SINGULAR CONTROL OF THE DRIFT OF A BROWNIAN SYSTEM

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Abstract. We consider a standard Brownian motion whose drift can be increased or decreased in a possibly singular manner. The objective is to minimize an expected functional involving the time-integral of a running cost and the proportional costs of adjusting the drift. The resulting two-dimensional degenerate singular stochastic control problem is solved by combining techniques of viscosity theory and free boundary problems. We provide a detailed description of the problem's value function and of the geometry of the state space, which is split into three regions by two monotone curves. Our main result shows that those curves are continuously differentiable with locally Lipschitz derivative and solve a system of nonlinear ordinary differential equations.

Keywords: singular stochastic control; Dynkin game; viscosity solution; free boundary; smooth-fit; Brownian motion; ordinary differential equation.

MSC2010 subject classification: 93E20, 91A55, 49L25, 49J40, 35R35, 91B64.

1. Introduction

Consider a system whose position or level is subject to random fluctuations and can be corrected by acting on its drift. The latter can be increased or decreased, and the actions affecting the drift's dynamics do not need to be performed at rates; also impulses or singularly continuous forces can be applied. The objective of the decision maker is to minimize a total expected functional consisting of the time-integral of a running cost and of the proportional costs of adjusting the drift.

We model this problem as a two-dimensional singular stochastic control problem (see, e.g., [20], [24], [25], and [39] as classical contributions to the theory of singular stochastic control). The system's position/level X evolves as

$$X_t = x + \alpha \int_0^t Y_s \, \mathrm{d}s + \eta W_t, \quad x \in \mathbb{R}, \tag{1.1}$$

for some positive constants α, η and for a given standard Brownian motion W, and the drift Y is such that

$$Y_t = y + \xi_t^+ - \xi_t^-, \quad y \in \mathbb{R}. \tag{1.2}$$

Here, ξ_t^+ (respectively, ξ_t^-) are the cumulative increase (respectively, decrease) of the drift up to time $t \geq 0$ and, as such, ξ^+ and ξ^- are nondecreasing processes, and $\xi := \xi^+ - \xi^-$ has finite variation. The process X might be thought of as a random demand/level of sales whose instantaneous trend Y can be affected via production, according to supply and demand rules, or through an inventory management policy (see, e.g., the review [38]). Alternatively, X could be the position of a satellite which is subject to random disturbances and can be adjusted by properly acting on its velocity. The decision maker aims at picking a control rule ξ that minimizes an expected cost functional. This consists of a term measuring the total cost of acting on the system, which is proportional to the total variation of ξ, and of a term involving a running convex cost function f of the current values (X_t, Y_t). For example, if X is a satellite position and Y its velocity, the decision maker might want to keep the satellite as close as possible to a given target level, say 0, while minimizing the system's kinetic energy; in such a case a possible choice of f might therefore be $f(x, y) = x^2 + y^2$.

Date: May 14, 2020.

Our problem might be seen as a generalization of the bounded-velocity control of a scalar Brownian motion introduced by V.E. Beneš in 1974 [2], which has stimulated a subsequent large literature allowing for different specifications of the performance criterion and incorporating also other features like discretionary stopping and partial observation (see [1], [3], [21], [22], [23], [28], among many others). However, while in the previous papers the decision maker tracks the position of the Brownian system by choosing the value of its drift within a bounded set (hence the term bounded-velocity control), in our problem the Brownian motion is only indirectly affected by the controller's actions that, in fact, can unlimitedly increase and decrease the Brownian's drift at proportional costs. As a result, in our case the optimal control rule is expected to be of singular type (see Section 6.1 below), rather than of so-called bang-bang type (cf. [2], [21], [22], [23], among others).

In [37] the optimal correction problem of a damped random oscillator is studied. Differently to us, in that paper the velocity is subject to random disturbances and it is linearly controlled via a process of bounded variation, while the oscillator's position is not affected by noise. The authors formulate the problem as a (cheap) degenerate two-dimensional singular stochastic control problem and a thorough study of the related dynamic programming equation is performed via analytic methods. In [11], it is provided a numerical analysis of the non-cheap linear version of the control problem of [37].

The two papers that are perhaps closest to ours are [26] and [18]. In [26] a singular stochastic control problem with monotone controls and with finite-fuel constraint is considered. The problem is motivated by the issue of irreversible installation of solar panels, where the price of solar electricity is mean-reverting, with drift affected by the cumulative amount of installed solar panels. The authors solve the problem via a guess-and-verify approach and characterize the free boundary as the unique solution to a first-order ordinary differential equation (ODE) complemented by a boundary condition directly implied by the finite-fuel constraint. In [18], it is studied a two-dimensional singular stochastic control problem with controls of bounded-variation and interconnected dynamics. The problem's characteristic is that the mean-reversion level of the diffusive component of the state process is an affine function of the purely controlled second component.

As in [18], the full degeneracy of our setting where the drift component is purely controlled, allows us to provide a detailed study of the value function and of the geometry of the problem's state space. In particular, we argue as in [18], and following a direct approach employing techniques from viscosity theory and free-boundary problems we show that: (i) the value function V is differentiable with first derivatives that are (locally) Lipschitz, and its y-derivative identifies with the value of an optimal stopping game (see also [9]); (ii) the state space is split into three connected regions (continuation and action regions) by two monotone curves (free boundaries); (iii) the expression of the value function in each of those regions is provided; (iii) the second order derivative V_{yx} is continuous in the whole space (second-order smooth-fit); (iv) the free boundaries solve a system of integral equations.

Furthermore, because the uncontrolled process is a Brownian motion (rather than a more complex Ornstein-Uhlenbeck process as in [18]) here we are able to push the analysis of [18] much further (see Sections 5 and 6 below). As a matter of fact, we can show that the limit of the third derivative V_{yxx} at the free boundaries along any sequence of points belonging to the (interior of the) continuation region exists and is nonzero. This allows us to suitably apply the implicit function theorem and to show that the free boundaries are locally Lipschitz functions of the y coordinate. Then, by exploiting such a property and differentiating the integral equations solved by the free boundaries, we can prove that the latter satisfy a system of (explicitly computable) first-order ODEs. The regularity of the forcing term appearing in the ODEs finally implies that the free boundaries are actually continuously differentiable with locally Lipschitz derivative (see Theorem 6.3). To the best of our knowledge, in the context of a fully degenerate two-dimensional singular stochastic control problem with interconnected dynamics, a similar result appears here for the first time.

Unfortunately, providing boundary conditions complementing the ODEs still remains an open problem. Indeed, it seems hard to identify a relevant value of y for which the values of the free boundaries can be determined, as well as some kind of asymptotic growth that might restrict the

functional class where to look for uniqueness of the ODEs' system. However, in Section 6.1 we propose a conjecture about the derivation of a Cauchy problem involving the first derivatives of the free boundaries with respect to the parameter α (cf. (1.1)), rather than y. A discussion on the structure of the optimal control is also presented in Section 6.1.

The rest of the paper is organized as follows. Problem formulation and preliminary results are provided in Section 2, while preliminary properties of the free boundaries in Section 3. Section 4 contains the structure of the value function and the second-order smooth-fit property, and most of the results of this section follow from them in [18]. Further important properties of the free boundaries - as their (locally) Lipschitz continuity - are proved in Section 5, while the system of ODEs for the free boundaries is finally obtained in Section 6.

1.1. **Notation.** In the rest of this paper, we adopt the following notation and functional spaces. We will use $|\cdot|$ for the Euclidean norm on any finite-dimensional space, without indicating the dimension each time for simplicity of exposition.

Given a smooth function $h:\mathbb{R}\to\mathbb{R}$, we shall write h', h'', etc. to denote its derivatives. If the function h admits k continuous derivatives, $k\geq 1$, we shall write $h\in C^k(\mathbb{R};\mathbb{R})$, while $h\in C(\mathbb{R};\mathbb{R})$ if such a function is only continuous.

For a smooth function $h:\mathbb{R}^2\to\mathbb{R}$, we denote by h_x, h_y, h_{xx}, h_{yy}, etc. its partial derivatives. Given $k,j\in\mathbb{N}$, we let $C^{k,j}(\mathbb{R}^2;\mathbb{R})$ be the class of functions $h:\mathbb{R}^2\to\mathbb{R}$ which are k-times continuously differentiable with respect to the first variable and j-times continuously differentiable with respect to the second variable. If $k=j$, we shall simply write $C^k(\mathbb{R}^2;\mathbb{R})$. Moreover, for a domain $\mathcal{O}\subseteq\mathbb{R}^d$, $d\in\{1,2\}$, we shall work with the space $C^{k,\mathrm{Lip}}_{\mathrm{loc}}(\mathcal{O};\mathbb{R})$, $k\geq 1$, which consists of all the functions $h:\mathcal{O}\to\mathbb{R}$ that are k times continuously differentiable, with locally-Lipschitz kth-derivative(s).

Also, for $p\geq 1$ we shall denote by $L^p(\mathcal{O};\mathbb{R})$ (resp. $L^p_{\mathrm{loc}}(\mathcal{O};\mathbb{R})$) the space of real-valued functions $h:\mathcal{O}\to\mathbb{R}$ such that $|h|^p$ is integrable with respect to the Lebesgue measure on $\mathcal{O}$ (resp. locally integrable on $\mathcal{O}$). Finally, for $k\geq 1$, we shall make use of the space $W^{k,p}(\mathcal{O};\mathbb{R})$ (resp. $W^{k,p}_{\mathrm{loc}}(\mathcal{O};\mathbb{R})$), which is the space of all the functions $h:\mathcal{O}\to\mathbb{R}$ that admit kth-order weak derivative(s) in $L^p(\mathcal{O};\mathbb{R})$ (resp. $L^p_{\mathrm{loc}}(\mathcal{O};\mathbb{R})$)).

2. Problem Formulation and Preliminary Results

Let $(\Omega,\mathcal{F},\mathbb{F}:=(\mathcal{F}_t)_{t\geq 0},\mathsf{P})$ be a complete filtered probability space rich enough to accommodate an $\mathbb{F}$-Brownian motion $W:=(W_t)_{t\geq 0}$. We assume that the filtration $\mathbb{F}$ satisfies the usual conditions.

We introduce the (nonempty) set

$$\mathcal{A}:=\{\xi:\Omega\times\mathbb{R}_+\to\mathbb{R}:\ (\xi_t)_{t\geq 0}\text{ is }\mathbb{F}\text{-adapted and such that }t\mapsto\xi_t\text{ is a.s. càdlàg and (locally) of finite variation}\},\tag{2.1}$$

and for any $\xi\in\mathcal{A}$ we denote by ξ^+ and ξ^- the two nondecreasing $\mathbb{F}$-adapted càdlàg processes providing the minimal decomposition of ξ; that is, such that $\xi=\xi^+-\xi^-$ and the (random) Borel-measures induced on $[0,\infty)$ by ξ^+ and ξ^- have disjoint supports. In the following, for any $\xi\in\mathcal{A}$, we set $\xi^\pm_{0^-}=0$ a.s. and we denote by $|\xi|_t:=\xi^+_t+\xi^-_t$, $t\geq 0$, its total variation.

For $\xi\in\mathcal{A}$, $(x,y)\in\mathbb{R}^2$, and $\alpha>0$, we then consider the purely controlled dynamics

$$Y^{y,\xi}_t=y+\xi^+_t-\xi^-_t,\quad t\geq 0,\qquad Y^{y,\xi}_{0^-}=y,\tag{2.2}$$

as well as the diffusive

$$\begin{cases}\mathrm{d}X^{x,y,\xi}_t=\alpha Y^{y,\xi}_t\mathrm{d}t+\eta\mathrm{d}W_t, & t>0,\\ X^{x,y,\xi}_0=x.\end{cases}\tag{2.3}$$

The unique strong solution to (2.3) is given by

$$X^{x,y,\xi}_t=x+\alpha\int_0^t Y^{y,\xi}_s\,\mathrm{d}s+\eta W_t,\quad\forall\xi\in\mathcal{A},\ t\geq 0.\tag{2.4}$$

The parameter α measures the strength of the interaction between the processes X and Y. Clearly, for $\alpha = 0$ the two dynamics are decoupled and X is a Brownian motion with volatility $\eta > 0$.

