OPTIMAL DIVIDEND PAYOUT UNDER STOCHASTIC DISCOUNTING

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Abstract. Adopting a probabilistic approach we determine the optimal dividend payout policy of a firm whose surplus process follows a controlled arithmetic Brownian motion and whose cash-flows are discounted at a stochastic dynamic rate. Dividends can be paid to shareholders at unrestricted rates so that the problem is cast as one of singular stochastic control. The stochastic interest rate is modelled by a Cox-Ingersoll-Ross (CIR) process and the firm's objective is to maximize the total expected flow of discounted dividends until a possible insolvency time.

We find an optimal dividend payout policy which is such that the surplus process is kept below an endogenously determined stochastic threshold expressed as a decreasing function $r \mapsto b(r)$ of the current interest rate value. We also prove that the value function of the singular control problem solves a variational inequality associated to a second-order, non-degenerate elliptic operator, with a gradient constraint.

1. Introduction

1.1. **The Problem.** In this paper we solve an optimal dividend problem in which dividends are discounted at a stochastic rate. In our model, the company pays dividends to shareholders at unrestricted rates and any dividend payment instantaneously reduces the company's surplus. The company's manager takes the point of view of a risk-neutral representative shareholder, and thus aims at maximizing the total expected discounted return of dividends' payments, up to a possible insolvency time. We assume that the representative shareholder discounts dividends exponentially at a stochastic rate given by a deterministic nondecreasing and nonnegative function of the short interest rate. Such a discounting force might be justified, e.g., by thinking that the personal time-preferences of the representative shareholder are linked to the financial market's evolution, and in particular to the interest paid by an alternative investment opportunity given by a safe asset like a bond. In this way, the discounting can be seen as an "opportunity cost" for the shareholder: the more the dividends' payments are delayed, the lower are the opportunities of investment in a safe asset. Alternatively, if we suppose that the evaluation of the dividends' returns is performed under an equivalent martingale measure arising in a financial market (that, however, we do not model here), we can view the discount factor as a classical deflator process. As we will discuss in detail in the literature review, in the classical dividend problem the discount rate is often deterministic (and constant) so that shareholders are only exposed to risks arising from the random profitability of the firm. On the contrary, in our setting shareholders are also exposed to uncertainty from the wider macro-economic activity via random fluctuations in the interest rate.

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From a mathematical point of view, we model the previous problem as a *two-dimensional singular stochastic control problem.* The two coordinates of the state process are the surplus process and the short interest rate. The surplus process evolves as a Brownian motion $(Z^D_t)_{t\geq 0}$ with drift μ and volatility σ, which is linearly controlled through a nondecreasing stochastic process $(D_t)_{t\geq 0}$ representing the cumulative amount of distributed dividends. The uncontrolled short interest rate $(R_t)_{t\geq 0}$ triggers the exponential discount factor appearing in the expected return of dividends' payments. The process $(R_t)_{t\geq 0}$ is assumed to be independent of the surplus' process, and to follow a mean-reverting dynamics specified by the Cox-Ingersoll-Ross (CIR) model. We require that the coefficients of the CIR process fulfill the so-called Feller condition (see (2.4) below), so that the short interest rate is strictly positive at any time with probability one. The discount rate at time t is of the form $\rho(R_t)$ (hence, total discounting up to time t is $e^{-\int_0^t \rho(R_s)ds}$), for some nonnegative and nondecreasing function ρ satisfying suitable growth conditions (see Assumption 2.1 below). Notice that our requirements on ρ are such that the cases of constant and linear discounting forces (i.e., like $\rho(r) = \rho_0 > 0$ or $\rho(r) = r$ for all $r \in \mathbb{R}_+$) are included in our setting. The aim is to maximize the total expected discounted value of dividends, up to the random time $\tau^D := \inf\{t \geq 0 : Z^D_t \leq \alpha\}$, for a given and fixed solvency level $\alpha \geq 0$. If $\alpha = 0$ we find the classical bankruptcy condition for this kind of models.

1.2. **Methodology and Results.** The key challenge in our work arises from the two-dimensional (non-degenerate) diffusive nature of the set-up. Indeed, dynamic programming ideas link the stochastic control problem to a variational problem involving an elliptic partial differential equation (PDE) with gradient constraint that is not amenable to an explicit solution. This stands in contrast with some of the more classical versions of the same problem where the state process is purely one-dimensional (see [24] for an early formulation and, for example, [30] and [38] among more recent contributions). Indeed, the dynamic programming equation arising in one-dimensional problems involves an ordinary differential equation (ODE) so that a so-called guess-and-verify approach can be implemented. The latter consists of an educated guess on the structure of the problem's solution, leading to an ODE for the value function with suitable boundary conditions (usually involving *smooth-fit*). The ODE can be solved explicitly and a verification theorem allows to prove that such solution is indeed the value function of the problem. That approach fails in our set-up since explicit solutions are not available.

In order to solve our two-dimensional optimal dividend problem, here we follow ideas developed in [9] and later extended in [8]. We link the optimal dividend problem to an auxiliary problem of optimal stopping whose underlying process is a two-dimensional reflecting diffusion (R, K) and whose payoff increases upon each new reflection of (R, K), but it is discounted with the same stochastic dynamic rate as in the original dividend problem. In both [9] and [8] the interest rate is constant although the state-space is two-dimensional. In [9] the problem is set on a finite-time horizon but the diffusive dynamics only affects one state variable. In [8] the time-horizon is infinite but there is partial information that leads to the same Brownian motion driving a two-dimensional SDE (hence degenerate). On the contrary, here we have a fully two-dimensional diffusive set-up so that the construction of the auxiliary optimal stopping problem is different to those in [9] and [8] (e.g., here it preserves the stochastic discounting) and the subsequent analysis of the optimal dividend policy must follow a different line of argument. In particular, the use of a stochastic discount rate with CIR dynamics leads to numerous technical complications. These arise, e.g., in the proof of a preliminary verification theorem for the dividend problem (Theorem 2.5), as well as in showing boundedness and regularity of the value in the optimal stopping problem (Propositions 3.4 and 3.11). Also it is worth noticing that the dynamic programming equation in [9] and [8] involves a one-dimensional parabolic PDE, while in our problem we have a two-dimensional elliptic PDE.

In the auxiliary optimal stopping problem that we consider (see the beginning of Section 3), the state variable consists of the original short interest rate R appearing in the discount factor, and of a Brownian motion K with drift $-\mu$ and volatility σ, which is reflected at the solvency level α. By making use of almost exclusively probabilistic arguments, we show that the optimal stopping time is expressed in terms of the hitting time of the process $t \mapsto K_t$ to a (stochastic) moving boundary $t \mapsto b(R_t)$, where b is a nonincreasing and right-continuous function on $[0, \infty)$ whose properties are collected in Lemma 3.8 and Proposition 3.13. Moreover, using that the underlying process (R, K) is strongly Feller and that the boundary points are regular (in the probabilistic sense) for the stopping region, we can show (Proposition 3.11) that the value function U of the stopping problem is everywhere continuously differentiable (see also [10] for general results in this direction).

The smoothness of the function U allows to construct the value function V of the dividend problem by a simple integration (formula (4.1) in Section 4) and provides nice regularity properties for V. Indeed, as a function of the state variables (r, z) associated to the process (R, Z^D), the mapping $(r, z) \mapsto V(r, z)$ is globally C^1, with second order derivatives $\partial_{zz}V$ and $\partial_{rz}V$ that are continuous everywhere. Furthermore, the second order derivative $\partial_{rr}V$ is locally bounded in the whole space and continuous away from the boundary $z = b(r)$ with well-defined limits up to the boundary (Propositions 4.1 and 4.2). A direct approach to the variational problem with gradient constraint for the function V is involved, especially because of an additional boundary condition along the solvency level, i.e. $V(r, \alpha) = 0$ (see, e.g., [20, 21, 22]). In this respect, our probabilistic approach overcomes the difficulties arising in the PDE arguments.

The main result of the paper is Theorem 4.3 which links, thanks to the verification Theorem 2.5 and to the regularity results mentioned above, the value functions U and V and provides an optimal dividend strategy as a Skorokhod reflection of the process $t \mapsto Z_t^D$ below the stochastic boundary $t \mapsto b(R_t)$.

1.3. **Related Literature.** The first version of an optimal dividend problem was formulated by Bruno de Finetti in 1957 in [11]. De Finetti proposed to measure the value of an insurance company in terms of the discounted value of its future dividend payments. Since then the optimal dividend problem has been studied extensively and it has become a cornerstone of the modern Mathematical Finance/Actuarial Mathematics literature. Early contributions addressing the dividend problem via control-theoretic techniques include, e.g., [24], where the authors consider several problem formulations, including controls with bounded-velocity and singular controls (see also [35], which appeared in the same years). A broad class of infinite-time horizon singular control problems for one-dimensional diffusions, inspired by the optimal dividend problem, were analysed in [40] who obtained general formulae. Numerous extensions and refinements of those early models have appeared in the literature; here we only mention a few of them and our review is certainly not exhaustive. For example, in [5] the cash reserve has a mean-reverting dynamics and lump sum dividend payments are made at optimally chosen discrete dates (i.e., impulsive controls are considered); [36] studies a model with stochastic drift in the dynamics of the company's surplus process; in [4] the surplus process evolves as a jump-diffusion so that the company faces two types of liquidity risk: a

Brownian risk and a Poisson risk. On an infinite-time horizon, [30] allows capital injections in order to avoid company's bankruptcy, whereas [17] considers a general diffusive model with "forced" capital injections (see also [18] for the finite-time horizon version). In the series of papers [20, 21, 22] the author solves the optimal dividend problem with finite-time horizon by means of purely PDE methods, whereas [9] addresses the problem probabilistically. Additional references can be also found in the review [3] and in the book [37].

More closely related to our work are the papers considering stochastic discounting, many of which have appeared in recent years. In a discrete-time setting, the analysis is typically considered in the context of risk models for insurance companies (see, e.g., [42] and the more recent [41]). In continuous-time we find, e.g., [1] and [25] where the wealth process is a drifted Brownian motion and the interest rate is modulated by a continuous-time Markov chain (more recently [26] extends [25] to the case of a jump-diffusive surplus process). Fixed-point methods are adopted in [25] and [26], whereas dynamic programming ideas appear in [1].

The papers [14] and [16] consider discounting factors of the form e^{-U_t}. In [14] the process $(U_t)_{t\geq 0}$ is either a drifted Brownian motion or an integrated Ornstein-Uhlenbeck process, while it is a CIR process in [16]. It is worth noticing that the CIR process in [16] does not mean-revert to a finite value but explodes as t diverges to infinity, in order to guarantee a finite value of the problem. With such specifications of the discount factor, the nature of the optimal dividend problems considered in [14] and [16] is very different from ours. In our paper indeed it is the *discount rate* - and not the cumulative discounting force - that takes a mean-reverting CIR dynamics. At the technical level, when $(U_t)_{t\geq 0}$ in [14] is a Brownian motion with drift, a change of measure allows a reduction to a one-dimensional diffusive set-up. When $(U_t)_{t\geq 0}$ is an integrated Ornstein-Uhlenbeck process a viscosity characterization of the value function is provided but without an optimal dividend policy. In [16], explicit solutions are obtained when the surplus process is deterministic; the case of a stochastic surplus is instead investigated only in a regime of small volatility. Extensions of [14] to the case in which $(U_t)_{t\geq 0}$ is a Lévy process can be found in [6], [15], and [27].

Compared to the existing literature we provide a detailed analysis of the value function and of the optimal dividend policy in a two-dimensional diffusive setting, under very mild assumptions on the discount rate (cf. Assumption 2.1 below), and under the Feller condition (2.4) that guarantees strictly positive interest rates.

1.4. **Plan of the paper.** The rest of the paper is organized as follows. In Section 2 we set up the problem and prove a preliminary verification theorem. The auxiliary optimal stopping problem is studied in Section 3, while in Section 4 we construct the value function of the optimal dividend problem together with its optimal dividend strategy. A financial interpretation of the the optimal dividend policy is given in Section 4.1.

2. Problem Setting and Verification Theorem

2.1. **Problem Formulation and Assumptions.** We consider a probability space $(\Omega, \mathcal{F}, \mathsf{P})$ that carries two *independent* Brownian motions $(B_t)_{t\geq 0}$ and $(W_t)_{t\geq 0}$. We denote by $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ the filtration generated by (B, W) and augmented with P-null sets. We fix $\alpha \geq 0$, representing a minimum capital requirement, and we assume that the cash reserve (or surplus) of a company follows the controlled dynamics

$$Z_t^D = z + \mu\, t + \sigma B_t - D_t, \qquad t \geq 0, \tag{2.1}$$

where $z \geq \alpha$, and $(D_t)_{t\geq 0}$ is right-continuous and nondecreasing. Indeed, D_t denotes the total amount of dividends paid to the shareholders up to time t. The set of admissible cumulative dividend payments is given by

$$\begin{aligned}\mathcal{A} := \{D\ : D \text{ is } \mathbb{F}\text{-adapted, nondecreasing, right-continuous and such that,} \\ \text{setting } D_{0-}=0, \text{ we have } D_t - D_{t-} \leq Z^D_{t-} - \alpha,\ \forall t \geq 0,\ \mathsf{P}\text{-a.s.}\},\end{aligned} \tag{2.2}$$

In the rest of the paper we denote by Z^0 the solution to (2.1) with $D \equiv 0$.

The interest rate follows a CIR dynamics and, in particular, we have, for all $t \geq 0$,

$$dR_t = k(\theta - R_t)\,dt + \gamma\sqrt{R_t}\,dW_t, \quad R_0 = r \geq 0, \tag{2.3}$$

where k, θ and γ are fixed constants. We assume the so-called Feller condition

$$2k\theta \geq \gamma^2 \tag{2.4}$$

so that $R_t > 0$, P-a.s. for all $t > 0$ (see, e.g., [23, p. 357 and Section 6.1.3]). In what follows we find sometimes convenient to use the notation R^r_t for the interest rate process that starts at time zero from $R_0 = r$. Similarly, we denote by $Z^{z,D}_t$ the surplus process started at time $0-$ (i.e., before any dividend payment) from the level $z \geq \alpha$, and by $Z^{z,0}_t$ the process $z + \mu t + \sigma B_t$. Accordingly, we will denote by $\mathsf{P}_{r,z}$ the probability measure on $(\Omega, \mathcal{F})$ such that $\mathsf{P}_{r,z}(\,\cdot\,) = \mathsf{P}(\,\cdot\,|R_0 = r,\ Z^D_{0-} = z)$, and we define $\mathsf{E}_{r,z}$ the corresponding expected value. Also, E_r will denote the expectation under $\mathsf{P}_r(\,\cdot\,) = \mathsf{P}(\,\cdot\,|R_0 = r)$ and E_z the expectation under $\mathsf{P}_z(\,\cdot\,) = \mathsf{P}(\,\cdot\,|Z^D_{0-} = z)$.

We assume that the firm's manager discounts dividends at a rate ρ that depends on the current level of the interest rate. The manager aims at maximizing the total expected discounted flow of dividends up to a possible insolvency time of the firm. Then the value function of the problem reads

$$V(r,z) = \sup_{D\in\mathcal{A}} \mathsf{E}_{r,z}\left[\int_{0-}^{\tau^D_\alpha} e^{-\int_0^t \rho(R_t)dt} dD_t\right], \tag{2.5}$$

where, for any $D \in \mathcal{A}$, the random time horizon

$$\tau^D_\alpha := \inf\{t \geq 0\ :\ Z^D_t \leq \alpha\} \tag{2.6}$$

enforces the solvency requirement $Z^D_t \geq \alpha$ for all $t \geq 0$. The notation $0-$ in the integral means that we include a possible jump $D_0 - D_{0-} \leq z - \alpha$ at time zero. If $\alpha = 0$ we recover the classical bankruptcy condition for this kind of models (see, e.g., [37, Chapter 2, Section 2.5]).

The following assumptions on the discount rate will be standing.

Assumption 2.1. *The discount rate $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function. Moreover*

(i) *it is nondecreasing;*
(ii) *there exist two non-negative constants c_1 and c_2 such that $c_1 + c_2 > 0$ and $\rho(r) \geq c_1 + c_2\, r$ for $r \geq 0$;*
(iii) *there exists $c_3 > 0$ and $q \in \mathbb{N}$ such that, for $r_1 > r_2 \geq 0$,*

$$\rho(r_1) - \rho(r_2) \leq c_3(1 + r_1^q)|r_1 - r_2|. \tag{2.7}$$

Remark 2.2. *We observe that (i) and (ii) of Assumption 2.1 above will be used to prove all the results below.*

- *Condition (i) enables to obtain monotonicity properties of the value function.*
- *Condition (ii) is a mild requirement which allows us to deal with the (possibly) infinite horizon in Problem (2.5).*

Assumption 2.1-(iii) above is only needed in order to prove the C^1 property of Proposition 3.11; hence all the results obtained before Proposition 3.11 do actually hold without Assumption 2.1-(iii).

Observe also that condition (2.7) is verified, e.g., when $\rho \in C^1(\mathbb{R}^+)$ and there exist $C > 0$ and $q \in \mathbb{N}$ such that $\rho'(r) \leq C\big(1 + r^q\big)$ for any $r \geq 0$.
Finally, notice that (i) + (ii) + (iii) is consistent with reasonable models for the discount rate, including $\rho(r) = r$ and $\rho(r) = const.$, which are canonical.

The next assumption on the drift rate μ is based on the following remark.

Remark 2.3. *If $\mu \leq 0$ it is intuitively clear that the firm's manager wants to liquidate the fund immediately, by paying dividends in a single transaction, i.e. $D_0 = z - \alpha$. It is indeed not difficult to check that, if $\mu \leq 0$, the couple $v(r, z) = z - \alpha$ and $a(r) \equiv \alpha$ solves problem (2.12) below, and it is equal to the value function.*

Hence, with no loss of generality, we make the following assumption.

