

A DECOMPOSITION OF GENERAL PREMIUM PRINCIPLES INTO RISK AND DEVIATION

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ABSTRACT. In this paper, we provide an axiomatic approach to general premium principles giving rise to a decomposition into risk, as a generalization of the expected value, and deviation, as a generalization of the variance. We show that, for every premium principle, there exists a maximal risk measure capturing all risky components covered by the insurance prices. In a second step, we consider dual representations of convex risk measures consistent with the premium principle. In particular, we show that the convex conjugate of the aforementioned maximal risk measure coincides with the convex conjugate of the premium principle on the set of all finitely additive probability measures. In a last step, we consider insurance prices in the presence of a not necessarily frictionless market, where insurance claims are traded. In this setup, we discuss premium principles that are consistent with hedging using securization products that are traded in the market.

Key words: Principles of premium calculation, risk measure, deviation measure, convex duality, superhedging

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1. INTRODUCTION

In classical risk theory, a premium principle is a map that assigns to a random variable X a real number $H(X)$, the price for insuring the claim X , see Bühlmann [3], Deprez and Gerber [5], Young [24] or, for a textbook reference Rolski et al. [19]. In this approach, it is assumed that the probability distribution of any loss is known. Frequently, however, the probability distribution is not known exactly. The issue of Knightian or model uncertainty has entered the center stage in recent years. The International Actuarial Association acknowledges the importance of such uncertainty in Chapter 17 of the risk book [11]: '*Risk* is the effect of variation that results from the random nature of the outcomes being studied (i.e., a quantity susceptible of measurement). *Uncertainty* involves the degree of confidence in understanding the effect of perils or hazards not easily susceptible to measurement.' Model uncertainty is also widely recognised, for example, in the context of life insurance, cf. Biagini et al. [2], Bauer et al. [1], Milevsky et al. [15], and Schmeck and Schmidli [20].

In this paper, we thus take a more general position and model insurance claims as measurable functions, thus being closer to the actual real world contract. In particular, we do not assume ex ante that the probability distributions of losses are known to the insurer. We impose only two very natural conditions on premium principles. We require that there is no unjustified risk loading, i.e. a shift of a loss by a known amount is priced correctly, or

$$H(X + m) = H(X) - m \quad \text{for all } X \in C \text{ and } m \in \mathbb{R}, \quad (\text{P1})$$

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compare Deprez and Gerber [5] and Young [24]. In the context of monetary risk measures, property (P1) is usually referred to as *cash additivity*, see e.g. Föllmer and Schied [8]. Our second natural requirement has the form

$$H(X) \geq H(0) = 0 \quad \text{for all } X \in C \text{ with } X \leq 0. \quad (\text{P2})$$

Condition (P2) simply states that an insurer will not be willing to pay money for insuring *pure losses*, i.e. claims with only negative outcomes. Since typically insurance claims have only negative outcomes, one could, loosely speaking, interpret (P2) as a condition stating that insurance premia are always nonnegative, a standard requirement, see e.g. Young [24].

Our first main result shows that every insurance premium can be written as

$$H(X) = R(X) + D(X) \quad \text{for all } X \in C,$$

where R is a monetary risk measure (compare, e.g., Föllmer and Schied [8]) and D is a deviation measure (compare Rockafellar and Uryasev [18]). Therefore, the simple axioms (P1) and (P2) immediately give a lot of structure and contain most known examples that are used in practice. In the classic case, when the probability distribution is known, a typical insurance premium consists of the sum of the fair premium and a multiple of the variance or standard deviation, compare [19]. As the expected loss is a risk measure and the variance a deviation measure, we thus show that one can think of insurance premia as generalizations of this basic approach in a very general way.

It is natural to ask in what sense the risk and the deviation measure can be identified uniquely. In general, this is not the case. However, we show that the premium principle can be uniquely decomposed into a *maximal* risk measure R_{Max} (capturing all risky components of the insurance claim) and a *minimal* deviation measure D_{Min} measuring the claim's pure fluctuations. Moreover, we show that R_{Max} can be explicitly read out of the premium principle H and, additionally, be extended from C to the space of *all* bounded random variables. That is, it is possible to explicitly filter the (maximal) risk contribution to the premium principle H and price other insurance contracts (that are not contained in C) in a consistent way. The minimal deviation, i.e. the difference between the premium and the maximal risk measure can be seen as a margin for compensating the parts of the claim that cannot be quantified as pure risk.

We show that the classic premium principles as the mentioned variance or standard deviation principle or the well-known economic principles can be subsumed under our framework. We also discuss generalizations of these classic premium principles to Knightian uncertainty, and we discuss the more modern notions of quantile-based premia involving Value at Risk or Expected Shortfall. Similar to Castagnoli et al. [4], we discuss the case, where

$$H(X) = \mathbb{E}_{\mathbb{P}}(-X) + \text{Amb}_{\mathcal{P}}(X) \quad (1.1)$$

with a fixed baseline model $\mathbb{P} \in \mathcal{P}$ and where

$$\text{Amb}_{\mathcal{P}}(X) := \frac{1}{2} \sup_{\mathbb{Q}, \mathbb{Q}' \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}}(X) - \mathbb{E}_{\mathbb{Q}'}(X)$$

measures the ambiguity of the model. Castagnoli et al. [4] consider premium principles of the form (1.1) together with a no-ripoff condition, which as we will show, implies that the set of priors \mathcal{P} is dominated by the reference measure \mathbb{P} .

In a second step, we assume that, in addition to (P1) and (P2), the premium principle H is convex or sublinear. We then derive a dual representation of R_{Max} in terms of the

Fenchel-Legendre transform of H . In the sublinear case, we show that there exists a maximal set \mathcal{P} of probability measures (priors) satisfying

$$H(X) \geq \mathbb{E}_{\mathbb{P}}(-X) \quad \text{for all } X \in C \quad \text{and } \mathbb{P} \in \mathcal{P}. \quad (1.2)$$

The latter can be seen as a generalized version of a safety loading, see Castagnoli et al. [4] and Young [24]. In the case, where $\mathcal{P} = \{\mathbb{P}\}$ consists of a single prior, one ends up with the classical condition to avoid bankruptcy according to the principle of pooling risk in a large group. We therefore see that, in the sublinear case, the notion a premium principle H covers a certain amount of model uncertainty or ambiguity in terms of the prior. In view of equation (1.2), the set \mathcal{P} can be seen as the set of all priors that are covered by the premium principle in the sense that the premium principle avoids bankruptcy under each model $\mathbb{P} \in \mathcal{P}$. We will therefore also refer to \mathcal{P} as the set of all plausible models. In a last step, we discuss the relation of the maximal risk measure to superhedging in presence of a competitive market without frictions, that is used by the insurer to hedge against certain risks using portfolios or securization products that are traded in the market. In the spirit of Föllmer and Schied [7], we derive equivalent conditions ensuring that the premium principle is consistent with superhedging.

The paper is structured as follows. In Section 2, we introduce the setup and notations, provide the decomposition of a premium principle into risk and deviation, give an explicit description of the maximal risk measure R_{Max} , and discuss various examples illustrating our notion of a premium principle. Section 3 is devoted to the study of convex and sublinear premium principles. In this context, we discuss dual representations, multiple priors, and baseline models. In Section 4, we address the connection between market consistency of insurance premia and hedging using securization products that are traded in a competitive frictionless market. The proofs can be found in the Appendix A.

