Large deviation principle for the two-dimensional stochastic Navier-Stokes equations with anisotropic viscosity *

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Abstract

In this paper we establish the large deviation principle for the the two-dimensional stochastic Navier-Stokes equations with anisotropic viscosity both for small noise and for short time. The proof for large deviation principle is based on the weak convergence approach. For small time asymptotics we use the exponential equivalence to prove the result.

Keywords: Large deviation principle; Stochastic Navier-Stokes equations; Anisotropic viscosity; Small time asymptotics; Weak convergence approach

1 Introduction

The main aim of this work is to establish large deviation principle and small time asymptotics for the stochastic Navier-Stokes equation with anisotropic viscosity. We consider the following stochastic Navier-Stokes equation with anisotropic viscosity on the two dimensional (2D) torus $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$:

$$\begin{aligned} &du = \partial_1^2 u dt - u \cdot \nabla u dt + \sigma(t,u) dW(t) - \nabla p dt, \\ &\text{div } u = 0, \\ &u(0) = u_0, \end{aligned} \tag{1.1}$$

where $u(t,x)$ denotes the velocity field at time $t \in [0,T]$ and position $x \in \mathbb{T}^2$, p denotes the pressure field, σ is the random external force and W is an l^2-cylindrical Wiener process.

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Let's first recall the classical Navier-Stokes (N-S) equation which is given by

$$
\begin{aligned}
&du = \nu\Delta u dt - u\cdot\nabla u dt - \nabla p dt,\\
&\mathrm{div}\ u = 0, \qquad\qquad\qquad\qquad\qquad (1.2)\\
&u(0) = u_0,
\end{aligned}
$$

where $\nu > 0$ is the viscosity of the fluid. (1.2) describes the time evolution of an incompressible fluid. In 1934, J. Leray proved global existence of finite energy weak solutions for the deterministic case in the whole space $\mathbb{R}^d$ for $d = 2, 3$ in the seminar paper [Ler33]. For more results on deterministic N-S equation, we refer to [CKN82], [Tem79], [Tem95], [KT01] and reference therein. For the stochastic case, there exists a great amount of literature too. The existence and uniqueness of solutions and ergodicity property to the stochatic 2D Navier-Stokes equation have been obtained (see e.g. [FG95], [MR05], [HM06]). Large deviation principles for the two-dimensional stochastic N-S equations have been established in [CM10] and [SS06].

Compared to (1.2), (1.1) only has partial dissipation, which can be viewed as an intermediate equation between N-S equation and Euler equation. System of this type appear in in geophysical fluids (see for instance [CDGG06] and [Ped79]). Instead of putting the classical viscosity $-\nu\Delta$ in (1.2), meteorologist often modelize turbulent diffusion by putting a viscosity of the form: $-\nu_h\Delta_h - \nu_3\partial_{x_3}^2$, where ν_h and ν_3 are empiric constants, and ν_3 is usually much smaller than ν_h. We refer to the book of J. Pedlovsky [Ped79, Chapter 4] for a more complete discussion. For the 3 dimensional case there is no result concerning global existence of weak solutions.

In the 2D case, [LZZ18] investigates both the deterministic system and the stochastic system (1.1) for $H^{0,1}$ initial value (For the definition of space see Section 2). The main difference in obtaining the global well-posedness for (1.1) is that the L^2-norm estimate is not enough to establish $L^2([0,T],L^2)$ strong convergence due to lack of compactness. In [LZZ18], the proof is based on an additional $H^{0,1}$-norm estimate. In this paper, we want to establish the large deviation principles for the two-dimensional stochastic Navier-Stokes equations with anisotropic viscosity both for small noise and for short time.

The large deviation theory concerns the asymptotic behavior of a family of random variables X_ε and we refer to the monographs [DPZ09] and [Str84] for many historical remarks and extensive references. It asserts that for some tail or extreme event A, $P(X_\varepsilon \in A)$ converges to zero exponentially fast as $\varepsilon \to 0$ and the exact rate of convergence is given by the so-called rate function. The large deviation principle was first established by Varadhan in [Var66] and he also studied the small time asymptotics of finite dimensional diffusion processes in [Var67]. Since then, many important results concerning the large deviation principle have been established. For results on the large deviation principle for stochastic differential equations in finite dimensional case we refer to [FW84]. For the extensions to infinite dimensional diffusions or SPDE, we refer the readers to [BDM08], [CM10], [DM09], [Liu09], [LRZ13], [RZ08], [XZ09], [Zha00] and the references therein.

We first study the small noise large deviations by using the weak convergence approach. This approach is mainly based on a variational representation formula for certain functionals of infinite dimensional Brownian Motion, which was established by Budhiraja and Dupuis in [BD00]. The main advantage of the weak convergence approach is that one can avoid some exponential probability estimates, which might be very difficult to derive for many infinite dimensional models. To use the weak convergence approach, we need to prove two conditions in Hypothesis 3.1. In [Liu09] and [LRZ13], the authors use integration by parts and lead to some extra condition on diffusion coefficient. In [CM10], the authors use time discretization and require time-regularity of diffusion coefficient. In this paper, we use the argument in [WZZ15], in which the authors prove a moderate deviation principle by this argument, i.e. we first establish the convergence in $L^2([0,T],L^2)$ and then by using this and Itô's formula, $L^\infty([0,T],L^2)\bigcap L^2([0,T],H^{1,0})$ convergence can be obtained. By this argument, we can drop the extra condition on diffusion coefficient in [Liu09] and [CM10].

For the small time asymptotics (large deviations) of the two-dimensional stochastic Navier-Stokes equations with anisotropic viscosity. This describes the limiting behaviour of the solution in time interval $[0,t]$ as t goes to zero. Another motivation will be to get the following Varadhan identity through the small time asymptotics:

$$\lim_{t\to 0} 2t\log P(u(0)\in B, u(t)\in C) = -d^2(B,C),$$

where d is an appropriate Riemannian distance associated with the diffusion generated by the solutions of the two-dimensional stochastic Navier-Stokes equations with anisotropic viscosity. The small time asymptotics is also theoretically interesting, since the study involves the investigation of the small noise and the effect of the small, but highly nonlinear drift.

To prove the small time asymptotics, we follow the idea of [XZ09] to prove the solution to (1.1) is exponentially equivalent to the solution to the linear equation. The main difference compared to [XZ09] is that similar to [LZZ18] L^2-norm estimate is not enough due to less dissipation and we have to do $H^{0,1}$-norm estimate.

Organization of the paper

In Section 2, we introduce the basic notation, definition and recall some preliminary results. In Section 3, we will build the small noise large deviation principle. In Section 4, we prove the small time asymptotics for the the two-dimensional stochastic Navier-Stokes equations with anisotropic viscosity.

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2 Preliminary

Function spaces on $\mathbb{T}^2$

We first recall some definitions of function spaces for the two dimensional torus $\mathbb{T}^2$.

Let $\mathbb{T}^2=\mathbb{R}/2\pi\mathbb{Z}\times\mathbb{R}/2\pi\mathbb{Z}=(\mathbb{T}_h,\mathbb{T}_v)$ where h stands for the horizonal variable x_1 and v stands for the vertical variable x_2. For exponents $p,q\in[1,\infty)$, we denote the space $L^p(\mathbb{T}_h,L^q(\mathbb{T}_v))$ by $L^p_h(L^q_v)$, which is endowed with the norm

$$\|u\|_{L^p_h(L^q_v)(\mathbb{T}^2)} := \{\int_{\mathbb{T}_h}(\int_{\mathbb{T}_v}|u(x_1,x_2)|^q dx_2)^{\frac{p}{q}}dx_1\}^{\frac{1}{p}}.$$

Similar notation for $L^p_v(L^q_h)$. In the case $p,q=\infty$, we denote L^∞ the essential supremum norm. Throughout the paper, we denote various positive constants by the same letter C.

For $u \in L^2(\mathbb{T}^2)$, we consider the Fourier expansion of u:

$$u(x) = \sum_{k\in\mathbb{Z}^2} \hat{u}_k e^{ik\cdot x} \text{ with } \hat{u}_k = \overline{\hat{u}_{-k}},$$

where $\hat{u}_k := \frac{1}{(2\pi)^2}\int_{[0,2\pi]\times[0,2\pi]} u(x)e^{-ik\cdot x}dx$ denotes the Fourier coefficient of u on $\mathbb{T}^2$.

Define the Sobolev norm:

$$\|u\|_{H^s}^2 := \sum_{k\in\mathbb{Z}^2}(1+|k|^2)^s|\hat{u}_k|^2,$$

and the anisotropic Sobolev norm:

$$\|u\|_{H^{s,s'}}^2 = \sum_{k\in\mathbb{Z}^2}(1+|k_1|^2)^s(1+|k_2|^2)^{s'}|\hat{u}_k|^2,$$

where $k=(k_1,k_2)$. We define the Sobolev spaces $H^s(\mathbb{T}^2)$, $H^{s,s'}(\mathbb{T}^2)$ as the completion of $C^\infty(\mathbb{T}^2)$ with the norms $\|\cdot\|_{H^s}$, $\|\cdot\|_{H^{s,s'}}$ respectively.

To formulate the stochastic Navier-Stokes equations with anisotropic viscosity, we need the following spaces:

$$\begin{aligned} H &:= \{u \in L^2(\mathbb{T}^2;\mathbb{R}^2); \text{div } u = 0\},\\ V &:= \{u \in H^1(\mathbb{T}^2;\mathbb{R}^2); \text{div } u = 0\},\\ \tilde{H}^{s,s'} &:= \{u \in H^{s,s'}(\mathbb{T}^2;\mathbb{R}^2); \text{div } u = 0\}. \end{aligned}$$

Moreover, we use $\langle\cdot,\cdot\rangle$ to denote the scalar product (which is also the inner product of L^2 and H)

$$\langle u, v\rangle = \sum_{j=1}^{2}\int_{\mathbb{T}^2} u^j(x)v^j(x)dx$$

and $\langle\cdot,\cdot\rangle_X$ to denote the inner product of Hilbert space X where $X = l^2$, V or $\tilde{H}^{s,s'}$.

Due to the divergence free condition, we need the Larey projection operator $P_H: \ L^2(\mathbb{T}^2) \to H$:

$$P_H : u \mapsto u - \nabla\Delta^{-1}(\text{div } u).$$

By applying the operator P_H to (1.1) we can rewrite the equation in the following form:

$$\begin{aligned} &du(t) = \partial_1^2 u(t)dt - B(u(t))dt + \sigma(t,u(t))dW(t),\\ &u(0) = u_0, \end{aligned} \tag{2.1}$$

where the nonlinear operator $B(u,v) = P_H(u\cdot\nabla v)$ with the notation $B(u) = B(u,u)$. Here we use the same symbol σ after projection for simplicity.

For $u,v,w\in V$, define

$$b(u,v,w) := \langle B(u,v), w\rangle.$$

We have $b(u,v,w) = -b(u,w,v)$ and $b(u,v,v) = 0$.

We put some estimates of b in the Appendix.

Large deviation principle

We recall the definition of the large deviation principle. For a general introduction to the theory we refer to [DPZ09], [DZ10].

Definition 2.1 (Large deviation principle)**.** *Given a family of probability measures $\{\mu_\varepsilon\}_{\varepsilon>0}$ on a metric space (E,ρ) and a lower semicontinuous function $I : E \to [0,\infty]$ not identically equal to $+\infty$. The family $\{\mu_\varepsilon\}$ is said to satisfy the large deviation principle(LDP) with respect to the rate function I if*
(U) for all closed sets $F \subset E$ we have

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(F) \leqslant -\inf_{x\in F} I(x),$$

(L) for all open sets $G \subset E$ we have

$$\liminf_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(G) \geqslant -\inf_{x\in G} I(x).$$

A family of random variable is said to satisfy large deviation principle if the law of these random variables satisfy large deviation princple.

Moreover, I is a good rate function if its level sets $I_r := \{x \in E : I(x) \leqslant r\}$ are compact for arbitrary $r \in (0,+\infty)$.

Definition 2.2 (Laplace principle)**.** *A sequence of random variables $\{X^\varepsilon\}$ is said to satisfy the Laplace principle with rate function I if for each bounded continuous real-valued function h defined on E*

$$\lim_{\varepsilon\to 0} \varepsilon \log E\left[e^{-\frac{1}{\varepsilon}h(X^\varepsilon)}\right] = -\inf_{x\in E}\{h(x)+I(x)\}.$$

Given a probabilty space $(\Omega,\mathcal{F},P)$, the random variables $\{Z_\varepsilon\}$ and $\{\overline{Z}_\varepsilon\}$ which take values in (E,ρ) are called exponentially equivalent if for each $\delta>0$,

$$\lim_{\varepsilon\to 0} \varepsilon \log P(\rho(Z_\varepsilon,\overline{Z}_\varepsilon)>\delta) = -\infty.$$

Lemma 2.1 ([DZ10, Theorem 4.2.13])**.** *If an LDP with a rate function $I(\cdot)$ holds for the random variables $\{Z_\varepsilon\}$, which are exponentially equivalent to $\{\overline{Z}_\varepsilon\}$, then the same LDP holds for $\{\overline{Z}_\varepsilon\}$.*

Existence and uniqueness of solutions

We introduce the precise assumptions on the diffusion coefficient σ. Given a complete probability space $(\Omega,\mathcal{F},P)$ with filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$. Let $L_2(l^2,U)$ denotes the Hilbert-Schmidt norms from l^2 to U for a Hilbert space U. We recall the following conditions for σ from [LZZ18]:

(i) **Growth condition**

There exists nonnegative constants K_i', K_i, $\tilde{K}_i$ $(i=0,1,2)$ such that for every $t\in[0,T]$:

(A0) $\|\sigma(t,u)\|^2_{L_2(l^2,H^{-1})} \leqslant K_0' + K_1'\|u\|^2_H$;
(A1) $\|\sigma(t,u)\|^2_{L_2(l^2,H)} \leqslant K_0 + K_1\|u\|^2_H + K_2\|\partial_1 u\|^2_H$;
(A2) $\|\sigma(t,u)\|^2_{L_2(l^2,H^{0,1})} \leqslant \tilde{K}_0 + \tilde{K}_1\|u\|^2_{H^{0,1}} + \tilde{K}_2(\|\partial_1 u\|^2_H + \|\partial_1\partial_2 u\|^2_H)$;

(ii)**Lipschitz condition**

There exists nonnegative constants L_1, L_2 such that:

(A3) $\|\sigma(t,u)-\sigma(t,v)\|^2_{L_2(l^2,H)} \leqslant L_1\|u-v\|^2_H + L_2\|\partial_1(u-v)\|^2_H$.

The following theorem from [LZZ18] gives the well-posedness of equation (2.1):

Lemma 2.2 ([LZZ18, Theorem 4.1, Theorem 4.2])**.** *Under the assumptions (A0)-(A3) with* $K_2 < \frac{2}{11}, \tilde{K}_2 < \frac{2}{5}, L_2 < \frac{2}{5}$*, equation (2.1) has a unique strong solution* $u \in L^\infty([0,T], \tilde{H}^{0,1}) \cap L^2([0,T], \tilde{H}^{1,1}) \cap C([0,T], H^{-1})$ *for* $u_0 \in \tilde{H}^{0,1}$*.*

A martingale lemma

The following remarkable result is from [BY82] and [Dav76]:

Lemma 2.3. *There exists a universal constant c such that, for any* $p \geqslant 2$ *and for all continuous martingale* (M_t) *with* $M_0 = 0$ *and stopping times* τ*,*

$$\|M_\tau^*\|_p \leqslant cp^{\frac{1}{2}} \|\langle M \rangle_\tau^{\frac{1}{2}}\|_p,$$

where $M_t^* = \sup_{0 \leqslant s \leqslant t} |M_s|$ *and* $\|\cdot\|_p$ *stands for the* L^p *norm with respect to the probability space.*

3 Large deviation principle

In this section, we consider the large deviation principle for the stochastic Navier-Stokes equations with anisotropic viscosity. We will use the weak convergence approach introduced by Budhiraja and Dupuis in [BD00]. First we recall it. The starting point is the equivalence between the large deviation principle and the Laplace principle. This result was first formulated in [Puk94] and it is essentially a consequence of Varadhan's lemma [Var66] and Bryc's converse theorem [Bry90].

Remark 3.1. *By [DZ10] we have the the equivalence between the large deviation principle and the Laplace principle in completely regular topological spaces. In [BD00] the authors give the weak convergence approach on a Polish space. Since the proof does not depend on the separability and the completeness, the result also holds in metric spaces.*

Let $\{W(t)\}_{t \geqslant 0}$ be a cylindrical Wiener process on l^2 w.r.t. a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ (i.e. the path of W take values in $C([0,T]; U)$, where U is another Hilbert space such that the embedding $l^2 \subset U$ is Hilbert-Schmidt). For $\varepsilon > 0$, suppose g^ε: $C([0,T], U) \to E$ is a measurable map and $u^\varepsilon := g^\varepsilon(W(\cdot))$. Let

$$\mathcal{A} := \left\{ v : v \text{ is } l^2\text{-valued } \mathcal{F}_t\text{-predictable process and } \int_0^T \|v(s)(\omega)\|_{l^2}^2 ds < \infty \text{ a.s.} \right\},$$

$$S_N := \left\{ \phi \in L^2([0,T], l^2) : \int_0^T \|\phi(s)\|_{l^2}^2 ds \leqslant N \right\},$$

$$\mathcal{A}_N := \{ v \in \mathcal{A} : v(\omega) \in S_N \text{ P-a.s.} \}.$$

Here we will always refer to the weak topology on S_N in the following if we do not state it explicitly.

Now we formulate the following sufficient conditions for the Laplace principle of u^ε as $\varepsilon \to 0$.

Hypothesis 3.1. *There exists a measurable map $g^0 : C([0,T],U) \to E$ such that the following two conditions hold:*
1. Let $\{v^\varepsilon : \varepsilon > 0\} \subset \mathcal{A}_N$ for some $N < \infty$. If v^ε converge to v in distribution as S_N-valued random elements, then

$$g^\varepsilon\left(W(\cdot) + \frac{1}{\sqrt{\varepsilon}}\int_0^\cdot v^\varepsilon(s)ds\right) \to g^0\left(\int_0^\cdot v(s)ds\right)$$

in distribution as $\varepsilon \to 0$.
2. For each $N < \infty$, the set

$$K_N = \left\{g^0\left(\int_0^\cdot \phi(s)ds\right) : \phi \in S_N\right\}$$

is a compact subset of E.

Lemma 3.1 ([BD00, Theorem 4.4])**.** *If $u^\varepsilon = g^\varepsilon(W)$ satisfies the Hypothesis 3.1, then the family $\{u^\varepsilon\}$ satisfies the Laplace principle (hence large deviation principle) on E with the good rate function I given by*

$$I(f) = \inf_{\{\phi \in L^2([0,T],l^2): f = g^0(\int_0^\cdot \phi(s)ds)\}} \left\{\frac{1}{2}\int_0^T \|\phi(s)\|_{l^2}^2 ds\right\}. \tag{3.1}$$

Consider the following equation:

$$\begin{aligned} &du^\varepsilon(t) = \partial_1^2 u^\varepsilon(t)dt - B(u^\varepsilon(t))dt + \sqrt{\varepsilon}\sigma(t,u^\varepsilon(t))dW(t),\\ &u^\varepsilon(0) = u_0. \end{aligned} \tag{3.2}$$

By Lemma 2.2, under the assumptions (A0)-(A3), (3.2) has a unique strong solution $u^\varepsilon \in L^\infty([0,T],\tilde{H}^{0,1})\bigcap L^2([0,T],\tilde{H}^{1,1})\bigcap C([0,T],H^{-1})$ for $u_0 \in \tilde{H}^{0,1}$. It follows from Yamada-Watanabe theorem (See [LR15, Appendix E]) that there exists a Borel-measurable function

$$g^\varepsilon : C([0,T],U) \to L^\infty([0,T],H)\bigcap L^2([0,T],\tilde{H}^{1,0})\bigcap C([0,T],H^{-1})$$

such that $u^\varepsilon = g^\varepsilon(W)$ a.s..

Let us introduce the following skeleton equation associated to (3.2), for $\phi \in L^2([0,T],l^2)$:

$$\begin{aligned} &dz^\phi(t) = \partial_1^2 z^\phi(t)dt - B(z^\phi(t))dt + \sigma(t,z^\phi(t))\phi(t)dt,\\ &\text{div } z^\phi = 0,\\ &z^\phi(0) = u_0. \end{aligned} \tag{3.3}$$

An element $z^\phi \in L^\infty([0,T],H)\bigcap L^2([0,T],\tilde{H}^{1,0})\bigcap C([0,T],H^{-1})$ is called a (weak) solution to (3.3) if for any $\varphi \in (C_0^\infty([0,T]\times\mathbb{T}^2))^2$ with $\text{div}\varphi = 0$, and $t > 0$,

$$\langle z^\phi(t),\varphi(t)\rangle = \langle u_0,\varphi(0)\rangle + \int_0^t \langle z^\phi,\partial_t\varphi\rangle - \langle\partial_1 z^\phi,\partial_1\varphi\rangle + \langle -B(z^\phi) + \sigma(s,z^\phi)\phi,\varphi\rangle ds.$$

The existence of the weak solution to (3.3) can be obtained by the same method as in [LZZ18] (see Lemma 3.2 in the following).

Define $g^0 : C([0,T],U) \to L^\infty([0,T],H)\bigcap L^2([0,T],\tilde{H}^{1,0})\bigcap C([0,T],H^{-1})$ by

$$g^0(h) := \begin{cases} z^\phi, & \text{if } h = \int_0^\cdot \phi(s)ds \text{ for some } \phi \in L^2([0,T],l^2); \\ 0, & \text{otherwise.} \end{cases}$$

Then the rate function can be written as

$$I(z) = \inf\left\{\frac{1}{2}\int_0^T \|\phi(s)\|_{l^2}^2 ds : \ z = z^\phi, \ \phi \in L^2([0,T],l^2)\right\}, \tag{3.4}$$

where $z \in L^\infty([0,T],H)\bigcap L^2([0,T],\tilde{H}^{1,0})\bigcap C([0,T],H^{-1})$.

