle B^i, B^j\rangle$ the cross-variation process. Then, for two processes $\eta \in M_G^2(0, T)$ and $\xi \in M_G^1(0, T)$, the G-Itô integrals $\int \eta_s \mathrm{d}B_s^i$ and $\int \xi_s \mathrm{d}\langle B^i, B^j\rangle_s$ are well defined; see Peng [14].

2.2 The Problem of Stochastic Control with Obstacle Constraints

Definition 2.1 For each given $t \geq 0$, $u : [t, T] \times \Omega_T \to U$ is said to be an admissible control on $[t, T]$, if $u \in M_G^2(t, T)$. The set of admissible controls on $[t, T]$ is denoted by $\mathcal{U}[t, T]$.

For each $t \geq 0$ and $\xi \in L_G^p(\Omega_t)$ with $p > 2$, assume that the agent can choose an admissible control $u \in \mathcal{U}[t, T]$ to obtain the following G-SDEs:

$$X_s^{t,\xi,u} = \xi + \int_t^s b(r, X_r^{t,\xi,u}, u_r)\mathrm{d}r + \int_t^s \sigma(r, X_r^{t,\xi,u}, u_r)\mathrm{d}B_r, \tag{1}$$

where $b : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$ and $\sigma : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^{n\times d}$ are continuous in t and Lipschitz continuous in (x, u), with Lipschitz constant L.

Then, recalling Chapter V of Peng [14], the G-SDE (1) admits a unique solution $X^{t,\xi,u} \in M_G^2(t, T)$.

Lemma 2.1 *For each $\xi, \xi' \in L_G^p(\Omega_t)$ with $p > 2$, and $\delta \in [0, T - t]$, we have*

(i) $\hat{\mathbb{E}}_t[\sup_{s\in[t,t+\delta]} |X_s^{t,\xi,u} - X_s^{t,\xi',u'}|^p] \leq C_T(|\xi - \xi'|^p + \hat{\mathbb{E}}_t[\int_t^{t+\delta} |u_s - u'_s|^p \mathrm{d}s])$;

(ii) $\hat{\mathbb{E}}_t[\sup_{s\in[t,T]} |X_s^{t,\xi,u}|^p] \leq C_T(1 + |\xi|^p)$;

(iii) $\hat{\mathbb{E}}_t[\sup_{s\in[t,t+\delta]} |X_s^{t,\xi,u} - \xi|^p] \leq C_T(1 + |\xi|^p)\delta^{p/2}$,

where the constant C_T depends on G, L, p, n, U and T.

Example 2.1 Consider a production planning problem of a factory. Assume that the inventory process is described by G-SDE (1) with an initial inventory x and a production plan $u \in M_G^2(t, T)$, which can be chosen by the production management, for example the production rate.

Suppose the cost per unit time is $f(x, u)$. Denote $\Phi(x)$ and a constant $\lambda > 0$ by the cost of retaining the leftover inventory x at time T and the discounted rate, respectively. Then, the expected discounted cost at time t is given by

$$\bar{J}(t, x, u) = \hat{\mathbb{E}}_t\left[e^{-\lambda(T-t)}\Phi(X_T^{t,x,u}) + \int_t^T e^{-\lambda(s-t)} f(X_s^{t,x,u}, u_s)\mathrm{d}s\right],$$

which can be regarded as the $\bar{Y}_t^{t,x,u}$ term of the solution of the following G-BSDE:

$$\begin{aligned}\bar{Y}_s^{t,x,u} &= \Phi(X_T^{t,x,u}) + \int_s^T (f(X_r^{t,x,u}, u_r) - \lambda\bar{Y}_r^{t,x,u})\mathrm{d}r - \int_s^T \bar{Z}_r^{t,x,u}\mathrm{d}B_r \\ &\quad - (\bar{K}_T^{t,x,u} - \bar{K}_s^{t,x,u}).\end{aligned}$$

In [7], the authors study the minimal cost $\bar{V}(t, x)$ over the set of production plans.

However, taking into account the occurrence of unexpected events, such as power cutoff or misoperation, the expected discounted cost function may be required to be bigger than one specific function $l(t,x)$ at time t. In this case, one can measure the expected discounted cost $J(t,x,u)$ by the following G-BSDE with reflection formulated by [12], i.e., $J(t,x,u) := Y_t^{t,x,u}$,

(i) $Y_s^{t,x,u} = \Phi(X_T^{t,x,u}) + \int_s^T (f(X_r^{t,x,u}, u_r) - \lambda Y_r^{t,x,u})\mathrm{d}r - \int_s^T Z_r^{t,x,u}\mathrm{d}B_r + (A_T^{t,x,u} - A_s^{t,x,u})$,
(ii) $Y_s^{t,x,u} \geq l(s, X_s^{t,x,u}),\ \forall t \leq s \leq T$,
(iii) $\{\int_t^s (l(s, X_r^{t,x,u}) - Y_r^{t,x,u})\mathrm{d}A_r^{t,x,u}\}_{s\in[t,T]}$ is a non-increasing G-martingale.

Note that the cost process $Y^{t,x,u}$ satisfies the management constraint at each time; see condition (ii), called obstacle constraints. The above condition (iii) is called "martingale condition" to ensure the minimality of solution $Y^{t,x,u}$.

In this typical case, our stochastic optimal control problem with the obstacle constraints is to find a production plan u to "minimize" the solution $Y_0^{0,x,u}$.

Remark 2.1 In Example 2.1, we use G-expectation to measure the expected cost due to the volatility uncertainty. More precisely, the volatility uncertainty is parametrized by a set of non-dominated probability measures; see Theorem 2.1. In the case without volatility uncertainty, the G-expectation and the reflected G-BSDE reduce to a linear expectation and a standard reflected BSDE, respectively.

Now, we assume that the cost function of the agent is defined by the solution of a G-BSDE with obstacle constraint associated with the controlled G-diffusion process (1). Specifically, for each $t \geq 0$, $u \in \mathcal{U}[t,T]$ and $\xi \in L_G^p(\Omega_t)$ with $p > 2$, the cost function $Y_t^{t,\xi,u}$ satisfies the following equation:

$$\begin{aligned}
&\text{(i) } Y_s^{t,\xi,u} = \Phi(X_T^{t,\xi,u}) + \int_s^T f\big(r, X_r^{t,\xi,u}, Y_r^{t,\xi,u}, Z_r^{t,\xi,u}, u_r\big)\mathrm{d}r \\
&\qquad\qquad - \int_s^T Z_r^{t,\xi,u}\mathrm{d}B_r + (A_T^{t,\xi,u} - A_s^{t,\xi,u}); \\
&\text{(ii) } Y_s^{t,\xi,u} \geq l\big(s, X_s^{t,\xi,u}\big), \quad \forall s \in [t,T]; \\
&\text{(iii)}\Big\{\int_t^s \big(l(s, X_r^{t,x,u}) - Y_r^{t,x,u}\big)\mathrm{d}A_r^{t,x,u}\Big\}_{s\in[t,T]} \text{ is a non-increasing } G\text{-martingale,}
\end{aligned} \tag{2}$$

where $\Phi : \mathbb{R}^n \to \mathbb{R}$, $f : [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times U \to \mathbb{R}$ and $l : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ are deterministic continuous functions satisfying the following conditions:

(A1) f is uniformly Lipschitz in (x,y,z,u), Φ, l are uniformly Lipschitz in x, with Lipschitz constant L,
(A2) there is a constant c such that $l \leq c$ and $l(T,x) \leq \Phi(x)$ for any $x \in \mathbb{R}^n$,
(A2') l belongs to $C_{Lip}^{1,2}([0,T] \times \mathbb{R}^n)$ and $l(T,x) \leq \Phi(x)$ for each $x \in \mathbb{R}^n$. Here, $C_{Lip}^{1,2}([0,T] \times \mathbb{R}^n)$ is the set of all functions of class $C^{1,2}([0,T] \times \mathbb{R}^n)$ whose partial derivatives of order less than or equal to 2 and itself are Lipschtiz continuous functions with respect to x.

