# Global well-posedness of the 3D Navier–Stokes equations perturbed by a deterministic vector field

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### Abstract

We are concerned with the problem of global well-posedness of the 3D Navier–Stokes equations on the torus with unitary viscosity. While a full answer to this question seems to be out of reach of the current techniques, we establish a regularization by a deterministic vector field. More precisely, we consider the vorticity form of the system perturbed by an additional transport type term. Such a perturbation conserves the enstrophy and therefore a priori it does not imply any smoothing. Our main result is a construction of a deterministic vector field v = v(t, x) which provides the desired regularization of the system and permits to obtain global well-posedness. The proof relies on probabilistic arguments developed by Flandoli and Luo, tools from rough path theory by Hofmanová, Leahy and Nilssen and a new Wong–Zakai approximation result, which itself combines probabilistic and rough path techniques.

**Keywords:** 3D Navier-Stokes equations, vorticity form, well-posedness, regularization by noise, Wong–Zakai principle

# **1** Introduction

The problem of global well-posedness of the three dimensional Navier–Stokes system describing flows of incompressible fluids remains an outstanding open problem of great interest. Recently, it experienced a major breakthrough due to Buckmaster and Vicol [BV19] and Buckmaster, Colombo and Vicol [BCV18], who were able to prove nonuniqueness in a class of weak solutions. The result followed a series of important works on the Euler equations by Buckmaster, De Lellis, Isett,

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Székelyhidi Jr. and Vicol [BDLSV19, DLS09, DLS10, DLS13, Ise18] proving among others the Onsager conjecture. The solutions to the Navier–Stokes system constructed by Buckmaster and Vicol [BCV18] do not satisfy the corresponding energy inequality, i.e., they are not the so-called Leray solutions. Therefore, they can be regarded as nonphysical. The problem of uniqueness of Leray solutions is one of the major challenges in the mathematical fluid dynamics research.

The systems of Navier–Stokes and Euler equations are derived from the basic physical principles. However, the derivation proceeds under a number of simplifying assumptions and in particular many physical parameters are neglected. In view of the above theoretical difficulties, one may therefore wonder whether the systems are not oversimplified so that the key physical properties are lost for their solutions. In the present paper we give an affirmative answer to the following question:

Is there an additional enstrophy preserving deterministic term which provides global well-posedness for the 3D Navier–Stokes equations?

Our guiding principle is the enstrophy conservation of the added term together with the fact that the equations shall remain deterministic. More precisely, even though our result is in the spirit of *regularization by noise* and our proof makes an essential use of probabilistic arguments, the constructed perturbation is deterministic. Furthermore, the perturbation is driven by a vector field which is a time derivative of a highly oscillatory but an explicit piecewise linear function. As a consequence, the result can be further underlined by numerical simulations, which is one of the reasons why we aimed for the piecewise linear setting and deterministic vector fields. However, by a slight modification, our proof yields regularization by smooth deterministic vector fields as well, see Remark 6.

Consider the 3D Navier–Stokes equations on the torus  $T^3$  in vorticity form

$$\partial_t \xi + \mathcal{L}_u \xi = \Delta \xi \tag{1}$$

with an initial condition  $\xi_0 \in H$ . Here *u* and  $\xi$  are the velocity and vorticity of the fluid, respectively, and  $\mathcal{L}_u \xi = u \cdot \nabla \xi - \xi \cdot \nabla u$  is the Lie derivative. We write *H* for the space of square integrable divergence free vector fields on  $\mathbf{T}^3$ , see Section 2.1 for details. Without loss of generality, we focus on unitary viscosity. On the formal level, one can study the time evolution of the enstrophy and derive the inequality

$$\frac{d}{dt} \|\xi(t)\|_{H}^{2} \le C_{1} \|\xi(t)\|_{H}^{3}.$$
(2)

This is the key inequality which provides a local bound yielding the maximal existence and uniqueness of a solution of class  $C([0, \tau); H)$ . The final time  $\tau$  is not known to be infinite or finite. We have denoted the constant by  $C_1$  to remind ourselves that this is a constant coming from unitary viscosity. The only available lower bound on  $\tau$  due to (2) is a finite value  $\tau^*(C_1, \|\xi_0\|_H)$  depending on  $C_1$  and  $\|\xi_0\|_H$ . Let v = v(t, x) be a given vector field, possibly random, periodic and divergence free; we consider the following modified model:

$$\partial_t \xi + \mathcal{L}_u \xi = \Delta \xi + \Pi (v \cdot \nabla \xi), \tag{3}$$

where  $\Pi$  is the Leray projection from  $L^2(\mathbf{T}^3, \mathbf{R}^3)$  to *H*. We stress that the added perturbation is of transport type. Therefore, for a general vector field *v*, it does not have any smoothing effect, nor does it extend the lifespan of solutions. Indeed, due to the divergence-free constraint for *v* it follows

$$\langle \Pi(v \cdot \nabla \xi), \xi \rangle_H = 0.$$

Hence, the energy type estimate on  $\|\xi(t)\|_{H}^{2}$  for this model is the same as above, namely (2) with the same constant  $C_{1}$ . Hence, a priori, the only available lower bound on the maximal time of well-posedness in H is the same value  $\tau^{*}(C_{1}, \|\xi_{0}\|_{H})$  as above.

The main result of this paper significantly improves what simple energy type estimates can do. In particular, we prove the following.

**Theorem 1.** Let K, T > 0 be given. There is a deterministic vector field v such that, for every  $\xi_0 \in H$  with  $\|\xi_0\|_H \leq K$ , the system (3) starting from the initial condition  $\xi_0$  is well-posed on [0, T].

Theorem 1 is the first result proving a regularization by *deterministic* vector fields for the 3D Navier–Stokes equations. By a different method the authors in [IXZ] obtained a regularization by a deterministic transport term for several classes of equations. However, due to the Leray projection, the Navier–Stokes system is more delicate and the method does not apply.

Several results exist on regularization by noise where the noise is white in time and multiplication operations are of Stratonovich type. Such results are especially appreciated when Stratonovich models are accepted as an idealization of real models, that is, a fast varying term is replaced by white noise in time. However, for practical purposes as well as numerical simulations this is not satisfactory since one is forced to go back and replace white noise by a suitable smooth approximation. In other words, certain stability with respect to the driving noise is necessary which translates to the so-called Wong–Zakai principle. Our proof of Theorem 1 is motivated by these considerations. More precisely, two ingredients are needed: a stochastic regularization by noise and a Wong–Zakai principle.

On the one hand, in a previous work of two members of our team, namely Flandoli and Luo [FL19], regularization by a transport noise of Stratonovich type was established for the 3D Navier–Stokes system. Remarkably, by a very delicate argument it was possible to show that a suitable noise increases the dissipation of the system with a large probability. Practically, this translates to an increased viscosity which in turn extends the lifespan of the solution. We refer to Section 3 for a more detailed discussion of the results of [FL19].

On the other hand, the Wong–Zakai principle for stochastic partial differential equations became significantly more accessible by the recent advances in the theory of rough paths. In the context of the Navier–Stokes system, the theory was developed by the other two members of our author team, namely Hofmanová and Nilssen (together with Leahy) [HLN19a, HLN19b]. In these works, certain rough perturbations of transport type were included in the model and existence of solutions was proved, proceeding by a mollification of the noise. Additional results including the Wong–Zakai principle were obtained in the two dimensional setting.

It turns out that the noise which provides regularization in [FL19] was not treated in [HLN19a, HLN19b]. More importantly, it is not clear how to obtain the necessary Wong–Zakai principle in the three dimensional setting directly by the techniques of [HLN19a, HLN19b]. Indeed, to this end it would be necessary to establish at least local-in-time uniqueness of strong solutions to the rough path formulation of the equations. Otherwise one could only formulate a certain Wong–Zakai principle up to a subsequence which depends on the randomness variable  $\omega$ . Thus, measurability with respect to  $\omega$  may be lost. We refer to Remark 13 for more details.

To this time we are not able to prove the necessary uniqueness in the rough path setting. Thus, we proceed differently. The idea is to make use of the corresponding uniqueness result in the stochastic setting, which is surprisingly easy to establish. To this end, a further combination of rough path techniques with the stochastic compactness method based on the Skorokhod representation theorem is necessary. Especially the identification of the limiting equation in our main Proposition 14 below manifests the nice interplay between probabilistic arguments based on the martingale theory and pathwise arguments relying on the theory of rough paths.

As an intermediate result towards the proof of Theorem 1 we obtain the following statement which reflects the probabilistic nature of our construction and which is interesting in its own right. To this end, let W be a complex Brownian motion on some probability space as defined in Section 2.2.

**Theorem 2.** Given  $K, T, \varepsilon > 0$ , there is a random vector field v = v(W(t), x), such that for every  $\xi_0 \in H$  with  $\|\xi_0\|_H \leq K$ , the maximal time of well-posedness in H for equation (3) with initial condition  $\xi_0$  is greater than T, with probability  $1 - \varepsilon$ .

We point out that the time regularity of the constructed vector field v in Theorem 1 is such that (3) is understood and solved in the classical deterministic sense. In other words, it is *not* a stochastic partial differential equation. It will be seen in the proof below that v involves the time derivative of a suitable regularization of W.

The paper is organized as follows. In Section 2 we make some preparations concerning the functional analytical setting, the explicit choice of a complete orthonormal system of H, and the elements of rough path theory. In Section 3 we first recall the main results of [FL19], and then state in a more precise way the equations and a series of intermediate results needed for proving Theorem 1. Section 4 is devoted to the proof of the Wong–Zakai approximation: Theorem 8, which is the main ingredient in the proof of Theorem 1.