2. PREMIUM PRINCIPLES AND THEIR DECOMPOSITIONS

2.1. Model and Notation. Let (Ω, \mathcal{F}) be a measurable space. Denote the space of all bounded, real-valued measurable functions by $B_b = B_b(\Omega, \mathcal{F})$. Let $C \subset B_b$ represent the set of insurance claims covered by a premium policy. We assume that $0 \in C$ and that $X + m \in C$ for all $X \in C$ and $m \in \mathbb{R}$, where, in the notation, we do not differentiate between real constants and constant functions (with real values). We call every measurable function $X \in B_b$ a *claim*. We use the notation

$$\max X := \max_{\omega \in \Omega} X(\omega) \quad \text{and} \quad \min X = \min_{\omega \in \Omega} X(\omega).$$

We denote by \leq , both, the usual order on the reals and the pointwise order on B_b .

2.2. Premium Principles and a Basic Decomposition. The central object in our analysis is the following notion of a premium principle.

Definition 2.1. We say that a map $H: C \rightarrow \mathbb{R}$ is a *premium principle* on C if

- (P1) $H(X + m) = H(X) - m$ for all $X \in C$ and $m \in \mathbb{R}$.
- (P2) $H(X) \geq H(0) = 0$ for all $X \in C$ with $X \leq 0$.

Notice that (P1) together with $H(0) = 0$, implies that $H(-m) = m$ for all constant claims leading to the common assumption of *no unjustified risk loading*, cf. Deprez and Gerber [5] and Young [24]. Concerning Property (P2), note that the condition $H(0) = 0$ is natural for insurance claims. A typical policy insures *damages* in the sense

that the claim is either zero or negative. (P2) ensures that the company or the market take a nonnegative premium for sure damages. It is thus a minimal requirement for a sensible notion of premium policy.

Recall that a map $R: C \rightarrow \mathbb{R}$ is a (*monetary*) *risk measure* (see e.g. Föllmer and Schied [8]) if

- (R1) $R(0) = 0$ and $R(X + m) = R(X) - m$ for all $X \in C$ and $m \in \mathbb{R}$,
- (R2) $R(X) \leq R(Y)$ for all $X, Y \in C$ with $X \geq Y$.

A map $D: C \rightarrow \mathbb{R}$ is a *deviation measure* (cf. Rockafellar-Uryasev [18]) if

- (D1) $D(X + m) = D(X)$ for all $X \in C$ and $m \in \mathbb{R}$,
- (D2) $D(0) = 0$ and $D(X) \geq 0$ for all $X \in C$.

Let $R: C \rightarrow \mathbb{R}$ be a risk measure and $D: C \rightarrow \mathbb{R}$ be a deviation measure. Then, one readily verifies that the sum

$$H(X) := R(X) + D(X), \quad \text{for } X \in C,$$

defines a premium principle on C . It is quite remarkable that this decomposition into a monetary risk measure and a deviation measure characterizes all premium principles.

Theorem 2.2. *A map $H: C \rightarrow \mathbb{R}$ is a premium principle if and only if $H = R + D$, where $R: C \rightarrow \mathbb{R}$ is a risk measure and $D: C \rightarrow \mathbb{R}$ is a deviation measure.*

The theorem shows that premium principles can be decomposed into a *net premium* or *safety loading* that takes care of the claim's risk and a *fluctuation loading* that prices the variability of the damage. In the classic case when a prior probability distribution \mathbb{P} is given, the typical premium consisting of the sum of the expected loss $E_{\mathbb{P}}(-X)$ and (a multiple of) the variance of X under \mathbb{P} is a case in point. Note that the expected loss is a risk measure and the variance a deviation measure.

It is natural to ask in what sense the risk and the deviation measure can be identified uniquely. The following theorem provides a partial answer in the sense that it decomposes the premium principle into a maximal risk measure R_{Max} (capturing all risky components of the insurance claim) and a minimal deviation measure D_{Min} measuring the claim's fluctuations that cannot be captured by any risk measure.

Theorem 2.3. *Let $H: C \rightarrow \mathbb{R}$ be a premium principle. Define*

$$R_{\text{Max}}(X) := \inf \{H(X_0) \mid X_0 \in C, X_0 \leq X\}, \quad \text{for } X \in B_b.$$

The map $R_{\text{Max}}: B_b \rightarrow \mathbb{R}$ defines a risk measure on B_b with $R_{\text{Max}}(0) = H(0)$ and $R_{\text{Max}}(X) \leq H(X)$ for all $X \in C$. Moreover, $D_{\text{Min}}(X) := H(X) - R_{\text{Max}}(X)$ defines a deviation measure on C , and

$$H(X) = R_{\text{Max}}(X) + D_{\text{Min}}(X) \quad \text{for all } X \in C.$$

For every other decomposition of the form $H = R + D$ with a risk measure R and a deviation measure D , we have $R \leq R_{\text{Max}}$ and $D \geq D_{\text{Min}}$.

Theorem 2.3 shows that one can identify uniquely a *maximal* risk measure and a *minimal* deviation measure whose sum forms the premium principle. The risk measure R_{Max} solves a variational problem that is, at least in spirit, akin to the idea of superhedging in finance, it computes the minimal premium that one has to pay for an insurance policy $X_0 \in C$ that occurs at least as high a damage as the given claim X in every state of the world. Note that this risk measure is defined on the whole space of

claims B_b , the theorem thus provides a natural extension of the premium principle H to the whole space of claims. In particular, we obtain an algorithm to extend a given premium principle to the set of all claims.

2.3. Examples. We illustrate how classic and new approaches of insurance pricing can be subsumed under our framework.

2.3.1. *Classic Premium Principles under a Given Probabilistic Model.*

Example 2.4 (Ad hoc premium principles under a given model). The benchmark premium principle is the *fair premium principle* given by

$$H(X) = \mathbb{E}_{\mathbb{P}}(-X), \quad \text{for } X \in B_b,$$

where \mathbb{P} is a fixed probability measure on (Ω, \mathcal{F}) . Here, $R_{\text{Max}} = \mathbb{E}_{\mathbb{P}}(-\cdot)$ and $D_{\text{Min}} = 0$. In practice, since the fair premium contains no premium for taking risk, insurers usually add a safety loading, e.g. in terms of the variance

$$H(X) = \mathbb{E}_{\mathbb{P}}(-X) + \frac{\theta}{2} \text{var}_{\mathbb{P}}(X), \quad \text{for } X \in B_b,$$

with a constant $\theta \geq 0$. Here, $R = \mathbb{E}_{\mathbb{P}}(-\cdot)$, and $D = \frac{\theta}{2} \text{var}_{\mathbb{P}}(\cdot)$ is a decomposition of H into risk and deviation. However, as we will see in Example 3.4, for $\theta > 0$, the maximal risk measure R_{Max} is given by

$$R_{\text{Max}}(X) = \max_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}(-X) - \frac{1}{2\theta} C(\mathbb{Q}|\mathbb{P}),$$

where \mathcal{P} consists of all probability measures \mathbb{Q} , which are absolutely continuous w.r.t. \mathbb{P} and satisfy

$$C(\mathbb{Q}|\mathbb{P}) := \text{var}_{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right) < \infty.$$

Recall that C is the so-called *Gini concentration index*, see e.g. Maccheroni et al. [13],[14].