Assumption 2.4. *We have $\mu > 0$.*

For frequent future use we recall that for any $\beta > 0$ one has (see, e.g., [23], Corollary 6.3.4.3, p. 362)

$$\mathsf{E}_r\left[e^{-\beta\int_0^t R_u du}\right] = e^{-A_\beta(t) - rG_\beta(t)}, \tag{2.8}$$

with

$$\begin{aligned} G_\beta(t) &:= \frac{2\beta\left(e^{\eta_\beta t} - 1\right)}{\eta_\beta\left(e^{\eta_\beta t} + 1\right) + k\left(e^{\eta_\beta t} - 1\right)}, \\ A_\beta(t) &:= -\frac{2k\theta}{\gamma^2}\ln\left[\frac{2\eta_\beta e^{(\eta_\beta + k)\frac{t}{2}}}{(\eta_\beta + k)\left(e^{\eta_\beta t} - 1\right) + 2\eta_\beta}\right], \end{aligned} \tag{2.9}$$

and $\eta_\beta := \sqrt{k^2 + 2\gamma^2\beta}$.

2.2. **Verification Theorem.** The infinitesimal generator $\mathcal{L}$ of the pair (Z^0, R) is defined by its action on twice-continuously differentiable functions f as

$$(\mathcal{L}f)(r, z) := \frac{1}{2}\sigma^2 f_{zz}(r, z) + \mu f_z(r, z) + \frac{1}{2}\gamma^2 r f_{rr}(r, z) + k(\theta - r) f_r(r, z), \tag{2.10}$$

where we adopt the notation $f_r := \frac{\partial}{\partial r}f$, $f_z := \frac{\partial}{\partial z}f$, $f_{rr} := \frac{\partial^2}{\partial r^2}f$, $f_{rz} := \frac{\partial^2}{\partial r \partial z}f$ and $f_{zz} := \frac{\partial^2}{\partial z^2}f$.

The financial intuition suggests that the firm's manager is more likely to pay dividends when the firm performs well. We thus expect that for each value r of the interest rate, there should be a critical value of the surplus process, such that dividends are paid if z is larger than such a value. Motivated by this intuition and by the idea that a dynamic programming principle should also hold, we formulate the following verification theorem.

For the ease of notation we introduce the sets

$$\mathcal{O} := (0, \infty) \times (\alpha, \infty) \text{ and } \overline{\mathcal{O}} := [0, \infty) \times [\alpha, \infty).$$

Theorem 2.5. *Let Assumptions 2.1, 2.4 and condition (2.4) hold. Assume that there exists functions $a : (0, +\infty) \to [\alpha, +\infty)$ and $v : \overline{\mathcal{O}} \to \mathbb{R}_+$ with the following properties.*

(i) *The mapping $r \mapsto a(r)$ is right-continuous and non-increasing.*

(ii) *The function* v *is such that* $v \in C^1(\mathcal{O}) \cap C(\overline{\mathcal{O}})$ *with* $v_{zz}, v_{rz} \in C(\mathcal{O})$ *and* $v_{rr} \in L^\infty_{loc}(\mathcal{O}) \cap C(\bar{\mathcal{I}} \cap \mathcal{O})$*, where*

$$\mathcal{I} := \{(r,z) \in \mathcal{O} : v_z(r,z) > 1\}. \tag{2.11}$$

(iii) *The couple* (v, a) *solves the free-boundary problem*

$$\begin{cases} \mathcal{L}v(r,z) - \rho(r)\, v(r,z) \le 0, & a.e.\ (r,z) \in \mathcal{O} \\ \mathcal{L}v(r,z) - \rho(r)\, v(r,z) = 0, & \alpha < z < a(r),\ r > 0 \\ v_z(r,z) > 1, & \alpha < z < a(r),\ r > 0 \\ v_z(r,z) = 1, & z \ge a(r),\ r > 0 \\ v(r,\alpha) = 0, & r \ge 0. \end{cases} \tag{2.12}$$

Then, $v \ge V$ *on* $\overline{\mathcal{O}}$*.*

In addition, if $v(r,z) \le c(z-\alpha)$ *for all* $(r,z) \in \overline{\mathcal{O}}$ *and some* $c > 0$*, then for every* $(r,z) \in \overline{\mathcal{O}}$ *we have* $v(r,z) = V(r,z)$ *and the process*

$$D^a_t := \sup_{0 \le s \le t} \left[Z^{z,0}_s - a(R^r_s) \right]^+, \qquad t \ge 0 \tag{2.13}$$

is optimal at (r,z)*; i.e.,*

$$v(r,z) = V(r,z) = \mathsf{E}_{r,z}\left[\int_{0-}^{\tau_\alpha^{D^a}} e^{-\int_0^t \rho(R_t)dt} dD^a_t \right]. \tag{2.14}$$

Proof. **Part 1: Proof that $v \ge V$ on $\overline{\mathcal{O}}$.**

We start arguing as in [19], Chapter VIII, Theorem 4.1. More precisely, for each $k \ge 1$, we introduce the standard mollifier $\phi_k(z) = k^{-2}\phi(kz)$ with $\phi \in C^\infty_c(B_1(0))$, $\phi \ge 0$, $\int_{\mathbb{R}^2} \phi(z)dz = 1$ (where $B_1(0)$ is the ball in $\mathbb{R}^2$ centered in zero with radius one), so that $\phi_k(z) \in C^\infty_c(B_{1/k}(0))$. Then we define $(v^k)_{k\ge1} \in C^\infty(\overline{\mathcal{O}})$ by convolution as $v^k := v * \phi_k$. Thanks to the regularity assumptions on v, for any compact set $K \subset \mathcal{O}$ we have

$$\lim_{k\to\infty} ||v^k - v||_{L^\infty(K)} = 0, \tag{2.15}$$

$$\lim_{k\to\infty} ||v^k_z - v_z||_{L^\infty(K)} = 0, \qquad \lim_{k\to\infty} ||v^k_r - v_r||_{L^\infty(K)} = 0, \tag{2.16}$$

$$\lim_{k\to\infty} ||v^k_{zz} - v_{zz}||_{L^\infty(K)} = 0, \quad \lim_{k\to\infty} ||v^k_{rz} - v_{rz}||_{L^\infty(K)} = 0. \tag{2.17}$$

In general v^k_{rr} will not converge to v_{rr} uniformly on every compact subset of $\mathcal{O}$, since v_{rr} is not continuous. Therefore we cannot expect that $\mathcal{L}v^k$ converges to $\mathcal{L}v$ uniformly on compact sets. However, by the definition of weak derivative and since $v_{rr} \in L^\infty_{loc}(\mathcal{O})$, we have $(v^k)_{rr} = (v_{rr} * \phi_k)$. Then, thanks to the continuity of the coefficients in $\mathcal{L}$ we have

$$\lim_{k\to\infty} ||(\mathcal{L}v^k) - [(\mathcal{L}v) * \phi_k]||_{L^\infty(K)} = 0, \tag{2.18}$$

for every compact $K \subset \mathcal{O}$, using that the minimal distance from K to $\mathcal{O}$ is strictly positive. Recalling that $\mathcal{L}v - \rho(\cdot)\, v \le 0$ a.e. in $\mathcal{O}$, then it also holds that $(\mathcal{L}v - \rho(\cdot)\, v) * \phi_k \le 0$ everywhere in $\mathcal{O}$. Hence (2.18) yields

$$\limsup_{k\to\infty} \sup_{(r,z)\in K} (\mathcal{L}v^k - \rho(r)v^k)(r,z) \le 0. \tag{2.19}$$

Let now $(r,z) \in \mathcal{O}$ be given and fixed, and consider an arbitrary admissible dividend strategy $D \in \mathcal{A}$. For $0 < \varepsilon < z - \alpha$, set

$$\eta^{Z^D}_\varepsilon := \inf\{t \ge 0 : \alpha \le Z^{z,D}_t \le \alpha + \varepsilon\}.$$

Notice that when $\tau_\alpha^D(\omega) = 0$ (recall that τ_α^D is defined in (2.6)) also $\eta_\varepsilon^{Z^D}(\omega) = 0$ for every $\varepsilon \in (0, z-\alpha)$. Moreover, if $\tau_\alpha^D(\omega) > 0$, for every $\delta > 0$ sufficiently small we have

$$\inf_{0 \le t \le \tau_\alpha^D(\omega) - \delta} Z_t^{z,D}(\omega) > \alpha,$$

hence for every $\delta > 0$ we find $\varepsilon > 0$ such that

$$\inf_{0 \le t \le \tau_\alpha^D(\omega) - \delta} Z_t^{z,D}(\omega) > \alpha + \varepsilon \implies \tau_\alpha^D(\omega) - \delta \le \eta_\varepsilon^{Z^D}(\omega) \le \tau_\alpha^D(\omega).$$

Since $\eta_\varepsilon^{Z^D}(\omega)$ is increasing in ε we conclude that $\eta_\varepsilon^{Z^D}(\omega) \uparrow \tau_\alpha^D(\omega)$, $\mathsf{P}_{r,z}$ a.s., as $\varepsilon \downarrow 0$.

Let us also define

$$\tau_\varepsilon^{Z^D} := \inf\left\{t \ge 0 : \ Z_t^{z,D} \ge \frac{1}{\varepsilon}\right\}, \quad \eta_\varepsilon^R := \inf\left\{t \ge 0 : R_t^r \notin \left(\varepsilon, \frac{1}{\varepsilon}\right)\right\},$$

and

$$\vartheta_\varepsilon^D := \eta_\varepsilon^{Z^D} \wedge \eta_\varepsilon^R \wedge \tau_\varepsilon^{Z^D}.$$

We have $\vartheta_\varepsilon^D = \inf\{t \ge 0 : \ (R_t^r, Z_t^{z,D}) \notin K_\varepsilon\}$, where $K_\varepsilon = (\varepsilon, \frac{1}{\varepsilon}) \times (\alpha + \varepsilon, \frac{1}{\varepsilon})$. Since $+\infty$ is unattainable for the processes R and Z^D and 0 is unattainable for R, we also have $\vartheta_\varepsilon^D \uparrow \tau_\alpha^D$ $\mathsf{P}_{r,z}$ a.s., as $\varepsilon \downarrow 0$.

Let us now fix $t > 0$. The Dynkin formula applied to the process $e^{-\int_0^s \rho(R_u)du} v^k(R_s, Z_s^D)$ on the (random) time interval $[0, \vartheta_\varepsilon^D \wedge t]$ gives

$$\begin{aligned} v^k(r,z) = \, & \mathsf{E}_{r,z}\Big[e^{-\int_0^{\vartheta_\varepsilon^D \wedge t} \rho(R_u)du} v^k(R_{\vartheta_\varepsilon^D \wedge t}, Z^D_{\vartheta_\varepsilon^D \wedge t})\Big] \\ & - \mathsf{E}_{r,z}\Big[\int_0^{\vartheta_\varepsilon^D \wedge t} e^{-\int_0^s \rho(R_u)du} \big(\mathcal{L} - \rho(R_s)\big) v^k(R_s, Z_s^D) ds\Big] \\ & + \mathsf{E}_{r,z}\Big[\int_0^{\vartheta_\varepsilon^D \wedge t} e^{-\int_0^s \rho(R_u)du} v_z^k(R_s, Z_s^D) dD_s^c\Big] \\ & - \mathsf{E}_{r,z}\Big[\sum_{0 \le s \le \vartheta_\varepsilon^D \wedge t} e^{-\int_0^s \rho(R_u)du} \big(v^k(R_s, Z_s^D) - v^k(R_s, Z_{s-}^D)\big)\Big], \end{aligned} \tag{2.20}$$

where D^c denotes the continuous part of D and the final sum is non-zero only for (at most countably many) times s such that $\Delta D_s := D_s - D_{s-} > 0$. Notice that

$$\begin{aligned} & \sum_{0 \le s \le \vartheta_\varepsilon^D \wedge t} e^{-\int_0^s \rho(R_u)du} \big(v^k(R_s, Z_s^D) - v^k(R_s, Z_{s-}^D)\big) \\ & = - \sum_{0 \le s \le \vartheta_\varepsilon^D \wedge t} e^{-\int_0^s \rho(R_u)du} \int_0^{\Delta D_s} v_z^k(R_s, Z_{s-}^D - y) dy, \end{aligned}$$

Since $(Z_s^{z,D}, R_s^r)_{0 \le s \le \vartheta_\varepsilon^D \wedge t} \in K_\varepsilon$, using (2.15)-(2.16)-(2.17) and (2.19), (2.20) we obtain, sending $k \to +\infty$,

$$\begin{aligned} v(r,z) \ge \, & \mathsf{E}_{r,z}\Big[e^{-\int_0^{\vartheta_\varepsilon^D \wedge t} \rho(R_u)du} v(R_{\vartheta_\varepsilon^D \wedge t}, Z^D_{\vartheta_\varepsilon^D \wedge t})\Big] \\ & + \mathsf{E}_{r,z}\Big[\int_0^{\vartheta_\varepsilon^D \wedge t} e^{-\int_0^s \rho(R_u)du} v_z(R_s, Z_s^D) dD_s^c\Big] \\ & + \mathsf{E}_{r,z}\Big[\sum_{0 \le s \le \vartheta_\varepsilon^D \wedge t} e^{-\int_0^s \rho(R_u)du} \int_0^{\Delta D_s} v_z(R_s, Z_{s-}^D - y) dy\Big]. \end{aligned} \tag{2.21}$$

Recalling that $v_z \geq 1$ on $\mathcal{O}$ by (2.12) (hence $v \geq 0$ too, since $v(r,\alpha) = 0$ for any $r \geq 0$) we obtain from (2.21) that

$$v(r,z) \geq \mathsf{E}_{r,z}\Big[e^{-\int_0^{\vartheta_\varepsilon^D \wedge t} \rho(R_u)du} v(R_{\vartheta_\varepsilon^D \wedge t}, Z^D_{\vartheta_\varepsilon^D \wedge t})\Big] + \mathsf{E}_{r,z}\bigg[\int_0^{\vartheta_\varepsilon^D \wedge t} e^{-\int_0^s \rho(R_u)du}\, dD_s\bigg] \tag{2.22}$$
$$\geq \mathsf{E}_{r,z}\bigg[\int_0^{\vartheta_\varepsilon^D \wedge t} e^{-\int_0^s \rho(R_u)du}\, dD_s\bigg].$$

Then, we can take limits first as $t \uparrow \infty$, and then as $\varepsilon \downarrow 0$, and employ monotone convergence to obtain

$$v(r,z) \geq \mathsf{E}_{r,z}\bigg[\int_0^{\tau_\alpha^D} e^{-\int_0^s \rho(R_u)du}\, dD_s\bigg]. \tag{2.23}$$

Since $v \in C(\overline{\mathcal{O}})$ and $r \mapsto \rho(R_t^r)$ is continuous and nondecreasing, P-a.s., an application of Fatou's lemma also gives

$$v(0,z) = \lim_{r\downarrow 0} v(r,z) \geq \mathsf{E}_z\bigg[\int_0^{\tau_\alpha^D} \liminf_{r\downarrow 0} e^{-\int_0^s \rho(R_u^r)du}\, dD_s\bigg]$$
$$= \mathsf{E}_z\bigg[\int_0^{\tau_\alpha^D} e^{-\int_0^s \rho(R_u^0)du}\, dD_s\bigg],$$

upon noticing that τ_α^D is independent of r.

Since (2.23) is true for any $D \in \mathcal{A}$ and for any $(r,z) \in [0,\infty) \times (\alpha,\infty)$, and $v(r,\alpha) = 0 = V(r,\alpha)$ (the last comes from the definition of V), we conclude that $v \geq V$ on $\overline{\mathcal{O}}$.

Part 2: Proof of $v = V$ and (2.14). We divide this part of the proof into three steps.

Step 1. Fix $(r,z) \in [0,+\infty) \times (\alpha,+\infty)$. We are going to prove that the process D^a in (2.13) belongs to $\mathcal{A}$ and, $\mathsf{P}_{r,z}$-a.s.,

$$Z_t^{D^a} \leq a(R_t) \quad \text{for all} \quad 0 \leq t \leq \tau_\alpha^{D^a}. \tag{2.24}$$

Moreover, we show the Skorokhod minimality condition:

$$\int_0^{\tau_\alpha^{D^a}} \mathbb{1}_{\{Z_{t-}^{D^a} < a(R_t)\}}\, dD_t^a = \sum_{0 \leq t \leq \tau_\alpha^{D^a}} \int_0^{\Delta D_t^a} \mathbb{1}_{\{Z_{t-}^{D^a} - \zeta < a(R_t)\}} d\zeta = 0. \tag{2.25}$$

To prove these facts observe first that D^a is by construction $\mathbb{F}$-adapted and nondecreasing. Moreover, by definition of D^a we easily get, for $0 \leq t \leq \tau_\alpha^{D^a}$,

$$D_t^a - D_{t-}^a = \max\{0, (Z_t^0 - a(R_t))^+ - D_{t-}^a\} = \max\{0, Z_{t-}^{D^a} - a(R_t)\} \leq Z_{t-}^{D^a} - \alpha,$$

where in the last inequality we used that $a \geq \alpha$. The second equality above also implies

$$Z_{t-}^{D^a} - \Delta D_t^a = \min\{Z_{t-}^{D^a}, a(R_t)\},$$

which guarantees that the second integral in (2.25) equals zero. Condition (2.24) follows by definition of D^a, upon noticing that

$$Z_t^{D^a} = Z_t^0 - D_t^a \leq a(R_t) \quad \text{for } 0 \leq t \leq \tau_\alpha^{D^a}, \quad \mathsf{P}_{r,z}\text{-a.s.}$$

It remains to show that D^a is right-continuous and that the first integral in (2.25) is also zero. Fix $\omega \in \Omega$ (outside of a null set so that $t \mapsto (Z_t^0(\omega), R_t(\omega))$ are continuous) and $t \in (0, \tau_\alpha^{D^a}(\omega)]$. Using that D^a is nondecreasing, we get that, if $Z_{t-}^{D^a}(\omega) = Z_t^0(\omega) - D_{t-}^a(\omega) < a(R_t(\omega))$, then also $Z_t^{D^a}(\omega) = Z_t^0(\omega) - D_t^a(\omega) < a(R_t(\omega))$, i.e. $Z_t^0(\omega) - a(R_t(\omega)) < D_t^a(\omega)$. Recalling that $r \mapsto a(r)$ is right-continuous and non-increasing, then it is also lower semi-continuous. Hence $t \mapsto Z^0_t(\omega) - a(R_t(\omega))$ is upper semi-continuous. Then there exists some $\varepsilon := \varepsilon(\omega, t) > 0$ such that

$$\sup_{s\in[t,t+\varepsilon]} \big[Z^0_s(\omega) - a(R_s(\omega))\big]^+ \le D^a_t(\omega).$$

It thus follows that for all $s \in [t, t+\varepsilon]$ we have

$$D^a_s(\omega) = D^a_t(\omega) \vee \sup_{u\in(t,s]} \big[Z^0_u(\omega) - a(R_u(\omega))\big]^+ = D^a_t(\omega). \tag{2.26}$$

In particular, this proves the right-continuity of D^a, so that the process D^a belongs to $\mathcal{A}$. Moreover, since (2.26) holds for any $0 < t \le \tau^{D^a}_\alpha$ such that $Z^{D^a}_{t-}(\omega) < a(R_t(\omega))$, the first integral in (2.25) is zero.