# 2 Preliminaries

### 2.1 Function spaces

For a given  $m \in \mathbf{R}$  and  $d, D \in \mathbf{N}$ , we denote  $W^{m,2}(\mathbf{T}^d; \mathbf{R}^D) = (I - \Delta)^{-\frac{m}{2}} L^2(\mathbf{T}^d; \mathbf{R}^D)$ . We denote by  $H^m$  the subspace of  $W^{m,2}(\mathbf{T}^d; \mathbf{R}^d)$  consisting of divergence free vector fields, i.e.,

$$H^m = \left\{ f \in W^{m,2}(\mathbf{T}^d; \mathbf{R}^d); \ \nabla \cdot f = 0 \right\},\$$

and let  $\|\cdot\|_{H^m}$  be the corresponding norm. We write H for  $H^0$ . In order to analyze the convective term in the Navier–Stokes system, we employ the classical notation and bounds. Owing to Lemma 2.1 in [Tem83], the trilinear form

$$b(u,v,w) = \int_{\mathbf{T}^d} ((u \cdot \nabla)v) \cdot w \, dx = \sum_{i,j=1}^d \int_{\mathbf{T}^d} u^i \partial_{x_i} v^j w^j \, dx$$

satisfies the continuity property

 $|b(u,v,w)| \leq_{m_1,m_2,m_3,d} ||u||_{H^{m_1}} ||v||_{H^{m_2+1}} ||w||_{H^{m_3}}, \qquad m_1 + m_2 + m_3 > \frac{d}{2}, \quad m_1, m_2, m_3 \ge 0.$ (4)

Moreover, for all  $u \in H^{m_1}$  and  $(v, w) \in W^{m_2+1,2} \times W^{m_3,2}$  such that  $m_1, m_2, m_3$  satisfy (4), we have

$$b(u, v, w) = -b(u, w, v)$$
 and  $b(u, v, v) = 0.$  (5)

### **2.2** A basis of *H* and complex Brownian motions

Recall that  $\mathbb{Z}_0^3 = \mathbb{Z}^3 \setminus \{0\}$  is the nonzero lattice points. Let  $\mathbb{Z}_0^3 = \mathbb{Z}_+^3 \cup \mathbb{Z}_-^3$  be a partition of  $\mathbb{Z}_0^3$  such that

$$\mathbb{Z}^3_+ \cap \mathbb{Z}^3_- = \emptyset, \quad \mathbb{Z}^3_+ = -\mathbb{Z}^3_-$$

Let  $L_0^2(\mathbf{T}^3, \mathbf{C})$  be the space of complex valued square integrable functions on  $\mathbf{T}^3$  with zero average. It has the complete orthonormal system:

$$e_k(x) = \mathrm{e}^{2\pi\mathrm{i}k\cdot x}, \quad x \in \mathbf{T}^3, \ k \in \mathbb{Z}_0^3,$$

where i is the imaginary unit. For any  $k \in \mathbb{Z}_+^3$ , let  $\{a_{k,1}, a_{k,2}\}$  be an orthonormal basis of  $k^{\perp} := \{x \in \mathbb{R}^3 : k \cdot x = 0\}$  such that  $\{a_{k,1}, a_{k,2}, \frac{k}{|k|}\}$  is right-handed. The choice of  $\{a_{k,1}, a_{k,2}\}$  is not unique. For  $k \in \mathbb{Z}_+^3$ , we define  $a_{k,\alpha} = a_{-k,\alpha}$ ,  $\alpha = 1, 2$ . Now we can define the divergence free vector fields:

$$\sigma_{k,\alpha}(x) = a_{k,\alpha}e_k(x), \quad x \in \mathbf{T}^3, \, k \in \mathbb{Z}_0^3, \, \alpha = 1, 2.$$
(6)

Then  $\{\sigma_{k,1}, \sigma_{k,2} : k \in \mathbb{Z}_0^3\}$  is a CONS of the subspace  $H_{\mathbb{C}} \subset L_0^2(\mathbb{T}^3, \mathbb{C}^3)$  of square integrable and divergence free vector fields with zero mean. A vector field

$$v = \sum_{k \in \mathbb{Z}_0^3} \sum_{\alpha=1}^2 v_{k,\alpha} \sigma_{k,\alpha} \in H_{\mathbf{C}}$$

has real components if and only if  $\overline{v_{k,\alpha}} = v_{-k,\alpha}$ . Next we introduce the family  $\{W^{k,\alpha} : k \in \mathbb{Z}_0^3, \alpha = 1, 2\}$  of complex Brownian motions. Let

$$\left\{B^{k,\alpha}: k \in \mathbb{Z}_0^3, \, \alpha = 1, 2\right\}$$

be a family of independent standard real Brownian motions; then the complex Brownian motions can be defined as

$$W^{k,\alpha} = \begin{cases} B^{k,\alpha} + \mathrm{i}B^{-k,\alpha}, & k \in \mathbb{Z}_+^3; \\ B^{-k,\alpha} - \mathrm{i}B^{k,\alpha}, & k \in \mathbb{Z}_-^3. \end{cases}$$

Note that  $\overline{W^{k,\alpha}} = W^{-k,\alpha}$   $(k \in \mathbb{Z}_0^3, \alpha = 1, 2)$ , and they have the following quadratic covariation:

$$\langle\!\langle W^{k,\alpha}, W^{l,\beta} \rangle\!\rangle_t = 2 t \,\delta_{k,-l} \,\delta_{\alpha,\beta}, \quad k,l \in \mathbb{Z}_0^3, \,\alpha,\beta \in \{1,2\}.$$

$$\tag{7}$$

#### **Elements of rough paths theory** 2.3

Let  $I \subset \mathbf{R}$  be a bounded interval and let E be a Banach space with a norm  $\|\cdot\|_{E}$ . For a path  $g: I \to E$  we define its increment as  $\delta g_{st} := g_t - g_s$ ,  $s, t \in I$ . Let  $\Delta_I := \{(s, t) \in I^2 : s \leq t\}$ . For a two-index map  $g: \Delta_I \to E$ , we define the second order increment operator  $\delta g_{s\theta t} = g_{st} - g_{\theta t} - g_{s\theta}$ ,  $s \leq \theta \leq t$ .

Let  $\alpha > 0$ . We denote by  $C_2^{\alpha}(I; E)$  the closure of the set of smooth 2-index maps  $g: \Delta_I \to E$ with respect to the seminorm

$$[g]_{\alpha} := [g]_{\alpha,I,E} := \sup_{s,t \in \Delta_I, s \neq t} \frac{||g_{st}||_E}{|t-s|^{\alpha}} < \infty.$$

By  $C_{2,\text{loc}}^{\alpha}(I; E)$  we denote the space of 2-index maps  $g : \Delta_I \to E$  such that there exists a covering  $\{I_k\}_k$  of the interval I so that  $g \in C_2^{\alpha}(I_k; E)$  for every k. By  $C^{\alpha}(I; E)$  we denote the closure of the set of smooth paths  $g: I \to E$  with respect to the seminorm  $[\delta g]_{\alpha}$ . Note that with this definition, the spaces  $C^{\alpha}(I; \mathbf{R}^m)$  and  $C^{\alpha}_2(I; \mathbf{R}^m)$ ,  $m \in \mathbf{N}$ , are Polish.

Next, we present the definition of a rough path. A detailed exposition of rough path theory can be found in [FH14].

**Definition 3.** Let  $T > 0, m \in \mathbb{N}$  and  $\alpha \in (\frac{1}{3}, \frac{1}{2}]$ . An  $\alpha$ -rough path is a pair

$$\mathbf{Z} = (Z, \mathbb{Z}) \in C_2^{\alpha}([0, T]; \mathbf{R}^m) \times C_2^{2\alpha}([0, T]; \mathbf{R}^{m \times m})$$
(8)

satisfying the Chen's relation

$$\delta \mathbb{Z}_{s\theta t} = Z_{s\theta} \otimes Z_{\theta t}, \qquad s \le \theta \le t.$$

Given a smooth path z, there is a canonical lift to a rough path  $(Z, \mathbb{Z})$  given by

$$Z_{st} := \delta z_{st}$$
 and  $\mathbb{Z}_{st} := \int_{s}^{t} \delta z_{sr} \otimes \dot{z}_{r} dr$ ,

for which Chen's relation is readily checked. An  $\alpha$ -rough path  $\mathbf{Z} = (Z, \mathbb{Z})$  is said to be geometric if it can be obtained as the limit in the product topology  $C_2^{\alpha}([0, T]; \mathbf{R}^m) \times C_2^{2\alpha}([0, T]; \mathbf{R}^{m \times m})$  of a sequence of rough paths  $\{(Z^n, \mathbb{Z}^n)\}_{n=1}^{\infty}$  which are canonical lifts of some smooth paths  $z^n : [0, T] \to \mathbf{R}^m$ .

We proceed with a definition of an unbounded rough driver, which can be regarded as an operator valued rough path taking values in unbounded operators. In view of the application to the Navier–Stokes system we work directly with the scale of Hilbert spaces  $H^n$  defined in Section 2.1.