Example 2.5 (Economic premium principles). Let \mathbb{P} be a probability measure on (Ω, \mathcal{F}) and $\ell: \mathbb{R} \rightarrow \mathbb{R}$ be a strictly increasing *loss function*. One can then consider, for example, the *safety equivalent*

$$H(X) := \ell^{-1}\left(\mathbb{E}_{\mathbb{P}}[\ell(\max X - X)]\right) - \max X, \quad \text{for } X \in B_b.$$

Often, one considers a random initial endowment $Z \in B_b$, which could be interpreted as an existing portfolio of insurance contracts. Assuming a continuously differentiable loss function ℓ , the premium $p := H(X)$ is then computed by requiring that the new insurance contract together with the premium p (infinitesimally) does not change the expected loss. That is,

$$0 = \lim_{h \rightarrow 0} \frac{\mathbb{E}_{\mathbb{P}}(\ell[-Z - h(p + X)]) - \mathbb{E}_{\mathbb{P}}(\ell(-Z))}{h} = -\mathbb{E}_{\mathbb{P}}(\ell'(-Z) \cdot (p + X)),$$

which leads to the so-called *Esscher transform* (cf. [6] or, in the context of option pricing, Gerber and Shiu [9])

$$H(X) := -\frac{\mathbb{E}_{\mathbb{P}}(X \ell'(-Z))}{\mathbb{E}_{\mathbb{P}}(\ell'(-Z))}, \quad \text{for } X \in B_b.$$

Notice that, for \mathbb{P} -a.s. constant Z , this leads to the mean value principle. In the case of an exponential utility function $\ell(x) := \frac{1}{\alpha}(1 - e^{-\alpha x})$ with $\alpha > 0$, this leads to

$$H(X) := -\frac{\mathbb{E}_{\mathbb{P}}(X e^{\alpha Z})}{\mathbb{E}_{\mathbb{P}}(e^{\alpha Z})}, \quad \text{for } X \in B_b,$$

see also Bühlmann [3]. For $Z = X$, we obtain the celebrated *Esscher principle* (see e.g. Bühlmann [3] and Deprez and Gerber [5])

$$H(X) := -\frac{\mathbb{E}_{\mathbb{P}}(X e^{\alpha X})}{\mathbb{E}_{\mathbb{P}}(e^{\alpha X})}, \quad \text{for } X \in B_b.$$

Another condition, when considering a random endowment Z , is given by the following modification of the safety equivalent (cf. Deprez and Gerber [5])

$$\mathbb{E}_{\mathbb{P}}(\ell[-(Z + p + X)]) = \mathbb{E}_{\mathbb{P}}(\ell(-Z)).$$

For the exponential utility function $\ell(x) := \frac{1}{\alpha}(1 - e^{-\alpha x})$ with $\alpha > 0$, this leads to

$$H(X) = -\frac{1}{\alpha} \log \frac{\mathbb{E}_{\mathbb{P}}(e^{\alpha(Z+X)})}{\mathbb{E}_{\mathbb{P}}(e^{\alpha Z})}, \quad \text{for } X \in B_b.$$

2.3.2. Model Uncertainty. The recent history brought the issue of model uncertainty to center stage; in particular, it has become clear that working under the assumption of a single probability distribution can be too optimistic for insurance companies. New regulations thus ask insurers to take various models into account (stress testing).

Example 2.6 (Model uncertainty). Instead of a single probability measure \mathbb{P} on (Ω, \mathcal{F}) , we now consider a nonempty set \mathcal{P} of probability measures on (Ω, \mathcal{F}) . The set \mathcal{P} can be seen as a set of plausible models, and we thus end up with a setup, where we have model uncertainty w.r.t. the models contained in \mathcal{P} . Then, one can consider robust versions of the aforementioned premium principles by regarding worst case scenarios. Examples include:

- (i) A robust variance principle

$$H(X) = \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}(-X) + \theta \sup_{\mathbb{P} \in \mathcal{P}} \text{var}_{\mathbb{P}}(X), \quad \text{for } X \in B_b,$$

with $\theta \geq 0$.

- (ii) A robust Esscher principle

$$H(X) := \sup_{\mathbb{P} \in \mathcal{P}} \frac{\mathbb{E}_{\mathbb{P}}(-X e^{\alpha X})}{\mathbb{E}_{\mathbb{P}}(e^{\alpha X})}, \quad \text{for } X \in B_b,$$

or a robust safety equivalent with exponential utility function and random endowment $Z \in B_b$

$$H(X) = \sup_{\mathbb{P} \in \mathcal{P}} \left(-\frac{1}{\alpha} \log \frac{\mathbb{E}_{\mathbb{P}}(e^{\alpha(Z+X)})}{\mathbb{E}_{\mathbb{P}}(e^{\alpha Z})} \right), \quad \text{for } X \in B_b,$$

for $\alpha > 0$.

- (iii) Maxmin expected loss (cf. Gilboa-Schmeidler [10])

$$H(X) := \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}(\ell(\max X - X)) - \max X, \quad \text{for } X \in B_b,$$

with a strictly increasing loss function $\ell: \mathbb{R} \rightarrow \mathbb{R}$.

- (iv) Alternatively and particularly in the case of parameter uncertainty, one can consider a probability measure $\mu: \Sigma \rightarrow [0, 1]$ (second-order prior), where $\Sigma = \Sigma(\mathcal{P})$ denotes the Borel σ -algebra on \mathcal{P} endowed with the vague topology, and take a mean value w.r.t. μ . In the simplest case, where $\ell(x) = \phi(x) = x$, this corresponds to a Bayesian prediction. This approach can be modified by considering an increasing loss function $\ell: \mathbb{R} \rightarrow \mathbb{R}$ and another increasing function $\phi: \mathbb{R} \rightarrow \mathbb{R}$

(second-order loss function). Then, one obtains the so-called *smooth ambiguity model* (cf. Klibanoff et al. [12])

$$H(X) := \int_{\mathcal{P}} \phi(\mathbb{E}_{\mathbb{P}}[\ell(\max X - X)]) \mu(d\mathbb{P}) - \max X, \quad \text{for } X \in B_b.$$

Example 2.7 (Ambiguity indices). Again, we consider a nonempty set \mathcal{P} of probability measures on (Ω, \mathcal{F}) . In contrast to the previous example, we now fix a baseline model $\mathbb{P} \in \mathcal{P}$, which can be seen as the (due to some case-dependent reasons) most plausible model. The idea is now to consider a safety loading, where we differentiate between risk and uncertainty. The risk premium can then be given, for example, by the variance or the (average) value at risk, and the premium for uncertainty is given by (cf. Castagnoli et al. [4])

$$\text{Amb}_{\mathcal{P}}(X) := \frac{1}{2} \sup_{\mathbb{Q}, \mathbb{Q}' \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}}(X) - \mathbb{E}_{\mathbb{Q}'}(X), \quad \text{for } X \in B_b.$$

Then, $\text{Amb}_{\mathcal{P}}$ as an uncertainty premium together with the variance as a risk premium leads to the premium principle

$$H(X) = \mathbb{E}_{\mathbb{P}}(-X) + \theta \sqrt{\text{var}_{\mathbb{P}}(X)} + \gamma \text{Amb}_{\mathcal{P}}(X), \quad \text{for } X \in B_b, \quad (2.1)$$

with $\gamma, \theta \geq 0$. We would like to point out that, in the setup chosen by Castagnoli et al. [4], it is only possible to consider premium principles with an ambiguity index for sets \mathcal{P} , which are symmetric to the baseline model \mathbb{P} , i.e. if $2\mathbb{P} - \mathbb{Q} \in \mathcal{P}$ for all $\mathbb{Q} \in \mathcal{P}$. However, our notion of a premium principle includes premium principles based on ambiguity indices for any nonempty set \mathcal{P} and every choice of the baseline model. Instead of $\text{Amb}_{\mathcal{P}}$ in (2.1), one could, for example, also consider the ambiguity index

$$\text{Amb}'_{\mathcal{P}}(X) := \frac{1}{2} \sup_{\mathbb{Q}, \mathbb{Q}' \in \mathcal{P}} d_{W_1}(\mathbb{Q} \circ X^{-1}, \mathbb{Q}' \circ X^{-1}),$$

where d_{W_1} denotes the Wasserstein distance.