The above implies that the processes (Z^{D^a}, R, D^a) solve the Skorokhod reflection problem for the process (Z^0, R) (with reflecting direction $(-1, 0)$) in the set $\{\alpha \le z < a(r),\, r \ge 0\}$, seen as a relatively open[1] subset of the orthant $\overline{\mathcal{O}}$. By construction, the process cannot jump into the set $\{\alpha \le z < a(r),\, r \ge 0\}$. Indeed jumps are allowed only at points of left discontinuity of a (hence when the boundary $\{z = a(r)\}$ contains a vertical segment) and cannot go out of this boundary.

Step 2. Here we show that $v = V$. Fix $(r, z) \in \mathcal{O}$. We know that (2.20) holds for the special choice of control D^a. The process (Z^{D^a}, R) is constrained to evolve in the set $\{\alpha \le z \le a(r),\, r \ge 0\} = \overline{\mathcal{I}}$ (cf. (2.11)), and v_{rr} is assumed to be continuous therein.

It follows that $(Z^{z,D}_s, R^r_s)_{0\le s\le\vartheta^D_\varepsilon} \in K_\varepsilon \cap \bar{\mathcal{I}}$. and, consequently, that $\mathcal{L}v^k \to \mathcal{L}v$ on $K_\varepsilon \cap \bar{\mathcal{I}}$. Exploiting the second of (2.12) and the continuity of $\mathcal{L}, \rho, v$, this implies that the second term of the right hand side of (2.20) converges to 0 as $k \to \infty$. The limit for the first, the third and the fourth term of (2.20) can be instead obtained as in Part 1, thus yielding (2.21) with equality for the control D^a. Now, recalling (2.24), we see that the random measure $t \mapsto dD^a_t$ is supported on the (random) set of times $t \in [0, \tau^{D^a}_\alpha]$ for which $Z^{D^a}_{t-} \ge a(R_t)$; hence, using the fourth of (2.12), also the inequality of the first line of (2.22) becomes equality when $D = D^a$.

Hence, for $r > 0$ we have

$$\begin{aligned} v(r,z) = \mathsf{E}_{r,z}&\Big[e^{-\int_0^{\vartheta^{D^a}_\varepsilon\wedge t}\rho(R_u)du}\, v(R_{\vartheta^{D^a}_\varepsilon\wedge t}, Z^{D^a}_{\vartheta^{D^a}_\varepsilon\wedge t})\Big] \\ &+ \mathsf{E}_{r,z}\Big[\int_0^{\vartheta^{D^a}_\varepsilon\wedge t} e^{-\int_0^s\rho(R_u)du}\, dD^a_s\Big] \end{aligned} \tag{2.27}$$

and it remains to take limits as $t \uparrow \infty$ and $\varepsilon \downarrow 0$. Assume for a moment that

$$\lim_{\varepsilon\downarrow 0}\lim_{t\uparrow\infty} \mathsf{E}_{r,z}\Big[e^{-\int_0^{\vartheta^{D^a}_\varepsilon\wedge t}\rho(R_u)du}\, v(R_{\vartheta^{D^a}_\varepsilon\wedge t}, Z^{D^a}_{\vartheta^{D^a}_\varepsilon\wedge t})\Big] = 0, \tag{2.28}$$

then the second term in (2.27) also converges by monotone convergence as in (2.23) and we have

$$v(r,z) = \mathsf{E}_{r,z}\Big[\int_0^{\tau^{D^a}_\alpha} e^{-\int_0^s\rho(R_u)du}\, dD^a_s\Big] \le V(r,z)$$

for all $(r, z) \in \mathcal{O}$. By the result in Part 1 of the proof we conclude that $v = V$ on $\mathcal{O}$ and $v(r, \alpha) = V(r, \alpha) = 0$ for all $r \ge 0$. The result extends to $r = 0$ by recalling that $r \mapsto \rho(r)$ is nondecreasing (hence $\rho(R^r_t) \ge \rho(R^0_t)$ for all $t \ge 0$, P-a.s.) and $v \in C(\overline{O})$.

[1] Note that this set is open since a is right-continuous.

Indeed we have

$$\begin{aligned} V(0,z) \le v(0,z) &= \lim_{r\downarrow 0} v(r,z) = \lim_{r\downarrow 0}\sup_{D\in\mathcal{A}} \mathsf{E}_{r,z}\Big[\int_0^{\tau_\alpha^D} e^{-\int_0^s \rho(R_u)du}\, dD_s\Big] \\ &\le \sup_{D\in\mathcal{A}} \mathsf{E}_{0,z}\Big[\int_0^{\tau_\alpha^D} e^{-\int_0^s \rho(R_u)du}\, dD_s\Big] = V(0,z), \end{aligned}$$

where the first inequality was proved in Part 1 above and the second inequality also uses that the set $\mathcal{A}$ does not depend on $r \ge 0$.

Step 3. In this step it only remains to prove (2.28). By using that, by assumption, $v(r,z) \le c(z-\alpha)$ for some $c>0$, we have

$$\begin{aligned} &\mathsf{E}_{r,z}\Big[e^{-\int_0^{\vartheta_\varepsilon^{D^a}\wedge t}\rho(R_u)du} v(R_{\vartheta_\varepsilon^{D^a}\wedge t}, Z^{D^a}_{\vartheta_\varepsilon^{D^a}\wedge t})\Big] \qquad (2.29)\\ &\le c\,\mathsf{E}_{r,z}\Big[e^{-\int_0^{\vartheta_\varepsilon^{D^a}}\rho(R_u)du}\big(Z^{D^a}_{\vartheta_\varepsilon^{D^a}} - \alpha\big)\mathbb{1}_{\{\vartheta_\varepsilon^{D^a}<t\}}\mathbb{1}_{\{\vartheta_\varepsilon^{D^a}=\eta_\varepsilon^{Z^{D^a}}\}}\Big] \\ &\quad + c\,\mathsf{E}_{r,z}\Big[e^{-\int_0^{\vartheta_\varepsilon^{D^a}}\rho(R_u)du}\big(Z^{D^a}_{\vartheta_\varepsilon^{D^a}} - \alpha\big)\mathbb{1}_{\{\vartheta_\varepsilon^{D^a}<t\}}\mathbb{1}_{\{\vartheta_\varepsilon^{D^a}\neq\eta_\varepsilon^{Z^{D^a}}\}}\Big] \\ &\quad + c\,\mathsf{E}_{r,z}\Big[e^{-\int_0^{t}\rho(R_u)du}\big(Z^{D^a}_{t} - \alpha\big)\mathbb{1}_{\{\vartheta_\varepsilon^{D^a}\ge t\}}\Big] \\ &\le c\varepsilon \mathsf{P}_{r,z}[\vartheta_\varepsilon^{D^a} = \eta_\varepsilon^{Z^{D^a}}] \\ &\quad + c\,\mathsf{E}_{r,z}\Big[e^{-\int_0^{\vartheta_\varepsilon^{D^a}}\rho(R_u)du}\big(z-\alpha+\mu\vartheta_\varepsilon^{D^a} + B_{\vartheta_\varepsilon^{D^a}}\big)\mathbb{1}_{\{\vartheta_\varepsilon^{D^a}<t\}}\mathbb{1}_{\{\vartheta_\varepsilon^{D^a}\neq\eta_\varepsilon^{Z^{D^a}}\}}\Big] \\ &\quad + c\,\mathsf{E}_{r,z}\Big[e^{-\int_0^{t}\rho(R_u)du}\big(z-\alpha+\mu t + \sigma B_t\big)\mathbb{1}_{\{\vartheta_\varepsilon^{D^a}\ge t\}}\Big] \end{aligned}$$

where we have used that $Z^{D^a}_{\vartheta_\varepsilon^{D^a}} \le \alpha+\varepsilon$ on the event $\{\vartheta_\varepsilon^{D^a} = \eta_\varepsilon^{Z^{D^a}}\}$, as well as that $Z_t^{D^a} \le Z_t^0 = z+\mu t+\sigma B_t$ for all $t\ge 0$, by (2.1). We now estimate the last two terms of (2.29). For the third one, the independence of B and W and standard inequalities give

$$\begin{aligned} &\mathsf{E}_{r,z}\Big[e^{-\int_0^{t}\rho(R_u)du}\big(z-\alpha+\mu t + \sigma B_t\big)\mathbb{1}_{\{\vartheta_\varepsilon^{D^a}\ge t\}}\Big] \qquad (2.30)\\ &\le \big(z-\alpha+\mu t + \mathsf{E}[|B_t|]\big)\mathsf{E}_r\Big[e^{-\int_0^{t}\rho(R_u)du}\Big] \\ &\le (z-\alpha+\mu t+\sqrt{t})\mathsf{E}_r\Big[e^{-\int_0^{t}\rho(R_u)du}\Big]. \end{aligned}$$

Now we look at the second term. Since $\vartheta_\varepsilon^{D^a} < t$ and B and W are independent, we have

$$\begin{aligned} &\mathsf{E}_{r,z}\Big[e^{-\int_0^{t}\rho(R_u)du}\big(z-\alpha+\mu\vartheta_\varepsilon^{D^a} + B_{\vartheta_\varepsilon^{D^a}}\big)\mathbb{1}_{\{\vartheta_\varepsilon^{D^a}<t\}}\mathbb{1}_{\{\vartheta_\varepsilon^{D^a}\neq\eta_\varepsilon^{Z^{D^a}}\}}\Big] \qquad (2.31)\\ &\le (z-\alpha+\mu t)\mathsf{E}_r\Big[e^{-\int_0^{t}\rho(R_u)du}\Big] + \mathsf{E}_r\Big[e^{-\int_0^{t}\rho(R_u)du}\Big]\mathsf{E}\big[\sup_{0\le s\le t} B_s\big] \\ &\le \mathsf{E}_r\Big[e^{-\int_0^{t}\rho(R_u)du}\Big]\Big(z-\alpha+\mu t+2\sqrt{t}\Big), \end{aligned}$$

where the final inequality follows by Jensen's and Doob's inequalities for B. Feeding (2.30) and (2.31) back into (2.29) we obtain, for a suitable constant $C>0$,

$$\begin{aligned} &\mathsf{E}_{r,z}\Big[e^{-\int_0^{\vartheta_\varepsilon^{D^a}\wedge t}\rho(R_u)du} v(R_{\vartheta_\varepsilon^{D^a}\wedge t}, Z^{D^a}_{\vartheta_\varepsilon^{D^a}\wedge t})\Big] \qquad (2.32)\\ &\le c\varepsilon + C(z-\alpha+\mu t+\sqrt{t})\mathsf{E}_r\Big[e^{-\int_0^{t}\rho(R_u)du}\Big]. \end{aligned}$$

We now distinguish two cases coming from Assumption 2.1-(ii). If $\rho(r) \geq c_1$ for any $r \geq 0$ and for some $c_1 > 0$ then (2.28) is immediately deduced from (2.32). If $\rho(r) \geq c_2 r$ for some $c_2 > 0$, then

$$\mathsf{E}\Big[e^{-\int_0^t \rho(R_u^r)du}\Big] \leq \mathsf{E}\Big[e^{-c_2\int_0^t R_u^r du}\Big] = e^{-A_{c_2}(t)-rG_{c_2}(t)},$$

where we used (2.8) and (2.9) for the equality.

Plugging the latter back into (2.32) we get

$$\mathsf{E}_{r,z}\Big[e^{-\int_0^{\vartheta_\varepsilon^{D^a}\wedge t}\rho(R_u)du}v(R_{\vartheta_\varepsilon^{D^a}\wedge t}, Z^{D^a}_{\vartheta_\varepsilon^{D^a}\wedge t})\Big] \leq c\varepsilon + C(z-\alpha+\mu t+\sqrt{t})e^{-A_{c_2}(t)-rG_{c_2}(t)}$$

and (2.28) holds since (cf. (2.9)) $G_{c_2}(t) \geq 0$ and $A_{c_2}(t) \approx \frac{k\theta}{\gamma^2}(\eta_{c_2}-k)t$ for t sufficiently large, with $\eta_{c_2} > k$. □

From now on we will assume that (2.4) and Assumptions 2.1 and 2.4 hold without repeating them.

3. An Auxiliary Two-Dimensional Optimal Stopping Problem

As we discussed in the Introduction, in order to tackle our singular control problem we follow the approach taken in [9]: we guess a link between the dividend problem and an optimal stopping problem, we then solve the latter by characterizing its optimal stopping boundary, and finally we go back to the original problem. The present section is devoted to introduce and study the optimal stopping problem "associated" to our original optimal dividend problem. After the formulation of such an optimal stopping problem (whose value function is denoted by U), we divide the section into two parts. First, in Section 3.1 we provide basic properties of U (monotonicity, boundedness and continuity, respectively in Lemma 3.3, Proposition 3.4, Proposition 3.6), which in turn allow us to show (Corollary 3.7) that U solves a suitable free boundary problem. Second, in Section 3.2 we prove the global regularity of U (i.e. even across the free boundary; cf. Proposition 3.11), and two additional results on a required boundary condition (Corollary 3.12) and on the optimal stopping boundary (Proposition 3.13).

We denote $\mathbb{F}^B := \sigma(B_t, t \geq 0)$. For $t \geq 0$, let

$$Y_t := -\mu t + \sigma B_t, \quad S_t := \sup_{0\leq u\leq t} Y_u, \quad \text{and} \quad K_t^z := (z-\alpha)\vee S_t - Y_t. \tag{3.1}$$

When clear from the context we will simply write K_t instead K_t^z. Notice that, according to the discussion at p. 2 of [32], the process K is a Markov process. Then, setting

$$\lambda = \frac{2\mu}{\sigma^2}, \tag{3.2}$$

we introduce the optimal stopping problem

$$U(r,z) = \sup_{\tau\geq 0}\mathsf{E}\left[e^{\lambda((z-\alpha)\vee S_\tau-(z-\alpha))-\int_0^\tau \rho(R_s^r)\,ds}\right], \quad (r,z) \in \overline{\mathcal{O}}, \tag{3.3}$$

where the optimization is taken over all the $\mathbb{F}^{K,W}$-stopping times, where $\mathbb{F}^{K,W} := (\mathcal{F}_t^{K,W})_{t\geq 0}$ is the filtration generated by K and W, augmented by the P-null sets. The latter is the optimal stopping problem that we expect to be associated to the original optimal dividend problem. A heuristic derivation of (3.3) can be obtained by employing arguments in the same spirit as those in Section 3 of [9].

Remark 3.1. *Due to the presence of the processes S_t and $\int_0^t \rho(R_s^r)ds$ in the exponential of the gain process, the optimal stopping problem (3.3) may appear non-standard in our Markovian set-up. Indeed, the standard form of a Markovian problem involves* *the expectation of a function of a Markov process, stopped at a stopping time, while the processes S_t and $\int_0^t \rho(R^r_s)ds$ are not Markovian. We now show that (3.3) can be rewritten easily as a standard optimal stopping problem.*

Denote $I^{i,r}_t := i+\int_0^t \rho(R^r_s)ds$, $Y^y_t := y-\mu t+\sigma B_t$ and notice that $K^z_t+Y^0_t = (z-\alpha)\vee S_t$ by (3.1) and that the process (K,Y) is Markovian. Then, it is easy to see that for U as in (3.3) we have

$$U(r,z) = e^{i-\lambda y}\widehat{U}(r,z,y,i), \tag{3.4}$$

where $\widehat{U}$ is the value function of the standard optimal stopping problem

$$\widehat{U}(r,z,y,i) = \sup_{\tau\geq 0} \mathsf{E}\left[e^{\lambda\left(K^z_\tau+Y^y_\tau-(z-\alpha)\right)-I^{i,r}_\tau}\right], \quad (r,z,y,i)\in\overline{\mathcal{O}}\times\mathbb{R}\times\mathbb{R}_+, \tag{3.5}$$

for the four-dimensional Markov process $(R_t,K_t,Y_t,I_t)_{t\geq 0}$. However, due to (3.4), we can abandon the general standard formulation (3.5) and just consider a problem of optimal stopping for the process $(R_t,K_t)_{t\geq 0}$ rather than for the process $(R_t,K_t,Y_t,I_t)_{t\geq 0}$.

Remark 3.2. *It is worth noticing that, for $r\geq 0$,*

$$L_r := \lim_{t\to\infty}\left(\lambda\, S_t - \int_0^t \rho(R^r_s)\,ds\right) \leq \lambda\, S_\infty$$

and by [28, Sec. 3.5.C, Eq. (5.13)]

$$\mathsf{P}(S_\infty > x) = e^{-\lambda x}.$$

Hence $\mathsf{P}(L_r=+\infty)\leq \mathsf{P}(S_\infty=+\infty)=0$ for all $r\geq 0$, since $\mu>0$ (Assumption 2.4).