The main result of this section is the following one:

Theorem 3.1. *Assume (A0)-(A3) hold with $L_2 = 0$ and $u_0 \in \tilde{H}^{0,1}$, then u^ε satisfies a large deviation principle on $L^\infty([0,T],H)\bigcap L^2([0,T],\tilde{H}^{1,0})\bigcap C([0,T],H^{-1})$ with the good rate function I given by (3.4).*

The proof is divided into the following lemmas.

Lemma 3.2. *Assume (A0)-(A3) hold with $L_2 = 0$. For all $u_0 \in \tilde{H}^{0,1}$ and $\phi \in L^2([0,T],l^2)$ there exists a unique solution*

$$z^\phi \in L^\infty([0,T],\tilde{H}^{0,1})\bigcap L^2([0,T],\tilde{H}^{1,1})\bigcap C([0,T],H^{-1})$$

to (3.3).

Proof. First we give some a priori estimates for z^ϕ. By taking H inner product of (3.3) with z^ϕ and using div $z^\phi = 0$, we have

$$\begin{aligned}
&\|z^\phi(t)\|_H^2 + 2\int_0^t \|\partial_1 z^\phi(s)\|_H^2 ds \\
=&\|u_0\|_H^2 + 2\int_0^t \langle z^\phi(s), \sigma(s,z^\phi(s))\phi(s)\rangle ds \\
\leqslant&\|u_0\|_H^2 + 2\int_0^t \|z^\phi(s)\|_H \|\sigma(s,z^\phi(s))\|_{L_2(l^2,H)}\|\phi(s)\|_{l^2} ds \\
\leqslant&\|u_0\|_H^2 + 2\int_0^t \left(\|z^\phi(s)\|_H^2\|\phi(s)\|_{l^2}^2 + K_0 + K_1\|z^\phi(s)\|_H^2 + K_2\|\partial_1 z^\phi(s)\|_H^2\right) ds,
\end{aligned}$$

where we used (A1) in the last inequality.

Hence by Gronwall's inequality, we have

$$\|z^\phi(t)\|_H^2 + \int_0^t \|\partial_1 z^\phi(s)\|_H^2 ds \leqslant (\|u_0\|_H^2 + C)e^{C\int_0^t(\|\phi(s)\|_{l^2}^2+1)ds}. \tag{3.5}$$

Similarly, we have

$$
\begin{aligned}
&\|z^\phi(t)\|^2_{\tilde H^{0,1}}+2\int_0^t(\|\partial_1 z^\phi(s)\|^2_H+\|\partial_1\partial_2 z^\phi(s)\|^2_H)ds\\
=&\|u_0\|^2_{\tilde H^{0,1}}-2\int_0^t\langle\partial_2 z^\phi(s),\partial_2(z^\phi\cdot\nabla z^\phi)(s)\rangle ds+2\int_0^t\langle z^\phi(s),\sigma(s,z^\phi(s))\phi(s)\rangle_{\tilde H^{0,1}}ds\\
\leqslant&\|u_0\|^2_{\tilde H^{0,1}}+\int_0^t(\frac{1}{5}\|\partial_1\partial_2 z^\phi(s)\|^2_H+C(1+\|\partial_1 z^\phi(s)\|^2_H)\|\partial_2 z^\phi(s)\|^2_H)ds\\
&+2\int_0^t(\|z^\phi(s)\|^2_{\tilde H^{0,1}}\|\phi(s)\|^2_{l^2}+\|\sigma(s,z^\phi(s))\|^2_{L_2(l^2,\tilde H^{0,1})})ds,
\end{aligned}
$$

where we used Lemma A.5 in the last inequality.

Hence by (A2) we deduce that

$$
\begin{aligned}
&\|z^\phi(t)\|^2_{\tilde H^{0,1}}+\int_0^t\|z^\phi(s)\|^2_{\tilde H^{1,1}}ds\\
\leqslant&\|u_0\|^2_{\tilde H^{0,1}}+C+C\int_0^t(1+\|\partial_1 z^\phi(s)\|^2_H+\|\phi(s)\|^2_{l^2})\|z^\phi(s)\|^2_{\tilde H^{0,1}}ds.
\end{aligned}
$$

Then by Gronwall's inequality and (3.5) we have

$$
\|z^\phi(t)\|^2_{\tilde H^{0,1}}+\int_0^t\|z^\phi(s)\|^2_{\tilde H^{1,1}}ds\leqslant(\|u_0\|^2_{\tilde H^{0,1}}+C)e^{C(t,\phi,u_0)},\tag{3.6}
$$

where

$$
C(t,\phi,u_0)=C\left(\int_0^t(1+\|\phi(s)\|^2_{l^2})ds+(\|u_0\|^2_H+1)e^{C\int_0^t(1+\|\phi(s)\|^2_{l^2})ds}\right).
$$

Now consider the following approximate equation:

$$
\left\{
\begin{array}{l}
dz^\phi_\epsilon(t)=\partial_1^2 z^\phi_\epsilon(t)dt+\epsilon^2\partial_2^2 z^\phi_\epsilon(t)dt-B(z^\phi_\epsilon(t))dt+\sigma(t,z^\phi_\epsilon(t))\phi(t)dt,\\
\mathrm{div} z^\phi_\epsilon=0,\\
z^\phi_\epsilon(0)=u_0*j_\epsilon,
\end{array}
\right.\tag{3.7}
$$

where j is a smooth function on $\mathbb{R}^2$ with

$$
j(x)=1,\ |x|\leqslant 1;\ j(x)=0,\ |x|\geqslant 2,
$$

and

$$
j_\epsilon(x)=\frac{1}{\epsilon^2}j(\frac{x}{\epsilon}).
$$

It follows from classical theory on Navier-Stokes system that (3.7) has a unique global smooth solution z^ϕ_ϵ for any fixed ϵ. Furthermore, along the same line to (3.5) and (3.6) we have

$$
\begin{aligned}
&\|z^\phi_\epsilon(t)\|^2_H+\int_0^t\|\partial_1 z^\phi_\epsilon(s)\|^2_H ds+\epsilon^2\int_0^t\|\partial_2 z^\phi_\epsilon(s)\|^2_H ds\leqslant(\|u_0\|^2_H+C)e^{C\int_0^t(\|\phi(s)\|^2_{l^2}+1)ds},\\
&\|\partial_2 z^\phi_\epsilon(t)\|^2_H+\int_0^t\|\partial_1\partial_2 z^\phi_\varepsilon(s)\|^2_H ds+\epsilon^2\int_0^t\|\partial_2^2 z^\phi_\epsilon(s)\|^2_H ds\leqslant(\|u_0\|^2_{\tilde H^{0,1}}+C)e^{C(t,\phi,u_0)},
\end{aligned}\tag{3.8}
$$

The following follows a similar argument as in the proof of [LZZ18, Theorem 3.1]. By (3.8), we have $\{z_\epsilon^\phi\}_{\epsilon>0}$ is uniformly bounded in $L^\infty([0,T],\tilde{H}^{0,1})\bigcap L^2([0,T],\tilde{H}^{1,1})$, hence bounded in $L^4([0,T],H^{\frac{1}{2}})$ (by interpolation) and $L^4([0,T],L^4(\mathbb{T}^2))$ (by Sobolev embedding). Thus $B(z_\epsilon^\phi)$ is uniformly bounded in $L^2([0,T],H^{-1})$. Let $p\in(1,\frac{4}{3})$, we have

$$\begin{aligned}\int_0^T \|\sigma(s,z_\epsilon^\phi(s))\phi(s)\|_{H^{-1}}^p ds \leqslant &\int_0^T \|\sigma(s,z_\epsilon^\phi(s))\|_{L_2(l^2,H^{-1})}^p \|\phi(s)\|_{l^2}^p ds\\ \leqslant &C\int_0^T (1+\|\sigma(s,z_\epsilon^\phi(s))\|_{L_2(l^2,H^{-1})}^4 + \|\phi(s)\|_{l^2}^2) ds\\ \leqslant &C\int_0^T (1+\|z_\epsilon^\phi(s))\|_H^4 + \|\phi(s)\|_{l^2}^2) ds < \infty,\end{aligned}$$

where we used Young's inequality in the second line and (A0) in the third line. It comes out that

$$\{\partial_t z_\epsilon^\phi\}_{\epsilon>0} \text{ is uniformly bounded in } L^p([0,T],H^{-1}). \tag{3.9}$$

Thus by Aubin-Lions lemma (see [LZZ18, Lemma 3.6]), there exists a $z^\phi\in L^2([0,T],H)$ such that

$$z_\epsilon^\phi \to z^\phi \text{ strongly in } L^2([0,T],H) \text{ as } \epsilon\to 0 \text{ (in the sense of subsequence).}$$

Since $\{z_\epsilon^\phi\}_{\epsilon>0}$ is uniformly bounded in $L^\infty([0,T],\tilde{H}^{0,1})\bigcap L^2([0,T],\tilde{H}^{1,1})$, there exists a $\tilde{z}\in L^\infty([0,T],\tilde{H}^{0,1})\bigcap L^2([0,T],\tilde{H}^{1,1})$ such that

$$z_\epsilon^\phi \to \tilde{z} \text{ weakly in } L^2([0,T],\tilde{H}^{1,1}) \text{ as } \epsilon\to 0 \text{ (in the sense of subsequence).}$$

$$z_\epsilon^\phi \to \tilde{z} \text{ weakly star in } L^\infty([0,T],\tilde{H}^{0,1}) \text{ as } \epsilon\to 0 \text{ (in the sense of subsequence).}$$

By the uniqueness of weak convergence limit, we deduce that $z^\phi=\tilde{z}$. By (3.9) and [FG95, Theorem 2.2], we also have for any $\delta>0$

$$z_\epsilon^\phi \to z^\phi \text{ strongly in } C([0,T],H^{-1-\delta}) \text{ as } \epsilon\to 0 \text{ (in the sense of subsequence).}$$

Now we use the above convergence to prove that z^ϕ is a solution to (3.3). Note that for any $\varphi\in C^\infty([0,T]\times\mathbb{T}^2)$ with $\text{div}\varphi=0$, for any $t\in[0,T]$, z_ϵ^ϕ satisfies

$$\langle z_\epsilon^\phi(t),\varphi(t)\rangle = \langle u_0,\varphi(0)\rangle + \int_0^t \langle z_\epsilon^\phi,\partial_t\varphi\rangle - \langle \partial_1 z_\epsilon^\phi,\partial_1\varphi\rangle - \epsilon^2\langle \partial_2 z_\epsilon^\phi,\partial_2\varphi\rangle + \langle -B(z_\epsilon^\phi)+\sigma(s,z_\epsilon^\phi)\phi,\varphi\rangle ds. \tag{3.10}$$

By [Tem79, Chapter 3, Lemma 3.2] we have

$$\int_0^t \langle -B(z_\epsilon^\phi),\varphi\rangle ds \to \int_0^t \langle -B(z^\phi),\varphi\rangle ds \text{ as } \epsilon\to 0.$$

For the last term in the right hand side of (3.10), we have

$$
\begin{aligned}
&\int_0^t \langle \sigma(s, z_\epsilon^\phi)\phi - \sigma(s, z^\phi)\phi, \varphi\rangle ds \\
\leqslant &\int_0^t \|(\sigma(s, z_\epsilon^\phi) - \sigma(s, z^\phi))\phi\|_H \|\varphi\|_H ds \\
\leqslant &C\int_0^t \|\sigma(s, z_\epsilon^\phi) - \sigma(s, z^\phi)\|_{L_2(l^2,H)} \|\phi\|_{l^2} ds \\
\leqslant &C\left(\int_0^t \|z_\epsilon^\phi - z^\phi\|_H^2 ds\right)^{\frac{1}{2}} \left(\int_0^t \|\phi(s)\|_{l^2}^2 ds\right)^{\frac{1}{2}},
\end{aligned}
$$

where we used Hölder's inequality and (A3) with $L_2 = 0$ in the last inequality.

Thus let $\epsilon \to 0$ in (3.10), we have $z^\phi \in L^\infty([0,T], \tilde{H}^{0,1}) \bigcap L^2([0,T], \tilde{H}^{1,1})$ and

$$
\partial_t z^\phi = \partial_1^2 z^\phi - B(z^\phi) + \sigma(t, z^\phi(t))\phi.
$$

Since the right hand side belongs to $L^p([0,T], H^{-1})$, we deduce that

$$
z^\phi \in L^\infty([0,T], \tilde{H}^{0,1}) \bigcap L^2([0,T], \tilde{H}^{1,1}) \bigcap C([0,T], H^{-1}).
$$

For uniqueness, let $z_1^\phi, z_2^\phi \in L^\infty([0,T], \tilde{H}^{0,1}) \bigcap L^2([0,T], \tilde{H}^{1,1}) \bigcap C([0,T], H^{-1})$ be two solutions to (3.3) and $w^\phi = z_1^\phi - z_2^\phi$. Then we have

$$
\begin{aligned}
&\|w^\phi(t)\|_H^2 + 2\int_0^t \|\partial_1 w^\phi(s)\|_H^2 ds \\
=&\|w^\phi(0)\|_H^2 - 2\int_0^t \langle w^\phi(s), B(z_1^\phi)(s) - B(z_2^\phi)(s)\rangle ds \\
&+ 2\int_0^t \langle w^\phi(s), \sigma(s, z_1^\phi(s))\phi(s) - \sigma(s, z_2^\phi(s))\phi(s)\rangle ds \\
\leqslant&\|w^\phi(0)\|_H^2 - 2\int_0^t b(w^\phi(s), z_2^\phi(s), w^\phi(s)) ds \\
&+ 2\int_0^t \|w^\phi(s)\|_H \|\sigma(s, z_1^\phi(s)) - \sigma(s, z_2^\phi(s))\|_{L_2(l^2,H)} \|\phi(s)\|_{l^2} ds \\
\leqslant&\|w^\phi(0)\|_H^2 + \int_0^t \frac{1}{5}\|\partial_1 w^\phi(s)\|_H^2 ds + C\int_0^t (1 + \|z_2^\phi(s)\|_{\tilde{H}^{1,1}}^2)\|w^\phi(s)\|_H^2 ds \\
&+ \int_0^t (\|w^\phi(s)\|_H^2 \|\phi(s)\|_{l^2}^2 + L_1\|w^\phi(s)\|_H^2) ds,
\end{aligned}
$$

where we used Lemma A.3 in the sixth line and (A3) with $L_2 = 0$ in the last line.

Then by Gronwall's inequality we have

$$
\|w^\phi(t)\|_H^2 \leqslant \|w^\phi(0)\|_H^2 e^{C\int_0^t (1 + \|z_2^\phi(s)\|_{\tilde{H}^{1,1}}^2 + \|\phi(s)\|_{l^2}^2) ds},
$$

which along with the fact that $z_2^\phi \in L^2([0,T], \tilde{H}^{1,1})$ and $\phi \in L^2([0,T], l^2)$ implies that $w^\phi(t) = 0$. That is: $z_1^\phi = z_2^\phi$. $\square$

The following Lemma shows that I is a good rate function. The proof follows essentially the same argument as in [WZZ15, Proposition 4.5].

Lemma 3.3. *Assume (A0)-(A3) hold with $L_2 = 0$. For all $N < \infty$, the set*

$$K_N = \left\{ g^0 \left(\int_0^\cdot \phi(s)ds \right) : \phi \in S_N \right\}$$

is a compact subset in $L^\infty([0,T],H) \bigcap L^2([0,T],\tilde{H}^{1,0}) \bigcap C([0,T],H^{-1})$.

Proof. By definition, we have

$$K_N = \left\{ z^\phi : \phi \in L^2([0,T],l^2), \ \int_0^T \|\phi(s)\|_{l^2}^2 ds \leqslant N \right\}.$$

Let $\{z^{\phi_n}\}$ be a sequence in K_N where $\{\phi_n\} \subset S_N$. Note that (3.6) implies that z^{ϕ_n} is uniformly bounded in $L^\infty([0,T],H^{1,0}) \cap L^2([0,T],H^{1,1})$. Thus by weak compactness of S_N, a similar argument as in the proof of Lemma 3.2 shows that there exists $\phi \in \mathcal{S}_N$ and $z' \in L^2([0,T],H)$ such that the following convergence hold as $n \to \infty$ (in the sense of subsequence):
$\phi_n \to \phi$ in $\mathcal{S}_N$ weakly,
$z^{\phi_n} \to z'$ in $L^2([0,T],H^{1,0})$ weakly,
$z^{\phi_n} \to z'$ in $L^\infty([0,T],H)$ weak-star,
$z^{\phi_n} \to z'$ in $L^2([0,T],H)$ strongly.
$z^{\phi_n} \to z'$ in $C([0,T],H^{-1-\delta})$ strongly for any $\delta > 0$.
Then for any $\varphi \in C^\infty([0,T] \times \mathbb{T}^2)$ with $\mathrm{div}\varphi = 0$ and for any $t \in [0,T]$, z^{ϕ_n} satisfies

$$\langle z^{\phi_n}(t), \varphi(t)\rangle = \langle u_0, \varphi(0)\rangle + \int_0^t \langle z^{\phi_n}, \partial_t\varphi\rangle - \langle \partial_1 z^{\phi_n}, \partial_1\varphi\rangle + \langle -B(z^{\phi_n}) + \sigma(s, z^{\phi_n})\phi_n, \varphi\rangle ds. \quad (3.11)$$

Let $n \to \infty$, we have

$$\begin{aligned}
&\int_0^t \langle \sigma(s, z^{\phi_n})\phi_n - \sigma(s,z')\phi, \varphi\rangle ds \\
=&\int_0^t \langle [\sigma(s, z^{\phi_n}) - \sigma(s,z')]\phi_n + \sigma(s,z')(\phi_n - \phi), \varphi\rangle ds \\
\leqslant&\int_0^t \|(\sigma(s, z^{\phi_n}) - \sigma(s,z'))\phi_n\|_H \|\varphi\|_H ds + \int_0^t \langle \sigma(s,z')(\phi_n - \phi), \varphi\rangle ds \\
\leqslant& C\int_0^t \|\sigma(s, z^{\phi_n}) - \sigma(s,z')\|_{L_2(l^2,H)} \|\phi_n\|_{l^2} ds + \int_0^t \langle \sigma(s,z')(\phi_n - \phi), \varphi\rangle ds \\
\leqslant& C\left(\int_0^t \|z^{\phi_n} - z'\|_H^2 ds\right)^{\frac{1}{2}} \left(\int_0^t \|\phi_n(s)\|_{l^2}^2 ds\right)^{\frac{1}{2}} + \int_0^t \langle \sigma(s,z')(\phi_n - \phi), \varphi\rangle ds \\
\to& 0,
\end{aligned}$$

where we used Hölder's inequality and (A3) with $L_2 = 0$ in the last inequality. By [Tem79, Chapter 3, Lemma 3.2] we also have

$$\int_0^t \langle -B(z^{\phi_n}), \varphi\rangle ds \to \int_0^t \langle -B(z'), \varphi\rangle ds.$$

Then we deduce that

$$\langle z'(t),\varphi(t)\rangle=\langle u_0,\varphi(0)\rangle+\int_0^t\langle z',\partial_t\varphi\rangle-\langle\partial_1 z',\partial_1\varphi\rangle+\langle -B(z')+\sigma(s,z')\phi,\varphi\rangle ds,$$

which implies that z' is a solution to (3.3). By the uniqueness of solution, we deduce that $z'=z^{\phi}$.

Our goal is to prove $z^{\phi_n}\to z^{\phi}$ in $L^\infty([0,T],H)\bigcap L^2([0,T],\tilde{H}^{1,0})\bigcap C([0,T],H^{-1})$.

Let $w^n=z^{\phi_n}-z^{\phi}$, by a direct calculation, we have

$$\begin{aligned}
&\|w^n(t)\|_H^2+2\int_0^t\|\partial_1 w^n(s)\|_H^2 ds\\
=&-2\int_0^t\langle w^n(s),B(z^{\phi_n})(s)-B(z^{\phi})(s)\rangle ds\\
&+2\int_0^t\langle w^n(s),\sigma(s,z^{\phi_n}(s))\phi_n(s)-\sigma(s,z^{\phi}(s))\phi(s)\rangle ds\\
=&-2\int_0^t b(w^n,z^{\phi},w^n)(s)ds+2\int_0^t\langle w^n(s),(\sigma(s,z^{\phi_n}(s))-\sigma(s,z^{\phi}(s)))\phi_n(s)\rangle ds\\
&+2\int_0^t\langle w^n(s),\sigma(s,z^{\phi}(s))(\phi_n(s)-\phi(s))\rangle ds\\
\leqslant&\int_0^t\frac{1}{5}\|\partial_1 w^n(s)\|_H^2 ds+C\int_0^t(1+\|z^{\phi}(s)\|_{\tilde{H}^{1,1}}^2)\|w^n(s)\|_H^2 ds\\
&+C\int_0^t\|w^n(s)\|_H^2\|\phi_n(s)\|_{l^2}ds\\
&+\int_0^t\|w^n(s)\|_H\|\phi_n(s)-\phi(s)\|_{l^2}(K_0+K_1\|z^{\phi}(s)\|_H^2+K_2\|\partial_1 z^{\phi}(s)\|_H^2)^{\frac{1}{2}}ds,
\end{aligned}$$

where we used Lemma A.3 in the sixth line, (A3) with $L_2=0$ in the seventh line and (A1) in the last line. Then we have

$$\begin{aligned}
&\sup_{t\in[0,T]}\|w^n(t)\|_H^2+\int_0^T\|\partial_1 w^n(s)\|_H^2 ds\\
\leqslant&C\int_0^T(1+\|z^{\phi}(s)\|_{\tilde{H}^{1,1}}^2)\|w^n(s)\|_H^2 ds\\
&+C(\sup_{t\in[0,T]}\|z^{\phi_n}(t)\|_H+\sup_{t\in[0,T]}\|z^{\phi}(t)\|_H)\left(\int_0^T\|\phi_n(s)\|_{l^2}^2 ds\right)^{\frac{1}{2}}\left(\int_0^T\|w^n(s)\|_H^2 ds\right)^{\frac{1}{2}}\\
&+C\left(\int_0^T\|\phi_n(s)-\phi(s)\|_{l^2}^2 ds\right)^{\frac{1}{2}}\left(\int_0^T(1+\|z^{\phi}(s)\|_H^2+\|\partial_1 z^{\phi}(s)\|_H^2)\|w^n(s)\|_H^2 ds\right)^{\frac{1}{2}}\\
\leqslant&C\int_0^T(1+\|z^{\phi}(s)\|_{\tilde{H}^{1,1}}^2)\|w^n(s)\|_H^2 ds+C(N)\left(\int_0^T\|w^n(s)\|_H^2 ds\right)^{\frac{1}{2}}\\
&+CN^{\frac{1}{2}}\left(\int_0^T(1+\|z^{\phi}(s)\|_H^2+\|\partial_1 z^{\phi}(s)\|_H^2)\|w^n(s)\|_H^2 ds\right)^{\frac{1}{2}},
\end{aligned}$$

where we used (3.5) and the fact that ϕ_n, ϕ are in $\mathcal{S}_N$.