Remark 2.1. *It is worth noticing that the restriction $\alpha > 0$ is not necessary for the subsequent analysis; in fact, all the results of this paper (up to obvious modifications) can be still deduced with the same techniques also in the case $\alpha < 0$. We have decided to consider only the case $\alpha > 0$ just in order to simplify the exposition.*

Controlling the dynamics (X, Y) gives rise to an instantaneous cost that is proportional – with marginal constant cost $K > 0$ – to the total variation of the exerted control. Moreover, the controller faces also a running cost depending on the current levels (X_t, Y_t). The aim is therefore to choose a control $\xi \in \mathcal{A}$ such that, for any $(x, y) \in \mathbb{R}^2$, and for a given $\rho > 0$, the cost functional

$$\mathcal{J}(x, y; \xi) := \mathsf{E}\Big[\int_0^\infty e^{-\rho t} f(X_t^{x,y,\xi}, Y_t^{y,\xi})\mathrm{d}t + \int_0^\infty e^{-\rho t} K\, \mathrm{d}|\xi|_t\Big] \tag{2.5}$$

is minimized; that is, to solve

$$V(x, y) := \inf_{\xi \in \mathcal{A}} \mathcal{J}(x, y; \xi), \qquad (x, y) \in \mathbb{R}^2. \tag{2.6}$$

In (2.5) and in the following, the integrals with respect to $\mathrm{d}|\xi|$ and $\mathrm{d}\xi^\pm$ are intended in the Lebesgue-Stieltjes' sense; in particular, for $\zeta \in \{|\xi|, \xi^+, \xi^-\}$, we set $\int_0^s (\,\cdot\,)\mathrm{d}\zeta_t := \int_{[0,s]} (\,\cdot\,)\mathrm{d}\zeta_t$ in order to take into account a possible mass at time zero of the Borel (random) measure $\mathrm{d}\zeta$. The function $f : \mathbb{R}^2 \to \mathbb{R}^+$ satisfies the following standing assumption.

Assumption 2.2. *There exists constants $p > 1$, and $C_0, C_1, C_2 > 0$ such that the following hold true:*

(i) $0 \le f(z) \le C_0\big(1 + |z|\big)^p$, *for every* $z = (x, y) \in \mathbb{R}^2$;
(ii) *for every* $z = (x, y), z' = (x', y') \in \mathbb{R}^2$,
$$|f(z) - f(z')| \le C_1\big(1 + f(z) + f(z')\big)^{1-\frac{1}{p}}|z - z'|;$$
(iii) *for every* $z = (x, y), z' = (x', y') \in \mathbb{R}^2$ *and* $\lambda \in (0, 1)$,
$$0 \le \lambda f(z) + (1-\lambda)f(z') - f(\lambda z + (1-\lambda)z') \le C_2\lambda(1-\lambda)(1 + f(z) + f(z'))^{\left(1-\frac{2}{p}\right)^+}|z - z'|^2;$$
(iv) $x \mapsto f_y(x, y)$ *is nondecreasing for any* $y \in \mathbb{R}$.

Remark 2.3. (i) *By Assumption 2.2-(iii), f is convex and locally semiconcave; then, by* [7, Cor. 3.3.8],
$$f \in C^{1,Lip}_{loc}(\mathbb{R}^2; \mathbb{R}) = W^{2,\infty}_{loc}(\mathbb{R}^2; \mathbb{R}).$$
(ii) *A function f satisfying Assumption 2.2 is, for example,*
$$f(x, y) = |x - \hat{x}|^p + |y - \hat{y}|^p,$$
with $p \ge 2$ ad for some $\hat{x}, \hat{y} \in \mathbb{R}$.

We now provide some preliminary properties of the value function, whose classical proof exploits the linear structure of the state equations.

Proposition 2.4. *Let Assumption 2.2 hold and let $p > 1$ be the constant appearing in such assumption. There exist constants $\widehat{C}_0, \widehat{C}_1, \widehat{C}_2 > 0$ such that the following hold:*

(i) $0 \le V(z) \le \widehat{C}_0\big(1 + |z|^p\big)$, *for every* $z = (x, y) \in \mathbb{R}^2$;
(ii) *for every* $z = (x, y), z' = (x', y') \in \mathbb{R}^2$,
$$|V(z) - V(z')| \le \widehat{C}_1\big(1 + |z| + |z'|\big)^{p-1}|z - z'|;$$

(iii) *for every* $z = (x, y), z' = (x', y') \in \mathbb{R}^2$ *and* $\lambda \in (0, 1)$,

$$0 \leq \lambda V(z) + (1-\lambda)V(z') - V(\lambda z + (1-\lambda)z') \leq \widehat{C}_2 \lambda(1-\lambda)(1+|z|+|z'|)^{(p-2)^+}|z-z'|^2.$$

In particular, by (iii), V *is convex and locally semiconcave, hence, by Corollary 3.3.8 in* [7],

$$V \in C^{1,Lip}_{loc}(\mathbb{R}^2;\mathbb{R}) = W^{2,\infty}_{loc}(\mathbb{R}^2;\mathbb{R}).$$

Proof. Due to (2.2) and (2.3), the properties of f required in (ii) and (iii) of Assumption 2.2 are straightly inherited by V (see, e.g., the proof of Theorem 1 of [10], that can easily adapted to our infinite time-horizon setting, or that of Theorem 2.1 in [8]). □

3. The Dynkin Game and Preliminary Properties of the Free Boundaries

In this section we show that V_y identifies with the value function of a suitable Dynkin game (a zero-sum game of optimal stopping), and we derive preliminary properties of the two curves (free boundaries) that delineate the region of the space where the $|V_y| < K$. In order to simplify the notation, in the following we write $X^{x,y}$, instead of $X^{x,y,0}$, to identify the solution to (2.3) for $\xi \equiv 0$. Most of the results of this section are close to those in Section 3 of [18], and their proof will be therefore omitted for the sake of brevity.

Theorem 3.1. *Let* $(x, y) \in \mathbb{R}^2$. *Denote by* $\mathcal{T}$ *the set of all* $\mathbb{F}$*-stopping times, and for* $(\sigma, \tau) \in \mathcal{T} \times \mathcal{T}$ *consider the stopping functional*

$$\begin{aligned}\Psi(\sigma,\tau;x,y) := \mathsf{E}\Bigg[\int_0^{\tau\wedge\sigma} e^{-\rho t}\Big(f_y(X^{x,y}_t, y) + \alpha V_x(X^{x,y}_t, y)\Big)\,\mathrm{d}t \\ - e^{-\rho\tau}K\mathbb{1}_{\{\tau<\sigma\}} + e^{-\rho\sigma}K\mathbb{1}_{\{\tau>\sigma\}}\Bigg],\end{aligned} \tag{3.1}$$

where V_x *is the partial derivative of* V *with respect to* x *(which exists continuous by Proposition 2.4). One has that the game has a value, i.e.*

$$\inf_{\sigma\in\mathcal{T}}\sup_{\tau\in\mathcal{T}}\Psi(\sigma,\tau;x,y) = \sup_{\tau\in\mathcal{T}}\inf_{\sigma\in\mathcal{T}}\Psi(\sigma,\tau;x,y),$$

and such a value is given by

$$V_y(x,y) = \inf_{\sigma\in\mathcal{T}}\sup_{\tau\in\mathcal{T}}\Psi(\sigma,\tau;x,y) = \sup_{\tau\in\mathcal{T}}\inf_{\sigma\in\mathcal{T}}\Psi(\sigma,\tau;x,y). \tag{3.2}$$

Moreover, the couple of $\mathbb{F}$*-stopping times* $(\tau^\star(x,y), \sigma^\star(x,y)) := (\tau^\star, \sigma^\star)$ *such that*

$$\sigma^\star := \inf\left\{t \geq 0 : V_y(X^{x,y}_t, y) \geq K\right\}, \quad \tau^\star := \inf\left\{t \geq 0 : V_y(X^{x,y}_t, y) \leq -K\right\} \tag{3.3}$$

(with the usual convention $\inf\emptyset = +\infty$*) form a saddle-point; that is,*

$$\forall\tau\in\mathcal{T} \quad \Psi(\sigma^\star,\tau;x,y) \leq V_y(x,y) = \Psi(\sigma^\star,\tau^\star;x,y) \leq \Psi(\sigma,\tau^\star;x,y) \quad \forall\sigma\in\mathcal{T}.$$

The proof of Theorem 3.1 can be obtained by arguing as in the proof of Theorem 3.1 of [18] (see Appendix A therein). In particular, it follows from Theorems 3.11 and 3.13 in [9], through a suitable (and not immediate) approximation procedure needed to accommodate our degenerate setting. Details are omitted.

From (3.2) it readily follows that $-K \leq V_y(x, y) \leq K$ for any $(x, y) \in \mathbb{R}^2$. Hence, defining

$$\begin{cases}\mathcal{I} := \left\{(x,y)\in\mathbb{R}^2 : \quad V_y(x,y) = -K\right\},\\ \mathcal{C} := \left\{(x,y)\in\mathbb{R}^2 : \quad -K < V_y(x,y) < K\right\},\\ \mathcal{D} := \left\{(x,y)\in\mathbb{R}^2 : \quad V_y(x,y) = K\right\},\end{cases} \tag{3.4}$$

we have that those regions provide a partition of $\mathbb{R}^2$.

By continuity of V_y (cf. Proposition 2.4), $\mathcal{C}$ is an open set, while $\mathcal{I}$ and $\mathcal{D}$ are closed sets. Moreover, convexity of V provides the representation

$$\mathcal{C} = \{(x,y) \in \mathbb{R}^2 : \ b_1(x) < y < b_2(x)\},$$
$$\mathcal{I} = \{(x,y) \in \mathbb{R}^2 : \ y \le b_1(x)\}, \quad \mathcal{D} = \{(x,y) \in \mathbb{R}^2 : \ y \ge b_2(x)\},$$

where the functions $b_1 : \mathbb{R} \to \overline{\mathbb{R}}$ and $b_2 : \mathbb{R} \to \overline{\mathbb{R}}$ are defined as

$$b_1(x) := \inf\{y \in \mathbb{R} \mid V_y(x,y) > -K\} = \sup\{y \in \mathbb{R} \mid V_y(x,y) = -K\}, \quad x \in \mathbb{R}, \tag{3.5}$$

$$b_2(x) := \sup\{y \in \mathbb{R} \mid V_y(x,y) < K\} = \inf\{y \in \mathbb{R} \mid V_y(x,y) = K\}, \quad x \in \mathbb{R}, \tag{3.6}$$

(with the usual conventions $\inf \emptyset = \infty$, $\inf \mathbb{R} = -\infty$, $\sup \emptyset = -\infty$, $\sup \mathbb{R} = \infty$).

Equation (3.2), together with the fact that $x \mapsto V_x(x,y)$ is nondecreasing for any $y \in \mathbb{R}$ by convexity of V (cf. Proposition 2.4) and $x \mapsto f_y(x,y)$ is nondecreasing by Assumption 2.2-(iv), easily imply the following result.

Lemma 3.2. *$V_y(\cdot, y)$ is nondecreasing for all $y \in \mathbb{R}$.*

We now move on by obtaining preliminary properties of b_1 and b_2. Its proof can be obtained by exploiting the continuity and the monotonicity of V_y, and easily adjusting the arguments of the proof of Proposition 3.3 of [18] to the present setting in which $V_y(\cdot, y)$ is nondecreasing.