**Definition 4.** Let T > 0 and  $\alpha \in (\frac{1}{3}, \frac{1}{2}]$ . An unbounded  $\alpha$ -rough driver is a pair  $\mathbf{A} = (A^1, A^2)$  of 2-index maps satisfying: there exists a constant  $C_A > 0$  such that for every  $0 \le s \le t \le T$ 

$$|A_{st}^{1}|_{\mathcal{L}(H^{-n},H^{-(n+1)})} \leq C_{A}|t-s|^{\alpha} \text{ for } n \in \{0,2\},$$

$$|A_{st}^{2}|_{\mathcal{L}(H^{-n},H^{-(n+2)})} \leq C_{A}|t-s|^{2\alpha} \text{ for } n \in \{0,1\},$$
(9)

and Chen's relation holds true, namely,

$$\delta A_{s\theta t}^{1} = 0, \qquad \delta A_{s\theta t}^{2} = A_{\theta t}^{1} A_{s\theta}^{1}, \qquad 0 \le s \le \theta \le t \le T.$$
<sup>(10)</sup>

The partial differential equations of interest in this paper can be written in the abstract form

$$dg_t = d\mu_t + \mathbf{A}(dt)g_t,\tag{11}$$

where  $\mu$  denotes the corresponding drift (of appropriate spatial regularity) and **A** is an unbounded  $\alpha$ -rough driver. We say that a path  $g : [0, T] \to H$  is a solution to (11) provided the 2-index map

$$g_{st}^{\natural} := \delta g_{st} - \delta \mu_{st} - A_{st}^1 g_s - A_{st}^2 g_s, \qquad 0 \le s \le t \le T,$$

belongs to  $C_{2,\text{loc}}^{3\alpha}([0,T];H^{-3})$ .

We conclude this section with the main a priori estimate which can be proved following the lines of [DGHT19, Corollary 2.11], cf. [HLN19a, HLN19b]. The bound which holds global in time is a consequence of the local estimate combined with [FH14, Exercise 4.24].

### **Theorem 5.** Assume

- $\mathbf{A} = (A^1, A^2)$  is an unbounded  $\alpha$ -rough driver for some  $\alpha \in (\frac{1}{3}, \frac{1}{2}]$ ;
- $\mu: [0,T] \to H^{-2}$  is Lipschitz continuous and  $\mu: [0,T] \to H^{-1}$  is  $\frac{1}{2}$ -Hölder continuous, i.e.,

$$\|\delta\mu_{st}\|_{H^{-2}} \le C_{\mu,1}|t-s|$$
 and  $\|\delta\mu_{st}\|_{H^{-1}} \le C_{\mu,2}|t-s|^{\frac{1}{2}}$ 

for some constants  $C_{\mu,1}$  and  $C_{\mu,2}$ ;

• a bounded path  $g : [0,T] \rightarrow H$  is a solution to (11).

Then there exists a constant L > 0 such that whenever  $C_A |t - s|^{\alpha} \leq L$  we have

$$\begin{split} \|g_{st}^{\natural}\|_{H^{-3}} &\lesssim \left(|g|_{L_{T}^{\infty}H}(1+C_{A}^{2})+C_{\mu,1}C_{A}\right)|t-s|^{3\alpha}, \\ \|\delta g_{st}\|_{H^{-1}} &\lesssim \left(|g|_{L_{T}^{\infty}H}(1+C_{A}^{2})+C_{\mu,2}+C_{\mu,1}C_{A}\right)|t-s|^{\alpha}, \end{split}$$

$$\|\delta g_{st}\|_{H^{-1}} \lesssim \left(|g|_{L^{\infty}_{T}H}(1+C^{2}_{A})+C_{\mu,2}+C_{\mu,1}C_{A}\right)\left(1+C^{\frac{1-\alpha}{\alpha}}_{A}\right)|t-s|^{\alpha}$$

# **3** Formulation of the main results

Let  $\ell^2 = \ell^2(\mathbb{Z}_0^3)$  be the space of square summable sequences indexed by  $\mathbb{Z}_0^3$ ; it is equipped with the norm  $\|\cdot\|_{\ell^2}$ . For any  $N \in \mathbb{Z}_+$ , take  $\theta^N \in \ell^2$  such that

$$heta_k^N = rac{\mathbf{1}_{\{N \leq |k| \leq 2N\}}}{|k|^\gamma}, \quad k \in \mathbb{Z}_0^3,$$

where  $\gamma > 0$  is some fixed constant. On a given time interval [0, T], we consider the stochastic 3D Navier–Stokes equations with a transport type noise:

$$d\xi^{N} + \mathcal{L}_{u^{N}}\xi^{N} dt = \Delta\xi^{N} dt + \frac{C_{\nu}}{\|\theta^{N}\|_{\ell^{2}}} \sum_{k \in \mathbb{Z}_{0}^{3}} \sum_{\alpha=1}^{2} \theta_{k}^{N} \Pi(\sigma_{k,\alpha} \cdot \nabla\xi^{N}) \circ dW_{t}^{k,\alpha},$$
(12)

with  $\xi^N(0) = \xi_0 \in H$ . Here  $C_\nu = \sqrt{3\nu/2}$  with some given  $\nu > 0$ , which will be chosen depending on the size of the initial conditions.

As already discussed in the introduction, due to the presence of the nonlinear term, these equations have only local solutions in *H*. Thus we make use of a cut-off technique. For R > 0, let  $f_R \in C_b^1([0,\infty), [0,1])$  be a non-increasing function such that it is identically 1 on [0, R] and vanishes on  $[R + 1, \infty)$ . Consider the equations with cut-off:

$$d\xi_R^N + f_R(\|\xi_R^N\|_{-\delta})\mathcal{L}_{u_R^N}\xi_R^N dt = \Delta\xi_R^N dt + \frac{C_\nu}{\|\theta^N\|_{\ell^2}} \sum_{k \in \mathbb{Z}_0^3} \sum_{\alpha=1}^2 \theta_k^N \Pi(\sigma_{k,\alpha} \cdot \nabla\xi_R^N) \circ dW_t^{k,\alpha},$$
(13)

where  $\|\cdot\|_{-\delta} = \|\cdot\|_{H^{-\delta}}$ . It was shown in Theorem 1.3 of [FL19] that, for every  $N \ge 1$  and  $\xi_R^N(0) = \xi_0 \in H$  with  $\|\xi_0\|_H \le K$ , the above equation has a pathwise unique solution satisfying

$$\sup_{t \in [0,T]} \|\xi_R^N(t)\|_H^2 + \int_0^T \|\nabla \xi_R^N(t)\|_H^2 \, dt \le C(R,K),\tag{14}$$

where C(R, K) > 0 is some constant.

Moreover, given K > 0, we deduce from [FL19, Theorem 1.4] that for all R > 0 and  $\nu > 0$  big enough, for any  $\varepsilon > 0$ , it holds

$$\lim_{N \to \infty} \sup_{\xi_0 \in H, \|\xi_0\|_H \le K} \mathbb{P} \left( \sup_{t \in [0,T]} \left\| \xi_R^N(t,\xi_0) - \xi(t,\xi_0) \right\|_{-\delta} > \varepsilon \right) = 0,$$
(15)

where  $\xi(t,\xi_0)$  is the unique solution to the following deterministic 3D Navier–Stokes equation

$$\partial_t \xi + \mathcal{L}_u \xi = \left(1 + \frac{3}{5}\nu\right) \Delta \xi, \quad \xi(0) = \xi_0.$$
(16)

It is well known that, for given K > 0, there exists  $\nu > 0$  such that for all  $\xi_0 \in H$  with  $\|\xi_0\|_H \leq K$ , the equation (16) admits a unique solution satisfying

$$\sup_{t \in [0,T]} \|\xi(t)\|_{H}^{2} + \int_{0}^{T} \|\nabla\xi(t)\|_{H}^{2} dt \le C(K)^{2}.$$
(17)

In the sequel, K and a corresponding v will be considered as fixed.

Choose  $R_K = C(K) + 2$ ; we deduce from the assertions (15) and (17) that, given  $\varepsilon > 0$ , there is  $N_0 = N_0(K, \varepsilon) \in \mathbb{Z}_+$  such that for all  $N \ge N_0$ , for all  $\xi_0 \in H$  with  $\|\xi_0\|_H \le K$ ,

$$\mathbb{P}\left(\sup_{t\in[0,T]}\left\|\xi_{R_{K}}^{N}(t,\xi_{0})\right\|_{-\delta}\leq R_{K}-1\right)\geq1-\varepsilon,$$
(18)

where we have used the fact that  $||y||_{-\delta} \le ||y||_H$ . This implies that, for every  $N \ge N_0$ ,  $\xi_{R_K}^N$  solves the equation (12) without cut-off with a probability greater than  $1 - \varepsilon$ . Next, for  $k \in \mathbb{Z}_0^3$  and  $\alpha = 1, 2$ , let  $\{W_t^{k,\alpha,n}\}_{n \ge 1}$  be a piecewise linear approximation of the

Brownian motion  $W_t^{k,\alpha}$ .

**Remark 6.** Alternatively, we may replace piecewise linear approximations by mollifications, in which case the obtained vector fields v in Theorem 1 and Theorem 2 are smooth.

We consider the 3D Navier-Stokes equations with smooth random force

$$\partial_t \xi^{N,n} + \mathcal{L}_{u^{N,n}} \xi^{N,n} = \Delta \xi^{N,n} + \frac{C_{\nu}}{\|\theta^N\|_{\ell^2}} \sum_{k \in \mathbb{Z}_0^3} \sum_{\alpha=1}^2 \theta_k^N \Pi(\sigma_{k,\alpha} \cdot \nabla \xi^{N,n}) \partial_t W_t^{k,\alpha,n}, \tag{19}$$

as well as the equations with cut-off

$$\partial_{t}\xi_{R}^{N,n} + f_{R}(\|\xi_{R}^{N,n}\|_{-\delta})\mathcal{L}_{u_{R}^{N,n}}\xi_{R}^{N,n} = \Delta\xi_{R}^{N,n} + \frac{C_{\nu}}{\|\theta^{N}\|_{\ell^{2}}} \sum_{k \in \mathbb{Z}_{0}^{3}} \sum_{\alpha=1}^{2} \theta_{k}^{N} \Pi(\sigma_{k,\alpha} \cdot \nabla\xi_{R}^{N,n}) \partial_{t}W_{t}^{k,\alpha,n}.$$
(20)

Similarly to (13), for any  $\xi_R^{N,n}(0) = \xi_0 \in H$  with  $\|\xi_0\|_H \leq K$ , the latter equation admits a unique solution verifying

$$\sup_{t \in [0,T]} \|\xi_R^{N,n}(t)\|_H^2 + \int_0^T \|\nabla \xi_R^{N,n}(t)\|_H^2 dt \le C(R,K).$$
(21)

Note that the estimate (21) depends only on *R* and the bound *K* of the initial condition  $\xi_0 \in H$  and is independent of *N*, *n*. The basis for our Wong–Zakai result is obtained by techniques from rough path theory developed in [HLN19a, HLN19b], which allow us to derive additional estimates uniform in *n* (see Section 2.3 and Proposition 12). However, rough path theory alone is not sufficient to conclude. In particular, the obtained bounds only permit to deduce *relative* compactness of realizations of the approximate sequence of solutions  $\{\xi_{R_K}^{N_0,n}(\omega)\}_{n\geq 1}$  and the convergence follows only for a subsequence which depends on  $\omega$ .