For any $\epsilon > 0$, let

$$A_\epsilon := \{s \in [0,T]; \|z^{\phi_n}(s) - z^\phi(s)\|_H > \epsilon\}.$$

Since $z^{\phi_n} \to z^\phi$ in $L^2([0,T],H)$ strongly, we have

$$\int_0^T \|w^n(s)\|_H^2 ds \to 0, \text{ as } n \to \infty$$

and $\lim_{n\to\infty} Leb(A_\epsilon) = 0$, where $Leb(B)$ means the Lebesgue measure of $B \in \mathcal{B}(\mathbb{R})$. Thus we have

$$\begin{aligned}
&\int_0^T (1 + \|z^\phi(s)\|_{\tilde{H}^{1,1}}^2)\|w^n(s)\|_H^2 ds \\
\leqslant &\left(\int_{A_\epsilon} + \int_{[0,T]\setminus A_\epsilon}\right)(1 + \|z^\phi(s)\|_{\tilde{H}^{1,1}}^2)\|w^n(s)\|_H^2 ds \\
\leqslant &C\epsilon + 2\int_{A_\epsilon}(1 + \|z^\phi(s)\|_{\tilde{H}^{1,1}}^2)(\|z^{\phi_n}(s)\|_H^2 + \|z^\phi(s)\|_H^2) ds \\
\leqslant &C\epsilon + C\int_{A_\epsilon}(1 + \|z^\phi(s)\|_{\tilde{H}^{1,1}}^2) ds \\
\to &\ C\epsilon \text{ as } n \to \infty,
\end{aligned}$$

where we used (3.5) in the forth line and (3.6) in the last line. A similar argument also implies that

$$\int_0^T (1 + \|z^\phi(s)\|_H^2 + \|\partial_1 z^\phi(s)\|_H^2)\|w^n(s)\|_H^2 ds \leqslant C\epsilon.$$

Hence we have

$$\sup_{t\in[0,T]} \|w^n(t)\|_H^2 + \int_0^T \|\partial_1 w^n(s)\|_H^2 ds \leqslant C\epsilon + C\sqrt{\epsilon} \text{ as } n \to \infty.$$

Since ϵ is arbitrary, we obtain that

$$z^{\phi^n} \to z^\phi \text{ strongly in } L^\infty([0,T],H)\bigcap L^2([0,T],\tilde{H}^{1,0})\bigcap C([0,T],H^{-1}).$$

□

For next step, consider the following equation:

$$\begin{aligned}
&dZ_v^\varepsilon(t) = \partial_1^2 Z_v^\varepsilon(t)dt - B(Z_v^\varepsilon(t))dt + \sigma(t, Z_v^\varepsilon(t))v^\varepsilon(t)dt + \sqrt{\varepsilon}\sigma(t, Z_v^\varepsilon(t))dW(t), \\
&\mathrm{div} Z_v^\varepsilon = 0, \\
&Z_v^\varepsilon(0) = u_0,
\end{aligned} \tag{3.12}$$

where $v^\varepsilon \in \mathcal{A}_N$ for some $N < \infty$. Here Z_v^ε should have been denoted $Z_{v^\varepsilon}^\varepsilon$ and the slight abuse of notation is for simplicity.

Lemma 3.4. *Assume (A0)-(A3) hold with $L_2 = 0$ and $v^\varepsilon \in \mathcal{A}_N$ for some $N < \infty$. Then $Z_v^\varepsilon = g^\varepsilon\left(W(\cdot) + \frac{1}{\sqrt{\varepsilon}}\int_0^\cdot v^\varepsilon(s)ds\right)$ is the unique strong solution to (3.12).*

Proof. Since $v^\varepsilon \in \mathcal{A}_N$, by the Girsanov theorem (see [LR15, Appendix I]), $\tilde{W}(\cdot) := W(\cdot) + \frac{1}{\sqrt{\varepsilon}}\int_0^\cdot v^\varepsilon(s)ds$ is an l^2-cylindrical Wiener-process under the probability measure

$$d\tilde{P} := \exp\left\{-\frac{1}{\sqrt{\varepsilon}}\int_0^T v^\varepsilon(s)dW(s) - \frac{1}{2\varepsilon}\int_0^T \|v^\varepsilon(s)\|_{l^2}^2 ds\right\} dP.$$

Then $(Z_v^\varepsilon, \tilde{W})$ is the solution to (3.2) on the stochastic basis $(\Omega, \mathcal{F}, \tilde{P})$. By (A0) we have

$$\int_0^T \|\sigma(s, Z_v^\varepsilon(s))\|_{H^{-1}} ds < \infty.$$

Then (Z_v^ε, W) satisfies the condition of the definition of weak solution (see [LZZ18, Definition 4.1]) and hence is a weak solution to (3.12) on the stochastic basis $(\Omega, \mathcal{F}, P)$ and $Z_v^\varepsilon = g^\varepsilon\left(W(\cdot) + \frac{1}{\sqrt{\varepsilon}}\int_0^\cdot v^\varepsilon(s)ds\right)$.

If $\tilde{Z}_v^\varepsilon$ and Z_v^ε are two weak solutions to (3.12) on the same stochastic basis $(\Omega, \mathcal{F}, P)$. Let $W^\varepsilon = Z_v^\varepsilon - \tilde{Z}_v^\varepsilon$ and $q(t) = k\int_0^t (\|Z_v^\varepsilon(s)\|_{\tilde{H}^{1,1}}^2 + \|v^\varepsilon(s)\|_{l^2}^2)ds$ for some constant k. Applying Itô's formula to $e^{-q(t)}\|W^\varepsilon(t)\|_H^2$, we have

$$\begin{aligned}
&e^{-q(t)}\|W^\varepsilon(t)\|_H^2 + 2\int_0^t e^{-q(s)}\|\partial_1 W^\varepsilon(s)\|_H^2 ds\\
=&-k\int_0^t e^{-q(s)}\|W^\varepsilon(s)\|_H^2(\|Z_v^\varepsilon(s)\|_{\tilde{H}^{1,1}}^2 + \|v^\varepsilon(s)\|_{l^2}^2)ds - 2\int_0^t e^{-q(s)} b(W^\varepsilon, Z_v^\varepsilon, W^\varepsilon)ds\\
&+ 2\int_0^t e^{-q(s)}\langle\sigma(s, Z_v^\varepsilon)v^\varepsilon - \sigma(s, \tilde{Z}_v^\varepsilon)v^\varepsilon, W^\varepsilon(s)\rangle ds\\
&+ 2\sqrt{\varepsilon}\int_0^t e^{-q(s)}\langle W^\varepsilon(s), (\sigma(s, Z_v^\varepsilon) - \sigma(s, \tilde{Z}_v^\varepsilon))dW(s)\rangle\\
&+ \varepsilon\int_0^t e^{-q(s)}\|\sigma(s, Z_v^\varepsilon) - \sigma(s, \tilde{Z}_v^\varepsilon)\|_{L_2(l^2,H)}^2 ds.
\end{aligned}$$

By Lemma A.3, there exists constants $\tilde{\alpha} \in (0,1)$ and $\tilde{C}$ such that

$$|b(W^\varepsilon, Z_v^\varepsilon, W^\varepsilon)| \leqslant \tilde{\alpha}\|\partial_1 W^\varepsilon\|_H^2 + \tilde{C}(1 + \|Z_v^\varepsilon\|_{\tilde{H}^{1,1}}^2)\|W^\varepsilon\|_H^2.$$

We also have

$$\begin{aligned}
2|\langle\sigma(s, Z_v^\varepsilon)v^\varepsilon - \sigma(s, \tilde{Z}_v^\varepsilon)v^\varepsilon, W^\varepsilon\rangle| &\leqslant 2\|(\sigma(s, Z_v^\varepsilon) - \sigma(s, \tilde{Z}_v^\varepsilon))v^\varepsilon\|_H\|W^\varepsilon\|_H\\
&\leqslant \|\sigma(s, Z_v^\varepsilon) - \sigma(s, \tilde{Z}_v^\varepsilon)\|_{L_2(l^2,H)}^2 + \|v^\varepsilon\|_{l^2}^2\|W^\varepsilon\|_H^2.
\end{aligned}$$

Let $k > 2\tilde{C}$ and we may assume $\varepsilon < \frac{16}{25}$, by (A3) with $L_2 = 0$ we have

$$\begin{aligned}
&e^{-q(t)}\|W^\varepsilon(t)\|_H^2 + (2 - 2\tilde{\alpha})\int_0^t e^{-q(s)}\|\partial_1 W^\varepsilon(s)\|_H^2 ds\\
\leqslant& C\int_0^t e^{-q(s)}\|W^\varepsilon(s)\|_H^2 ds + 2\sqrt{\varepsilon}\int_0^t e^{-q(s)}\langle W^\varepsilon(s), (\sigma(s, Z_v^\varepsilon) - \sigma(s, \tilde{Z}_v^\varepsilon))dW(s)\rangle.
\end{aligned}$$

By the Burkhölder-Davis-Gundy's inequality (see [LR15, Appendix D]), we have

$$\begin{aligned}
&2\sqrt{\varepsilon}|E[\sup_{r\in[0,t]}\int_0^r e^{-q(s)}\langle W^\varepsilon(s),(\sigma(s,Z_v^\varepsilon)-\sigma(s,\tilde{Z}_v^\varepsilon))dW(s)\rangle]|\\
\leqslant&6\sqrt{\varepsilon}E\left(\int_0^t e^{-2q(s)}\|\sigma(s,Z_v^\varepsilon)-\sigma(s,\tilde{Z}_v^\varepsilon)\|^2_{L_2(l^2,H)}\|W^\varepsilon(s)\|_H^2ds\right)^{\frac{1}{2}}\\
\leqslant&\sqrt{\varepsilon}E(\sup_{s\in[0,t]}(e^{-q(s)}\|W^\varepsilon(s)\|_H^2))+9\sqrt{\varepsilon}E\int_0^t e^{-q(s)}L_1\|W^\varepsilon(s)\|_H^2ds,
\end{aligned}$$

where we used (A3) with $L_2=0$ and assume that $\tilde{\alpha}<1$.

Thus we have

$$E(\sup_{s\in[0,t]}(e^{-q(s)}\|W^\varepsilon(s)\|_H^2))\leqslant CE\int_0^t e^{-q(s)}\|W^\varepsilon(s)\|_H^2ds.$$

By the Gronwall's inequality we obtain $W^\varepsilon=0$ P-a.s., i.e. $\tilde{Z}_v^\varepsilon=Z_v^\varepsilon$ P-a.s..

Then by the Yamada-Watanabe theorem, we have Z_v^ε is the unique strong solution to (3.12). □

Lemma 3.5. *Assume Z_v^ε is a solution to (3.12) with $v^\varepsilon\in\mathcal{A}_N$ and $\varepsilon<1$ small enough. Then we have*

$$E(\sup_{t\in[0,T]}\|Z_v^\varepsilon(t)\|_H^4)+E\int_0^T\|Z_v^\varepsilon(s)\|_H^2\|Z_v^\varepsilon(s)\|^2_{\tilde{H}^{1,0}}ds+E\int_0^T\|\partial_1 Z_v^\varepsilon(s)\|_H^2ds\leqslant C(N,u_0). \quad (3.13)$$

Moreover, there exists $k>0$ such that

$$E(\sup_{t\in[0,T]}e^{-kg(t)}\|Z_v^\varepsilon(t)\|^2_{\tilde{H}^{0,1}})+E\int_0^T e^{-kg(s)}\|Z_v^\varepsilon(s)\|^2_{\tilde{H}^{1,1}}ds\leqslant C(N,u_0), \quad (3.14)$$

where $g(t)=\int_0^t\|Z_v^\varepsilon(s)\|_H^2ds$ and $C(N,u_0)$ is a constant depend on N,u_0 but independent of ε.

Proof. We prove (3.13) by two parts of estimates. For first step, applying Itô's formula to $\|Z_v^\varepsilon(t)\|_H^2$, we have

$$\begin{aligned}
&\|Z_v^\varepsilon(t)\|_H^2+2\int_0^t\|\partial_1 Z_v^\varepsilon(s)\|_H^2ds\\
=&\|u_0\|_H^2+2\int_0^t\langle Z_v^\varepsilon(s),\sigma(s,Z_v^\varepsilon(s))v^\varepsilon(s)\rangle ds\\
&+2\sqrt{\varepsilon}\int_0^t\langle Z_v^\varepsilon(s),\sigma(s,Z_v^\varepsilon(s))dW(s)\rangle+\varepsilon\int_0^t\|\sigma(s,Z_v^\varepsilon(s))\|^2_{L_2(l^2,H)}ds\\
\leqslant&\|u_0\|_H^2+\int_0^t(\|Z^\varepsilon(s)\|_H^2\|v^\varepsilon(s)\|^2_{l^2}+\|\sigma(s,Z_v^\varepsilon(s))\|^2_{L_2(l^2,H)})ds\\
&+2\sqrt{\varepsilon}\int_0^t\langle Z_v^\varepsilon(s),\sigma(s,Z_v^\varepsilon(s))dW(s)\rangle+\varepsilon\int_0^t\|\sigma(s,Z_v^\varepsilon(s))\|^2_{L_2(l^2,H)}ds\\
\leqslant&\|u_0\|_H^2+\int_0^t\|Z_v^\varepsilon(s)\|_H^2\|v^\varepsilon(s)\|^2_{l^2}ds+(1+\varepsilon)\int_0^t(K_0+K_1\|Z_v^\varepsilon\|_H^2+K_2\|\partial_1 Z_v^\varepsilon\|_H^2)ds\\
&+2\sqrt{\varepsilon}\int_0^t\langle Z_v^\varepsilon(s),\sigma(s,Z_v^\varepsilon(s))dW(s)\rangle,
\end{aligned}$$

where we used (A1) in the last inequality.

By Gronwall's inequality and $v^\varepsilon \in \mathcal{A}_N$,

$$\begin{aligned}&\|Z_v^\varepsilon(t)\|_H^2 + (2-(1+\varepsilon)K_2)\int_0^t \|\partial_1 Z_v^\varepsilon(s)\|_H^2 ds\\ \leqslant&(\|u_0\|_H^2 + C + 2\sqrt{\varepsilon}\int_0^t \langle Z_v^\varepsilon(s), \sigma(s, Z_v^\varepsilon(s))dW(s)\rangle)e^{N+2K_1T}.\end{aligned}$$

For the term in the right hand side, by the Burkhölder-Davis-Gundy inequality we have

$$\begin{aligned}&2\sqrt{\varepsilon}e^{N+K_1T}E\left(\sup_{0\leqslant s\leqslant t}|\int_0^s \langle Z_v^\varepsilon(r), \sigma(r, Z_v^\varepsilon(r))dW(r)\rangle|\right)\\ \leqslant&6\sqrt{\varepsilon}e^{N+K_1T}E\left(\int_0^t \|Z_v^\varepsilon(r)\|_H^2\|\sigma(r, Z_v^\varepsilon(r))\|^2_{L_2(l^2,H)}ds\right)^{\frac{1}{2}}\\ \leqslant&\sqrt{\varepsilon}E[\sup_{0\leqslant s\leqslant t}(\|Z_v^\varepsilon(s)\|_H^2)] + 9\sqrt{\varepsilon}e^{2N+2K_1T}E\int_0^t [K_0 + K_1\|Z_v^\varepsilon(s)\|_H^2 + K_2\|\partial_1 Z_v^\varepsilon(s)\|_H^2]ds,\end{aligned}$$

where $(9\sqrt{\varepsilon}e^{2N+2K_1T} + 1 + \varepsilon)K_2 - 2 < 0$ (this can be done when $\varepsilon < (\frac{10}{9e^{2N+2K_1T}+1})^2$) and we used (A1) in the last inequality. Thus we have

$$\begin{aligned}&E[\sup_{s\in[0,t]}(\|Z_v^\varepsilon(t)\|_H^2)] + E\int_0^t \|\partial_1 Z_v^\varepsilon(s)\|_H^2 ds\\ \leqslant&C(\|u_0\|_H^2 + 1) + C\int_0^t E[\sup_{r\in[0,s]}(\|Z_v^\varepsilon(r)\|_H^2)]ds.\end{aligned}$$

Then by Gronwall's inequality we have

$$E(\sup_{0\leqslant t\leqslant T}\|Z_v^\varepsilon(t)\|_H^2) + E\int_0^T \|\partial_1 Z_v^\varepsilon(s)\|_H^2 ds \leqslant C(1+\|u_0\|_H^2). \tag{3.15}$$

The second step is similar to [LZZ18, Lemma 4.2]. By Itô's formula we have

$$\begin{aligned}\|Z_v^\varepsilon(t)\|_H^4 =&\|u_0\|_H^4 - 4\int_0^t \|Z_v^\varepsilon\|_H^2\|\partial_1 Z_v^\varepsilon(s)\|_H^2 ds\\ &+ 4\int_0^t \|Z_v^\varepsilon(s)\|_H^2\langle\sigma(s, Z_v^\varepsilon(s))v^\varepsilon(s), Z_v^\varepsilon(s)\rangle ds\\ &+ 2\varepsilon\int_0^t \|Z_v^\varepsilon(s)\|_H^2\|\sigma(s, Z_v^\varepsilon(s))\|^2_{L_2(l^2,H)}ds\\ &+ 4\varepsilon\int_0^t \|\sigma(s, Z_v^\varepsilon(s))^*(Z_v^\varepsilon)\|_{l^2}^2 ds\\ &+ 4\sqrt{\varepsilon}\int_0^t \|Z_v^\varepsilon(s)\|_H^2\langle Z_v^\varepsilon(s), \sigma(s, Z_v^\varepsilon(s))dW(s)\rangle_H\\ =:&\|u_0\|_H^4 - 4\int_0^t \|Z_v^\varepsilon\|_H^2\|\partial_1 Z_v^\varepsilon(s)\|_H^2 ds + I_1 + I_2 + I_3 + I_4.\end{aligned} \tag{3.16}$$

By (A1) we have

$$
\begin{aligned}
I_1(t) \leqslant & 4\int_0^t \|Z_v^\varepsilon(s)\|_H^2 \|\sigma(s,Z_v^\varepsilon(s))\|_{L_2(l^2,H)} \|v^\varepsilon(s)\|_{l^2} \|Z_v^\varepsilon(s)\|_H ds \\
\leqslant & 2\int_0^t \|Z_v^\varepsilon(s)\|_H^2 (K_0 + K_1\|Z_v^\varepsilon(s)\|_H^2 + K_2\|\partial_1 Z_v^\varepsilon(s)\|_H^2 + \|v^\varepsilon(s)\|_{l^2}^2 \|Z_v^\varepsilon(s)\|_H^2) ds,
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 + I_3 \leqslant & 6\varepsilon \int_0^t \|\sigma(s,Z_v^\varepsilon(s))\|_{L_2(l^2,H)}^2 \|Z_v^\varepsilon(s)\|_H^2 ds \\
\leqslant & 6\varepsilon \int_0^t (K_0 + K_1\|Z_v^\varepsilon(s)\|_H^2 + K_2\|\partial_1 Z_v^\varepsilon(s)\|_H^2) \|Z_v^\varepsilon(s)\|_H^2 ds.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
& \|Z_v^\varepsilon(t)\|_H^4 + (4 - 2K_2 - 6\varepsilon K_2) \int_0^t \|Z_v^\varepsilon(s)\|_H^2 \|\partial_1 Z_v^\varepsilon(s)\|_H^2 ds \\
\leqslant & \|u_0\|_H^4 + I_4 + (2+6\varepsilon)K_0 \int_0^t \|Z_v^\varepsilon(s)\|_H^2 ds + \int_0^t (2K_1 + 6\varepsilon K_1 + 2\|v^\varepsilon(s)\|_{l^2}^2) \|Z_v^\varepsilon(s)\|_H^4) ds.
\end{aligned}
$$

Since $v^\varepsilon \in \mathcal{A}_N$, by Gronwall's inequality we have

$$
\begin{aligned}
& \|Z_v^\varepsilon(t)\|_H^4 + (4 - 2K_2 - 6\varepsilon K_2) \int_0^t \|Z_v^\varepsilon(s)\|_H^2 \|\partial_1 Z_v^\varepsilon(s)\|_H^2 ds \\
\leqslant & \left(\|u_0\|_H^4 + I_4 + (2+6\varepsilon)K_0 \int_0^t \|Z_v^\varepsilon(s)\|_H^2 ds \right) e^{8K_1T+N}.
\end{aligned}
$$

The Burkhölder-Davis-Gundy inequality, the Young's inequality and (A1) imply that

$$
\begin{aligned}
E(\sup_{s\in[0,t]} I_4(s)) \leqslant & 12\sqrt{\varepsilon} E\left(\int_0^t \|\sigma(s,Z_v^\varepsilon(s))\|_{L_2(l^2,H)}^2 \|Z_v^\varepsilon(s)\|_H^6 ds \right)^{\frac{1}{2}} \\
\leqslant & \sqrt{\varepsilon} E(\sup_{s\in[0,t]} \|Z_v^\varepsilon(s)\|_H^4) \\
& + 36\sqrt{\varepsilon} E\int_0^t (K_0 + K_1\|Z_v^\varepsilon(s)\|_H^2 + K_2\|\partial_1 Z_v^\varepsilon(s)\|_H^2) \|Z_v^\varepsilon(s)\|_H^2 ds.
\end{aligned}
$$