Proposition 3.3. *The following hold:*

(i) *$b_1 : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$, $b_2 : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$;*
(ii) *b_1 and b_2 are nonincreasing;*
(iii) *$b_1(x) < b_2(x)$ for all $x \in \mathbb{R}$;*
(iv) *b_1 is left-continuous and b_2 is right-continuous.*

Let us now define

$$\bar{b}_1 := \sup_{x \in \mathbb{R}} b_1(x), \quad \underline{b}_1 := \inf_{x \in \mathbb{R}} b_1(x), \quad \bar{b}_2 := \sup_{x \in \mathbb{R}} b_2(x), \quad \underline{b}_2 := \inf_{x \in \mathbb{R}} b_2(x), \tag{3.7}$$

together with the pseudo-inverses of b_1 and b_2 by

$$g_1(y) := \sup\{x \in \mathbb{R} : b_1(x) \ge y\}, \quad g_2(y) := \inf\{x \in \mathbb{R} : b_2(x) \le y\} \tag{3.8}$$

(again, with the usual conventions $\inf \emptyset = \infty$, $\inf \mathbb{R} = -\infty$, $\sup \emptyset = -\infty$, $\sup \mathbb{R} = \infty$).

Also the next proposition can be proved by easily adapting to our setting the proof of Proposition 3.4 in [18].

Proposition 3.4. *The following holds:*

(i) $g_1(y) = \inf\{x \in \mathbb{R} : V_y(x,y) > -K\}, \quad g_2(y) = \sup\{x \in \mathbb{R} : V_y(x,y) < K\}$;
(ii) *the functions g_1, g_2 are nonincreasing and $g_1(y) < g_2(y)$ for any $y \in \mathbb{R}$;*
(iii) *If $\bar{b}_2 < \infty$, then $g_2(y) = -\infty$ for all $y \ge \bar{b}_2$ and if $\underline{b}_1 > -\infty$, then $g_1(r) = \infty$ for all $y \le \underline{b}_1$.*

4. The Structure of the Value Function and the Second-Order Smooth Fit

In this section, we exploit the results of the previous section in order to determine the structure of the value function V, and to show that V_{yx} is continuous on the whole state space (second-order smooth-fit property).

For any given and fixed $y \in \mathbb{R}$, denote by $\mathcal{L}^y$ the infinitesimal generator associated to the uncontrolled process $X^{x,y}$. Acting on $g \in C^2(\mathbb{R}; \mathbb{R})$ it yields

$$\big(\mathcal{L}^y g\big)(x) := \frac{\eta^2}{2} g''(x) + \alpha y g'(x), \quad x \in \mathbb{R}.$$

Any solution $\beta(\cdot, y)$ to the second-order ordinary differential equation (ODE)

$$\big(\mathcal{L}^y \beta(\cdot, y)\big)(x) - \rho \beta(x,y) = 0, \quad x \in \mathbb{R},$$

can be written as

$$\beta(x,y) = A(y)\psi(x,\alpha y) + B(y)\varphi(x,\alpha y), \quad x \in \mathbb{R},$$

where the strictly positive functions ψ and φ are given, for any $z \in \mathbb{R}$, by

$$\psi(x,z) := e^{r_1(z)x} \qquad \varphi(x,z) := e^{r_2(z)x}, \tag{4.1}$$

with

$$r_1(z) := \frac{-z + \sqrt{z^2 + 2\rho\eta^2}}{\eta^2} > 0, \tag{4.2}$$

$$r_2(z) := \frac{-z - \sqrt{z^2 + 2\rho\eta^2}}{\eta^2} < 0. \tag{4.3}$$

Notice that $\psi(\cdot, z)$ is strictly increasing while $\varphi(\cdot, z)$ is strictly decreasing for any $z \in \mathbb{R}$.

By the dynamic programming principle, we expect that V identifies with a suitable solution to the following variational inequality

$$\max\Big\{ -v_y(x,r) - K,\ v_y(x,r) - K,\ [(\rho - \mathcal{L}^y)v(\cdot,y)](x) - f(x,y)\Big\} = 0, \quad (x,y) \in \mathbb{R}^2. \tag{4.4}$$

We now show that V is a viscosity solution to (4.4). Later, this will enable us to determine the structure of V (see Theorem 4.5 below) and then to upgrade its regularity (cf. Theorem 4.7) in order to derive necessary optimality conditions for the boundaries splitting the state space (cf. Theorem 6.1).

Definition 4.1.

(i) *A function $v \in C^0(\mathbb{R}^2;\mathbb{R})$ is called a* viscosity subsolution *to (4.4) if, for every $(x,y) \in \mathbb{R}^2$ and every $\beta \in C^{2,1}(\mathbb{R}^2;\mathbb{R})$ such that $v - \beta$ attains a local maximum at (x,y), it holds*

$$\max\Big\{ -\beta_y(x,y) - K,\ \beta_y(x,y) - K,\ \rho\beta(x,y) - [\mathcal{L}^y\beta(\cdot,y)](x) - f(x,y)\Big\} \le 0.$$

(ii) *A function $v \in C^0(\mathbb{R}^2;\mathbb{R})$ is called a* viscosity supersolution *to (4.4) if, for every $(x,y) \in \mathbb{R}^2$ and every $\beta \in C^{2,1}(\mathbb{R}^2;\mathbb{R})$ such that $v - \beta$ attains a local minimum at (x,y), it holds*

$$\max\Big\{ -\beta_y(x,y) - K,\ \beta_y(x,y) - K,\ \rho\beta(x,y) - [\mathcal{L}^y\beta(\cdot,y)](x) - f(x,y)\Big\} \ge 0.$$

(iii) *A function $v \in C^0(\mathbb{R}^2;\mathbb{R})$ is called a* viscosity solution *to (4.4) if it is both a viscosity subsolution and supersolution.*

Following the arguments developed in Theorem 5.1 in Section VIII.5 of [19], one can show the following result (see also Proposition 4.2 in [18])

Proposition 4.2. *The value function V is a viscosity solution to (4.4).*

Remark 4.3. *Recall that by Proposition 2.4-(iii) our value function V lies in the class $W^{2,\infty}_{loc}(\mathbb{R}^2;\mathbb{R})$. Hence, by Lemma 5.4 in Chapter 4 of [40] it is also a* strong solution *to (4.4) (in the sense, e.g., of [5]; see the same reference also for relations between these notions of solutions); that is, it solves (4.4) in the pointwise sense almost everywhere.*

We have decided to employ the concept of viscosity solution since our analysis will later make use of the variational inequality (4.4) on sets of null Lebesgue measure (regular lines) (see Proposition 4.4 and Proposition 4.7 below). Because the viscosity property holds for all *(and not merely for a.e.) points of the state space $\mathbb{R}^2$, the concept of viscosity solution is still able to provide information on V on regular lines.*

For later use, notice that the function

$$\widehat{V}(x,y) := \mathcal{J}(x,y,0) = \mathsf{E}\Big[\int_0^\infty e^{-\rho t} f(X_t^{x,y},y)\,\mathrm{d}t\Big], \quad (x,y)\in\mathbb{R}^2, \tag{4.5}$$

is finite by Assumption 2.2-(i) and standard estimates, and continuously differentiable with respect to y and x, given the assumed regularity of f_x and f_y in Assumption 2.2-(iii). Moreover, for any given and fixed $y\in\mathbb{R}$, we introduce the scale function density of the process $X^{x,y}$

$$S_x(x,z) := \exp\Big\{-\frac{2zx}{\eta^2}\Big\}, \quad x\in\mathbb{R}, \tag{4.6}$$

the density of the speed measure

$$m_x(x,z) := \frac{2}{\eta^2 S_x(x,z)}, \quad x\in\mathbb{R}, \tag{4.7}$$

as well as the positive constant (normalized) Wronskian between ψ and φ

$$W := \frac{\psi_x(x,z)\varphi(x,z)-\varphi_x(x,z)\psi(x,z)}{S_x(x,z)}, \quad (x,z)\in\mathbb{R}^2. \tag{4.8}$$

Then, letting

$$G(x,z,u) := W^{-1}\cdot\begin{cases}\psi(x,u)\varphi(z,u), & x\le z,\\ \varphi(x,u)\psi(z,u), & x\ge z,\end{cases} \tag{4.9}$$

be the Green function, we have that $\widehat{V}$ admits the representation (cf., e.g., Ch. 2 of [4])

$$\widehat{V}(x,y) = \int_{-\infty}^{+\infty} f(z,y)G(x,z,\alpha y)m_x(z,\alpha y)\mathrm{d}z, \quad (x,y)\in\mathbb{R}^2; \tag{4.10}$$

that is, using (4.1) and (4.7),

$$\widehat{V}(x,y) = \frac{2}{\eta^2 W}\Big[e^{r_1(\alpha y)x}\int_x^\infty e^{-r_1(\alpha y)z}f(z,y)\mathrm{d}z + e^{r_2(\alpha y)x}\int_{-\infty}^x e^{-r_2(\alpha y)z}f(z,y)\mathrm{d}z\Big]. \tag{4.11}$$

By direct calculations, it thus follows from (4.11) that $\widehat{V}$ identifies with a classical particular solution to the inhomogeneous linear ODE

$$[(\mathcal{L}^y-\rho)\beta(\cdot,y)](x) + f(x,y) = 0, \quad x\in\mathbb{R}. \tag{4.12}$$

Recall now the regions $\mathcal{C}$, $\mathcal{I}$ and $\mathcal{D}$ from (3.4), and that $V_y=-K$ on $\mathcal{I}$, while $V_y=K$ on $\mathcal{D}$. The next proposition provides the structure of V inside $\mathcal{C}$. Its proof can be obtained by arguing exactly as in the proof of Proposition 4.4 of [18] (see also Remarks 4.3 and 4.5 therein), and it is therefore omitted.

Proposition 4.4. *Recall* (3.7) *and let* $y_o\in(\underline{b}_1,\bar{b}_2)$.

(i) *The function* $V(\cdot,y_o)$ *is a viscosity solution to*

$$\rho\beta(x,y_o) - [\mathcal{L}^{y_o}\beta(\cdot,y_o)](x) - f(x,y_o) = 0, \quad x\in(g_1(y_o),g_2(y_o)). \tag{4.13}$$

(ii) $V(\cdot,y_o)\in C^{3,Lip}_{loc}((g_1(y_o),g_2(y_o));\mathbb{R})$.

(iii) *There exist constants* $A(y_o)$ *and* $B(y_o)$ *such that for all* $x\in(g_1(y_o),g_2(y_o))$

$$V(x,y_o) = A(y_o)\psi(x,\alpha y_o) + B(y_o)\varphi(x,\alpha y_o) + \widehat{V}(x,y_o),$$

where the functions ψ *and* φ *have been defined in* (4.1) *and* $\widehat{V}$ *is as in* (4.5).

We can now determine the structure of the value function V. The proof of the next proposition is completely analogous to that of Theorem 4.6 in [18]; however, we provide it here since it will be useful in the proof of a subsequent result (cf. Proposition 5.8).

