Precise Limit in Wasserstein Distance for Conditional Empirical Measures of Dirichlet Diffusion Processes*

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July 1, 2020

Abstract

Let M be a d-dimensional connected compact Riemannian manifold with boundary ∂M, let $V \in C^2(M)$ such that $\mu(\mathrm{d}x) := \mathrm{e}^{V(x)}\mathrm{d}x$ is a probability measure, and let X_t be the diffusion process generated by $L := \Delta + \nabla V$ with $\tau := \inf\{t \ge 0 : X_t \in \partial M\}$. Consider the conditional empirical measure $\mu_t^\nu := \mathbb{E}^\nu\big(\frac{1}{t}\int_0^t \delta_{X_s}\mathrm{d}s\big|t < \tau\big)$ for the diffusion process with initial distribution ν such that $\nu(\partial M) < 1$. Then

$$\lim_{t\to\infty} \big\{t\mathbb{W}_2(\mu_t^\nu, \mu_0)\big\}^2 = \frac{1}{\{\mu(\phi_0)\nu(\phi_0)\}^2} \sum_{m=1}^\infty \frac{\{\nu(\phi_0)\mu(\phi_m) + \mu(\phi_0)\nu(\phi_m)\}^2}{(\lambda_m - \lambda_0)^3},$$

where $\nu(f) := \int_M f\mathrm{d}\nu$ for a measure ν and $f \in L^1(\nu)$, $\mu_0 := \phi_0^2\mu$, $\{\phi_m\}_{m\ge0}$ is the eigenbasis of $-L$ in $L^2(\mu)$ with the Dirichlet boundary, $\{\lambda_m\}_{m\ge0}$ are the corresponding Dirichlet eigenvalues, and $\mathbb{W}_2$ is the L^2-Wasserstein distance induced by the Riemannian metric.

AMS subject Classification: 60D05, 58J65.
Keywords: Conditional empirical measure, Dirichlet diffusion process, Wasserstein distance, eigenvalues, eigenfunctions.

1 Introduction

Let M be a d-dimensional connected complete Riemannian manifold with a smooth boundary ∂M. Let $V \in C^2(M)$ such that $\mu(\mathrm{d}x) = \mathrm{e}^{V(x)}\mathrm{d}x$ is a probability measure on M, where $\mathrm{d}x$ is

*Supported in part by NNSFC (11771326, 11831014, 11921001), and DFG through the CRC ?Taming uncertainty and profiting from randomness and low regularity in analysis, stochastics and their applications?.

the Riemannian volume measure. Let X_t be the diffusion process generated by $L := \Delta + \nabla V$ with hitting time

$$\tau := \inf\{t \geq 0 : X_t \in \partial M\}.$$

Denote by $\mathscr{P}$ the set of all probability measures on M, and let $\mathbb{E}^\nu$ be the expectation taken for the diffusion process with initial distribution $\nu \in \mathscr{P}$. Consider the conditional empirical measure

$$\mu_t^\nu := \mathbb{E}^\nu\left(\frac{1}{t}\int_0^t \delta_{X_s} \mathrm{d}s \middle| t < \tau\right), \quad t > 0, \nu \in \mathscr{P}.$$

Since $\tau = 0$ when $X_0 \in \partial M$, to ensure $\mathbb{P}^\nu(\tau > t) > 0$ we only consider

$$\nu \in \mathscr{P}_0 := \big\{\nu \in \mathscr{P} : \ \nu(M^\circ) > 0\big\}, \quad M^\circ := M \setminus \partial M.$$

Let $\{\phi_m\}_{m \geq 0}$ be the eigenbasis in $L^2(\mu)$ of $-L$ with the Dirichlet boundary such that $\phi_0 > 0$ in M°, and let $\{\lambda_m\}_{m \geq 0}$ be the associated eigenvalues listed in the increasing order counting multiplicities. Then $\mu_0 := \phi_0^2 \mu$ is a probability measure on M. It is easy to see from [5, Theorem 2.1] that for any probability measure ν supported on M°, we have

$$\lim_{t \to \infty} \|\mu_t^\nu - \mu_0\|_{var} = 0,$$

where $\|\cdot\|_{var}$ is the total variational norm.

In this paper, we investigate the convergence of μ_t^ν to μ_0 under the Wasserstein distance $\mathbb{W}_2$:

$$\mathbb{W}_2(\mu_1, \mu_2) := \inf_{\pi \in \mathscr{C}(\mu_1, \mu_2)} \left(\int_{M \times M} \rho(x,y)^2 \pi(\mathrm{d}x, \mathrm{d}y)\right)^{\frac{1}{2}}, \quad \mu_1, \mu_2 \in \mathscr{P},$$

where $\mathscr{C}(\mu_1, \mu_2)$ is the set of all probability measures on $M \times M$ with marginal distributions μ_1 and μ_2, and $\rho(x,y)$ is the Riemannian distance between x and y, i.e. the length of the shortest curve on M linking x and y.

Recently, the convergence rate under $\mathbb{W}_2$ has been characterized in [15] for the empirical measures of the L-diffusion processes without boundary (i.e. $\partial M = \emptyset$) or with a reflecting boundary. Since in the present setting the diffusion process is killed at time τ, it is reasonable to consider the conditional empirical measure μ_t^ν given $t < \tau$. This is a counterpart to the quasi-ergodicity for the convergence of the conditional distribution $\tilde{\mu}_t$ of X_t given $t < \tau$. Unlike in the case without boundary or with a reflecting boundary where both the distribution and the empirical measure of X_t converge to the unique invariant probability measure, in the present case the conditional distribution $\tilde{\mu}_t$ of X_t given $t < \tau$ converges to $\tilde{\mu}_0 := \frac{\phi_0}{\mu(\phi_0)}\mu$ rather than $\mu_0 := \phi_0^2 \mu$, and this convergence is called the quasi-ergodicity in the literature, see for instance [6] and references within.

Let $\nu(f) := \int_M f \mathrm{d}\nu$ for $\nu \in \mathscr{P}$ and $f \in L^1(\nu)$. The main result of this paper is the following.

Theorem 1.1. *For any $\nu \in \mathscr{P}_0$,*

$$\lim_{t \to \infty} \big\{t^2 \mathbb{W}_2(\mu_t^\nu, \mu_0)^2\big\} = I := \frac{1}{\{\mu(\phi_0)\nu(\phi_0)\}^2} \sum_{m=1}^\infty \frac{\{\nu(\phi_0)\mu(\phi_m) + \mu(\phi_0)\nu(\phi_m)\}^2}{(\lambda_m - \lambda_0)^3} > 0.$$

If either $d \le 5$ or $d \ge 6$ but $\nu = h\mu$ with $\mu(h^p) \wedge \mu_0(|h\phi_0^{-1}|^q) < \infty$ for some $p > \frac{2d}{d+6}$ and $q > \frac{2(d+2)}{d+6}$, then $I < \infty$.