In order to obtain convergence of the full sequence, one would need to establish uniqueness of the rough path formulation of the limiting (as  $n \to \infty$ ) equation (13). This is a very challenging problem which remains open. The main difficulty lies in the presence of the Leray projection which is not compatible with the tenzorization technique developed in [DGHT19] to prove uniqueness of variational rough PDEs.

To overcome this issue, we reach back to probability theory and proceed by a stochastic compactness argument relying on Skorokhod representation theorem. The point is that uniqueness for the stochastic formulation of (13) follows by classical arguments. Nevertheless, it is necessary to preserve the rough path formulation of the equations in the core of the proof, as this is the setting where we are able to rigorously obtain the convergence of (20) to (13). The final step entails a new identification of the limit procedure which combines martingale and rough path arguments.

More precisely, first we prove the following result (see *Step 1* of the proof of Proposition 14).

**Lemma 7.** Assume that the sequence  $\{\xi_0^n\}_{n\geq 1} \subset H$  satisfies  $\|\xi_0^n\|_H \leq K$  for all  $n \geq 1$ . Let  $\xi_{R_K}^{N_0,n}$  be the unique solution to (20) with  $N = N_0$ ,  $R = R_K$  and  $\xi_{R_K}^{N_0,n}(0) = \xi_0^n$ . Then the family of laws of  $\xi_{R_K}^{N_0,n}$  is tight in  $C([0,T]; H^{-\delta}) \cap L^2(0,T; H)$ .

Consequently, our main Wong–Zakai approximation result proved in Section 4 reads as follows. **Theorem 8.** Let  $\xi_{R_K}^{N_0,n}(t,\xi_0)$  (resp.  $\xi_{R_K}^{N_0}(t,\xi_0)$ ) be the unique solution to (20) (resp. (13)) with the initial value  $\xi_0 \in H$ . Then for any  $\varepsilon > 0$ ,

$$\lim_{n\to\infty}\sup_{\xi_0\in H, \|\xi_0\|_H\leq K}\mathbb{P}\bigg(\sup_{t\in[0,T]}\big\|\xi_{R_K}^{N_0,n}(t,\xi_0)-\xi_{R_K}^{N_0}(t,\xi_0)\big\|_{-\delta}>\varepsilon\bigg)=0.$$

Combining this assertion with (18), we obtain the following.

**Corollary 9.** Given  $\varepsilon > 0$  there is  $n_0 = n_0(K, \varepsilon)$  such that for every  $n \ge n_0$ , for all  $\xi_0 \in H$  with  $\|\xi_0\|_H \le K$ ,

$$\mathbb{P}\left(\sup_{t\in[0,T]}\left\|\xi_{R_{K}}^{N_{0},n}(t,\xi_{0})\right\|_{-\delta}\leq R_{K}\right)\geq 1-2\varepsilon.$$

In particular,  $\xi_{R_K}^{N_0,n_0}$  is a global (namely on [0,T]) solution of equation (19) with large probability.

Remark that  $\xi_{R_K}^{N_0,n_0}$  satisfies also the bound (21). Since the involved rough paths are smooth, one can follow the classical arguments to prove the uniqueness of solutions with such bounds, see for instance [FL19, Theorem 1.3].

**Lemma 10.** Consider equation (19) for some fixed values of the parameters N, n. Assume it has a weak solution  $\xi^{N,n}$  on some interval [0,T] so that,  $\mathbb{P}$ -a.s.,

$$\sup_{t \in [0,T]} \left\| \xi^{N,n}(t) \right\|_{H}^{2} + \int_{0}^{T} \left\| \nabla \xi^{N,n}(t) \right\|_{H}^{2} dt \leq C < \infty.$$

Then this solution is pathwise unique.

Next, we derive the following consequence.

**Corollary 11.** Given K > 0 and  $\varepsilon > 0$ , there are  $N_0$  and  $n_0$  such that for all  $\xi_0 \in H$  with  $\|\xi_0\|_H \leq K$ , the maximal time  $\tau_{N_0,n_0}(\xi_0)$  of existence and uniqueness for equation (19) in H satisfies

$$\mathbb{P}(\tau_{N_0,n_0}(\xi_0) \ge T) \ge 1 - \varepsilon$$

And this permits to conclude the proof of Theorem 2 since it now suffices to define

$$v(t,x) := \frac{C_{\nu}}{\|\theta^{N_0}\|_{\ell^2}} \sum_{k \in \mathbb{Z}_0^3} \sum_{\alpha=1}^2 \theta_k^{N_0} \sigma_{k,\alpha}(x) \partial_t W_t^{k,\alpha,n_0}.$$

Finally, choosing one trajectory  $\omega$  from the set  $\{\tau_{N_0,n_0}(\xi_0) \ge T\}$  completes the proof of Theorem 1.

# 4 Wong–Zakai result: the proof of Theorem 8

Throughout this section, the parameter N is kept fixed and omitted for notational simplicity. In order to simplify the notations, we will also modify the sub/superscrips from the notations of the previous section.

Fix a finite dimensional Brownian motion  $W = (W^{k,\alpha})_{k,\alpha}$ , with  $k \in \operatorname{supp} \theta^N$ ,  $\alpha = 1, 2$ , on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and let  $(\mathcal{F}_t)_{t\geq 0}$  be its normal filtration. Let  $W^n = (W^{k,\alpha,n})_{k,\alpha}$  be its piecewise linear approximation based on a sequence of partitions  $(\pi^n)_{n\in\mathbb{N}}$  of the interval [0, T] with vanishing mesh size  $h^n = O(\frac{1}{n})$ . In other words, the sample paths of  $W^n$  are of bounded variation and the noise term in (20) is given by the classical Riemann integral. We note that the piecewise linear approximation is not adapted to  $(\mathcal{F}_t)_{t\geq 0}$ , but it is adapted to  $(\mathcal{F}_{t+h^n})_{t\geq 0}$ .

The proof of Theorem 8 relies on the framework of unbounded rough drivers as developed in [BG17], [DGHT19] and in the context of the Navier–Stokes system in [HLN19a], [HLN19b]. To be more precise, recalling that due to the cut-off  $\theta^N$ , the considered noise is finite dimensional, integrating (20) in time over an interval  $[s, t] \subset [0, T]$  and iterating the equation into itself we may rewrite (20) as

$$\delta\xi_{st}^{R,n} = \int_{s}^{t} \left[\Delta\xi_{r}^{R,n} - f_{R}(\|\xi_{r}^{R,n}\|_{-\delta})\mathcal{L}_{u_{r}^{R,n}}\xi_{r}^{R,n}\right]dr + A_{st}^{n,1}\xi_{s}^{R,n} + A_{st}^{n,2}\xi_{s}^{R,n} + \xi_{st}^{R,n,\natural}$$
(22)

with unbounded rough drivers

$$A_{st}^{n,1}\phi = \frac{C_{\nu}}{\|\theta^N\|_{\ell^2}} \sum_{k \in \mathbb{Z}_0^3} \sum_{\alpha=1}^2 \theta_k^N \Pi(\sigma_{k,\alpha} \cdot \nabla \phi) \delta W_{st}^{k,\alpha,n}$$
(23)

$$A_{st}^{n,2}\phi = \frac{C_{\nu}^2}{\|\theta^N\|_{\ell^2}^2} \sum_{k,l \in \mathbb{Z}_0^3} \sum_{\alpha,\beta=1}^2 \theta_k^N \Pi(\sigma_{k,\alpha} \cdot \nabla[\theta_\ell^N \Pi(\sigma_{\ell,\beta} \cdot \nabla\phi)]) \mathbb{W}_{st}^{\ell,k,\beta,\alpha,n}$$
(24)

where

$$\mathbb{W}_{st}^{\ell,k,\beta,\alpha,n} = \int_{s}^{t} \delta W_{sr}^{\ell,\beta,n} \dot{W}_{r}^{k,\alpha,n} dr.$$

A detailed discussion of this step can be found in Section 2.5 in [HLN19a]. We recall that the term  $\xi^{R,n,\natural}$  is defined through (22) and shall be a remainder in the sense that it has sufficient time regularity, namely,  $\xi^{R,n,\natural} \in C_{2,\text{loc}}^{3\alpha}([0,T]; H^{-3})$ .