Proposition 4.5. *Define the sets*

$$\mathcal{O}_1 := \{x \in \mathbb{R} : \, b_1(x) > -\infty\} \quad \mathcal{O}_2 := \{x \in \mathbb{R} : \, b_2(x) < \infty\}. \tag{4.14}$$

There exist functions

$$A, B \in W^{2,\infty}_{loc}((\underline{b}_1, \bar{b}_2); \mathbb{R}) = C^{1,Lip}_{loc}((\underline{b}_1, \bar{b}_2); \mathbb{R}), \quad z_{1,2} : \mathcal{O}_{1,2} \to \mathbb{R}$$

such that the value function defined in (2.6) can be written as

$$V(x,y) = \begin{cases} A(y)\psi(x,\alpha y) + B(y)\varphi(x,\alpha y) + \widehat{V}(x,y) & \text{on } \bar{\mathcal{C}}, \\ z_1(x) - Ky & \text{on } \mathcal{I}, \\ z_2(x) + Ky & \text{on } \mathcal{D}, \end{cases} \tag{4.15}$$

where $\bar{\mathcal{C}}$ denotes the closure of $\mathcal{C}$,

$$z_1(x) := V(x, b_1(x)) + Kb_1(x), \quad x \in \mathcal{O}_1 \tag{4.16}$$

and

$$z_2(x) := V(x, b_2(x)) - Kb_2(x), \quad x \in \mathcal{O}_2. \tag{4.17}$$

Proof. We start by deriving the structure of V within $\mathcal{C}$. Using Lemma 4.4, we already know the existence of functions $A, B : (\underline{b}_1, \bar{b}_2) \to \mathbb{R}$ such that

$$V(x,y) = A(y)\psi(x,\alpha y) + B(y)\varphi(x,\alpha y) + \widehat{V}(x,y), \quad (x,y) \in \mathcal{C}. \tag{4.18}$$

Take now $y_o \in (\underline{b}_1, \bar{b}_2)$. Since $g_1(y) < g_2(y)$ for any $y \in \mathbb{R}$ (cf. Proposition 3.4-(ii)), we can find x and $\tilde{x}$, $x \neq \tilde{x}$, such that $(x,y), (\tilde{x},y) \in \mathcal{C}$ for any given $y \in (y_o - \varepsilon, y_o + \varepsilon)$, for a suitably small $\varepsilon > 0$. Now, by evaluating (4.18) at the points (x,y) and $(\tilde{x},y)$, we obtain a linear algebraic system that we can solve with respect to $A(y)$ and $B(y)$ so to obtain

$$A(y) = \frac{(V(x,y) - \widehat{V}(x,y))\varphi(\tilde{x},\alpha y) - (V(\tilde{x},y) - \widehat{V}(\tilde{x},y)\varphi(x,\alpha y)}{\psi(x,\alpha y)\varphi(\tilde{x},\alpha y) - \psi(\tilde{x},\alpha y)\varphi(x,\alpha y)}, \tag{4.19}$$

$$B(y) = \frac{(V(\tilde{x},y) - \widehat{V}(\tilde{x},y)\psi(x,\alpha y) - (V(x,y) - \widehat{V}(x,y))\psi(\tilde{x},\alpha y)}{\psi(x,\alpha y)\varphi(\tilde{x},\alpha y) - \psi(\tilde{x},\alpha y)\varphi(x,\alpha y)}. \tag{4.20}$$

The denominators of the last two expressions do not vanish due to the strict monotonicity of ψ and φ, and to the fact that $x \neq \tilde{x}$. Since y_o was arbitrary and V, $\widehat{V}$, V_y, and $\widehat{V}_y$ are continuous with respect to y, we therefore obtain that A and B belong to $W^{2,\infty}_{\text{loc}}((\underline{b}_1, \bar{b}_2); \mathbb{R}) = C^{1,Lip}_{\text{loc}}((\underline{b}_1, \bar{b}_2); \mathbb{R})$. The structure of V in the closure of $\mathcal{C}$, denoted by $\overline{\mathcal{C}}$, is then obtained by Proposition 4.4 and by recalling that V is continuous on $\mathbb{R}^2$ and that A, B, and $\widehat{V}$ are also continuous.

Given the definition of z_1 and z_2, the structure of V inside the regions $\mathcal{I}$ and $\mathcal{D}$ follow by (3.4) and the continuity of V. □

Remark 4.6. *Actually, by (4.19) and (4.20) one has that A and B belong to $W^{2,\infty}$ up to $\underline{b}_1$ (resp. $\bar{b}_2$) if $\underline{b}_1$ (resp. $\bar{b}_2$) is finite (cf. also Remark 4.7 in [18]).*

Notice that

$$V_{yx}(x,y) = 0 \qquad \forall (x,y) \in \mathbb{R}^2 \setminus \overline{\mathcal{C}}.$$

The next result shows that one actually has continuity of V_{yx} on the whole $\mathbb{R}^2$. Its proof can be obtained by following that of Theorem 5.1 in [18] (see also Proposition 5.3 in [17]), upon recalling that in our setting $V_y(\cdot, y)$ is nondecreasing (cf. Lemma 3.2).

Proposition 4.7. *One has that*

$$\lim_{\substack{(x,y)\to (x_o,y_o)\\ (x,y)\in\mathcal{C}}} V_{yx}(x,y)=0 \qquad \forall (x_o,y_o)\in\partial\mathcal{C}. \tag{4.21}$$

Hence, $V_{yx}\in C(\mathbb{R}^2;\mathbb{R})$.

Lemma 4.8. *It holds* $V_{yxx}\in L^\infty_{loc}(\mathbb{R}\times(\underline{b}_1,\bar{b}_2);\mathbb{R})$.

Proof. Notice that by (4.1) one has $\psi_{xx}(x,\alpha y)=r_1^2(\alpha y)\psi(x,\alpha y)$, $\varphi_{xx}(x,\alpha y)=r_2^2(\alpha y)\varphi(x,\alpha y)$, and $\psi_{zxx}(x,\alpha y)=xr_1'(\alpha y)r_1^2(\alpha y)\psi(x,\alpha y)$, $\varphi_{zxx}(x,\alpha y)=xr_2'(\alpha y)r_2^2(\alpha y)\varphi(x,\alpha y)$. Moreover, $\widehat{V}_{yxx}\in L^\infty_{\text{loc}}(\mathbb{R}^2)$ by direct calculations on (4.11), and $A_y,B_y\in W^{1,\infty}_{\text{loc}}((\underline{b}_1,\bar{b}_2);\mathbb{R})$ by Proposition 4.5. Hence, $V_{yxx}\in L^\infty_{\text{loc}}(\mathbb{R}\times(\underline{b}_1,\bar{b}_2);\mathbb{R})$ by (4.15). □

5. Further Properties of the Free Boundaries

In this section we move on by proving further properties of the free boundaries under additional mild requirements on f.

Assumption 5.1.

(i) $\lim_{x\to\pm\infty} f_x(x,y)=\pm\infty$.
(ii) f_{yx} *exists continuous.*
(iii) *One of the following holds true:*
  (a) $x\mapsto f_y(x,y)$ *is strictly increasing for any* $y\in\mathbb{R}$*;*
  (b) $f_{yx}\equiv 0$ *and* $f(\cdot,y)$ *is strictly convex for any* $y\in\mathbb{R}$.

Remark 5.2. *The functions* f *discussed in Remark 2.3 satisfy the previous assumptions.*

Under these assumptions, we will show that g_i, $i=1,2$ are locally Lipschitz functions (cf. Proposition 5.8 below). We start by studying the limiting behavior of the functions b_i and some natural bounds for g_i.

Proposition 5.3.

(i) *Let Assumption 5.1-(i) hold. Then*

$$\bar{b}_1=\lim_{x\downarrow-\infty} b_1(x)=\infty,\quad \underline{b}_2=\lim_{x\uparrow\infty} b_2(x)=-\infty;$$

hence, by Proposition 3.3-(iii), one also has $\underline{b}_1=-\infty$ *and* $\bar{b}_2=\infty$.

(ii) *Define*

$$\zeta_1(y):=\sup\{x\in\mathbb{R}: -\alpha V_x(x,y)-f_y(x,y)-\rho K\geq 0\},\quad y\in\mathbb{R},$$
$$\zeta_2(y):=\inf\{x\in\mathbb{R}: -\alpha V_x(x,y)-f_y(x,y)+\rho K\leq 0\},\quad y\in\mathbb{R}.$$

Then, for any $y\in\mathbb{R}$*, we have*

$$g_1(y)<\zeta_1(y)<\zeta_2(y)<g_2(y).$$

Proof. Proof of (i). Here we show that $\lim_{x\downarrow-\infty} b_1(x)=\infty$. The fact that $\lim_{x\uparrow\infty} b_2(x)=-\infty$ can be proved by similar arguments. We argue by contradiction assuming $\bar{b}_1:=\lim_{x\downarrow-\infty} b_1(x)<\infty$. Take $y_o>\bar{b}_1$, so that $\tau^\star=\tau^\star(x,y_o)=\infty$ for all $x\in\mathbb{R}$, the latter being the stopping time defined in (3.3). Then, take $x_o<g_2(y_o)$ such that $(x_o,y_o)\in\mathcal{C}$. Clearly, every $x<x_o$ belongs to $\mathcal{C}$, and therefore, by the representation (4.15), we obtain that it must be $B(y_o)=0$; indeed, otherwise, by taking limits as $x\to-\infty$ and using (4.1), we would contradict Proposition 2.4. Moreover, since for any $y\in\mathbb{R}$ one has $\psi_x(x,\alpha y)\to 0$ when $x\to-\infty$ (cf. (4.1)), we then have by dominated convergence

$$\lim_{x\to-\infty} V_x(x,y_0)=\lim_{x\to-\infty}\widehat{V}_x(x,y_o)=\lim_{x\to-\infty}\mathsf{E}\left[\int_0^\infty e^{-\rho t}f_x(X_t^{x,y_o},y_o)\mathrm{d}t\right]=-\infty. \tag{5.1}$$

Now, setting

$$\hat{\sigma}_x:=\inf\{t\geq 0: X_t^{x,y_o}\geq x_o\},$$

for $x < x_o$, we have by monotonicity of $f_y(\cdot, y)$ (cf. Assumption 2.2-(iv))

$$
\begin{aligned}
-K < V_y(x, y_o) &= \inf_{\sigma \in \mathcal{T}} \mathsf{E}\Big[\int_0^{\sigma} e^{-\rho t}\Big(\alpha V_x(X_t^{x,y_o}, y_o) + f_y(X_t^{x,y_o}, y_o)\Big)\, \mathrm{d}t + e^{-\rho\sigma} K\Big] \\
&\leq \mathsf{E}\Big[\int_0^{\hat{\sigma}_x} e^{-\rho t}\Big(\alpha V_x(X_t^{x,y_o}, y_o) + f_y(x_o, y_o)\Big)\, \mathrm{d}t + K\Big]. \qquad (5.2)
\end{aligned}
$$

The latter implies

$$
2K + \frac{|f_y(x_o, y_o)|}{\rho} \geq -\alpha \mathsf{E}\Big[\int_0^{\hat{\sigma}_x} e^{-\rho t} V_x(X_t^{x,y_o}, y_o)\, \mathrm{d}t\Big] = -\alpha \mathsf{E}\Big[\int_0^{\hat{\sigma}_x} e^{-\rho t} V_x(x + \alpha y_o t + \eta W_t, y_o)\, \mathrm{d}t\Big].
$$

Hence, letting $x \downarrow -\infty$, using (5.1), and invoking the dominated convergence theorem we get a contradiction.

Proof of (ii). The fact that for any $y \in \mathbb{R}$ we have $g_1(y) \leq \zeta_1(y)$ and $g_2(y) \geq \zeta_2(y)$ can be obtained as in the proof of item (ii) of Proposition 6.1 in [18], by employing the proved regularity of $V_y(\cdot, y)$ and the semiharmonic characterization of [32] (see equations (2.27)–(2.29) therein, suitably adjusted to take care of the integral term appearing in (3.2)). Moreover, $\zeta_1(y) < \zeta_2(y)$ for any $y \in \mathbb{R}$ by definition. It thus remains to show that one actually has $g_1(y) < \zeta_1(y)$ and $g_2(y) > \zeta_2(y)$ for any $y \in \mathbb{R}$.