It follows from Lemma A.1 that the reflected G-BSDE (2) has a unique solution $(Y^{t,\xi,u}, Z^{t,\xi,u}, A^{t,\xi,u})$ under conditions (A1), (A2) or (A2'). For convenience, for each $(x,u) \in \mathbb{R}^n \times \mathcal{U}[0,T]$, set

$$(X^{x,u}, Y^{x,u}, Z^{x,u}, A^{x,u}) = (X^{0,x,u}, Y^{0,x,u}, Z^{0,x,u}, A^{0,x,u}).$$

Then, the objective of the agent is to find some $u \in \mathcal{U}[0, T]$ so as to minimize the cost function $Y_0^{x,u}$ for each $x \in \mathbb{R}^n$. For this purpose, it is necessary to introduce the definition of essential infimum of $\{Y_t^{t,\xi,u} \mid u \in \mathcal{U}[t, T]\}$ in the G-expectation framework.

Definition 2.2 For each $\xi \in L_G^p(\Omega_t)$ with $p > 2$, the essential infimum of $\{Y_t^{t,\xi,u} : u \in \mathcal{U}[t, T]\}$, denoted by $\operatorname*{ess\,inf}_{u(\cdot)\in\mathcal{U}[t,T]} Y_t^{t,\xi,u}$, is an element $\varsigma \in L_G^2(\Omega_t)$ satisfying:

(i) $\forall u \in \mathcal{U}[t, T]$, $\varsigma \leq Y_t^{t,\xi,u}$ q.s.;
(ii) if η is a random variable satisfying $\eta \leq Y_t^{t,\xi,u}$ q.s. for any $u \in \mathcal{U}[t, T]$, then $\varsigma \geq \eta$ q.s.

Remark 2.2 To our best knowledge, Cohen [16] first introduced the essential supremum in the quasi-surely sense. He showed that, if the family of probability measures satisfies the Hahn property, the essential supremum exists. However, the resulting essential supremum may not be quasi-continuous, which cannot be applied directly to our case.

We can now define our stochastic optimal control problem with the obstacle constraints precisely as the following.

Definition 2.3 For each $t \in [0, T]$ and $x \in \mathbb{R}^n$, the value function is defined as the following:

$$V(t, x) := \operatorname*{ess\,inf}_{u\in\mathcal{U}[t,T]} Y_t^{t,x,u} \quad \text{for } (t, x) \in [0, T] \times \mathbb{R}^n, \tag{3}$$

where $Y^{t,x,u}$ is the solution to Eq. (2)

Remark 2.3 In the linear case, Wu and Yu [13] consider a maximum utility problem, where the utility is defined by the solution of classical reflected BSDEs. We refer the readers to [17–19] for a closest related approach on this research.

3 Regularity of the Value Function

Theorem 3.1 *Assume* (A1), (A2) *or* (A2') *hold. Then, the value function* $V(t, x)$ *is a deterministic function and*

$$V(t, x) = \inf_{u\in\mathcal{U}^t[t,T]} Y_t^{t,x,u} = \operatorname*{ess\,inf}_{u\in\mathbb{U}[t,T]} Y_t^{t,x,u}.$$

In order to prove the above theorem, we need to state some useful lemmas.

Lemma 3.1 *([7]) Let $u \in \mathcal{U}[t,T]$ be given. Then, for each $p \geq 2$, there exists a sequence $(u^k)_{k\geq 1}$ in $\mathbb{U}[t,T]$ such that*

$$\lim_{k\to\infty} \hat{\mathbb{E}}\left[\int_t^T |u_s - u_s^k|^p ds\right] = 0.$$

Lemma 3.2 *Assume that $\xi, \xi' \in L_G^p(\Omega_t;\mathbb{R}^n)$ with $p > 3$ and $u, u' \in \mathcal{U}[t,T]$. Let conditions* (A1)–(A2) *hold. Then, there exists a constant $\hat{C}_1$ depending on T, G, n, c and L such that*

(i) $|Y_t^{t,\xi,u}| \leq \hat{C}_1(1+|\xi|)$,

(ii) $$|Y_t^{t,\xi,u} - Y_t^{t,\xi',u'}|^2 \leq \hat{C}_1\left\{|\hat{\xi}|^2 + \hat{\mathbb{E}}_t\left[\int_t^T |\hat{u}_s|^2 \mathrm{d}s\right] +(1+|\xi|^{\frac{3}{2}}+|\xi'|^{\frac{3}{2}})(|\hat{\xi}|^2 + \hat{\mathbb{E}}_t\left[\int_t^T |\hat{u}_s|^2\mathrm{d}s\right])^{\frac{1}{2}}\right\},$$

where $\hat{u}_s = u_s - u_s'$ and $\hat{\xi} = \xi - \xi'$.

Lemma 3.3 *Assume that $\xi, \xi' \in L_G^p(\Omega_t;\mathbb{R}^n)$ with $p > 8$ and $u, u' \in \mathcal{U}[t,T]$. Let conditions* (A1) *and* (A2') *hold. Then, there exists a constant $\bar{C}_1$ depending on T, G, n and L such that*

(i) $|Y_t^{t,\xi,u}| \leq \bar{C}_1(1+|\xi|^3)$,

(ii) $$|Y_t^{t,\xi,u} - Y_t^{t,\xi',u'}|^2 \leq \bar{C}_1(1+|\xi|^4+|\xi'|^4)\left(|\hat{\xi}|^2 + \hat{\mathbb{E}}_t\left[\int_t^T |\hat{u}_s|^4\mathrm{d}s\right]^{\frac{1}{2}}\right),$$

where $\hat{u}_s = u_s - u_s'$ and $\hat{\xi} = \xi - \xi'$.

The proof of Lemmas 3.2 and 3.3 will be stated in "Appendix B." Now, we are ready to give the proof of Theorem 3.1.

Proof It suffices to prove the case where (A1)–(A2) hold, since the other case can be proved in a similar way. Note that for each $u \in \mathcal{U}^t[t,T]$, the G-diffusion process $X^{t,x,u} \in M_G^{2,t}(t,T)$. Then, it follows from Lemma A.1 that $Y^{t,x,u} \in S_G^{2,t}(t,T)$ (see also [12]). In particular, $Y_t^{t,x,u} \in L_G^2(\Omega_t^t)$ is a constant.

