Precise Limit in Wasserstein Distance for Conditional Empirical Measures of Dirichlet Diffusion Processes

Feng-Yu Wang\textsuperscript{(a),\textit{b})}

\textsuperscript{a)} Center for Applied Mathematics, Tianjin University, Tianjin 300072, China
\textsuperscript{b)} Department of Mathematics, Swansea University, Bay Campus, Swansea, SA1 8EN, United Kingdom

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Abstract

Let $M$ be a $d$-dimensional connected compact Riemannian manifold with boundary $\partial M$, let $V \in C^2(M)$ such that $\mu(dx) := e^{V(x)}dx$ is a probability measure, and let $X_t$ be the diffusion process generated by $L := \Delta + \nabla V$ with $\tau := \inf\{t \geq 0 : X_t \in \partial M\}$. Consider the conditional empirical measure $\mu^n_t := \mathbb{E}^\nu\left(\frac{1}{t} \int_0^t \delta_{X_s} ds \mid t < \tau\right)$ for the diffusion process with initial distribution $\nu$ such that $\nu(\partial M) < 1$. Then

$$\lim_{t \to \infty} \left\{t \mathbb{W}_2^2(\mu^n_t, \mu_0)\right\}^2 = \frac{1}{\{\mu(\phi_0)\nu(\phi_0)\}^2} \sum_{m=1}^{\infty} \frac{\{\nu(\phi_0)\mu(\phi_m) + \mu(\phi_0)\nu(\phi_m)\}^2}{(\lambda_m - \lambda_0)^3},$$

where $\nu(f) := \int_M f d\nu$ for a measure $\nu$ and $f \in L^1(\nu)$, $\mu_0 := \phi_0^2 \mu$, $\{\phi_m\}_{m \geq 0}$ is the eigenbasis of $-L$ in $L^2(\mu)$ with the Dirichlet boundary, $\{\lambda_m\}_{m \geq 0}$ are the corresponding Dirichlet eigenvalues, and $\mathbb{W}_2$ is the $L^2$-Wasserstein distance induced by the Riemannian metric.

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1 Introduction

Let $M$ be a $d$-dimensional connected complete Riemannian manifold with a smooth boundary $\partial M$. Let $V \in C^2(M)$ such that $\mu(dx) = e^{V(x)}dx$ is a probability measure on $M$, where $dx$ is

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the Riemannian volume measure. Let $X_t$ be the diffusion process generated by $L := \Delta + \nabla V$ with hitting time

$$\tau := \inf\{t \geq 0 : X_t \in \partial M\}.$$  

Denote by $\mathcal{P}$ the set of all probability measures on $M$, and let $\mathbb{E}^\nu$ be the expectation taken for the diffusion process with initial distribution $\nu \in \mathcal{P}$. Consider the conditional empirical measure

$$\mu_t^\nu := \mathbb{E}^\nu\left(\frac{1}{t} \int_0^t \delta_{X_s} ds \big| t < \tau\right), \quad t > 0, \nu \in \mathcal{P}.$$  

Since $\tau = 0$ when $X_0 \in \partial M$, to ensure $\mathbb{E}^\nu(\tau > t) > 0$ we only consider

$$\nu \in \mathcal{P}_0 := \{\nu \in \mathcal{P} : \nu(M^0) > 0\}, \quad M^0 := M \setminus \partial M.$$  

Let $\{\phi_m\}_{m \geq 0}$ be the eigenbasis in $L^2(\mu)$ of $-L$ with the Dirichlet boundary such that $\phi_0 > 0$ in $M^0$, and let $\{\lambda_m\}_{m \geq 0}$ be the associated eigenvalues listed in the increasing order counting multiplicities. Then $\mu_0 := \phi_0^2 \mu$ is a probability measure on $M$. It is easy to see from [5, Theorem 2.1] that for any probability measure $\nu$ supported on $M^0$, we have

$$\lim_{t \to \infty} \|\mu_t^\nu - \mu_0\|_{\text{var}} = 0,$$

where $\| \cdot \|_{\text{var}}$ is the total variational norm.

In this paper, we investigate the convergence of $\mu_t^\nu$ to $\mu_0$ under the Wasserstein distance $\mathbb{W}_2$:

$$\mathbb{W}_2(\mu_1, \mu_2) := \inf_{\pi \in \mathcal{C}(\mu_1, \mu_2)} \left(\int_{M \times M} \rho(x, y)^2 \pi(dx, dy)\right)^{\frac{1}{2}}, \quad \mu_1, \mu_2 \in \mathcal{P},$$

where $\mathcal{C}(\mu_1, \mu_2)$ is the set of all probability measures on $M \times M$ with marginal distributions $\mu_1$ and $\mu_2$, and $\rho(x, y)$ is the Riemannian distance between $x$ and $y$, i.e. the length of the shortest curve on $M$ linking $x$ and $y$.

Recently, the convergence rate under $\mathbb{W}_2$ has been characterized in [15] for the empirical measures of the $L$-diffusion processes without boundary (i.e. $\partial M = \emptyset$) or with a reflecting boundary. Since in the present setting the diffusion process is killed at time $\tau$, it is reasonable to consider the conditional empirical measure $\mu_t^\nu$ given $t < \tau$. This is a counterpart to the quasi-ergodicity for the convergence of the conditional distribution $\tilde{\mu}_t$ of $X_t$ given $t < \tau$. Unlike in the case without boundary or with a reflecting boundary where both the distribution and the empirical measure of $X_t$ converge to the unique invariant probability measure, in the present case the conditional distribution $\tilde{\mu}_t$ of $X_t$ given $t < \tau$ converges to $\tilde{\mu}_0 := \frac{\phi_0}{\mu(\phi_0)} \mu$ rather than $\mu_0 := \phi_0^2 \mu$, and this convergence is called the quasi-ergodicity in the literature, see for instance [6] and references within.

Let $\nu(f) := \int_M f d\nu$ for $\nu \in \mathcal{P}$ and $f \in L^1(\nu)$. The main result of this paper is the following.

**Theorem 1.1.** For any $\nu \in \mathcal{P}_0$,

$$\lim_{t \to \infty} \left\{t^2 \mathbb{W}_2(\mu_t^\nu, \mu_0)^2\right\} = I \left(\frac{1}{\mu(\phi_0)\nu(\phi_0)}\right) \sum_{m=1}^{\infty} \frac{\nu(\phi_0)\mu(\phi_m) + \mu(\phi_0)\nu(\phi_m)}{\lambda_m - \lambda_0}^2 > 0.$$
If either \( d \leq 5 \) or \( d \geq 6 \) but \( \nu = h\mu \) with \( \mu(h^p) \wedge \mu_0(|h\phi_0^{-1}|^q) < \infty \) for some \( p > \frac{2d}{d+6} \) and \( q > \frac{2(d+2)}{d+6} \), then \( I < \infty \).

**Remark 1.1.** (1) Let \( X_t \) be the (reflecting) diffusion process generated by \( L \) on \( M \) where \( \partial M \) may be empty. We consider the mean empirical measure \( \hat{\mu}_t := \mathbb{E}(\frac{1}{t} \int_0^t \delta_{X_s} \, ds) \), where \( \nu \) is the initial distribution of \( X_t \). Then

\[
\lim_{t \to \infty} \{ t^2 \mathbb{W}_2(\hat{\mu}_t, \nu) \} = \sum_{m=1}^{\infty} \frac{\{\nu(\phi_m)\}^2}{\lambda_m} < \infty,
\]

where \( \{\phi_m\}_{m \geq 1} \) is the eigenbasis of \( -L \) in \( L^2(\mu) \) with the Neumann boundary condition if \( \partial M \) exists, \( \{\lambda_m\}_{m \geq 1} \) are the corresponding non-trivial (Neumann) eigenvalues, and the limit is zero if and only if \( \nu = \mu \). This can be confirmed by the proof of Theorem 1.1 with \( \phi_0 = 1, \lambda_0 = 0 \) and \( \mu(\phi_m) = 0 \) for \( m \geq 1 \). In this case, \( \mu \) is the unique invariant probability measure of \( X_t \), so that \( \hat{\mu}_t = \mu \) for \( t \geq 0 \) and hence the limit in (1.1) is zero for \( \nu = \mu \).

However, in the Dirichlet diffusion case, the conditional distribution of \( (X_s)_{0 \leq s \leq t} \) given \( t < \tau \) is no longer stationary, so that even starting from the limit distribution \( \mu_0 \) we do not have \( \mu_t^{\mu_0} = \mu_0 \) for \( t > 0 \). This leads to a non-zero limit in Theorem 1.1 even for \( \nu = \mu_0 \).

(2) It is also interesting to investigate the convergence of \( \mathbb{E}^\nu(\mathbb{W}_2(\mu_t, \mu_0)^2|t < \tau) \) for \( \mu_t := \frac{1}{t} \int_0^t \delta_{X_s} ds \), which is the counterpart to the study of [15] where the case without boundary or with a reflecting boundary is considered. According to [15], the convergence rate of \( \mathbb{E}^\nu(\mathbb{W}_2(\mu_t, \mu_0)^2|t < \tau) \) will be at most \( t^{-1} \), which is slower than the rate \( t^{-2} \) for \( \mathbb{W}_2(\mu_t^{\nu}, \mu_0)^2 \) as shown in Theorem 1.1. As the study of this convergence has essential difference from the present one, we leave it to a forthcoming paper.

In Section 2, we first recall some well known facts on the Dirichlet semigroup, then present an upper bound estimate on \( \|\nabla(\phi_m \phi_0^{-1})\|_\infty \). The latter is non-trivial when \( \partial M \) is non-convex, and should be interesting by itself. With these preparations, we prove upper and lower bound estimates in Sections 3 and 4 respectively.