Remark 1.1. (1) Let X_t be the (reflecting) diffusion process generated by L on M where ∂M may be empty. We consider the mean empirical measure $\hat{\mu}_t^\nu := \mathbb{E}(\frac{1}{t}\int_0^t \delta_{X_s} \mathrm{d}s)$, where ν is the initial distribution of X_t. Then

$$\lim_{t\to\infty} \left\{ t^2 \mathbb{W}_2(\hat{\mu}_t^\nu, \mu_0)^2 \right\} = \sum_{m=1}^{\infty} \frac{\{\nu(\phi_m)\}^2}{\lambda_m^3} < \infty, \tag{1.1}$$

where $\{\phi_m\}_{m\ge1}$ is the eigenbasis of $-L$ in $L^2(\mu)$ with the Neumann boundary condition if ∂M exists, $\{\lambda_m\}_{m\ge1}$ are the corresponding non-trivial (Neumann) eigenvalues, and the limit is zero if and only if $\nu = \mu$. This can be confirmed by the proof of Theorem 1.1 with $\phi_0 = 1, \lambda_0 = 0$ and $\mu(\phi_m) = 0$ for $m \ge 1$. In this case, μ is the unique invariant probability measure of X_t, so that $\hat{\mu}_t^\mu = \mu$ for $t \ge 0$ and hence the limit in (1.1) is zero for $\nu = \mu$. However, in the Dirichlet diffusion case, the conditional distribution of $(X_s)_{0\le s\le t}$ given $t < \tau$ is no longer stationary, so that even starting from the limit distribution μ_0 we **do not have** $\mu_t^{\mu_0} = \mu_0$ for $t > 0$. This leads to a non-zero limit in Theorem 1.1 even for $\nu = \mu_0$.

(2) It is also interesting to investigate the convergence of $\mathbb{E}^\nu(\mathbb{W}_2(\mu_t, \mu_0)^2 | t < \tau)$ for $\mu_t := \frac{1}{t}\int_0^t \delta_{X_s} \mathrm{d}s$, which is the counterpart to the study of [15] where the case without boundary or with a reflecting boundary is considered. According to [15], the convergence rate of $\mathbb{E}^\nu(\mathbb{W}_2(\mu_t, \mu_0)^2 | t < \tau)$ will be at most t^{-1}, which is slower than the rate t^{-2} for $\mathbb{W}_2(\mu_t^\nu, \mu_0)^2$ as shown in Theorem 1.1. As the study of this convergence has essential difference from the present one, we leave it to a forthcoming paper.

In Section 2, we first recall some well known facts on the Dirichlet semigroup, then present an upper bound estimate on $\|\nabla(\phi_m\phi_0^{-1})\|_\infty$. The latter is non-trivial when ∂M is non-convex, and should be interesting by itself. With these preparations, we prove upper and lower bound estimates in Sections 3 and 4 respectively.

2 Some preparations

We first recall some well known facts on the Dirichlet semigroup, see for instances [4, 7, 8, 13]. Let $\{\phi_m\}_{m\ge0}$ be the eigenbasis of the Dirichlet operator L in $L^2(\mu)$, with Dirichlet eigenvalues $\{\lambda_m\}_{m\ge0}$ of $-L$ listed in the increasing order counting multiplicities. Then $\lambda_0 > 0$ and

$$\|\phi_m\|_\infty \le \alpha_0\sqrt{m}, \quad \alpha_0^{-1} m^{\frac{2}{d}} \le \lambda_m - \lambda_0 \le \alpha_0 m^{\frac{2}{d}}, \quad m \ge 1 \tag{2.1}$$

holds for some constant $\alpha_0 > 1$. Let ρ_∂ be the Riemannian distance function to the boundary ∂M. Then $\phi_0^{-1}\rho_\partial$ is bounded such that

$$\|\phi_0^{-1}\|_{L^p(\mu_0)} < \infty, \quad p \in [1, 3). \tag{2.2}$$

The Dirichlet heat kernel has the representation

$$p_t^D(x,y) = \sum_{m=0}^{\infty} \mathrm{e}^{-\lambda_m t}\phi_m(x)\phi_m(y), \quad t>0, x,y\in M.$$

Let $\mathbb{E}^x$ denote the expectation for the L-diffusion process starting at point x. Then Dirichlet diffusion semigroup generated by L is given by

$$\begin{aligned} P_t^D f(x) &:= \mathbb{E}^x[f(X_t)1_{\{t<\tau\}}] = \int_M p_t^D(x,y)f(y)\mu(\mathrm{d}y) \\ &= \sum_{m=0}^{\infty} \mathrm{e}^{-\lambda_m t}\mu(\phi_m f)\phi_m(x), \quad t>0, f\in L^2(\mu). \end{aligned} \tag{2.3}$$

There exists a constant $c>0$ such that

$$\|P_t^D\|_{L^p(\mu)\to L^q(\mu)} := \sup_{\mu(|f|^p)\le 1} \|P_t^D f\|_{L^q(\mu)} \le c\mathrm{e}^{-\lambda_0 t}(1\wedge t)^{-\frac{d(q-p)}{2pq}}, \quad t>0, q\ge p\ge 1. \tag{2.4}$$

Next, let $L_0 = L + 2\nabla\log\phi_0$. Then L_0 is a self-adjoint operator in $L^2(\mu_0)$ with semigroup $P_t^0 := \mathrm{e}^{tL_0}$ satisfying

$$P_t^0 f = \mathrm{e}^{\lambda_0 t}\phi_0^{-1}P_t^D(f\phi_0), \quad f\in L^2(\mu_0), \quad t\ge 0. \tag{2.5}$$

So, $\{\phi_0^{-1}\phi_m\}_{m\ge 0}$ is an eigenbasis of L_0 in $L^2(\mu_0)$ with

$$L_0(\phi_m\phi_0^{-1}) = -(\lambda_m-\lambda_0)\phi_m\phi_0^{-1}, \quad P_t^0(\phi_m\phi_0^{-1}) = \mathrm{e}^{-(\lambda_m-\lambda_0)t}\phi_m\phi_0^{-1}, \quad m\ge 0, t\ge 0. \tag{2.6}$$

Consequently,

$$P_t^0 f = \sum_{m=0}^{\infty} \mu_0(f\phi_m\phi_0^{-1})\mathrm{e}^{-(\lambda_m-\lambda_0)t}\phi_m\phi_0^{-1}, \quad f\in L^2(\mu_0), \tag{2.7}$$

and the heat kernel of P_t^0 with respect to μ_0 is given by

$$p_t^0(x,y) = \sum_{m=0}^{\infty} (\phi_m\phi_0^{-1})(x)(\phi_m\phi_0^{-1})(y)\mathrm{e}^{-(\lambda_m-\lambda_0)t}, \quad x,y\in M, t>0. \tag{2.8}$$