From Lemma 3.1, we could find a sequence $u^k = \sum_{i=1}^{N_k} I_{A_i^k} u^{i,k} \in \mathbb{U}[t,T]$, $k = 1, 2, \ldots$, so that $\lim_{k\to\infty} \hat{\mathbb{E}}_t[\int_t^T |u_s - u_s^k|^2 \mathrm{d}s] = 0$ for each given $u \in \mathcal{U}[t,T]$. By the uniqueness of reflected G-BSDE, we have that $\sum_{i=1}^{N_k} I_{A_i^k} Y_t^{t,x,u^{i,k}} = Y_t^{t,x,u^k}$, which together with Lemma 3.2 indicates that $\sum_{i=1}^{N_k} I_{A_i^k} Y_t^{t,x,u^{i,k}}$ converges to $Y_t^{t,x,u}$. Therefore, one can easily get that $Y_t^{t,x,u} \geq \inf_{v\in\mathcal{U}^t[t,T]} Y_t^{t,x,v}$.

Consequently, in spirit of the fact that $\mathcal{U}^t[t,T] \subset \mathcal{U}[t,T]$, we derive that $\underset{u\in\mathcal{U}[t,T]}{ess\inf}\, Y_t^{t,x,u} = \inf_{u\in\mathcal{U}^t[t,T]} Y_t^{t,x,u}$, which completes the proof. □

Lemma 3.4 *For any $x, y \in \mathbb{R}^n$, we have the following:*

(i) Assume (A1)–(A2) *hold. Then,* $|V(t,x)| \leq \hat{C}_1(1+|x|)$ *and*

$$|V(t,x) - V(t,y)| \leq \hat{C}_1|x-y| + \hat{C}_1(1+|x|^{\frac{3}{4}} + |y|^{\frac{3}{4}})|x-y|^{\frac{1}{2}}.$$

(ii) Assume (A1) *and* (A2') *hold. Then,* $|V(t,x)| \leq \bar{C}_1(1+|x|^3)$ *and*

$$|V(t,x) - V(t,y)| \leq \bar{C}_1(1+|x|^2+|y|^2)|x-y|.$$

Proof The proof follows from Theorem 3.1, Lemmas 3.2 and 3.3. □

Theorem 3.2 *Assume that* (A1), (A2) *or* (A2') *are satisfied. Then, for any* $\xi \in L_G^p(\Omega_t;\mathbb{R}^n)$ *with* $p > 8$*, we have*

$$V(t,\xi) = \underset{u\in\mathcal{U}[t,T]}{ess\inf}\, Y_t^{t,\xi,u}.$$

Proof The proof is similar to the one of Theorem 20 in [7] and we omit it. □

In the rest of this section, we shall discuss the dynamic programming principle for our stochastic optimal control problem. Firstly, we introduce a family of backward semigroups, established by Peng [20]. For a given initial data (t,x), a positive real number $\delta \leq T-t$, $u \in \mathcal{U}[t,t+\delta]$ and $\eta \in L_G^p(\Omega_{t+\delta})$ with $p > 8$, we define

$$\mathbb{G}_{t,t+\delta}^{t,x,u}[\eta] := Y_t^{t,t+\delta,x,u},$$

where $(Y_s^{t,t+\delta,x,u}, Z_s^{t,t+\delta,x,u}, A_s^{t,t+\delta,x,u})_{t\leq s\leq t+\delta}$ is the solution of the following reflected G-BSDE with obstacle process $l(s,X_s^{t,x,u})$:

$$\begin{aligned} Y_s^{t,t+\delta,x,u} &= \eta + \int_s^{t+\delta} f\big(r, X_r^{t,x,u}, Y_r^{t,t+\delta,x,u}, Z_r^{t,t+\delta,x,u}, u_r\big)\mathrm{d}r \\ &\quad - \int_s^{t+\delta} Z_r^{t,t+\delta,x,u}\mathrm{d}B_r + \big(A_{t+\delta}^{t,t+\delta,x,u} - A_s^{t,t+\delta,x,u}\big). \end{aligned}$$

Then, we have the following dynamic programming principle.

Theorem 3.3 *Suppose that* (A1), (A2) *or* (A2') *hold. Then, for each* $s \in [t,T]$ *and* $x \in \mathbb{R}^n$*, we have*

$$V(t,x) = \underset{u\in\mathcal{U}[t,s]}{ess\inf}\,\mathbb{G}_{t,s}^{t,x,u}[V(s,X_s^{t,x,u})] = \inf_{u\in\mathcal{U}^t[t,s]} \mathbb{G}_{t,s}^{t,x,u}[V(s,X_s^{t,x,u})]. \tag{4}$$

Proof Note that $X_s^{t,x,u} \in L_G^p(\Omega_s)$ for any $p \geq 2$ by Lemma 2.1. Then, with the help of Theorem 3.2, we conclude that $Y_s^{s,X_s^{t,x,u},u} \geq V(s,X_s^{t,x,u})$, where $Y_s^{s,X_s^{t,x,u},u} = Y_s^{t,x,u}$ due to the uniqueness of solution to reflected G-BSDE. Consequently, it follows from the comparison theorem of reflected G-BSDE (Theorem 5.3 in [12]) that $Y_t^{t,x,u} \geq \mathbb{G}_{t,s}^{t,x,u}[V(s, X_s^{t,x,u})]$. Therefore, we get that $V(t,x) \geq \underset{u\in\mathcal{U}[t,s]}{ess\inf}\mathbb{G}_{t,s}^{t,x,u}[V(s, X_s^{t,x,u})]$.

By a similar analysis as in Lemma 22 in [7] or Lemma 5.2 in [21] , we can obtain that

$$V(t,x) \leq \inf_{u\in\mathcal{U}^t[t,s]} \mathbb{G}_{t,s}^{t,x,u}[V(s, X_s^{t,x,u})] = \underset{u\in\mathcal{U}[t,s]}{ess\inf}\mathbb{G}_{t,s}^{t,x,u}[V(s, X_s^{t,x,u})],$$

which completes the proof. □

Finally, we shall prove the continuity property of $V(t,x)$ with respect to t.

Lemma 3.5 *Assume* (A1), (A2) *or* (A2') *hold. Then, the value function V is continuous in t.*

Proof We shall only prove the case where (A1)–(A2) hold as above. For each $x \in \mathbb{R}^n$, $0 \leq t_1 \leq t_2 \leq T$, applying Theorem 3.3 yields that

$$|V(t_1,x) - V(t_2,x)| \leq \sup_{u\in\mathcal{U}^{t_1}[t_1,t_2]} |\mathbb{G}_{t_1,t_2}^{t_1,x,u}[V(t_2, X_{t_2}^{t_1,x,u})] - V(t_2,x)|.$$

On the other hand, it is easy to check that $V(t_2,x) \geq l(t_2,x)$. Thus, we can get that $(V(t_2,x), 0, 0)$ is the solution to reflected G-BSDE with data $(V(t_2,x), 0, l(t_2,x))$ in the interval $[t_1,t_2]$. Therefore, applying Lemmas A.2, 3.4 and B.1 and by a simple calculation, we obtain that for each $u \in \mathcal{U}^{t_1}[t_1,t_2]$,

$$\begin{aligned}|\mathbb{G}_{t_1,t_2}^{t_1,x,u}[V(t_2, X_{t_2}^{t_1,x,u})] - V(t_2,x)|^2 \leq C_2(1 + |x|^{\frac{5}{2}})(|t_2 - t_1|^{\frac{1}{2}} \\ + \sup_{t_1\leq s\leq t_2} |l(s,x) - l(t_2,x)|),\end{aligned}$$

which implies the desired result. □

4 Stochastic Representation for HJBI Equations

In this section, we shall show that the value function $V(t,x)$ is the viscosity solution to the related HJBI equation. In the sequel, we always assume (A1), (A2) or (A2') hold.