From now on we focus on the study of problem (3.3). We will then prove in Section 4 how such an optimal stopping problem is related to the original optimal dividend problem.

3.1. **Basic Properties of U and a Free Boundary Problem.**

It is not hard to verify that, P-almost surely, the map

$$(r,z)\mapsto \lambda\left[(z-\alpha)\vee S_\tau-(z-\alpha)\right]-\int_0^\tau \rho(R^r_s)\,ds \tag{3.6}$$

is nonincreasing in z. Moreover, using comparison theorems for (2.3), we also have that the map in (3.6) is nonincreasing in r since $\rho(\,\cdot\,)$ is nondecreasing. These facts imply the next simple result, whose proof is omitted for brevity.

Lemma 3.3. *The map $z\mapsto U(r,z)$ is nonincreasing for each $r\in\mathbb{R}_+$. Moreover, the map $r\mapsto U(r,z)$ is nonincreasing for each $z\in[\alpha,+\infty)$.*

The next proposition gives us an important bound on U, and estimates obtained in its proof will be used several times in the rest of the paper. It is useful to introduce here the random variables

$$H^r := 1+\int_0^\infty e^{-c_2\int_0^t R^r_s\,ds}\,\lambda\, e^{\lambda\, S_t}\,dS_t \tag{3.7}$$

and

$$S^p := \sup_{0\leq t<\infty}\left(B_t-pt\right), \tag{3.8}$$

where $p:=\mu/\sigma+c_1\sigma/2\mu$ and the constants $c_1,c_2\geq 0$ are as in (ii) of Assumption 2.1.

Proposition 3.4. *Recall c_1 and c_2 from (ii) in Assumption 2.1. We have*

$$0 \le U(r,z) \le h_0, \qquad \text{for all } (r,z) \in \overline{\mathcal{O}}, \tag{3.9}$$

where

$$h_0 := \mathsf{E}\big[e^{\lambda\sigma S^p}\big] < +\infty \text{ if } c_1 > 0 \qquad \text{and} \qquad h_0 := \sup_{r\in\mathbb{R}_+} \mathsf{E}[H^r] < +\infty \text{ if } c_2 > 0.$$

Proof. The lower bound in (3.9) is trivial. For the upper bound instead we use Assumption 2.1 to write

$$\mathsf{E}\left[e^{\lambda\,((z-\alpha)\vee S_\tau-(z-\alpha))-\int_0^\tau \rho(R^r_s)\,ds}\right] \le \mathsf{E}\left[e^{\lambda\,S_\tau - c_1\tau - c_2\int_0^\tau R^r_s\,ds}\right].$$

Now, if $c_1 > 0$ we have, by using (3.2),

$$\begin{aligned} U(r,z) &\le \sup_\tau \mathsf{E}\left[e^{\lambda\,S_\tau - c_1\tau}\right] \le \mathsf{E}\big[e^{\lambda\sigma S^p}\big] \\ &= 2p\int_0^\infty e^{\frac{2\mu}{\sigma}y}e^{-2py}dy = 2p\int_0^\infty e^{-\frac{c_1\sigma}{\mu}y}dy < +\infty, \end{aligned} \tag{3.10}$$

where we used that $\mathsf{P}(S^p > x) = \exp(-2px)$ (see [28, Sec. 3.5.C, Eq. (5.13)]).

If instead $c_2 > 0$ (and in particular when $c_1 = 0$) calculations are a bit more involved. Noticing that the process S is of finite variation, we first use an integration by parts to obtain

$$\begin{aligned} U(r,z) \le &\sup_\tau \mathsf{E}\left[e^{\lambda\,S_\tau - c_2\int_0^\tau R^r_s\,ds}\right] \\ =&1 + \sup_\tau \mathsf{E}\left[\int_0^\tau e^{-c_2\int_0^t R^r_s\,ds}\,\lambda\,e^{\lambda\,S_t}\,dS_t - c_2\int_0^\tau e^{-c_2\int_0^t R^r_s\,ds}R^r_t\,e^{\lambda\,S_t}\,dt\right] \\ \le&\mathsf{E}[H^r] \le \sup_{r\in\mathbb{R}_+}\mathsf{E}[H^r], \end{aligned} \tag{3.11}$$

where in the last inequality we used that $R^r_t \ge 0$ for all $t \ge 0$. It remains to prove that $h_0 = \sup_{r\in\mathbb{R}_+}\mathsf{E}[H^r] < +\infty$. Letting

$$H^r_T := \int_0^T e^{-c_2\int_0^t R^r_s\,ds}\,\lambda\,e^{\lambda\,S_t}\,dS_t \tag{3.12}$$

we have $\mathsf{E}[H^r] = \lim_{T\to\infty}\mathsf{E}[H^r_T]$ by monotone convergence. It is therefore sufficient to find a bound for $\mathsf{E}[H^r_T]$ which is independent of T and r. Using independence of B and W, Fubini's theorem and explicit formulae for CIR model (see, e.g., [23], p. 361), we obtain

$$\begin{aligned} \mathsf{E}[H^r_T] =&\mathsf{E}\left[\mathsf{E}\left(\int_0^T e^{-c_2\int_0^t R^r_s\,ds}\,\lambda\,e^{\lambda\,S_t}\,dS_t\,\middle|\,\mathbb{F}^B\right)\right] \\ =&\mathsf{E}\left[\int_0^T \mathsf{E}\left(e^{-c_2\int_0^t R^r_s\,ds}\,\middle|\,\mathbb{F}^B\right)\lambda\,e^{\lambda\,S_t}\,dS_t\right] \\ =&\mathsf{E}\left[\int_0^T \mathsf{E}\left(e^{-c_2\int_0^t R^r_s\,ds}\right)\lambda\,e^{\lambda\,S_t}\,dS_t\right] \\ =&\mathsf{E}\left[\int_0^T e^{-A_{c_2}(t)-rG_{c_2}(t)}\,\lambda\,e^{\lambda\,S_t}\,dS_t\right] \end{aligned} \tag{3.13}$$

where G_{c_2} and A_{c_2} are as in (2.9) with $\beta = c_2$, and where $\eta_{c_2} := \sqrt{k^2 + 2\,\gamma^2\,c_2}$. Setting $f(t) := \mathsf{E}[e^{\lambda S_t}]$, integrating by parts in (3.13), using Fubini and undoing the integration by parts we get

$$\mathsf{E}[H_T^r] = e^{-A_{c_2}(T)-rG_{c_2}(T)} f(T) - e^{-A_{c_2}(0)-rG_{c_2}(0)} - \int_0^T \mathsf{E}\big[e^{\lambda S_t}\big] d\big(e^{-A_{c_2}(t)-rG_{c_2}(t)}\big)$$

$$= \int_0^T e^{-A_{c_2}(t)-rG_{c_2}(t)} f'(t)dt, \tag{3.14}$$

where by Sec. 3.5.C in [28] (upon using equations (5.11) and (5.12) therein, and noticing that our $\mathsf{P}(S_t > b)$ is equal to $\mathsf{P}^{(-\mu)}(T_b \le t)$ in the notation of [28]) we have

$$f(t) = \int_0^\infty e^{\lambda z} \left(\int_0^t \frac{1}{\sqrt{2\pi\sigma^2 s^3}} \Big(\frac{z+\mu s}{\sigma^2 s} z - 1 \Big) e^{-\frac{(z+\mu s)^2}{2\sigma^2 s}} ds \right) dz, \tag{3.15}$$

$$f'(t) = \frac{1}{\sqrt{2\pi\sigma^2 t^3}} \int_0^\infty e^{\lambda z - \frac{(z+\mu t)^2}{2\sigma^2 t}} \Big(\frac{z+\mu t}{\sigma^2 t} z - 1 \Big) dz. \tag{3.16}$$

Recalling that $\lambda = 2\mu/\sigma^2$, straightforward algebra gives

$$\lambda z - \frac{(z+\mu t)^2}{2\sigma^2 t} = -\frac{(z-\mu t)^2}{2\sigma^2 t}.$$

Changing variable in the integral (3.16) we obtain

$$\begin{aligned} f'(t) =& \frac{1}{\sqrt{2\pi\sigma^2 t^3}} \int_0^\infty e^{-\frac{(z-\mu t)^2}{2\sigma^2 t}} \Big(\frac{z+\mu t}{\sigma^2 t} z - 1 \Big) dz \\ =& \frac{1}{t} \int_{-\mu t}^\infty \Big(\frac{y+2\mu t}{\sigma^2 t}(y+\mu t) - 1 \Big) \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{y^2}{2\sigma^2 t}} dy \\ =& \frac{1}{t} \mathsf{E}\left[\mathbb{1}_{\{\sigma B_t \ge -\mu t\}} \Big(\frac{\sigma B_t + 2\mu t}{\sigma^2 t}(\sigma B_t + \mu t) - 1 \Big) \right] \\ =& \frac{1}{t} \mathsf{E}\left[\mathbb{1}_{\{\sigma B_t \ge -\mu t\}} \Big(\frac{B_t^2}{t} - 1 + \frac{3\mu}{\sigma} B_t + \frac{2\mu^2}{\sigma^2} t \Big) \right] \\ \le& \frac{2\mu^2}{\sigma^2} + \frac{3\mu}{\sigma\sqrt{t}} + \frac{1}{t} \mathsf{E}\left[\mathbb{1}_{\{\sigma B_t \ge -\mu t\}} \Big(\frac{B_t^2}{t} - 1 \Big) \right]. \end{aligned} \tag{3.17}$$

The last term above may be evaluated as follows:

$$\begin{aligned} \frac{1}{t} \mathsf{E}\left[\mathbb{1}_{\{\sigma B_t \ge -\mu t\}} \Big(\frac{B_t^2}{t} - 1 \Big) \right] &= \frac{1}{t} \int_{-\frac{\mu t}{\sigma}}^\infty \frac{1}{\sqrt{2\pi t}} \Big(\frac{y^2}{t} - 1 \Big) e^{-\frac{y^2}{2t}} dy \\ =& \frac{1}{t} \left(\int_{-\frac{\mu t}{\sigma}}^\infty \frac{1}{\sqrt{2\pi t}} y \Big(-e^{-\frac{y^2}{2t}} \Big)' dy - \int_{-\frac{\mu t}{\sigma}}^\infty \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}} dy \right) = -\frac{\mu}{\sigma\sqrt{2\pi}} \frac{1}{\sqrt{t}} e^{-\frac{\mu^2 t}{2\sigma^2}} < 0, \end{aligned} \tag{3.18}$$

where, in the last equality, we have used the integration by parts. Using (3.17)-(3.18) above in (3.14) we then conclude

$$\begin{aligned} \mathsf{E}[H_T^r] \le& \int_0^T e^{-A_{c_2}(t)-rG_{c_2}(t)} \Big(\frac{2\mu^2}{\sigma^2} + \frac{3\mu}{\sigma\sqrt{t}} \Big) dt \\ \le& \int_0^\infty e^{-A_{c_2}(t)} \Big(\frac{2\mu^2}{\sigma^2} + \frac{3\mu}{\sigma\sqrt{t}} \Big) dt < +\infty, \end{aligned} \tag{3.19}$$

where the last integral is finite because $A_{c_2}(t) \approx \frac{k\theta}{\gamma^2}(\eta_{c_2} - k)t$ as $t \to \infty$, $\eta_{c_2} > k$, and $A_{c_2}(0) = 0$. □

An important consequence of the proof of Proposition 3.4 is that

$$\mathsf{E}\left[\sup_{0\le t<\infty} e^{\lambda[(z-\alpha)\vee S_t]-\int_0^t \rho(R^r_s)ds}\right] < +\infty, \quad \text{for all } (r,z)\in\overline{\mathcal{O}}. \tag{3.20}$$

Moreover, it is not hard to verify that the Markov process $(K_t, S_t, Y_t, R_t, \int_0^t \rho(R_s)ds)_{t\ge0}$ is also of Feller type. Then, [39, Lemma 3, Sec. 3.2.3 and Lemma 4, Sec. 3.2.4] guarantees that there exists a lower semi-continuous function u which is the smallest superharmonic function bigger than one (see Remark 3.1 for a detailed comparison with [39]). Here, superharmonic refers to the property

$$u(r,z) \ge \mathsf{E}\left[e^{\lambda[(z-\alpha)\vee S_\tau]-\int_0^\tau \rho(R^r_s)ds} u\big(R^r_\tau, K^z_\tau\big)\right]$$

for any stopping time τ and any $(r,z)\in\overline{\mathcal{O}}$. Now, let us introduce the sets

$$\mathcal{C} := \{(r,z)\in\overline{\mathcal{O}} \,:\, U(r,z) > 1\}, \tag{3.21}$$

$$\mathcal{S} := \{(r,z)\in\overline{\mathcal{O}} \,:\, U(r,z) = 1\}, \tag{3.22}$$

known in the literature as *continuation* and *stopping* sets, respectively. Thanks to [39, Thm. 1, Sec. 3.3.1 and Thm. 3, Sec. 3.3.3], and the fact that U is lower semi-continuous, we have that $U = u$ and that

$$\tau_* := \inf\{t\ge 0 \,:\, (R_t, K_t)\in\mathcal{S}\} \tag{3.23}$$

is the smallest optimal stopping time for (3.3), provided that $\mathsf{P}_{r,z}(\tau_* < +\infty) = 1$, otherwise it is an optimal Markov time. In some instances below we will stress the dependence on the data (r,z) of the optimal stopping time, i.e.,

$$\tau_*(r,z) := \inf\{t\ge 0 \,:\, (R^r_t, K^z_t)\in\mathcal{S}\}. \tag{3.24}$$

Moreover, recalling again that U is lower semi-continuous and given the process

$$\Lambda_t := e^{\lambda((z-\alpha)\vee S_t-(z-\alpha))-\int_0^t \rho(R_s)ds} U(R_t, K_t), \qquad t\ge 0,$$

then

$$(\Lambda_t)_{t\ge0} \text{ is a } \mathsf{P}_{r,z}\text{-supermartingale} \tag{3.25}$$

and

$$(\Lambda_{t\wedge\tau_*})_{t\ge0} \text{ is a } \mathsf{P}_{r,z}\text{-martingale} \tag{3.26}$$

for all $(r,z)\in\overline{\mathcal{O}}$ (see [34, Thm. 2.4, Sec. 2, Chapter I] or [39, Sec. 3.4]).

Next we provide a technical lemma which is useful to prove continuity of U later on.

Lemma 3.5. *For $n>0$, let us denote*

$$U^n(r,z) = \sup_{0\le\tau\le n} \mathsf{E}\left[e^{\lambda\,((z-\alpha)\vee S_\tau-(z-\alpha))-\int_0^\tau \rho(R^r_s)\,ds}\right], \quad (r,z)\in\overline{\mathcal{O}}.$$

Then for all $(r,z)\in\overline{\mathcal{O}}$ we have

$$\lim_{n\to\infty} U^n(r,z) = U(r,z).$$

Proof. Clearly $(U^n)_{n>0}$ is an increasing sequence and $U^n \le U$ for all $n>0$. Therefore we denote its limit $U^\infty := \lim_{n\to\infty} U^n \le U$. Let us now fix $(r,z)\in\mathbb{R}_+\times[\alpha,+\infty)$ and let $\tau_* = \tau_*(r,z)$ be optimal for $U(r,z)$. Then

$$U^n(r,z) \ge \mathsf{E}_{r,z}\left[e^{\lambda L^\alpha_{\tau_*\wedge n}(\xi)-\int_0^{\tau_*\wedge n}\rho(R_t)dt}\right]$$

and using Fatou's lemma we conclude

$$
\begin{aligned}
U^\infty(r,z) = \liminf_{n\to\infty} U^n(r,z) \geq & \mathsf{E}_{r,z}\left[\liminf_{n\to\infty} e^{\lambda L^\alpha_{\tau_*\wedge n}(\xi) - \int_0^{\tau_*\wedge n}\rho(R_t)dt}\right] \\
= & \mathsf{E}_{r,z}\left[e^{\lambda L^\alpha_{\tau_*}(\xi) - \int_0^{\tau_*}\rho(R_t)dt}\right] = U(r,z).
\end{aligned}
$$

□

We close this section by proving that U is indeed continuous. It is worth remarking that all our results hold without any restriction on μ, σ, and the only requirement is $2k\theta \geq \gamma^2$ to guarantee strictly positive rates.

Proposition 3.6. (Continuity of U) *The function U is continuous on $\overline{\mathcal{O}}$ and $z \mapsto U(r,z)$ is convex for each $r \in \mathbb{R}_+$.*

Proof. First we show convexity. Since

$$
z \mapsto e^{\lambda\,[(z-\alpha)\vee S_\tau - (z-\alpha)] - \int_0^\tau \rho(R^r_s)\,ds}
$$

is convex and $\sup(f+g) \leq \sup(f) + \sup(g)$, we easily obtain

$$
\begin{aligned}
& U(r, \beta z_1 + (1-\beta)z_2) \\
& \leq \sup_{\tau\geq 0} \mathsf{E}\left[\Big(\beta e^{\lambda\,[(z_1-\alpha)\vee S_\tau - (z_1-\alpha)]} + (1-\beta)e^{\lambda\,[(z_2-\alpha)\vee S_\tau-(z_2-\alpha)]}\Big)e^{-\int_0^\tau \rho(R^r_s)\,ds}\right] \\
& \leq \beta U(r,z_1) + (1-\beta)U(r,z_2)
\end{aligned}
$$

for all $\beta \in (0,1)$.