Example 2.8 (Quantile-based premium principles). Let $\varepsilon \in (0, 1)$, \mathbb{P} be a probability measure on (Ω, \mathcal{F}) , and

$$\mathbb{P}_X^{-1}(\lambda) := \inf\{a \in \mathbb{R} \mid \mathbb{P}(X \leq a) \leq \lambda\}, \quad \text{for } X \in B_b \text{ and } \lambda \in (0, 1).$$

Then, we could consider the ε -*quantile principle*, cf. Rolski et al. [19, Section 3.1.3],

$$H(X) = \text{V@R}_{\mathbb{P}}^{\varepsilon}(X) := \mathbb{P}_X^{-1}(1 - \varepsilon), \quad \text{for } X \in B_b,$$

as a possible premium principle, where $\text{V@R}_{\mathbb{P}}^{\varepsilon}$ is also known as the *value at risk* under \mathbb{P} at level ε , cf. Föllmer and Schied [8]. Here $R = \text{V@R}_{\mathbb{P}}^{\varepsilon}(\cdot)$ and $D = 0$. A major drawback, when considering the value at risk is that the value at risk is typically not convex and thus does not reflect diversification effects. Therefore, one often considers the *expected shortfall* or *average value at risk* $\text{AV@R}_{\mathbb{P}}^{\varepsilon}$, given by

$$\text{AV@R}_{\mathbb{P}}^{\varepsilon}(X) := \frac{1}{\varepsilon} \int_0^{\varepsilon} \text{V@R}_{\mathbb{P}}^{\gamma}(X) d\gamma, \quad \text{for } X \in B_b,$$

instead of $\text{V@R}_{\mathbb{P}}^{\varepsilon}$. Notice that $\text{AV@R}_{\mathbb{P}}^{\varepsilon}$ is convex and positive homogeneous (of degree 1). Alternatively, for $\theta \geq 0$, one can consider the so-called *absolute deviation principle*, cf. Rolski et al. [19, Section 3.1.3],

$$H(X) = \mathbb{E}_{\mathbb{P}}(-X) + \theta \mathbb{E}_{\mathbb{P}}(|X - \mathbb{P}_X^{-1}(\frac{1}{2})|), \quad \text{for } X \in B_b,$$

as a modification of the standard deviation principle. In this case, $R(X) = \mathbb{E}_{\mathbb{P}}(-X)$ and $D(X) = \theta \mathbb{E}_{\mathbb{P}}(|X - \mathbb{P}_X^{-1}(1/2)|)$ for $X \in B_b$. Notice that

$$D(X) = \frac{\theta}{2} \left(\text{AV@R}_{\mathbb{P}}^{\frac{1}{2}}(X) - \text{AV@R}_{\mathbb{P}}^{\frac{1}{2}}(-X) \right) = \theta \text{Amb}_{\mathcal{Q}_2}(X)$$

is (up to a constant) an ambiguity index, where \mathcal{Q}_2 consists of all probability measures $\mathbb{Q} \ll \mathbb{P}$ whose density $\frac{d\mathbb{Q}}{d\mathbb{P}}$ is \mathbb{P} -a.s. bounded by 2, cf. Example 3.5. In Example 3.5, we further show that, for $\theta \geq 1$, the maximal risk measure R_{Max} is given by

$$R_{\text{Max}}(X) = \text{AV@R}_{\mathbb{P}}^{\frac{1}{1+\theta}}(X), \quad \text{for } X \in B_b.$$

Example 2.9 (Choquet integrals). Wang, Young, and Panjer [23] derive an axiomatic characterization of premium principles in a competitive market setting that results in a representation using Choquet integrals. Consider a set function $\gamma: \mathcal{F} \rightarrow [0, 1]$ with $\gamma(\emptyset) = 0$, $\gamma(\Omega) = 1$, and $\gamma(A) \leq \gamma(B)$ for all $A, B \in \mathcal{F}$ with $A \subset B$. Then, we consider the premium principle given by the *Choquet integral* w.r.t. γ

$$H(X) := \int_{-\infty}^{\max X} \gamma(\{X \leq t\}) dt - \max X \quad \text{for } X \in B_b.$$

Wang, Young, and Panjer [23] show that, under certain axioms, every premium principle H can be represented as a Choquet integral w.r.t. a distorted probability $\gamma = g \circ \mathbb{P}$ for a probability measure \mathbb{P} and a *distortion function* g (a nondecreasing function on $[0, 1]$ with $g(0) = 0$ and $g(1) = 1$). In this case,

$$H(X) = \int_{-\infty}^{\max X} g(\mathbb{P}_X(t)) dt - \max X,$$

where, for $t \geq 0$, $\mathbb{P}_X(t) := \mathbb{P}(X \leq t)$. Wir könnten z.B. nichtmonotone Funktionen $g: [0, 1] \rightarrow [0, 1]$ betrachten.

3. DUAL REPRESENTATION OF CONVEX PREMIUM PRINCIPLES AND BASELINE MODELS

Premium principles should generally reflect the benefits of diversification and the aversion to uncertainty. In this section, we thus consider *convex* premium principles, generalizing the approach of [5] who assume that the probability distribution of claims is known. We identify the maximal risk measure in the premium's decomposition as a convex risk measure ([7]). Throughout this section, we assume that C is a linear space with $\mathbb{R} \subset C$. We denote the set of all finitely additive probability measures on (Ω, \mathcal{F}) by ba_+^1 . We say that a premium principle $H: C \rightarrow \mathbb{R}$ is *convex* if

$$H(\lambda X + (1 - \lambda)Y) \leq \lambda H(X) + (1 - \lambda)H(Y) \quad \text{for all } \lambda \in [0, 1] \text{ and } X, Y \in C.$$

In this case, we denote the *convex dual* of H by

$$H^*(\mathbb{P}) := \inf_{X \in C} \mathbb{E}_{\mathbb{P}}(X) + H(X) \in [-\infty, 0] \quad \text{for } \mathbb{P} \in \text{ba}_+^1.$$

Theorem 3.1. *Let $H: C \rightarrow \mathbb{R}$ a convex premium principle. Then, the maximal risk measure R_{Max} in the decomposition of H satisfies*

$$R_{\text{Max}}(X) = \max_{\mathbb{P} \in \text{ba}_+^1} \mathbb{E}_{\mathbb{P}}(-X) + H^*(\mathbb{P}) \quad \text{for all } X \in B_b.$$

Moreover,

$$H^*(\mathbb{P}) = \inf_{X \in B_b} \mathbb{E}_{\mathbb{P}}(X) + R_{\text{Max}}(X) \quad \text{for all } \mathbb{P} \in \text{ba}_+^1, \quad (3.1)$$

By the previous theorem, the convex dual H^* of the premium principle corresponds to the penalty function of its maximal risk measure. H^* thus represents the confidence that the insurer puts on a particular model \mathbb{P} in the class of all possible models. In the sequel, we will refer to

$$\mathcal{P} := \{\mathbb{P} \in \text{ba}_+^1 \mid H^*(\mathbb{P}) < \infty\}$$

as the set of all *plausible models*.

If the premium principle is also scalable in the sense that it is positively homogeneous, R_{Max} is even a coherent risk measure.