### 2 Some preparations

We first recall some well known facts on the Dirichlet semigroup, see for instances [4, 7, 8, 13]. Let \( \{\phi_m\}_{m \geq 0} \) be the eigenbasis of the Dirichlet operator \( L \) in \( L^2(\mu) \), with Dirichlet eigenvalues \( \{\lambda_m\}_{m \geq 0} \) of \( -L \) listed in the increasing order counting multiplicities. Then \( \lambda_0 > 0 \) and

\[
\|\phi_m\|_\infty \leq \alpha_0 \sqrt{m}, \quad \alpha_0^{-1} m^{\frac{2}{3}} \leq \lambda_m - \lambda_0 \leq \alpha_0 m^{\frac{2}{3}}, \quad m \geq 1
\]

holds for some constant \( \alpha_0 > 1 \). Let \( \rho_0 \) be the Riemannian distance function to the boundary \( \partial M \). Then \( \phi_0^{-1} \rho_0 \) is bounded such that

\[
\|\phi_0^{-1}\|_{L^p(\mu_0)} < \infty, \quad p \in [1, 3).
\]
The Dirichlet heat kernel has the representation
\[ p_t^D(x, y) = \sum_{m=0}^{\infty} e^{-\lambda_m t} \phi_m(x) \phi_m(y), \quad t > 0, x, y \in M. \]

Let \( \mathbb{E}^x \) denote the expectation for the \( L \)-diffusion process starting at point \( x \). Then Dirichlet diffusion semigroup generated by \( L \) is given by
\[
P_t^D f(x) := \mathbb{E}^x[f(X_t) 1_{\{t < \tau\}}] = \int_M p_t^D(x, y) f(y) \mu(dy)
\]
\[ = \sum_{m=0}^{\infty} e^{-\lambda_m t} \mu(\phi_m f) \phi_m(x), \quad t > 0, f \in L^2(\mu). \]

There exists a constant \( c > 0 \) such that
\[
\|P_t^D\|_{L^p(\mu) \to L^q(\mu)} := \sup_{\mu(\|f\|) \leq 1} \|P_t^D f\|_{L^q(\mu)} \leq ce^{-\lambda_0 t} (1 \wedge t)^{-\frac{d(q-p)}{2pq}}, \quad t > 0, q \geq p \geq 1.
\]

Next, let \( L_0 = L + 2\nabla \log \phi_0 \). Then \( L_0 \) is a self-adjoint operator in \( L^2(\mu_0) \) with semigroup \( P_t^0 := e^{tL_0} \) satisfying
\[
P_t^0 f = e^{\lambda_0 t} \phi_0^{-1} P_t^D (f \phi_0), \quad f \in L^2(\mu_0), \quad t \geq 0.
\]

So, \( \{\phi_m^{-1} \phi_0\}_{m \geq 0} \) is an eigenbasis of \( L_0 \) in \( L^2(\mu_0) \) with
\[
L_0(\phi_m \phi_0^{-1}) = -(\lambda_m - \lambda_0) \phi_m \phi_0^{-1}, \quad P_t^0(\phi_m \phi_0^{-1}) = e^{-\lambda_m t} \phi_m \phi_0^{-1}, \quad m \geq 0, t \geq 0.
\]

Consequently,
\[
P_t^0 f = \sum_{m=0}^{\infty} \mu_0(f \phi_m \phi_0^{-1}) e^{-\lambda_m t} \phi_m \phi_0^{-1}, \quad f \in L^2(\mu_0),
\]
and the heat kernel of \( P_t^0 \) with respect to \( \mu_0 \) is given by
\[
p_t^0(x, y) = \sum_{m=0}^{\infty} (\phi_m \phi_0^{-1})(x)(\phi_m \phi_0^{-1})(y) e^{-\lambda_m t}, \quad x, y \in M, t > 0.
\]

By the intrinsic ultracontractivity, see for instance [9], there exists a constant \( \alpha_1 \geq 1 \) such that
\[
\|P_t^0 - \mu_0\|_{L^1(\mu_0) \to L^\infty(\mu_0)} := \sup_{\mu_0(\|f\|) \leq 1} \|P_t^0 f - \mu_0(f)\|_\infty \leq \frac{\alpha_1 e^{-\lambda_1 t}}{(1 \wedge t)^{\frac{d+2}{2}}}, \quad t > 0.
\]

Combining this with the semigroup property and the contraction of \( P_t^0 \) in \( L^p(\mu) \) for any \( p \geq 1 \), we find a constant \( \alpha_2 \geq 1 \) such that
\[
\|P_t^0 - \mu_0\|_{L^p(\mu_0)} := \sup_{\mu_0(\|f\|) \leq 1} \|P_t^0 f - \mu_0(f)\|_{L^p(\mu_0)} \leq \alpha_2 e^{-\lambda_1 t}, \quad t \geq 0, p \geq 1.
\]
By the interpolation theorem, (2.9) and (2.10) yield

\[ \| P_t^0 - \mu_0 \|_{L^p(\mu_0) \to L^q(\mu_0)} \leq \alpha_3 e^{-(\lambda_1 - \lambda_0)t} \{ 1 \land t \}^{-(\frac{(d+2)(q-p)}{2pq})}, \quad t > 0, \infty \geq q > p \geq 1. \]

Since \( \mu_0(\phi_m^2 \phi_0^{-2}) = 1 \), (2.11) for \( p = 2 \) implies

\[ \| \phi_m \phi_0^{-1} \|_{\infty} = e^{(\lambda_m - \lambda_0)t} \| P_t^0(\phi_m \phi_0^{-1}) \|_{\infty} \leq \frac{c_0(\lambda_m - \lambda_0)t}{(1 \land t)^{\frac{d+2}{4}}}, \quad t > 0. \]

Taking \( t = (\lambda_m - \lambda_0)^{-1} \) and applying (2.1), we find a constant \( \alpha_2 > 0 \) such that

\[ \| \phi_m \phi_0^{-1} \|_{\infty} \leq \alpha_2 m^{\frac{d+2}{4}}, \quad m \geq 1. \]

In the remainder of this section, we investigate gradient estimates on \( P_t^0 \) and \( \phi_m \phi_0^{-1} \), which will be used in Section 4 for the study of the lower bound estimate on \( \mathbb{W}_2(\mu^\nu_t, \mu_0) \). To this end, we need to estimate the Hessian tensor of \( \log \phi_0 \).

Let \( \rho_0 \) be the distance function to the boundary \( \partial M \) and \( T \partial M \) is the tangent bundle of the \((d - 2)\)-dimensional manifold \( \partial M \). When \( d = 1 \), the boundary \( \partial M \) degenerates to a set of two end points, such that \( \partial M = \emptyset \) and the condition (2.13) trivially holds; that is, \( M \) is convex for \( d = 1 \). Recall that \( M^\circ := M \setminus \partial M \) is the interior of \( M \).

**Lemma 2.1.** If \( \partial M \) is convex, then there exists a constant \( K_0 \geq 0 \) such that

\[ \text{Hess}_{\log \phi_0}(u, u) \leq K_0 |u|^2, \quad u \in TM^\circ. \]

**Proof.** Since \( M \) is compact with smooth boundary, there exists a constant \( r_0 > 0 \) such that \( \rho_0 \) is smooth on the set

\[ \partial_0 M := \{ x \in M : \rho_0(x) \leq r_0 \}. \]

Since \( \phi_0 \) is smooth and satisfies \( \phi_0 \geq c \rho_0 \) for some constant \( c > 0 \), we have \( \log(\phi_0 \rho_0^{-1}) \in C^2(\partial_0 M) \). So, it suffices to find a constant \( c > 0 \) such that

\[ \text{Hess}_{\log \rho_0}(u, u) \leq c |u|^2, \quad u \in TM^\circ. \]

To this end, we first estimate \( \text{Hess}_{\rho_0} \) on the boundary \( \partial M \). For any \( x \in \partial M \) and \( u \in T_x M \), consider the orthogonal decomposition \( u = u_1 + u_2 \), where

\[ u_1 = (N, u)N, \quad u_2 := u - u_1 \in T \partial M. \]

Since \( |\nabla \rho_0| = 1 \) on \( \partial_0 M \), we have

\[ \text{Hess}_{\rho_0}(X, N) = \text{Hess}_{\rho_0}(X, \nabla \rho_0) = \frac{1}{2} (X, \nabla |\nabla \rho_0|^2) = 0, \quad X \in T_x M. \]
On the other hand, since $u_2 \in T\partial M$ and $\nabla \rho_0 = N$ on $\partial M$, (2.13) implies
\[
\text{Hess}_{\rho_0}(u_2, u_2) = \langle \nabla u_2 N, u_2 \rangle \leq 0.
\]
Combining this with (2.15) we obtain
\[
\text{Hess}_{\rho_0}(u, u) = \langle N, u \rangle^2 \text{Hess}_{\rho_0}(N, N) + 2\langle N, u \rangle \text{Hess}_{\rho_0}(u_2, N) + \text{Hess}_{\rho_0}(u_2, u_2) \leq 0
\]
for $u \in \cup_{x \in \partial M} T_x M$. Since $\text{Hess}_{\rho_0}$ is smooth on the compact set $\partial_0 M$, this implies
\[
\text{Hess}_{\rho_0}(u, u) \leq c|u|^2 \rho_0(x), \quad x \in M, u \in T_x M
\]
for some constant $c > 0$. Then the desired estimate (2.14) follows from
\[
\text{Hess}_{\log \rho_0}(u, u) = \rho_0^{-1} \text{Hess}_{\rho_0}(u, u) - \rho_0^{-2} \langle \nabla \rho_0, u \rangle^2 \leq c|u|^2, \quad u \in TM^0.
\]