Now we show that $z \mapsto U(r,z)$ is continuous uniformly with respect to $r \in \mathbb{R}_+$. Recall that $U(r,\,\cdot\,)$ is decreasing (Lemma 3.3), let $z_2 > z_1$ and denote by $\tau_1 := \tau_*(r,z_1)$ the optimal stopping time for $U(r,z_1)$. Since τ_1 is suboptimal in $U(r,z_2)$ we get

$$
\begin{aligned}
0 \leq & U(r,z_1) - U(r,z_2) \\
\leq & \mathsf{E}\left[e^{-\int_0^{\tau_1}\rho(R_t)dt}\Big(e^{\lambda((z_1-\alpha)\vee S_{\tau_1}-(z_1-\alpha)} - e^{\lambda((z_2-\alpha)\vee S_{\tau_1}-(z_2-\alpha)}\Big)\right] \\
\leq & \mathsf{E}\left[\mathbb{1}_{\{S_{\tau_1}>z_1-\alpha\}}e^{\lambda S_{\tau_1}-\int_0^{\tau_1}\rho(R_t)dt}\Big(e^{-\lambda(z_1-\alpha)} - e^{-\lambda(z_2-\alpha)}\Big)\right] \\
\leq & h_0\Big(e^{-\lambda(z_1-\alpha)} - e^{-\lambda(z_2-\alpha)}\Big)
\end{aligned}
\tag{3.27}
$$

where h_0 is as in Proposition 3.4 and we have also used that

$$
e^{\lambda(S_{\tau_1}-(z_2-\alpha))} \leq e^{\lambda((z_2-\alpha)\vee S_{\tau_1}-(z_2-\alpha))}.
$$

It only remains to prove that $r \mapsto U(r,z)$ is continuous for each $z \in [\alpha,+\infty)$ given and fixed.

Since ρ is nondecreasing (cf. (i) in Assumption 2.1), then $r \mapsto U(r,z)$ is nonincreasing (Lemma 3.3) and lower semi-continuous (see the discussion above Lemma 3.5). Hence $r \mapsto U(r,z)$ is right-continuous for each $z \in [\alpha,+\infty)$. Recalling U^n from Lemma 3.5, and noticing that $U(r,z) - U(r-h,z) \leq 0$ increases as $h \downarrow 0$, we have

$$
\begin{aligned}
0 \geq \lim_{h\to 0}\big[U(r,z) - U(r-h,z)\big] = & \lim_{h\to 0}\lim_{n\to\infty}\Big[U^n(r,z) - U(r-h,z)\Big] \\
= \lim_{n\to\infty}\lim_{h\to 0}\Big[U^n(r,z) - U(r-h,z)\Big],
\end{aligned}
$$

where we are allowed to swap the limits as both sequences are increasing (as $n\to\infty$ and $h\to 0$). Now we set $\tau_h := \tau_*(r-h,z)$, which is optimal for $U(r-h,z)$, and consider the suboptimal stopping time $\tau_h \wedge n$ inside U^n. With no loss of generality we assume

$r - h \geq r_0$ for some $r_0 > 0$. Then, using that $\rho(R^{r-h}_\cdot) \geq \rho(R^{r_0}_\cdot)$ (in the last term of the expression below), we obtain

$$
\begin{aligned}
(3.28) \qquad & U^n(r,z) - U(r-h,z) \\
\geq & \mathsf{E}\left[\mathbb{1}_{\{\tau_h \leq n\}} e^{\lambda((z-\alpha)\vee S_{\tau_h} - (z-\alpha)) - \int_0^{\tau_h} \rho(R^r_t)dt} \left(1 - e^{-\int_0^{\tau_h}[\rho(R^{r-h}_t) - \rho(R^r_t)]dt}\right)\right] \\
& + \mathsf{E}\Big[\mathbb{1}_{\{\tau_h > n\}} e^{\lambda((z-\alpha)\vee S_n - (z-\alpha)) - \int_0^n \rho(R^r_t)dt} \cdot \\
& \cdot \left(1 - e^{\lambda((z-\alpha)\vee S_{\tau_h} - (z-\alpha)\vee S_n) - \int_0^n [\rho(R^{r-h}_t) - \rho(R^r_t)]dt} e^{-\int_n^{\tau_h} \rho(R^{r_0}_t)dt}\right)\Big].
\end{aligned}
$$

We make a number of observations: (i) since $\tau_h = \inf\{t \geq 0 \;\; U(K^z_t, R^{r-h}_t) \leq 1\}$, and $U(z, \cdot)$ is nonincreasing, we have $\tau_h \downarrow \sigma$, P-a.s. as $h \to 0$ with σ a stopping time; (ii) the latter implies that P-a.s. we have

$$
\lim_{h\to 0} S_{\tau_h} = S_\sigma \quad \text{and} \quad \lim_{h\to 0} \int_n^{\tau_h} \rho(R^r_t)dt = \int_n^\sigma \rho(R^r_t)dt \quad \text{for all } r > 0;
$$

(iii) by dominated convergence and continuity of ρ we have, P-a.s.

$$
\lim_{h\to 0} \int_0^n \left|\rho(R^{r-h}_t) - \rho(R^r_t)\right| dt = 0,
$$

which also implies

$$
\lim_{h\to 0} \left(\mathbb{1}_{\{\tau_h \leq n\}} \int_0^{\tau_h} [\rho(R^{r-h}_t) - \rho(R^r_t)]dt\right) = 0.
$$

Recalling (3.20) we can use dominated convergence in (3.28) to obtain

$$
\begin{aligned}
(3.29) \qquad 0 \geq & \lim_{n\to\infty} \lim_{h\to 0} \left[U^n(r,z) - U(r-h,z)\right] \\
\geq & \lim_{n\to\infty} \mathsf{E}\Big[\mathbb{1}_{\{\sigma \geq n\}} e^{\lambda((z-\alpha)\vee S_{\sigma\wedge n} - (z-\alpha)) - \int_0^{\sigma\wedge n} \rho(R^r_t)dt} \cdot \\
& \cdot \left(1 - e^{\lambda((z-\alpha)\vee S_\sigma - (z-\alpha)\vee S_{\sigma\wedge n}) - \int_{\sigma\wedge n}^\sigma \rho(R^{r_0}_t)dt}\right)\Big].
\end{aligned}
$$

It is now easy to check that, P-a.s.

$$
\lim_{n\to\infty} \left[\lambda((z-\alpha)\vee S_\sigma - (z-\alpha)\vee S_{\sigma\wedge n}) - \int_{\sigma\wedge n}^\sigma \rho(R^{r_0}_t)dt\right] = 0.
$$

Hence, using dominated convergence once again in (3.29), gives

$$
0 \geq \lim_{h\to 0} \left[U(r,z) - U(r-h,z)\right] = \lim_{n\to\infty} \lim_{h\to 0} \left[U^n(r,z) - U(r-h,z)\right] \geq 0
$$

as claimed. □

Continuity of U immediately implies that $\mathcal{S}$ is closed and that $\mathcal{C}$ is relatively open in $\overline{\mathcal{O}}$: indeed, by its definition, $\mathcal{C}$ may not be open in $\mathbb{R}^2$ since it may include a portion of the lines $\{r = 0\}$ and $\{z = \alpha\}$. For this reason we will use the notation $\partial\mathcal{C}$ for the boundary of $\mathcal{C}$ in $\mathbb{R}^2$ and $\partial_{\overline{\mathcal{O}}}\mathcal{C}$ for the relative boundary in $\overline{\mathcal{O}}$. Moreover $\operatorname{Int}\mathcal{C}$ will denote the interior of $\mathcal{C}$ in $\mathbb{R}^2$.

Observe now that the (super)martingale property of the process Λ (see (3.25) and (3.26)), along with standard arguments (see, e.g., [29, Theorem 2.7.7]) gives the following corollary.

Corollary 3.7. (Free boundary problem) *The function U belongs to C^2 separately in the interior of $\mathcal{C}$ and in the interior of $\mathcal{S}$ (so away from $\partial\mathcal{C}$), and it satisfies*

$$\mathcal{L}U(r,z) - \rho(r)U(r,z) = 0, \qquad \textit{for } (r,z) \in \operatorname{Int}\mathcal{C} \tag{3.30}$$

$$\mathcal{L}U(r,z) - \rho(r)U(r,z) = -\rho(r), \qquad \textit{for } (r,z) \in \operatorname{Int}\mathcal{S} \tag{3.31}$$

$$U(r,z) = 1, \qquad \textit{for } (r,z) \in \partial_{\overline{\mathcal{O}}}\mathcal{C}. \tag{3.32}$$

Refined regularity of U and its behaviour at $\mathbb{R}_+ \times \{\alpha\}$ will be provided in the next section.

3.2. **Differentiability of U.** In order to obtain higher regularity properties for U we need some information on the shape of the stopping region $\mathcal{S}$. Recalling Lemma 3.3 (in particular the fact that U is nonincreasing in z) and defining, for $r \geq 0$,

$$b(r) := \sup\{z \in [\alpha, +\infty) \,:\, U(r,z) > 1\} \tag{3.33}$$

with the convention that $\sup\varnothing = \alpha$, we immediately find, for $r \in \mathbb{R}_+$,

$$\mathcal{S}_r := \{z \in [\alpha, +\infty) \,:\, (r,z) \in \mathcal{S}\} = [b(r), +\infty). \tag{3.34}$$

This means that the r-section of the stopping set is connected and the graph of the map $r \mapsto b(r)$ describes the boundary (that we decided above to denote by $\partial_{\overline{\mathcal{O}}}\mathcal{C}$) that separates $\mathcal{S}$ from $\mathcal{C}$. Next we state few important properties of the optimal boundary.

Lemma 3.8. *Consider the map $\mathbb{R}_+ \to [\alpha, +\infty]$, $r \mapsto b(r)$, where $b(r)$ is defined in (3.33). Then*

$$r \mapsto b(r) \textit{ is nonincreasing and right-continuous.} \tag{3.35}$$

Moreover, $b(r) > \alpha$ for all $r \geq 0$.

Proof. The fact that $\mathcal{S}$ is closed and (3.34) imply that $r \mapsto b(r)$ is lower semi-continuous. Indeed take any sequence $(r_n)_{n\geq 1}$ converging to some $r_0 \geq 0$. Then

$$(r_n, b(r_n)) \in \mathcal{S} \implies \mathcal{S} \ni \liminf_{n\to\infty}(r_n, b(r_n)) = (r_0, \liminf_{n\to\infty} b(r_n))$$

and by (3.33) we have $\liminf_{n\to\infty} b(r_n) \geq b(r_0)$. Using again Lemma 3.3 (in particular the fact that U is nonincreasing in r) we have

$$(r,z) \in \mathcal{S} \implies [r, +\infty) \times \{z\} \in \mathcal{S}, \tag{3.36}$$

i.e., $r \mapsto b(r)$ is nonincreasing. Since $b(\cdot)$ is also lower semi-continuous, then (3.35) holds.

It only remains to prove the final statement. Take any $r_0 \geq 0$, fix $\varepsilon > 0$ and denote $\tau_\varepsilon = \inf\{t \geq 0 : R_t^{r_0} \geq r_0 + \varepsilon\}$. For any $t > 0$ the stopping time $\tau_\varepsilon \wedge t$ is admissible and suboptimal for $U(r_0, \alpha)$ so that

$$U(r_0,\alpha) \geq \mathsf{E}\left[e^{\lambda S_{\tau_\varepsilon\wedge t} - \int_0^{\tau_\varepsilon\wedge t}\rho(R_s^{r_0})ds}\right] \geq \exp\left(\mathsf{E}\left[\lambda S_{\tau_\varepsilon\wedge t} - \int_0^{\tau_\varepsilon\wedge t}\rho(R_s^{r_0})ds\right]\right), \tag{3.37}$$

where the final inequality is due to Jensen's inequality. Recalling that ρ is nondecreasing (Assumption 2.1) we have

$$\int_0^{\tau_\varepsilon\wedge t}\rho(R_s^{r_0})ds \leq \bar{\rho}_\varepsilon(\tau_\varepsilon \wedge t),$$

with $\bar{\rho}_\varepsilon := \rho(r_0+\varepsilon) = \sup_{0\le r\le r_0+\varepsilon}\rho(r)$. Now we use estimates as in [33, Lemma 15]. In particular, we have

$$\begin{aligned}(3.38)\qquad \mathsf{E}\left[\lambda S_{\tau_\varepsilon\wedge t} - \int_0^{\tau_\varepsilon\wedge t}\rho(R^{r_0}_s)ds\right] &\ge \mathsf{E}\left[\lambda\sigma \sup_{0\le s\le \tau_\varepsilon\wedge t} B_s - (\mu+\bar{\rho}_\varepsilon)(\tau_\varepsilon\wedge t)\right]\\ &\ge \lambda\sigma\mathsf{E}\left[\sup_{0\le s\le t} B_s - \mathbb{1}_{\{\tau_\varepsilon\le t\}}\sup_{0\le s\le t} B_s\right] - (\mu+\bar{\rho}_\varepsilon)t\\ &\ge \lambda\sigma\mathsf{E}\left[\sup_{0\le s\le t} B_s\right] - \lambda\sigma\mathsf{P}(\tau_\varepsilon\le t)^{\frac12}\mathsf{E}\left[\big(\sup_{0\le s\le t} B_s\big)^2\right]^{\frac12} - (\mu+\bar{\rho}_\varepsilon)t\\ &= \lambda\sigma\sqrt{t}\left(1-\mathsf{P}(\tau_\varepsilon\le t)^{\frac12}\right) - (\mu+\bar{\rho}_\varepsilon)t\end{aligned}$$

where in the final inequality we used that $\sup_{0\le s\le t} B_s = |B_t|$ in law. Since $\mathsf{P}(\tau_\varepsilon > 0) = 1$ and, consequently, $\mathsf{P}(\tau_\varepsilon \le t)\to 0$ as we let $t\to 0$, we have that the term involving $\sqrt{t}$ dominates. Hence, plugging (3.38) in (3.37) and choosing t sufficiently small we reach $U(r_0,\alpha) > 1$ which implies $b(r_0) > \alpha$. Since $r_0 \ge 0$ was arbitrary, the proof is complete. □

The simple properties that we have obtained above are crucial to guarantee global C^1 regularity of U. We start by noticing that K and R are independent and have transition densities $p^K(t,z;z')$ and $p^R(t,r;r')$, respectively, which are continuous with respect to the initial point, i.e. $z\mapsto p^K(t,z;z')$ and $r\mapsto p^R(t,r;r')$ are continuous for all $t>0$ $z'\in[\alpha,+\infty)$, $r'\in[0,+\infty)$. Then it is not hard to verify that the process $(R_t,K_t)_{t\ge0}$ is strong Feller, i.e. for any Borel measurable and bounded function $f:\mathbb{R}_+\times\mathbb{R}_+$ and any $t>0$, it holds that $(r,z)\mapsto \mathsf{E}_{r,z}[f(R_t,K_t)]$ is continuous. We then have the following important result.

Lemma 3.9. *For any $(r_0,z_0)\in\partial_{\overline{\mathcal{O}}}\mathcal{C}$ and any sequence $(r_n,z_n)_{n\ge1}\subset\mathcal{C}$ such that $(r_n,z_n)\to(r_0,z_0)$ as $n\to\infty$, we have*

$$(3.39)\qquad \lim_{n\to\infty}\tau_*(r_n,z_n) = 0 \quad \mathsf{P}\text{-a.s.}$$

Proof. Let us denote by σ_* the first hitting time of (K,R) to $\mathcal{S}$:

$$\sigma_*(r,z) := \inf\{t>0\,:\,(R^r_t,K^z_t)\in\mathcal{S}\}.$$

It is well known (see [12, Chapter 13.1-2, Vol. II]) that since $(R_t,K_t)_{t\ge0}$ is a strong Feller process, (3.39) holds if and only if all the boundary points are regular for $\mathcal{S}$, namely

$$(3.40)\qquad \mathsf{P}_{r,z}(\sigma_*=0)=1 \qquad \forall (r,z)\in\partial_{\overline{\mathcal{O}}}\mathcal{C}.$$

(For further details on the above statement the reader may consult, e.g., [28, Theorem 2.12, Ch. 4.2] and [10, pp. 4-5 and Corollary 2].)

Denoting by

$$\widehat{\sigma}_*(r,z) := \inf\{t\ge0\,:\,(R^r_t,K^z_t)\in\operatorname{Int}\mathcal{S}\}$$

the first entry time of $(R^r_t,K^z_t)_{t\ge0}$ to the interior of $\mathcal{S}$, and noticing that $\sigma_*\le\widehat{\sigma}_*$, we now prove (3.40) by showing that

$$(3.41)\qquad \mathsf{P}_{r,z}(\widehat{\sigma}_*=0)=1 \qquad \forall (r,z)\in\partial_{\overline{\mathcal{O}}}\mathcal{C}.$$

Let $(r_0,z_0)\in\partial_{\overline{\mathcal{O}}}\mathcal{C}$. Define $\mathcal{R} := [r_0,\infty)\times[z_0,\infty)$, and denote by $\operatorname{Int}\mathcal{R}$ and $\partial\mathcal{R}$ respectively its interior and its boundary in $\mathbb{R}^2$. Since $r\mapsto b(r)$ is nonincreasing, we have $\mathcal{R}\subseteq\mathcal{S}$. Also, let $\mathcal{K}$ be a compact neighbourhood of r_0 and let $\operatorname{Int}\mathcal{K}$ and $\partial\mathcal{K}$ denote respectively its interior and its boundary in $\mathbb{R}^2$. Since $(r_0, z_0) \in \partial_{\overline{\mathcal{O}}}\mathcal{C}$ then $r_0 > 0$ and we assume that $\mathcal{K} \cap \{r = 0\} = \varnothing$. Then there exists some $\eta_{\mathcal{K}} > 0$ such that

$$\eta_{\mathcal{K}}^{-1} \geq \gamma^2 r \geq \eta_{\mathcal{K}} \quad \text{on } \mathcal{K} \tag{3.42}$$

so that the diffusion coefficient of the process $(R_t)_{t\geq 0}$ is uniformly non degenerate over $\mathcal{K}$. Let us define an auxiliary process $(\widetilde{R}_t)_{t\geq 0}$ with dynamics

$$\begin{aligned} d\widetilde{R}_t &= b_{\mathcal{K}}(\widetilde{R}_t)dt + \gamma_{\mathcal{K}}(\widetilde{R}_t)dW_t, & \widetilde{R}_0 &= r, \\ d\widetilde{K}_t &= \mu dt + \sigma dB_t, & \widetilde{K}_0 &= z, \end{aligned}$$

where $b_{\mathcal{K}}(r) = \kappa(\theta - r)$ and $\gamma_{\mathcal{K}}(r) = \gamma\sqrt{r}$ on $\mathcal{K}$, and are continuously extended to be constant outside $\mathcal{K}$. Notice that the uniform ellipticity condition (3.42) holds for $\gamma_{\mathcal{K}}$ on the whole $\mathbb{R}$.