By the intrinsic ultracontractivity, see for instance [9], there exists a constant $\alpha_1\ge 1$ such that

$$\|P_t^0-\mu_0\|_{L^1(\mu_0)\to L^\infty(\mu_0)} := \sup_{\mu_0(|f|)\le 1} \|P_t^0 f - \mu_0(f)\|_\infty \le \frac{\alpha_1 \mathrm{e}^{-(\lambda_1-\lambda_0)t}}{(1\wedge t)^{\frac{d+2}{2}}}, \quad t>0. \tag{2.9}$$

Combining this with the semigroup property and the contraction of P_t^0 in $L^p(\mu)$ for any $p\ge 1$, we find a constant $\alpha_2\ge 1$ such that

$$\|P_t^0-\mu_0\|_{L^p(\mu_0)} := \sup_{\mu_0(|f|^p)\le 1} \|P_t^0 f - \mu_0(f)\|_{L^p(\mu_0)} \le \alpha_2\mathrm{e}^{-(\lambda_1-\lambda_0)t}, \quad t\ge 0, p\ge 1. \tag{2.10}$$

By the interpolation theorem, (2.9) and (2.10) yield

$$(2.11)\qquad \|P_t^0-\mu_0\|_{L^p(\mu_0)\to L^q(\mu_0)}\le \alpha_3 \mathrm{e}^{-(\lambda_1-\lambda_0)t}\{1\wedge t\}^{-\frac{(d+2)(q-p)}{2pq}},\quad t>0, \infty\ge q>p\ge 1.$$

Since $\mu_0(\phi_m^2\phi_0^{-2})=1$, (2.11) for $p=2$ implies

$$\|\phi_m\phi_0^{-1}\|_\infty = \mathrm{e}^{(\lambda_m-\lambda_0)t}\|P_t^0(\phi_m\phi_0^{-1})\|_\infty\le \frac{c\mathrm{e}^{(\lambda_m-\lambda_0)t}}{(1\wedge t)^{\frac{d+2}{4}}},\quad t>0.$$

Taking $t=(\lambda_m-\lambda_0)^{-1}$ and applying (2.1), we find a constant $\alpha_2>0$ such that

$$(2.12)\qquad \|\phi_m\phi_0^{-1}\|_\infty\le \alpha_2 m^{\frac{d+2}{2d}},\quad m\ge 1.$$

In the remainder of this section, we investigate gradient estimates on P_t^0 and $\phi_m\phi_0^{-1}$, which will be used in Section 4 for the study of the lower bound estimate on $\mathbb{W}_2(\mu_t^\nu,\mu_0)$. To this end, we need to estimate the Hessian tensor of $\log\phi_0$.

Let N be the inward unit normal vector field of ∂M. We call M (or ∂M) convex if

$$(2.13)\qquad \langle\nabla_u N,u\rangle = \mathrm{Hess}_{\rho_\partial}(u,u)\le 0,\quad u\in T\partial M,$$

where ρ_∂ is the distance function to the boundary ∂M, and $T\partial M$ is the tangent bundle of the $(d-2)$-dimensional manifold ∂M. When $d=1$, the boundary ∂M degenerates to a set of two end points, such that $\partial M=\emptyset$ and the condition (2.13) trivially holds; that is, M is convex for $d=1$. Recall that $M^\circ := M\setminus\partial M$ is the interior of M.

Lemma 2.1. *If ∂M is convex, then there exists a constant $K_0\ge 0$ such that*

$$\mathrm{Hess}_{\log\phi_0}(u,u)\le K_0|u|^2,\quad u\in TM^\circ.$$

Proof. Since M is compact with smooth boundary, there exists a constant $r_0>0$ such that ρ_∂ is smooth on the set

$$\partial_0 M := \{x\in M : \rho_\partial(x)\le r_0\}.$$

Since ϕ_0 is smooth and satisfies $\phi_0\ge c\rho_\partial$ for some constant $c>0$, we have $\log(\phi_0\rho_\partial^{-1})\in C_b^2(\partial_0 M)$. So, it suffices to find a constant $c>0$ such that

$$(2.14)\qquad \mathrm{Hess}_{\log\rho_\partial}(u,u)\le c|u|^2,\quad u\in TM^\circ.$$

To this end, we fisrt estimate $\mathrm{Hess}_{\rho_\partial}$ on the boundary ∂M. For any $x\in\partial M$ and $u\in T_xM$, consider the orthogonal decomposition $u=u_1+u_2$, where

$$u_1=\langle N,u\rangle N,\quad u_2:=u-u_1\in T\partial M.$$

Since $|\nabla\rho_\partial|=1$ on $\partial_0 M$, we have

$$(2.15)\qquad \mathrm{Hess}_{\rho_\partial}(X,N)=\mathrm{Hess}_{\rho_\partial}(X,\nabla\rho_\partial)=\frac{1}{2}\langle X,\nabla|\nabla\rho_\partial|^2\rangle=0,\quad X\in T_xM.$$

On the other hand, since $u_2 \in T\partial M$ and $\nabla \rho_\partial = N$ on ∂M, (2.13) implies

$$\mathrm{Hess}_{\rho_\partial}(u_2, u_2) = \langle \nabla_{u_2} N, u_2 \rangle \le 0.$$

Combining this with (2.15) we obtain

$$\mathrm{Hess}_{\rho_\partial}(u,u) = \langle N, u\rangle^2 \mathrm{Hess}_{\rho_\partial}(N,N) + 2\langle N, u\rangle \mathrm{Hess}_{\rho_\partial}(u_2, N) + \mathrm{Hess}_{\rho_\partial}(u_2, u_2) \le 0$$

for $u \in \cup_{x \in \partial M} T_x M$. Since $\mathrm{Hess}_{\rho_\partial}$ is smooth on the compact set $\partial_0 M$, this implies

$$\mathrm{Hess}_{\rho_\partial}(u,u) \le c|u|^2 \rho_\partial(x), \quad x \in M, u \in T_x M$$

for some constant $c > 0$. Then the desired estimate (2.14) follows from

$$\mathrm{Hess}_{\log \rho_\partial}(u,u) = \rho_\partial^{-1} \mathrm{Hess}_{\rho_\partial}(u,u) - \rho_\partial^{-2} \langle \nabla \rho_\partial, u\rangle^2 \le c|u|^2, \quad u \in TM^\circ.$$

□

By Lemma 2.1, when ∂M is convex, there exists a constant $K \ge 0$ such that

$$\mathrm{Ric} - \mathrm{Hess}_{V + 2\log \phi_0} \ge -K. \tag{2.16}$$

Since the diffusion process generated by $L_0 := \Delta + \nabla(V + 2\log\phi_0)$ is non-explosive in M°, by (2.16) and Bakry-Emery's semigroup calculus, (see for instance [3] or [13, Theorem 2.3.3]), we have

$$|\nabla P_t^0 g| \le \mathrm{e}^{Kt} P_t^0 |\nabla g|, \quad t \ge 0, g \in C_b^1(M) \tag{2.17}$$

and for any $p > 1$, there exists a constant $c(p) > 0$ such that

$$\begin{aligned} |\nabla P_t^0 g|^2 &\le \frac{2K\{(P_t^0|g|^{p\wedge 2})(P_t^0|g|)^{(2-p)^+} - (P_t^0|g|)^2\}}{(p\wedge 2)(p \wedge 2 - 1)(1 - \mathrm{e}^{-2Kt})} \\ &\le \frac{c(p)}{1 \wedge t} (P_t^0 |g|^p)^{\frac{2}{p}}, \quad t > 0, g \in \mathscr{B}_b(M). \end{aligned} \tag{2.18}$$

When ∂M is non-convex, we take as in [12] a conformal change of metric to make it convex under the new metric. More precisely, we have the following result.