Corollary 3.2. *Let $H: C \rightarrow \mathbb{R}$ be a sublinear premium principle, i.e. H is a convex premium principle, and $H(\lambda X) = \lambda H(X)$ for all $X \in C$ and $\lambda > 0$. Then, the representing maximal risk measure R_{Max} is a coherent risk measure, i.e.*

$$R_{\text{Max}}(X) = \max_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}(-X) \quad \text{for all } X \in B_b,$$

where the set of plausible models is given by

$$\mathcal{P} = \{\mathbb{P} \in \text{ba}_+^1 \mid \forall X \in C: \mathbb{E}_{\mathbb{P}}(-X) \leq H(X)\}.$$

Proof. This follows directly from Theorem 3.1 together with the observation that sublinearity implies that $H^*(\mathbb{P}) \in \{-\infty, 0\}$ for all $\mathbb{P} \in \text{ba}_+^1$. \square

Notice that, in the sublinear case, for all probabilistic models $\mathbb{P} \in \mathcal{P}$ and all claims $X \in B_b$,

$$H(X) \geq \mathbb{E}_{\mathbb{P}}(-X).$$

In other words, the premium principle H incorporates a so-called *safety loading* under each plausible model $\mathbb{P} \in \mathcal{P}$, cf. Castagnoli et al. [4] and Young [24]. In the next step, we analyze in more detail the minimal deviation measure of the premium's decomposition. For $\mathbb{P} \in \mathcal{P}$,

$$D_{\mathbb{P}}(X) := H(X - \mathbb{E}_{\mathbb{P}}(X)), \quad \text{for } X \in C, \quad (3.2)$$

defines a deviation measure, that we call the *fluctuation loading* under \mathbb{P} . We have

$$H(X) = \mathbb{E}_{\mathbb{P}}(-X) + D_{\mathbb{P}}(X) \quad \text{for all } X \in C. \quad (3.3)$$

The deviation measures $D_{\mathbb{P}}$ can thus be seen as the profit for accepting the aleatoric uncertainty of X under the model $\mathbb{P} \in \mathcal{P}$. Equation (3.3) is a model-dependent decomposition into a risk measure and a deviation measure. We have the following relation between the minimal deviation measure D_{Min} and the family of model-dependent deviation measures $(D_{\mathbb{P}})_{\mathbb{P} \in \mathcal{P}}$.

Corollary 3.3. *Let $H: C \rightarrow \mathbb{R}$ be a convex premium principle. Then,*

$$D_{\text{Min}}(X) = \min_{\mathbb{P} \in \mathcal{P}} D_{\mathbb{P}}(X) - H^*(\mathbb{P}) \quad \text{for all } X \in C.$$

Proof. By Theorem 3.1,

$$D_{\text{Min}}(X) = H(X) - R_{\text{Max}}(X) = \min_{\mathbb{P} \in \mathcal{P}} H(X - \mathbb{E}_{\mathbb{P}}(X)) - H^*(\mathbb{P}) \quad \text{for all } X \in C.$$

The statement now follows from Equation (3.2). \square

Example 3.4. Let \mathbb{P} be a probability measure, $\theta > 0$, and consider

$$H(X) := \mathbb{E}_{\mathbb{P}}(-X) + \frac{\theta}{2} \text{var}_{\mathbb{P}}(X) \quad \text{for all } X \in B_b.$$

Let $\mathbb{Q} \in \mathcal{P}$. Then, \mathbb{Q} is countably additive and absolutely continuous w.r.t. \mathbb{P} . Let $Z := \frac{d\mathbb{Q}}{d\mathbb{P}}$. Then, using the Cauchy-Schwarz inequality, we obtain that

$$\begin{aligned} H^*(\mathbb{Q}) &= \inf_{X \in \mathbb{B}_b} \mathbb{E}_{\mathbb{Q}}(X) + H(X) = \inf_{X \in \mathbb{B}_b} \mathbb{E}_{\mathbb{Q}} \left[X \left(Z - 1 + \frac{\theta}{2}(X - \mathbb{E}_{\mathbb{P}}(X)) \right) \right] \\ &= \inf \left\{ \alpha \mathbb{E}(X^2) \mid \alpha \leq 0, \alpha X = Z - 1 + \frac{\theta}{2}(X - \mathbb{E}_{\mathbb{P}}(X)) \right\}. \end{aligned}$$

Notice that $\alpha X = Z - 1 + \frac{\theta}{2}(X - \mathbb{E}_{\mathbb{P}}(X))$ implies that $\mathbb{E}_{\mathbb{P}}(X) = 0$, which, in turn, implies that $X = (\alpha - \frac{\theta}{2})^{-1}(Z - 1)$. Hence,

$$H^*(\mathbb{Q}) = \inf_{\alpha \leq 0} \alpha \left(\alpha - \frac{\theta}{2} \right)^{-2} \text{var}_{\mathbb{P}} \left(\frac{d\mathbb{Q}}{d\mathbb{P}} \right).$$

Notice that $\frac{d}{d\alpha} \alpha \left(\alpha - \frac{\theta}{2} \right)^{-2} = 0$ if and only if $\alpha = -\frac{\theta}{2}$. We therefore obtain that

$$H^*(\mathbb{Q}) = -\frac{1}{2\theta} \text{var}_{\mathbb{P}} \left(\frac{d\mathbb{Q}}{d\mathbb{P}} \right).$$

is (up to the factor $-\frac{1}{2\theta}$) the *Gini concentration index*. By Theorem 3.1,

$$R_{\text{Max}}(X) = \max_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}}(-X) - \frac{1}{2\theta} \text{var}_{\mathbb{P}} \left(\frac{d\mathbb{Q}}{d\mathbb{P}} \right) \quad \text{for all } X \in \mathbb{B}_b.$$

Example 3.5. Let \mathbb{P} be a probability measure, $\theta \geq 0$, and consider

$$H(X) = \mathbb{E}_{\mathbb{P}}(-X) + \theta \mathbb{E}_{\mathbb{P}} \left(|X - \mathbb{P}_X^{-1}(\frac{1}{2})| \right), \quad \text{for } X \in \mathbb{B}_b.$$

Then, by [8, Lemma 4.46],

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} \left(|X - \mathbb{P}_X^{-1}(\frac{1}{2})| \right) &= \mathbb{E}_{\mathbb{P}} \left((X - \mathbb{P}_X^{-1}(\frac{1}{2}))^+ \right) - \mathbb{E}_{\mathbb{P}} \left((X - \mathbb{P}_X^{-1}(\frac{1}{2}))^- \right) \\ &= \frac{1}{2} (\text{AV@R}_{\mathbb{P}}^{\frac{1}{2}}(X) - \text{AV@R}_{\mathbb{P}}^{\frac{1}{2}}(-X)) \end{aligned}$$

Recall that, for $\varepsilon \in (0, 1)$,

$$\text{AV@R}_{\mathbb{P}}^{\varepsilon}(X) = \max_{\mathbb{Q} \in \mathcal{Q}_{1/\varepsilon}} \mathbb{E}_{\mathbb{Q}}(-X) \quad \text{for all } X \in \mathbb{B}_b,$$

where, for $\alpha \geq 1$, \mathcal{Q}_{α} denotes the set of all probability measures $\mathbb{Q} \ll \mathbb{P}$ whose density is \mathbb{P} -a.s. bounded by α , cf. [8, Theorem 4.47]. Therefore, the set \mathcal{P} related to \mathbb{R}_{Max} is given by the set of all probability measures \mathbb{Q}^* of the form

$$\mathbb{Q}^* = \mathbb{P} + \frac{\theta}{2}(\mathbb{Q} - \mathbb{Q}') \tag{3.4}$$

with $\mathbb{Q}, \mathbb{Q}' \in \mathcal{Q}_2$. Therefore, \mathcal{P} consists of all probability measures $\mathbb{Q}^* \ll \mathbb{P}$ with

$$1 - \theta \leq \frac{d\mathbb{Q}^*}{d\mathbb{P}} \leq 1 + \theta \quad \mathbb{P}\text{-a.s.} \tag{3.5}$$

In particular, for $\theta \geq 1$, $\mathcal{P} = \mathcal{Q}_{1+\theta}$, which implies that

$$R_{\text{Max}}(X) = \text{AV@R}_{\mathbb{P}}^{\frac{1}{1+\theta}}(X) \quad \text{for all } X \in \mathbb{B}_b.$$