We only prove that $g_2(y) > \zeta_2(y)$ for any $y \in \mathbb{R}$, as the other case can be treated similarly. Suppose that there exists some y_o such that $g_2(y_o) = \zeta_2(y_o)$. Then $V_y(\zeta_2(y_o), y_o) = K$. Let now $\tau^\star := \tau^\star(\zeta_2(y_o), y_o)$ be the optimal stopping time for the sup player when the Dynkin game (3.2) starts at time zero from the point $(\zeta_2(y_o), y_o)$, and for $\varepsilon > 0$ define

$$
q_\varepsilon := q_\varepsilon(\zeta_2(y_o), y_o) := \inf\{t \geq 0 : \, X_t^{\zeta_2(y_o), y_o} \geq \zeta_2(y_o) + \varepsilon\}.
$$

Then by using that $f_y(\cdot, y_o) + \alpha V_x(\cdot, y_o)$ is nondecreasing and locally Lipschitz by Assumption 2.2-(iii) and Proposition 2.4(iii), we have from (3.2) for some constant $C(y_o) > 0$

$$
\begin{aligned}
K = V_y(\zeta_2(y_o), y_o) &\leq \mathsf{E}\Big[\int_0^{\tau^\star \wedge q_\varepsilon} e^{-\rho t}\big(f_y + \alpha V_x\big)(X_s^{\zeta_2(y_o), y_o}, y_o)\mathrm{d}s\Big] \\
&+ \mathsf{E}\Big[K e^{-\rho q_\varepsilon} \mathbb{1}_{\{\tau^\star > q_\varepsilon\}} - K e^{-\rho \tau^\star} \mathbb{1}_{\{\tau^\star < q_\varepsilon\}}\Big] \\
&\leq \big(f_y + \alpha V_x\big)(\zeta_2(y_o) + \varepsilon, y_o) \frac{1}{\rho} \mathsf{E}\Big[1 - e^{-\rho(\tau^\star \wedge q_\varepsilon)}\Big] \\
&+ \mathsf{E}\Big[K e^{-\rho q_\varepsilon} \mathbb{1}_{\{\tau^\star > q_\varepsilon\}} - K e^{-\rho \tau^\star} \mathbb{1}_{\{\tau^\star < q_\varepsilon\}}\Big] \qquad (5.3) \\
&\leq \frac{1}{\rho}\Big[\big(f_y + \alpha V_x\big)(\zeta_2(y_o), y_o) + \varepsilon C(y_o)\Big] \mathsf{E}\Big[1 - e^{-\rho(\tau^\star \wedge q_\varepsilon)}\Big] \\
&+ \mathsf{E}\Big[K e^{-\rho q_\varepsilon} \mathbb{1}_{\{\tau^\star > q_\varepsilon\}} - K e^{-\rho \tau^\star} \mathbb{1}_{\{\tau^\star < q_\varepsilon\}}\Big].
\end{aligned}
$$

Using now that, by definition of ζ_2, it must be $\big(f_y + \alpha V_x\big)(\zeta_2(y_o), y_o) = \rho K$, and rearranging terms, we get that

$$
0 \leq \frac{\varepsilon C(y_o)}{\rho} \mathsf{E}\Big[1 - e^{-\rho(\tau^\star \wedge q_\varepsilon)}\Big] - 2K \mathsf{E}\Big[e^{-\rho \tau^\star} \mathbb{1}_{\{\tau^\star < q_\varepsilon\}}\Big]. \qquad (5.4)
$$

Notice now that (cf. eq. (4.3) in [13], among others)

$$
\mathsf{E}\Big[e^{-\rho \tau^\star} \mathbb{1}_{\{\tau^\star < q_\varepsilon\}}\Big] = \frac{\psi(\zeta_2(y_o), \alpha y_o)\varphi(\zeta_2(y_o) + \varepsilon, \alpha y_o) - \psi(\zeta_2(y_o) + \varepsilon, \alpha y_o)\varphi(\zeta_2(y_o), \alpha y_o)}{\psi(g_1(y_o), \alpha y_o)\varphi(\zeta_2(y_o) + \varepsilon, \alpha y_o) - \psi(\zeta_2(y_o) + \varepsilon, \alpha y_o)\varphi(g_1(y_o), \alpha y_o)}
$$

and

$$
\mathsf{E}\Big[e^{-\rho q_\varepsilon} \mathbb{1}_{\{\tau^\star > q_\varepsilon\}}\Big] = \frac{\psi(g_1(y_o), \alpha y_o)\varphi(\zeta_2(y_o), \alpha y_o) - \psi(\zeta_2(y_o), \alpha y_o)\varphi(g_1(y_o), \alpha y_o)}{\psi(g_1(y_o), \alpha y_o)\varphi(\zeta_2(y_o) + \varepsilon, \alpha y_o) - \psi(\zeta_2(y_o) + \varepsilon, \alpha y_o)\varphi(g_1(y_o), \alpha y_o)}.
$$

Then, because

$$1-e^{-\rho(\tau^\star\wedge q_\varepsilon)}=1-e^{-\rho\tau^*}\mathbb{1}_{\{\tau^\star<q_\varepsilon\}}-e^{-\rho q_\varepsilon}\mathbb{1}_{\{\tau^\star>q_\varepsilon\}},$$

using the last two formulas in (5.4) and performing a first-order Taylor's expansion around $\varepsilon=0$ of the terms on the right-hand side of (5.4), one finds that the first term on the right-hand side of (5.4) is positive and converges to zero as $\varepsilon\downarrow 0$ with order ε^2, while the second term is negative and converges to zero with order ε. We thus reach a contradiction in (5.4) for ε small enough, and therefore it cannot exist y_o at which $g_2(y_o)=\zeta_2(y_o)$. □

The next result readily follows from Proposition 5.3-(i).

Corollary 5.4. *Let Assumption 5.1-(i) hold. Then the functions g_1,g_2 defined in (3.8) are finite.*

Proposition 5.5. *Let Assumption 5.1 hold. Then the functions b_1,b_2 are strictly decreasing.*

Proof. We prove the claim only for b_1, since analogous arguments apply to prove it for b_2.

Case (a). We assume here that item (a) of Assumption 5.1-(iii) holds, i.e. that $x\mapsto f_y(x,y)$ is strictly increasing for any $y\in\mathbb{R}$. By Proposition 4.5, we can differentiate the first line of (4.15) with respect to y and get by Proposition 4.4-(i) that V_y solves inside $\mathcal{C}$ the equation

$$\frac{1}{2}\eta^2V_{yxx}(x,y)+\alpha yV_{yx}(x,y)-\rho V_y(x,y)=-f_y(x,y)-\alpha V_x(x,y). \tag{5.5}$$

By continuity, (5.5) also holds on $\overline{\mathcal{C}}$, i.e.

$$\frac{1}{2}\eta^2V_{yxx}(x,y)+\alpha yV_{yx}(x,y)-\rho V_y(x,y)=-f_y(x,y)-\alpha V_x(x,y),\quad \forall(x,y)\in\overline{\mathcal{C}}. \tag{5.6}$$

In particular it holds on $\partial^1\mathcal{C}:=\overline{\mathcal{C}}\cap\mathcal{I}$. Assume now, by contradiction, that the boundary b_1 is constant on $(x_o,x_o+\varepsilon)$, for some $x_o\in\mathbb{R}$ and some $\varepsilon>0$. Then, setting $y_o:=b_1(x_o)$, we will have $V_{yxx}(\cdot,y_o)=V_{yx}(\cdot,y_o)=0$ and $V_y(\cdot,y_o)=-K$ on $(x_o,x_o+\varepsilon)$. Hence, we obtain from (5.5) that

$$-\rho K=f_y(x,y_o)+\alpha V_x(x,y_o),\quad \forall x\in(x_o,x_o+\varepsilon), \tag{5.7}$$

and thus

$$-f_{yx}(x,y_o)=\alpha V_{xx}(x,y_o),\quad \forall x\in(x_o,x_o+\varepsilon). \tag{5.8}$$

But now $\alpha V_{xx}(x,y_o)\geq 0$ for any $x\in(x_o,x_o+\varepsilon)$ by convexity of $V(\cdot,y_o)$, while, by assumption, f_{yx} must be strictly positive on a subset of $(x_o,x_o+\varepsilon)$ with positive measure. Hence a contradiction is reached.

Case (b). We assume here that item (b) of Assumption 5.1-(iii) holds, i.e. that $f_{yx}\equiv 0$ and that $f(\cdot,y)$ is strictly convex for any $y\in\mathbb{R}$. In such a case the claim can be proved by employing the same arguments of the proof of Proposition 6.3 in [18]. □

From the result above, it immediately follows the following corollary.

Corollary 5.6. *Let Assumption 5.1 hold. Then the functions g_1,g_2 defined in (3.8) are continuous.*

The next result will be of fundamental importance to show the locally Lipschitz property of g_i, $i=1,2$ and, in the next section, to determine a system of differential equations for those curves.

Proposition 5.7. *Let Assumption 5.1 hold. Then*

$$\exists\lim_{\substack{(x,y)\to(x_o,y_o)\\(x,y)\in\mathcal{C}}}V_{yxx}(x,y)\neq 0\quad\forall\,(x_o,y_o)\in\partial\mathcal{C}. \tag{5.9}$$

Proof. We provide the proof only for any $(x_o,y_o)\in\partial^2\mathcal{C}:=\overline{\mathcal{C}}\cap\mathcal{D}$, as the other case can be treated similarly.

First of all, we notice that the limit in (5.9) exists since, by Proposition 4.5, the function $V:\mathcal{C}\to\mathbb{R}$ can be differentiated twice with respect to x and once with respect to y with continuity up to the boundary $\partial\mathcal{C}$.

Case (a). We assume here that item (a) of Assumption 5.1-(iii) holds, i.e. that $x \mapsto f_y(x,y)$ is strictly increasing for any $y \in \mathbb{R}$. Suppose, by contradiction, that for some $y_o \in \mathbb{R}$ one has

$$\lim_{\substack{(x,y)\to\ (g_2(y_o),y_o)\\ (x,y)\in\mathcal{C}}} V_{yxx}(x,y) = 0. \tag{5.10}$$

Then taking limits as $(x,y) \to\ (g_2(y_o), y_o)$ for $(x,y) \in \mathcal{C}$ in (4.13) we find, using that $V_{yx}(g_2(y_o), y_o) = 0$ by Proposition 4.7 and that $V_y(g_2(y_o), y_o) = K$,

$$-\rho K + f_y(g_2(y_o), y_o) = -\alpha V_x(g_2(y_o), y_o). \tag{5.11}$$

Since $g_2(y_o) > \zeta_2(y_o)$ by Proposition 5.3, and by definition of ζ_2, it must be

$$-\rho K + f_y(x, y_o) = -\alpha V_x(x, y_o) \quad \forall\, x \in (\zeta_2(y_o), g_2(y_o)),$$

which also implies that $-\alpha V_{xx}(x, y_o) = f_{yx}(x, y_o)$ for any $x \in (\zeta_2(y_o), g_2(y_o))$. We then conclude as in Step 1 of the proof of Proposition 5.5.