Since the process $(\widetilde{R}_t, \widetilde{K}_t)_{t\geq 0}$ is non degenerate over the whole $\mathbb{R}^2$, it admits a continuous transition density $\widetilde{p}(\cdot,\cdot,\cdot; r, z)$ such that, for any $t > 0$

$$\frac{M}{t}e^{-\lambda_0\frac{|r-\bar{r}|^2+|z-\bar{z}|^2}{t}} \geq \widetilde{p}(t, \bar{r}, \bar{z}; r, z) \geq \frac{m}{t}e^{-\Lambda_0\frac{|r-\bar{r}|^2+|z-\bar{z}|^2}{t}} \tag{3.43}$$

for some constants $M > m > 0$, $\Lambda_0 > \lambda_0 > 0$ (see, e.g., [2, Theorem 1]). Moreover, denoting

$$\tau_{\mathcal{K}} := \inf\{t \geq 0 : (R_t, K_t) \notin \operatorname{Int}\mathcal{K} \times (\alpha, \infty)\}$$

and

$$\widetilde{\tau}_{\mathcal{K}} := \inf\{t \geq 0 : (\widetilde{R}_t, \widetilde{K}_t) \notin \operatorname{Int}\mathcal{K} \times (\alpha, \infty)\},$$

we have that

$$(R_{t\wedge\tau_{\mathcal{K}}}, K_{t\wedge\tau_{\mathcal{K}}}) = (\widetilde{R}_{t\wedge\widetilde{\tau}_{\mathcal{K}}}, \widetilde{K}_{t\wedge\widetilde{\tau}_{\mathcal{K}}}), \quad \mathsf{P}_{r_0,z_0}\text{-a.s.} \tag{3.44}$$

by uniqueness of the solution of the SDE (recall that the reflected process K is just a Brownian motion with drift away from the reflection point α).

Now, let $\mathcal{R}'$ be a (half) cone with vertex in (r_0, z_0), whose closure is contained in $\operatorname{Int}\mathcal{R} \cup (r_0, z_0)$, and denote by $\widehat{\sigma}'_{\mathcal{R}}$ and $\widetilde{\sigma}'_{\mathcal{R}}$ the corresponding entry times of (R, K) and $(\widetilde{R}, \widetilde{K})$, respectively, into the interior of $\mathcal{R}'$, denoted by $\partial\mathcal{R}'$. Notice that this additional cone is needed in the argument that follows because (t_0, z_0) may lie on a horizontal/vertical stretch of the boundary $\partial_{\overline{\mathcal{O}}}\mathcal{C}$, in which case $(\partial_{\overline{\mathcal{O}}}\mathcal{C} \cap \partial\mathcal{R}) \setminus (r_0, z_0) \neq \varnothing$ whereas $(\partial_{\overline{\mathcal{O}}}\mathcal{C} \cap \partial\mathcal{R}') \setminus (r_0, z_0) = \varnothing$ always holds. Fixing $t > 0$ we then have

$$\begin{aligned} \mathsf{P}_{r_0,z_0}(\widehat{\sigma}_* \leq t) &\geq \mathsf{P}_{r_0,z_0}(\widehat{\sigma}'_{\mathcal{R}} \leq t) \geq \mathsf{P}_{r_0,z_0}(\widehat{\sigma}'_{\mathcal{R}} \leq t, \tau_{\mathcal{K}} > t) \\ &= \mathsf{P}_{r_0,z_0}(\widetilde{\sigma}'_{\mathcal{R}} \leq t, \widetilde{\tau}_{\mathcal{K}} > t) = \mathsf{P}_{r_0,z_0}(\widetilde{\sigma}'_{\mathcal{R}} \leq t) - \mathsf{P}_{r_0,z_0}(\widetilde{\tau}_{\mathcal{K}} \leq t), \end{aligned} \tag{3.45}$$

where the first equality holds by (3.44). Thanks to (3.43)

$$\mathsf{P}_{r_0,z_0}(\widetilde{\sigma}'_{\mathcal{R}} \leq t) = \int_{\mathcal{R}'} \widetilde{p}(t, r_0, z_0; r, z)drdz \geq \int_{\mathcal{R}'} \frac{m}{t}e^{-\Lambda_0\frac{|r-r_0|^2+|z-z_0|^2}{t}}drdz. \tag{3.46}$$

Using the fact that the change of variable $s = \frac{r-r_0}{\sqrt{t}}$, $\zeta = \frac{z-z_0}{\sqrt{t}}$ maps the cone $\mathcal{R}'$ into a cone $\mathcal{R}'_0$ with the same aperture but vertex in $(0, 0)$, we get

$$\mathsf{P}_{r_0,z_0}(\widetilde{\sigma}'_{\mathcal{R}} \leq t) \geq \int_{\mathcal{R}'_0} m\, e^{-\Lambda_0(s^2+\zeta^2)}dsd\zeta =: q > 0.$$

Letting $t \to 0$ we obtain $\mathsf{P}_{r_0,z_0}(\widetilde{\sigma}'_{\mathcal{R}} = 0) \geq q > 0$ and therefore, by (3.45), also that $\mathsf{P}_{r_0,z_0}(\widehat{\sigma}_* = 0) \geq q > 0$ upon noting that $\mathsf{P}_{r_0,z_0}(\widetilde{\tau}_{\mathcal{K}} \leq t) \to 0$ as $t \to 0$.

Since $\{\widehat{\sigma}_* = 0\}$ is measurable with respect to the trivial σ-algebra $\mathcal{F}_0^{K,W}$, by the Blumenthal's 0-1 Law we obtain $\mathsf{P}_{r_0,z_0}(\widehat{\sigma}_* = 0) = 1$, which completes the proof. □

Lemma 3.10. *There is a constant $c>0$ such that, for all $\mathbb{F}^{K,W}$-stopping times τ, and any $(r,z)\in\mathbb{R}_+\times[\alpha,+\infty)$, it holds*

$$\mathsf{E}\left[e^{\lambda S_\tau-\int_0^\tau \rho(R_t^r)dt}\int_0^\tau e^{-\frac{k}{2}t}\left[1+(R_t^r)^{1+q}\right]\sqrt{R_t}\,dt\right]\le c. \tag{3.47}$$

Moreover the family

$$\left\{e^{\lambda S_\tau-\int_0^\tau \rho(R_t^r)dt}\int_0^\tau e^{-\frac{k}{2}t}\left[1+(R_t^r)^{1+q}\right]\sqrt{R_t}\,dt,\ \ \tau\ge 0\right\}$$

is uniformly integrable.

Proof. Using that S is of finite variation we integrate by parts to get a first, convenient, upper bound

$$\begin{aligned}
&e^{\lambda S_\tau-\int_0^\tau \rho(R_t^r)dt}\int_0^\tau e^{-\frac{k}{2}t}\left[1+(R_t^r)^{1+q}\right]\sqrt{R_t^r}dt\\
\le&\lambda\int_0^\tau e^{\lambda S_t-\int_0^t \rho(R_s^r)ds}\left(\int_0^t e^{-\frac{k}{2}s}\left[1+(R_s^r)^{1+q}\right]\sqrt{R_s^r}ds\right)dS_t\\
&\qquad+\int_0^\tau e^{\lambda S_t-\int_0^t \rho(R_s^r)ds-\frac{k}{2}t}\left[1+(R_t^r)^{1+q}\right]\sqrt{R_t^r}dt\\
\le&\lambda\int_0^\infty e^{\lambda S_t-\int_0^t \rho(R_s^r)ds}\left(\int_0^t e^{-\frac{k}{2}s}\left[1+(R_s^r)^{1+q}\right]\sqrt{R_s^r}ds\right)dS_t\\
&\qquad+\int_0^\infty e^{\lambda S_t-\int_0^t \rho(R_s^r)ds-\frac{k}{2}t}\left[1+(R_t^r)^{1+q}\right]\sqrt{R_t^r}dt\\
=:&\,A+B.
\end{aligned}$$

Hence, to prove both claims of this lemma it is enough to show that $\mathsf{E}[A]+\mathsf{E}[B]<+\infty$.

We start by proving that $\mathsf{E}[B]<+\infty$. Using that $\rho\ge 0$ (see Assumption 2.1), that $\sqrt{r}\le 1+r$, Fubini's theorem and independence of S_t and R_t we obtain

$$\mathsf{E}[B]\le c\int_0^\infty \mathsf{E}\left[e^{\lambda S_t-\frac{k}{4}t}\right]e^{-\frac{k}{4}t}\mathsf{E}\left[1+(R_t^r)^{1+q}\right]dt$$

for some constant $c>0$, which will vary from line to line. Observe now that (recall (3.8))

$$\lambda S_t-\frac{k}{4}t\le\lambda\sigma\sup_{0\le s\le t}\left(B-\frac{\mu}{\sigma}s-\frac{k}{4\lambda\sigma}s\right)\le\lambda\sigma S^p,\qquad\text{with } p=\tfrac{\mu}{\sigma}+\tfrac{k}{4\lambda\sigma}.$$

Since $\mathsf{P}(S^p_\infty>x)=\exp(-2px)$ for $p>0$ (see Remark 3.2), as in (3.10) we easily get $\mathsf{E}\left[\exp(\lambda S_t-kt/4)\right]\le c'$ for some $c'>0$. Hence

$$\mathsf{E}[B]\le c\int_0^\infty e^{-\frac{k}{4}t}\mathsf{E}\left[1+(R_t^r)^{1+q}\right]dt. \tag{3.48}$$

Now we recall [13, Thm. 2.1], which states that there is a constant $C_q>0$, only depending on q and the coefficients of the SDE (2.3), such that

$$\mathsf{E}\left[1+(R_t^r)^{1+q}\right]\le C_q,\quad\text{for all } t\ge 0. \tag{3.49}$$

Plugging the latter bound in (3.48) we get $\mathsf{E}[B]<+\infty$.

Next we show that $\mathsf{E}[A]<+\infty$. We only provide full details in the case $\rho(r)\ge c_2 r$ (see Assumption 2.1), since the case $\rho(r)\ge c_1$ is easier and can be dealt with in the same way. Below we use $\mathsf{E}[A] = \mathsf{E}[\mathsf{E}(A|\mathbb{F}^B)]$ and independence of R from $\mathbb{F}^B$. Then, recalling that $\sqrt{r} \le 1 + r$, by Fubini's theorem we obtain

$$
\begin{aligned}
\mathsf{E}[A] \le & c\,\mathsf{E}\left[\int_0^\infty e^{\lambda S_t}\mathsf{E}\left(e^{-c_2\int_0^t R_s^r ds}\int_0^t (1+(R_s^r)^{1+q})e^{-\frac{k}{2}s}ds\big|\mathbb{F}^B\right)dS_t\right] \\
\le & c\,\mathsf{E}\left[\int_0^\infty e^{\lambda S_t}\mathsf{E}\left(e^{-c_2\int_0^t R_s^r ds}\int_0^t (1+(R_s^r)^{1+q})e^{-\frac{k}{2}s}ds\right)dS_t\right]
\end{aligned}
\tag{3.50}
$$

for some constant $c > 0$, which will vary from line to line. Repeated use of Hölder inequality and (2.8) give

$$
\begin{aligned}
&\mathsf{E}\left(e^{-c_2\int_0^t R_s^r ds}\int_0^t (1+(R_s^r)^{1+q})e^{-\frac{k}{2}s}ds\right) \\
&\le \mathsf{E}\left(e^{-2c_2\int_0^t R_s^r ds}\right)^{\frac{1}{2}}\mathsf{E}\left[\left(\int_0^t (1+(R_s^r)^{1+q})e^{-\frac{k}{2}s}ds\right)^2\right]^{\frac{1}{2}} \\
&\le e^{-\frac{1}{2}A_{2c_2}(t)-\frac{r}{2}G_{2c_2}(t)}\mathsf{E}\left[\int_0^t e^{-\frac{k}{2}s}ds\int_0^t (1+(R_s^r)^{1+q})^2 e^{-\frac{k}{2}s}ds\right]^{\frac{1}{2}} \\
&\le C_q'\, e^{-\frac{1}{2}A_{2c_2}(t)-\frac{r}{2}G_{2c_2}(t)},
\end{aligned}
$$

where the final inequality follows from (3.49), with some $C_q' > 0$.

Plugging the last expression above in (3.50) gives

$$
\mathsf{E}[A] \le c\,\mathsf{E}\left[\int_0^\infty e^{\lambda S_t}e^{-\frac{1}{2}A_{2c_2}(t)-\frac{r}{2}G_{2c_2}(t)}dS_t\right].
$$

The latter can be treated exactly by the same methods that we used to estimate (3.13), hence $\mathsf{E}[A] < +\infty$. □

The methodology that we adopt to prove C^1 regularity of the value function was developed in [10] for general multi-dimensional, finite-time and infinite-time horizon, optimal stopping problems. However, due to the square root in the diffusion coefficient of the CIR dynamics, some of the integrability conditions required in [10] seem difficult to verify directly. So in the proof of Proposition 3.11 below we adapt the method to our setting.

Proposition 3.11. **(C^1 regularity of U)** *One has that $U \in C^1(\mathcal{O})$. Moreover*

$$
U_z(r,z) = -\lambda\mathsf{E}_{r,z}\left[\mathbb{1}_{\{S_{\tau_*}>z-\alpha\}}e^{\lambda(S_{\tau_*}-(z-\alpha))-\int_0^{\tau_*}\rho(R_t)dt}\right]
\tag{3.51}
$$

for all $(r,z) \in \mathcal{O}$.

Proof. The proof is organized in two steps.

Step 1. We start by noticing that (3.51) trivially holds in the interior of $\mathcal{S}$ with $U_z = 0$. Further, we know that U_z is continuous in $\operatorname{Int}\mathcal{C}$, so that if we can prove (3.51) in $\operatorname{Int}\mathcal{C}$, then Lemma 3.9 and the use of dominated convergence will also imply continuity of U_z across $\partial_{\overline{\mathcal{O}}}\mathcal{C}$. Finally, to show that (3.51) holds in $\operatorname{Int}\mathcal{C}$ we can repeat the same steps as in the proof of [9, Thm. 5.3], upon replacing the discount factor therein by $\int_0^{\tau_*}\rho(R_s)ds$. We omit further details in the interest of brevity.

Step 2. Here we prove that $U_r \in C(\mathcal{O})$. We know that U_r is continuous separately in $\operatorname{Int}\mathcal{C}$ and $\operatorname{Int}\mathcal{S}$. Then, it suffices to prove continuity across the boundary $\partial_{\overline{\mathcal{O}}}\mathcal{C}$. We start finding bounds on U_r.

Fix $(r,z)\in \operatorname{Int}\mathcal{C}$, $\varepsilon\in(0,\varepsilon_0)$, and denote $\tau_*:=\tau_*(r,z)$. Recalling Lemma 3.3 and optimality of τ_* for $U(r,z)$, we obtain

$$
\begin{aligned}
(3.52)\qquad 0 &\geq \frac{U(r+\varepsilon,z)-U(r,z)}{\varepsilon}\\
&\geq \frac{1}{\varepsilon}\mathsf{E}\left[e^{\lambda((z-\alpha)\vee S_{\tau_*}-(z-\alpha))-\int_0^{\tau_*}\rho(R_s^{r+\varepsilon})ds}\left(1-e^{\int_0^{\tau_*}(\rho(R_s^{r+\varepsilon})-\rho(R_s^r))ds}\right)\right]\\
&\geq \frac{1}{\varepsilon}\mathsf{E}\left[e^{\lambda((z-\alpha)\vee S_{\tau_*}-(z-\alpha))-\int_0^{\tau_*}\rho(R_s^{r})ds}\left(1-e^{\int_0^{\tau_*}c_3(1+(R_s^{r+\varepsilon})^q)(R_s^{r+\varepsilon}-R_s^r)ds}\right)\right],
\end{aligned}
$$

where in the last inequality we have used Assumption 2.1, (i) and (iii), and the fact that $r\mapsto R^r$ is nondecreasing.