In fact, by the previous argumentation, it follows that every $\mathbb{Q}^* \in \mathcal{P}$ is of the form (3.4), which, in turn, implies that it satisfies (3.5). Now, assume that $\mathbb{Q}^* \ll \mathbb{P}$ is a probability measure, which satisfies (3.5). For $\theta = 0$, it follows that $\mathbb{Q}^* = \mathbb{P} \in \mathcal{P}$. Hence, assume that $\theta > 0$, and define

$$Z := \frac{2}{\theta} \left(\frac{d\mathbb{Q}^*}{d\mathbb{P}} - 1 \right).$$

Then, $|Z| \leq 2$ \mathbb{P} -a.s., $\mathbb{E}_{\mathbb{P}}(Z) = 0$, and, by Hölder's inequality, $\alpha := \frac{\mathbb{E}_{\mathbb{P}}(|Z|)}{2} \leq 1$. Define

$$Y := Z^+ + 1 - \frac{|Z|}{2} \quad \text{and} \quad Y' := Z^- + 1 - \frac{|Z|}{2}.$$

Then, $0 \leq Y \leq 2$ and $0 \leq Y' \leq 2$ \mathbb{P} -a.s. Moreover, $Y - Y' = Z$ \mathbb{P} -a.s. and

$$\mathbb{E}_{\mathbb{P}}(Y) = \mathbb{E}_{\mathbb{P}}(Z^+) + 1 - \frac{\mathbb{E}_{\mathbb{P}}(|Z|)}{2} = 1 = \mathbb{E}_{\mathbb{P}}(Z^-) + 1 - \frac{\mathbb{E}_{\mathbb{P}}(|Z|)}{2} = \mathbb{E}_{\mathbb{P}}(Y').$$

Hence, Equation (3.4) is satisfied with $\mathbb{Q} := Y d\mathbb{P}$ and $\mathbb{Q}' := Y' d\mathbb{P}$

We say that a premium principle $H: C \rightarrow \mathbb{R}$ is *monotone* if $H(X) \geq H(Y)$ for all $X, Y \in C$ with $X \geq Y$. Throughout the remainder of this section, we discuss the relation to monotone sublinear premium principles that Castagnoli et al. consider in [4]. More precisely, we show that, in the convex case, replacing Axiom (P2) in the definition of a premium principle by a so-called *internality condition*, cf. [4], implies the monotonicity of the premium principle, and thus, together with positive homogeneity, leads to the objects considered in [4]. A similar result can be found in Deprez and Gerber [5, Theorem 3].

Proposition 3.6. *Let $H: C \rightarrow \mathbb{R}$ a convex map that satisfies (P1). Then, the following statements are equivalent:*

- (i) H is a monotone premium principle,
- (ii) $H(X) \leq H(0) = 0$ for all $X \in C$ with $X \geq 0$, i.e. H is internal.

The following proposition is a partial extension of Theorem 3 in Castagnoli et al. [4], where statement (i) is a reformulation of Axiom P.7 in [4].

Proposition 3.7. *Let $H: C \rightarrow \mathbb{R}$ be a sublinear premium principle, and define*

$$\text{Amb}_{\mathcal{P}}(X) := \frac{1}{2} (R_{\text{Max}}(X) + R_{\text{Max}}(-X)) = \frac{1}{2} \max_{\mathbb{Q}, \mathbb{Q}' \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}}(X) - \mathbb{E}_{\mathbb{Q}'}(X)$$

for $X \in B_b$. Then, for every $\mathbb{P} \in \mathcal{P}$, the following statements are equivalent:

- (i) $\mathbb{E}_{\mathbb{P}}(-X) = \frac{1}{2} (R_{\text{Max}}(X) - R_{\text{Max}}(-X))$ for all $X \in B_b$,
- (ii) \mathcal{P} is symmetric with center \mathbb{P} , i.e. $2\mathbb{P} - \mathbb{Q} \in \mathcal{P}$ for all $\mathbb{Q} \in \mathcal{P}$,
- (iii) $\text{Amb}_{\mathcal{P}}(X) = \max_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}(X) - \mathbb{E}_{\mathbb{Q}}(X)$,
- (iv) $D_{\mathbb{P}}(X) = D_{\text{Min}}(X) + \text{Amb}_{\mathcal{P}}(X)$ for all $X \in C$.

In this case, R_{Max} is dominated by \mathbb{P} , i.e. every $\mathbb{Q} \in \mathcal{P}$ is absolutely continuous w.r.t. \mathbb{P} , and \mathbb{P} is countably additive if and only if every $\mathbb{Q} \in \mathcal{P}$ is countably additive.

Remark 3.8. We would like to point out what are the implications of Proposition 3.7 in view of [4]. Notice that, in contrast to assumption (P2) in our definition of a premium principle (Definition 2.1), the internality axiom P.1 together with the subadditivity requirement P.3 in [4] already implies the monotonicity of the premium functional H . Therefore, Proposition 3.7 shows that in [4] only very particular types of ambiguity sets \mathcal{P} , namely symmetric ones, and only a very particular choice of the baseline model \mathbb{P} can be considered for the premium calculation. That is, the case, where \mathcal{P} is not symmetric or \mathbb{P} is not the center of \mathcal{P} does not fall into the setup chosen in [4]. Moreover, the symmetry of \mathcal{P} implies that all elements of \mathcal{P} are absolutely continuous w.r.t. the baseline model \mathbb{P} excluding setups, where the set \mathcal{P} is undominated. However undominated sets of plausible models appear quite naturally, for example, when considering a Brownian Motion with uncertainty in the volatility parameter, see e.g. Peng [16],[17] or Soner et al. [21],[22]. Hence, replacing the internality axiom P.1 in [4] by the apparently

similar assumption (P2) in Definition 2.1 has a huge impact. In the following example, we describe a basic setup that leads to a nonsymmetric set \mathcal{P} .

Example 3.9. Consider the setup (2.1) from [4] with $\Omega = \mathbb{N}$, endowed with the σ -algebra $\mathcal{F} = 2^{\mathbb{N}}$ (power set). For $n \in \mathbb{N}$, we consider the measure

$$\mathbb{P}_n := \frac{1}{n} \sum_{k=1}^n \delta_k,$$

where δ_k denotes the Dirac measure with barycenter $k \in \mathbb{N}$. We then consider the monotone premium principle

$$H(X) := \sup_{n \in \mathbb{N}} \mathbb{E}_{\mathbb{P}_n}(-X) = \sup_{n \in \mathbb{N}} \frac{1}{n} \sum_{k=1}^n -X(k), \quad \text{for } X \in \mathcal{B}_b.$$

One readily verifies that the set \mathcal{P} consists only of probability measures \mathbb{P} of the form

$$\mathbb{P} = \sum_{n \in \mathbb{N}} \lambda_n \delta_n \tag{3.6}$$

with a nonincreasing sequence $(\lambda_n)_{n \in \mathbb{N}} \subset [0, 1]$ summing up to 1. Assume that there existed some $\mathbb{P} \in \mathcal{P}$ with $2\mathbb{P} - \mathbb{P}_n \in \mathcal{P}$ for all $n \in \mathbb{N}$. Then, \mathbb{P} is of the form (3.6) with a nonincreasing sequence $(\lambda_n)_{n \in \mathbb{N}} \subset [0, 1]$. On the other hand, $2\mathbb{P} - \mathbb{P}_n \in \mathcal{P}$ for all $n \in \mathbb{N}$, which, in particular, means that

$$2\lambda_n - \frac{1}{n} \geq 2\lambda_{n+1} \quad \text{for all } n \in \mathbb{N}. \tag{3.7}$$

However, Equation (3.7) implies that

$$\lambda_{n+1} = \lambda_1 + \sum_{k=1}^n (\lambda_{k+1} - \lambda_k) \leq \lambda_1 - \sum_{k=1}^n \frac{1}{2k} \rightarrow -\infty, \quad \text{as } n \rightarrow \infty,$$

leading to a contradiction. By Proposition 3.7, we may therefore conclude that there exists no $\mathbb{P} \in \mathcal{P}$ with

$$H(X) = \mathbb{E}_{\mathbb{P}}(-X) + \text{Amb}_{\mathcal{P}}(X) \quad \text{for all } X \in \mathcal{B}_b.$$

That is, the left right-hand side of the previous equation does not define a premium principle in the sense of Castagnoli et al. [4], whereas it defines a premium principle in the sense of Definition 2.1.