Let ε small enough such that $2K_2 + 6\varepsilon K_2 + 36\sqrt{\varepsilon} K_2 e^{8K_1T+N} < 4$ and $\sqrt{\varepsilon} e^{8K_1T+N} < 1$ (for instance $\varepsilon < (\frac{10}{3+18e^{8K_1T+N}})^2$). Then the above estimates and (3.13) imply that

$$
\begin{aligned}
& E(\sup_{s\in[0,t]} \|Z_v^\varepsilon(s)\|_H^4) + \int_0^t \|Z_v^\varepsilon(s)\|_H^2 \|Z_v^\varepsilon(s)\|_{\tilde{H}^{1,0}}^2 ds \\
\leqslant & C(N,u_0) + CE\int_0^t \|Z_v^\varepsilon(s)\|_H^4 ds,
\end{aligned}
$$

which by Gronwall's inequality yields that

$$
E(\sup_{s\in[0,t]} \|Z_v^\varepsilon(s)\|_H^4) + \int_0^t \|Z_v^\varepsilon(s)\|_H^2 \|Z_v^\varepsilon(s)\|_{\tilde{H}^{1,0}}^2 ds \leqslant C(N,u_0).
$$

For (3.14), let $h(t) = kg(t) + \int_0^t \|v^\varepsilon(s)\|_{l^2}^2 ds$ for some universal constant k. Applying Itô's formula to $e^{-h(t)}\|Z_v^\varepsilon(t)\|_{\tilde{H}^{0,1}}^2$, we have

$$
\begin{aligned}
&e^{-h(t)}\|Z_v^\varepsilon(t)\|_{\tilde{H}^{0,1}}^2 + 2\int_0^t e^{-h(s)}(\|\partial_1 Z_v^\varepsilon(s)\|_H^2 + \|\partial_1\partial_2 Z_v^\varepsilon(s)\|_H^2)ds \\
=&\|u_0\|_{\tilde{H}^{0,1}}^2 - \int_0^t e^{-h(s)}(k\|\partial_1 Z_v^\varepsilon(s)\|_H^2 + \|v^\varepsilon(s)\|_{l^2}^2)\|Z_v^\varepsilon(s)\|_{\tilde{H}^{0,1}}^2 ds \\
&+ 2\int_0^t e^{-h(s)}\langle \partial_2 Z_v^\varepsilon(s), \partial_2(Z_v^\varepsilon \cdot \nabla Z_v^\varepsilon)(s)\rangle ds + 2\int_0^t e^{-h(s)}\langle Z_v^\varepsilon(s), \sigma(s, Z_v^\varepsilon(s))v^\varepsilon(s)\rangle_{\tilde{H}^{0,1}} ds \\
&+ 2\sqrt{\varepsilon}\int_0^t e^{-h(s)}\langle Z_v^\varepsilon(s), \sigma(s, Z_v^\varepsilon(s))dW(t)\rangle_{\tilde{H}^{0,1}} + \varepsilon\int_0^t e^{-h(s)}\|\sigma(s, Z_v^\varepsilon(s))\|_{L_2(l^2,\tilde{H}^{0,1})}^2 ds.
\end{aligned}
$$

By Lemma A.5, there exists a constant C_1 such that

$$
|\langle \partial_2 Z_v^\varepsilon, \partial_2(Z_v^\varepsilon \cdot \nabla Z_v^\varepsilon)\rangle| \leqslant \frac{1}{2}\|\partial_1\partial_2 Z_v^\varepsilon\|_H^2 + C_1(1 + \|\partial_1 Z_v^\varepsilon\|_H^2)\|\partial_2 Z_v^\varepsilon\|_H^2.
$$

By Young's inequality,

$$
2|\langle Z_v^\varepsilon(s), \sigma(s, Z_v^\varepsilon(s))v^\varepsilon(s)\rangle_{\tilde{H}^{0,1}}| \leqslant \|Z_v^\varepsilon\|_{\tilde{H}^{0,1}}^2\|v^\varepsilon\|_{l^2}^2 + \|\sigma(s, Z_v^\varepsilon)\|_{L_2(l^2,\tilde{H}^{0,1})}^2.
$$

Choosing $k > 2C_1$, we have

$$
\begin{aligned}
&e^{-h(t)}\|Z_v^\varepsilon(t)\|_{\tilde{H}^{0,1}}^2 + \int_0^t e^{-h(s)}(\|\partial_1 Z_v^\varepsilon(s)\|_H^2 + \|\partial_1\partial_2 Z_v^\varepsilon(s)\|_H^2)ds \\
\leqslant&\|u_0\|_{\tilde{H}^{0,1}}^2 + C\int_0^t e^{-h(s)}\|\partial_2 Z_v^\varepsilon(s)\|_H^2 ds + (1+\varepsilon)\int_0^t e^{-h(s)}\|\sigma(s, Z_v^\varepsilon(s))\|_{L_2(l^2,\tilde{H}^{0,1})}^2 ds \\
&+ 2\sqrt{\varepsilon}\int_0^t e^{-h(s)}\langle Z_v^\varepsilon(s), \sigma(s, Z_v^\varepsilon(s))dW(t)\rangle_{\tilde{H}^{0,1}}.
\end{aligned}
$$

By the Burkhölder-Davis-Gundy inequality we have

$$
\begin{aligned}
&2\sqrt{\varepsilon}E\left(\sup_{s\in[0,t]}|\int_0^s e^{-h(r)}\langle Z_v^\varepsilon(r), \sigma(r, Z_v^\varepsilon(r))dW(r)\rangle_{\tilde{H}^{0,1}}|\right) \\
\leqslant&6\sqrt{\varepsilon}E\left(\int_0^t e^{-2h(s)}\|Z_v^\varepsilon(r)\|_{\tilde{H}^{0,1}}^2\|\sigma(r, Z_v^\varepsilon(r))\|_{L_2(l^2,\tilde{H}^{0,1})}^2 ds\right)^{\frac{1}{2}} \\
\leqslant&\sqrt{\varepsilon}E[\sup_{s\in[0,t]}(e^{-h(s)}\|Z_v^\varepsilon(s)\|_{\tilde{H}^{0,1}}^2)] \\
&+ 9\sqrt{\varepsilon}E\int_0^t e^{-h(s)}[\tilde{K_0} + \tilde{K_1}\|Z_v^\varepsilon(s)\|_{\tilde{H}^{0,1}}^2 + \tilde{K_2}(\|\partial_1 Z_v^\varepsilon(s)\|_H^2 + \|\partial_1\partial_2 Z_v^\varepsilon(s)\|_H^2)]ds,
\end{aligned}
$$

where $(9\sqrt{\varepsilon} + 1 + \varepsilon)\tilde{K_2} - 1 < 0$ (this can be done if $\varepsilon < \frac{9}{400}$) and we used (A2) in the last inequality.

Combine the above estimates, we have

$$
\begin{aligned}
&E(\sup_{s\in[0,t]} e^{-h(s)}\|Z_v^\varepsilon(s)\|^2_{\tilde{H}^{0,1}}) + E\int_0^t e^{-h(s)}\|Z_v^\varepsilon(s)\|^2_{\tilde{H}^{1,1}}ds\\
\leqslant& C(\|u_0\|^2_{\tilde{H}^{0,1}} + 1 + E\int_0^t e^{-h(s)}\|Z_v^\varepsilon(s)\|^2_{\tilde{H}^{0,1}}ds)
\end{aligned}
$$

Then Gronwall's inequality implies that

$$
E(\sup_{0\leqslant t\leqslant T} e^{-h(t)}\|Z_v^\varepsilon(t)\|^2_{\tilde{H}^{0,1}}) + E\int_0^T e^{-h(s)}\|Z_v^\varepsilon(s)\|^2_{\tilde{H}^{1,1}}ds \leqslant C(1+\|u_0\|^2_{\tilde{H}^{0,1}}).
$$

Since $v^\varepsilon \in \mathcal{S}_N$, we deduce that

$$
E(\sup_{t\in[0,T]} e^{-kg(t)}\|Z_v^\varepsilon(t)\|^2_{\tilde{H}^{0,1}}) + E\int_0^T e^{-kg(s)}\|Z_v^\varepsilon(s)\|^2_{\tilde{H}^{1,1}}ds \leqslant C(1+\|u_0\|^2_{\tilde{H}^{0,1}})e^N. \tag{3.17}
$$

$\square$

Similar as [LZZ18, lemma 4.3], we have the following tightness lemma:

Lemma 3.6. *Assume Z_v^ε is a solution to (3.12) with $v^\varepsilon \in \mathcal{A}_N$ and $\varepsilon < 1$ small enough. There exists $\varepsilon_0 > 0$, such that $\{Z_v^\varepsilon\}_{\varepsilon\in(0,\varepsilon_0)}$ is tight in the space*

$$
\chi = C([0,T],H^{-1})\bigcap L^2([0,T],H)\bigcap L^2_w([0,T],H^{1,1})\bigcap L^\infty_{w^*}([0,T],H^{0,1}),
$$

where L^2_w denotes the weak topology and $L^\infty_{w^*}$ denotes the weak star topology.

Proof. Let k be the same constant as in the proof of (3.14) and let

$$
\begin{aligned}
K_R :=& \Big\{u\in C([0,T],H^{-1}) : \sup_{t\in[0,T]}\|u(t)\|^2_H + \int_0^T \|u(t)\|^2_{\tilde{H}^{1,0}}dt + \|u\|_{C^{\frac{1}{16}}([0,T],H^{-1})}\\
&+ \sup_{t\in[0,T]} e^{-k\int_0^t \|\partial_1 u(s)\|^2_H ds}\|u(t)\|^2_{\tilde{H}^{0,1}} + \int_0^T e^{-k\int_0^t \|\partial_1 u(s)\|^2_H ds}\|u(t)\|^2_{\tilde{H}^{1,1}}dt \leqslant R\Big\},
\end{aligned}
$$

where $C^{\frac{1}{16}}([0,T],H^{-1})$ is the Hölder space with the norm:

$$
\|f\|_{C^{\frac{1}{16}}([0,T],H^{-1})} = \sup_{0\leqslant s<t\leqslant T}\frac{\|f(t)-f(s)\|_{H^{-1}}}{|t-s|^{\frac{1}{16}}}.
$$

Then from the proof of [LZZ18, Lemma 4.3], we know that for any $R>0$, K_R is relatively compact in χ.

Now we only need to show that for any $\delta > 0$, there exists $R>0$, such that $P(Z_v^\varepsilon \in K_R) > 1-\delta$ for any $\varepsilon \in (0,\varepsilon_0)$, where ε_0 is the constant such that Lemma 3.5 hold.

By Lemma 3.5 and Chebyshev inequality, we can choose R_0 large enough such that

$$
P\left(\sup_{t\in[0,T]}\|Z_v^\varepsilon(t)\|^2_H + \int_0^T \|Z_v^\varepsilon(t)\|^2_{\tilde{H}^{1,0}}dt > \frac{R_0}{3}\right) < \frac{\delta}{4},
$$

and

$$P\left(\sup_{t\in[0,T]} e^{-k\int_0^t \|\partial_1 u(s)\|_H^2 ds}\|u(t)\|^2_{\tilde{H}^{0,1}} + \int_0^T e^{-k\int_0^t \|\partial_1 u(s)\|_H^2 ds}\|u(t)\|^2_{\tilde{H}^{1,1}}dt > \frac{R_0}{3}\right) < \frac{\delta}{4},$$

where k is the same constant as in (3.14).

Fix R_0 and let

$$\begin{aligned}\hat{K}_{R_0} =& \Big\{u \in C([0,T],H^{-1}) : \sup_{t\in[0,T]}\|u(t)\|_H^2 + \int_0^T \|u(t)\|^2_{\tilde{H}^{1,0}}dt \leqslant \frac{R_0}{3} \text{ and} \\ & \sup_{t\in[0,T]} e^{-k\int_0^t \|\partial_1 u(s)\|_H^2 ds}\|u(t)\|^2_{\tilde{H}^{0,1}} + \int_0^T e^{-k\int_0^t \|\partial_1 u(s)\|_H^2 ds}\|u(t)\|^2_{\tilde{H}^{1,1}}dt \leqslant \frac{R_0}{3}\Big\}.\end{aligned}$$

Then $P(Z_v^\varepsilon \in C([0,T],H^{-1}) \setminus \hat{K}_{R_0}) < \frac{\delta}{2}$.

Now for $Z_v^\varepsilon \in \hat{K}_{R_0}$, we have $\partial_1^2 Z_v^\varepsilon$ is uniformly bounded in $L^2([0,T],H^{-1})$. Similar as in Lemma 3.2, Z_v^ε is uniformly bounded in $L^4([0,T],H^{\frac{1}{2}})$ and $L^4([0,T],L^4(\mathbb{T}^2))$, thus $B(Z_v^\varepsilon)$ is uniformly bounded in $L^2([0,T],H^{-1})$. By Hölder's inequality, we have

$$\sup_{s,t\in[0,T],s\neq t}\frac{\|\int_s^t \partial_1^2 Z_v^\varepsilon(r) + B(Z_v^\varepsilon(r))dr\|^2_{H^{-1}}}{|t-s|} \leqslant \int_0^T \|\partial_1^2 Z_v^\varepsilon(r) + B(Z_v^\varepsilon(r))\|^2_{H^{-1}}dr \leqslant C(R_0),$$

where $C(R_0)$ is a constant depend on R_0. For any $p\in(1,\frac{4}{3})$, by Hölder's inequality, we have

$$\begin{aligned}\sup_{s,t\in[0,T],s\neq t}\frac{\|\int_s^t \sigma(r,Z_v^\varepsilon(r))v^\varepsilon(r)dr\|^p_{H^{-1}}}{|t-s|^{p-1}} \leqslant& \int_0^T \|\sigma(r,Z_v^\varepsilon(r))v^\varepsilon(r)\|^p_{H^{-1}}dr \\ \leqslant& \int_0^T \|\sigma(r,Z_v^\varepsilon(r))\|^p_{L_2(l^2,H^{-1})}\|v^\varepsilon(r)\|^p_{l^2}dr \\ \leqslant& C\int_0^T (1+\|Z_v^\varepsilon(r)\|_H^4 + \|v^\varepsilon(r)\|^4_{l^2})dr \\ \leqslant& C(R_0),\end{aligned}$$

where we used Young's inequality and (A0) in the third inequality.

Moreover, for any $0\leqslant s\leqslant t\leqslant T$, by Hölder's inequality we have

$$\begin{aligned}E\|\int_s^t \sigma(r,Z_v^\varepsilon(r))dW(r)\|^4_{H^{-1}} \leqslant& CE\left(\int_s^t \|\sigma(r,Z_v^\varepsilon(r))\|^2_{L_2(l^2,H^{-1})}dr\right)^2 \\ \leqslant& C|t-s|E\int_s^t \|\sigma(r,Z_v^\varepsilon(r))\|^4_{L_2(l^2,H^{-1})}dr \\ \leqslant& C|t-s|^2(1+E(\sup_{t\in[0,T]}\|Z_v^\varepsilon(t)\|_H^4)) \\ \leqslant& C|t-s|^2,\end{aligned}$$

where we used (A0) in the third inequality and (3.13) in the last inequality. Then by Kolmogorov's continuity criterion, for any $\alpha\in(0,\frac{1}{4})$, we have

$$E\left(\sup_{s,t\in[0,T],s\neq t}\frac{\|\int_s^t \sigma(r,Z_v^\varepsilon(r))dW(r)\|^4_{H^{-1}}}{|t-s|^{2\alpha}}\right) \leqslant C.$$

Choose $p = \frac{8}{7}, \alpha = \frac{1}{8}$ in the above estimates, we deduce that there exists $R > R_0$ such that

$$P\left(\|Z_v^\varepsilon\|_{C^{\frac{1}{16}}([0,T],H^{-1})} > \frac{R}{3}, Z_v^\varepsilon \in \hat{K}_{R_0}\right) \leqslant \frac{E\left(\sup_{s,t\in[0,T],s\neq t} \frac{\|Z_v^\varepsilon(t)-Z_v^\varepsilon(s)\|_{H^{-1}}}{|t-s|^{\frac{1}{16}}} 1_{\{Z_v^\varepsilon \in \hat{K}_{R_0}\}}\right)}{\frac{R}{3}} < \frac{\delta}{2}.$$

Combining the fact that $P(Z_v^\varepsilon \in C([0,T],H^{-1}) \setminus \hat{K}_{R_0}) < \frac{\delta}{2}$, we finish the proof.

□

Lemma 3.7. *Assume (A0)-(A3) hold with $L_2 = 0$. Let $\{v^\varepsilon\}_{\varepsilon>0} \subset \mathcal{A}_N$ for some $N < \infty$. Assume v^ε converge to v in distribution as S_N-valued random elements, then*

$$g^\varepsilon\left(W(\cdot) + \frac{1}{\sqrt{\varepsilon}}\int_0^\cdot v^\varepsilon(s)ds\right) \to g^0\left(\int_0^\cdot v(s)ds\right)$$

in distribution as $\varepsilon \to 0$.

Proof. The proof follows essentially the same argument as in [WZZ15, Proposition 4.7].

By Lemma 3.4, we have $Z_v^\varepsilon = g^\varepsilon\left(W(\cdot) + \frac{1}{\sqrt{\varepsilon}}\int_0^\cdot v^\varepsilon(s)ds\right)$. By a similar but simple argument as in the proof of Lemmas 3.2 and 3.5, there exists a unique strong solution $Y^\varepsilon \in L^\infty([0,T],\tilde{H}^{0,1}) \bigcap L^2([0,T],\tilde{H}^{1,1}) \bigcap C([0,T],H^{-1})$ satisfying

$$\begin{aligned} dY^\varepsilon(t) =& \partial_1^2 Y^\varepsilon(t)dt + \sqrt{\varepsilon}\sigma(t, Z_v^\varepsilon(t))dW(t), \\ \text{div } Y^\varepsilon =& 0, \\ Y^\varepsilon(0) =& 0, \end{aligned}$$

and

$$\lim_{\varepsilon\to 0}\left[E\sup_{t\in[0,T]}\|Y^\varepsilon(t)\|_H^2 + E\int_0^T \|Y^\varepsilon(t)\|_{\tilde{H}^{1,0}}^2 dt\right] = 0,$$

$$\lim_{\varepsilon\to 0}\left[E\sup_{t\in[0,T]}(e^{-kg(t)}\|Y^\varepsilon(t)\|_{\tilde{H}^{0,1}}^2) + E\int_0^T e^{-kg(t)}\|Y^\varepsilon(t)\|_{\tilde{H}^{1,1}}^2 dt\right] = 0,$$

where $g(t) = \int_0^t \|Z_v^\varepsilon(s)\|_H^2 ds$ and k are the same as in (3.14).

Set

$$\Xi := \left(\chi, \mathcal{S}_N, L^\infty([0,T],H)\bigcap L^2([0,T],\tilde{H}^{1,0})\bigcap C([0,T].H^{-1})\right).$$

The above limit implies that $Y^\varepsilon \to 0$ a.s. in $L^\infty([0,T],H)\bigcap L^2([0,T],\tilde{H}^{1,0})\bigcap C([0,T].H^{-1})$ as $\varepsilon \to 0$ (in the sense of subsequence). By Lemma 3.6 the family $\{(Z_v^\varepsilon, v^\varepsilon)\}_{\varepsilon\in(0,\varepsilon_0)}$ is tight in $(\chi, \mathcal{S}_N)$. Let $(Z_v, v, 0)$ be any limit point of $\{(Z_v^\varepsilon, v^\varepsilon, Y^\varepsilon)\}_{\varepsilon\in(0,\varepsilon_0)}$. Our goal is to show that Z_v has the same law as $g^0\left(\int_0^\cdot v(s)ds\right)$ and Z_v^ε convergence in distribution to Z_v in the space $L^\infty([0,T],H)\bigcap L^2([0,T],\tilde{H}^{1,0})\bigcap C([0,T],H^{-1})$.

By the Skorokhod Theorem, there exists a stochastic basis $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}_{t\in[0,T]}, \tilde{P})$ and, on this basis, Ξ-valued random variables $(\tilde{Z}_v, \tilde{v}, 0)$, $(\tilde{Z}_v^\varepsilon, \tilde{v}^\varepsilon, \tilde{Y}^\varepsilon)$, such that $(\tilde{Z}_v^\varepsilon, \tilde{v}^\varepsilon, \tilde{Y}^\varepsilon)$ (respectively

$(\tilde{Z}_v, \tilde{v}, 0))$ has the same law as $(Z_v^\varepsilon, v^\varepsilon, Y^\varepsilon)$ (respectively $(Z_v, v, 0)$), and $(\tilde{Z}_v^\varepsilon, \tilde{v}^\varepsilon, \tilde{Y}^\varepsilon) \to (\tilde{Z}_v, \tilde{v}, 0)$, $\tilde{P}$-a.s.