According to Exercise 10.14 in [FH14], the approximate rough path  $(W^n, W^n)$  converges to (W, W) in the rough path topology  $\mathbb{P}$ -a.s. and in every  $L^q(\Omega)$  for  $q \in [1, \infty)$ . Consequently, we deduce the convergence of the associated unbounded rough drivers  $(A^{n,1}, A^{n,2})$  to the limit unbounded rough driver given by

$$A_{st}^{1}\phi = \frac{C_{\nu}}{\|\theta^{N}\|_{\ell^{2}}} \sum_{k \in \mathbb{Z}_{0}^{3}} \sum_{\alpha=1}^{2} \theta_{k}^{N} \Pi(\sigma_{k,\alpha} \cdot \nabla \phi) \delta W_{st}^{k,\alpha}$$
(25)

$$A_{st}^{2}\phi = \frac{C_{\nu}^{2}}{\|\theta^{N}\|_{\ell^{2}}^{2}} \sum_{k,l \in \mathbb{Z}_{0}^{3}} \sum_{\alpha,\beta=1}^{2} \theta_{k}^{N} \Pi(\sigma_{k,\alpha} \cdot \nabla[\theta_{\ell}^{N}\Pi(\sigma_{\ell,\beta} \cdot \nabla\phi)]) \mathbb{W}_{st}^{\ell,k,\beta,\alpha}$$
(26)

where the associated rough path (W, W) corresponds to the Stratonovich lift, i.e.,

$$\mathbb{W}_{st}^{\ell,k,\beta,lpha} = \int_{s}^{t} \delta W_{sr}^{\ell,\beta} \circ dW_{r}^{k,lpha}.$$

This means that the operators  $(A^{n,1}, A^{n,2})$  satisfy the bounds

$$\|A_{st}^{n,1}\|_{\mathcal{L}(H^{-k};H^{-(k+1)})} \lesssim C_{A^n}|t-s|^{\alpha}, \qquad \|A_{st}^{n,2}\|_{\mathcal{L}(H^{-k};H^{-(k+2)})} \leq C_{A^n}|t-s|^{2\alpha},$$

where the first bound holds for  $k \in \{0, 2\}$  whereas the second one for  $k \in \{0, 1\}$ . In addition, by Exercise 10.14 in [FH14] we have for all  $q \in [1, \infty)$ 

$$\sup_{n \in \mathbb{N}} \mathbb{E}[C_{A^n}^q] < \infty \tag{27}$$

and  $\mathbb{P}$ -a.s.

$$\sup_{n \in \mathbf{N}} C_{A^n}(\omega) \le C(\omega) \tag{28}$$

for some random constant  $C(\omega)$ . Accordingly, the rough path formulation of (13) reads as

$$\delta\xi_{st}^{R} = \int_{s}^{t} [\Delta\xi_{r}^{R} - f_{R}(\|\xi_{r}^{R}\|_{-\delta})\mathcal{L}_{u_{r}^{R}}\xi_{r}^{R}]dr + A_{st}^{1}\xi_{s}^{R} + A_{st}^{2}\xi_{s}^{R} + \xi_{st}^{R,\natural}.$$
(29)

In view of Corollary 5.2 in [FH14], we aim to conclude that an adapted rough path solution to (13) is also a solution in the classical (stochastic) sense. Since the adaptedness is the key point needed for the construction of the stochastic integral, we have to make sure that our Wong–Zakai convergence result produces adapted solutions. By merely pathwise arguments we are not able to construct adapted solutions to (29). The principal difficulty is that we are not able to prove uniqueness of rough path solutions to (29) and therefore a pathwise compactness argument does not preserve measurability in  $\omega$ . To overcome this obstacle, we combine rough path techniques together with probabilistic arguments, namely, the stochastic compactness method based on Skorokhod representation theorem. This permits to make use of the uniqueness for the stochastic version of (29), i.e., the equation (13), and eventually construct adapted solutions to (29).

An additional technical difficulty follows from the fact that the approximation  $W^n$  is not adapted to  $(\mathcal{F}_t)_{t\geq 0}$  but only to  $(\mathcal{F}_{t+h^n})_{t\geq 0}$ . While this point can be fixed for instance by replacing piecewise linear approximations by one-sided mollifications which remain adapted, we choose to work with piecewise linear approximations as they are better suited for applications in numerical analysis.

As the first step, we establish the necessary uniform estimates.

**Proposition 12.** There exists a unique solution  $\xi^{R,n}$  to (22) and it is adapted to  $(\mathcal{F}_{t+h^n})_{t\geq 0}$ . Moreover, it holds

$$\begin{aligned} \|\xi^{R,n}\|_{L^{\infty}_{T}H}^{2} + \|\xi^{R,n}\|_{L^{2}_{T}H^{1}}^{2} &\leq C(R,K), \\ \|\xi^{R,n}\|_{C^{\alpha}_{T}H^{-1}} &\leq (1+C(R,K))(1+C^{2}_{A^{n}}). \end{aligned}$$

and there exists a deterministic constant L > 0 such that whenever  $C_{A^n} |t - s|^{3\alpha} \leq L$  we have

$$\|\xi_{st}^{R,n,\natural}\|_{H^{-3}} \lesssim (1+C(R,K))(1+C_{A^n}^2)|t-s|^{3\alpha},$$

for some deterministic implicit constant independent of n.

*Proof.* Existence and uniqueness of a solution  $\xi^{R,n}$  follows by classical arguments since the driver  $(A^{n,1}, A^{n,2})$  is smooth. Since  $W^n$  is adapted to  $(\mathcal{F}_{t+h^n})_{t\geq 0}$ , the same remains valid for the solution  $\xi^{R,n}$ .

The first bound in the statement of the proposition follows from (21). For the other two estimates, we intend to apply Theorem 5. Thus, we shall derive the necessary bounds for the drift term

$$\delta\mu_{st} = \int_{s}^{t} \left[\Delta\xi_{r}^{R,n} - f_{R}\left(\|\xi_{r}^{R,n}\|_{H^{-\delta}}\right) \mathcal{L}_{u_{r}^{R,n}}\xi_{r}^{R,n}\right] dr$$

in  $H^{-2}$  and  $H^{-1}$ . To this end, we observe that since  $u^{R,n}$  is divergence free we have

$$\begin{aligned} \langle \mathcal{L}_{u_r^{R,n}} \xi_r^{R,n}, \phi \rangle &= \langle (u_r^{R,n} \cdot \nabla) \xi_r^{R,n}, \phi \rangle - \langle (\xi_r^{R,n} \cdot \nabla) u_r^{R,n}, \phi \rangle \\ &= -\langle (u_r^{R,n} \cdot \nabla) \phi, \xi_r^{R,n} \rangle - \langle (\xi_r^{R,n} \cdot \nabla) u_r^{R,n}, \phi \rangle. \end{aligned}$$

Hence in view of (4) with  $m_1 = m_2 = 1$ ,  $m_3 = 0$  for the first term and  $m_1 = m_2 = 0$ ,  $m_3 = 2$  for the second term we obtain

$$\|\mathcal{L}_{u_r^{R,n}}\xi_r^{R,n}\|_{H^{-2}} \lesssim \|\xi_r^{R,n}\|_H (1+\|u_r^{R,n}\|_{H^1}).$$

Therefore due to (21)

$$\|\delta\mu_{st}\|_{H^{-2}} \lesssim \int_{s}^{t} \|\xi_{r}^{R,n}\|_{H} (1+\|u_{r}^{R,n}\|_{H^{1}}) dr \lesssim (t-s)(1+C(R,K)),$$

with a deterministic implicit constant and Theorem 5 implies

$$\|\xi_{st}^{R,n,\natural}\|_{H^{-3}} \lesssim (1+C(R,K))(1+C_{A^n}^2)|t-s|^{3\alpha},$$

which gives the desired bound of the remainder.

Finally, we observe that by (4) the drift can be estimated in  $H^{-1}$  as follows

$$\begin{split} \|\delta\mu_{st}\|_{H^{-1}} &\lesssim \int_{s}^{t} \|\xi_{r}^{R,n}\|_{H^{1}} (1+\|u_{r}^{R,n}\|_{H^{1}}) dr \\ &\lesssim (t-s)^{\frac{1}{2}} |\xi^{R,n}|_{L^{2}_{T}H^{1}} (1+|u^{R,n}|_{L^{\infty}_{T}H^{1}}) \lesssim |t-s|^{\frac{1}{2}} (1+C(R,K)). \end{split}$$

Hence Theorem 5 implies

$$|\xi^{R,n}|_{C^{\alpha}_{T}H^{-1}} \leq (1 + C(R,K))(1 + C^{2}_{A^{n}})$$

and the proof is complete.

**Remark 13.** In view of the above uniform bound which holds true for  $\mathbb{P}$ -a.e.  $\omega$ , it is tempting to apply the Aubin–Lions compactness theorem for every such  $\omega$  and take the limit to immediately get  $\mathbb{P}$ -a.s. convergence to the solution of (29). However, the Aubin–Lions theorem gives only a converging subsequence,  $\{\xi^{R,n_k}(\omega)\}_k \subset \{\xi^{R,n}(\omega)\}_n$  and we note that this subsequence depends on  $\omega$ . The choice of subsequence is in general not a measurable map, and it is thus not clear whether one can obtain measurability of the limit. To circumvent this problem, we use the Skorokhod representation theorem to infer almost sure convergence, but at the price of changing the underlying probability space.

Now, we have all in hand to prove the following result (recall that we fix  $N \in \mathbb{Z}_+$  in this section).