Consider the following HJBI equation:

$$\begin{aligned}&\min\{V(t,x) - l(t,x),\ -\partial_t V - \inf_{u\in U} H(t,x,V,\partial_x V, \partial_{xx}^2 V, u)\} = 0, \\ &V(T,x) = \Phi(x), \quad x \in \mathbb{R}^n,\end{aligned} \tag{5}$$

where function H is given by

$$H(t,x,v,p,A,u) = G(\sigma^\top A\sigma(t,x,u)) + \langle p, b(t,x,u)\rangle \\ + f(t,x,v,\sigma^\top(t,x,u)p,u).$$

Firstly, we state the main result of this section.

Theorem 4.1 *The value function V defined by Eq.* (3) *is the unique viscosity solution to Hamilton–Jacobi–Bellman–Isaac Eq.* (5).

In order to prove Theorem 4.1, we need to introduce an auxiliary stochastic optimal control problem without obstacle constraints. Indeed, for any positive integer N, the following standard G-BSDE in time interval $[t,T]$

$$Y_s^{N,t,x,u} = \Phi(X_T^{t,x,u}) + \int_s^T f^N\big(r, X_r^{t,x,u}, Y_r^{N,t,x,u}, Z_r^{N,t,x,u}\big)\mathrm{d}r \\ - \int_s^T Z_r^{N,t,x,u}\mathrm{d}B_r - \big(K_T^{N,t,x,u} - K_s^{N,t,x,u}\big)$$

admits a unique solution $(Y^{N,t,x,u}, Z^{N,t,x,u}, K^{N,t,x,u})$, where

$$f^N(t,x,y,z) = f(t,x,y,z,u_t) + N(y - l(t,x))^-.$$

Furthermore, we have $Y_s^{N,t,x,u} \uparrow Y_s^{t,x,u}$.

Next, we consider a stochastic optimal control problem, in which the cost function is defined by $Y^{N,t,x,u}$, i.e., the value function $V^N(t,x)$ is given by $\underset{u\in\mathcal{U}[t,T]}{ess\inf} Y_t^{N,t,x,u}$. It follows from [7] that V^N is the viscosity solution of the following fully nonlinear partial differential equations (PDEs):

$$-\partial_t V^N - \inf_{u\in U} H^N(t,x,V^N,\partial_x V^N,\partial_{xx}^2 V^N,u) = 0, \\ V^N(T,x) = \Phi(x), \quad x \in \mathbb{R}^n, \tag{6}$$

where $H^N(t,x,v,p,A,u) = H(t,x,v,p,A,u) + N(v - l(t,x))^-$. Then, we have the following relation between V and V^N.

Lemma 4.1 *For any $(t,x) \in [0,T]\times\mathbb{R}^n$, $V^N(t,x) \uparrow V(t,x)$.*

Proof It follows from the monotonicity of $Y_t^{N,t,x,u}$ that $V^N(t,x) \uparrow \bar{V}(t,x)$. Thus, we only need to prove $\bar{V}(t,x) \geq V(t,x)$.

We claim that

$$\lim_{N,M\to\infty} \sup_{u\in\mathcal{U}^t[t,T]} \hat{\mathbb{E}}\left[\sup_{s\in[t,T]} |Y_s^{N,t,x,u} - Y_s^{M,t,x,u}|^2\right] = 0, \tag{7}$$

whose proof will be given in "Appendix B." Therefore, for each $\varepsilon > 0$, we can find some N_1 so that $Y_t^{t,x,u} \leq Y_t^{N_1,t,x,u} + \frac{\varepsilon}{2}$, for any $u \in \mathcal{U}^t[t,T]$. Recalling Theorem 17 in [7], there exists a $u^{N_1} \in \mathcal{U}^t[t, T]$ such that

$$Y_t^{N_1,t,x,u^{N_1}} \leq V^{N_1}(t,x) + \frac{\varepsilon}{2},$$

which implies that $V(t,x) \leq Y_t^{t,x,u^{N_1}} \leq V^{N_1}(t,x) + \varepsilon \leq \bar{V}(t,x) + \varepsilon$. Letting ε converge to 0, we can get the desired result. □

Now, we are in a position to complete the proof of Theorem 4.1.

Proof In spirit of the fact that G is sublinear function, the PDE (5) can be seen as a special case of Equation (4.2) in [22]. Thus, we can get the uniqueness by Theorem 5.1 and Remark 5.1 of [22]. In the following, we only prove that V is the viscosity subsolution, since the other case can be proved similarly.

Given $(t,x) \in]0,T[\times\mathbb{R}^n$ and $(a,p,X) \in \mathcal{P}^{2,+}V(t,x)$. Without loss of generality, we assume that $V(t,x) > l(t,x)$. Then, from Lemma 6.1 in [23], there exist sequences $N_j \to \infty$, $(t_j,x_j) \to (t,x)$ and $(a_j,p_j,X_j) \in \mathcal{P}^{2,+}V^{N_j}(t_j,x_j)$, such that $(a_j,p_j,X_j) \to (a,p,X)$. Since V^N is the viscosity solution to Eq. (6), we derive that

$$-a_j - \inf_{u\in U} H^{N_j}(t_j,x_j,V^{N_j}(t_j,x_j),p_j,X_j,u) \leq 0,$$

Note that $V(t,x) > l(t,x)$ and V^N uniformly converges to V on each compact subset. Then, it is easy to check that $V^{N_j}(t_j,x_j) > l(t_j,x_j)$ for j large enough. Therefore, we get that for j large enough

$$-a_j - \inf_{u\in U} H(t_j,x_j,V^{N_j}(t_j,x_j),p_j,X_j,u) \leq 0.$$

By a similar analysis as in [13], we derive that

$$\begin{aligned}
&\lim_{j\to\infty} \inf_{u\in U} H(t_j,x_j,V^{N_j}(t_j,x_j),p_j,X_j,u)\\
&= \inf_{u\in U} \lim_{j\to\infty} H(t_j,x_j,V^{N_j}(t_j,x_j),p_j,X_j,u).
\end{aligned}$$

Combining the above two equations, we could obtain the desired result. □

5 Conclusions

In this article, we investigate the stochastic optimal control problem with obstacle constraints in G-framework. This problem arises in minimizing the cost of an agent subject to some constraints under volatility uncertainty. In order to solve this problem, we first show that the value function is a deterministic continuous function by the approximation method for admissible controls. Then, we establish the dynamic programming principle for the value function via the "implied partition" approach. Finally, we show that the value function is the unique viscosity solution of the associated HJBI equation.