4. SUPERHEDGING AND MARKET CONSISTENCY

The integration of insurance and finance has been a central issue of research in the last years. In this section, we consider insurance premia that are consistent with a given financial market (or liquidly traded insurance contracts). We will identify the maximal risk measure in the premium's decomposition as the so-called superhedging risk measure.

The financial market is modeled by a linear subspace $M \subset C$ and a nonnegative linear pricing functional $F: M \rightarrow \mathbb{R}$. Assuming M to be a linear space and $F: M \rightarrow \mathbb{R}$ to be linear corresponds to a competitive market without frictions. We would like to point out that, without further modifications, our model can be used for markets with frictions. That is, the linearity of the price functional F can be replaced by sublinearity without further modifications. In this case, F would resemble the ask price for securization products that are traded in the market or, in other words, the price an insurer has to

pay for “selling” the risk of a claim to the market. Hierzu müsste nur $-F$ durch $F(-\cdot)$ ersetzt werden. Nonnegativity is a no arbitrage condition as it requires

$$F(X) \geq 0 \quad (4.1)$$

for nonnegative claims $X \geq 0$. Without loss of generality, we assume that $F(1) = 1$, i.e. the interest rate that is implicit in F is zero. We call

$$\mathcal{M} = \{\mathbb{P} \in \text{ba}_+^1 \mid \forall X_0 \in M: \mathbb{E}_{\mathbb{P}}(X_0) = F(X_0)\}$$

the set of martingale measures for the financial market.

Throughout this section, we consider a sublinear premium principle. We assume that the premia charged by our insurer coincide with market prices on M , i.e. $H|_M = -F$. The condition $H|_M = -F$ expresses the fact that the insurer cannot charge a premium above market prices due to competition. We introduce the set

$$M_0 := \{X_0 \in M \mid F(-X_0) = 0\}$$

of all claims that are traded on the market with price 0. In the sequel, we consider the superhedging risk measure

$$R_*(X) := \inf \{m \in \mathbb{R} \mid \exists X_0 \in M_0: -X \leq m + X_0\} \quad \text{for all } X \in B_b.$$

The superhedging risk measure amounts to the cost of staying on the safe side with the help of the products that are already being traded liquidly in the market. Notice that R_* is well-defined, since M_0 is nonempty. Moreover, $R_{\text{Max}} \leq R_*$ since $H|_M = -F$.

Proposition 4.1. *Let H be sublinear. Then the following statements are equivalent:*

- (i) *The maximal risk measure in the decomposition of H is the superhedging functional R_* , i.e. $R_{\text{Max}} = R_*$.*
- (ii) *The premium principle H is based on the use of securization products, i.e., for all $X \in C$, there exists some $X_0 \in M$ with $-X \leq X_0$ and $H(X) \geq F(X_0)$.*
- (iii) *The plausible models for H coincide with the martingale measures, i.e. $\mathcal{P} = \mathcal{M}$.*

APPENDIX A. PROOFS

Proof of Theorem 2.3. First, notice that $R_{\text{Max}}: B_b \rightarrow \mathbb{R}$ is well-defined since $\min X \in C$ with $\min X \leq X$ and $H(X_0) \geq H(\max X)$ for all $X_0 \in C$ with $X_0 \leq X$. By definition, $R_{\text{Max}}(X) \leq H(X)$ for all $X \in C$. Hence, $D_{\text{Min}}(X) = H(X) - R_{\text{Max}}(X) \geq 0$ for all $X \in C$. Moreover, $H(X_0) \geq H(0) = 0$ for all $X_0 \in C$ with $X_0 \leq 0$, which implies that $R_{\text{Max}}(0) = 0$. In particular, $D_{\text{Min}}(0) = H(0) - R_{\text{Max}}(0) = 0$. We will now show that R_{Max} defines a risk measure. First, observe that $R_{\text{Max}}(X) \leq H(Y_0)$ for $X, Y \in B_b$ with $X \geq Y$ and $Y_0 \in C$ with $Y_0 \leq Y$. Taking the infimum over all $Y_0 \in C$ with $Y_0 \leq Y$, it follows that $R_{\text{Max}}(X) \leq R_{\text{Max}}(Y)$. Now, let $X \in B_b$, $m \in \mathbb{R}$ and $X_0 \in C$ with $X_0 \leq X$. Then,

$$R_{\text{Max}}(X + m) \leq H(X_0 + m) = H(X_0) - m.$$

Taking the infimum over all $X_0 \in C$ with $X_0 \leq X$ implies that $R_{\text{Max}}(X + m) \leq R_{\text{Max}}(X) - m$. On the other hand,

$$R_{\text{Max}}(X) - m = R_{\text{Max}}(X + m - m) - m \leq R_{\text{Max}}(X + m).$$

This also shows that $D_{\text{Min}}(X + m) = H(X + m) - R_{\text{Max}}(X + m) = H(X) - R_{\text{Max}}(X) = D_{\text{Min}}(X)$ for all $X \in C$ and $m \in \mathbb{R}$.

Let $R: B_b \rightarrow \mathbb{R}$ be a risk measure with $R(X) \leq H(X)$ for all $X \in C$. Then, for all $X \in B_b$ and $X_0 \in C$ with $X_0 \leq X$,

$$R(X) \leq R(X_0) \leq H(X_0).$$

Taking the infimum over all $X_0 \in C$ with $X_0 \leq X$, we may conclude that $R(X) \leq R_{\text{Max}}(X)$ for all $X \in B_b$. \square

Proof of Theorem 3.1. We first show that $R_{\text{Max}}: B_b \rightarrow \mathbb{R}$ is convex. Let $X, Y \in B_b$ and $\lambda \in [0, 1]$. Then, for $X_0, Y_0 \in C$ with $X_0 \leq X$ and $Y_0 \leq Y$,

$$R_{\text{Max}}(\lambda X + (1 - \lambda)Y) \leq H(\lambda X_0 + (1 - \lambda)Y_0) \leq \lambda H(X_0) + (1 - \lambda)H(Y_0).$$

Taking the infimum over all $X_0, Y_0 \in C$ with $X_0 \leq X$ and $Y_0 \leq Y$, we obtain that R_{Max} is convex. Since R_{Max} is a convex risk measure, it follows that, see e.g. Föllmer and Schied [8, Theorem 4.12],

$$R_{\text{Max}}(X) = \max_{\mathbb{P} \in \text{ba}_+^1} \mathbb{E}_{\mathbb{P}}(-X) + R_{\text{Max}}^*(\mathbb{P}) \quad \text{for all } X \in B_b,$$

where $R_{\text{Max}}^*(\mathbb{P}) := \inf_{X \in B_b} \mathbb{E}_{\mathbb{P}}(X) + R_{\text{Max}}(X)$ for $\mathbb{P} \in \text{ba}_+^1$. It remains to show (3.1), i.e. $H^*(\mathbb{P}) = R_{\text{Max}}^*(\mathbb{P})$ for all $\mathbb{P} \in \text{ba}_+^1$. Since $R_{\text{Max}}(X) \leq H(X)$ for all $X \in C$, it follows that

$$R_{\text{Max}}^*(\mathbb{P}) \leq \inf_{X \in C} \mathbb{E}_{\mathbb{P}}(X) - R_{\text{Max}}(X) \leq H^*(\mathbb{P}) \quad \text{for all } \mathbb{P} \in \text{ba}_+^1.$$