We have

$$\begin{aligned} d(\tilde{Z}_v^\varepsilon(t) - \tilde{Y}^\varepsilon(t)) =& \partial_1^2(\tilde{Z}_v^\varepsilon(t) - \tilde{Y}^\varepsilon(t))dt - B(\tilde{Z}_v^\varepsilon(t))dt + \sigma(t, \tilde{Z}_v^\varepsilon(t))\tilde{v}^\varepsilon(t)dt, \\ \tilde{Z}_v^\varepsilon(0) - \tilde{Y}^\varepsilon(0) =& u_0, \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} & P(\tilde{Z}_v^\varepsilon - \tilde{Y}^\varepsilon \in L^\infty([0,T], H) \bigcap L^2([0,T], \tilde{H}^{1,0}) \bigcap C([0,T], H^{-1})) \\ =& P(Z_v^\varepsilon - Y^\varepsilon \in L^\infty([0,T], H) \bigcap L^2([0,T], \tilde{H}^{1,0}) \bigcap C([0,T], H^{-1})) \\ =& 1. \end{aligned}$$

Let $\tilde{\Omega}_0$ be the subset of $\tilde{\Omega}$ such that for $\omega \in \tilde{\Omega}_0$,

$$(\tilde{Z}_v^\varepsilon, \tilde{v}^\varepsilon, \tilde{Y}^\varepsilon)(\omega) \to (\tilde{Z}_v, \tilde{v}, 0)(\omega) \text{ in } \Xi,$$

and

$$e^{-k\int_0^\cdot \|\tilde{Z}_v^\varepsilon(\omega,s)\|_H^2 ds}\tilde{Y}^\varepsilon(\omega) \to 0 \text{ in } L^\infty([0,T], \tilde{H}^{0,1}) \bigcap L^2([0,T], \tilde{H}^{1,1}) \bigcap C([0,T], H^{-1}),$$

then $P(\tilde{\Omega}_0) = 1$. For any $\omega \in \tilde{\Omega}_0$, fix ω, we have $\sup_\varepsilon \int_0^T \|\tilde{Z}_v^\varepsilon(\omega, s)\|_H^2 ds < \infty$, then we deduce that

$$\lim_{\varepsilon \to 0} \left(\sup_{t \in [0,T]} \|\tilde{Y}^\varepsilon(\omega, t)\|_{\tilde{H}^{0,1}} + \int_0^T \|\tilde{Y}^\varepsilon(\omega, t)\|_{\tilde{H}^{1,1}}^2 dt \right) = 0. \tag{3.19}$$

Now we show that

$$\sup_{t \in [0,T]} \|\tilde{Z}_v^\varepsilon(\omega, t) - \tilde{Z}_v(\omega, t)\|_H^2 + \int_0^T \|\tilde{Z}_v^\varepsilon(\omega, t) - \tilde{Z}_v(\omega, t)\|_{\tilde{H}^{1,0}}^2 dt \to 0 \text{ as } \varepsilon \to 0. \tag{3.20}$$

Let $Z^\varepsilon = \tilde{Z}_v^\varepsilon(\omega) - \tilde{Y}^\varepsilon(\omega)$, then by (3.18) we have

$$dZ^\varepsilon(t) = \partial_1^2 Z^\varepsilon(t)dt - B(Z^\varepsilon(t) + \tilde{Y}^\varepsilon(t))dt + \sigma(t, Z^\varepsilon(t) + \tilde{Y}^\varepsilon(t))\tilde{v}^\varepsilon(t)dt. \tag{3.21}$$

Since $Z^\varepsilon(\omega) \to \tilde{Z}_v(\omega)$ in χ, by a very similar argument as in Lemma 3.3 we deduce that $\tilde{Z}_v = z^{\tilde{v}} = g^0\left(\int_0^\cdot \tilde{v}(s)ds\right)$. Moreover, note that $\tilde{Z}_v^\varepsilon(\omega) \to z^{\tilde{v}}(\omega)$ weak star in $L^\infty([0,T], \tilde{H}^{0,1})$, then the uniform boundedness principle implies that

$$\sup_\varepsilon \sup_{t \in [0,T]} \|\tilde{Z}_v^\varepsilon(\omega)\|_{\tilde{H}^{0,1}} < \infty. \tag{3.22}$$

Let $w^\varepsilon = Z^\varepsilon - z^{\tilde{v}}$, then we have

$$\begin{aligned} & \|w^\varepsilon(t)\|_H^2 + 2\int_0^t \|\partial_1 w^\varepsilon(s)\|_H^2 ds \\ =& -2\int_0^t \langle w^\varepsilon(s), B(Z^\varepsilon + \tilde{Y}^\varepsilon) - B(z^{\tilde{v}})\rangle ds + 2\int_0^t \langle w^\varepsilon(s), \sigma(s, Z^\varepsilon + \tilde{Y}^\varepsilon)\tilde{v}^\varepsilon(s) - \sigma(s, z^{\tilde{v}})\tilde{v}(s)\rangle ds. \end{aligned}$$

By Lemmas A.3 and A.4, we have

$$
\begin{aligned}
&\int_0^t \langle w^\varepsilon(s), B(Z^\varepsilon+\tilde{Y}^\varepsilon)-B(z^{\tilde{v}})\rangle ds\\
=&\int_0^t b(\tilde{Y}^\varepsilon, z^{\tilde{v}}, w^\varepsilon)+b(\tilde{Y}^\varepsilon,\tilde{Y}^\varepsilon,w^\varepsilon)+b(w^\varepsilon,\tilde{Y}^\varepsilon+z^{\tilde{v}},w^\varepsilon)+b(z^{\tilde{v}},\tilde{Y}^\varepsilon,w^\varepsilon)ds\\
\leqslant&\int_0^t [\frac{1}{2}\|\partial_1 w^\varepsilon(s)\|_H^2+\frac{1}{2}\|\tilde{Y}^\varepsilon(s)\|_{\tilde{H}^{1,1}}^2+C(1+\|z^{\tilde{v}}(s)\|_{\tilde{H}^{1,1}}^2+\|\tilde{Y}^\varepsilon(s)\|_{\tilde{H}^{1,1}}^2)\|w^\varepsilon(s)\|_H^2]ds\\
&+C\int_0^t \|\tilde{Y}^\varepsilon(s)\|_{\tilde{H}^{1,1}}^2\|w^\varepsilon(s)\|_H ds\\
\leqslant&\int_0^t \frac{1}{2}\|\partial_1 w^\varepsilon(s)\|_H^2 ds+C\int_0^t \|\tilde{Y}^\varepsilon(s)\|_{\tilde{H}^{1,1}}^2 ds+C\int_0^t (1+\|z^{\tilde{v}}(s)\|_{\tilde{H}^{1,1}}^2)\|w^\varepsilon(s)\|_H^2 ds,
\end{aligned}
$$

where we used the fact that by (3.19) and (3.22) w^ε are uniformly bounded in $L^\infty([0,T],H)$ in the last inequality. By (A1) and (A3) with $L_2=0$ we have

$$
\begin{aligned}
&\int_0^t \langle w^\varepsilon(s), \sigma(s,Z^\varepsilon+\tilde{Y}^\varepsilon)v^\varepsilon(s)-\sigma(s,z^{\tilde{v}})\tilde{v}(s)\rangle ds\\
=&\int_0^t \langle w^\varepsilon(s),(\sigma(s,Z^\varepsilon+\tilde{Y}^\varepsilon)-\sigma(s,z^{\tilde{v}}))\tilde{v}^\varepsilon(s)\rangle ds+\int_0^t \langle w^\varepsilon(s),\sigma(s,z^{\tilde{v}})(\tilde{v}^\varepsilon(s)-\tilde{v}(s))\rangle ds\\
\leqslant& C\int_0^t (\|w^\varepsilon(s)\|_H\|\tilde{v}^\varepsilon(s)\|_{l^2}(\|w^\varepsilon(s)\|_H^2+\|\tilde{Y}^\varepsilon(s)\|_H^2)^{\frac{1}{2}}ds\\
&+\int_0^t \|w^\varepsilon(s)\|_H\|\tilde{v}^\varepsilon(s)-\tilde{v}(s)\|_{l^2}(K_0+K_1\|z^{\tilde{v}}(s)\|_H^2+K_2\|\partial_1 z^{\tilde{v}}(s)\|_H^2)^{\frac{1}{2}}ds\\
\leqslant& CN^{\frac{1}{2}}\left(\int_0^t (\|w^\varepsilon(s)\|_H^2+\|\tilde{Y}^\varepsilon(s)\|_H^2 ds\right)^{\frac{1}{2}}\\
&+CN^{\frac{1}{2}}\left(\int_0^t \|w^\varepsilon(s)\|_H^2(K_0+K_1\|z^{\tilde{v}}(s)\|_H^2+K_2\|\partial_1 z^{\tilde{v}}(s)\|_H^2)ds\right)^{\frac{1}{2}},
\end{aligned}
$$

where we used the fact that w^ε are uniformly bounded in $L^\infty([0,T],H)$ and that $\tilde{v}^\varepsilon$, $\tilde{v}$ are in $\mathcal{A}_N$. Thus we have

$$
\begin{aligned}
&\|w^\varepsilon(t)\|_H^2+\int_0^t \|\partial_1 w^\varepsilon(s)\|_H^2 ds\\
\leqslant& C\int_0^t (1+\|z^{\tilde{v}}(s)\|_{\tilde{H}^{1,1}}^2)\|w^\varepsilon(s)\|_H^2 ds+C\int_0^t \|\tilde{Y}^\varepsilon(s)\|_{\tilde{H}^{1,1}}^2 ds\\
&+CN^{\frac{1}{2}}\left(\int_0^t (\|w^\varepsilon(s)\|_H^2+\|\tilde{Y}^\varepsilon(s)\|_H^2)ds\right)^{\frac{1}{2}}+CN^{\frac{1}{2}}\left(\int_0^t (1+\|z^{\tilde{v}}(s)\|_{\tilde{H}^{1,1}}^2)\|w^\varepsilon(s)\|_H^2 ds\right)^{\frac{1}{2}}.
\end{aligned}
$$

Since $Z^\varepsilon(\omega)\to z^{\tilde{v}}(\omega)$ strongly in $L^2([0,T],H)$ and $\tilde{Y}^\varepsilon\to 0$ in $L^2([0,T],\tilde{H}^{1,1})$, the same argument used in Lemma 3.3 implies

$$
\sup_{t\in[0,T]}\|\tilde{Z}_v^\varepsilon(\omega,t)-z^{\tilde{v}}(\omega,t)\|_H^2+\int_0^T \|\tilde{Z}_v^\varepsilon(\omega,t)-z^{\tilde{v}}(\omega,t)\|_{\tilde{H}^{1,0}}^2 dt\to 0 \text{ as } \varepsilon\to 0. \qquad (3.23)
$$

The proof is thus complete.

$\square$

Proof of Theorem 3.1. The result holds from Lemmas 3.1, 3.3 and 3.7. $\square$

4 Small time asymptotics

In this section, we consider the small time behaviour. We need the following additional assumption (A3') and (A4). Note that (A3') is stronger than (A3).

(A3') $\|\sigma(t,u)-\sigma(s,v)\|^2_{L_2(l^2,H)} \leqslant L_0|t-s|^\alpha + L_1\|u-v\|^2_H$.

(A4) $\|\sigma(t,u)\|^2_{L_2(l^2,V)} \leqslant \overline{K}_0 + \overline{K}_1\|u\|^2_V$.

Remark 4.1. *A typical example of σ is similar as in [LZZ18, Remark 4.2]. For $u=(u^1,u^2)\in H^{1,1}$ and $y\in l^2$, let*

$$\sigma(t,u)y = \sum_{k=1}^{\infty} b_k g(u)\langle y,\psi_k\rangle_{l^2},$$

where $\{\psi_k\}_{k\geqslant 0}$ is the orthonormal basis of l^2, $\{b_k\}_{k\geqslant 0}$ are functions from $\mathbb{T}^2$ to $\mathbb{R}$ and g is a differentiable function from $\mathbb{R}^2$ to $\mathbb{R}$. Assume that $|g(x)-g(y)|\leqslant C|x-y|$ for all $x,y\in\mathbb{R}^2$ and some constant C depends on g. Also suppose that $\mathrm{div}(b_k g(u))=0$ and $b_k,\partial_1 b_k,\partial_2 b_k\in L^\infty$, $\sum_{k=1}^\infty\|b_k\|^2_{L^\infty}\leqslant M$, $\sum_{k=1}^\infty\|\partial_1 b_k\|^2_{L^\infty}\leqslant M$ and $\sum_{k=1}^\infty\|\partial_2 b_k\|^2_{L^\infty}\leqslant M$. From the conditions of g, it is easy to obtain $|g(u)|\leqslant C|u|+C$, $|\partial_1 g(u)|\leqslant C$ and $|\partial_2 g(u)|\leqslant C$. In this case, σ satisfies (A0)-(A4) and (A3'):

$$\begin{aligned}
\|\sigma(t,u)\|^2_{L_2(l^2,H)} &\leqslant \sum_{k=1}^{\infty}\|b_k g(u)\|^2_H \leqslant CM(\|u\|^2_H+1);\\
\|\sigma(t,u)\|^2_{L_2(l^2,H^{0,1})} &\leqslant \sum_{k=1}^{\infty}\|b_k g(u)\|^2_H + \sum_{k=1}^{\infty}\|\partial_2(b_k g(u))\|^2_H\\
&\leqslant CM(\|u\|^2_H+1) + \sum_{k=1}^{\infty}\|\partial_2 b_k g(u) + b_k(\partial_1 g(u)\partial_2 u^1 + \partial_2 g(u)\partial_2 u^2)\|^2_H\\
&\leqslant CM(1+\|u\|^2_H+\|\partial_2 u\|^2_H);\\
\|\sigma(t,u)\|^2_{L_2(l^2,V)} &\leqslant CM(\|u\|^2_H+1) + \sum_{k=1}^{\infty}\|\partial_1(b_k g(u))\|^2_H + \sum_{k=1}^{\infty}\|\partial_2(b_k g(u))\|^2_H\\
&\leqslant CM(1+\|u\|^2_H+\|\partial_1 u\|^2_H+\|\partial_2 u\|^2_H);\\
\|\sigma(t,u)-\sigma(s,v)\|^2_{L_2(l^2,H)} &\leqslant MC\|u-v\|^2_H.
\end{aligned}$$

Let $\varepsilon>0$ and u be the solution to (2.1), by the scaling property of the Brownian motion, $u(\varepsilon t)$ coincides in law with the solution to the following equation:

$$\begin{aligned}
&du_\varepsilon = \varepsilon\partial_1^2 u_\varepsilon dt - \varepsilon B(u_\varepsilon)dt + \sqrt{\varepsilon}\sigma(\varepsilon t,u_\varepsilon)dW(t),\\
&u_\varepsilon(0)=u_0.
\end{aligned} \tag{4.1}$$

Define a functional I^{u_0} on $L^\infty([0,T],H)\bigcap C([0,T],H^{-1})$ by

$$I^{u_0}(g)=\inf_{h\in\Gamma_g}\{\frac{1}{2}\int_0^T\|h(t)\|_{l^2}^2dt\},$$

where

$$\Gamma_g=\{h\in L^2([0,T],l^2):g(t)=u_0+\int_0^t\sigma(0,g(s))h(s)ds,\ t\in[0,T]\}.$$

The main theorem of this section is the following one:

Theorem 4.1. *Assume (A0), (A1), (A2), (A3'), (A4) hold with $K_2=\tilde{K}_2=0$ and $u_0\in\tilde{H}^{0,1}$, then u_ε satisfies a large deviation principle on the space $L^\infty([0,T],H)\bigcap C([0,T],H^{-1})$ with the good rate function I^{u_0}.*

We aim to prove that u_ε is exponentially equivalent to the solution to the following equation:

$$v_\varepsilon(t)=u_0+\sqrt{\varepsilon}\int_0^t\sigma(\varepsilon s,v_\varepsilon(s))dW(s). \tag{4.2}$$

Because of the non-linear form $b(\cdot,\cdot,\cdot)$ and the anisotropic viscosity, we split the proof into several lemmas.

Lemma 4.1. *Assume $u_0\in\tilde{H}^{0,1}$, then v_ε satisfies a large deviation principle on the space $L^\infty([0,T],H)\bigcap C([0,T],H^{-1})$ with the good rate function I^{u_0}.*

Proof. Let z_ε be the solution to the stochastic equation:

$$z_\varepsilon(t)=u_0+\sqrt{\varepsilon}\int_0^t\sigma(0,z_\varepsilon(s))dW(s).$$

By [DPZ09, Theorem 12.11], we know that z_ε satisfies a large deviation principle with the good rate function I^{u_0}. Applying Itô's formula to $\|v_\varepsilon-z_\varepsilon\|_H^2$, we obtain

$$\begin{aligned}\|v_\varepsilon(t)-z_\varepsilon(t)\|_H^2=&2\sqrt{\varepsilon}\int_0^t\langle v_\varepsilon(s)-z_\varepsilon(s),[\sigma(\varepsilon s,v_\varepsilon(s))-\sigma(0,z_\varepsilon(s))]dW(s)\rangle\\&+\varepsilon\int_0^t\|\sigma(\varepsilon s,v_\varepsilon(s))-\sigma(0,z_\varepsilon(s))\|_{L_2(l^2,H)}^2ds.\end{aligned}$$

Then by (A3') and Lemma 2.3, we get for $p\geqslant 2$,

$$
\begin{aligned}
&\left(E[\sup_{0\leqslant t\leqslant T}\|v_\varepsilon(t)-z_\varepsilon(t)\|_H^{2p}]\right)^{\frac{2}{p}}\\
\leqslant& C\varepsilon\left(E[\sup_{0\leqslant t\leqslant T}\int_0^t\langle v_\varepsilon(s)-z_\varepsilon(s),(\sigma(\varepsilon s,v_\varepsilon(s))-\sigma(0,z_\varepsilon(s)))dW(s)\rangle]^p\right)^{\frac{2}{p}}\\
&+C\varepsilon^2\left(E[\int_0^T\|\sigma(\varepsilon s,v_\varepsilon(s))-\sigma(0,z_\varepsilon(s))\|^2_{L_2(l^2,H)}ds]^p\right)^{\frac{2}{p}}\\
\leqslant& C\varepsilon p\left(E\left[\int_0^T\|v_\varepsilon(s)-z_\varepsilon(s)\|_H^2\|\sigma(\varepsilon s,v_\varepsilon(s))-\sigma(0,z_\varepsilon(s))\|^2_{L_2(l^2,H)}ds\right]^{\frac{p}{2}}\right)^{\frac{2}{p}}\\
&+C\varepsilon^2\left(\varepsilon^{2\alpha}T^{2+2\alpha}+T\int_0^T\left(E[\sup_{0\leqslant l\leqslant s}\|v_\varepsilon(l)-z_\varepsilon(l)\|_H^{2p}]\right)^{\frac{2}{p}}ds\right)\\
\leqslant& C\varepsilon p\left(\varepsilon^{2\alpha}+\int_0^T\left(E[\sup_{0\leqslant l\leqslant s}\|v_\varepsilon(l)-z_\varepsilon(l)\|_H^{2p}]\right)^{\frac{2}{p}}ds\right)\\
&+C\varepsilon^2\left(\varepsilon^{2\alpha}+\int_0^T\left(E[\sup_{0\leqslant l\leqslant s}\|v_\varepsilon(l)-z_\varepsilon(l)\|_H^{2p}]\right)^{\frac{2}{p}}ds\right).
\end{aligned}
$$

By Gronwall's inequality, we have

$$
\left(E[\sup_{0\leqslant t\leqslant T}\|v_\varepsilon(t)-z_\varepsilon(t)\|_H^{2p}]\right)^{\frac{2}{p}}\leqslant C(\varepsilon^{1+2\alpha}p+\varepsilon^{2+2\alpha})e^{C(\varepsilon p+\varepsilon^2)}.
$$

Then Chebyshev's inequality implies that

$$
\begin{aligned}
\varepsilon\log P(\sup_{0\leqslant t\leqslant T}\|v_\varepsilon(t)-z_\varepsilon(t)\|_H^2>\delta)\leqslant&\varepsilon\log E[\sup_{0\leqslant t\leqslant T}\|v_\varepsilon(t)-z_\varepsilon(t)\|_H^{2p}]-\varepsilon p\log\delta\\
\leqslant&\frac{\varepsilon p}{2}(C+C\varepsilon p+C\varepsilon^2+\log(\varepsilon^{1+2\alpha}p+\varepsilon^{2+2\alpha})-2\log\delta).
\end{aligned}
$$

Let $p=\frac{1}{\varepsilon}$ and $\varepsilon\to0$, we get that v_ε and z_ε are exponentially equivalent, which by Lemma 2.1 implies the result. □

Lemma 4.2. *Let $F_{u_\varepsilon}(t)=\sup_{0\leqslant s\leqslant t}\|u_\varepsilon(s)\|_H^2+\varepsilon\int_0^t\|\partial_1u_\varepsilon(s)\|_H^2ds$, then*

$$
\lim_{M\to\infty}\sup_{0<\varepsilon\leqslant1}\varepsilon\log P(F_{u_\varepsilon}(T)>M)=-\infty.
$$

Proof. Since $b(u_\varepsilon,u_\varepsilon,u_\varepsilon)=0$, applying Itô's formula to $\|u_\varepsilon(t)\|_H^2$, we have

$$
\begin{aligned}
&\|u_\varepsilon(t)\|_H^2+2\varepsilon\int_0^t\|\partial_1u_\varepsilon(s)\|_H^2ds\\
=&\|u_0\|_H^2+2\sqrt{\varepsilon}\int_0^t\langle u_\varepsilon(s),\sigma(\varepsilon s,u_\varepsilon(s)dW(s)\rangle+\varepsilon\int_0^t\|\sigma(\varepsilon s,u_\varepsilon(s))\|^2_{L_2(l^2,H)}ds.
\end{aligned}
$$

Then it follows from (A1) with $K_2 = 0$ that

$$\|u_\varepsilon(t)\|_H^2 + \varepsilon \int_0^t \|\partial_1 u_\varepsilon(s)\|_H^2 ds \leqslant \|u_0\|_{\tilde{H}^{0,1}}^2 + C\varepsilon t + C\varepsilon \int_0^t \|u_\varepsilon(s)\|_H^2 ds$$
$$+ 2\sqrt{\varepsilon} \int_0^t \langle u_\varepsilon, \sigma(\varepsilon s, u_\varepsilon(s)) dW(s) \rangle.$$

Take supremum over t, for $p \geqslant 2$, we have

$$(E[F_{u_\varepsilon}(T)]^p)^{\frac{1}{p}} \leqslant \|u_0\|_{\tilde{H}^{0,1}}^2 + C\varepsilon T + C\varepsilon \int_0^T (E[F_{u_\varepsilon}(t)]^p)^{\frac{1}{p}} dt$$
$$+ 2\sqrt{\varepsilon}(E[\sup_{0\leqslant t\leqslant T} | \int_0^t \langle u_\varepsilon, \sigma(\varepsilon s, u_\varepsilon(s)) dW(s) \rangle |]^p)^{\frac{1}{p}}.$$

For the term in the last line, by Lemma 2.3 and [XZ09, (3.12)], we have

$$2\sqrt{\varepsilon}(E[\sup_{0\leqslant t\leqslant T} | \int_0^t \langle u_\varepsilon, \sigma(\varepsilon s, u_\varepsilon(s)) dW(s) \rangle |]^p)^{\frac{1}{p}}$$
$$\leqslant C\sqrt{\varepsilon p} \left[\int_0^T 1 + (E\|u_\varepsilon(s)\|_H^{2p})^{\frac{2}{p}} ds \right]^{\frac{1}{2}}.$$

Combining the above estimate, we arrive at

$$(E[F_{u_\varepsilon}(T)]^p)^{\frac{2}{p}} \leqslant C \left(\|u_0\|_{\tilde{H}^{0,1}}^2 + \varepsilon T \right)^2 + C\varepsilon^2 \int_0^T (E[F_{u_\varepsilon}(t)]^p)^{\frac{2}{p}} ds$$
$$+ C\varepsilon p T + C\varepsilon p \int_0^T (E[F_{u_\varepsilon}(t)]^p)^{\frac{2}{p}} dt.$$

Then the Gronwall's inequality implies

$$(E[F_{u_\varepsilon}(T)]^p)^{\frac{2}{p}} \leqslant C \left[\|u_0\|_{\tilde{H}^{0,1}}^4 + \varepsilon^2 + \varepsilon p \right] e^{C\varepsilon^2 + C\varepsilon p}.$$

Let $p = \frac{1}{\varepsilon}$, by Chebyshev's inequality, we have

$$\begin{aligned}
&\varepsilon \log P(F_{u_\varepsilon}(T) > M) \\
\leqslant& -\log M + \log (E[F_{u_\varepsilon}(T)]^p)^{\frac{1}{p}} \\
\leqslant& -\log M + \log \sqrt{\|u_0\|_{\tilde{H}^{0,1}}^4 + \varepsilon^2 + 1} + C(\varepsilon^2 + 1).
\end{aligned}$$

Take supremum over ε and let $M \to \infty$, we finish the proof.