By Lemma 2.1, when $\partial M$ is convex, there exists a constant $K \geq 0$ such that
\[
(2.16) \quad \text{Ric} - \text{Hess}_{V + 2 \log \phi_0} \geq -K.
\]

Since the diffusion process generated by $L_0 := \Delta + \nabla (V + 2 \log \phi_0)$ is non-explosive in $M^0$, by (2.16) and Bakry-Emery’s semigroup calculus, (see for instance [3] or [13, Theorem 2.3.3]), we have
\[
(2.17) \quad |\nabla P^0_t g| \leq e^{Kt} P^0_t |\nabla g|, \quad t \geq 0, g \in C^1_b(M)
\]
and for any $p > 1$, there exists a constant $c(p) > 0$ such that
\[
(2.18) \quad |\nabla P^0_t g|^2 \leq \frac{2K\{(P^0_t |g|^{p+2})(P^0_t |g|)^{(2-p)+} - (P^0_t |g|)^2\}}{(p \wedge 2)(p \wedge 2 - 1)(1 - e^{-2Kt})}

\leq \frac{c(p)}{1 \wedge t} (P^0_t |g|^p)^{\frac{2}{p}}, \quad t > 0, g \in B_b(M).
\]

When $\partial M$ is non-convex, we take as in [12] a conformal change of metric to make it convex under the new metric. More precisely, we have the following result.

**Lemma 2.2.** There exists a function $1 \leq \phi \in C^\infty_b(M)$ such that $\partial M$ is convex under the metric $\langle \cdot, \cdot \rangle_\phi := \phi^{-2} \langle \cdot, \cdot \rangle$. Moreover, there exists a smooth vector field $Z_\phi$ on $M$ such that
\[
(2.19) \quad L_0 = \phi^{-2} \Delta_\phi + Z_\phi + 2\phi^{-1} \nabla_\phi \log \phi_0,
\]
where $\nabla_\phi$ and $\Delta_\phi$ are the gradient and Laplace-Beltrami operators induced by $\langle \cdot, \cdot \rangle_\phi$ respectively.
Proof. Let $\delta > 0$ such that the second fundamental form of $\partial M$ is bounded below by $-\delta$. Take $1 \leq \phi \in C^\infty_0(M)$ such that $\phi = 1 + \delta \rho_\partial$ in a neighborhood of $\partial M$ in which the distance function $\rho_\partial$ to $\partial M$ is smooth. By [14, Lemma 2.1] (see also [12]), $\partial M$ is convex under the metric $\langle \cdot, \cdot \rangle_\phi := \phi^{-2} \langle \cdot, \cdot \rangle$. Next, according to the proof of [14, Lemma 2.2], there exists a smooth vector field $Z_\phi$ on $M$ such that (2.19) holds.

Let $1 \leq \phi \in C^\infty_0(M)$ be in Lemma 2.2, and let $P^\phi_t$ be the diffusion semigroup generated by

$$L^\phi := \phi L_0 = \phi^{-1} \Delta^\phi + \phi Z_\phi + 2\nabla^\phi \log \phi_0.$$ 

We have the following result.

**Lemma 2.3.** Let $1 \leq \phi \in C^\infty_0(M)$ be in Lemma 2.2.

1. For any $p \in (1, \infty]$, there exists a constant $c > 0$ such that

$$|\nabla^\phi P^\phi_t f|_\phi \leq \frac{c(q)}{\sqrt{t}} (P^\phi_t |f|^q)^{\frac{1}{2}}, \quad t > 0, f \in C^1_b(M).$$

Moreover, there exists a constant $K > 0$ such that

$$|\nabla^\phi P^\phi_t f|_\phi \leq e^{Kt} P^\phi_t |\nabla^\phi f|_\phi, \quad t > 0, f \in C^1_b(M).$$

2. There exists a constant $c > 0$ such that

$$\|P^\phi_t\|_{L^p(\mu_0) \to L^\infty(\mu_0)} \leq \kappa'(1 \wedge t)^{-\frac{d+2}{dp}}, \quad t > 0, p \in [1, \infty].$$

Proof. (1) Since $\partial M$ is convex under the metric $\langle \cdot, \cdot \rangle_\phi$, by Lemma 2.1, we find a constant $K^\phi_0 > 0$ such that

$$2\text{Hess}^\phi_{\log \phi_0}(u, u) \leq K^\phi_0 |u|^2, \quad u \in TM^\circ,$$

where $\text{Hess}^\phi$ is the Hessian tensor induced by the metric $\langle \cdot, \cdot \rangle_\phi$. Since the operator $A^\phi := \phi^{-1} \Delta^\phi + \phi Z_\phi$ is a $C^2$-smooth strictly elliptic second order differential operator on the compact manifold $M$, it has bounded below Bakry-Emery curvature; that is, there exists a constant $K^\phi_1 > 0$ such that

$$A^\phi |\nabla^\phi f|^2_\phi \geq -K^\phi_1 |\nabla^\phi f|^2_\phi, \quad f \in C^\infty(M), |u|^2_\phi := \langle u, u \rangle_\phi.$$

Combining this with (2.23) we obtain

$$L^\phi |\nabla^\phi f|^2_\phi \geq -K^\phi_1 |\nabla^\phi f|^2_\phi =: -K^\phi |\nabla^\phi f|^2_\phi, \quad f \in C^\infty(M^\circ),$$

which means that the Bakry-Emery curvature of $L^\phi$ is bounded below by $-K^\phi$. By the same reason leading to (2.17) and (2.18), this implies (2.20) and (2.21).
(2) To estimate \( \|P_t^\phi\|_{L^p(\mu_0)\to L^\infty(\mu_0)} \), we make use of [10, Theorem 4.5(b)] or [11, Theorem 3.3.15(2)], which says that (2.9) implies the super Poincaré inequality
\[
\mu_0(f^2) \leq r\mu_0(|\nabla f|^2) + \beta(1 + r^{-\frac{d+2}{2}})\mu_0(|f|)^2, \quad f \in C_b^1(M)
\]
for some constant \( \beta > 0 \). Let \( \mu^\phi = \frac{\phi\mu_0}{\mu_0(\phi)} \). By \( L^\phi = \phi L_0 \) we obtain
\[
\mathcal{E}^\phi(f, g) := -\int_M fL^\phi g d\mu^\phi = -\frac{1}{\mu_0(\phi)}\int_M fL_0gd\mu_0 = \frac{1}{\mu(\phi)}\mu_0(\langle \nabla f, \nabla g \rangle), \quad f, g \in C_b^2(M).
\]
Then the above super Poincaré inequality implies
\[
\mu^\phi(f^2) \leq r\mathcal{E}^\phi(f, f) + \beta'(1 + r^{-\frac{d+2}{2}})\mu^\phi(|f|)^2, \quad f \in C_b^1(M)
\]
for some constant \( \beta' > 0 \). Using [10, Theorem 4.5(b)] or [11, Theorem 3.3.15(2)] again, this implies
\[
\|P_t^\phi\|_{L^p(\mu^\phi)\to L^\infty(\mu^\phi)} \leq \kappa(1 \wedge t)^{-\frac{d+2}{2p}}, \quad t > 0, p \in [1, \infty]
\]
for some constant \( \kappa > 0 \). Noting that
\[
\|\phi\|_{L^\infty}^{-1}\mu_0 \leq \mu^\phi \leq \|\phi\|_{L^\infty}\mu_0,
\]
we find a constant \( c > 0 \) such that (2.22) holds. \( \square \)

**Lemma 2.4.** For any \( p \in (1, \infty) \), there exists a constant \( c > 0 \) such that for any \( f \in D(L_0) \),
\[
\|\nabla P_t^0 f\|_\infty \leq ce^{-\lambda_0 t}\left(1 + t \right)^{-\frac{1}{2} - \frac{d+2}{2p}}\|\mu_0(f)\|_{L^p(\mu_0)} + (1 + t)^{\frac{1}{2} - \frac{d+2}{2p}}\|L_0 f\|_{L^p(\mu_0)}\}; \quad t > 0.
\]

Consequently, there exists a constant \( c > 0 \) such that
\[
\|\nabla(\phi_\infty^m\phi_0^{-1})\|_\infty \leq cm^\frac{d+4}{2p}, \quad m \geq 1.
\]

**Proof.** (a) By the semigroup property and the \( L^p(\mu_0) \) contraction of \( P_t^0 \), for the proof of (2.24) it suffices to consider \( t \in (0, 1] \). Since \( 1 \leq \phi \in C^\infty_b(M) \), we have \( \mathcal{D}(L_0) = \mathcal{D}(L^\phi) \) and
\[
P_t^\phi f = P_t^\phi f - \int_0^t P_s^\phi \{(\phi - 1)P_{t-s}^0 L_0 f\} ds, \quad t \geq 0, f \in \mathcal{D}(L_0).
\]

Next, by (2.20) and (2.22), we find constants \( c_1, c_2 > 0 \) such that
\[
\|\nabla P_t^\phi f\|_\infty = \|\nabla P_t^\phi (P_{t/2}^\phi f)\|_\infty \leq c_1 t^{-\frac{1}{2}}\|P_{t/2}^\phi f\|_\infty \leq c_2 t^{-\frac{1}{2} - \frac{d+2}{2p}}\|f\|_{L^p(\mu_0)}, \quad t \in (0, 1].
\]

Combining this with (2.11) and (2.20), we find constants \( c_3, c_4 > 0 \) such that
\[
\int_0^t \|\nabla P_s^\phi \{(\phi - 1)P_{t-s}^0 L_0 f\}\|_\infty ds \leq c_3 \int_0^t s^{-\frac{1}{2}}\|\{P_s^\phi P_{t-s}^0 L_0 f\|_\infty) \frac{1}{2}\|_\infty ds
\]

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Substituting this and (2.27) into (2.26), we prove (2.24).