Acknowledgements Li's research was supported by the German Research Foundation (DFG) via CRC 1283. Wang's research was supported by the National Natural Science Foundation of China (Nos. 11601282, 11871310 and 11871458) and the Natural Science Foundation of Shandong Province (No. ZR2016AQ10). The authors wish to thank the editor and the referees for their valuable comments and constructive suggestions which improved the presentation of this manuscript.

Appendix A: Reflected *G*-BSDE

The definition of reflected G-BSDEs Given an obstacle process $\{S_t\}_{t\in[0,T]}$, a terminal value $\zeta \in L_G^{\beta}(\Omega_T)$ with $\zeta \geq S_T$ for $\beta > 2$ and generator $f(t,\omega,y,z) : [0,T]\times \Omega\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$, a triple of processes $(Y,Z,A)\in \mathfrak{S}_G^2(0,T)$ is called a solution of reflected G-BSDE with data (ζ, f, S) if the following properties hold:

(i) $Y_t = \zeta + \int_t^T f(s,Y_s,Z_s)\mathrm{d}s - \int_t^T Z_s\mathrm{d}B_s + (A_T - A_t)$;
(ii) $Y_t \geq S_t$, and $\{-\int_0^t (Y_s - S_s)\mathrm{d}A_s\}_{t\in[0,T]}$ is a non-increasing G-martingale,

where $\mathfrak{S}_G^2(0,T)$ is the collection of processes (Y,Z,A) such that $Y\in S_G^2(0,T)$, $Z\in M_G^2(0,T)$ and $A\in S_G^2(0,T)$ is a continuous non-decreasing process starting from origin.

The well-posedness of reflected G-BSDEs Consider the following assumption:

(H1) there exists a constant $\beta > 2$ such that for any y, z, $f(\cdot,\cdot,y,z)\in M_G^{\beta}(0,T)$;
(H2) there exists a constant $L_1 > 0$ such that $|f(t,y,z) - f(t,y',z')| \leq L_1(|y - y'| + |z - z'|)$;
(H3) there exists a constant c such that $\{S_t\}_{t\in[0,T]}\in S_G^{\beta}(0,T)$ and $S_t \leq c$ for each $t\in[0,T]$;
(H3') $\{S_t\}_{t\in[0,T]}$ has the following form:

$$S_t = S_0 + \int_0^t b_s\mathrm{d}s + \sum_{i,j=1}^d \int_0^t \gamma_s^{ij} d\langle B^i, B^j\rangle_s + \sum_{j=1}^d \int_0^t \kappa_s^j \mathrm{d}B_s^j,$$

where the processes $b_s, \gamma_s^{ij} = \gamma_s^{ji} \in M_G^{\beta}(0,T)$ and $\kappa_s^j \in H_G^{\beta}(0,T)$, $1\leq i,j\leq d$.

Lemma A.1 ([12]) *Assume that f satisfies* (H1)-(H2) *for some $\beta > 2$ and let* (H3) *or* (H3') *hold. Then, the reflected G-BSDE has a unique solution $(Y,Z,K)\in\mathfrak{S}_G^2(0,T)$.*

Lemma A.2 ([12]) *Let $\zeta^{\nu}\in L_G^{\beta}(\Omega_T)$, $\nu = 1,2$ and f^{ν}, S^{ν} satisfy* (H1)-(H3) *for some $\beta > 2$. Assume that $(Y^{\nu},Z^{\nu},K^{\nu})\in\mathfrak{S}_G^2(0,T)$, $\nu = 1,2$ are the solutions of the reflected G-BSDE corresponding to data $(\zeta^{\nu}, f^{\nu}, S^{\nu})$. Set $\hat{Y}_t = Y_t^1 - Y_t^2$, $\hat{S}_t = S_t^1 - S_t^2$ and $\hat{\zeta} = \zeta^1 - \zeta^2$. Then, there exists a constant $\hat{C}$ depending on T, G, β, c and L_1 such that*

$$|Y_t^{\nu}|^2 \leq \hat{C}\hat{\mathbb{E}}_t\left[1 + |\zeta^{\nu}|^2 + \int_t^T |\lambda_s^{\nu,0}|^2\mathrm{d}s\right],$$

$$|\hat{Y}_t|^2 \leq \hat{C}\left\{\hat{\mathbb{E}}_t\left[|\hat{\xi}|^2 + \int_t^T |f^1(s, Y_s^2, Z_s^2) - f^2(s, Y_s^2, Z_s^2)|^2 \mathrm{d}s\right]\right.$$
$$\left.+\hat{\mathbb{E}}_t\left[\sup_{s\in[t,T]} |\hat{S}_s|^2\right]^{1/2} \Psi_{t,T}^{1/2}\right\},$$

where $\Psi_{t,T} = \sum_{\nu=1}^2 \hat{\mathbb{E}}_t[\sup_{s\in[t,T]} \hat{\mathbb{E}}_s[1 + |\zeta^\nu|^2 + \int_t^T |f^\nu(r, 0, 0)|^2 \mathrm{d}r]]$.

Lemma A.3 ([12]) *Let* $\zeta^\nu \in L_G^\beta(\Omega_T)$, $\nu = 1, 2$ *and* f^ν, S^ν *satisfy* (H1), (H2), (H3') *for some* $\beta > 2$. *Assume that* $(Y^\nu, Z^\nu, K^\nu) \in \mathfrak{S}_G^2(0, T)$, $\nu = 1, 2$ *are the solutions of the reflected G-BSDE with data* $(\zeta^\nu, f^\nu, S^\nu)$. *Set* $\bar{Y}_t = (Y_t^1 - S_t^1) - (Y_t^2 - S_t^2)$ *and* $\hat{S}_t = S_t^1 - S_t^2$. *Then, there exists a constant* $\bar{C}$ *depending on* T, G, β *and* L_1 *such that*

$$|Y_t^\nu|^2 \leq \bar{C}\hat{\mathbb{E}}_t\left[|\zeta^\nu|^2 + \sup_{s\in[t,T]} |S_s^\nu|^2 + \int_t^T |\bar{\lambda}_s^{\nu,0}|^2 \mathrm{d}s\right],$$
$$|\bar{Y}_t|^2 \leq \bar{C}\left\{\hat{\mathbb{E}}_t\left[|\zeta^1 - S_T^1 - \zeta^2 + S_T^2|^2 + \int_t^T (|\hat{\lambda}_s|^2 + |\hat{\rho}_s|^2 + |\hat{S}_s|^2)\mathrm{d}s\right]\right\},$$

where $\hat{\lambda}_s = |f^1(s, Y_s^2, Z_s^2) - f^2(s, Y_s^2, Z_s^2)|$, $\bar{\lambda}_s^{\nu,0} = |f^\nu(s, 0, 0)| + |b_s^\nu| + \sum_{i,j=1}^d |\gamma_s^{\nu,ij}| + \sum_{j=1}^d |\kappa_s^{\nu,j}|$ *and* $\hat{\rho}_s = |b_s^1 - b_s^2| + \sum_{i,j=1}^d |\gamma_s^{1,ij} - \gamma_s^{2,ij}| + \sum_{j=1}^d |\kappa_s^{1,j} - \kappa_s^{2,j}|$.