Still using that $r\mapsto R^r$ is nondecreasing, we can estimate the difference $\Delta_\varepsilon R_s := R_s^{r+\varepsilon}-R_s^r$ as

$$
(3.53)\qquad \Delta_\varepsilon R_s = \left(\sqrt{R_s^{r+\varepsilon}}-\sqrt{R_s^r}\right)\left(\sqrt{R_s^{r+\varepsilon}}+\sqrt{R_s^r}\right)\leq 2\sqrt{R_s^{r+\varepsilon_0}}\left(\sqrt{R_s^{r+\varepsilon}}-\sqrt{R_s^r}\right).
$$

Next, we notice that by Tanaka formula and Yamada-Watanabe's theorem, the process $A:=\sqrt{R}$ is the unique solution to

$$
dA_t=\left[\left(\frac{k\theta}{2}-\frac{\gamma^2}{8}\right)\frac{1}{A_t}-\frac{k}{2}A_t\right]dt+\frac{\gamma}{2}dW_t,\quad A_0=\sqrt{R_0}.
$$

We then have

$$
\begin{aligned}
d(A_t\,e^{\frac{k}{2}t}) &= e^{\frac{k}{2}t}\,dA_t+A_t\,\frac{k}{2}e^{\frac{k}{2}t}\,dt\\
&= e^{\frac{k}{2}t}\left[\left(\frac{k\theta}{2}-\frac{\gamma^2}{8}\right)\frac{1}{A_t}-\frac{k}{2}A_t\right]dt+e^{\frac{k}{2}t}\frac{\gamma}{2}dW_t+A_t\,\frac{k}{2}e^{\frac{k}{2}t}\,dt\\
&= e^{\frac{k}{2}t}\left[\left(\frac{k\theta}{2}-\frac{\gamma^2}{8}\right)\frac{1}{A_t}\right]dt+e^{\frac{k}{2}t}\frac{\gamma}{2}dW_t,
\end{aligned}
$$

which gives in the integral form

$$
A_s\,e^{\frac{k}{2}s}=A_0+\left(\frac{k\theta}{2}-\frac{\gamma^2}{8}\right)\int_0^s e^{\frac{k}{2}t}\frac{1}{A_t}dt+\int_0^s e^{\frac{k}{2}t}\frac{\gamma}{2}dW_t.
$$

Hence, using the above formula, we obtain

$$
\begin{aligned}
&\left(\sqrt{R_s^{r+\varepsilon}}-\sqrt{R_s^r}\right)e^{\frac{k}{2}s}\\
&=\sqrt{r+\varepsilon}-\sqrt{r}-\left(\frac{k\theta}{2}-\frac{\gamma^2}{8}\right)\int_0^s e^{\frac{k}{2}t}\frac{\sqrt{R_t^{r+\varepsilon}}-\sqrt{R_t^r}}{\sqrt{R_t^{r+\varepsilon}}\sqrt{R_t^r}}dt\\
&\leq \sqrt{r+\varepsilon}-\sqrt{r},
\end{aligned}
$$

where the inequality follows from $R^r\leq R^{r+\varepsilon}$, upon recalling that $2k\theta\geq\gamma^2$. Therefore,

$$
(3.54)\qquad \left(\sqrt{R_s^{r+\varepsilon}}-\sqrt{R_s^r}\right)\leq(\sqrt{r+\varepsilon}-\sqrt{r})e^{-\frac{k}{2}s}
$$

and plugging (3.54) into (3.53), we find

$$
\Delta_\varepsilon R_s\leq 2\sqrt{R_s^{r+\varepsilon_0}}\,(\sqrt{r+\varepsilon}-\sqrt{r})\,e^{-\frac{k}{2}s}.
$$

Hence, substituting in the last integral of (3.52) we get

$$\begin{aligned}&\int_0^{\tau_*} c_3(1+(R_s^{r+\varepsilon})^q)(R_s^{r+\varepsilon}-R_s^r)ds\\&\le 2\left(\sqrt{r+\varepsilon}-\sqrt{r}\right)\int_0^{\tau_*} e^{-\frac{k}{2}s}c_3(1+(R_s^{r+\varepsilon})^q)\sqrt{R_s^{r+\varepsilon_0}}ds.\end{aligned}$$

Plugging this expression in (3.52) and using that

$$\begin{aligned}1-e^{2(\sqrt{r+\varepsilon}-\sqrt{r})C}&=-\varepsilon C\int_0^1\frac{1}{\sqrt{r+\varepsilon u}}e^{2(\sqrt{r+\varepsilon u}-\sqrt{r})C}du\\&\ge-\varepsilon Ce^{2(\sqrt{r+\varepsilon_0}-\sqrt{r})C}\int_0^1\frac{1}{\sqrt{r+\varepsilon u}}du\\&=-Ce^{2(\sqrt{r+\varepsilon_0}-\sqrt{r})C}2(\sqrt{r+\varepsilon}-\sqrt{r}),\end{aligned}$$

for any $C\ge 0$ independent of ε, we continue with the chain of inequalities

$$\begin{aligned}0&\ge\frac{U(r+\varepsilon,z)-U(r,z)}{\varepsilon}\\&\ge-\frac{2(\sqrt{r+\varepsilon}-\sqrt{r})}{\varepsilon}\\&\quad\cdot\mathsf{E}\bigg[e^{\lambda((z-\alpha)\vee S_{\tau_*}-(z-\alpha))-\int_0^{\tau_*}\rho(R_s^r)ds}\int_0^{\tau_*}e^{-\frac{k}{2}s}c_3(1+(R_s^{r+\varepsilon})^q)\sqrt{R_s^{r+\varepsilon_0}}ds\\&\qquad\cdot\exp\left(2\left(\sqrt{r+\varepsilon_0}-\sqrt{r}\right)\int_0^{\tau_*}e^{-\frac{k}{2}s}c_3(1+(R_s^{r+\varepsilon})^q)\sqrt{R_s^{r+\varepsilon_0}}ds\right)\bigg].\end{aligned}$$

Now we let $\varepsilon\to 0$ first, and then we also let $\varepsilon_0\to 0$. Thanks to monotone convergence we obtain

$$\begin{aligned}0&\ge U_r(r,z)\\&\ge-\frac{1}{\sqrt{r}}\mathsf{E}\left[e^{\lambda((z-\alpha)\vee S_{\tau_*}-(z-\alpha))-\int_0^{\tau_*}\rho(R_s^r)ds}\int_0^{\tau_*}c_3(1+(R_s^r)^q)\sqrt{R_s^r}\,e^{-\frac{k}{2}s}ds\right].\end{aligned}\tag{3.55}$$

We notice that the right-hand side above is bounded by a constant, thanks to Lemma 3.10.

Now, fix $(r_0,z_0)\in\partial_{\overline{\mathcal{O}}}\mathcal{C}$ and take a sequence $\operatorname{Int}\mathcal{C}\ni(\mathrm{r_n},\mathrm{z_n})\to(\mathrm{r_0},\mathrm{z_0})$, as $n\to\infty$. Using (3.55) with (r_n,z_n) in place of (r,z), recalling that $\tau_*(r_n,z_n)\to 0$ by Lemma 3.9, and using dominated convergence (justified by the second claim of Lemma 3.10), we get

$$0\ge\limsup_{n\to\infty}U_r(r_n,z_n)\ge\liminf_{n\to\infty}U_r(r_n,z_n)\ge 0.$$

Since the boundary point was arbitrary we conclude that U_r is continuous across $\partial_{\overline{\mathcal{O}}}\mathcal{C}$. □

An immediate consequence of the above proposition is the following.

Corollary 3.12. *For all $r\in\mathbb{R}_+$, we have*

$$U_z(r,\alpha+)=-\lambda\,U(r,\alpha).\tag{3.56}$$

Proof. Fix $r\ge 0$ and let $z_n\downarrow\alpha$ as $n\to\infty$. Then, if

$$\tau_*^n:=\tau_*(r,z_n)\to\tau_*^\alpha=\tau_*(r,\alpha)\ \text{ as } n\to\infty,\quad \mathsf{P}\text{-a.s.},\tag{3.57}$$

it suffices to take limits in (3.51). As a matter of fact, by dominated convergence (recall (3.20)) we obtain

$$U_z(r,\alpha+) = -\lambda \mathsf{E}\left[e^{\lambda S_{\tau_*^\alpha} - \int_0^{\tau_*^\alpha} \rho(R_t^r)dt}\right] = -\lambda\, U(r,\alpha),$$

where, in order to remove the indicator function in the limit of (3.51), we have also used that $\mathsf{P}(S_{\tau_*^\alpha} > 0) = 1$, being $\mathsf{P}(\tau_*^\alpha > 0) = 1$ since $b(r) > \alpha$ by Lemma 3.8. So it only remains to prove convergence of the stopping times in (3.57).

The sequence $(K^{z_n})_{n\geq 1}$ is decreasing and therefore the sequence of stopping times $(\tau_*^n)_{n\geq 1}$ is increasing with $\tau_*^n \leq \tau_*^\alpha$ for all $n \geq 1$. Hence, $\tau_*^n \uparrow \tau^\infty \leq \tau_*^\alpha$, P-a.s., for some stopping time τ^∞. Now we show that $\tau^\infty = \tau_*^\alpha$ as needed, using an argument similar to those used in [7, Lem. 4.17] and [31, Lem. 1.2] but under different conditions.

Recall that $(t,r,z) \mapsto (R_t^r(\omega), K_t^z(\omega))$ is continuous for all $\omega \in \Omega \setminus N$ and some universal null set N by Kolmogorov-Chentsov continuity theorem. Fix $\omega \in \Omega \setminus N$. Let $\delta > 0$ be such that $\tau_*^\alpha(\omega) > \delta$, then by continuity of paths there exists $c_\delta > 0$ such that

$$\inf_{0\leq t\leq\delta}\Big(U(R_t^r(\omega), K_t^\alpha(\omega)) - 1\Big) \geq c_\delta.$$

Thanks to (3.27) and the explicit dynamics of $(K_t)_{t\geq 0}$ in (3.1), we have

$$\sup_{0\leq t\leq\delta}\Big|U(R_t^r(\omega), K_t^\alpha(\omega)) - U(R_t^r(\omega), K_t^{z_n}(\omega))\Big| \leq h_0(z_n - \alpha).$$

Then there is $n_{\delta,\omega} \geq 1$ such that

$$\inf_{0\leq t\leq\delta}\Big(U(R_t^r(\omega), K_t^{z_n}(\omega)) - 1\Big) \geq \frac{c_\delta}{2}$$

for all $n \geq n_{\delta,\omega}$. Hence $\lim_{n\to\infty}\tau_*^n(\omega) > \delta$ and, since δ was arbitrary

$$\lim_{n\to\infty}\tau_*^n(\omega) \geq \tau_*^\alpha(\omega).$$

Recalling that $\omega \in \Omega\setminus N$ was also arbitrary, we conclude. □

We close this section providing further results on the optimal boundary.

Proposition 3.13. *One has:*

(i) $b(r) < +\infty$ *for all* $r > 0$*;*

(ii) *if* $\rho(r) \geq c_1$ *for some* $c_1 > 0$*, then there exists* $z_{c_1}^\star \in (\alpha,\infty)$ *such that* $b(r) \leq z_{c_1}^\star$ *for all* $r \geq 0$*;*

(iii) *if* $\rho(r) \geq c_2 r$ *for some* $c_2 > 0$*, then* $\lim_{r\uparrow\infty} b(r) = \alpha$.

Proof. We prove each item separately.

(i) Suppose that there exists $r_o > 0$ such that $b(r_o) = +\infty$. Then, by monotonicity, $b(r) = +\infty$ for all $r \in [0, r_o)$. Then take $r \in [0, r_o)$ and set $\widehat{\tau} := \inf\{t \geq 0 : R_t^r \geq r_o\}$, P-a.s. Clearly, $\widehat{\tau} \leq \tau_*$ $\mathsf{P}_{r,z}$-a.s. for all $z \geq \alpha$, and therefore the superharmonic property property of the value U (cf. (3.25) and (3.26)) implies that

$$\begin{aligned}
(3.58)\qquad 1 < U(r,z) &= \mathsf{E}\left[e^{\lambda((z-\alpha)\vee S_{\widehat{\tau}} - (z-\alpha)) - \int_0^{\widehat{\tau}}\rho(R_s^r)\,ds}U(R_{\widehat{\tau}}^r, K_{\widehat{\tau}}^z)\right] \\
&\leq \mathsf{E}\left[\mathbb{1}_{\{S_{\widehat{\tau}} \geq z-\alpha\}} e^{\lambda(S_{\widehat{\tau}} - (z-\alpha)) - \int_0^{\widehat{\tau}}\rho(R_s^r)ds} h_0\right] \\
&\quad + \mathsf{E}\left[\mathbb{1}_{\{S_{\widehat{\tau}} < z-\alpha\}} e^{-\int_0^{\widehat{\tau}}\rho(R_s^r)\,ds}U(R_{\widehat{\tau}}^r, K_{\widehat{\tau}}^z)\right] \\
&\leq e^{-\lambda(z-\alpha)}\mathsf{E}\left[e^{\lambda S_{\widehat{\tau}} - \int_0^{\widehat{\tau}}\rho(R_s^r)ds}h_0\right] + \mathsf{E}\left[e^{-\int_0^{\widehat{\tau}}\rho(R_s^r)\,ds}U(R_{\widehat{\tau}}^r, K_{\widehat{\tau}}^z)\right].
\end{aligned}$$

By noticing that $\widehat{\tau}$ does not depend on z, recalling (3.20), and taking limits as $z \uparrow \infty$ we obtain

$$\lim_{z\to\infty} e^{-\lambda\,(z-\alpha)}\mathsf{E}\left[e^{\lambda\,S_{\widehat{\tau}}-\int_0^{\widehat{\tau}}\rho(R^r_s)ds}h_0\right]=0.$$

On the other hand, for any $r\in[0,\,r_o]$ we have

$$\begin{aligned}1<U(r,z)&=\sup_{\tau\geq 0}\mathsf{E}\left[e^{\lambda\,((z-\alpha)\vee S_\tau-(z-\alpha))-\int_0^\tau\rho(R^r_s)\,ds}\right]\\&\leq\sup_{\tau\geq 0}\mathsf{E}\left[\mathbb{1}_{\{S_\tau\geq z-\alpha\}}\,e^{\lambda\,(S_\tau-(z-\alpha))-\int_0^\tau\rho(R^r_s)\,ds}\right]+\sup_{\tau\geq 0}\mathsf{E}\left[\mathbb{1}_{\{S_\tau<z-\alpha\}}\right]\\&\leq e^{-\lambda(z-\alpha)}\sup_{\tau\geq 0}\mathsf{E}\left[e^{\lambda\,S_\tau-\int_0^\tau\rho(R^r_s)\,ds}\right]+1\\&\leq h_0\,e^{-\lambda(z-\alpha)}+1.\end{aligned}$$

It follows that $\lim_{z\to+\infty}U(r,z)=1$ for any $r\in[0,\,r_o]$. Recalling that $\lim_{z\to+\infty}K^z_t=+\infty$ a.s., and noticing that the CIR process is positively recurrent, this in turn yields

$$\lim_{z\to\infty}U(R^r_{\widehat{\tau}},K^z_{\widehat{\tau}})=1\text{ a.s.}$$

Thus, applying the Lebesgue dominated convergence theorem in (3.58), we get

$$1\leq\mathsf{E}\left[e^{-\int_0^{\widehat{\tau}}\rho(R^r_s)\,ds}\right].$$

Being $\mathsf{P}_r(\widehat{\tau}>0)>0$ for any $r\in[0,r_o)$, we reach a contradiction.

(ii) Assume that $\rho(r)\geq c_1$ for some $c_1>0$. Because

$$U(r,z)\leq\sup_{\tau\geq 0}\mathsf{E}\left[e^{\lambda\,((z-\alpha)\vee S_{\widehat{\tau}}-(z-\alpha))-c_1\tau}\right]=:v(z;c_1),$$

one has for any $r\geq 0$ that

$$\{z>\alpha:\;z\geq b(r)\}=\{z>\alpha:\;U(r,z)=1\}\supseteq\{z>\alpha:\;v(z;c_1)=1\}.$$

Notice now that $v(z;c_1)\leq e^{\lambda(z-\alpha)}\overline{v}$ for some constant $\overline{v}>0$ for all $z\geq 0$ (cf. (3.10)), and that $\{z>\alpha:\;v(z;c_1)=1\}=\{z>\alpha:\;z\geq z^\star_{c_1}\}$ for some $z^\star_{c_1}\in(\alpha,\infty)$. Hence we conclude that $b(r)\leq z^\star_{c_1}$.

(iii) Assume that $\rho(r)\geq c_2 r$ for some $c_2>0$. To prove that $\lim_{r\uparrow\infty}b(r)=\alpha$ we argue by contradiction and we suppose that $b_\infty:=\lim_{r\uparrow\infty}b(r)>\alpha$. Then take z_1,z_2 such that $\alpha<z_1<z_2<b_\infty$ and for $z\in(z_1,z_2)$ and $r\geq 0$ set $\widehat{\sigma}:=\inf\{t\geq 0:K_t\notin(z_1,z_2)\}$ P_z-a.s. Clearly, $\mathbb{R}_+\times(z_1,z_2)\subset\mathcal{C}$, and therefore $\widehat{\sigma}\leq\tau_*$ $\mathsf{P}_{r,z}$-a.s., and this fact implies that (see (3.26))

$$\begin{aligned}1<U(r,z)&=\mathsf{E}\left[e^{\lambda\,((z-\alpha)\vee S_{\widehat{\sigma}}-(z-\alpha))-\int_0^{\widehat{\sigma}}\rho(R^r_s)\,ds}U(R^r_{\widehat{\sigma}},K^z_{\widehat{\sigma}})\right]\\&\leq h_0\mathsf{E}\left[e^{\lambda\,((z-\alpha)\vee S_{\widehat{\sigma}}-(z-\alpha))-c_2\int_0^{\widehat{\sigma}}R^r_s\,ds}\right]\\&=h_0\mathsf{E}\left[e^{\lambda S_{\widehat{\sigma}}-A_{c_2}(\widehat{\sigma})-rG_{c_2}(\widehat{\sigma})}\right].\end{aligned}\tag{3.59}$$

Here, (3.9) has been used for the penultimate step, while the independence of the Brownian motions W and B led to the last equality, together (2.8) and (2.9). Since the last expectation on the right-hand side of (3.59) can be made arbitrarily small by taking r sufficiently large, we reach a contradiction and we have thus proved that $\lim_{r\uparrow\infty}b(r)=\alpha$. □

4. Solution to the dividend problem

In this section we show that we can find a couple (v, a) that satisfies all the assumptions in Theorem 2.5, hence we obtain a full solution to problem (2.5).

Let us define the function $v : \overline{\mathcal{O}} \to \mathbb{R}_+$ as follows

$$v(r,z) := \int_\alpha^z U(r,y)dy. \tag{4.1}$$

Using Proposition 3.11 we obtain that the functions v_z, v_{zz}, v_r and v_{zr} are continuous on $\mathcal{O}$.

Proposition 4.1. *The function v has a weak derivative $v_{rr} \in L^\infty_{loc}(\mathcal{O})$. Moreover, we can select an element of the equivalence class of $v_{rr} \in L^\infty_{loc}(\mathcal{O})$ (still denoted by v_{rr}) such that*

$$\begin{aligned}
&v_{rr}(r,z) \\
&= \mathbb{1}_{\{b_-(r)\geq\alpha\}} \frac{2}{\gamma^2}\left(\int_\alpha^{b_-(r)\wedge z} \Big[\rho(r)U(r,y) - \mu U_z(r,y) - k(\theta-r)U_r(r,y)\Big]dy\right) r^{-1} \\
&\quad - \mathbb{1}_{\{b_-(r)\geq\alpha\}} \frac{\sigma^2}{\gamma^2}\left(U_z(r, z\wedge b_-(r)) - U_z(r,\alpha+)\right) r^{-1},
\end{aligned} \tag{4.2}$$

where $b_-(\cdot) := \lim_{\varepsilon\downarrow 0} b(\cdot - \varepsilon)$.