(b) Applying (2.24) to \( p = \infty, f = \phi_m \phi_0^{-1}, t = (\lambda_m - \lambda_0)^{-1} \) and using (2.6), we obtain

\[
e^{-1} \| \nabla (\phi_m \phi_0^{-1}) \|_\infty \leq c_1 (\lambda_m - \lambda_0)^{\frac{1}{2}} \| \phi_m \phi_0^{-1} \|_\infty, \quad m \geq 1
\]

for some constant \( c_1 > 0 \). This together with (2.1) and (2.12) implies (2.25) for some constant \( c > 0 \). \( \square \)

### 3 Upper bound estimate

According to [15, Lemma 2.3], we have

\[
\mathbb{W}_2(\mu_{\nu}, \mu_0)^2 \leq \int_M \frac{|\nabla L^{-1}_0(h_{\nu} - 1)|^2}{\mathcal{M}(h_{\nu}, 1)} \, d\mu,
\]

where

\[
h_{\nu} := \frac{d\mu_{\nu}}{d\mu_0}, \quad \mathcal{M}(a, b) := 1_{\{a \land b > 0\}} \frac{a - b}{\log a - \log b}.
\]

So, to investigate the upper bound estimate, we first calculate \( h_{\nu} \).

By (2.8), we have

\[
(3.2) \quad \psi_{\nu} := \int_M \phi_0(x)p_s^0(x, \cdot)\nu(dx) = \nu(\phi_0) + \sum_{m=1}^\infty \nu(\phi_m)e^{-(\lambda_m - \lambda_0)s}\phi_m\phi_0^{-1}, \quad s > 0.
\]

Next, (2.5) and (2.8) imply

\[
(3.3) \quad \nu(P_s^D f) = e^{-\lambda_0 s}\nu(\phi_0 P_s^0(f\phi_0^{-1})) = e^{-\lambda_0 s}\int_M \psi_{\nu} \phi_0^{-1} f \, d\mu_0, \quad f \in \mathcal{B}^+(M),
\]

where \( \mathcal{B}^+(M) \) is the class of nonnegative measurable functions on \( M \). Moreover, for any \( t \geq s > 0 \), by the Markov property, (2.3), (2.5) and (3.3), we obtain

\[
\int_M f d\mathbb{E}^\nu[\delta_{X_s} 1_{\{t < \tau\}}] = \mathbb{E}^\nu[f(X_s) 1_{\{s < \tau\}}(P_{t-s}^D 1)(X_s)] = \nu(P_s^D \{P_{t-s}^D 1\})
\]

\[
e^{-\lambda_0 t}\int_M (\psi_{\nu} P_{t-s}^0 \phi_0^{-1}) f \, d\mu_0, \quad f \in \mathcal{B}^+(M).
\]
Noting that (3.3) implies
\[ \mathbb{E}[1_{t<\tau}] = \nu(P_t^D) = e^{-\lambda_0 t} \mu_0(\psi^t \phi_0^{-1}) = e^{-\lambda_0 t} \nu(\phi_0 P_t^0 \phi_0^{-1}), \]
we arrive at
\[ h_t^{\nu} := \frac{d\mu_t^{\nu}}{d\mu_0} = \frac{1}{t \mathbb{E}[1_{t<\tau}]} \int_0^t \frac{d\mathbb{E}[\delta_X 1_{t<\tau}]}{d\mu_0} \, ds = 1 + \rho_t^{\nu}, \]
(3.4)
\[ \rho_t^{\nu} := \frac{1}{t \nu(\phi_0 P_t^0 \phi_0^{-1})} \int_0^t \{ \psi_s^t P_t^0 \phi_0^{-1} - \nu(\phi_0 P_t^0 \phi_0^{-1}) \} \, ds. \]

By (2.11), \( \|\phi_0\|_\infty < \infty \) and \( \|\phi_0^{-1}\|_{L^2(\mu_0)} = 1 \), we find a constant \( c > 0 \) such that
\[ \|\nu(\phi_0 P_t^0 \phi_0^{-1}) - \nu(\phi_0)\mu(\phi_0)\| \leq \nu(\phi_0)\|P_t^0 \phi_0^{-1} - \mu_0(\phi_0^{-1})\| \leq ce^{-(\lambda_1 - \lambda_0)t}, \quad t \geq 1, \nu \in \mathcal{P}_0. \]

Due to the lack of simple representation of the product \( \psi^t_s P_{t-s}^0 \phi_0^{-1} \) in terms of the eigenbasis \( \{\phi_m \phi_0^{-1}\}_{m \geq 0} \), it is inconvenient to estimate the upper bound in (3.1). To this end, below we reduce this product to a linear combination of \( \psi^t_s \) and \( P_{t-s}^0 \phi_0^{-1} \), for which the spectral representation works. Write
\[ \psi^t_s P_{t-s}^0 \phi_0^{-1} - \nu(\phi_0 P_{t-s}^0 \phi_0^{-1}) = I_1(s) + I_2(s), \]
(3.6)
\[ I_1(s) := \{ \psi^t_s - \nu(\phi_0) \} \cdot \{ P_{t-s}^0 \phi_0^{-1} - \mu(\phi_0) \} + \nu(\phi_0) \{ \mu(\phi_0) - P_{t-s}^0 \phi_0^{-1} \}, \]
\[ I_2(s) := \mu(\phi_0) \{ \psi^t_s - \nu(\phi_0) \} + \nu(\phi_0) \{ P_{t-s}^0 \phi_0^{-1} - \mu(\phi_0) \}. \]

By (2.7), (2.8) and (3.2), we have
\[ P_{t-s}^0 \phi_0^{-1} - \mu(\phi_0) = \sum_{m=1}^{\infty} \mu(\phi_m) e^{-\lambda_m - \lambda_0}(t-s) \phi_m \phi_0^{-1}, \]
(3.7)
\[ \psi^t_s - \nu(\phi_0) = \sum_{m=1}^{\infty} \nu(\phi_m) e^{-\lambda_m - \lambda_0} s \phi_m \phi_0^{-1}, \quad t > s > 0. \]

Then
\[ \rho_t^{\nu} = \tilde{\rho}_t^{\nu} + \frac{1}{t \nu(\phi_0 P_t^0 \phi_0^{-1})} \int_0^t I_1(s) \, ds - A_t, \]
(3.8)
\[ \tilde{\rho}_t^{\nu} := \frac{1}{t \nu(\phi_0 P_t^0 \phi_0^{-1})} \sum_{m=1}^{\infty} \frac{\mu(\phi_0) \nu(\phi_m) + \nu(\phi_0) \mu(\phi_m)}{\lambda_m - \lambda_0} \phi_m \phi_0^{-1}, \]
\[ A_t := \frac{1}{t \nu(\phi_0 P_t^0 \phi_0^{-1})} \sum_{m=1}^{\infty} \frac{\{ \mu(\phi_0) \nu(\phi_m) + \nu(\phi_0) \mu(\phi_m) \} e^{-(\lambda_m - \lambda_0)t}}{\lambda_m - \lambda_0} \phi_m \phi_0^{-1}. \]

Since \( \rho_t^{\nu} \in L^1(\mu_0) \), the following lemma implies \( \tilde{\rho}_t^{\nu} \in L^1(\mu_0) \) for \( t > 0 \).
Lemma 3.1. There exists a constant $c > 0$ such that

\begin{equation}
\mu_0(|\rho_t^\nu - \tilde{\rho}_t^\nu|) \leq c \|h\|_{L^2(\mu)} e^{-(\lambda_1 - \lambda_0)t}, \quad t > 0, \nu = h \mu \in \mathcal{P}_0.
\end{equation}

Proof. By (2.1) and (2.12), for any $t_0 > 0$ we find a constant $c_0 > 0$ such that

\begin{equation}
\sum_{m=1}^\infty \|\phi_m\|_\infty e^{-(\lambda_m - \lambda_0)t} \leq c_0 e^{-(\lambda_1 - \lambda_0)t}, \quad t \geq t_0.
\end{equation}

Combining this with (3.8) and (3.5), and noting that $\|h\phi_0^{-1}\|_{L^2(\mu_0)} = \|h\|_{L^2(\mu)}$, it suffices to find a constant $c_1 > 0$ such that

\begin{equation}
B := \frac{1}{t} \int_0^t \|P_{t-s}\phi_0^{-1} - \mu(\phi_0)\|_{L^2(\mu_0)} ds \leq c_1 \|h\|_{L^2(\mu)} e^{-(\lambda_1 - \lambda_0)t}, \quad t \geq t_0.
\end{equation}

Since $\|\phi_0^{-1}\|_{L^2(\mu_0)} = 1$ and $\psi_s^\nu = P_s^0(h\phi_0^{-1})$ for $\nu = h \mu$, by (2.10), we find a constant $c_1 > 0$ such that

\begin{align*}
B &\leq \frac{1}{t} \int_0^t \|P_{t-s}\phi_0^{-1} - \mu_0(\phi_0^{-1})\|_{L^2(\mu_0)} \|P_s^0(h\phi_0^{-1}) - \mu_0(h\phi_0^{-1})\|_{L^2(\mu_0)} ds \\
&\leq \frac{1}{t} \int_0^t \|P_{t-s} - \mu_0\|_{L^2(\mu_0)} \|P_s^0 - \mu_0\|_{L^2(\mu_0)} \|h\|_{L^2(\mu)} ds \\
&\leq c_1 \|h\|_{L^2(\mu)} e^{-(\lambda_1 - \lambda_0)t}, \quad t > 0.
\end{align*}

\[\square\]

Lemma 3.2. For any $\alpha > 0$, there exist constants $c_0, t_0 > 0$ such that

\begin{equation}
\tilde{\rho}_t^\nu \geq -\frac{c_0}{\nu(\phi_0)t}, \quad t \geq t_0, \quad \nu \in \mathcal{P}_0, \nu \in \mathcal{P}_0.
\end{equation}

Consequently, if $\nu = h \mu$ with $h \in L^2(\mu)$, then $\tilde{\rho}_t^\nu := (1 + \tilde{\rho}_t^\nu)\mu_0$ is a probability measure for $t > t_0(1 + c_0)$.