Lemma 2.2. *There exists a function $1 \le \phi \in C_b^\infty(M)$ such that ∂M is convex under the metric $\langle\cdot,\cdot\rangle_\phi := \phi^{-2}\langle\cdot,\cdot\rangle$. Moreover, there exists a smooth vector field Z_ϕ on M such that*

$$L_0 = \phi^{-2}\Delta^\phi + Z_\phi + 2\phi^{-1}\nabla^\phi \log \phi_0, \tag{2.19}$$

where ∇^ϕ and Δ^ϕ are the gradient and Lapalce-Beltrami operators induced by $\langle\cdot,\cdot\rangle_\phi$ respectively.

Proof. let $\delta > 0$ such that the second fundamental form of ∂M is bounded below by $-\delta$. Take $1 \le \phi \in C_b^\infty(M)$ such that $\phi = 1 + \delta\rho_\partial$ in a neighborhood of ∂M in which the distance function ρ_∂ to ∂M is smooth. By [14, Lemma 2.1](see also [12]), ∂M is convex under the metric $\langle\cdot,\cdot\rangle_\phi := \phi^{-2}\langle\cdot,\cdot\rangle$. Next, according to the proof of [14, Lemma 2.2], there exists a smooth vector field Z_ϕ on M such that (2.19) holds. $\square$

Let $1 \le \phi \in C_b^\infty(M)$ be in Lemma 2.2, and let P_t^ϕ be the diffusion semigroup generated by

$$L^\phi := \phi L_0 = \phi^{-1}\Delta^\phi + \phi Z_\phi + 2\nabla^\phi \log \phi_0.$$

We have the following result.

Lemma 2.3. *Let $1 \le \phi \in C_b^\infty(M)$ be in Lemma 2.2.*

(1) *For any $p \in (1,\infty]$, there exists a constant $c > 0$ such that*

$$|\nabla^\phi P_t^\phi f|_\phi \le \frac{c(q)}{\sqrt{t}}(P_t^\phi |f|^q)^{\frac{1}{q}}, \quad t > 0, f \in C_b^1(M). \tag{2.20}$$

Moreover, there exists a constant $K > 0$ such that

$$|\nabla^\phi P_t^\phi f|_\phi \le \mathrm{e}^{Kt} P_t^\phi |\nabla^\phi f|_\phi, \quad t > 0, f \in C_b^1(M). \tag{2.21}$$

(2) *There exists a constant $c > 0$ such that*

$$\|P_t^\phi\|_{L^p(\mu_0)\to L^\infty(\mu_0)} \le \kappa'(1\wedge t)^{-\frac{d+2}{2p}}, \quad t > 0, p \in [1,\infty]. \tag{2.22}$$

Proof. (1) Since ∂M is convex under the metric $\langle\cdot,\cdot\rangle_\phi$, by Lemma 2.1, we find a constant $K_0^\phi > 0$ such that

$$2\mathrm{Hess}^\phi_{\log\phi_0}(u,u) \le K_0^\phi |u|^2, \quad u \in TM^\circ, \tag{2.23}$$

where Hess^ϕ is the Hessian tensor induced by the metric $\langle\cdot,\cdot\rangle_\phi$. Since the operator $A^\phi := \phi^{-1}\Delta^\phi + \phi Z_\phi$ is a C^2-smooth strictly elliptic second order differential operator on the compact manifold M, it has bounded below Bakry-Emery curvature; that is, there exists a constant $K_1^\phi > 0$ such that

$$A^\phi|\nabla^\phi f|_\phi^2 - 2\langle\nabla^\phi A^\phi f, \nabla^\phi f\rangle_\phi \ge -K_1^\phi|\nabla^\phi f|_\phi^2, \quad f \in C^\infty(M), |u|_\phi^2 := \langle u,u\rangle_\phi.$$

Combining this with (2.23) we obtain

$$L^\phi|\nabla^\phi f|_\phi^2 - 2\langle\nabla^\phi L^\phi f, \nabla^\phi f\rangle_\phi \ge -(K_0^\phi + K_1^\phi)|\nabla^\phi f|_\phi^2 =: -K^\phi|\nabla^\phi f|_\phi^2, \quad f \in C^\infty(M^\circ),$$

which means that the Bakry-Emery curvature of L^ϕ is bounded below by $-K^\phi$. By the same reason leading to (2.17) and (2.18), this implies (2.20) and (2.21).

(2) To estimate $\|P_t^\phi\|_{L^p(\mu_0)\to L^\infty(\mu_0)}$, we make use of [10, Theorem 4.5(b)] or [11, Theorem 3.3.15(2)], which says that (2.9) implies the super Poincaré inequality

$$\mu_0(f^2) \le r\mu_0(|\nabla f|^2) + \beta(1 + r^{-\frac{d+2}{2}})\mu_0(|f|)^2, \quad f \in C_b^1(M)$$

for some constant $\beta > 0$. Let $\mu^\phi = \frac{\phi\mu_0}{\mu_0(\phi)}$. By $L^\phi = \phi L_0$ we obtain

$$\mathscr{E}^\phi(f,g) := -\int_M fL^\phi g \mathrm{d}\mu^\phi = -\frac{1}{\mu_0(\phi)}\int_M fL_0 g\mathrm{d}\mu_0 = \frac{1}{\mu(\phi)}\mu_0(\langle\nabla f, \nabla g\rangle), \quad f, g \in C_b^2(M).$$

Then the above super Poincaré inequality implies

$$\mu^\phi(f^2) \le r\mathscr{E}^\phi(f,f) + \beta'(1 + r^{-\frac{d+2}{2}})\mu^\phi(|f|)^2, \quad f \in C_b^1(M)$$

for some constant $\beta' > 0$. Using [10, Theorem 4.5(b)] or [11, Theorem 3.3.15(2)] again, this implies

$$\|P_t^\phi\|_{L^p(\mu^\phi)\to L^\infty(\mu^\phi)} \le \kappa(1\wedge t)^{-\frac{d+2}{2p}}, \quad t > 0, p \in [1,\infty]$$

for some constant $\kappa > 0$. Noting that

$$\|\phi\|_\infty^{-1}\mu_0 \le \mu^\phi \le \|\phi\|_\infty \mu_0,$$

we find a constant $c > 0$ such that (2.22) holds. □

Lemma 2.4. *For any $p \in (1,\infty]$, there exists a constant $c > 0$ such that for any $f \in D(L_0)$,*

$$\|\nabla P_t^0 f\|_\infty \le c\mathrm{e}^{-\lambda_0 t}\big\{(1\wedge t)^{-\frac{1}{2}-\frac{d+2}{2p}}\|f\|_{L^p(\mu_0)} + (1\wedge t)^{\frac{1}{2}-\frac{d+2}{2p}}\|L_0 f\|_{L^p(\mu_0)}\big\}, \quad t > 0. \tag{2.24}$$