**Proposition 14.** Let  $\{\xi_0^n\}_{n\geq 1} \subset H$  be a sequence satisfying  $\|\xi_0^n\|_H \leq K$  for all  $n \geq 1$ , and  $\xi^{R,n}$  the unique solution to (20) with  $\xi^{R,n}(0) = \xi_0^n$ . Assume that  $\xi_0^n$  converges weakly in H to some  $\xi_0$  as  $n \to \infty$ ; then  $\xi^{R,n}$  converge in probability in the topology of  $C([0,T]; H^{-\delta})$  to  $\xi^R$ , the solution to (13) with initial value  $\xi_0$ .

|  | - | - | - |
|--|---|---|---|

Proof. Step 1: Tightness. We define the space

$$\begin{split} & \mathcal{X} := \mathcal{X}_{\xi} \times \mathcal{X}_{\mathrm{RP}}, \\ & \mathcal{X}_{\xi} := L^2(0,T;H) \cap C([0,T];H^{-\delta}), \qquad \mathcal{X}_{\mathrm{RP}} := C_2^{\alpha}([0,T];\mathbf{R}^m) \times C_2^{2\alpha}([0,T];\mathbf{R}^{m \times m}), \end{split}$$

where  $m \in \mathbb{N}$  is the dimension of the Brownian motion W. A version of the Aubin–Lions compactness theorem shows that  $L^{\infty}(0, T; H) \cap L^2(0, T; H^1) \cap C^{\alpha}([0, T]; H^{-1})$  is compactly embedded into  $X_{\xi}$ . Consequently, due to Proposition 12 and in particular due to the fact that the right hand sides of the estimates are uniformly bounded in expectation due to (27), the family of the pushforward measures  $(\xi^{R,n})_{\sharp}\mathbb{P}$  is tight on  $X_{\xi}$ . In order to apply the theory of rough paths for the passage limit we shall also need the structure of the noise. Since due to Exercise 10.14 in [FH14], the approximate rough path  $(W^n, W^n)$  converges  $\mathbb{P}$ -a.s. in the rough path topology to the Stratonovich lift of a Brownian motion W, the family of joint laws of  $(W^n, W^n)$  is tight on  $X_{RP}$  which is separable. Thus, we deduce that the joint laws  $(\xi^{R,n}, W^n, W^n)_{\sharp}\mathbb{P}$  are tight as a family of probability measures on X.

From the Skorokhod representation theorem which applies to Polish spaces (see e.g. Section 2.6 in [BFH18]) there exists a probability space  $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$  and random variables

$$(\tilde{\xi}^{R,n}, \tilde{W}^n, \tilde{W}^n) : \tilde{\Omega} \to \mathcal{X}, \quad n \in \mathbf{N}, \qquad (\tilde{\xi}^R, \tilde{W}, \tilde{W}) : \tilde{\Omega} \to \mathcal{X},$$

such that (up to a subsequence)

- (i)  $(\tilde{\xi}^{R,n}, \tilde{W}^n, \tilde{W}^n) \to (\tilde{\xi}^R, \tilde{W}, \tilde{W})$  in  $\mathcal{X} \tilde{\mathbb{P}}$ -a.s. as  $n \to \infty$ ,
- (*ii*)  $(\tilde{\xi}^{R,n}, \tilde{W}^n, \tilde{W}^n)_{\sharp} \tilde{\mathbb{P}} = (\xi^{R,n}, W^n, W^n)_{\sharp} \mathbb{P}$  for all  $n \in \mathbb{N}$ .

We define  $(\tilde{\mathcal{F}}_t)_{t \ge 0}$  as the augmented canonical filtration generated by  $(\tilde{\xi}^R, \tilde{W}, \tilde{W})$ , that is, we let

$$\tilde{\mathcal{F}}_t := \sigma(\sigma(\tilde{\xi}_s^R, \tilde{W}_{rs}, \tilde{W}_{rs}; 0 \le r \le s \le t) \cup \{N; \tilde{\mathbb{P}}(N) = 0\}), \qquad t \ge 0.$$

Step 2: Passage to the limit. As the next step, we shall prove that  $(\tilde{\xi}^{R,n}, \tilde{W}^n, \tilde{W}^n)$  gives rise to a solution of (22) on the new probability space. First, we shall identify the corresponding rough path. To this end, we observe that Chen's relation giving the necessary compatibility condition between components of a rough path holds for  $(\tilde{W}^n, \tilde{W}^n)$  as well. Indeed, it follows from the equality of laws

$$\tilde{\mathbb{P}}\left(\delta \tilde{\mathbb{W}}_{rst}^{n} = \tilde{W}_{rs}^{n} \otimes \tilde{W}_{st}^{n} \quad \text{for all} \quad 0 \leq r \leq s \leq t \leq T\right) \\ = \mathbb{P}\left(\delta \mathbb{W}_{rst}^{n} = W_{rs}^{n} \otimes W_{st}^{n} \quad \text{for all} \quad 0 \leq r \leq s \leq t \leq T\right) = 1.$$

In other words,  $(\tilde{W}^n, \tilde{W}^n)$  is a well-defined rough path  $\tilde{\mathbb{P}}$ -a.s. Hence we may define the unbounded rough drivers  $(\tilde{A}^{n,1}, \tilde{A}^{n,2})$  through the formulas (23), (24) with  $(W^n, W^n)$  replaced by  $(\tilde{W}^n, \tilde{W}^n)$ .

Let us now define

$$\tilde{\xi}_{st}^{R,n,\natural} := \delta \tilde{\xi}_{st}^{R,n} - \int_{s}^{t} \left[ \Delta \tilde{\xi}_{r}^{R,n} - f_{R} \big( \| \tilde{\xi}_{r}^{R,n} \|_{H^{-\delta}} \big) \mathcal{L}_{u_{r}^{R,n}} \tilde{\xi}_{r}^{R,n} \right] dr - \tilde{A}_{st}^{n,1} \tilde{\xi}_{s}^{R,n} - \tilde{A}_{st}^{n,2} \tilde{\xi}_{s}^{R,n}.$$
(30)

Recall that (22) was satisfied on the original probability space even in the classical formulation (20) as the driving path  $W^n$  is regular. In addition, the right hand side of (30) is a measurable function of  $(\tilde{\xi}^{R,n}, \tilde{W}^n, \tilde{W}^n)$ , we deduce again by the equality of joint laws that  $\tilde{\xi}^{R,n}$  is  $\tilde{\mathbb{P}}$ -a.s. a solution to

$$\delta \tilde{\xi}_{st}^{R,n} = \int_{s}^{t} [\Delta \tilde{\xi}_{r}^{R,n} - f_{R}(\|\tilde{\xi}_{r}^{R,n}\|_{H^{-\delta}}) \mathcal{L}_{u_{r}^{R,n}} \tilde{\xi}_{r}^{R,n}] dr + \tilde{A}_{st}^{n,1} \tilde{\xi}_{s}^{R,n} + \tilde{A}_{st}^{n,2} \tilde{\xi}_{s}^{R,n} + \tilde{\xi}_{st}^{R,n,\natural},$$
(31)

which is the rough path formulation of (20) on the probability space  $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ .

Our goal is to pass to the limit in (31) and identify the limit  $\xi^R$  as a solution to (29) where the rough driver is given by the Stratonovich lift of a Brownian motion. To this end, we note that the approach of Proposition 12 giving the uniform bounds can be applied to (31) as well. In particular, in view of (28) we obtain a pathwise uniform bound for the remainders  $\xi^{R,n,\natural}$ . Together with the  $\tilde{\mathbb{P}}$ -a.s. convergence of  $(\tilde{W}^n, \tilde{W}^n) \rightarrow (\tilde{W}, \tilde{W})$  in  $\chi_{RP}$ , this permits to pass to the limit in (31). Note in particular that passing to the limit in the Chen's relation guarantees that the limit  $(\tilde{W}, \tilde{W})$  is a rough path itself. Therefore, we obtain the convergence of the rough driver  $(\tilde{A}^{n,1}, \tilde{A}^{n,2})$  to  $(\tilde{A}^1, \tilde{A}^2)$  given by (25), (26) with (W, W) replaced by  $(\tilde{W}, \tilde{W})$ . Hence the limit satisfies

$$\delta \tilde{\xi}_{st}^{R} = \int_{s}^{t} \left[ \Delta \tilde{\xi}_{r}^{R} - f_{R} (\|\tilde{\xi}_{r}^{R}\|_{H^{-\delta}}) \mathcal{L}_{u_{r}^{R}} \tilde{\xi}_{r}^{R} \right] dr + \tilde{A}_{st}^{1} \tilde{\xi}_{s}^{R} + \tilde{A}_{st}^{2} \tilde{\xi}_{s}^{R} + \tilde{\xi}_{st}^{R,\natural}$$
(32)

for some remainder  $\tilde{\xi}^{R,\natural}$  which belongs  $\tilde{\mathbb{P}}$ -a.s. to  $C_{2,\text{loc}}^{3\alpha}([0,T];H^{-3})$ .

Step 3: Identification of the limiting driver. We have shown that the limit  $\tilde{\xi}^R$  solves the rough path formulation of (13) and it only remains to prove that it is also a solution of (13) in the classical stochastic sense. To this end, it is necessary to identify  $(\tilde{W}, \tilde{W})$  as the Stratonovich lift of a Brownian motion.