Case (b). We assume here that item (b) of Assumption 5.1-(iii) holds, which implies that there exists q such that $f_y(x,y) = q(y)$ for any $(x,y) \in \mathbb{R}^2$. Suppose again, with the aim of reaching a contradiction, that for some $y_o \in \mathbb{R}$ one has (5.10). Then taking limits as $(x,y) \to\ (g_2(y_o), y_o)$ for $(x,y) \in \mathcal{C}$ in (4.13) we find, using that $V_{yx}(g_2(y_o), y_o) = 0$ by Proposition 4.7 and that $V_y(g_2(y_o), y_o) = K$,

$$-\rho K + q(y_o) = -\alpha V_x(g_2(y_o), y_o).$$

As before, because $g_2(y_o) > \zeta_2(y_o)$ by Proposition 5.3, and by definition of ζ_2, it must be

$$-\rho K + q(y_o) = -\alpha V_x(x, y_o) \quad \forall\, x \in (\zeta_2(y_o), g_2(y_o));$$

that is, V is an affine function of x in that interval. However, using the latter and (4.13), we also have

$$\frac{1}{\alpha}\alpha y_o\big(\rho K - q(y_o)\big) - \rho V(x, y_o) = -f(x, y_o) \quad \forall\, x \in (\zeta_2(y_o), g_2(y_o)),$$

and we reach a contradiction since $f(\cdot, y_o)$ is strictly convex by assumption, while $V(\cdot, y_o)$ is affine. □

Proposition 5.8. *Let Assumption 5.1 hold. Then the functions g_1, g_2 are locally Lipschitz.*

Proof. Define the function

$$\bar{V}(x,y) := A(y)\psi(x,\alpha y) + B(y)\varphi(x,\alpha y) + \widehat{V}(x,y), \quad (x,y) \in \mathbb{R}^2, \tag{5.12}$$

where A, B are the functions of Proposition 4.5. Then, one clearly has that $\bar{V} \in C^{2,1}(\mathbb{R}^2;\mathbb{R})$, and $\bar{V} = V$ in $\mathbb{R}^2 \cap \bar{\mathcal{C}}$. Moreover, the mixed derivative $\bar{V}_{yx}$ exists and is continuous, and standard differentiation yield

$$\bar{V}_{yx}(x,y) = A_y(y)\psi_x(x,\alpha y) + B_y(y)\varphi_x(x,\alpha y) + \alpha\big(A(y)\psi_{zx}(x,\alpha y) + B(y)\varphi_{zx}(x,\alpha y)\big) + \widehat{V}_{yx}(x,y). \tag{5.13}$$

Since A_y and B_y are locally Lipschitz by Proposition 4.5, and ψ and φ are smooth (cf. (4.1)), we deduce that $\bar{V}_{yx}(x,\cdot)$ is locally Lipschitz.

Let now $y_o \in \mathbb{R}$. Then, for any given $x_o \in \mathbb{R}$ such that $(x_o, y_o) \in \partial\mathcal{C}$, we know by Proposition 5.7 that $\bar{V}_{yxx}(x_o, y_o) \neq 0$, while $\bar{V}_{yx}(x_o, y_o) = 0$. By the implicit function theorem (see, e.g., the Corollary at p. 256 in [12] or Theorem 3.1 in [30]) we therefore gain that for any $i = 1, 2$ there exists a unique continuous function $\bar{g}_i : (y_o - \delta, y_o + \delta) \to (x_o - \delta', x_o + \delta')$, for suitable $\delta, \delta' > 0$, such that $\bar{V}_{yx}(\bar{g}_i(y), y) = 0$ in $(y_o - \delta, y_o + \delta)$. Also, the aforementioned properties of $\bar{V}_{yxy}$ and $\bar{V}_{yxx}$ imply that there exists $C(y_o) > 0$ such that

$$|\bar{g}_i(y_2) - \bar{g}_i(y_1)| \leq C(y_o)|y_2 - y_1|, \quad \forall\, y_1, y_2 \in (y_o - \delta, y_o + \delta).$$

Recalling now that $\bar{V}_{yx}(g_i(y), y) = 0$, we can identify $\bar{g}_i = g_i$, $i = 1, 2$, in $(y_o - \delta, y_o + \delta)$ and therefore g_i is locally Lipschitz therein. Given the arbitrariness of the point (x_o, y_o) the proof is complete. □

6. A System of Differential Equations for the Free Boundaries

In this section we derive a first-order system of nonlinear differential equations for the free boundaries g_1 and g_2, i.e. we will be able to write

$$\begin{cases} g_1'(y) = G_1(g_1(y), g_2(y), y), \\ g_2'(y) = G_2(g_1(y), g_2(y), y), \end{cases}$$

for some explicitly determined maps G_1, G_2, whose regularity will allow also to establish a $C^{1,\mathrm{Lip}}$ regularity for g_1, g_2. To the best of our knowledge, for a two-dimensional degenerate singular stochastic control problem with interconnected dynamics as ours, a similar result appears here for the first time.

We first move on by establishing four equations relating g_1, g_2 and A, B. Recall (4.6), (4.7), and (4.9). We also denote by p the transition density of $X^{x,y}$ with respect to the speed measure; then, letting $A \mapsto \mathsf{P}_t(x, A, y)$, $A \in \mathcal{B}(\mathbb{R})$, $t > 0$ and $y \in \mathbb{R}$, be the probability of starting at time 0 from level $x \in \mathbb{R}$ and reaching the set $A \in \mathcal{B}(\mathbb{R})$ in t units of time, we have (cf., e.g., p. 13 in [4])

$$\mathsf{P}_t(x, A, y) = \int_A p(t, x, z, y) m_x(z, \alpha y) \mathrm{d}z. \tag{6.1}$$

The density p can be taken positive, jointly continuous in all variables and symmetric (i.e. $p(t, x, z, y) = p(t, z, x, y)$).

Theorem 6.1. *Let Assumption 5.1 hold. Recall (4.1), (4.15), and for any $(x, y) \in \mathbb{R}^2$ define*

$$H(x, y) := f_y(x, y) + \alpha V_x(x, y). \tag{6.2}$$

Moreover, for $z \in \mathbb{R}$ let

$$\lambda(z) := \sqrt{z^2 + 2\rho\eta^2}, \tag{6.3}$$

and for $y \in \mathbb{R}$, $i, j = 1, 2$, $j \neq i$,

$$\gamma_i(y) := -\frac{2}{\eta^2} \int_{g_1(y)}^{g_2(y)} e^{-r_j(\alpha y)u} \big(f_y(u, y) + \alpha \widehat{V}_x(u, y)\big) \mathrm{d}u + K r_i(\alpha y) \Big(e^{-r_j(\alpha y) g_1(y)} + e^{-r_j(\alpha y) g_2(y)}\Big), \tag{6.4}$$

The free boundaries g_1 and g_2 as in (3.8), and the coefficients A and B are such that

$$0 = \int_{g_1(y)}^{g_2(y)} \psi(z, \alpha y) H(z, y) m_x(z, \alpha y) \, \mathrm{d}z - K \frac{\psi_x(g_1(y), \alpha y)}{S_x(g_1(y), \alpha y)} - K \frac{\psi_x(g_2(y), \alpha y)}{S_x(g_2(y), \alpha y)}, \tag{6.5}$$

$$0 = \int_{g_1(y)}^{g_2(y)} \varphi(z, \alpha y) H(z, y) m_x(z, \alpha y) \, \mathrm{d}z - K \frac{\varphi_x(g_1(y), \alpha y)}{S_x(g_1(y), \alpha y)} - K \frac{\varphi_x(g_2(y), \alpha y)}{S_x(g_2(y), \alpha y)}, \tag{6.6}$$

$$\begin{aligned} &A'(y)\psi_x(g_1(y), \alpha y) + B'(y)\varphi_x(g_1(y), \alpha y) + \widehat{V}_{yx}(g_1(y), y) \\ &\quad + \alpha\big[A(y)\psi_{zx}(g_1(y), \alpha y) + B(y)\varphi_{zx}(g_1(y), \alpha y)\big] = 0 \end{aligned} \tag{6.7}$$

$$\begin{aligned} &A'(y)\psi_x(g_2(y), \alpha y) + B'(y)\varphi_x(g_2(y), \alpha y)\widehat{V}_{yx}(g_2(y), y) \\ &\quad + \alpha\big[A(y)\psi_{zx}(g_2(y), \alpha y) + B(y)\varphi_{zx}(g_2(y), \alpha y)\big] = 0. \end{aligned} \tag{6.8}$$

Hence,

$$A(y) = -\frac{r_2(\alpha y)\eta^2\lambda(\alpha y)}{8\alpha\rho} \left[\frac{\gamma_1(y)\Big(e^{-\frac{2}{\eta^2}\lambda(\alpha y) g_1(y)} - e^{-\frac{2}{\eta^2}\lambda(\alpha y) g_2(y)}\Big) - \frac{2\lambda(\alpha y)\gamma_2(y)}{\eta^2}\big(g_2(y) - g_1(y)\big)}{\sinh^2\Big(\frac{\lambda(\alpha y)}{\eta^2}\big(g_2(y) - g_1(y)\big)\Big) - \Big(\frac{\lambda(\alpha y)}{\eta^2}\big(g_2(y) - g_1(y)\big)\Big)^2} \right] \tag{6.9}$$

$$B(y) = -\frac{r_1(\alpha y)\eta^2\lambda(\alpha y)}{8\alpha\rho}\left[\frac{\gamma_2(y)\Big(e^{\frac{2}{\eta^2}\lambda(\alpha y)g_2(y)} - e^{\frac{2}{\eta^2}\lambda(\alpha y)g_1(y)}\Big) - \frac{2\lambda(\alpha y)\gamma_1(y)}{\eta^2}\big(g_2(y)-g_1(y)\big)}{\sinh^2\Big(\frac{\lambda(\alpha y)}{\eta^2}\big(g_2(y)-g_1(y)\big)\Big) - \Big(\frac{\lambda(\alpha y)}{\eta^2}\big(g_2(y)-g_1(y)\big)\Big)^2}\right], \tag{6.10}$$

$$A'(y) = \frac{\eta^2}{2\rho}e^{\frac{2\alpha y}{\eta^2}g_1(y)}\left[\frac{M(g_1(y),g_2(y),y)}{e^{r_1(\alpha y)(g_2(y)-g_1(y))} - e^{r_2(\alpha y)(g_2(y)-g_1(y))}}\right] \tag{6.11}$$

$$B'(y) = \frac{\eta^2}{2\rho}e^{\frac{2\alpha y}{\eta^2}g_1(y)}\left[\frac{N(g_1(y),g_2(y),y)}{e^{r_1(\alpha y)(g_2(y)-g_1(y))} - e^{r_2(\alpha y)(g_2(y)-g_1(y))}}\right], \tag{6.12}$$

where

$$\begin{aligned}
&M(x_1,x_2,y) := r_2(\alpha y)\big(e^{r_2(\alpha y)x_1}\widehat{V}_{yx}(x_2,y) - e^{r_2(\alpha y)x_2}\widehat{V}_{yx}(x_1,y)\big)\\
&+\alpha A(y)r_2'(\alpha y)r_2(\alpha y)\Big(e^{r_1(\alpha y)x_2+r_2(\alpha y)x_1}\big(r_2(\alpha y)x_2+1\big) - e^{r_1(\alpha y)x_1+r_2(\alpha y)x_2}\big(r_1(\alpha y)x_1+1\big)\Big)\\
&+\alpha B(y)r_2^2(\alpha y)r_2'(\alpha y)e^{r_2(\alpha y)(x_1+x_2)}(x_2-x_1)
\end{aligned}$$

and

$$\begin{aligned}
&N(x_1,x_2,y) := r_1(\alpha y)\big(e^{r_1(\alpha y)x_2}\widehat{V}_{yx}(x_1,y) - e^{r_1(\alpha y)x_1}\widehat{V}_{yx}(x_2,y)\big)\\
&+\alpha B(y)r_2'(\alpha y)r_1(\alpha y)\Big(e^{r_1(\alpha y)x_2+r_2(\alpha y)x_1}\big(r_2(\alpha y)x_1+1\big) - e^{r_1(\alpha y)x_1+r_2(\alpha y)x_2}\big(r_2(\alpha y)x_2+1\big)\Big)\\
&-\alpha A(y)r_1^2(\alpha y)r_1'(\alpha y)e^{r_1(\alpha y)(x_1+x_2)}(x_2-x_1).
\end{aligned}$$

Proof. To obtain equations (6.5) and (6.6) we exploit the proved regularity of $V(\cdot,y)$ (cf. Propositions 2.4, 4.7, and Lemma 4.8) in order to follow the proof of Theorem 6.5 in [18]. This is based on an application of the local time-space calculus of [31] to the process $(e^{-\rho s}V_y(X_s,y))_{s\geq 0}$ and the use of the Green function (4.9), the transition probability (6.1), and Fubini's theorem. Alternatively (and equivalently) they can be derived from (4.15) by imposing that $V_y(\cdot,y)$ and $V_{yx}(\cdot,y)$ are continuous on $\partial\mathcal{C}$ and proceeding via the more analytical direct approach of the proof of Proposition 5.5 in [17]. In particular, the second-order smooth fit $V_{yx}(g_i(y),y)=0$, $i=1,2$, easily gives (6.7) and (6.8).