$\square$

Lemma 4.3. *For M>0, define a random time*

$$\tau_{M,\varepsilon} = T \wedge \inf\{t : \|u_\varepsilon(t)\|_H^2 > M, \text{ or } \varepsilon \int_0^t \|\partial_1 u_\varepsilon(s)\|_H^2 ds > M\}.$$

Then $\tau_{M,\varepsilon}$ is a stopping time with respect to $\mathcal{F}_{t+} = \cap_{s>t}\mathcal{F}_s$.

Similarly, Let

$$\tau'_{M,\varepsilon} = T \wedge \inf\{t : \|u_\varepsilon(t)\|^2_{\tilde{H}^{0,1}} > M, \ \text{or}\ \varepsilon \int_0^t \|u_\varepsilon(s)\|^2_{\tilde{H}^{1,1}} ds > M\},$$

then $\tau'_{M,\varepsilon}$ is a stopping time with respect to $\mathcal{F}_{t+}$.

Proof. The problem comes with the continuity of $u_\varepsilon(t)$. More precisely, since $\int_0^t \|\partial_1 u_\varepsilon(s)\|^2_H ds$ is a continuous adapted process, we only need to prove that $\hat{\tau} = \inf\{t > 0 : \|u_\varepsilon(t)\|^2_H > M\}$ is a stopping time.

Since $u_\varepsilon \in L^\infty([0,T], H) \bigcap C([0,T], H^{-1})$, $u_\varepsilon(t)$ is weakly continuous on H, which implies the lower semi-continuity of u_ε on H.

By definition of $\hat{\tau}$, for $t > 0$

$$\bigcap_{s\in(0,t]} \{\|u_\varepsilon(s)\|^2_H \leqslant M\} \subset \{\hat{\tau} \geqslant t\} \subset \bigcap_{s\in(0,t)} \{\|u_\varepsilon(s)\|^2_H \leqslant M\}.$$

On the contrary, if $\omega \in \{\hat{\tau} \geqslant t\}$, for any $s < t$, $\|u_\varepsilon(s)(\omega)\|^2_H \leqslant M$. Then lower semi-continuity implies

$$\|u_\varepsilon(t)(\omega)\|^2_H \leqslant \liminf_{s<t, s\to t} \|u_\varepsilon(s)\|^2_H \leqslant M.$$

Hence we have

$$\{\hat{\tau} \geqslant t\} = \bigcap_{s\in(0,t]} \{\|u_\varepsilon(s)\|^2_H \leqslant M\}.$$

Note that for $\omega \in \bigcap_{s\in(0,t]\cap\mathbb{Q}} \{\|u_\varepsilon(s)\|^2_H \leqslant M\}$, we have for any $s \in (0,t]$, by the lower semi-continuity,

$$\|u_\varepsilon(s)(\omega)\|^2_H \leqslant \liminf_{s'\to s} \|u_\varepsilon(s')\|^2_H \leqslant \liminf_{s'\to s, s'\in\mathbb{Q}} \|u_\varepsilon(s')\|^2_H \leqslant M,$$

which means

$$\bigcap_{s\in(0,t]} \{\|u_\varepsilon(s)\|^2_H \leqslant M\} = \bigcap_{s\in(0,t]\cap\mathbb{Q}} \{\|u_\varepsilon(s)\|^2_H \leqslant M\}.$$

Then we have for $t > 0$

$$\{\hat{\tau} \geqslant t\} = \bigcap_{s\in(0,t]} \{\|u_\varepsilon(s)\|^2_H \leqslant M\} = \bigcap_{s\in(0,t]\cap\mathbb{Q}} \{\|u_\varepsilon(s)\|^2_H \leqslant M\} \in \mathcal{F}_t,$$

which implies the result.

For $\tau'_{M,\varepsilon}$, the result follows from the fact that u_ε is weakly continuous in $\tilde{H}^{0,1}$ since $u_\varepsilon \in L^\infty([0,T], \tilde{H}^{0,1}) \bigcap C(0,T], H^{-1})$.

$\square$

Lemma 4.4. *Let $G_{u_\varepsilon}(t) = \sup_{0\leqslant s\leqslant t} \|u_\varepsilon(s)\|^2_{\tilde{H}^{0,1}} + \varepsilon \int_0^t \|u_\varepsilon(s)\|^2_{\tilde{H}^{1,1}} ds$. For fixed M_1, we have*

$$\lim_{M\to\infty} \sup_{0<\varepsilon\leqslant 1} \varepsilon \log P(G_{u_\varepsilon}(\tau_{M_1,\varepsilon}) > M) = -\infty.$$

Proof. Let k be a positive constant and $f_\varepsilon(t) = 1 + \|\partial_1 u_\varepsilon(t)\|_H^2$. Applying Itô's formula to $e^{-k\varepsilon\int_0^t f_\varepsilon(s)ds}\|u_\varepsilon(t)\|_{\tilde{H}^{0,1}}^2$, we obtain

$$\begin{aligned}
&e^{-k\varepsilon\int_0^t f_\varepsilon(s)ds}\|u_\varepsilon(t)\|_{\tilde{H}^{0,1}}^2 + 2\varepsilon\int_0^t e^{-k\varepsilon\int_0^s f_\varepsilon(r)dr}(\|\partial_1 u_\varepsilon(s)\|_H^2 + \|\partial_1\partial_2 u_\varepsilon(s)\|_H^2)ds\\
=&\|u_0\|_{\tilde{H}^{0,1}}^2 - k\varepsilon\int_0^t e^{-k\varepsilon\int_0^s f_\varepsilon(r)dr} f_\varepsilon(s)\|u_\varepsilon(s)\|_{\tilde{H}^{0,1}}^2 ds\\
&- 2\varepsilon\int_0^t e^{-k\varepsilon\int_0^s f_\varepsilon(r)dr}\langle\partial_2 u_\varepsilon(s), \partial_2(u_\varepsilon\cdot\nabla u_\varepsilon)(s)\rangle ds\\
&+ 2\sqrt{\varepsilon}\int_0^t e^{-k\varepsilon\int_0^s f_\varepsilon(r)dr}\langle u_\varepsilon(s), \sigma(\varepsilon s, u_\varepsilon(s))dW(s)\rangle_{\tilde{H}^{0,1}}\\
&+ \varepsilon\int_0^t e^{-k\varepsilon\int_0^s f_\varepsilon(r)dr}\|\sigma(\varepsilon s, u_\varepsilon(s))\|_{L_2(l^2,\tilde{H}^{0,1})}^2 ds.
\end{aligned}$$

The fourth and the fifth line can be dealt in the same way as in the proof of Lemma 4.2. For the third line, by Lemma A.5, we have

$$|\langle\partial_2 u_\varepsilon, \partial_2(u_\varepsilon\cdot\nabla u_\varepsilon)\rangle| \leqslant \frac{1}{2}\|\partial_1\partial_2 u_\varepsilon\|_H^2 + C_1 f_\varepsilon\|\partial_2 u_\varepsilon\|_H^2,$$

where C_1 is a constant. Therefore by (A2) with $\tilde{K}_2 = 0$ we get

$$\begin{aligned}
&e^{-k\varepsilon\int_0^t f_\varepsilon(s)ds}\|u_\varepsilon(t)\|_{\tilde{H}^{0,1}}^2 + \varepsilon\int_0^t e^{-k\varepsilon\int_0^s f_\varepsilon(r)dr}\|u_\varepsilon(s)\|_{\tilde{H}^{1,1}}^2 ds\\
\leqslant&\|u_0\|_{\tilde{H}^{0,1}}^2 - k\varepsilon\int_0^t e^{-k\varepsilon\int_0^s f_\varepsilon(r)dr} f_\varepsilon(s)\|u_\varepsilon(s)\|_{\tilde{H}^{0,1}}^2 ds\\
&+ 2C_1\varepsilon\int_0^t e^{-k\varepsilon\int_0^s f_\varepsilon(r)dr} f_\varepsilon(s)\|u_\varepsilon(s)\|_{\tilde{H}^{0,1}}^2 ds\\
&+ 2\sqrt{\varepsilon}\int_0^t e^{-k\varepsilon\int_0^s f_\varepsilon(r)dr}\langle u_\varepsilon(s), \sigma(\varepsilon s, u_\varepsilon(s))dW(s)\rangle_{\tilde{H}^{0,1}}\\
&+ \varepsilon\int_0^t e^{-k\varepsilon\int_0^s f_\varepsilon(r)dr}[\tilde{K}_0 + (\tilde{K}_1 + 1)\|u_\varepsilon(s)\|_{\tilde{H}^{0,1}}^2]ds.
\end{aligned}$$

For the last second line, similar to [XZ09, (3.12)], we have

$$
\begin{aligned}
&2\sqrt{\varepsilon}(E[\sup_{0\leqslant s\leqslant t}|\int_0^s e^{-k\varepsilon\int_0^r f_\varepsilon(l)dl}\langle u_\varepsilon(r),\sigma(\varepsilon r,u_\varepsilon(r))dW(r)\rangle_{\tilde{H}^{0,1}}|]^p)^{\frac{1}{p}}\\
\leqslant& C\sqrt{\varepsilon p}(E[\int_0^t e^{-2k\varepsilon\int_0^r f_\varepsilon(l)dl}\|u_\varepsilon(r)\|^2_{\tilde{H}^{0,1}}\|\sigma(\varepsilon r,u_\varepsilon(r))\|^2_{L_2(l^2,\tilde{H}^{0,1})}dr]^{\frac{p}{2}})^{\frac{1}{p}}\\
\leqslant& C\sqrt{\varepsilon p}(E[\int_0^t e^{-2k\varepsilon\int_0^r f_\varepsilon(l)dl}\|u_\varepsilon(r)\|^2_{\tilde{H}^{0,1}}(1+\|u_\varepsilon(r)\|^2_{\tilde{H}^{0,1}})dr]^{\frac{p}{2}})^{\frac{1}{p}}\\
\leqslant& C\sqrt{\varepsilon p}(E[\int_0^t e^{-2k\varepsilon\int_0^r f_\varepsilon(l)dl}(1+\|u_\varepsilon(r)\|^4_{\tilde{H}^{0,1}})dr]^{\frac{p}{2}})^{\frac{1}{p}}\\
\leqslant& C\sqrt{\varepsilon p}\left[\int_0^t 1+(E[e^{-pk\varepsilon\int_0^r f_\varepsilon(l)dl}\|u_\varepsilon(s)\|^{2p}_{\tilde{H}^{0,1}}])^{\frac{2}{p}}ds\right]^{\frac{1}{2}},
\end{aligned}
$$

where we used (A2) with $K_2=0$ in the third line.

Let $k>2C_1$ and using Lemma 2.3, we have for $p\geqslant 2$

$$
\begin{aligned}
&\left(E\left[\sup_{0\leqslant s\leqslant t\wedge\tau_{M_1,\varepsilon}} e^{-k\varepsilon\int_0^s f_\varepsilon(r)dr}\|u_\varepsilon(s)\|^2_{\tilde{H}^{0,1}}+\varepsilon\int_0^{t\wedge\tau_{M_1,\varepsilon}} e^{-k\varepsilon\int_0^s f_\varepsilon(r)dr}\|u_\varepsilon(s)\|^2_{\tilde{H}^{1,1}}ds\right]^p\right)^{\frac{2}{p}}\\
\leqslant& C(\|u_0\|^2_{\tilde{H}^{0,1}}+\varepsilon)^2+C\varepsilon^2\int_0^t\left(E\left[\sup_{0\leqslant r\leqslant s\wedge\tau_{M_1,\varepsilon}} e^{-k\varepsilon\int_0^r f_\varepsilon(l)dl}\|u_\varepsilon(r)\|^2_{\tilde{H}^{0,1}}\right]^p\right)^{\frac{2}{p}}ds\\
&+C\varepsilon p+C\varepsilon p\int_0^t\left(E\left[\sup_{0\leqslant r\leqslant s\wedge\tau_{M_1,\varepsilon}} e^{-k\varepsilon\int_0^r f_\varepsilon(l)dl}\|u_\varepsilon(r)\|^2_{\tilde{H}^{0,1}}\right]^p\right)^{\frac{2}{p}}ds.
\end{aligned}
$$

Applying Gronwall's inequality, we obtain

$$
\begin{aligned}
&\left(E\left[\sup_{0\leqslant t\leqslant\tau_{M_1,\varepsilon}} e^{-k\varepsilon\int_0^t f_\varepsilon(s)ds}\|u_\varepsilon(t)\|^2_{\tilde{H}^{0,1}}+\varepsilon\int_0^{\tau_{M_1,\varepsilon}} e^{-k\varepsilon\int_0^s f_\varepsilon(r)dr}\|u_\varepsilon(s)\|^2_{\tilde{H}^{1,1}}ds\right]^p\right)^{\frac{2}{p}}\\
\leqslant& C\left[\|u_0\|^4_{\tilde{H}^{0,1}}+\varepsilon^2+\varepsilon p\right]e^{C(\varepsilon^2+\varepsilon p)}.
\end{aligned}
$$

Hence by the definition of $\tau_{M_1,\varepsilon}$, we have

$$
\begin{aligned}
&(E\left[G_{u_\varepsilon}(\tau_{M_1,\varepsilon})\right]^p)^{\frac{2}{p}}\\
\leqslant&\left(E\left[\left(\sup_{0\leqslant t\leqslant\tau_{M_1,\varepsilon}} e^{-k\varepsilon\int_0^t f_\varepsilon(s)ds}\|u_\varepsilon(t)\|^2_{\tilde{H}^{0,1}}+\varepsilon\int_0^{\tau_{M_1,\varepsilon}} e^{-k\varepsilon\int_0^s f_\varepsilon(r)dr}\|u_\varepsilon(s)\|^2_{\tilde{H}^{1,1}}ds\right)^p e^{pk\varepsilon\int_0^t f_\varepsilon(s)ds}\right]\right)^{\frac{2}{p}}\\
\leqslant& e^{C(M_1+\varepsilon)}\left(E\left[\sup_{0\leqslant t\leqslant\tau_{M_1,\varepsilon}} e^{-k\varepsilon\int_0^t f_\varepsilon(s)ds}\|u_\varepsilon(t)\|^2_{\tilde{H}^{0,1}}+\varepsilon\int_0^{\tau_{M_1,\varepsilon}} e^{-k\varepsilon\int_0^s f_\varepsilon(r)dr}\|u_\varepsilon(s)\|^2_{\tilde{H}^{1,1}}ds\right]^p\right)^{\frac{2}{p}}\\
\leqslant& Ce^{C(M_1+\varepsilon)}\left[\|u_0\|^4_{\tilde{H}^{0,1}}+\varepsilon^2+\varepsilon p\right]e^{C(\varepsilon^2+\varepsilon p)}.
\end{aligned}
$$

Let $p = \frac{2}{\varepsilon}$, by Chebyshev's inequality, we have

$$
\begin{aligned}
&\varepsilon \log P(G_{u_\varepsilon}(\tau_{M_1,\varepsilon}) > M) \\
\leqslant &\varepsilon \log \frac{E\left[G_{u_\varepsilon}(\tau_{M_1,\varepsilon})\right]^p}{M^p} \\
\leqslant &-2\log M + C + C(M_1+\varepsilon) + C(\varepsilon^2 + \varepsilon p) + \log[\|u_0\|^4_{\tilde{H}^{0,1}} + \varepsilon^2 + \varepsilon p].
\end{aligned}
$$

Take supremum over ε and let $M \to \infty$, we finish the proof. □

Since V is dense in $\tilde{H}^{0,1}$, there exists a sequence $\{u_0^n\} \subset V$ such that

$$\lim_{n\to+\infty} \|u_0^n - u_0\|_{\tilde{H}^{0,1}} = 0.$$

Let $u_{n,\varepsilon}$ be the solution to (4.1) with the initial data u_0^n. Similarly, let $v_{n,\varepsilon}$ be the solution to (4.2) with the initial data u_0^n.

For $M > 0$, define a random time (which is also a stopping time with respect to $\mathcal{F}_{t+}$ by Lemma 4.3)

$$\tau^n_{M,\varepsilon} := T \wedge \inf\{t : \|u_{n,\varepsilon}(t)\|_H^2 > M, \text{ or } \varepsilon \int_0^t \|\partial_1 u_{n,\varepsilon}(s)\|^2_H ds > M\}.$$

From the proof of Lemma 4.2 and Lemma 4.4, it follows that

Lemma 4.5.

$$\lim_{M\to\infty} \sup_n \sup_{0<\varepsilon\leqslant 1} \varepsilon \log P(F_{u_{n,\varepsilon}}(T) > M) = -\infty.$$

For fixed M_1, we have

$$\lim_{M\to\infty} \sup_n \sup_{0<\varepsilon\leqslant 1} \varepsilon \log P(G_{u_{n,\varepsilon}}(\tau^n_{M_1,\varepsilon}) > M) = -\infty.$$

The following lemma for $v_{n,\varepsilon}$ is from [XZ09]:

Lemma 4.6 ([XZ09, Lemma 3.2])**.**

$$\lim_{M\to\infty} \sup_{0<\varepsilon\leqslant 1} \varepsilon \log P\left(\sup_{0\leqslant t\leqslant T} \|v_{n,\varepsilon}(t)\|^2_V > M\right) = -\infty.$$

Lemma 4.7. *For any $\delta > 0$,*

$$\lim_{n\to\infty} \sup_{0<\varepsilon\leqslant 1} \varepsilon \log P\left(\sup_{0\leqslant t\leqslant T} \|u_{n,\varepsilon}(t) - u_\varepsilon(t)\|^2_H > \delta\right) = -\infty.$$

Proof. Clearly, for $M_1, M_2 > 0$

$$
\begin{aligned}
&P\left(\sup_{0\leqslant t\leqslant T} \|u_{n,\varepsilon}(t) - u_\varepsilon(t)\|^2_H > \delta\right) \\
\leqslant &P\left(\sup_{0\leqslant t\leqslant T} \|u_{n,\varepsilon}(t) - u_\varepsilon(t)\|^2_H > \delta, F_{u_\varepsilon}(T) \leqslant M_1, G_{u_\varepsilon}(T) \leqslant M_2\right) \\
&+ P\left(F_{u_\varepsilon}(T) > M_1\right) + P\left(F_{u_\varepsilon}(T) \leqslant M_1, G_{u_\varepsilon}(T) > M_2\right) \\
\leqslant &P\left(\sup_{0\leqslant t\leqslant \tau_{M_1,\varepsilon}\wedge\tau'_{M_2,\varepsilon}} \|u_{n,\varepsilon}(t) - u_\varepsilon(t)\|^2_H > \delta\right) \\
&+ P\left(F_{u_\varepsilon}(T) > M_1\right) + P\left(G_{u_\varepsilon}(\tau_{M_1,\varepsilon}) > M_2\right),
\end{aligned}
\tag{4.3}
$$

where $\tau_{M_1,\varepsilon}$ and $\tau'_{M_2,\varepsilon}$ are introduced in Lemma 4.3.