Consequently, there exists a constant $c > 0$ such that

$$\|\nabla(\phi_m\phi_0^{-1})\|_\infty \le cm^{\frac{d+4}{2d}}, \quad m \ge 1. \tag{2.25}$$

Proof. (a) By the semigroup property and the $L^p(\mu_0)$ contraction of P_t^0, for the proof of (2.24) it suffices to consider $t \in (0,1]$. Since $1 \le \phi \in C_b^\infty(M)$, we have $\mathscr{D}(L_0) = \mathscr{D}(L^\phi)$ and

$$P_t^0 f = P_t^\phi f - \int_0^t P_s^\phi\{(\phi-1)P_{t-s}^0 L_0 f\}\mathrm{d}s, \quad t \ge 0, f \in \mathscr{D}(L_0). \tag{2.26}$$

Next, by (2.20) and (2.22), we find constants $c_1, c_2 > 0$ such that

$$\begin{aligned}\|\nabla P_t^\phi f\|_\infty &= \|\nabla P_{t/2}^\phi(P_{t/2}^\phi f)\|_\infty\\ &\le c_1 t^{-\frac{1}{2}}\|P_{t/2}^\phi f\|_\infty \le c_2 t^{-\frac{1}{2}-\frac{d+2}{2p}}\|f\|_{L^p(\mu_0)}, \quad t \in (0,1].\end{aligned} \tag{2.27}$$

Combining this with (2.11) and (2.20), we find constants $c_3, c_4 > 0$ such that

$$\int_0^t \|\nabla P_s^\phi\{(\phi-1)P_{t-s}^0 L_0 f\}\|_\infty \mathrm{d}s \le c_3\int_0^t s^{-\frac{1}{2}}\big\|\{P_s^\phi|P_{t-s}^0 L_0 f|^p\}^{\frac{1}{p}}\big\|_\infty \mathrm{d}s$$

$$
\begin{aligned}
&\le c_3 \int_0^{\frac{t}{2}} s^{-\frac12} \|P^0_{t-s} L_0 f\|_\infty \mathrm{d}s + c_3 \int_{\frac{t}{2}}^t s^{-\frac12} \big\|\{P^\phi_s |P^0_{t-s} L_0 f|^p\}^{\frac1p}\big\|_\infty \mathrm{d}s \\
&\le c_3 \int_0^{\frac{t}{2}} s^{-\frac12} \|P^0_{t-s}\|_{L^p(\mu_0)\to L^\infty(\mu_0)} \|L_0 f\|_{L^p(\mu_0)} \mathrm{d}s + c_3 \int_{\frac{t}{2}}^t s^{-\frac12} \|P^\phi_s\|_{L^p(\mu_0)\to L^\infty(\mu_0)} \|L_0 f\|_{L^p(\mu_0)} \\
&\le c_4 t^{\frac12 - \frac{d+2}{2p}} \|L_0 f\|_{L^p(\mu_0)}.
\end{aligned}
$$

Substituting this and (2.27) into (2.26), we prove (2.24).

(b) Applying (2.24) to $p=\infty$, $f=\phi_m\phi_0^{-1}, t=(\lambda_m-\lambda_0)^{-1}$ and using (2.6), we obtain

$$
\mathrm{e}^{-1}\|\nabla(\phi_m\phi_0^{-1})\|_\infty \le c_1(\lambda_m-\lambda_0)^{\frac12}\|\phi_m\phi_0^{-1}\|_\infty, \quad m\ge 1
$$

for some constant $c_1>0$. This together with (2.1) and (2.12) implies (2.25) for some constant $c>0$. □

3 Upper bound estimate

According to [15, Lemma 2.3], we have

$$
\mathbb{W}_2(\mu_t^\nu, \mu_0)^2 \le \int_M \frac{|\nabla L_0^{-1}(h_t^\nu - 1)|^2}{\mathscr{M}(h_t^\nu, 1)} \mathrm{d}\mu, \tag{3.1}
$$

where

$$
h_t^\nu := \frac{\mathrm{d}\mu_t^\nu}{\mathrm{d}\mu_0}, \quad \mathscr{M}(a,b) := 1_{\{a\wedge b>0\}} \frac{a-b}{\log a - \log b}.
$$

So, to investigate the upper bound estimate, we first calculate h_t^ν.

By (2.8), we have

$$
\psi_s^\nu := \int_M \phi_0(x) p_s^0(x,\cdot)\nu(\mathrm{d}x) = \nu(\phi_0) + \sum_{m=1}^\infty \nu(\phi_m)\mathrm{e}^{-(\lambda_m-\lambda_0)s}\phi_m\phi_0^{-1}, \quad s>0. \tag{3.2}
$$

Next, (2.5) and (2.8) imply

$$
\nu(P_s^D f) = \mathrm{e}^{-\lambda_0 s}\nu(\phi_0 P_s^0(f\phi_0^{-1})) = \mathrm{e}^{-\lambda_0 s}\int_M \psi_s^\nu \phi_0^{-1} f \mathrm{d}\mu_0, \quad f\in\mathscr{B}^+(M), \tag{3.3}
$$

where $\mathscr{B}^+(M)$ is the class of nonnegative measurable functions on M. Moreover, for any $t\ge s>0$, by the Markov property, (2.3), (2.5) and (3.3), we obtain

$$
\begin{aligned}
&\int_M f \mathrm{d}\mathbb{E}^\nu[\delta_{X_s} 1_{\{t<\tau\}}] = \mathbb{E}^\nu\big[f(X_s)1_{\{s<\tau\}}(P^D_{t-s}1)(X_s)\big] = \nu\big(P_s^D\{fP^D_{t-s}1\}\big) \\
&= \mathrm{e}^{-\lambda_0 t}\int_M (\psi_s^v P^0_{t-s}\phi_0^{-1}) f \mathrm{d}\mu_0, \quad f\in\mathscr{B}^+(M).
\end{aligned}
$$

Then

$$\frac{\mathrm{d}\mathbb{E}^\nu[\delta_{X_s} 1_{\{t<\tau\}}]}{\mathrm{d}\mu_0} = \mathrm{e}^{-\lambda_0 t} \psi_s^v P_{t-s}^0 \phi_0^{-1}.$$