Appendix B: The Complement Proofs

The following maximal inequality for G-martingale has been firstly established by Song [24].

Lemma B.1 *Assume* $\alpha \geq 1$ *and* $\delta > 0$. *Set*

$$C_G = 2\inf\left\{\frac{\gamma}{\gamma - 1}\left(1 + 14\sum_{i=1}^\infty i^{-\frac{\alpha+\delta}{\gamma}}\right) : 1 < \gamma < \alpha + \delta, \gamma \leq 2\right\}.$$

Then, we have

$$\hat{\mathbb{E}}_t\left[\sup_{s\in[t,T]} \hat{\mathbb{E}}_s[|\xi|^\alpha]\right] \leq C_G\{(\hat{\mathbb{E}}_t[|\xi|^{\alpha+\delta}])^{\alpha/(\alpha+\delta)} + \hat{\mathbb{E}}_t[|\xi|^{\alpha+\delta}]\}.$$

Proof The proof is immediate from the definition of conditional G-expectation and Theorem 3.4 in [24]. □

Now, we are going to state the proof of Lemmas 3.2 and 3.3.

Proof It is sufficient to prove the second inequalities in both cases, since the first ones can be proved similarly. For convenience, we omit superscripts t. We fist prove the second inequality in Lemma 3.2. Set $\hat{X}_s = X_s^{\xi,u} - X_s^{\xi',u'}$, $\hat{\Phi}(X_T) = \Phi(X_T^{\xi,u}) - \Phi(X_T^{\xi',u'})$ and $\hat{\lambda}_s = f(s, X_s^{\xi',u'}, Y_s^{\xi',u'}, Z_s^{\xi',u'}, u_s') - f(s, X_s^{\xi,u}, Y_s^{\xi',u'}, Z_s^{\xi',u'}, u_s)$. Applying Lemma A.2 and Lemma B.1 yields that

$$|\hat{Y}_t|^2 \le C_2 \left\{ \hat{\mathbb{E}}_t \left[|\hat{X}_T|^2 + \int_t^T (|\hat{X}_s|^2 + |\hat{u}_s|^2) \mathrm{d}s \right] + \hat{\mathbb{E}}_t \left[\sup_{s\in[t,T]} |\hat{X}_s|^2 \right]^{1/2} \Psi_{t,T} \right\},$$

where C_2 is a generic constant depending on T, G, c, L and n (may vary from line to line), and $\Psi_{t,T}^2 \le C_2(1 + |\xi|^3 + |\xi'|^3)$. From Lemma 2.1, we could get the desired result.

Then, we prove the second inequality in Lemma 3.3. Applying G-Itô's formula (see Theorem 6.5 of Chap. III in [14]) to $l(s, X_s^{\xi,u})$ yields that

$$l(s, X_s^{\xi,u}) = l(t,\xi) + \int_t^s b_r^{\xi,u} \mathrm{d}r + \int_t^s \gamma_r^{\xi,u,ij} d\langle B^i, B^j\rangle_r + \int_t^s \kappa_r^{\xi,u,j} \mathrm{d}B_r^j,$$

where $b^{\xi,u}$, $\gamma^{\xi,u}$ and $\kappa^{\xi,u}$ are given by

$$\begin{aligned} b_s^{\xi,u} &= \partial_s l(s, X_s^{\xi,u}) + \langle \partial_x l(s, X_s^{\xi,u}), b(s, X_s^{\xi,u}, u_s)\rangle, \\ \gamma_s^{\xi,u,ij} &= \langle \partial_x l(s, X_s^{\xi,u}), h_{ij}(s, X_s^{\xi,u}, u_s)\rangle \\ &\quad + \frac{1}{2}(\sigma^\top(s, X_s^{\xi,u}, u_s)\partial_{xx}^2 l(s, X_s^{\xi,u})\sigma(s, X_s^{\xi,u}, u_s))_{ij}, \\ \kappa_s^{\xi,u,j} &= \langle \partial_x l(s, X_s^{\xi,u}), \sigma_j^\top(s, X_s^{t,\xi,u}, u_s)\rangle, \ \ \sigma_j^\top \text{ is the } j\text{-th row of } \sigma^\top. \end{aligned}$$

Denote by C_3 a generic constant depending on T, G, n and L, which may vary from line to line. Then, recalling Lemma A.3, we deduce that

$$|\bar{Y}_t|^2 \le C_3 \left\{ \hat{\mathbb{E}}_t \left[|\bar{\xi}|^2 + \int_t^T (|\hat{\lambda}_s|^2 + |\hat{\rho}_s|^2 + |\hat{S}_s|^2) \mathrm{d}s \right] \right\},$$

where

$$\begin{aligned} &\bar{Y}_t = (Y_t^{\xi,u} - l(t,\xi)) - (Y_t^{\xi',u'} - l(t,\xi')), \ \hat{S}_s = l(s, X_s^{\xi,u}) - l(s, X_s^{\xi',u'}), \\ &\bar{\xi} = \Phi(X_T^{\xi,u}) - l(T, X_T^{\xi,u}) - \Phi(X_T^{\xi',u'}) + l(T, X_T^{\xi',u'}), \\ &\hat{\lambda}_s = |f(s, X_s^{\xi',u'}, Y_s^{\xi',u'}, Z_s^{\xi',u'}, u_s') - f(s, X_s^{\xi,u}, Y_s^{\xi',u'}, Z_s^{\xi',u'}, u_s)|, \\ &\hat{\rho}_s = |b_s^{\xi,u} - b_s^{\xi',u'}| + \sum_{i,j=1}^d |\gamma_s^{\xi,u,ij} - \gamma_s^{\xi',u',ij}| + \sum_{j=1}^d |\kappa_s^{\xi,u,j} - \kappa_s^{\xi',u',j}|. \end{aligned}$$

Set $\bar{\Psi}_{t,T}=\sup_{s\in[t,T]}(1+|X_s^{\xi,u}|+|X_s^{\xi',u'}|+|X_s^{\xi,u}|^2+|X_s^{\xi',u'}|^2)$ and $\hat{X}_s=X_s^{\xi,u}-X_s^{\xi',u'}$. Then, recalling assumptions (A1), (A2′) and Lemma B.1, we derive that

$$|\bar{Y}_t|^2\le C_3\left\{\hat{\mathbb{E}}_t\left[|\hat{X}_T|^2+\int_t^T(|\hat{X}_s|^2+|\hat{u}_s|^2)\mathrm{d}s\right]+\hat{\mathbb{E}}_t[\bar{\Psi}_{t,T}^4]^{\frac{1}{2}}\hat{\mathbb{E}}_t\left[\int_t^T(|\hat{X}_s|^4\right.\right.$$
$$\left.\left.+|\hat{u}_s|^4)\mathrm{d}s\right]^{\frac{1}{2}}\right\}.$$

Consequently, in spirit of Lemma 2.1, we get

$$|\bar{Y}_t|^2\le C_3(1+|\xi|^4+|\xi'|^4)\left(|\xi-\xi'|^2+\hat{\mathbb{E}}_t\left[\int_t^T|u_s-u_s'|^4\mathrm{d}s\right]^{\frac{1}{2}}\right),$$

which, together with $|Y_t^{\xi,u}-Y_t^{\xi',u'}|^2\le 2(|l(t,\xi)-l(t,\xi')|^2+|\bar{Y}_t|^2)$, implies the inequality (ii). The proof is complete. □

Finally, we are ready to state the proof of Eq. (7).