In particular, there exists some $\mathbb{P} \in \text{ba}_+^1$ with $H^*(\mathbb{P}) > -\infty$. Therefore,

$$R(X) := \sup_{\mathbb{P} \in \text{ba}_+^1} \mathbb{E}_{\mathbb{P}}(-X) + H^*(\mathbb{P}), \quad \text{for } X \in B_b,$$

defines a risk measure. Since $R_{\text{Max}}^*(\mathbb{P}) \leq H^*(\mathbb{P})$ for all $\mathbb{P} \in \text{ba}_+^1$, it follows that

$$R_{\text{Max}}(X) \leq R(X) \leq H(X) \quad \text{for all } X \in C.$$

By the maximality of R_{Max} , we may conclude that $R_{\text{Max}} = R$. In particular,

$$H^*(\mathbb{P}) \leq \mathbb{E}_{\mathbb{P}}(X) + R(X) = \mathbb{E}_{\mathbb{P}}(X) + R_{\text{Max}}(X) \quad \text{for all } X \in C \text{ and } \mathbb{P} \in \text{ba}_+^1.$$

By definition of R_{Max}^* , it follows that $H^*(\mathbb{P}) \leq R_{\text{Max}}^*(\mathbb{P})$ for all $\mathbb{P} \in \text{ba}_+^1$. \square

Proof of Proposition 3.6. Trivially, (i) implies (ii). We first show that (ii) implies (P2), let $X \in C$ with $X \leq 0$. Then, by Condition (ii),

$$0 \leq -H(-X) \leq H(X),$$

where the second inequality follows from $0 = 2H(0) \leq H(X) + H(-X)$. In order to prove the monotonicity, first notice that, due to (P1) and (ii),

$$H(X) = H(X - \min X) - \min X \leq -\min X = \max(-X) \quad \text{for all } X \in C. \quad (\text{A.1})$$

Now, let $X, Y \in C$ with $X \leq Y$. Then, by (A.1), for all $\lambda \in (0, 1)$,

$$\begin{aligned} H(Y) &\leq \lambda H(X) + (1 - \lambda)H\left(\frac{Y - \lambda X}{1 - \lambda}\right) \leq \lambda H(X) + \max(\lambda X - Y) \\ &\leq \lambda H(X) + (1 - \lambda) \max(-X). \end{aligned}$$

Letting $\lambda \rightarrow 1$, we obtain that $H(Y) \leq H(X)$. \square

Proof of Proposition 3.7. For all $X \in B_b$,

$$\text{Amb}_{\mathcal{P}}(X) = R_{\text{Max}}(-X) + \frac{1}{2}(R_{\text{Max}}(-X) - R_{\text{Max}}(X))$$

and

$$\max_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}(X) - \mathbb{E}_{\mathbb{Q}}(X) = R_{\text{Max}}(X) - \mathbb{E}_{\mathbb{P}}(-X).$$

Therefore, $\text{Amb}_{\mathcal{P}}(X) = \max_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}}(X) - \mathbb{E}_{\mathbb{P}}(X)$ for all $X \in B_b$ if and only if $\mathbb{E}_{\mathbb{P}}(-X) = \frac{1}{2}(R_{\text{Max}}(X) - R_{\text{Max}}(-X))$ for all $X \in B_b$. On the other hand, if $\mathbb{E}_{\mathbb{P}}(-X) = \frac{1}{2}(R_{\text{Max}}(X) - R_{\text{Max}}(-X))$ for all $X \in B_b$, then, for all $X \in B_b$ and $\mathbb{Q} \in \mathcal{P}$,

$$2\mathbb{E}_{\mathbb{P}}(-X) - \mathbb{E}_{\mathbb{Q}}(-X) \leq 2\mathbb{E}_{\mathbb{P}}(-X) + R_{\text{Max}}(-X) \leq R_{\text{Max}}(X),$$

i.e. $2\mathbb{P} - \mathbb{Q} \in \mathcal{P}$. Next, assume that $2\mathbb{P} - \mathbb{Q} \in \mathcal{P}$ for all $\mathbb{Q} \in \mathcal{P}$. Then, for all $X \in B_b$,

$$\begin{aligned} \frac{1}{2}(R_{\text{Max}}(-X) - R_{\text{Max}}(X)) &= \frac{1}{2}(\max_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}}(X) + \min_{\mathbb{Q}' \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}'}(X)) \\ &\leq \frac{1}{2} \max_{\mathbb{Q} \in \mathcal{P}} (\mathbb{E}_{\mathbb{Q}}(X) + (2\mathbb{E}_{\mathbb{P}}(X) - \mathbb{E}_{\mathbb{Q}}(X))) = \mathbb{E}_{\mathbb{P}}(X). \end{aligned}$$

Using a symmetry argument, this implies that $\frac{1}{2}(R_{\text{Max}}(-X) - R_{\text{Max}}(X)) = \mathbb{E}_{\mathbb{P}}(X)$ for all $X \in B_b$. We have therefore established the equivalence (i) - (iii). In order to establish the remaining equivalence, first observe that, for all $\mathbb{P} \in \mathcal{P}$,

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}(-X) + D_{\mathbb{P}}(X) &= H(X) = R_{\text{Max}}(X) + D_{\text{Min}}(X) \\ &= \frac{1}{2}(R_{\text{Max}}(X) - R_{\text{Max}}(-X)) + D_{\text{Min}}(X) + \text{Amb}_{\mathcal{P}}(X) \end{aligned}$$

The equivalence between (i) and (iv) is now an immediate consequence of the previous equation. Under (ii), it follows that $\mathbb{E}_{\mathbb{Q}}(X) \leq 2\mathbb{E}_{\mathbb{P}}(X)$ for all $X \in B_b$ and all $\mathbb{Q} \in \mathcal{P}$. Choosing $X = 1_N$ for $N \in \mathcal{F}$ with $\mathbb{P}(N) = 0$, it follows that every $\mathbb{Q} \in \mathcal{P}$ is absolutely continuous w.r.t. \mathbb{P} . On the other hand, let $\mathbb{Q} \in \mathcal{P}$ and $(X_n)_{n \in \mathbb{N}} \subset B_b$ with $X_{n+1} \leq X_n$ for all $n \in \mathbb{N}$ and $\inf_{n \in \mathbb{N}} X_n = 0$. If \mathbb{P} is countably additive, then

$$0 \leq \mathbb{E}_{\mathbb{Q}}(X_n) \leq 2\mathbb{E}_{\mathbb{P}}(X_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

which shows that \mathbb{Q} is countably additive. \square

Proof of Proposition 4.1. First observe that, since $H|_M = -F$

$$R_*(X) := \inf \{H(X_0) \mid X_0 \in M, X_0 \leq X\} \quad \text{for all } X \in B_b.$$

Therefore, by Theorem 3.1, statement (i) is equivalent to statement (iii). It remains to show the equivalence of (i) and (ii). In order to establish this equivalence, first assume that $R_{\text{Max}} = R_*$. Then, $R_*(X) \leq H(X)$ for all $X \in C$. By definition of R_* , this means that, for all $X \in C$, there exists some $X_0 \in M$ with $-X \leq X_0$ and $F(X_0) \leq H(X)$. Now, assume that, for all $X \in C$, there exists some $X_0 \in M$ with $-X \leq X_0$ and $F(X_0) \leq F(X)$. Then, by definition of R_* , it follows that $R_*(X) \leq H(X)$ for all $X \in M$. The maximality of R_{Max} together with $R_{\text{Max}} \leq R_*$, which is due to $H|_M = -F$, thus implies that $R_{\text{Max}} = R_*$. \square

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