We now move on by deriving (6.9) and (6.10). Notice that, given g_1 and g_2, and exploiting (4.15), one has from (6.5) and (6.6) that A and B solve the linear system

$$\begin{aligned}
&A(y)\left[\alpha\int_{g_1(y)}^{g_2(y)}\psi(z,\alpha y)\psi_x(z,\alpha y)m_x(z,\alpha y)\,\mathrm{d}z\right] + B(y)\left[\alpha\int_{g_1(y)}^{g_2(y)}\psi(z,\alpha y)\varphi_x(z,\alpha y)m_x(z,\alpha y)\,\mathrm{d}z\right]\\
&= K\left[\frac{\psi_x(g_1(y),\alpha y)}{S_x(g_1(y),\alpha y)} + \frac{\psi_x(g_2(y),\alpha y)}{S_x(g_2(y),\alpha y)}\right] - \int_{g_1(y)}^{g_2(y)}\psi(z,\alpha y)\Big(f_y(z,y)+\alpha\widehat{V}_x(z,y)\Big)m_x(z,\alpha y)\,\mathrm{d}z,
\end{aligned} \tag{6.13}$$

$$\begin{aligned}
&A(y)\left[\alpha\int_{g_1(y)}^{g_2(y)}\varphi(z,\alpha y)\psi_x(z,\alpha y)m_x(z,\alpha y)\,\mathrm{d}z\right] + B(y)\left[\alpha\int_{g_1(y)}^{g_2(y)}\varphi(z,\alpha y)\varphi_x(z,\alpha y)m_x(z,\alpha y)\,\mathrm{d}z\right]\\
&= K\left[\frac{\varphi_x(g_1(y),\alpha y)}{S_x(g_1(y),\alpha y)} + \frac{\varphi_x(g_2(y),\alpha y)}{S_x(g_2(y),\alpha y)}\right] - \int_{g_1(y)}^{g_2(y)}\varphi(z,\alpha y)\Big(f_y(z,y)+\alpha\widehat{V}_x(z,y)\Big)m_x(z,\alpha y)\,\mathrm{d}z.
\end{aligned} \tag{6.14}$$

By using expressions for ψ, φ, S_x and m_x (cf. (4.1), (4.6) and (4.7)) one can explicitly evaluate the integrals appearing on the left-hand sides of (6.13) and (6.14). Then, solving the latter two equations with respect to A and B one finds after some simple but tedious algebra (6.9) and (6.10).

Notice indeed that the denominator appearing in (6.9) and (6.10) is nonzero since $g_1 \neq g_2$ and one has $\sinh^2(z) - z^2 > 0$ for any $z \neq 0$.

In order to find (6.11) and (6.12) we solve (6.7) and (6.8) with respect to $A'(y)$ and $B'(y)$, and use (4.1), (4.2), (4.3), and (6.3).

□

Remark 6.2. *In [18] a system of equations like (6.5) and (6.6) has also been obtained (see eqs. (6.11) and (6.12) therein). However, in [18] the uncontrolled process is of Ornstein-Uhlenbeck type and this made it not possible to determine explicit expressions for $A(y)$ and $B(y)$ as in (6.9) and (6.10) above. Indeed, the complex form of the functions ψ and φ associated to the Ornstein-Uhlenbeck process does not allow to conclude that the determinant of the coefficients' matrix arising when one tries to solve (the analogous of) (6.13) and (6.13) with respect to $A(y)$ and $B(y)$ is nonzero.*

We can now state the main result of this paper.

Theorem 6.3. *Let $D := \{(x_1, x_2, y) \in \mathbb{R}^3 : x_1 \neq x_2\} \times \mathbb{R}$. There exist explicitly computable[1] functions $G_i \in C^{0,Lip}_{loc}(D;\mathbb{R})$, $i = 1, 2$ such that*

$$\begin{cases} g_1'(y) = G_1(g_1(y), g_2(y), y) \\ g_2'(y) = G_2(g_1(y), g_2(y), y). \end{cases} \tag{6.15}$$

In particular, $g_i \in C^{1,Lip}_{loc}(\mathbb{R};\mathbb{R})$ for $i = 1, 2$.

Proof. Recall Theorem 4.5 and (6.2). In particular, for any (x, y) such that $g_1(y) \leq x \leq g_2(y)$ – i.e. for any $(x, y) \in \bar{\mathcal{C}}$ – we have by (4.15)

$$V_x(x, y) = A(y)\psi_x(x, \alpha y) + B(y)\varphi_x(x, \alpha y) + \widehat{V}_x(x, y),$$

with A, B belonging to $W^{2,\infty}_{\text{loc}}(\mathbb{R};\mathbb{R})$. Defining then the function

$$\bar{H}(x, y) = f_y(x, y) + \alpha\big(A(y)\psi_x(x, \alpha y) + B(y)\varphi_x(x, \alpha y) + \widehat{V}_x(x, y)\big), \quad (x, y) \in \mathbb{R}^2,$$

one has $\bar{H} = H$ on $\bar{\mathcal{C}}$.

Introduce now $\Phi_i : D \to \mathbb{R}$ defined as

$$\begin{aligned} \Phi_1(x_1, x_2, y) := &\int_{x_1}^{x_2} \psi(z, \alpha y)\bar{H}(z, y)m_x(z, \alpha y)\ \mathrm{d}z \\ &- K\rho \int_{-\infty}^{x_1} \psi(z, \alpha y)m_x(z, \alpha y)\ \mathrm{d}z - K\rho \int_{-\infty}^{x_2} \psi(z, \alpha y)m_x(z, \alpha y)\ \mathrm{d}z \end{aligned} \tag{6.16}$$

$$\begin{aligned} \Phi_2(x_1, x_2, y) := &\int_{x_1}^{x_2} \varphi(z, \alpha y)\bar{H}(z, y)m_x(z, \alpha y)\ \mathrm{d}z \\ &+ K\rho \int_{x_2}^{\infty} \varphi(z, \alpha y)m_x(z, \alpha y)\ \mathrm{d}z + K\rho \int_{x_1}^{\infty} \varphi(z, \alpha y)m_x(z, \alpha y)\ \mathrm{d}z. \end{aligned} \tag{6.17}$$

Observing that (cf. Chapter II in [4])

$$\frac{\psi_x(\cdot, \alpha y)}{S_x(\cdot, \alpha y)} = \rho \int_{-\infty}^{\cdot} \psi(z, \alpha y)m_x(z, \alpha y)\ \mathrm{d}z, \qquad \frac{\varphi_x(\cdot, \alpha y)}{S_x(\cdot, \alpha y)} = -\rho \int_{\cdot}^{\infty} \varphi(z, \alpha y)m_x(z, \alpha y)\ \mathrm{d}z,$$

one can readily see that, by (6.5)-(6.6), for any $y \in \mathbb{R}$ one has

$$\Phi_1(g_1(y), g_2(y), y) = 0 \quad \text{and} \quad \Phi_2(g_1(y), g_2(y), y) = 0. \tag{6.18}$$

Thanks to Assumption 2.2 and Theorem 4.5, one has that $\bar{H} \in C^{1,\text{Lip}}_{\text{loc}}(\mathbb{R}^2;\mathbb{R})$. Hence, for any $i = 1, 2$, the map $(x_1, x_2) \mapsto \Phi_i(x_1, x_2, y)$ belongs to $C^2(D;\mathbb{R})$ for each $y \in \mathbb{R}$ and the map $y \mapsto$

[1] Cf. Remark 6.4.

$\Phi_i(x_1,x_2,y)$ belongs to $C^{1,\mathrm{Lip}}_{\mathrm{loc}}(D;\mathbb{R})$ for each $(x_1,x_2)\in\mathbb{R}^2$. Recalling Proposition 5.8 we can take the total derivative on both terms appearing in (6.18) we obtain for a.e. $y\in\mathbb{R}$ that

$$\underbrace{\begin{pmatrix} \frac{\partial\Phi_1}{\partial x_1}(g_1(y),g_2(y),y) & \frac{\partial\Phi_1}{\partial x_2}(g_1(y),g_2(y),y) \\ \\ \frac{\partial\Phi_2}{\partial x_1}(g_1(y),g_2(y),y) & \frac{\partial\Phi_2}{\partial x_2}(g_1(y),g_2(y),y)\end{pmatrix}}_{=:\Lambda(g_1(y),g_2(y),y)}\begin{pmatrix} g_1'(y) \\ \\ g_2'(y)\end{pmatrix} = -\begin{pmatrix} \frac{\partial\Phi_1}{\partial y}(g_1(y),g_2(y),y) \\ \\ \frac{\partial\Phi_2}{\partial y}(g_1(y),g_2(y),y)\end{pmatrix}. \tag{6.19}$$

The determinant of the matrix Λ, denoted by $|\Lambda|$, is given by

$$\begin{aligned}|\Lambda|(g_1(y),g_2(y),y) =&\big(\bar H(g_1(y),y)+K\rho\big)\big(\bar H(g_2(y),y)-K\rho\big)m_x(g_1(y),\alpha y)m_x(g_2(y),\alpha y)\cdot\\ &\cdot\Big(\psi(g_2(y),\alpha y)\varphi(g_1(y),\alpha y)-\psi(g_1(y),\alpha y)\varphi(g_2(y),\alpha y)\Big).\end{aligned} \tag{6.20}$$

We now aim at showing that $|\Lambda|(g_1(y),g_2(y),y)$ does not vanish for any $y\in\mathbb{R}$ under Assumption 5.1-(iii). On the one hand, if item (a) of that assumption holds, i.e. $x\mapsto f_y(x,y)$ is strictly increasing, then we have that $x\mapsto H(x,y)$ is such as well. Since $\bar H=H$ on $\bar{\mathcal{C}}$ and $g_2(y)>\zeta_2(y)>\zeta_1(y)>g_1(y)$ by Proposition 5.3-(ii), we have

$$\bar H(g_1(y),y)+K\rho<0,\quad \bar H(g_2(y),y)-K\rho>0,$$

and

$$\psi(g_2(y),\alpha y)\varphi(g_1(y),\alpha y)-\psi(g_1(y),\alpha y)\varphi(g_2(y),\alpha y)>0;$$

therefore, $|\Lambda|(g_1(y),g_2(y),y)<0$. On the other hand, if item (b) of Assumption 5.1-(iii) holds, i.e. if $f_{yx}\equiv 0$ and $f(\cdot,y)$ is strictly convex for any $y\in\mathbb{R}$, we can argue by contradiction as in Case (b) of the proof of Corollary 5.6. To this end, suppose, for example, that $\bar H(g_1(y_o),y_o)+K\rho = H(g_1(y_o),y_o)+K\rho=0$, for some $y_o\in\mathbb{R}$. Denoting $f_y(x,y)=q(y)$ it then follows that

$$-\rho K+q(y_o)=-\alpha V_x(x,y_o)\quad\forall\, x\in(g_1(y_o),\zeta_1(y_o)),$$

by definition of ζ_1 (cf. Proposition 5.3); that is, V is an affine function of x in that interval. However, using the latter and (4.13), we also have

$$\frac{1}{\alpha}\alpha y_o\big(\rho K-q(y_o)\big)-\rho V(x,y_o)=-f(x,y_o)\quad\forall\, x\in(g_1(y_o),\zeta_1(y_o)),$$

and we reach a contradiction since f is strictly convex in x by assumption while V is affine. The same argument also implies that $\bar H(g_1(y_o),y_o)+K\rho\neq 0$. We have then proved that in any case one has $|\Lambda|(g_1(y),g_2(y),y)\neq 0$ under Assumption 5.1-(iii).