Proof For readers' convenience, we shall give the sketch of the proof. For simplicity, we omit the superscripts (t,x).

From the proof of Lemma 4.4 in [12], it suffices to prove that

$$\lim_{N\to\infty}\sup_{u\in\mathcal{U}^t[t,T]}\hat{\mathbb{E}}\left[\sup_{s\in[t,T]}|(Y_s^{N,u}-l(s,X_s^u))^-|^2\right]=0.$$

For simplicity, set $\Theta_r^{N,u}=(X_r^u,Y_r^{N,u},Z_r^{N,u})$. Now, recalling Lemma 4.3 in [12], we derive that for each $(t,x)\in[0,T]\times\mathbb{R}^n$ and $u\in\mathcal{U}^t[t,T]$,

$$(Y_s^{N,u}-l(s,X_s^u))^-\le\hat{\mathbb{E}}_s\left[|\tilde{S}_s^{N,u}|+|\int_s^T e^{N(s-r)}f(r,\Theta_r^{N,u},u_r)\mathrm{d}r|\right],$$

where $\tilde{S}_s^{N,u}=e^{N(s-T)}(\Phi(X_T^u)-l(s,X_s^u))+\int_s^T Ne^{N(s-r)}(l(r,X_r^u)-l(s,X_s^u))\mathrm{d}r$.

In spirit of Lemma B.1 and using a similar analysis as Equation (4.3) in [12], we conclude that

$$\lim_{N\to\infty}\sup_{u\in\mathcal{U}^t[t,T]}\hat{\mathbb{E}}\left[\sup_{s\in[t,T]}\hat{\mathbb{E}}_s\left[|\int_s^T e^{N(s-r)}f(r,\Theta_r^{N,u},u_r)\mathrm{d}r|\right]^2\right]=0.\tag{8}$$

Next, we shall deal with the term $\tilde{S}_s^{N,u}$. Let $C_5>0$ be a generic constant. It follows from Equation (4.4) in [12] that for each fixed $\varepsilon,\delta>0$ and $\beta>2$,

$$\lim_{N\to\infty}\sup_{u\in\mathcal{U}^t[t,T]}\hat{\mathbb{E}}\left[\sup_{s\in[t,T-\delta]}|\tilde{S}_s^{N,u}|^\beta\right]$$

$$\leq C_5 \sup_{u\in\mathcal{U}^t[t,T]} \left\{ \hat{\mathbb{E}}\left[\sup_{s\in[t,T]} \sup_{r\in[s,s+\varepsilon]} |l(r, X_s^u) - l(s, X_s^u)|^\beta \right] + \hat{\mathbb{E}}\left[\sup_{s\in[t,T]} \sup_{r\in[s,s+\varepsilon]} |X_r^u - X_s^u|^\beta \right] \right\}.$$

Noting that for each $u \in \mathcal{U}^t[t, T]$ and any integer $\rho > 0$, we have

$$\begin{aligned}
&\hat{\mathbb{E}}\left[\sup_{s\in[t,T]} \sup_{r\in[s,s+\varepsilon]} |l(r, X_s^u) - l(s, X_s^u)|^\beta \right] \\
&\leq \hat{\mathbb{E}}\left[\sup_{s\in[t,T]} \sup_{r\in[s,s+\varepsilon]} |l(r, X_s^u) - l(s, X_s^u)|^\beta I_A \right] + \frac{C_5}{\rho} \hat{\mathbb{E}}\left[\sup_{s\in[t,T]} |X_s^u| + \sup_{s\in[t,T]} |X_s^u|^{1+\beta} \right] \\
&\leq \sup_{s\in[t,T],|x|\leq\rho} \sup_{r\in[s,s+\varepsilon]} |l(r, x) - l(s, x)|^\beta + \frac{C_5}{\rho}.
\end{aligned}$$

where $A = \{\sup_{s\in[t,T]} |X_s^u| \leq \rho\}$. Denote $t_i^\rho = \frac{i}{\rho}(T-t)$, $i = 0, \ldots, \rho$. Then, we deduce that, for each $\varepsilon \leq \frac{T-t}{\rho}$,

$$\hat{\mathbb{E}}\left[\sup_{s\in[t,T]} \sup_{r\in[s,s+\varepsilon]} |X_r^u - X_s^u|^\beta \right] \leq C_5 \sum_{i=0}^{\rho-1} \hat{\mathbb{E}}\left[\sup_{s\in[t_i^\rho, t_{i+1}^\rho]} |X_{t_i^\rho}^u - X_s^u|^\beta \right] \leq C_5 \rho^{1-\frac{\beta}{2}}.$$

Consequently, letting $\varepsilon \to 0$ and sending $\rho \to \infty$ we obtain that

$$\lim_{N\to\infty} \sup_{u\in\mathcal{U}^t[t,T]} \hat{\mathbb{E}}\left[\sup_{t\in[0,T-\delta]} |\tilde{S}_s^{N,u}|^\beta \right] = 0,$$

which, together with Eq. (8) and (4.5) in [12], implies that for each $t < \delta < T$

$$\begin{aligned}
&\lim_{N\to\infty} \sup_{u\in\mathcal{U}^t[t,T]} \hat{\mathbb{E}}\left[\sup_{s\in[t,T]} |(Y_s^{N,u} - l(s, X_s^u))^-|^2 \right] \\
&\leq \sup_{u\in\mathcal{U}^t[t,T]} \hat{\mathbb{E}}\left[\sup_{s\in[T-\delta,T]} |(Y_s^{1,u} - l(s, X_s^u))^-|^2 \right].
\end{aligned}$$

By the definition of $Y^{1,u}$, we have

$$Y_s^{1,u} = \hat{\mathbb{E}}_s\left[\Phi(X_T^u) + \int_s^T f(r, \Theta_r^{1,u}, u_r)\mathrm{d}r + \int_s^T (Y_r^{1,u} - l(r, X_r^u))^- \mathrm{d}r \right].$$

It is easy to check that for some $\beta > 2$,

$$\lim_{\delta\to 0}\sup_{u\in\mathcal{U}^t[t,T]}\hat{\mathbb{E}}\left[\sup_{s\in[T-\delta,T]}\int_s^T\{|f(r,\Theta_r^{1,u},u_r)|^\beta+|Y_r^{1,u}-l(r,X_r^u)|^\beta\}\mathrm{d}r\right]=0.$$