Noting that (3.3) implies

$$\mathbb{E}^\nu[1_{\{t<\tau\}}] = \nu(P_t^D 1) = \mathrm{e}^{-\lambda_0 t} \mu_0(\psi_t^\nu \phi_0^{-1}) = \mathrm{e}^{-\lambda_0 t} \nu(\phi_0 P_t^0 \phi_0^{-1}),$$

we arrive at

$$\begin{aligned} h_t^\nu &:= \frac{\mathrm{d}\mu_t^\nu}{\mathrm{d}\mu_0} = \frac{1}{t\mathbb{E}^\nu 1_{\{t<\tau\}}} \int_0^t \frac{\mathrm{d}\mathbb{E}^\nu[\delta_{X_s} 1_{\{t<\tau\}}]}{\mathrm{d}\mu_0} \mathrm{d}s = 1 + \rho_t^\nu, \\ \rho_t^\nu &:= \frac{1}{t\nu(\phi_0 P_t^0 \phi_0^{-1})} \int_0^t \big\{ \psi_s^\nu P_{t-s}^0 \phi_0^{-1} - \nu(\phi_0 P_t^0 \phi_0^{-1}) \big\} \mathrm{d}s. \end{aligned} \tag{3.4}$$

By (2.11), $\|\phi_0\|_\infty < \infty$ and $\|\phi_0^{-1}\|_{L^2(\mu_0)} = 1$, we find a constant $c > 0$ such that

$$\begin{aligned} &|\nu(\phi_0 P_t^0 \phi_0^{-1}) - \nu(\phi_0)\mu(\phi_0)| \le \nu(\phi_0) \|P_t^0 \phi_0^{-1} - \mu_0(\phi_0^{-1})\|_\infty \\ &\le c\mathrm{e}^{-(\lambda_1 - \lambda_0)t}, \quad t \ge 1, \nu \in \mathscr{P}_0. \end{aligned} \tag{3.5}$$

Due to the lack of simple representation of the product $\psi_s^\nu P_{t-s}^0 \phi_0^{-1}$ in terms of the eigenbasis $\{\phi_m \phi_0^{-1}\}_{m\ge 0}$, it is inconvenient to estimate the upper bound in (3.1). To this end, below we reduce this product to a linear combination of ψ_s^ν and $P_{t-s}^0 \phi_0^{-1}$, for which the spectral representation works. Write

$$\begin{aligned} &\psi_s^\nu P_{t-s}^0 \phi_0^{-1} - \nu(\phi_0 P_t^0 \phi_0^{-1}) = I_1(s) + I_2(s), \\ &I_1(s) := \{\psi_s^\nu - \nu(\phi_0)\} \cdot \{P_{t-s}^0 \phi_0^{-1} - \mu(\phi_0)\} + \nu(\phi_0 \{\mu(\phi_0) - P_t^0 \phi_0^{-1}\}), \\ &I_2(s) := \mu(\phi_0)\{\psi_s^\nu - \nu(\phi_0)\} + \nu(\phi_0)\{P_{t-s}^0 \phi_0^{-1} - \mu(\phi_0)\}. \end{aligned} \tag{3.6}$$

By (2.7), (2.8) and (3.2), we have

$$\begin{aligned} P_{t-s}^0 \phi_0^{-1} - \mu(\phi_0) &= \sum_{m=1}^\infty \mu(\phi_m) \mathrm{e}^{-(\lambda_m - \lambda_0)(t-s)} \phi_m \phi_0^{-1}, \\ \psi_s^\nu - \nu(\phi_0) &= \sum_{m=1}^\infty \nu(\phi_m) \mathrm{e}^{-(\lambda_m - \lambda_0)s} \phi_m \phi_0^{-1}, \quad t > s > 0. \end{aligned} \tag{3.7}$$

Then

$$\begin{aligned} &\rho_t^\nu = \tilde{\rho}_t^\nu + \frac{1}{t\nu(\phi_0 P_t^0 \phi_0^{-1})} \int_0^t I_1(s) \mathrm{d}s - A_t, \\ &\tilde{\rho}_t^\nu := \frac{1}{t\nu(\phi_0 P_t^0 \phi_0^{-1})} \sum_{m=1}^\infty \frac{\mu(\phi_0)\nu(\phi_m) + \nu(\phi_0)\mu(\phi_m)}{\lambda_m - \lambda_0} \phi_m \phi_0^{-1} \\ &A_t := \frac{1}{t\nu(\phi_0 P_t^0 \phi_0^{-1})} \sum_{m=1}^\infty \frac{\{\mu(\phi_0)\nu(\phi_m) + \nu(\phi_0)\mu(\phi_m)\} \mathrm{e}^{-(\lambda_m - \lambda_0)t}}{\lambda_m - \lambda_0} \phi_m \phi_0^{-1}. \end{aligned} \tag{3.8}$$

Since $\rho_t^\nu \in L^1(\mu_0)$, the following lemma implies $\tilde{\rho}_t^\nu \in L^1(\mu_0)$ for $t > 0$.

Lemma 3.1. *There exists a constant $c > 0$ such that*

$$\mu_0(|\rho_t^\nu - \tilde\rho_t^\nu|) \le c\|h\|_{L^2(\mu)}\mathrm{e}^{-(\lambda_1-\lambda_0)t}, \quad t > 0, \nu = h\mu \in \mathscr{P}_0. \tag{3.9}$$

Proof. By (2.1) and (2.12), for any $t_0 > 0$ we find a constant $c_0 > 0$ such that

$$\sum_{m=1}^\infty \|\phi_m\|_\infty \mathrm{e}^{-(\lambda_m-\lambda_0)t} \le c_0\mathrm{e}^{-(\lambda_1-\lambda_0)t}, \quad t \ge t_0. \tag{3.10}$$

Combining this with (3.8) and (3.5), and noting that $\|h\phi_0^{-1}\|_{L^2(\mu_0)} = \|h\|_{L^2(\mu)}$, it suffices to find a constant $c_1 > 0$ such that

$$B := \frac{1}{t}\int_0^t \big\|\{\psi_s^\nu - \nu(\phi_0)\}\cdot\{P_{t-s}^0\phi_0^{-1} - \mu(\phi_0)\}\big\|_{L^1(\mu_0)}\mathrm{d}s \le c_1\|h\|_{L^2(\mu)}\mathrm{e}^{-(\lambda_1-\lambda_0)t}, \quad t \ge t_0. \tag{3.11}$$

Since $\|\phi_0^{-1}\|_{L^2(\mu_0)} = 1$ and $\psi_s^\nu = P_0^s(h\phi_0^{-1})$ for $\nu = h\mu$, by (2.10), we find a constant $c_1 > 0$ such that

$$\begin{aligned} B &\le \frac{1}{t}\int_0^t \|P_{t-s}^0\phi_0^{-1} - \mu_0(\phi_0^{-1})\|_{L^2(\mu_0)}\|P_s^0(h\phi_0^{-1}) - \mu_0(h\phi_0^{-1})\|_{L^2(\mu_0)}\mathrm{d}s \\ &\le \frac{1}{t}\int_0^t \|P_{t-s}^0 - \mu_0\|_{L^2(\mu_0)}\|P_s^0 - \mu_0\|_{L^2(\mu_0)}\|h\|_{L^2(\mu)}\mathrm{d}s \\ &\le c_1\|h\|_{L^2(\mu)}\mathrm{e}^{-(\lambda_1-\lambda_0)t}, \quad t > 0. \end{aligned}$$

□

Lemma 3.2. *For any $\alpha > 0$, there exist constants $c_0, t_0 > 0$ such that*

$$\tilde\rho_t^\nu \ge -\frac{c_0}{\nu(\phi_0)t}, \quad t \ge t_0, \quad \nu \in \mathscr{P}_0, \nu \in \mathscr{P}_0. \tag{3.12}$$

Consequently, if $\nu = h\mu$ with $h \in L^2(\mu)$, then $\tilde\mu_t^\nu := (1 + \tilde\rho_t^\nu)\mu_0$ is a probability measure for $t > t_0(1 + c_0)$.