Since  $(\tilde{W}^n, \tilde{W}^n)$  is equal in law to  $(W^n, W^n)$  which converges  $\mathbb{P}$ -a.s. to the Stratonovich lift of the Brownian motion W, we deduce that  $(\tilde{W}^n, \tilde{W}^n)$  converges in law to (W, W). As a consequence,  $\tilde{W}$  has the same law as W and therefore it is an increment of a Brownian motion. Next, we show that it is a Brownian motion with respect to  $(\tilde{\mathcal{F}}_t)_{t\geq 0}$ . To this end, fix arbitrary times  $0 \leq r \leq s < t \leq T$  and an arbitrary continuous function  $\gamma : C([0, s]; H^{-\delta}) \times C_2^0([r, s]; \mathbf{R}^m) \times C_2^0([r, s]; \mathbf{R}^{m \times m}) \to [0, 1]$ . Due to equality of joint laws it holds

$$\begin{split} \tilde{\mathbb{E}}[\gamma(\tilde{\xi}^{R}|_{[0,s]},\tilde{W}|_{[r,s]},\tilde{W}|_{[r,s]})\tilde{W}_{st}] &= \lim_{n \to \infty} \tilde{\mathbb{E}}[\gamma(\tilde{\xi}^{R,n}|_{[0,s]},\tilde{W}^{n}|_{[r,s]},\tilde{W}^{n}|_{[r,s]})\tilde{W}_{st}^{n}] \\ &= \lim_{n \to \infty} \mathbb{E}[\gamma(\xi^{R,n}|_{[0,s]},W^{n}|_{[r,s]},\mathbb{W}^{n}|_{[r,s]})W_{st}^{n}]. \end{split}$$

Since for every  $n \in \mathbb{N}$  the random variable  $(\xi^{R,n}|_{[0,s]}, W^n|_{[r,s]}, \mathbb{W}^n|_{[r,s]})$  is measurable with respect to the  $\sigma$ -algebra  $\mathcal{F}_{s+h^n}$ , it can be written as a measurable function of  $W|_{[0,s+h^n]}$ , say

$$(\xi^{R,n}|_{[0,s]}, W^n|_{[r,s]}, \mathbb{W}^n|_{[r,s]}) = F^n(W|_{[0,s+h^n]}).$$

Consequently,

$$\tilde{\mathbb{E}}[\gamma(\tilde{\xi}^{R}|_{[0,s]}, \tilde{W}|_{[r,s]}, \tilde{W}|_{[r,s]})\tilde{W}_{st}] = \lim_{n \to \infty} \mathbb{E}[\gamma(F^{n}(W|_{[0,s+h^{n}]}))W^{n}_{st}].$$
(33)

Since the function  $\gamma$  is bounded, the sequence  $\gamma(F^n(W|_{[0,s+h^n]}))$  is uniformly bounded in *n*. Hence there is a subsequence converging weak star in  $L^{\infty}(\Omega)$ . In addition, the limit denoted by  $\Gamma_s$  is  $\mathcal{F}_{s+}$ -measurable since for every  $h \in (0, 1)$  it is a weak star limit of  $\mathcal{F}_{s+h}$ -measurable functions, i.e., the weak star limit can be taken in  $L^{\infty}(\Omega, \mathcal{F}_{s+h}, \mathbb{P})$ . Due to right continuity of the filtration  $(\mathcal{F}_t)_{t\geq 0}$ it follows that  $\Gamma_s$  is  $\mathcal{F}_s$ -measurable.

On the other hand, since  $W^n$  is defined as the increment of a piecewise linear approximation of W, it follows that  $W^n$  is bounded uniformly in n in every  $L^q(\Omega)$  for  $q \in [1, \infty)$ . Since in addition  $W^n$  converges in every  $L^q(\Omega)$ , we obtain from (33) by weak-strong convergence

$$\mathbb{\tilde{E}}[\gamma(\tilde{\xi}^{R}|_{[0,s]},\tilde{W}|_{[r,s]},\tilde{W}|_{[r,s]})\tilde{W}_{st}] = \mathbb{E}[\Gamma_{s}W_{st}] = 0,$$

where the last equality follows from the martingale property of W with respect to  $(\mathcal{F}_t)_{t \ge 0}$ . This shows that  $t \mapsto \tilde{W}_{0t}$  is a  $(\tilde{\mathcal{F}}_t)_{t \ge 0}$ -martingale and hence a  $(\tilde{\mathcal{F}}_t)_{t \ge 0}$ -Brownian motion.

It remains to identify  $\tilde{W}$  as the Stratonovich lift of  $\tilde{W}$ . More precisely, we want to prove that

$$\tilde{\mathbb{W}}_{st}^{\ell,k,\beta,\alpha} = \int_{s}^{t} \tilde{W}_{r}^{\ell,\beta} \circ d\tilde{W}_{r}^{k,\alpha} - \tilde{W}_{s}^{\ell,\beta}\tilde{W}_{st}^{k,\alpha}$$

holds  $\tilde{\mathbb{P}}$ -a.s. for all  $0 \le s \le t \le T$ . The right hand side can be rewritten in terms of an Itô integral and the corresponding cross variation as follows

$$\tilde{W}_{st}^{\ell,k,\beta,\alpha} = \int_{s}^{t} \tilde{W}_{r}^{\ell,\beta} d\tilde{W}_{r}^{k,\alpha} + \frac{1}{2} \delta \langle\!\langle \tilde{W}^{\ell,\beta}, \tilde{W}^{k,\alpha} \rangle\!\rangle_{st} - \tilde{W}_{s}^{\ell,\beta} \tilde{W}_{st}^{k,\alpha} = \int_{s}^{t} \tilde{W}_{sr}^{\ell,\beta} d\tilde{W}_{r}^{k,\alpha} + \frac{1}{2} (t-s) \delta_{k=\ell,\alpha=\beta}.$$

In other words, regarding *s* as an initial time the above says that  $t \mapsto \tilde{W}_{st}^{\ell,k,\beta,\alpha}$  should solve an Itô stochastic differential equation. Let us define the process

$$t \mapsto \tilde{M}_t := \tilde{W}_{st}^{\ell,k,\beta,\alpha} - \frac{1}{2}(t-s)\delta_{k=\ell,\alpha=\beta}$$

Once we prove that

$$\tilde{M}_t = \int_s^t \tilde{W}_{sr}^{\ell,\beta} d\tilde{W}_r^{k,\alpha},\tag{34}$$

holds  $\tilde{\mathbb{P}}$ -a.s. for all  $0 \leq s \leq t \leq T$ , the identification of  $\tilde{\mathbb{W}}$  is complete.

To this end, since we already know that  $(\tilde{W}, \tilde{W})$  equals in law to the Stratonovich lift (W, W), we define analogously on the original probability space

$$t \mapsto M_t := \mathbb{W}_{st}^{\ell,k,\beta,\alpha} - \frac{1}{2}(t-s)\delta_{k=\ell,\alpha=\beta}$$

and here we know that  $\mathbb{P}$ -a.s.

$$M_t = \int_s^t W_{sr}^{\ell,\beta} dW_r^{k,\alpha}.$$
(35)

We will use martingale arguments to deduce (34) from (35) and from the equality of joint laws. For  $0 \le s \le \tau \le \sigma < t \le T$  and an arbitrary continuous function  $\gamma : C_2^0([\tau, \sigma]; \mathbf{R}^m) \times C_2^0([\tau, \sigma]; \mathbf{R}^{m \times m}) \to [0, 1]$ , we obtain from the equality of joint laws of  $(\tilde{W}, \tilde{W})$  and (W, W)

$$\mathbb{\tilde{E}}[\gamma(\tilde{W}|_{[\tau,\sigma]}, \mathbb{\tilde{W}}|_{[\tau,\sigma]})(\tilde{M}_t - \tilde{M}_{\sigma})] = \mathbb{E}[\gamma(W|_{[\tau,\sigma]}, \mathbb{W}|_{[\tau,\sigma]})(M_t - M_{\sigma})].$$

The right hand side vanishes due to (35) and accordingly,  $\tilde{M}$  is a martingale with respect to the filtration generated by  $(\tilde{W}, \tilde{W})$ . In order to deduce that  $\tilde{M}$  is equal to the stochastic integral (34), it remains to identify its quadratic variation as the cross variation with the driving process  $\tilde{W}^{k,\alpha}$ . Proceeding by the same arguments we deduce

$$\begin{split} &\tilde{\mathbb{E}}\left[\gamma(\tilde{W}|_{[\tau,\sigma]},\tilde{W}|_{[\tau,\sigma]})\left(\tilde{M}_{t}^{2}-\tilde{M}_{\sigma}^{2}-\int_{\sigma}^{t}(\tilde{W}_{sr}^{\ell,\beta})^{2}dr\right)\right]=0,\\ &\tilde{\mathbb{E}}\left[\gamma(\tilde{W}|_{[\tau,\sigma]},\tilde{W}|_{[\tau,\sigma]})\left(\tilde{M}_{t}\tilde{W}_{t}^{k,\alpha}-\tilde{M}_{\sigma}\tilde{W}_{\sigma}^{k,\alpha}-\int_{\sigma}^{t}\tilde{W}_{sr}^{\ell,\beta}dr\right)\right]=0, \end{split}$$

in other words,

$$\langle\!\langle \tilde{M} \rangle\!\rangle_t = \int_s^t (\tilde{W}_{sr}^{\ell,\beta})^2 dr, \qquad \langle\!\langle \tilde{M}, \tilde{W}^{k,\alpha} \rangle\!\rangle_t = \int_s^t \tilde{W}_{sr}^{\ell,\beta} dr$$

Therefore,

$$\langle\!\langle \tilde{M} - \int_{s}^{\cdot} W_{sr}^{\ell,\beta} dW_{r}^{k,\alpha} \rangle\!\rangle_{t} = \langle\!\langle \tilde{M} \rangle\!\rangle_{t} - 2\langle\!\langle \tilde{M}, \int_{s}^{\cdot} W_{sr}^{\ell,\beta} dW_{r}^{k,\alpha} \rangle\!\rangle_{t} + \langle\!\langle \int_{s}^{\cdot} W_{sr}^{\ell,\beta} dW_{r}^{k,\alpha} \rangle\!\rangle_{t} = 0,$$

which finally implies that for every  $0 \le s \le T$  the equality (34) holds  $\tilde{\mathbb{P}}$ -a.s. for all  $s \le t \le T$ . So far the associated set of full probability depends on *s*, however, by continuity of the involved quantities in *s* the desired result follows.