By Lemma B.1, it follows that

$$\lim_{\delta\to 0}\sup_{u\in\mathcal{U}^t[t,T]}\hat{\mathbb{E}}\left[\sup_{s\in[T-\delta,T]}\hat{\mathbb{E}}_s\left[\int_s^T|f(r,\Theta_r^{1,u},u_r)|+|Y_r^{1,u}-l(r,X_r^u)|\mathrm{d}r\right]^2\right]=0.$$

Note that $(Y_T^{1,u}-l(T,X_T^u))^-=0$. Then, it holds that

$$(Y_s^{1,u}-l(s,X_s^u))^-\leq|Y_s^{1,u}-\Phi(X_T^u)|+|l(s,X_s^u)-l(T,X_T^u)|.$$

Thus, by a similar analysis as the above we derive that for each integer $\rho>0$,

$$\begin{aligned}\hat{\mathbb{E}}[\sup_{s\in[T-\delta,T]}|(Y_s^{1,u}-l(s,X_s^u))^-|^2]\leq C_5\Bigg\{&\hat{\mathbb{E}}\left[\sup_{s\in[T-\delta,T]}|\hat{\mathbb{E}}_s[\Phi(X_T^u)]-\Phi(X_T^u)|^2\right]\\&+\sup_{s\in[T-\delta,T],|x|\leq\rho}|l(s,x)-l(T,x)|^2\\&+\frac{C_5}{\rho}+m(\delta)\Bigg\},\end{aligned}$$

where $m(\cdot)$ is a nonnegative continuous function satisfying $\lim_{\delta\to 0}m(\delta)=0$.

On the other hand, it follows from Lemmas 2.1 and B.1 that

$$\lim_{\delta\to 0}\sup_{u\in\mathcal{U}^t[t,T]}\hat{\mathbb{E}}\left[\sup_{s\in[T-\delta,T]}|\hat{\mathbb{E}}_s[\Phi(X_T^u)]-\Phi(X_T^u)|^2\right]=0.$$

Therefore, by the above analysis, it holds that for each integer $\rho>0$

$$\begin{aligned}&\lim_{N\to\infty}\sup_{u\in\mathcal{U}^t[t,T]}\hat{\mathbb{E}}\left[\sup_{s\in[t,T]}|(Y_s^{N,u}-l(s,X_s^u))^-|^2\right]\\&\leq C_5\left(\sup_{s\in[T-\delta,T],|x|\leq\rho}|l(s,x)-l(T,x)|^2+\frac{1}{\rho}+m(\delta)\right).\end{aligned}$$

Sending $\delta\to 0$ and then $\rho\to\infty$ yields the desired result. □

References

1. Peng, S.G.: G-Expectation, G-Brownian motion and related stochastic calculus of Itô type. In: Stochastic Analysis and Applications, Abel Symposium, vol. 2, pp. 541–567. Springer, Berlin (2007)

2. Epstein, L.G., Ji, S.L.: Ambiguous volatility and asset pricing in continuous time. Rev. Financ. Stud. **26**, 1740–1786 (2013)
3. Epstein, L.G., Ji, S.L.: Ambiguous volatility, possibility and utility in continuous time. J. Math. Econ. **50**, 269–282 (2014)
4. Hu, M.S., Ji, S.L., Peng, S.G., Song, Y.S.: Backward stochastic differential equations driven by G-Brownian motion. Stoch. Proc. Appl. **124**, 759–784 (2014)
5. Hu, M.S., Ji, S.L., Peng, S.G., Song, Y.S.: Comparison theorem, Feynman–Kac formula and Girsanov transformation for BSDEs driven by G-Brownian motion. Stoch. Proc. Appl. **124**, 1170–1195 (2014)
6. Soner, H.M., Touzi, N., Zhang, J.F.: Wellposedness of second order backward SDEs. Probab. Theory Relat. Fields **153**, 149–190 (2012)
7. Hu, M.S., Ji, S.L.: Dynamic programming principle for stochastic recursive optimal control problem driven by a G-Brownian motion. Stoch. Proc. Appl. **127**, 107–134 (2017)
8. Fouque, J.-P., Pun, C.S., Wong, H.Y.: Portfolio optimization with ambiguous correlation and stochastic volatilities. SIAM. J. Control. Optim. **54**, 2309–2338 (2016)
9. Matoussi, A., Possamaï, D., Zhou, C.: Robust Utility maximization in non-dominated models with 2BSDEs. Math. Finance **25**(2), 258–287 (2015)
10. Neufeld, A., Nutz, M.: Robust utility maximization with Lévy processes. Math. Finance **28**(1), 82–105 (2018)
11. El Karoui, N., Pardoux, E., Quenez, M.C.: Reflected Backward SDEs and American Options. In: Rogers, L.C.G., Talay, D. (eds.) Numerical Methods in Finance, pp. 215–231 (1997)
12. Li, H.W., Peng, S.G., Soumana Hima, A.: Reflected solutions of BSDE driven by G-Brownian motion. Sci. China Math. **61**(1), 1–26 (2018)
13. Wu, Z., Yu, Z.Y.: Dynamic programming principle for one kind of stochastic recursive optimal control problem and Hamilton–Jacobi–Bellman equation. SIAM J Control Optim. **47**(5), 2616–2641 (2008)
14. Peng, S.G.: Nonlinear expectations and stochastic calculus under uncertainty. arXiv:1002.4546v1
15. Denis, L., Hu, M.S., Peng, S.G.: Function spaces and capacity related to a sublinear expectation: application to G-Brownian motion paths. Potential Anal. **34**, 139–161 (2011)
16. Cohen, S.: Quasi-sure analysis, aggregation and dual representations of sublinear expectations in general spaces. Electron. J. Probab. **17**, 1–15 (2012)
17. Peng, S.G.: A generalized dynamic programming principle and Hamilton–Jacobi–Bellmen equation. Stoch. Stoch. Rep. **38**, 119–134 (1992)
18. Peng, S.G.: Backward stochastic differential equation and application to optimal control. Appl. Math. Optim. **27**, 125–144 (1993)
19. Buckdahn, R., Nie, T.Y.: Generalized Hamilton–Jacobi–Bellman equations with Dirichlet boundary and stochastic exit time optimal control problem. SIAM. J. Control. Optim. **54**(2), 602–631 (2016)
20. Peng, S.G.: Backward stochastic differential equations-stochastic optimization theory and viscosity solutions of HJB equations. In: Yan, J., Peng, S., Fang, S., Wu, L. (eds.) Topics on Stochastic Analysis, pp. 85–138. Science Press, Beijing (1997). (in Chinese)
21. Hu, M.S., Wang, F.L.: Stochastic optimal control problem with infinite horizon driven by G-Brownian motion. ESAIM Control Optim. Calc. Var. **24**(2), 873–899 (2018)
22. Buckdahn, R., Li, J.: Stochastic differential games with reflection and related obstacle problems for Isaacs equations. Acta Math. Appl. Sin. English Ser. **27**(4), 647–678 (2011)
23. Crandall, M.G., Ishii, H., Lions, P.L.: User's guide to viscosity solutions of second order partial differential equations. Bull. Am. Math. Soc. **27**(1), 1–67 (1992)
24. Song, Y.S.: Some properties on G-evaluation and its applications to G-martingale decomposition. Sci. China Math. **54**, 287–300 (2011)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.