Proof. The main idea in this proof is to compute explicitly the weak derivative v_{rr}.

Since $v_r(\cdot, z)$ is a continuous function for all $z > \alpha$, we say that its weak derivative with respect to r is a function $f \in L^1(\mathcal{O})$ such that, for any $\varphi \geq 0$ with $\varphi \in C_c^\infty(\mathbb{R}_+)$, it holds

$$\int_0^\infty v_r(\eta,z)\varphi'(\eta)d\eta = -\int_0^\infty f(\eta,z)\varphi(\eta)d\eta, \quad \text{for } z \in (\alpha,+\infty).$$

We denote by g the generalised, right-continuous, inverse of the decreasing function b and, for future frequent use, we also define $g_\varepsilon(\,\cdot\,) := g(\,\cdot\,) - \varepsilon$ for $\varepsilon > 0$.

Using that U_r is continuous, with $U_r(\eta, y) = 0$ for $\eta \geq g(y)$, and employing Fubini's theorem we can write

$$\begin{aligned}
&\int_0^\infty v_r(\eta,z)\varphi'(\eta)d\eta \\
&= \int_0^\infty\left(\int_\alpha^z U_r(\eta,y)dy\right)\varphi'(\eta)d\eta = \int_\alpha^z\left(\int_0^\infty U_r(\eta,y)\varphi'(\eta)d\eta\right)dy \\
&= \int_\alpha^z\left(\int_0^{g(y)} U_r(\eta,y)\varphi'(\eta)d\eta\right)dy = \int_\alpha^z\left(\lim_{\varepsilon\to 0}\int_0^{g_\varepsilon(y)} U_r(\eta,y)\varphi'(\eta)d\eta\right)dy \\
&= \lim_{\varepsilon\to 0}\int_\alpha^z\left(\int_0^{g_\varepsilon(y)} U_r(\eta,y)\varphi'(\eta)d\eta\right)dy,
\end{aligned} \tag{4.3}$$

where in the last line we used dominated convergence. We now recall that

$$\tfrac{\gamma^2}{2} r U_{rr} = \rho(r)U - \tfrac{\sigma^2}{2}U_{zz} - \mu U_z - k(\theta - r)U_r$$

in $\mathcal{C}$ and that U_{rr} is continuous away from $\partial\mathcal{C}$. This implies that for fixed $\varepsilon > 0$ we can write (recalling that $\varphi(0) = 0$)

$$\int_0^{g_\varepsilon(y)} U_r(\eta, y)\varphi'(\eta)d\eta = U_r(g_\varepsilon(y), y)\varphi(g_\varepsilon(y)) - \int_0^{g_\varepsilon(y)} U_{rr}(\eta, y)\varphi(\eta)d\eta$$
$$= U_r(g_\varepsilon(y), y)\varphi(g_\varepsilon(y))$$
$$- \frac{2}{\gamma^2}\left(\int_0^{g_\varepsilon(y)} \eta^{-1}\Big[\rho(\eta)U(\eta, y) - \tfrac{\sigma^2}{2}U_{zz}(\eta, y) - \mu U_z(\eta, y) - k(\theta-\eta)U_r(\eta, y)\Big]\varphi(\eta)d\eta\right).$$

Plugging the latter in (4.3) we find

$$\lim_{\varepsilon\to 0}\int_\alpha^z \left(\int_0^{g_\varepsilon(y)} U_r(\eta, y)\varphi'(\eta)d\eta\right)dy \tag{4.4}$$
$$= \lim_{\varepsilon\to 0}\int_\alpha^z U_r(g_\varepsilon(y), y)\varphi(g_\varepsilon(y))dy$$
$$- \lim_{\varepsilon\to 0}\int_\alpha^z \frac{2}{\gamma^2}\left(\int_0^{g_\varepsilon(y)} \eta^{-1}\Big[\rho(\eta)U(\eta, y) - \mu U_z(\eta, y) - k(\theta-\eta)U_r(\eta, y)\Big]\varphi(\eta)d\eta\right)dy$$
$$+ \lim_{\varepsilon\to 0}\int_\alpha^z \frac{\sigma^2}{\gamma^2}\left(\int_0^{g_\varepsilon(y)} \eta^{-1}U_{zz}(\eta, y)\varphi(\eta)d\eta\right)dy.$$

For the first two limits on the right-hand side of (4.4) we can use dominated convergence and recall that $U_r(g(y), y) = 0$ to get

$$\int_0^\infty v_r(\eta, z)\varphi'(\eta)d\eta \tag{4.5}$$
$$= -\int_\alpha^z \frac{2}{\gamma^2}\left(\int_0^{g(y)} \eta^{-1}\Big[\rho(\eta)U(\eta, y) - \mu U_z(\eta, y) - k(\theta-\eta)U_r(\eta, y)\Big]\varphi(\eta)d\eta\right)dy$$
$$+ \lim_{\varepsilon\to 0}\int_\alpha^z \frac{\sigma^2}{\gamma^2}\left(\int_0^{g_\varepsilon(y)} \eta^{-1}U_{zz}(\eta, y)\varphi(\eta)d\eta\right)dy.$$

For the remaining term on the right-hand side of (4.4), we set $b_\varepsilon(\eta)$ as the generalized inverse of $g_\varepsilon(\eta)$, use Fubini's theorem and obtain

$$\lim_{\varepsilon\to 0}\int_\alpha^z \frac{\sigma^2}{\gamma^2}\left(\int_0^{g_\varepsilon(y)} \eta^{-1}U_{zz}(\eta, y)\varphi(\eta)d\eta\right)dy \tag{4.6}$$
$$= \frac{\sigma^2}{\gamma^2}\lim_{\varepsilon\to 0}\int_0^{g_\varepsilon(\alpha)}\left(\int_\alpha^{z\wedge b_\varepsilon(\eta)} U_{zz}(\eta, y)dy\right)\eta^{-1}\varphi(\eta)d\eta$$
$$= \frac{\sigma^2}{\gamma^2}\int_0^{g(\alpha)} \left(U_z(\eta, z\wedge b(\eta)) - U_z(\eta, \alpha+)\right)\eta^{-1}\varphi(\eta)d\eta,$$

where in the last line we also used $b_\varepsilon \to b$ and $g_\varepsilon \to g$. Combining (4.5) and (4.6), and using Fubini's theorem once more we find

$$\begin{aligned}
&\int_0^\infty v_r(\eta,z)\varphi'(\eta)d\eta \\
&= -\int_0^{g(\alpha)} \frac{2}{\gamma^2}\left(\int_\alpha^{b(\eta)\wedge z}\Big[\rho(\eta)U(\eta,y)-\mu U_z(\eta,y)-k(\theta-\eta)U_r(\eta,y)\Big]dy\right)\eta^{-1}\varphi(\eta)d\eta \\
&\quad + \frac{\sigma^2}{\gamma^2}\int_0^{g(\alpha)}\left(U_z(\eta,z\wedge b(\eta))-U_z(\eta,\alpha+)\right)\eta^{-1}\varphi(\eta)d\eta = -\int_0^\infty f(\eta,z)\varphi(\eta)d\eta,
\end{aligned}$$

where, noticing that $\{\eta \le g(\alpha)\} = \{b(\eta) \ge \alpha\}$, we have defined

$$\begin{aligned}
f(\eta,z) :=& \mathbb{1}_{\{b(\eta)\ge\alpha\}}\frac{2}{\gamma^2}\left(\int_\alpha^{b(\eta)\wedge z}\Big[\rho(\eta)U(\eta,y)-\mu U_z(\eta,y)-k(\theta-\eta)U_r(\eta,y)\Big]dy\right)\eta^{-1} \\
&- \mathbb{1}_{\{b(\eta)\ge\alpha\}}\frac{\sigma^2}{\gamma^2}\left(U_z(\eta,z\wedge b(\eta))-U_z(\eta,\alpha+)\right)\eta^{-1}.
\end{aligned}$$

It follows that $v_{rr} = f$ in the weak sense. However, it is not hard to verify that $f \in L^\infty_{loc}(\mathcal{O})$ thanks to Proposition 3.11 and Proposition 3.13. Hence $v_{rr} \in L^\infty_{loc}(\mathcal{O})$, as claimed.

Finally, notice that since $r \mapsto b(r)$ is nonincreasing and right-continuous, then it has at most countably many jumps for $r \in (0,\infty)$, hence $f(r,z) = \lim_{\varepsilon\downarrow 0} f(r-\varepsilon,z)$ for a.e. $r \in (0,\infty)$ (here the null set depends on $z \ge \alpha$). Let also $(r^J_k)_{k\ge1}$ be the collection of jump points of the free boundary b, and set

$$\mathcal{N} := \bigcup_{k\ge1}\left([b(r^J_k),\infty)\times\{r^J_k\}\right).$$

Then $f(r,z) = \lim_{\varepsilon\downarrow0} f(r-\varepsilon,z)$ for $(r,z) \in \mathcal{O}\setminus\mathcal{N}$. Since $\mathcal{N}$ is a subset of $\mathcal{O}$ with null Lebesgue measure, we conclude that (4.2) holds true. □

In order to use Theorem 2.5 we need to show that v_{rr} is continuous as well in the closure $\overline{\mathcal{C}}$ of the continuation set $\mathcal{C}$, and we accomplish that in the next proposition. We remark that global C^2 regularity of a solution to (2.12) is far from being a trivial result and, in particular, we are not aware of any probabilistic proof of this fact.

Proposition 4.2. *One has that v_{rr} is continuous in $\overline{\mathcal{C}} \cap \mathcal{O}$.*

Proof. It suffices to observe that for any $(r,z) \in \overline{\mathcal{C}}\cap\mathcal{O}$ we have $z \le b_-(r)$. Hence

$$\begin{aligned}
v_{rr}(r,z) =& \mathbb{1}_{\{b_-(r)\ge\alpha\}}\frac{2}{\gamma^2}\left(\int_\alpha^{z}\Big[\rho(r)U(r,y)-\mu U_z(r,y)-k(\theta-r)U_r(r,y)\Big]dy\right)r^{-1} \\
&- \mathbb{1}_{\{b_-(r)\ge\alpha\}}\frac{\sigma^2}{\gamma^2}\left(U_z(r,z)-U_z(r,\alpha+)\right)r^{-1},
\end{aligned}$$

and the claimed continuity follows from Proposition 3.11. Notice that $\mathbb{1}_{\{b_-(r)\ge\alpha\}} = 1$ for all $r < r_\alpha$, where $r_\alpha := \sup\{r > 0 : b_-(r) > \alpha\}$. □

We conclude this section by proving that indeed $V = v$ and by providing an optimal dividend strategy.

Theorem 4.3. *Recall b from (3.33), V from (2.5) and v from (4.1). Then $V(r,z) = v(r,z)$ for all $(r,z) \in \overline{\mathcal{O}}$ and the process*

$$D^*_t := \sup_{0\le s\le t}\left[Z^0_s - b(R_s)\right]^+, \qquad t \ge 0 \tag{4.7}$$

is an optimal dividend strategy; i.e., for all $(r, z) \in \overline{\mathcal{O}}$ *we have*

$$v(r, z) = V(r, z) = \mathsf{E}_{r,z}\left[\int_{0-}^{\tau_\alpha^{D^*}} e^{-\int_0^t \rho(R_t)dt} dD_t^*\right].$$

Proof. It suffices to check that v of (4.1) satisfies all the conditions in Theorem 2.5. The function v is continuous everywhere. Moreover, by Proposition 3.11, v_z, v_{zz}, v_r and v_{zr} are continuous on $\mathcal{O}$, and, by Proposition 4.2, v_{rr} is continuous in $\overline{\mathcal{C}} \cap \mathcal{O}$.

Since $U \geq 1$ we have that $v_z \geq 1$, with equality for $z \geq b(r)$, $r > 0$. Moreover, by (3.51) we see that $v_{zz} = U_z < 0$ for all $(r, z) \in \mathcal{O}$ such that $\alpha \leq z < b(r)$. Hence $v_z > 1$ for such values of (z, r) because $v_z(r, b(r)) = U(r, b(r)) = 1$. Also, $0 \leq v(r, z) \leq h_0(z-\alpha)$ for any $(r, z) \in \mathcal{O}$ due to (3.9).

For $r \in \mathbb{R}_+$ and $\alpha < z < b(r)$ we have by Corollary 3.7 and the dominated convergence therorem that

$$\begin{aligned}0 = &\int_\alpha^z \big(\mathcal{L} - \rho(r)\big)U(r, y)dy\\ =&\frac{1}{2}\sigma^2 v_{zz}(r, z) + \mu v_z(r, z) - \big(\frac{1}{2}\sigma^2 v_{zz}(r, \alpha+) + \mu v_z(r, \alpha+)\big)\\ &+ \frac{1}{2}\gamma^2 r v_{rr}(r, z) + \kappa(\theta - r)v_r(r, z) - \rho(r)v(r, z) = \big(\mathcal{L} - \rho(r)\big)v(r, z),\end{aligned}$$

upon observing that $\frac{1}{2}\sigma^2 v_{zz}(r, \alpha+) + \mu v_z(r, \alpha+) = 0$ by Corollary 3.12. Repeating the same calculations for $z > b(r)$, $r > 0$, we find that $\big(\mathcal{L} - \rho(r)\big)v(r, z) \leq 0$. Hence, $\big(\mathcal{L} - \rho(r)\big)v(r, z) \leq 0$ for a.e. $(r, z) \in \mathcal{O}$.

Therefore we have verified all the conditions in (2.12), and it thus follows that $v = V$ and $D^* \equiv D^b$ is optimal. □

4.1. **Some Comments on the Optimal Dividend Policy.** The optimal control from (4.7) prescribes to pay dividends in such a way to keep the surplus process below the stochastic threshold $t \mapsto b(R_t)$ at all times. In particular, the company distributes the minimum amount of dividends that prevents the current surplus level from exceeding the current optimal ceiling $b(R_t)$. Any excess of the surplus is paid as a lump sum. Figure 1 below provides an illustration of the curve $r \mapsto b(r)$, of the process (Z, R), and of the optimal dividend payout. The optimal dividends distribution is therefore of *barrier type* but, differently to classical models with constant discount rate and constant optimal barrier (see, e.g., Section 2.5.2 in Chapter 2 of [37]), here we observe dynamic (stochastic) adjustments of the barrier. This strategy shows how the firm's manager responds to the fluctuations of the spot rate and allows to draw some economic/financial conclusions in a dynamic (random) macro-economic set-up. In particular, since the free boundary b is a decreasing function, we observe that in scenarios where the interest rate tends to increase, the firm manager will pay dividends more frequently because the expected present value of future dividend payments decays. Of course this behaviour also increases the probability of an early insolvency of the firm since in our model the growth rate of the surplus process is constant and independent of the current spot rate on the market. Despite this general trend, we also observe that no matter how large the spot rate, an immediate liquidation of the firm can never be optimal (final claim in Lemma 3.8). The combined uncertainty on the future moves of the spot rate and the surplus process indeed encourage gradual liquidation in light of a possible reversion of the spot rate towards lower values and/or upwards excursions of the surplus process.

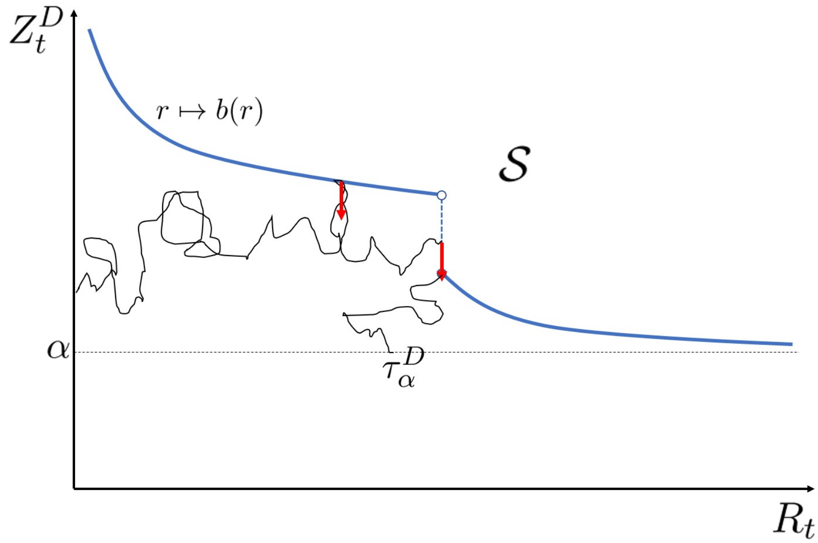

FIGURE 1. An illustrative drawing of the free boundary $r \mapsto b(r)$ and of the optimal dividend payout. The red arrows illustrate the vertical push that is needed to keep the surplus process below the interest-rate dependent boundary b. In particular, when the boundary is continuous the optimal dividend process defines a continuous measure $t \mapsto dD_t^*$ on $\mathbb{R}_+$ which is completely singular with respect to the Lebesgue measure.

We also see from Proposition 3.13-(ii) that if the discount rate is such that $\rho(r) \geq \rho_0$ for some constant $\rho_0 > 0$ (e.g., it is of linear form $\rho(r) = \rho_0 + r$), then the interest-rate dependent barrier b is uniformly bounded from above. As a consequence, in our model optimal lump sum payments are larger than those in the problem with constant discount rate, although the value function is actually smaller. In particular, according to Proposition 3.13-(iii), we see that in the linear case $\rho(r) = \rho_0 + r$ the size of the lump sum payments increases with the value of the interest rate (and indeed it attains its maximum when $r \uparrow \infty$), whereas it is constant if $\rho(r) = \rho_0$.

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