We can therefore invert the matrix Λ appearing in (6.19) and obtain that for a.e. $y\in\mathbb{R}$

$$\begin{cases} g_1'(y)=\dfrac{1}{|\Lambda|(g_1(y),g_2(y),y)}\Big[\dfrac{\partial\Phi_1}{\partial x_2}\dfrac{\partial\Phi_2}{\partial y}-\dfrac{\partial\Phi_2}{\partial x_2}\dfrac{\partial\Phi_1}{\partial y}\Big](g_1(y),g_2(y),y)=:G_1(g_1(y),g_2(y),y)\\ \\ g_2'(y)=\dfrac{1}{|\Lambda|(g_1(y),g_2(y),y)}\Big[\dfrac{\partial\Phi_2}{\partial x_1}\dfrac{\partial\Phi_1}{\partial y}-\dfrac{\partial\Phi_1}{\partial x_1}\dfrac{\partial\Phi_2}{\partial y}\Big](g_1(y),g_2(y),y)=:G_2(g_1(y),g_2(y),y)\end{cases} \tag{6.21}$$

Observe now that, given the aforementioned regularity of $\frac{\partial\Phi_i}{\partial x_j}$, $i,j=1,2$, and of $\frac{\partial\Phi_i}{\partial y}$, $i=1,2$, we have $G_i\in C^{0,\mathrm{Lip}}_{\mathrm{loc}}(D;\mathbb{R})$; hence, $g_i\in C^{1,\mathrm{Lip}}_{\mathrm{loc}}(\mathbb{R};\mathbb{R})$. □

Remark 6.4. *Notice that the right-hand sides of (6.21) are indeed functions only of $(g_1(y),g_2(y),y)$. To see that, it is enough to feed (6.9) and (6.10), and (6.11) and (6.12) in the right-hand sides of (6.21), upon noticing that for any $i,j=1,2$, $\frac{\partial\Phi_i}{\partial x_j}$ depend on $A(y),B(y)$, while, for any $i=1,2$, $\frac{\partial\Phi_i}{\partial y}$ depend on $A'(y),B'(y)$.*

Remark 6.5. *In the proof of Proposition 5.6 of* [17] *(see page 2213 therein; see also Step 4 in the proof of Lemma 7 in* [29] *and the proof of Proposition 6 in* [14]*), a system of ODEs for the free boundaries is determined with the aim of proving that the free boundaries belong to* C^1 *and are strictly monotone. In our problem, proving strict monotonicity of* g_1 *and* g_2 *would require to establish a strict sign for* G_1 *and* G_2 *(cf.* (6.21)*). However, the interaction between our dynamics – and the consequent dependency of* ψ*,* φ*, and* m_x *on* y *– makes the partial derivatives* $\frac{\partial \Phi_i}{\partial y}$ *appearing in* (6.21) *much more complex than the analogous quantities in* [17] *or* [29]*, and this in turn makes it unclear that* $G_i < 0$*,* $i = 1.2$ *(although expected).*

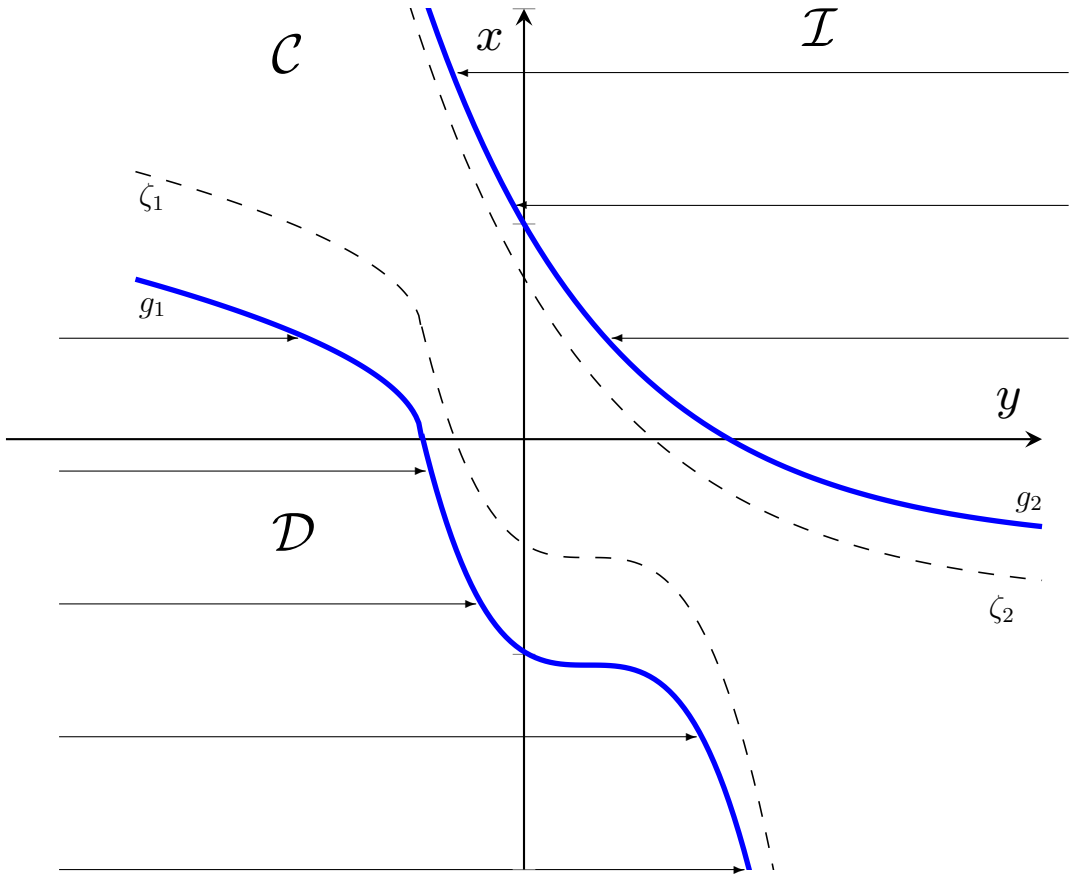
6.1. A Discussion on Theorem 6.3 and on the Optimal Control.

6.1.1. *On Theorem 6.3.* Given the full degeneracy of our setting, the fact that the free boundaries g_i, $i = 1, 2$, belong to the class $C^{1,\mathrm{Lip}}_{\mathrm{loc}}(\mathbb{R};\mathbb{R})$ is, to the best of our knowledge, a remarkable result. Indeed, the lack of uniform ellipticity of the diffusion coefficient makes it already difficult to obtain a preliminary (locally) Lipschitz property of g_is by invoking results from PDE theory ([6] and [33], among others) or techniques as those in [34], [35], and [36]. Also the probabilistic approach developed in [15] is not directly applicable since our free boundaries are associated to a Dynkin game rather than to an optimal stopping problem.

It is also worth stressing that Theorem 6.3 not only provides regularity of the free boundaries, but also a system of ODEs. To the best of our knowledge, a similar result appears here for the first time. Clearly, in order to provide a complete characterization of g_is, (6.15) should be complemented by boundary conditions. The determination of those is a non trivial task. As a matter of fact, we have not been able to identify a relevant value of y for which the values of the free boundaries can be determined. The only information available is that the free boundaries diverge for large (in absolute value) levels of y; but this is clearly not enough. Even enforcing a finite-fuel constraint like $\underline{y} \leq Y^{y,\xi}_t \leq \overline{y}$ a.s. for any $t \geq 0$ would not help in order to obtain boundary conditions. Indeed, differently to the case with monotone controls (see [26]), here the drift process Y can be pushed back into $(\underline{y}, \overline{y})$ once any of the boundary points of that interval is reached. Also, it is not clear to us how to obtain some kind of asymptotic growth of the free boundaries in order to restrict the functional class where to look for uniqueness of (6.15).

A possible way to obtain a complete implementable characterization of the free boundaries might be the following. Instead of thinking of g_1 and g_2 as functions of y, for a fixed parameter α, one could look at those as functions of α, for any given and fixed y. Bearing this in mind, one might try to prove that $\alpha \mapsto g_i(\alpha; y)$ are (at least) locally Lipschitz on $[0, \infty)$, and then follow the approach developed in this section in order to obtain a system of ODEs involving $\partial_\alpha g_i(\alpha; y)$, $i = 1, 2$, rather than $\partial_y g_i(\alpha; y)$. Those ODEs would then be complemented by a natural boundary condition since, by continuity, $g_1(0+; y)$ and $g_2(0+; y)$ would coincide with the free boundaries uniquely determined in Proposition 5.5 of [17]. However, it is not straightforward to prove the aforementioned Lipschitz regularity of $\alpha \mapsto g_i(\alpha; y)$; indeed, a preliminary analysis shows that this is related to that of $\alpha \mapsto V_x(x, y; \alpha)$, and how to prove the latter is not clear to us. The investigation of such an interesting conjecture is therefore left for future research.

6.1.2. *On The Optimal Control.* The following picture provides an illustrative description of the expected behavior of the optimal control rule $\xi^\star$. This should be such that the jumps of the two-dimensional process $(X^{x,y,\xi^\star}_t, Y^{y,\xi^\star}_t)_{t\geq 0}$ are induced by the optimal control only at initial time, if the initial data (x, y) lie in the interior of $\mathcal{I}$ or $\mathcal{D}$, or at those times at which the process meets jumps of the free boundaries. The size of those interventions should be such that the process is immediately brought to the closest point on $\partial\mathcal{C}$, from where it evolves according to (2.2) and (2.3) and in such a way that it is kept inside the closure of $\mathcal{C}$ in a minimal way. Mathematically, this amounts to construct $(X^{x,y,\xi^\star}_t, Y^{y,\xi^\star}_t)_{t\geq 0}$ as a (degenerate) diffusion that is reflected at $\partial\mathcal{C}$.



The latter is per se an interesting and not trivial problem, whose solution in multi-dimensional settings strongly hinges on the smoothness of the reflection boundary itself; sufficient conditions can be found in the seminal papers [16] and [27]. Unfortunately, our information on $\partial\mathcal{C}$ do not suffice to apply the results of the aforementioned works since we are not able to exclude horizontal segments of the free boundaries g_1 and g_2 (cf. Case (1) and Case (2) in [16]). Indeed, although we can provide explicit formulas for the maps G_1 and G_2 appearing in (6.15), their complex expressions makes it hard to show that they are strictly negative (see also Remark 6.5). On the other hand, also the more constructive approach followed in Section 5 of [9] seems not to apply in general to our case, unless we assume (as the authors of [9] do) a linear growth of the free boundaries b_i, $i = 1, 2$, or further requirements on f leading to a weak solution to the reflection problem as in Proposition 7.3 of [18]. We therefore leave for future research the general study of the intricate and intriguing problem of constructing the optimal control.

Acknowledgments. Financial support by the German Research Foundation (DFG) through the Collaborative Research Centre 1283 is gratefully acknowledged by the authors.

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