Proof. By Lemma 3.1, if $\nu = h \mu$ with $h \in L^2(\mu)$, we have $\tilde{\rho}_t^\nu \in L^1(\mu_0)$ for $t > 0$, and it is easy to see that $\mu_0(\tilde{\rho}_t^\nu) = 0$. Since (3.12) implies $1 + \tilde{\rho}_t^\nu > 0$ for $t > t_0(1 + c_0)$, $\tilde{\rho}_t^\nu$ is a probability measure. It remains to prove (3.12).

By (3.5) and (3.8), it suffices to find a constant $c_1 > 0$ such that

\begin{equation}
g := \sum_{m=1}^\infty \frac{\mu(\phi_0)\nu_0 + \nu(\phi_0)\mu_0}{\lambda_m - \lambda_0} \phi_m \phi_0^{-1} \geq -c_1.
\end{equation}

By (2.1) and (2.12), we have

\begin{equation}
\|P_t^0 g\|_\infty \leq c_2 := \sum_{m=1}^\infty \frac{2\|\phi_0\|_\infty \|\phi_m\|_\infty \|\phi_m \phi_0^{-1}\|_\infty}{(\lambda_m - \lambda_0)e^{\lambda_m - \lambda_0}} < \infty.
\end{equation}
Next, by (3.7) and the same formula for \( \mu = \nu \), we obtain

\[
P_s^0 g = (-L_0)^{-1}\{\mu(\phi_0)(\psi_s - \nu(\phi_0)) + \nu(\phi_0)(\psi_s - \mu(\phi_0))\} = (-L_0)^{-1}g_s, \quad s > 0,
\]

where by \( \phi_0, \psi_s^\nu, \psi_s^\mu \geq 0, \)

\[
g_s := \mu(\phi_0)(\psi_s^\nu - \nu(\phi_0)) + \nu(\phi_0)(\psi_s^\mu - \mu(\phi_0)) \geq -2\mu(\phi_0)\nu(\phi_0) \geq -2\nu(\phi_0), \quad s > 0.
\]

This together with (3.15) yields

\[-L_0 P_s^0 g \geq -2\nu(\phi_0), \quad s > 0.\]

Therefore, it follows from (3.14) that

\[g = P_1^0 g - \int_0^1 L_0 P_r^0 g dr \geq -c_2 - 2\nu(\phi_0) \geq -c_2 - 2\|\phi_0\|_\infty.\]

So, (3.13) holds for \( c_1 = c_2 + 2\|\phi_0\|_\infty. \)

**Lemma 3.3.** There exist constants \( c, t_0 > 0 \) such that for any \( t \geq t_0 \), and any \( \nu \in \mathcal{P}_0 \) with \( \nu = h\mu \) such that \( h \in L^2(\mu) \), we have \( \tilde{\mu}^\nu_t \in \mathcal{P}_0 \) and

\[
t^{2\mathcal{W}_2(\tilde{\mu}^\nu_t, \mu_0)}^2 \leq \frac{1 + ct^{-1}}{\{\mu(\phi_0)\nu(\phi_0)\}^2} \sum_{m=1}^\infty \frac{\nu(\phi_0)\mu_0(\phi_m) + \mu(\phi_0)\nu(\phi_m)}{(\lambda_m - \lambda_0)^3}.
\]

**Proof.** By Lemma 3.2, there exist constants \( c, t_0 > 0 \) such that \( \tilde{\mu}^\nu_t \in \mathcal{P}_0 \) for \( t \geq t_0 \), and

\[
\mathcal{M}(1 + \tilde{\rho}^\nu_t, 1) \geq 1 \land (1 + \tilde{\rho}^\nu_t) \geq \frac{1}{1 + ct^{-1}}, \quad t \geq t_0.
\]

So, [15, Lemma 2.3] implies

\[
\mathcal{W}_2(\tilde{\mu}^\nu_t, \mu_0)^2 \leq \int_M \frac{\|\nabla L_0^{-1}\tilde{\rho}^\nu_t\|^2}{\mathcal{M}(1 + \tilde{\rho}^\nu_t, 1)} d\mu_0 \leq (1 + ct^{-1})\mu_0(\|\nabla L_0^{-1}\tilde{\rho}^\nu_t\|^2), \quad t \geq t_0.
\]

Next, (2.6) and (3.8) yield

\[
t^2\mu_0(\|\nabla L_0^{-1}\tilde{\rho}^\nu_t\|^2) = \frac{1}{\{\nu(\phi_0^0\phi_0^0)\}^2} \sum_{m=1}^\infty \frac{\mu(\phi_0)\mu_0(\phi_m) + \nu(\phi_0)\mu_0(\phi_m)}{(\lambda_m - \lambda_0)^3}.
\]

Combining this with (3.5) and (3.17), we finish the proof.

We are now ready to prove the following result.

**Proposition 3.4.** For any \( \nu \in \mathcal{P}_0 \),

\[
\limsup_{t \to \infty} \{t^2\mathcal{W}_2(\tilde{\mu}^\nu_t, \mu_0)^2\} \leq I.
\]
Proof. (1) We first consider $\nu = h\mu$ with $h \in L^2(\mu)$. Let $D$ be the diameter of $M$. By Lemma 3.1, there exist constants $c_1, t_0 > 0$ such that $\tilde{\mu}_t^\nu$ is probability measure for $t \geq t_0$ and

$$W_2(\mu_t^\nu, \tilde{\mu}_t^\nu)^2 \leq D^2\|\mu_t^\nu - \tilde{\mu}_t^\nu\| \leq D^2\|\mu_0^\nu(\rho_t^\nu - \tilde{\rho}_t^\nu)| \leq c_1\|h\|_{L^2(\mu)}e^{-(\lambda_1-\lambda_0)t}, \quad t \geq t_0.$$  

Combining this with Lemma 3.3 and the triangle inequality of $W_2$, we obtain

$$t^2W_2(\mu_t^\nu, \mu_0)^2 \leq (1 + \delta^{-1})c_1t^2e^{-(\lambda_1-\lambda_0)t}\|h\|_{L^2(\mu)} + (1 + \delta)(1 + ct^{-1})I, \quad \delta > 0.$$

(2) In general, we may go back to the first situation by shifting a small time $\varepsilon > 0$. More precisely, by the Markov property, (2.3), (2.5) and (3.2), for any $t \geq s \geq \varepsilon > 0$, we have

$$\mathbb{E}^\nu[f(X_s)|t < \tau] = \mathbb{E}^\nu[\psi_{\varepsilon, \phi_0}^\nu(y)\mathbb{P}(t - \varepsilon < \tau)\mu(dy)]$$

With $f = 1$ this implies

$$\mathbb{P}^\nu(t < \tau) = e^{-\lambda_0\varepsilon}\int_M (\psi_{\varepsilon, \phi_0}^\nu(y)\mathbb{P}(t - \varepsilon < \tau)\mu(dy))\mu(dy).$$

So, letting

$$\nu_\varepsilon = \frac{\psi_{\varepsilon, \phi_0}^\nu}{\mu(\psi_{\varepsilon, \phi_0}^\nu)} = \varepsilon h\mu,$$

we arrive at

$$\mathbb{E}^\nu[f(X_s)|t < \tau] = \frac{\mathbb{E}^\nu[f(X_s)|t < \tau]}{\mathbb{P}^\nu(t < \tau)} = \frac{\mathbb{E}^\nu[f(X_{s-\varepsilon})|t - \varepsilon < \tau]}{\mathbb{P}^\nu(t - \varepsilon < \tau)} = \mathbb{E}^\nu[f(X_{s-\varepsilon})|t - \varepsilon < \tau].$$

Therefore,

$$\mu_{t, \varepsilon}^\nu := \frac{1}{t - \varepsilon}\int_\varepsilon^t \mathbb{E}^\nu(\delta X_s|t < \tau)ds = \mu_{t-\varepsilon}^\nu, \quad t > \varepsilon.$$  