For the first term on the right hand of (4.3), let k be a positive constant and

$$U_\varepsilon = 1 + \|u_\varepsilon\|^2_{\tilde{H}^{1,1}}.$$

Applying Itô's formula to $e^{-\varepsilon k\int_0^t U_\varepsilon(s)ds}\|u_\varepsilon(t) - u_{n,\varepsilon}(t)\|_H^2$, we get

$$\begin{aligned}
&e^{-\varepsilon k\int_0^t U_\varepsilon(s)ds}\|u_\varepsilon(t) - u_{n,\varepsilon}(t)\|_H^2 + 2\varepsilon\int_0^t e^{-\varepsilon k\int_0^s U_\varepsilon(r)dr}\|\partial_1(u_\varepsilon(s) - u_{n,\varepsilon}(s))\|_H^2 ds\\
=&\|u_0 - u_{n,0}\|_H^2 - k\varepsilon\int_0^t e^{-\varepsilon k\int_0^s U_\varepsilon(r)dr}U_\varepsilon(s)\|u_\varepsilon(s) - u_{n,\varepsilon}(s)\|_H^2 ds\\
&- 2\varepsilon\int_0^t e^{-\varepsilon k\int_0^s U_\varepsilon(r)dr}\left(b(u_\varepsilon, u_\varepsilon, u_\varepsilon - u_{n,\varepsilon})(s) - b(u_{n,\varepsilon}, u_{n,\varepsilon}, u_\varepsilon - u_{n,\varepsilon})(s)\right)ds\\
&+ \varepsilon\int_0^t e^{-\varepsilon k\int_0^s U_\varepsilon(r)dr}\|\sigma(\varepsilon s, u_\varepsilon(s)) - \sigma(\varepsilon s, u_{n,\varepsilon}(s))\|^2_{L_2(l^2,H)}ds\\
&+ 2\sqrt{\varepsilon}\int_0^t e^{-\varepsilon k\int_0^s U_\varepsilon(r)dr}\langle u_\varepsilon(s) - u_{n,\varepsilon}(s), (\sigma(\varepsilon s, u_\varepsilon(s)) - \sigma(\varepsilon s, u_{n,\varepsilon}(s)))dW(s)\rangle.
\end{aligned}$$

Notice that by the property of the trilinear form b and Lemma A.3, we have

$$\begin{aligned}
&|b(u_\varepsilon, u_\varepsilon, u_\varepsilon - u_{n,\varepsilon}) - b(u_{n,\varepsilon}, u_{n,\varepsilon}, u_\varepsilon - u_{n,\varepsilon})|\\
=&|b(u_\varepsilon, u_\varepsilon, u_\varepsilon - u_{n,\varepsilon}) - b(u_{n,\varepsilon}, u_\varepsilon, u_\varepsilon - u_{n,\varepsilon})|\\
=&|b(u_\varepsilon - u_{n,\varepsilon}, u_\varepsilon, u_\varepsilon - u_{n,\varepsilon})|\\
\leqslant&\frac{1}{2}\|\partial_1(u_\varepsilon - u_{n,\varepsilon})\|_H^2 + C_1 U_\varepsilon\|u_\varepsilon - u_{n,\varepsilon}\|_H^2,
\end{aligned}$$

where C_1 is a constant.

Therefore,

$$\begin{aligned}
&e^{-\varepsilon k\int_0^t U_\varepsilon(s)ds}\|u_\varepsilon(t) - u_{n,\varepsilon}(t)\|_H^2\\
\leqslant&\|u_0 - u_{n,0}\|^2_{\tilde{H}^{0,1}} - k\varepsilon\int_0^t e^{-\varepsilon k\int_0^s U_\varepsilon(r)dr}U_\varepsilon(s)\|u_\varepsilon(s) - u_{n,\varepsilon}(s)\|_H^2 ds\\
&+ 2\varepsilon C_1\int_0^t e^{-\varepsilon k\int_0^s U_\varepsilon(r)dr}U_\varepsilon(s)\|u_\varepsilon(s) - u_{n,\varepsilon}(s)\|_H^2 ds\\
&+ L\varepsilon\int_0^t e^{-\varepsilon k\int_0^s U_\varepsilon(r)dr}\|u_\varepsilon(s) - u_{n,\varepsilon}(s)\|_H^2 ds\\
&+ 2\sqrt{\varepsilon}\int_0^t e^{-\varepsilon k\int_0^s U_\varepsilon(r)dr}\langle u_\varepsilon(s) - u_{n,\varepsilon}(s), (\sigma(\varepsilon s, u_\varepsilon(s)) - \sigma(\varepsilon s, u_{n,\varepsilon}(s)))dW(s)\rangle,
\end{aligned}$$

where we used (A3') in the forth line.

Choosing $k > 2C_1$ and using Lemma 2.3 and (A3'), by the similar calculation as in the proof of Lemma 4.4 we have for $p \geqslant 2$

$$
\begin{aligned}
&\left(E\left[\sup_{0\leqslant s\leqslant t\wedge\tau_{M_1,\varepsilon}\wedge\tau'_{M_2,\varepsilon}} e^{-\varepsilon k\int_0^s U_\varepsilon(r)dr}\|u_\varepsilon(s)-u_{n,\varepsilon}(s)\|_H^2\right]^p\right)^{\frac{2}{p}}\\
\leqslant& 2\|u_0-u_{n,0}\|_{\tilde{H}^{0,1}}^4 + C\varepsilon^2\int_0^t\left(E\left[\sup_{0\leqslant r\leqslant s\wedge\tau_{M_1,\varepsilon}\wedge\tau'_{M_2,\varepsilon}} e^{-\varepsilon k\int_0^r U_\varepsilon(l)dl}\|u_\varepsilon(r)-u_{n,\varepsilon}(r)\|_H^2\right]^p\right)^{\frac{2}{p}}ds\\
&+C\varepsilon p\int_0^t\left(E\left[\sup_{0\leqslant r\leqslant s\wedge\tau_{M_1,\varepsilon}\wedge\tau'_{M_2,\varepsilon}} e^{-\varepsilon k\int_0^r U_\varepsilon(l)dl}\|u_\varepsilon(r)-u_{n,\varepsilon}(r)\|_H^2\right]^p\right)^{\frac{2}{p}}ds.
\end{aligned}
$$

Applying Gronwall's inequality, we obtain

$$
\left(E\left[\sup_{0\leqslant s\leqslant t\wedge\tau_{M_1,\varepsilon}\wedge\tau'_{M_2,\varepsilon}} e^{-\varepsilon k\int_0^s U_\varepsilon(r)dr}\|u_\varepsilon(s)-u_{n,\varepsilon}(s)\|_H^2\right]^p\right)^{\frac{2}{p}} \leqslant C\|u_0-u_{n,0}\|_{\tilde{H}^{0,1}}^4 e^{C(\varepsilon^2+\varepsilon p)}.
$$

Hence, by the definition of the stopping times,

$$
\begin{aligned}
&\left(E\left[\sup_{0\leqslant s\leqslant \tau_{M_1,\varepsilon}\wedge\tau'_{M_2,\varepsilon}} \|u_\varepsilon(s)-u_{n,\varepsilon}(s)\|_H^2\right]^p\right)^{\frac{2}{p}}\\
\leqslant&\left(E\left[\Big(\sup_{0\leqslant s\leqslant \tau_{M_1,\varepsilon}\wedge\tau'_{M_2,\varepsilon}} e^{-\varepsilon k\int_0^s U_\varepsilon(r)dr}\|u_\varepsilon(s)-u_{n,\varepsilon}(s)\|_H^2\Big)^p e^{kp\varepsilon\int_0^{\tau_{M_1,\varepsilon}\wedge\tau'_{M_2,\varepsilon}}U_\varepsilon(s)ds}\right]\right)^{\frac{2}{p}}\\
\leqslant& e^{C(\varepsilon+M_2)k}\left(E\left[\sup_{0\leqslant s\leqslant \tau_{M_1,\varepsilon}\wedge\tau'_{M_2,\varepsilon}} e^{-\varepsilon k\int_0^s U_\varepsilon(r)dr}\|u_\varepsilon(s)-u_{n,\varepsilon}(s)\|_H^2\right]^p\right)^{\frac{2}{p}}\\
\leqslant& Ce^{C(\varepsilon+M_2)k}\|u_0-u_{n,0}\|_{\tilde{H}^{0,1}}^4 e^{C(\varepsilon^2+\varepsilon p)}.
\end{aligned}
$$

Fix M_1, M_2, let $p=\frac{2}{\varepsilon}$, then Chebyshev's inequality implies that

$$
\begin{aligned}
&\sup_{0<\varepsilon\leqslant 1}\varepsilon\log P\left(\sup_{0\leqslant t\leqslant \tau_{M_1,\varepsilon}\wedge\tau'_{M_2,\varepsilon}}\|u_{n,\varepsilon}(t)-u_\varepsilon(t)\|_H^2>\delta\right)\\
\leqslant&\sup_{0<\varepsilon\leqslant 1}\varepsilon\log\frac{E\left[\sup_{0\leqslant t\leqslant \tau_{M_1,\varepsilon}\wedge\tau'_{M_2,\varepsilon}}\|u_{n,\varepsilon}(t)-u_\varepsilon(t)\|_H^{2p}\right]}{\delta^p}\\
\leqslant& C(\varepsilon+M_2)-2\log\delta+\log\|u_0-u_{n,0}\|_{\tilde{H}^{0,1}}^4+C(\varepsilon^2+\varepsilon p)+C\\
\to&-\infty,\ \text{ as } n\to\infty.
\end{aligned}
$$

By Lemma 4.2, for any $R>0$, there exists a constant M_1 such that for any $\varepsilon\in(0,1]$,

$$
P\left(F_{u_\varepsilon}(T)>M_1\right)\leqslant e^{-\frac{R}{\varepsilon}}.
$$

For such a M_1, by Lemma 4.4, there exists a constant M_2 such that for any $\varepsilon \in (0,1]$,

$$P\left(G_{u_\varepsilon}(\tau_{M_1,\varepsilon}) > M_2\right) \leqslant e^{-\frac{R}{\varepsilon}}.$$

For such M_1, M_2, there exists a positive integer N, such that for any $n \geqslant N$ and $\varepsilon \in (0,1]$,

$$P\left(\sup_{0\leqslant t\leqslant \tau_{M_1,\varepsilon}\wedge\tau'_{M_2,\varepsilon}} \|u_{n,\varepsilon}(t) - u_\varepsilon(t)\|_H^2 > \delta\right) \leqslant e^{-\frac{R}{\varepsilon}}.$$

Then by (4.3), we see that there exists a positive integer N, such that for any $n \geqslant N$, $\varepsilon \in (0,1]$,

$$P\left(\sup_{0\leqslant t\leqslant T} \|u_{n,\varepsilon}(t) - u_\varepsilon(t)\|_H^2 > \delta\right) \leqslant 3e^{-\frac{R}{\varepsilon}}.$$

Since R is arbitrary, the lemma follows. $\square$

The following lemma for v_ε is from [XZ09]:

Lemma 4.8 ([XZ09, Lemma 3.4])**.** *For any $\delta > 0$,*

$$\lim_{n\to\infty}\sup_{0<\varepsilon\leqslant 1} \varepsilon \log P\left(\sup_{0\leqslant t\leqslant T} \|v_{n,\varepsilon}(t) - v_\varepsilon(t)\|_H^2 > \delta\right) = -\infty.$$

Lemma 4.9. *For any $\delta > 0$, and every positive integer n,*

$$\lim_{\varepsilon\to 0} \varepsilon \log P\left(\sup_{0\leqslant t\leqslant T} \|u_{n,\varepsilon}(t) - v_{n,\varepsilon}(t)\|_H^2 > \delta\right) = -\infty.$$

Proof. For $M > 0$, recall the definition of $\tau^n_{M,\varepsilon}$ and define the following random time:

$$\tau^{2,n}_{M,\varepsilon} := T \wedge \inf\{t : \|u_{n,\varepsilon}(t)\|^2_{\tilde{H}^{0,1}} > M, \text{ or } \varepsilon\int_0^t \|u_{n,\varepsilon}(s)\|^2_{\tilde{H}^{1,1}} ds > M\},$$

which is a stopping time with respect to $\mathcal{F}_{t+}$ by Lemma 4.3.

Moreover, define

$$\tau^{3,n}_{M,\varepsilon} := T \wedge \inf\{t : \|v_{n,\varepsilon}(t)\|^2_V > M\},$$

$$\tau^{1,n}_{M,\varepsilon} := \tau^n_{M,\varepsilon} \wedge \tau^{3,n}_{M,\varepsilon}.$$

We should point out that $\tau^{3,n}_{M,\varepsilon}$ is a stopping time with respect to $\mathcal{F}_t$ under the condition $v_{n,\varepsilon} \in C([0,T], V)$. Now we prove that $v_{n,\varepsilon} \in C([0,T], V)$.

By Itô's formula and Gronwall's inequality there exists a constant $C(\varepsilon)$ such that

$$E(\sup_{s\in[0,t]} \|v_{n,\varepsilon}(s)\|^2_V) \leqslant C(\varepsilon).$$

For $0 \leqslant s < t \leqslant T$, by (A4) we have

$$
\begin{aligned}
E\|v_{n,\varepsilon}(t) - v_{n,\varepsilon}(s)\|_V^2 \leqslant & \varepsilon E \int_s^t \|\sigma(\varepsilon r, v_{n,\varepsilon}(r))\|_{L_2(l^2,V)}^2 dr \\
\leqslant & \varepsilon \int_s^t (\overline{K}_0 + \overline{K}_1 E(\sup_{l\in[0,r]} \|v_{n,\varepsilon}(l)\|_V^2)) dr \\
\leqslant & \varepsilon(\overline{K}_0 + \overline{K}_1 C(\varepsilon))|t-s|.
\end{aligned}
$$

Then Kolmogorov's continuity criterion implies that $v_{n,\varepsilon} \in C([0,T], V)$.

Now for $M_1, M_2 > 0$, similarly to (4.3), we have

$$
\begin{aligned}
& P\left(\sup_{0\leqslant t\leqslant T} \|u_{n,\varepsilon}(t) - v_{n,\varepsilon}(t)\|_H^2 > \delta\right) \\
\leqslant & P\left(\sup_{0\leqslant t\leqslant \tau^{1,n}_{M_1,\varepsilon}\wedge\tau^{2,n}_{M_2,\varepsilon}} \|u_{n,\varepsilon}(t) - v_{n,\varepsilon}(t)\|_H^2 > \delta\right) \\
& + P(F_{u_{n,\varepsilon}}(T) > M_1) + P(G_{u_{n,\varepsilon}}(\tau^n_{M_1,\varepsilon}) > M_2) + P\left(\sup_{0\leqslant t\leqslant T} \|v_{n,\varepsilon}(t)\|_V^2 > M_1\right)
\end{aligned}
\tag{4.4}
$$

Let $U_{n,\varepsilon} = 1 + \|u_{n,\varepsilon}\|_{\tilde{H}^{1,1}}^2$, applying Itô's formula to $e^{-k\varepsilon\int_0^t U_{n,\varepsilon}(s)ds}\|u_{n,\varepsilon}(t) - v_{n,\varepsilon}(t)\|_H^2$ for some constant $k > 0$, we get

$$
\begin{aligned}
& e^{-k\varepsilon\int_0^t U_{n,\varepsilon}(s)ds}\|u_{n,\varepsilon}(t) - v_{n,\varepsilon}(t)\|_H^2 + 2\varepsilon \int_0^t e^{-k\varepsilon\int_0^s U_{n,\varepsilon}(r)dr} \|\partial_1(u_{n,\varepsilon}(s) - v_{n,\varepsilon}(s))\|_H^2 ds \\
= & -k\varepsilon \int_0^t e^{-k\varepsilon\int_0^s U_{n,\varepsilon}(r)dr} U_{n,\varepsilon}(s)\|u_{n,\varepsilon}(s) - v_{n,\varepsilon}(s)\|_H^2 ds \\
& + 2\varepsilon \int_0^t e^{-k\varepsilon\int_0^s U_{n,\varepsilon}(r)dr} \langle u_{n,\varepsilon}(s) - v_{n,\varepsilon}(s), \partial_1^2 v_{n,\varepsilon}(s)\rangle ds \\
& - 2\varepsilon \int_0^t e^{-k\varepsilon\int_0^s U_{n,\varepsilon}(r)dr} b(u_{n,\varepsilon}(s), u_{n,\varepsilon}(s), u_{n,\varepsilon}(s) - v_{n,\varepsilon}(s)) ds \\
& + \varepsilon \int_0^t e^{-k\varepsilon\int_0^s U_{n,\varepsilon}(r)dr} \|\sigma(\varepsilon s, u_{n,\varepsilon}(s)) - \sigma(\varepsilon s, v_{n,\varepsilon}(s))\|_{L_2(l^2,H)}^2 ds \\
& + 2\sqrt{\varepsilon} \int_0^t e^{-k\varepsilon\int_0^s U_{n,\varepsilon}(r)dr} \langle u_{n,\varepsilon}(s) - v_{n,\varepsilon}(s), (\sigma(\varepsilon s, u_{n,\varepsilon}(s)) - \sigma(\varepsilon s, v_{n,\varepsilon}(s)))dW(s)\rangle.
\end{aligned}
\tag{4.5}
$$

For the second term on the right hand side of (4.5), we have

$$
\begin{aligned}
& \Big|\int_0^t e^{-k\varepsilon\int_0^s U_{n,\varepsilon}(r)dr} \langle u_{n,\varepsilon}(s) - v_{n,\varepsilon}(s), \partial_1^2 v_{n,\varepsilon}(s)\rangle ds\Big| \\
\leqslant & \int_0^t e^{-k\varepsilon\int_0^s U_{n,\varepsilon}(r)dr} \|\partial_1(u_{n,\varepsilon}(s) - v_{n,\varepsilon}(s))\|_H \|\partial_1 v_{n,\varepsilon}(s)\|_H ds \\
\leqslant & \frac{1}{4}\int_0^t e^{-k\varepsilon\int_0^s U_{n,\varepsilon}(r)dr} \|\partial_1(u_{n,\varepsilon}(s) - v_{n,\varepsilon}(s))\|_H^2 ds + C\int_0^t e^{-k\varepsilon\int_0^s U_{n,\varepsilon}(r)dr} \|v_{n,\varepsilon}(s)\|_V^2 ds,
\end{aligned}
$$

where we use Young's inequality in the last inequality.

For the third term on the right hand side of (4.5), by Lemmas A.3 and A.4 we have

$$
\begin{aligned}
&|b(u_{n,\varepsilon}, u_{n,\varepsilon}, u_{n,\varepsilon} - v_{n,\varepsilon})| \\
=&|b(u_{n,\varepsilon} - v_{n,\varepsilon}, u_{n,\varepsilon}, u_{n,\varepsilon} - v_{n,\varepsilon}) + b(v_{n,\varepsilon}, u_{n,\varepsilon}, u_{n,\varepsilon} - v_{n,\varepsilon}| \\
\leqslant&\frac{1}{4}\|\partial_1(u_{n,\varepsilon} - v_{n,\varepsilon})\|_H^2 + CU_{n,\varepsilon}\|u_{n,\varepsilon} - v_{n,\varepsilon}\|_H^2 + C\|v_{n,\varepsilon}\|_V\|u_{n,\varepsilon}\|_{\tilde{H}^{1,1}}\|u_{n,\varepsilon} - v_{n.\varepsilon}\|_H \\
\leqslant&\frac{1}{4}\|\partial_1(u_{n,\varepsilon} - v_{n,\varepsilon})\|_H^2 + C\|v_{n,\varepsilon}\|_V^2 + C_1U_{n,\varepsilon}\|u_{n,\varepsilon} - v_{n,\varepsilon}\|_H^2,
\end{aligned}
\tag{4.6}
$$

where C_1 is a constant.

Thus we obtain

$$
\begin{aligned}
&e^{-k\varepsilon\int_0^t U_{n,\varepsilon}(s)ds}\|u_{n,\varepsilon}(t) - v_{n,\varepsilon}(t)\|_H^2 + \varepsilon\int_0^t e^{-k\varepsilon\int_0^s U_{n,\varepsilon}(r)dr}\|\partial_1(u_{n,\varepsilon}(s) - v_{n,\varepsilon}(s))\|_H^2 ds \\
\leqslant& -k\varepsilon\int_0^t e^{-k\varepsilon\int_0^s U_{n,\varepsilon}(r)dr}U_{n,\varepsilon}(s)\|u_{n,\varepsilon}(s) - v_{n,\varepsilon}(s)\|_H^2 ds + C\varepsilon\int_0^t e^{-k\varepsilon\int_0^s U_{n,\varepsilon}(r)dr}\|v_{n,\varepsilon}(s)\|_V^2 ds \\
&+ C_1\varepsilon\int_0^t e^{-k\varepsilon\int_0^s U_{n,\varepsilon}(r)dr}U_{n,\varepsilon}(s)\|u_{n,\varepsilon}(s) - v_{n,\varepsilon}(s)\|_H^2 ds \\
&+ L_1\varepsilon\int_0^t e^{-k\varepsilon\int_0^s U_{n,\varepsilon}(r)dr}\|u_{n,\varepsilon}(s) - v_{n,\varepsilon}(s)\|_H^2 ds \\
&+ 2\sqrt{\varepsilon}\int_0^t e^{-k\varepsilon\int_0^s U_{n,\varepsilon}(r)dr}\langle u_{n,\varepsilon}(s) - v_{n,\varepsilon}(s), (\sigma(\varepsilon s, u_{n,\varepsilon}(s)) - \sigma(\varepsilon s, v_{n,\varepsilon}(s)))dW(s)\rangle,
\end{aligned}
$$

where we used (A3') in the fourth line.