Proof. By Lemma 3.1, if $\nu = h\mu$ with $h \in L^2(\mu)$, we have $\tilde\rho_t^\nu \in L^1(\mu_0)$ for $t > 0$, and it is easy to see that $\mu_0(\tilde\rho_t^\nu) = 0$. Since (3.12) implies $1 + \tilde\rho_t^\nu > 0$ for $t > t_0(1 + c_0)$, $\tilde\mu_t^\nu$ is a probability measure. It remains to prove (3.12).

By (3.5) and (3.8), it suffices to find a constant $c_1 > 0$ such that

$$g := \sum_{m=1}^\infty \frac{\mu(\phi_0)\nu(\phi_m) + \nu(\phi_0)\mu(\phi_m)}{\lambda_m - \lambda_0}\phi_m\phi_0^{-1} \ge -c_1. \tag{3.13}$$

By (2.1) and (2.12), we have

$$\|P_1^0 g\|_\infty \le c_2 := \sum_{m=1}^\infty \frac{2\|\phi_0\|_\infty\|\phi_m\|_\infty\|\phi_m\phi_0^{-1}\|_\infty}{(\lambda_m - \lambda_0)\mathrm{e}^{\lambda_m-\lambda_0}} < \infty. \tag{3.14}$$

Next, by (3.7) and the same formula for $\mu = \nu$, we obtain

$$P_s^0 g = (-L_0)^{-1}\big\{\mu(\phi_0)(\psi_s^\nu - \nu(\phi_0)) + \nu(\phi_0)(\psi_s^\mu - \mu(\phi_0))\big\} = (-L_0)^{-1} g_s, \quad s > 0, \tag{3.15}$$

where by $\phi_0, \psi_s^\nu, \psi_s^\mu \geq 0$,

$$g_s := \mu(\phi_0)(\psi_s^\nu - \nu(\phi_0)) + \nu(\phi_0)(\psi_s^\mu - \mu(\phi_0)) \geq -2\mu(\phi_0)\nu(\phi_0) \geq -2\nu(\phi_0), \quad s > 0.$$

This together with (3.15) yields

$$-L_0 P_s^0 g \geq -2\nu(\phi_0), \quad s > 0.$$

Therefore, it follows from (3.14) that

$$g = P_1^0 g - \int_0^1 L_0 P_r^0 g \mathrm{d}r \geq -c_2 - 2\nu(\phi_0) \geq -c_2 - 2\|\phi_0\|_\infty.$$

So, (3.13) holds for $c_1 = c_2 + 2\|\phi_0\|_\infty$. □

Lemma 3.3. *There exist constants $c, t_0 > 0$ such that for any $t \geq t_0$, and any $\nu \in \mathscr{P}_0$ with $\nu = h\mu$ such that $h \in L^2(\mu)$, we have $\tilde{\mu}_t^\nu \in \mathscr{P}_0$ and*

$$t^2 \mathbb{W}_2(\tilde{\mu}_t^\nu, \mu_0)^2 \leq \frac{1 + ct^{-1}}{\{\mu(\phi_0)\nu(\phi_0)\}^2} \sum_{m=1}^\infty \frac{\{\nu(\phi_0)\mu_0(\phi_m) + \mu(\phi_0)\nu(\phi_m)\}^2}{(\lambda_m - \lambda_0)^3}. \tag{3.16}$$

Proof. By Lemma 3.2, there exist constants $c, t_0 > 0$ such that $\tilde{\mu}_t^\nu \in \mathscr{P}_0$ for $t \geq t_0$, and

$$\mathscr{M}(1 + \tilde{\rho}_t^\nu, 1) \geq 1 \wedge (1 + \tilde{\rho}_t^\nu) \geq \frac{1}{1 + ct^{-1}}, \quad t \geq t_0.$$

So, [15, Lemma 2.3] implies

$$\mathbb{W}_2(\tilde{\mu}_t^\nu, \mu_0)^2 \leq \int_M \frac{|\nabla L_0^{-1} \tilde{\rho}_t^\nu|^2}{\mathscr{M}(1 + \tilde{\rho}_t^\nu, 1)} \mathrm{d}\mu_0 \leq (1 + ct^{-1})\mu_0(|\nabla L_0^{-1} \tilde{\rho}_t^\nu|^2), \quad t \geq t_0. \tag{3.17}$$

Next, (2.6) and (3.8) yield

$$t^2 \mu_0(|\nabla L_0^{-1} \tilde{\rho}_t^\nu|^2) = \frac{1}{\{\nu(\phi_0 P_t^0 \phi_0^{-1})\}^2} \sum_{m=1}^\infty \frac{\{\mu(\phi_0)\nu(\phi_m) + \nu(\phi_0)\mu(\phi_m)\}^2}{(\lambda_m - \lambda_0)^3}.$$

Combining this with (3.5) and (3.17), we finish the proof. □

We are now ready to prove the following result.

Proposition 3.4. *For any $\nu \in \mathscr{P}_0$,*

$$\limsup_{t \to \infty} \big\{t^2 \mathbb{W}_2(\mu_t^\nu, \mu_0)^2\big\} \leq I. \tag{3.18}$$

Proof. (1) We first consider $\nu = h\mu$ with $h \in L^2(\mu)$. Let D be the diameter of M. By Lemma 3.1, there exist constants $c_1, t_0 > 0$ such that $\tilde\mu_t^\nu$ is probability measure for $t \geq t_0$ and

$$\mathbb{W}_2(\mu_t^\nu, \tilde\mu_t^\nu)^2 \leq D^2 \|\mu_t^\nu - \tilde\mu_t^\nu\|_{var} = D^2 \mu_0(|\rho_t^\nu - \tilde\rho_t^\nu|) \leq c_1 \|h\|_{L^2(\mu)} \mathrm{e}^{-(\lambda_1-\lambda_0)t}, \quad t \geq t_0. \tag{3.19}$$