Since

$$\mu(\psi_{\varepsilon, \phi_0}^\nu) = \int_M p_\varepsilon^0(x, y)\phi_0(x)\phi_0(y)\mu(dx)\mu(dy) = \nu(\phi_0P_\varepsilon^0\phi_0^{-1}) \geq \nu(\phi_0)\|\phi_0\|_\infty^{-1} =: \alpha > 0,$$

by (2.9) we find a constant $c_2 > 0$ such that

$$\|h_\varepsilon\phi_0^{-1}\|_{L^2(\mu_0)} \leq \alpha^{-1}\|\psi_{\varepsilon, \phi_0}^\nu\|_{L^2(\mu_0)} \leq \alpha^{-1}\|\phi_0\|_\infty\|p_\varepsilon^0\|_{L^\infty(\mu_0)} \leq c_2e^{-\frac{\delta^2}{2}}, \quad \varepsilon \in (0, 1).$$
Then (3.20) and (3.21) yield
\[
(3.23) \quad t^2W_2(\mu_{\varepsilon}, \mu_0)^2 \\
\leq (1 + \delta^{-1})c_1c_2(\alpha^{-1}t^2e^{-(\lambda_1-\lambda_0)t_{\varepsilon^{-\frac{1}{2}}}2} + (1 + \delta)(1 + ct_{\varepsilon})I_{\varepsilon}, \quad \delta > 0, \varepsilon \in (0, 1),
\]
where
\[
I_{\varepsilon} := \frac{1}{\mu(\phi_0)\nu_{\varepsilon}(\phi_0)} \sum_{m=1}^{\infty} \frac{\{\nu_{\varepsilon}(\phi_0)\mu(\phi_m) + \mu(\phi_0)\nu_{\varepsilon}(\phi_m)\}^2}{(\lambda_m - \lambda_0)^3}.
\]
By (2.5), (2.6) and (3.2), we have
\[
\mu(\psi_{\varepsilon}^\nu\phi_0) = \nu(\phi_0P_{\varepsilon}^{-1}\phi_0^{-1}) = e^{\lambda_0\varepsilon}\nu(P_{\varepsilon}^D1), \\
\mu(\psi_{\varepsilon}\phi_0) = \nu(\phi_0P_{\varepsilon}^D(\phi_m\phi_0^{-1})) = e^{-(\lambda_m-\lambda_0)^{\varepsilon}}\nu(\phi_m),
\]
so that
\[
\nu_{\varepsilon}(\phi_m) = \frac{e^{-\lambda_m\varepsilon}\nu(\phi_m)}{\nu(P_{\varepsilon}^D1)}, \quad m \geq 0.
\]
Thus, \(\lim_{\varepsilon \to 0} \nu_{\varepsilon}(\phi_0) = \nu(\phi_0)\) and there exists a constant \(C > 1\) such that
\[
(3.24) \quad C^{-1}e^{-\lambda_m\varepsilon}\nu(\phi_m) \leq \nu_{\varepsilon}(\phi_m) \leq C\nu(\phi_m), \quad m \geq 1, \varepsilon \in (0, 1).
\]
Therefore, if \(I < \infty\), by this and
\[
(3.25) \quad \sum_{m=1}^{\infty} \mu(\phi_m)^2 \leq \mu(1) = 1,
\]
we may apply the dominated convergence theorem to derive \(\lim_{\varepsilon \to 0} I_{\varepsilon} = I\). On the other hand, if \(I = \infty\), which is equivalent to
\[
\sum_{m=1}^{\infty} \nu(\phi_m)^2 = \sum_{m=1}^{\infty} \frac{\nu(\phi_m)^2}{(\lambda_m - \lambda_0)^3} = \infty,
\]
then by (3.24) and the monotone convergence theorem we get
\[
\liminf_{\varepsilon \to 0} \sum_{m=1}^{\infty} \frac{\nu_{\varepsilon}(\phi_m)^2}{(\lambda_m - \lambda_0)^3} \geq C^{-2}\liminf_{\varepsilon \to 0} \sum_{m=1}^{\infty} \frac{e^{-2\lambda_m\varepsilon}\nu(\phi_m)^2}{(\lambda_m - \lambda_0)^3} = \infty,
\]
which together with (3.25) and \(\nu_{\varepsilon}(\phi_0) \to \nu(\phi_0)\) implies
\[
\liminf_{\varepsilon \to 0} I_{\varepsilon} = \frac{1}{\{\mu(\phi_0)\nu(\phi_0)\}^2} \liminf_{\varepsilon \to 0} \sum_{m=1}^{\infty} \frac{\{\nu_{\varepsilon}(\phi_0)\mu(\phi_m) + \mu(\phi_0)\nu_{\varepsilon}(\phi_m)\}^2}{(\lambda_m - \lambda_0)^3} \\
\geq \frac{1}{\{\mu(\phi_0)\nu(\phi_0)\}^2} \liminf_{\varepsilon \to 0} \frac{1}{2}\{\mu(\phi_0)\nu_{\varepsilon}(\phi_m)\}^2 - \|\phi_0\|^2 \mu(\phi_m)^2 = \infty.
\]
In conclusion, we have

\[(3.26) \lim_{\varepsilon \to 0} I_\varepsilon = I.\]

This together with (3.23) for \(\varepsilon = t^{-2}\) gives

\[(3.27) \limsup_{t \to \infty} \left\{ t^2 \mathbb{W}_2(\mu_{t,t-2}, \mu_0)^2 \right\} \leq I.\]

On the other hand, it is easy to see that

\[\|\mu_{t,\varepsilon}' - \mu_t'\|_{\text{var}} \leq \frac{2\varepsilon}{t}, \quad 0 < \varepsilon < t,\]

so that

\[(3.28) \mathbb{W}_2(\mu_t', \mu_{t,t-2})^2 \leq D^2 \|\mu_{t,t-2} - \mu_t'\|_{\text{var}} \leq 2D^2 t^{-3}, \quad t > 1.\]

Combining this with (3.27), we prove (3.18). \(\square\)

4 Lower bound estimate and the finiteness of the limit

We will follow the idea of [1, 15], for which we need to modify \(\tilde{\mu}_t'\) as follows. For any \(\beta > 0\), consider

\[\tilde{\mu}_{t,\beta} := (1 + \tilde{\rho}_{t,\beta}')\mu_0, \quad \tilde{\rho}_{t,\beta} := P_{t,\beta}^0 \tilde{\rho}_t', \quad t > 0.\]

According to Lemma 3.2, there exists \(t_0 > 0\) such that

\[(4.1) \tilde{h}_t' := 1 + \tilde{\rho}_t' \geq \frac{1}{2}, \quad \tilde{h}_{t,\beta} := 1 + \tilde{\rho}_{t,\beta}' \geq \frac{1}{2}, \quad \beta > 0, t \geq t_0.\]

Consequently, \(\tilde{\mu}_{t,\beta}'\) and \(\tilde{\mu}_t'\) are probability measures for any \(\beta > 0, t \geq t_0\).

**Lemma 4.1.** For any \(\beta > 0\), there exists a constant \(c > 0\) such that \(f_{t,\beta} := L_0^{-1}\tilde{\rho}_{t,\beta}'\) satisfies

\[\|f_{t,\beta}\|_\infty + \|L_0 f_{t,\beta}\|_\infty + \|\nabla f_{t,\beta}\|_\infty \leq ct^{\frac{5d+4}{4} - 1}, \quad t \geq 1.\]

**Proof.** By (2.6) and (3.8), we have

\[f_{t,\beta} = -\sum_{m=1}^{\infty} \frac{\{\mu(\phi_0)\nu(\phi_m) + \nu(\phi_0)\mu(\phi_m)\}e^{-(\lambda_m - \lambda_0)t}\beta}{t(\lambda_m - \lambda_0)^2\nu(\phi_0 P^0_\phi_0^{-1})} (\phi_m \phi_0^{-1}),\]

\[L_0 f_{t,\beta} = \sum_{m=1}^{\infty} \frac{\{\mu(\phi_0)\nu(\phi_m) + \nu(\phi_0)\mu(\phi_m)\}e^{-(\lambda_m - \lambda_0)t}\beta}{t(\lambda_m - \lambda_0)^2\nu(\phi_0 P^0_\phi_0^{-1})} (\phi_m \phi_0^{-1}).\]

Combining these with (2.1), (2.12), (3.5), and

\[|\mu(\phi_0)\nu(\phi_m) + \nu(\phi_0)\mu(\phi_m)| \leq \|\phi_0\|_\infty + \|\phi_m\|_\infty \leq c_0 m, \quad m \geq 1\]
for some constant $c_0 > 0$, we find constants $c_1, c_2, c_3, c_4, c_5 > 0$ such that
\[
t\|f_{t, \beta}\|_{\infty} + \|L_0 f_{t, \beta}\|_{\infty} \leq c_1 \sum_{m=1}^{\infty} e^{-c_2 m \frac{2}{m} t} \frac{m^{\frac{3d-2}{2}}} {\lambda_m - \lambda_0} \\
\leq c_2 \sum_{m=1}^{\infty} e^{-c_3 m \frac{\lambda_m - \lambda_0}{m^{\frac{3d-2}{2}}}} \leq c_4 \int_{0}^{\infty} e^{-c_3 s \frac{\lambda_m - \lambda_0}{m^{\frac{3d-2}{2}}}} ds \leq c_5 t^{\frac{\beta(5d-2)}{4}}, \quad t \geq 1.
\]

Similarly, by (2.25) we find constants $c_1', c_2', c_3' > 0$ such that
\[
t\|\nabla f_{t, \beta}\|_{\infty} \leq c_1' \sum_{m=1}^{\infty} e^{-c_2 m \frac{2}{m} t} \frac{m^{\frac{3d-4}{2}}} {\lambda_m - \lambda_0} \\
\leq c_2' \sum_{m=1}^{\infty} e^{-c_3 m \frac{\lambda_m - \lambda_0}{m^{\frac{3d-4}{2}}}} \leq c_3' t^{\frac{\beta(5d-4)}{4}}, \quad t \geq 1.
\]

Then the proof is finished.  