Therefore, we have proved that  $(\tilde{W}^n, \tilde{W}^n)$  converges to  $(\tilde{W}, \tilde{W})$  in the rough path topology  $\tilde{\mathbb{P}}$ a.s., where the iterated integral  $\tilde{W}$  is the Stratonovich lift of the Brownian motion  $\tilde{W}$ . This permits to identify  $\tilde{\xi}^R$  as a solution to the stochastic equation (13). Indeed, fix a test function  $\phi \in C^{\infty} \cap H$ and  $\tilde{\omega} \in \tilde{\Omega}$  from the set of full probability  $\tilde{\mathbb{P}}$  where the convergences above hold. In particular, we have  $(\tilde{W}(\tilde{\omega}), \tilde{W}(\tilde{\omega})) \in X_{\text{RP}}$ . From (32) we see that  $(\langle \tilde{\xi}^R(\tilde{\omega}), \phi \rangle, \langle \tilde{\xi}^R(\tilde{\omega}), -\operatorname{div}(\sigma_{k,a}\phi) \rangle)$  is a controlled rough path corresponding to  $(\tilde{W}(\tilde{\omega}), \tilde{W}(\tilde{\omega}))$ . Hence in view of the adaptedness of  $\tilde{\xi}^R$ to  $(\tilde{\mathcal{F}}_t)_{t\geq 0}$ , it follows from Corollary 5.2 in [FH14] that  $\tilde{\mathbb{P}}$ -a.s.

$$\frac{C_{\nu}}{\|\theta^{N}\|_{\ell^{2}}} \sum_{k \in \mathbb{Z}_{0}^{3}} \sum_{\alpha=1}^{2} \theta_{k}^{N} \int_{s}^{t} \left\langle \tilde{\xi}_{r}^{R}, -\operatorname{div}(\sigma_{k,\alpha}\phi) \right\rangle \circ d\tilde{W}_{r}^{k,\alpha}(\tilde{\omega})$$
$$= \left\langle \tilde{\xi}_{r}^{R}(\tilde{\omega}), \tilde{A}_{st}^{1,*}(\tilde{\omega})\phi + \tilde{A}_{st}^{2,*}(\tilde{\omega})\phi \right\rangle + \left\langle \tilde{\xi}_{st}^{R,\natural}(\tilde{\omega}), \phi \right\rangle$$

so that  $\tilde{\xi}^R$  satisfies (13).

Step 4: Convergence on the original probability space. Since pathwise uniqueness holds true for (13), it is possible to deduce that the original sequence of approximate solutions  $\xi^{R,n}$  converges in probability on the original probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . This is the classical Yamada–Watanabe argument, which can be established using the Gyöngy–Krylov lemma, see e.g. Section 2.10 in [BFH18] and an application of this method in Section 5.2.6 in [BFH18]. Then, by repeating the above limiting procedure on the original probability space, we deduce that the limit in probability, denoted by  $\xi^R$ , solves (13) on  $(\Omega, \mathcal{F}, \mathbb{P})$ . In fact, contrary to the identification of the limit on  $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ , the arguments simplifies significantly since no identification of the limiting rough path is necessary. This concludes the proof of the Wong–Zakai result.

Thanks to Proposition 14, we can finally prove Theorem 8.

*Proof of Theorem 8.* We follow the idea of the proof of Theorem 1.4 in [FL19] and argue by contradiction. Assume that there exist an  $\varepsilon_0 > 0$  and a subsequence  $\{n_i\}_{i\geq 1} \subset \mathbb{Z}_+$  such that

$$\lim_{i \to \infty} \sup_{\xi_0 \in H, \|\xi_0\|_{H} \le K} \mathbb{P}\Big( \Big\| \xi_{R_K}^{N_0, n_i}(\cdot, \xi_0) - \xi_{R_K}^{N_0}(\cdot, \xi_0) \Big\|_{C([0,T], H^{-\delta})} > \varepsilon_0 \Big) > 0.$$

Then we can find a sequence  $\{\xi_0^{n_i}\}_{i\geq 1} \subset H$  such that  $\|\xi_0^{n_i}\|_H \leq K$  for all  $i \geq 1$ , and (choosing a smaller  $\varepsilon_0 > 0$  if necessary)

$$\mathbb{P}\left(\left\|\xi_{R_{K}}^{N_{0},n_{i}}(\cdot,\xi_{0}^{n_{i}})-\xi_{R_{K}}^{N_{0}}(\cdot,\xi_{0}^{n_{i}})\right\|_{C\left([0,T],H^{-\delta}\right)}>\varepsilon_{0}\right)\geq\varepsilon_{0}>0.$$
(36)

Since the sequence  $\{\xi_0^{n_i}\}_{i\geq 1} \subset H$  is bounded, up to a subsequence, it converges weakly to some  $\xi_0 \in H$ . We can repeat the proof of Proposition 14 to show that, as  $i \to \infty$ , the sequence  $\xi_{R_K}^{N_0,n_i}(\cdot,\xi_0^{n_i})$  converges in probability in the topology of  $C([0,T]; H^{-\delta})$  to the solution  $\xi_{R_K}^{N_0}(\cdot,\xi_0)$  of (13) with initial value  $\xi_0$ . Similarly, the other sequence  $\xi_{R_K}^{N_0}(\cdot,\xi_0^{n_i})$  converges also in probability in the topology of  $C([0,T]; H^{-\delta})$  to  $\xi_{R_K}^{N_0}(\cdot,\xi_0)$ , see for instance Lemma 4.1 in [FL19] (or Corollary 3.5 therein). From these results and the following simple inequality:

$$\mathbb{P}\Big(\Big\|\xi_{R_{K}}^{N_{0},n_{i}}(\cdot,\xi_{0}^{n_{i}})-\xi_{R_{K}}^{N_{0}}(\cdot,\xi_{0}^{n_{i}})\Big\|_{C([0,T],H^{-\delta})} > \varepsilon_{0}\Big) \\
\leq \mathbb{P}\Big(\Big\|\xi_{R_{K}}^{N_{0},n_{i}}(\cdot,\xi_{0}^{n_{i}})-\xi_{R_{K}}^{N_{0}}(\cdot,\xi_{0})\Big\|_{C([0,T],H^{-\delta})} > \frac{\varepsilon_{0}}{2}\Big) + \mathbb{P}\Big(\Big\|\xi_{R_{K}}^{N_{0}}(\cdot,\xi_{0}^{n_{i}})-\xi_{R_{K}}^{N_{0}}(\cdot,\xi_{0})\Big\|_{C([0,T],H^{-\delta})} > \frac{\varepsilon_{0}}{2}\Big),$$

we immediately get a contradiction with (36). Thus we complete the proof of Theorem 8.

## References

[BCV18] T. Buckmaster, M. Colombo, and V. Vicol. Wild solutions of the Navier–Stokes equations whose singular sets in time have Hausdorff dimension strictly less than 1. *arXiv e-prints*, arXiv:1809.00600, 2018.

- [BDLSV19] T. Buckmaster, C. De Lellis, L. Székelyhidi, Jr., and V. Vicol. Onsager's conjecture for admissible weak solutions. *Comm. Pure Appl. Math.*, 72(2):229–274, 2019.
- [BFH18] D. Breit, E. Feireisl, and M. Hofmanová. Stochastically forced compressible fluid flows. De Gruyter Series in Applied and Numerical Mathematics 3. De Gruyter, Berlin, 2018.
- [BG17] I. Bailleul and M. Gubinelli. Unbounded rough drivers. *Annales de la Faculté des Sciences de Toulouse. Mathématiques.*, 26(4):795–830, 2017.
- [BV19] T. Buckmaster and V. Vicol. Nonuniqueness of weak solutions to the Navier-Stokes equation. *Ann. of Math.* (2), 189(1):101–144, 2019.
- [DGHT19] A. Deya, M. Gubinelli, M. Hofmanová, and S. Tindel. A priori estimates for rough PDEs with application to rough conservation laws. *J. Funct. Anal.*, 276(12):3577–3645, 2019.
- [DLS09] C. De Lellis and L. Székelyhidi, Jr. The Euler equations as a differential inclusion. *Ann. of Math.* (2), 170(3):1417–1436, 2009.
- [DLS10] C. De Lellis and L. Székelyhidi, Jr. On admissibility criteria for weak solutions of the Euler equations. *Arch. Ration. Mech. Anal.*, 195(1):225–260, 2010.
- [DLS13] C. De Lellis and L. Székelyhidi, Jr. Dissipative continuous Euler flows. *Invent. Math.*, 193(2):377–407, 2013.
- [FH14] P. K. Friz and M. Hairer. *A course on rough paths: with an introduction to regularity structures.* Universitext. Springer, Cham, 2014.
- [FL19] Franco Flandoli and Dejun Luo. High mode transport noise improves vorticity blowup control in 3d navier-stokes equations. *arXiv preprint: 1910.05742*, 2019.
- [HLN19a] M. Hofmanová, J.-M. Leahy, and T. Nilssen. On the Navier-Stokes equation perturbed by rough transport noise. *J. Evol. Equ.*, 19(1):203–247, 2019.
- [HLN19b] Martina Hofmanova, James-Michael Leahy, and Torstein Nilssen. On a rough perturbation of the navier-stokes system and its vorticity formulation. *arXiv preprint:* 1902.09348, 2019.
- [Ise18] P. Isett. A proof of Onsager's conjecture. Ann. of Math. (2), 188(3):871–963, 2018.
- [IXZ] G. Iyer, X. Xu, and A. Zlatoš. Convection-induced singularity suppression in the Keller-Segel and other non-linear pdes. *http://www.math.ucsd.edu/zlatos/Papers/mix-blowup.pdf*.

[Tem83] R. Temam. *Navier-Stokes equations and nonlinear functional analysis*, volume 41 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1983.