Hence, choosing $k > C_1 + C_2$, by Lemma 2.3 and the similar techniques in the previous lemma and the definition of stopping times, we deduce that for $p \geqslant 2$

$$
\begin{aligned}
&\left(E\left[\sup_{0\leqslant s\leqslant t\wedge\tau^{1,n}_{M_1,\varepsilon}\wedge\tau^{2,n}_{M_2,\varepsilon}} e^{-k\varepsilon\int_0^s U_{n,\varepsilon}(r)dr}\|u_{n,\varepsilon}(s) - v_{n,\varepsilon}(s)\|_H^2\right]^p\right)^{\frac{2}{p}} \\
\leqslant& CM_1^2\varepsilon^2 + C(\varepsilon^2 + \varepsilon p)\int_0^t\left(E\left[\sup_{0\leqslant r\leqslant s\wedge\tau^{1,n}_{M_1,\varepsilon}\wedge\tau^{2,n}_{M_2,\varepsilon}} e^{-k\varepsilon\int_0^r U_{n,\varepsilon}(l)dl}\|u_{n,\varepsilon}(r) - v_{n,\varepsilon}(r)\|_H^2\right]^p\right)^{\frac{2}{p}} ds.
\end{aligned}
$$

Then Gronwall's inequality implies that

$$
\begin{aligned}
&\left(E\left[\sup_{0\leqslant t\leqslant\tau^{1,n}_{M_1,\varepsilon}\wedge\tau^{2,n}_{M_2,\varepsilon}} \|u_{n,\varepsilon}(t) - v_{n,\varepsilon}(t)\|_H^2\right]^p\right)^{\frac{2}{p}} \\
\leqslant&\left(E\left[\sup_{0\leqslant t\leqslant\tau^{1,n}_{M_1,\varepsilon}\wedge\tau^{2,n}_{M_2,\varepsilon}} (e^{-k\varepsilon\int_0^t U_{n,\varepsilon}(s)ds}\|u_{n,\varepsilon}(t) - v_{n,\varepsilon}(t)\|_H^2)^p e^{kp\varepsilon\int_0^{\tau^{1,n}_{M_1,\varepsilon}\wedge\tau^{2,n}_{M_2,\varepsilon}} U_{n,\varepsilon}(s)ds}\right]\right)^{\frac{2}{p}} \\
\leqslant& e^{C(\varepsilon+M_2)}CM_1^2\varepsilon^2 e^{C(\varepsilon^2+\varepsilon p)}.
\end{aligned}
\tag{4.7}
$$

By Lemmas 4.5 and 4.6, we know that for any $R > 0$, there exists M_1 such that

$$\sup_{0<\varepsilon\leqslant 1} \varepsilon \log P\left(F_{u_{n,\varepsilon}}(T) > M_1\right) \leqslant -R,$$

$$\sup_{0<\varepsilon\leqslant 1} \varepsilon \log P\left(\sup_{0\leqslant t\leqslant T} \|v_{n,\varepsilon}(t)\|_V^2 > M_1\right) \leqslant -R.$$

For such a constant M_1, by Lemma 4.5, there exists M_2 such that

$$\sup_{0<\varepsilon\leqslant 1} \varepsilon \log P\left(G_{u_{n,\varepsilon}}(\tau^n_{M_1,\varepsilon}) > M_2\right) \leqslant -R.$$

Then for such M_1, M_2, let $p = \frac{2}{\varepsilon}$ in (4.7), we obtain

$$\begin{aligned}
&\varepsilon \log P\left(\sup_{0\leqslant t\leqslant \tau^{1,n}_{M_1,\varepsilon}\wedge\tau^{2,n}_{M_2,\varepsilon}} \|u_{n,\varepsilon}(t) - v_{n,\varepsilon}(t)\|_H^2 > \delta\right)\\
\leqslant &\log\left(E\left[\sup_{0\leqslant t\leqslant \tau^{1,n}_{M_1,\varepsilon}\wedge\tau^{2,n}_{M_2,\varepsilon}} \|u_{n,\varepsilon}(t) - v_{n,\varepsilon}(t)\|_H^2\right]^p\right)^{\frac{2}{p}} - \log\delta^2\\
\leqslant &C(\varepsilon + M_2) + \log[CM_1^2\varepsilon^2] + C(\varepsilon^2+1) - \log\delta^2\\
\to &-\infty \quad \text{as } \varepsilon \to 0,
\end{aligned}$$

where we used Chebyshev's inequality in the first inequality. Thus there exists a $\varepsilon_0 \in (0,1)$ such that for any $\varepsilon \in (0, \varepsilon_0)$,

$$P\left(\sup_{0\leqslant t\leqslant \tau^{1,n}_{M_1,\varepsilon}\wedge\tau^{2,n}_{M_2,\varepsilon}} \|u_{n,\varepsilon}(t) - v_{n,\varepsilon}(t)\|_H^2 > \delta\right) \leqslant e^{-\frac{R}{\varepsilon}}.$$

Putting the above estimate together, by (4.4) we see that for $\varepsilon \in (0, \varepsilon_0)$

$$P\left(\sup_{0\leqslant t\leqslant T} \|u_{n,\varepsilon}(t) - v_{n,\varepsilon}(t)\|_H^2 > \delta\right) \leqslant 4e^{-\frac{R}{\varepsilon}}.$$

Since R is arbitrary, we finish the proof.

□

Proof of Theorem 4.1. By Lemma 4.1, v_ε satisfies a large deviation principle with the rate function I^{u_0}. Our task remain is to show that u_ε and v_ε are exponentially equivalent, then the result follows from Lemma 2.1.

By Lemmas 4.7 and 4.8, for any $R > 0$, there exists a N_0 such that for any $\varepsilon \in (0, 1]$,

$$P\left(\sup_{0\leqslant t\leqslant T} \|u_\varepsilon(t) - u_{N_0,\varepsilon}(t)\|_H^2 > \frac{\delta}{3}\right) \leqslant e^{-\frac{R}{\varepsilon}},$$

and

$$P\left(\sup_{0\leqslant t\leqslant T} \|v_\varepsilon(t) - v_{N_0,\varepsilon}(t)\|_H^2 > \frac{\delta}{3}\right) \leqslant e^{-\frac{R}{\varepsilon}}.$$

Then by Lemma 4.9, for such N_0, there exists a ε_0 such that for any $\varepsilon \in (0, \varepsilon_0)$,

$$P\left(\sup_{0\leqslant t\leqslant T} \|u_{N_0,\varepsilon}(t) - v_{N_0,\varepsilon}(t)\|_H^2 > \frac{\delta}{3}\right) \leqslant e^{-\frac{R}{\varepsilon}}.$$

Therefore we deduce that for $\varepsilon \in (0, \varepsilon_0)$

$$P\left(\sup_{0\leqslant t\leqslant T} \|u_\varepsilon(t) - v_\varepsilon(t)\|_H^2 > \delta\right) \leqslant 3e^{-\frac{R}{\varepsilon}}.$$

Since R is arbitrary, we finish the proof. $\square$

A Appendix

We now present several lemmas from [LZZ18]. It follows from Minkowski inequality that

Lemma A.1. *For $1 \leqslant q \leqslant p \leqslant \infty$, we have*

$$\|u\|_{L_h^p(L_v^q)} \leqslant \|u\|_{L_v^q(L_h^p)},$$

$$\|u\|_{L_v^p(L_h^q)} \leqslant \|u\|_{L_h^q(L_v^p)}.$$

Lemma A.2 ([LZZ18, Lemma 3.4])**.** *Let u be a smooth function from $\mathbb{T}^2$ to $\mathbb{R}$, we have*

$$\|u\|^2_{L_v^2(L_h^\infty)} \leqslant C(\|u\|_{L^2}\|\partial_1 u\|_{L^2} + \|u\|^2_{L^2}),$$

$$\|u\|^2_{L_h^2(L_v^\infty)} \leqslant C(\|u\|_{L^2}\|\partial_2 u\|_{L^2} + \|u\|^2_{L^2}).$$

The following anisotropic estimate is from the proof of [LZZ18, Theorem 3.1]:

Lemma A.3. *For smooth functions u, v from $\mathbb{T}^2$ to $\mathbb{R}$ with u satisfies the divergence free condition, we have*

$$\begin{aligned}|b(u,v,u)| \leqslant a\|\partial_1 u\|^2_{L^2} + C\|u\|^2_{L^2}\Big(&\|\partial_1 v\|^{\frac{2}{3}}_{L^2}\|\partial_1\partial_2 v\|^{\frac{2}{3}}_{L^2} + \|\partial_2 v\|^{\frac{2}{3}}_{L^2}\|\partial_1\partial_2 v\|^{\frac{2}{3}}_{L^2}\\ &+ \|\partial_1 v\|^2_{L^2} + \|\partial_1 v\|_{L^2} + \|\partial_2 v\|^2_{L^2} + \|\partial_2 v\|_{L^2}\\ &+ \|\partial_1 v\|^{\frac{1}{2}}_{L^2}\|\partial_1\partial_2 v\|^{\frac{1}{2}}_{L^2} + \|\partial_2 v\|^{\frac{1}{2}}_{L^2}\|\partial_1\partial_2 v\|^{\frac{1}{2}}_{L^2}\Big),\end{aligned}$$

where $a > 0$ is a constant small enough.

In particular, we have

$$|b(u,v,u)| \leqslant a\|\partial_1 u\|^2_{L^2} + C\|u\|^2_{L^2}(1 + \|v\|^2_{H^{1,1}}).$$

Proof. We have

$$\begin{aligned}|b(u,v,u)| &= |\langle u^1\partial_1 v + u^2\partial_2 v, u\rangle|\\ &\leqslant (\|u^1\|_{L_h^\infty(L_v^2)}\|\partial_1 v\|_{L_h^2(L_v^\infty)} + \|u^2\|_{L_h^2(L_v^\infty)}\|\partial_2 v\|_{L_h^\infty(L_v^2)})\|u\|_{L^2},\end{aligned}$$

where $u=(u^1,u^2)$. Now we show the calculation of two terms in the right hand side separately.

For the first term, by Lemmas A.1 and A.2, we have

$$\begin{aligned}
&\|u^1\|_{L_h^\infty(L_v^2)}\|\partial_1 v\|_{L_h^2(L_v^\infty)}\|u\|_{L^2}\\
\leqslant & C\|u\|_{L^2}\left(\|u^1\|_{L^2}\|\partial_1 u^1\|_{L^2}+\|u^1\|_{L^2}^2\right)^{\frac12}\left(\|\partial_1 v\|_{L^2}\|\partial_1\partial_2 v\|_{L^2}+\|\partial_1 v\|_{L^2}^2\right)^{\frac12}\\
\leqslant & C\|u\|_{L^2}\left(\|u^1\|_{L^2}\|\partial_1 u^1\|_{L^2}\|\partial_1 v\|_{L^2}\|\partial_1\partial_2 v\|_{L^2}\right)^{\frac12}+C\|u\|_{L^2}\|u^1\|_{L^2}\|\partial_1 v\|_{L^2}\\
&+C\|u\|_{L^2}(\|u^1\|_{L^2}+\|\partial_1 u^1\|_{L^2})\|\partial_1 v\|_{L^2}+C\|u\|_{L^2}\|u^1\|_{L^2}\|\partial_1 v\|_{L^2}^{\frac12}\|\partial_1\partial_2 v\|_{L^2}^{\frac12}.
\end{aligned}$$

Then Young's inequality implies that

$$\begin{aligned}
&C\|u\|_{L^2}\left(\|u^1\|_{L^2}\|\partial_1 u^1\|_{L^2}\|\partial_1 v\|_{L^2}\|\partial_1\partial_2 v\|_{L^2}\right)^{\frac12}\\
\leqslant&\frac{a}{4}\|\partial_1 u\|_{L^2}^2+C\|\partial_1 v\|_{L^2}^{\frac23}\|\partial_1\partial_2 v\|_{L^2}^{\frac23}\|u\|_{L^2}^2,
\end{aligned}$$

and

$$C\|u\|_{L^2}\|\partial_1 u^1\|_{L^2}\|\partial_1 v\|_{L^2}\leqslant\frac{a}{4}\|\partial_1 u\|_{L^2}^2+C\|\partial_1 v\|_{L^2}^2\|u\|_{L^2}^2.$$

Thus we have

$$\begin{aligned}
&\|u^1\|_{L_h^\infty(L_v^2)}\|\partial_1 v\|_{L_h^2(L_v^\infty)}\|u\|_{L^2}\\
\leqslant&\frac{a}{2}\|\partial_1 u\|_{L^2}^2+C\|u\|_{L^2}^2\Big(\|\partial_1 v\|_{L^2}^{\frac23}\|\partial_1\partial_2 v\|_{L^2}^{\frac23}+\|\partial_1 v\|_{L^2}^2+\|\partial_1 v\|_{L^2}+\|\partial_1 v\|_{L^2}^{\frac12}\|\partial_1\partial_2 v\|_{L^2}^{\frac12}\Big).
\end{aligned}$$

Do the same calculation for the second term and combine the divergence free condition $\partial_2 u^2=-\partial_1 u^1$, we have

$$\begin{aligned}
&\|u^2\|_{L_h^2(L_v^\infty)}\|\partial_2 v\|_{L_h^\infty(L_v^2)}\|u\|_{L^2}\\
\leqslant&\frac{a}{2}\|\partial_1 u\|_{L^2}^2+C\|u\|_{L^2}^2\Big(\|\partial_2 v\|_{L^2}^{\frac23}\|\partial_1\partial_2 v\|_{L^2}^{\frac23}+\|\partial_2 v\|_{L^2}^2+\|\partial_2 v\|_{L^2}+\|\partial_2 v\|_{L^2}^{\frac12}\|\partial_1\partial_2 v\|_{L^2}^{\frac12}\Big),
\end{aligned}$$

which implies the first inequality.

The second inequality holds from the first one and Young's Inequality.

□

Similar to the proof of Lemma A.3, by Lemmas A.1 and A.2, we also have

Lemma A.4. *For smooth functions u,v,w form $\mathbb{T}^2$ to $\mathbb{R}^2$ with divergence free condition, we have*

$$|b(u,v,w)|\leqslant C\|u\|_{H^1}\|v\|_{H^{1,1}}\|w\|_{L^2}.$$

Proof.

$$\begin{aligned}
&|b(u,v,w)|\\
\leqslant&(\|u^1\|_{L_h^\infty(L_v^2)}\|\partial_1 v\|_{L_h^2(L_v^\infty)}+\|u^2\|_{L_h^2(L_v^\infty)}\|\partial_2 v\|_{L_h^\infty(L_v^2)})\|w\|_{L^2}\\
\leqslant&C\Big((\|u^1\|_{L^2}\|\partial_1 u^1\|_{L^2}+\|u^1\|_{L^2}^2)^{\frac12}(\|\partial_1 v\|_{L^2}\|\partial_1\partial_2 v\|_{L^2}+\|\partial_1 v\|_{L^2}^2)^{\frac12}\\
&+(\|u^2\|_{L^2}\|\partial_2 u^2\|_{L^2}+\|u^2\|_{L^2}^2)^{\frac12}(\|\partial_2 v\|_{L^2}\|\partial_1\partial_2 v\|_{L^2}+\|\partial_2 v\|_{L^2}^2)^{\frac12}\Big)\|w\|_{L^2}\\
\leqslant&C\|u\|_{H^1}\|v\|_{H^{1,1}}\|w\|_{L^2}.
\end{aligned}$$

□

The next lemma is from the proof of [LZZ18, Lemma 3.5], which plays an important role in $H^{0,1}$-estimate.

Lemma A.5. *For smooth function u form $\mathbb{T}^2$ to $\mathbb{R}^2$ with divergence free condition, we have*

$$|\langle \partial_2 u, \partial_2(u\cdot\nabla u)\rangle| \leqslant a\|\partial_1\partial_2 u\|_{L^2}^2 + C(1+\|\partial_1 u\|_{L^2}^2)\|\partial_2 u\|_{L^2}^2,$$

where $a>0$ is a constant small enough.

Proof. We have

$$\langle \partial_2 u, \partial_2(u\cdot\nabla u)\rangle = \langle \partial_2 u^1, \partial_2(u\cdot\nabla u^1)\rangle + \langle \partial_2 u^2, \partial_2(u\cdot\nabla u^2)\rangle,$$

where $u=(u^1,u^2)$.

For the first term on the right hand side, we have

$$\begin{aligned}
\langle \partial_2 u^1, \partial_2(u\cdot\nabla u^1)\rangle =&\langle \partial_2 u^1, \partial_2(u^1\partial_1 u^1 + u^2\partial_2 u^1)\rangle\\
=&\langle \partial_2 u^1, \partial_2 u^1\partial_1 u^1\rangle + \langle \partial_2 u^1, u^1\partial_2\partial_1 u^1\rangle\\
&+\langle \partial_2 u^1, \partial_2 u^2\partial_2 u^1\rangle + \langle \partial_2 u^1, u^2\partial_2^2 u^1\rangle\\
=&\langle \partial_2 u^1, u^1\partial_2\partial_1 u^1\rangle + \langle \partial_2 u^1, u^2\partial_2^2 u^1\rangle\\
=&\langle \partial_2 u^1, u\cdot\nabla\partial_2 u^1\rangle\\
=&-\frac{1}{2}\int \mathrm{div}\ u|\partial_2 u^1|^2dx\\
=&0,
\end{aligned}$$

where we use the fact div $u=0$ in the third and sixth equality.

Similarly, for the second term, we have

$$\begin{aligned}
\langle \partial_2 u^2, \partial_2(u\cdot\nabla u^2)\rangle =&\langle \partial_2 u^2, \partial_2 u^1\partial_1 u^2\rangle + \langle \partial_2 u^2, u^1\partial_2\partial_1 u^2\rangle\\
&+\langle \partial_2 u^2, \partial_2 u^2\partial_2 u^2\rangle + \langle \partial_2 u^2, u^2\partial_2^2 u^2\rangle\\
=&\langle \partial_2 u^2, \partial_2 u^1\partial_1 u^2\rangle + \frac{1}{2}\int u^1\partial_1(\partial_2 u^2)^2dx\\
&+\langle \partial_2 u^2, \partial_2 u^2\partial_2 u^2\rangle + \frac{1}{2}\int u^2\partial_2(\partial_2 u^2)^2dx\\
=&\langle \partial_2 u^2, \partial_2 u^1\partial_1 u^2\rangle + \langle \partial_2 u^2, \partial_2 u^2\partial_2 u^2\rangle\\
&-\frac{1}{2}\langle \partial_2 u^2, \partial_1 u^1\partial_2 u^2\rangle - \frac{1}{2}\langle \partial_2 u^2, \partial_2 u^2\partial_2 u^2\rangle\\
=&\langle \partial_2 u^2, \partial_2 u^1\partial_1 u^2\rangle + \langle \partial_2 u^2, \partial_2 u^2\partial_2 u^2\rangle,
\end{aligned}$$

where we use div $u=0$ in the last equality.

Then by Lemma A.2 we have

$$
\begin{aligned}
&|\langle \partial_2 u, \partial_2(u\cdot\nabla u)\rangle| \\
=&|\langle \partial_2 u^2, \partial_2 u^1 \partial_1 u^2\rangle + \langle \partial_2 u^2, \partial_2 u^2\partial_2 u^2\rangle| \\
\leqslant& \left(\|\partial_2 u^1\|_{L_h^\infty(L_v^2)}\|\partial_1 u^2\|_{L_h^2(L_v^\infty)} + \|\partial_1 u^1\|_{L_h^2(L_v^\infty)}\|\partial_2 u^2\|_{L_h^\infty(L_v^2)}\right)\|\partial_2 u^2\|_{L^2} \\
\leqslant& C\left(\|\partial_2 u\|_{L^2} + \|\partial_2 u\|_{L^2}^{\frac12}\|\partial_1\partial_2 u\|_{L^2}^{\frac12}\right)\left(\|\partial_1 u\|_{L^2} + \|\partial_1 u\|_{L^2}^{\frac12}\|\partial_1\partial_2 u\|_{L^2}^{\frac12}\right)\|\partial_2 u^2\|_{L^2} \\
\leqslant& C\|\partial_1 u\|_{L^2}\|\partial_2 u\|_{L^2}^2 + C\|\partial_1\partial_2 u\|_{L^2}\|\partial_1 u\|_{L^2}\|\partial_2 u\|_{L^2} \\
&+ C\|\partial_1\partial_2 u\|_{L^2}^{\frac12}\left(\|\partial_1 u\|_{L^2}\|\partial_2 u\|_{L^2}^{\frac12} + \|\partial_2 u\|_{L^2}\|\partial_1 u\|_{L^2}^{\frac12}\right)\|\partial_2 u^2\|_{L^2},
\end{aligned}
$$

where we use the following inequality in the last inequality:

$$
\begin{aligned}
&\|\partial_2 u\|_{L^2}^{\frac12}\|\partial_1\partial_2 u\|_{L^2}\|\partial_1 u\|_{L^2}^{\frac12}\|\partial_2 u^2\|_{L^2} \\
=&\|\partial_2 u\|_{L^2}^{\frac12}\|\partial_1\partial_2 u\|_{L^2}\|\partial_1 u\|_{L^2}^{\frac12}\|\partial_1 u^1\|_{L^2}^{\frac12}\|\partial_2 u^2\|_{L^2}^{\frac12} \\
\leqslant&\|\partial_1\partial_2 u\|_{L^2}\|\partial_1 u\|_{L^2}\|\partial_2 u\|_{L^2},
\end{aligned}
$$

where we use div $u = 0$ in the first equality.

By Young's inequality, we have

$$
C\|\partial_1\partial_2 u\|_{L^2}\|\partial_1 u\|_{L^2}\|\partial_2 u\|_{L^2} \leqslant \frac{a}{2}\|\partial_1\partial_2 u\|_{L^2}^2 + C\|\partial_1 u\|_{L^2}^2\|\partial_2 u\|_{L^2}^2,
$$

and

$$
\begin{aligned}
&C\|\partial_1\partial_2 u\|_{L^2}^{\frac12}\left(\|\partial_1 u\|_{L^2}\|\partial_2 u\|_{L^2}^{\frac12} + \|\partial_2 u\|_{L^2}\|\partial_1 u\|_{L^2}^{\frac12}\right)\|\partial_2 u^2\|_{L^2} \\
\leqslant&\frac{a}{2}\|\partial_1\partial_2 u\|_{L^2}^2 + C\left(\|\partial_1 u\|_{L^2}^{\frac43}\|\partial_2 u\|_{L^2}^{\frac23} + \|\partial_2 u\|_{L^2}^{\frac43}\|\partial_1 u\|_{L^2}^{\frac23}\right)\|\partial_2 u^2\|_{L^2}^{\frac43} \\
\leqslant&\frac{a}{2}\|\partial_1\partial_2 u\|_{L^2}^2 + C\|\partial_1 u\|_{L^2}^{\frac43}\|\partial_2 u\|_{L^2}^2 + C\|\partial_2 u\|_{L^2}^{\frac43}\|\partial_1 u\|_{L^2}^{\frac23}\|\partial_1 u^1\|_{L^2}^{\frac23}\|\partial_2 u^2\|_{L^2}^{\frac23} \\
\leqslant&\frac{a}{2}\|\partial_1\partial_2 u\|_{L^2}^2 + C(1+\|\partial_1 u\|_{L^2}^2)\|\partial_2 u\|_{L^2}^2,
\end{aligned}
$$

where we use div $u = 0$ in the second inequality.

Thus we deduce that

$$
|\langle \partial_2 u, \partial_2(u\cdot\nabla u)\rangle| \leqslant a\|\partial_1\partial_2 u\|_{L^2}^2 + C(1+\|\partial_1 u\|_{L^2}^2)\|\partial_2 u\|_{L^2}^2.
$$

□

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