**Lemma 4.2.** For any $\beta \in (0, \frac{1}{2d})$, there exists a constant $c > 0$ such that
\[
t^2 W_2(\tilde{\mu}^{\nu}_{t, \beta}, \mu_0)^2 \geq 1 - ct^{-\frac{1}{4}} \frac{1}{\{\mu(\nu(\phi_0))^{\nu(\phi_0)}\}^{\frac{1}{2}}} \sum_{m=1}^{\infty} \frac{\{\mu(h\phi_0)\mu_0(\phi_m) + \mu(\phi_0)\nu(\phi_m)\}^2} {\lambda_m - \lambda_0} - ct^{-\frac{1}{4}}.
\]

**Proof.** To estimate $W_2(\tilde{\mu}^{\nu}_{t, \beta}, \mu_0)$ from below by using the argument in [1, 15], we take
\[
\varphi_{0}^{\nu} := -\varepsilon \log P_{t, \beta}^{0} e^{-\varepsilon^{-1} f_{t, \beta}}, \quad \theta \in [0, 1], \varepsilon > 0.
\]

We have $\varphi_{0}^{\nu} = f_{t, \beta}, \|\varphi_{0}^{\nu}\|_{\infty} \leq \|f_{t, \beta}\|_{\infty}$, and by [15, Lemma 2.9], there exists a constant $c_1 > 0$ such that for any $\varepsilon \in (0, 1),$
\[
\varphi_{1}^{\nu}(y) - \varphi_{0}^{\nu}(x) \leq \frac{1}{2} \{\rho(x, y)^{2} + \varepsilon \|L_0 f_{t, \beta}\|_{\infty} + c_1 \varepsilon \|\nabla f_{t, \beta}\|^{2}_{\infty}\}, \quad x, y \in M.
\]

Therefore, by the Kantorovich dual formula, $\varphi_{0}^{\nu} = f_{t, \beta}$ and the integration by parts formula
\[
\int_{M} f_{t, \beta} \delta_{t, \beta}^{\nu} d\mu_{0} = \int_{M} f_{t, \beta} L_{0} f_{t, \beta} d\mu_{0} = -\int_{M} |\nabla f_{t, \beta}|^{2} d\mu_{0},
\]
we find a constant $c > 0$ such that
\[
c(\varepsilon \|L_0 f_{t, \beta}\|_{\infty} + \varepsilon^{\frac{1}{2}} \|\nabla f_{t, \beta}\|_{\infty}^{\frac{1}{2}}) + \frac{1}{2} W_{2}(\tilde{\mu}^{\nu}_{t, \beta}, \mu_0)^{2} \geq \int_{M} \varphi_{1}^{\nu} d\mu_{0} - \int_{M} \varphi_{0}^{\nu} d\tilde{\mu}_{t, \beta}^{\nu} \\
\geq \frac{1}{2} \int_{M} |\nabla f_{t, \beta}|^{2} d\mu_{0} - c\varepsilon^{-1} \|\nabla f_{t, \beta}\|^{4}_{\infty}.
\]
Taking $\varepsilon = t^{-\frac{3}{2}}$ and applying Lemma 4.1, when $\beta \leq \frac{1}{20d}$ we find a constant $c' > 0$ such that

$$
(4.3) \quad t^2\mathcal{W}_2(\tilde{\mu}_{t,\beta}', \mu_0)^2 \geq t^2\mu_0(|\nabla f_{t,\beta}|^2) - c't^{-\frac{1}{2}}, \quad t \geq t_0.
$$

Combining this with (3.5) and (4.3), we complete the proof. \hfill \Box

**Lemma 4.3.** There exist constants $c, t_0 > 0$ such that for any $\nu = h\mu \in \mathcal{P}_0$ with $h \in L^2(\mu)$, $\tilde{\mu}_{t,\beta}', \tilde{\mu}_t' \in \mathcal{P}_0$ for $t \geq t_0$ and

$$
(4.4) \quad t\mathcal{W}_2(\tilde{\mu}_{t,\beta}', \tilde{\mu}_t') \leq c\|h\|_{L^2(\mu)}t^{-\beta}, \quad t \geq t_0.
$$

**Proof.** $\tilde{\mu}_{t,\beta}', \tilde{\mu}_t' \in \mathcal{P}_0$ for large $t$ is implied by Lemma 3.2. Next, by (4.1), we have

$$
\mathcal{M}(\tilde{h}_t, \tilde{h}_{t,\beta}) \geq \frac{1}{2},
$$

so that [15, Lemma 2.3] implies

$$
(4.5) \quad \mathcal{W}_2(\tilde{\mu}_{t,\beta}', \tilde{\mu}_t')^2 \leq \int_M \frac{[\nabla L^{-1}_0(\tilde{h}_t' - \tilde{\mu}_t')]}{\mathcal{M}(\tilde{h}_t', \tilde{h}_{t,\beta})} \, d\mu_0 \leq 2\mu_0(|\nabla L^{-1}_0(\tilde{\mu}_t' - \tilde{\mu}_{t,\beta}')|^2).
$$

To estimate the upper bound in this inequality, we first observe that by (3.7) and (3.8), when $\nu = h\mu$ we have

$$
L^{-1}_0(\tilde{\mu}_t' - \tilde{\mu}_{t,\beta}') = L^{-1}_0(P^0_{t-\beta}\tilde{\mu}_t' - \tilde{\mu}_t') = \int_0^{t-\beta} P^0_r \tilde{\mu}_r' \, dr
$$

$$
= \frac{1}{t\nu(\phi_0 P^0_{t-\beta})} \int_0^{t-\beta} (-L_0)^{-1}_0(P^0_r - \mu_0)g \, dr,
$$

where

$$
g := \mu(\phi_0)h\phi_0^{-1} + \nu(\phi_0)\phi_0^{-1}.
$$

Since $\|h\|_{L^2(\mu)} \geq \mu(h) = 1$,

$$
(4.6) \quad \|g\|_{L^2(\mu_0)} \leq \|\phi_0\|_\infty(1 + \|h\|_{L^2(\mu)}) \leq 2\|\phi_0\|_\infty\|h\|_{L^2(\mu)}.
$$

By (2.10), (4.6) and the fact that $(-L_0)^{-\frac{1}{2}} = c\int_0^\infty P^0_{s^2}ds$ for some constant $c > 0$, we find a constants $c_1, c_2 > 0$ such that

$$
\|\nabla L^{-1}_0(P^0_r - \mu_0)g\|_{L^2(\mu_0)} = \|L^{-\frac{1}{2}}_0(\mu_0 - \mu_0)g\|_{L^2(\mu_0)} \leq \int_0^\infty \|(P^0_r + s^2 - \mu_0)g\|_{L^2(\mu_0)} \, ds
$$

$$
\leq c_1\|h\|_{L^2(\mu)} \int_1^\infty e^{-(\lambda_1 - \lambda_0)(s^2 + r)} \, ds \leq c_2\|h\|_{L^2(\mu)}, \quad r \in [0, 1].
$$

Therefore, by (3.5) and (4.5), we find constants $c > 0$ such that

$$
\|\nabla L^{-1}_0(\tilde{\mu}_{t,\beta}' - \tilde{\mu}_t')\|_{L^2(\mu_0)} \leq c' t^{-\beta} \int_0^{t-\beta} \|\nabla L^{-1}_0(P^0_r - \mu_0)g\|_{L^2(\mu_0)} \, dr \leq ct^{-(1+\beta)}\|h\|_{L^2(\mu)}, \quad t \geq t_0.
$$

Combining this with (4.4) we finish the proof. \hfill \Box

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We are now ready to prove the following result.

**Proposition 4.4.** For any \( \nu \in \mathcal{P}_0 \),

\[
\lim_{t \to \infty} \left\{ t^2 \mathbb{W}_2^2(\mu_t^\nu, \mu_0) \right\} \geq I > 0,
\]

and \( I < \infty \) provided either \( d \leq 5 \), or \( d \geq 6 \) but \( \nu = h\mu \) with \( \mu(\phi_0^1) \wedge \mu_0(\abs{h\phi_0^1}^q) < \infty \) for some \( p > \frac{2d}{d+6} \) and \( q > \frac{2(d+2)}{d+6} \).

**Proof.** Let \( \beta \in (0, \frac{1}{2d}] \). By (3.19), Lemma 4.2 and Lemma 4.3, there exist constants \( c,t_0 > 0 \) such that for \( \nu = h\mu \in \mathcal{P}_0 \) and \( t \geq t_0 \),

\[
\begin{align*}
&\mathbb{W}_2(\mu_t^\nu, \tilde{\mu}_t^\nu) \leq c\|h\|_{L^2(\mu)} t^{-\beta t}, \\
&\mathbb{W}_2(\tilde{\mu}_{t,\beta}^\nu, \mu_0) \geq \left\{ (1 - ct^{-\frac{1}{2}})I - ct^{-\frac{1}{4}} \right\}^\frac{1}{2}, \\
&\mathbb{W}_2(\mu_t^\nu, \tilde{\mu}_t^\nu) \leq cte^{-\left(\lambda_1 - \lambda_0\right)t/2}\|h\|_{L^2(\mu)}^\frac{1}{2}.
\end{align*}
\]

Then

\[
\begin{align*}
\mathbb{W}_2(\mu_t^\nu, \mu_0) &\geq \left\{ (1 - ct^{-\frac{1}{2}})I - ct^{-\frac{1}{4}} \right\}^\frac{1}{2} - c\|h\|_{L^2(\mu)} t^{-\beta t} - cte^{-\left(\lambda_1 - \lambda_0\right)t/2}\|h\|_{L^2(\mu)}^\frac{1}{2}. \\
&\text{In general, let } \mu_{t,\varepsilon}^\nu = \mu_{t-\varepsilon}^\nu \text{ be in the proof of Proposition 3.4. Applying (4.8) to } \mu_{t,t-\varepsilon}^\nu \text{ replacing } \mu_{t-\varepsilon}^\nu \text{ and using (3.22), (3.26), we obtain}
\end{align*}
\]

\[
\lim_{t \to \infty} \left\{ \mathbb{W}_2(\mu_{t,t-\varepsilon}^\nu, \mu_0) \right\} \geq \sqrt{I},
\]

which together with (3.28) proves (4.7).

It remains to prove \( I > 0 \) and \( I < \infty \) the under given conditions, where due to (3.25), \( I < \infty \) is equivalent to

\[
I' := \sum_{m=0}^{\infty} \frac{\nu(\phi_m)^2}{(\lambda_m - \lambda_0)^2} < \infty.
\]

Below we first prove \( I > 0 \) then shown \( I' < \infty \) under the given conditions.

(a) \( I > 0 \). If this is not true, then

\[
\mu(h\phi_0)\mu(\phi_m) = -\mu(\phi_0)\mu(h\phi_m), \quad m \geq 1.
\]

Combining this with the representation in \( L^2(\mu) \)

\[
f = \sum_{m=0}^{\infty} \mu(f\phi_m)\phi_m, \quad f \in L^2(\mu),
\]

where the equation holds point-wisely if \( f \in C_b(M) \) by the continuity, we obtain

\[
\mu(\phi_0)\nu(f) = \sum_{m=0}^{\infty} \mu(f\phi_m)\mu(\phi_0)\nu(\phi_m) = 2\mu(f\phi_0)\nu(\phi_0)\mu(\phi_0) - \sum_{m=0}^{\infty} \mu(f\phi_m)\mu(\phi_m)\nu(\phi_m)
\]

\]


\[ = 2\mu(f \phi_0)\nu(\phi_0)\mu(\phi_0) - \nu(\phi_0)\mu(f), \quad f \in C_b(M). \]

Consequently,
\[ 0 \leq \mu(\phi_0) \frac{d\nu}{d\mu} = 2\phi_0\nu(\phi_0)\mu(\phi_0) - \nu(\phi_0), \]
which is however impossible since the upper bound is negative in a neighborhood of \( \partial M \),
because \( \nu(M^c) > 0 \) implies \( \nu(\phi) > 0 \) for \( \phi > 0 \) in \( M^c \), and \( \phi_0 \) is continuous with \( \phi_0|_{\partial M} = 0 \).
Therefore, we must have \( I > 0 \).

(b) \( I' < \infty \) for \( d \leq 5 \). By (2.6), (3.2), and \((-L_0)^{-\frac{1}{2}} = c \int_0^\infty P_s^0 ds \) for some constant \( c > 0 \), we obtain

\[ \sqrt{I'} = \left\| \int_0^\infty (-L_0)^{-\frac{1}{2}} \{ \psi_r^\nu - \nu(\phi_0) \} dr \right\|_{L^2(\mu_0)} \]
\[ \leq c \int_0^\infty dr \int_0^\infty \| (P_{s^2+r/2}^0 - \mu_0) \psi_r^\nu \|_{L^2(\mu_0)} ds. \]

Noting that (3.2) and (2.8) imply \( \| \psi_r^\nu \|_{L^1(\mu_0)} = \nu(\phi_0) = \infty \) and

\[ \mu(\psi_r^\nu) = e^{\lambda_0 r/2} \int_{M \times M} P_r^D(x,y) \nu(dx)\mu(dy) \leq e^{\lambda_0 r/2}, \]

by (2.4) and (2.5), we find a constant \( c_1 > 0 \) such that

\[ \| (P_{s^2+r/2}^0 - \mu_0) \psi_r^\nu \|_{L^2(\mu_0)} \leq \| \psi_r^\nu \|_{L^1(\mu_0)} + e^{(s^2+r)\lambda_0} \| P_{s^2+r/2}^D \|_{L^1(\mu_0) \rightarrow L^2(\mu)} \]
\[ \leq c_1 (s^2 + r)^{-\frac{d}{4}} \leq c_1 (s^2 + r)^{-\frac{5}{4}}, \quad s^2 + r/2 \leq 1, \quad d \leq 5, \]

and due to (2.11)

\[ \| (P_{s^2+r/2}^0 - \mu_0) \psi_r^\nu \|_{L^2(\mu_0)} \leq \| P_{s^2+r/2}^0 - \mu_0 \|_{L^1(\mu_0) \rightarrow L^2(\mu_0)} \| \psi_r^\nu \|_{L^1(\mu_0)} \]
\[ \leq c_2 e^{-(\lambda_1 - \lambda_2)(s^2 + r/2)}, \quad s^2 + r \geq 1 \]

holds for some constant \( c_2 > 0 \). Combining these with (4.10), we prove \( I' < \infty \).

(c) \( I' < \infty \) for \( d \geq 6 \) and \( \nu = h\mu \) with \( h \in L^p(\mu) \) for some \( p > \frac{2}{d+6} \). Since \( \{\phi_m\phi_0^{-1}\}_{m \geq 0} \) is an orthonormal basis of \( L^2(\mu_0) \) and \( \mu_0(\phi_0^{-1} - \mu(\phi_0)) = 0 \), we have

\[ h\phi_0^{-1} - \mu(\phi_0) = \sum_{m=1}^\infty \mu_0 \{ h\phi_0^{-1} - \mu(\phi_0) \} \phi_m\phi_0^{-1}, \]

so that (2.6) and \( \mu_0(\phi_m\phi_0^{-1}) = 0 \) for \( m \geq 1 \) yield

\[ (-L_0)^{-\frac{3}{2}} (h\phi_0^{-1} - \mu(\phi_0)) = \sum_{m=1}^\infty \frac{\mu_0 \{ h\phi_0^{-1} - \mu(\phi_0) \} \phi_m\phi_0^{-1}}{(\lambda_m - \lambda_0)^{\frac{3}{2}}} \phi_m\phi_0^{-1}, \]
\[ = \sum_{m=1}^\infty \frac{\mu(h\phi_m)}{(\lambda_m - \lambda_0)^{\frac{3}{2}}} \phi_m\phi_0^{-1}. \]
Thus,

\[(4.11)\quad I' = \|(-L_0)^{-\frac{3}{2}}(h\phi_0^{-1} - \mu(h\phi_0))\|_{L^2(\mu_0)^2}^2.\]

Noting that $\mu_0((h\phi_0^{-1} - \mu(h\phi_0)) = 0$ and $(-L_0)^{-\frac{3}{2}} = c\int_0^\infty P_{t^3}^0 dt$ for some constant $c > 0$, combining this with (2.4), (2.5), (2.11), $\|h\phi_0^{-1}\|_{L^1(\mu_0)} < \infty$, and $\|h\|_{L^p(\mu_0)} < \infty$ for some $p \in \left(\frac{2(d+2)}{d+8}, 2\right)$ as we have assumed, we find constants $c_1, c_2 > 0$ such that

\[
\|(-L_0)^{-\frac{3}{2}}(h\phi_0^{-1} - \mu(h\phi_0))\|_{L^2(\mu_0)} \leq \int_0^\infty \|(P_{t^3}^0 - \mu_0)\{h\phi_0^{-1}\}\|_{L^2(\mu_0)} dt
\]

\[
\leq \|h\phi_0^{-1}\|_{L^1(\mu_0)} \int_1^\infty \|P_{t^3}^0 - \mu_0\|_{L^1(\mu_0) \to L^2(\mu_0)} dt + \int_0^1 \|\phi_0^{-1} P_{t^3}D \{h - \nu(\phi_0)\phi_0\}\|_{L^2(\mu_0)} dt
\]

\[
= \|h\phi_0^{-1}\|_{L^1(\mu_0)} \int_1^\infty \|P_{t^3}^0 - \mu_0\|_{L^1(\mu_0) \to L^2(\mu_0)} dt + \int_0^1 \|\phi_0^{-1} P_{t^3}D \{h - \nu(\phi_0)\phi_0\}\|_{L^2(\mu_0)} dt
\]

\[
\leq c_1 \int_1^\infty e^{-(\lambda_1 - \lambda_0)t} dt + c_1 \int_0^1 \|P_{t^3}D\|_{L^p(\mu) \to L^2(\mu)} dt \leq \frac{c_1}{\lambda_1 - \lambda_0} + c_2 \int_0^1 t^{-\frac{d(2-p)}{6q}} dt < \infty,
\]

since $p > \frac{2d}{d+6}$ implies $\frac{d(2-p)}{6q} < 1$. Combining this with (4.11) we prove (4.9).

(d) $I' < \infty$ for $d \geq 6$ and $\nu = h\mu$ with $h\phi_0^{-1} \in L^q(\mu_0)$ for some $q > \frac{2(d+2)}{d+8}$. By (2.11) we find constants $c_1, c_2 > 0$ such that

\[
\|(-L_0)^{-\frac{3}{2}}(h\phi_0^{-1} - \mu(h\phi_0))\|_{L^2(\mu_0)} \leq \int_0^\infty \|(P_{t^3}^0 - \mu_0)\{h\phi_0^{-1}\}\|_{L^2(\mu_0)} dt
\]

\[
\leq \int_0^\infty \|P_{t^3}^0 - \mu_0\|_{L^q(\mu_0) \to L^2(\mu_0)} \|h\phi_0^{-1}\|_{L^q(\mu_0)} dt
\]

\[
\leq c_1 \int_0^\infty \{1 \land t\}^{-\frac{(d+2)(2-q)}{6q}} e^{-(\lambda_1 - \lambda_0)t^\frac{3}{2}} dt < \infty
\]

since $q > \frac{2(d+2)}{d+8}$ implies $\frac{(d+2)(2-q)}{6q} < 1$.

\[\square\]

References