Combining this with Lemma 3.3 and the triangle inequality of $\mathbb{W}_2$, we obtain

$$t^2 \mathbb{W}_2(\mu_t^\nu, \mu_0)^2 \leq (1+\delta^{-1}) c_1 t^2 \mathrm{e}^{-(\lambda_1-\lambda_0)t} \|h\|_{L^2(\mu)} + (1+\delta)(1+ct^{-1}) I, \quad \delta > 0. \tag{3.20}$$

(2) In general, we may go back to the first situation by shifting a small time $\varepsilon > 0$. More precisely, by the Markov property, (2.3), (2.5) and (3.2), for any $f \in \mathscr{B}_b(M)$ and $t \geq s \geq \varepsilon > 0$, we have

$$\begin{aligned}
&\mathbb{E}^\nu[f(X_s) 1_{\{t<\tau\}}] = \mathbb{E}^\nu\big[1_{\{\varepsilon<\tau\}} \mathbb{E}^{X_\varepsilon}(f(X_{s-\varepsilon}) 1_{\{t-\varepsilon<\tau\}})\big] \\
&= \int_M p_\varepsilon^D(x,y) \mathbb{E}^y[f(X_{s-\varepsilon}) 1_{\{t-\varepsilon<\tau\}}] \nu(\mathrm{d}x) \mu(\mathrm{d}y) \\
&= \mathrm{e}^{-\lambda_0 \varepsilon} \int_M (\psi_\varepsilon^\nu \phi_0)(y) \mathbb{E}^y[f(X_{s-\varepsilon}) 1_{\{t-\varepsilon<\tau\}}] \nu(\mathrm{d}x) \mu(\mathrm{d}y).
\end{aligned}$$

With $f = 1$ this implies

$$\mathbb{P}^\nu(t < \tau) = \mathrm{e}^{-\lambda_0 \varepsilon} \int_M (\psi_\varepsilon^\nu \phi_0)(y) \mathbb{P}^y(t - \varepsilon < \tau) \mu(\mathrm{d}y) \mu(\mathrm{d}y).$$

So, letting

$$\nu_\varepsilon = \frac{\psi_\varepsilon^\nu \phi_0}{\mu(\psi_\varepsilon^\nu \phi_0)} =: h_\varepsilon \mu,$$

we arrive at

$$\mathbb{E}^\nu[f(X_s) | t < \tau] = \frac{\mathbb{E}^\nu[f(X_s) 1_{\{t<\tau\}}]}{\mathbb{P}^\nu(t<\tau)} = \frac{\mathbb{E}^{\nu_\varepsilon}[f(X_{s-\varepsilon}) 1_{\{t-\varepsilon<\tau\}}]}{\mathbb{P}^{\nu_\varepsilon}(t-\varepsilon<\tau)} = \mathbb{E}^{\nu_\varepsilon}[f(X_{s-\varepsilon}) | t - \varepsilon < \tau].$$

Therefore,

$$\mu_{t,\varepsilon}^\nu := \frac{1}{t-\varepsilon} \int_\varepsilon^t \mathbb{E}^\nu(\delta_{X_s} | t < \tau) \mathrm{d}s = \mu_{t-\varepsilon}^{\nu_\varepsilon}, \quad t > \varepsilon. \tag{3.21}$$

Since

$$\mu(\psi_\varepsilon^\nu \phi_0) = \int_M p_\varepsilon^0(x,y) \phi_0(x) \phi_0(y) \nu(\mathrm{d}x) \mu(\mathrm{d}y) = \nu(\phi_0 P_\varepsilon^0 \phi_0^{-1}) \geq \nu(\phi_0) \|\phi_0\|_\infty^{-1} =: \alpha > 0,$$

by (2.9) we find a constant $c_2 > 0$ such that

$$\|h_\varepsilon \phi_0^{-1}\|_{L^2(\mu_0)} \leq \alpha^{-1} \|\psi_\varepsilon^\nu\|_{L^2(\mu_0)} \leq \alpha^{-1} \|\phi_0\|_\infty \|p_\varepsilon^0\|_{L^\infty(\mu_0)} \leq c_2 \varepsilon^{-\frac{d+2}{2}}, \quad \varepsilon \in (0,1). \tag{3.22}$$

Then (3.20) and (3.21) yield

$$
\begin{aligned}
&t^2\mathbb{W}_2(\mu_{t,\varepsilon}^\nu,\mu_0)^2\\
&\le (1+\delta^{-1})c_1c_2\alpha^{-1}t^2\mathrm{e}^{-(\lambda_1-\lambda_0)t}\varepsilon^{-\frac{d+2}{2}}+(1+\delta)(1+ct^{-1})I_\varepsilon,\quad \delta>0,\varepsilon\in(0,1),
\end{aligned}
\tag{3.23}
$$

where

$$I_\varepsilon:=\frac{1}{\{\mu(\phi_0)\nu_\varepsilon(\phi_0)\}^2}\sum_{m=1}^\infty\frac{\{\nu_\varepsilon(\phi_0)\mu(\phi_m)+\mu(\phi_0)\nu_\varepsilon(\phi_m)\}^2}{(\lambda_m-\lambda_0)^3}.$$

By (2.5), (2.6) and (3.2), we have

$$
\begin{aligned}
&\mu(\psi_\varepsilon^\nu\phi_0)=\nu(\phi_0P_\varepsilon^{-1}\phi_0^{-1})=\mathrm{e}^{\lambda_0\varepsilon}\nu(P_\varepsilon^D 1),\\
&\mu(\psi_\nu\phi_0)=\nu(\phi_0P_\varepsilon^0(\phi_m\phi_0^{-1}))=\mathrm{e}^{-(\lambda_m-\lambda_0)\varepsilon}\nu(\phi_m),
\end{aligned}
$$

so that

$$\nu_\varepsilon(\phi_m)=\frac{\mathrm{e}^{-\lambda_m\varepsilon}\nu(\phi_m)}{\nu(P_\varepsilon^D 1)},\quad m\ge 0.$$

Thus, $\lim_{\varepsilon\to0}\nu_\varepsilon(\phi_0)=\nu(\phi_0)$ and there exists a constant $C>1$ such that

$$C^{-1}\mathrm{e}^{-\lambda_m\varepsilon}|\nu(\phi_m)|\le|\nu_\varepsilon(\phi_m)|\le C|\nu(\phi_m)|,\quad m\ge1,\varepsilon\in(0,1). \tag{3.24}$$

Therefore, if $I<\infty$, by this and

$$\sum_{m=1}^\infty\mu(\phi_m)^2\le\mu(1)=1, \tag{3.25}$$

we may apply the dominated convergence theorem to derive $\lim_{\varepsilon\to0}I_\varepsilon=I$. On the other hand, if $I=\infty$, which is equivalent to

$$\sum_{m=1}^\infty\frac{\nu(\phi_m)^2}{(\lambda_m-\lambda_0)^3}=\infty,$$

then by (3.24) and the monotone convergence theorem we get

$$\liminf_{\varepsilon\to0}\sum_{m=1}^\infty\frac{\nu_\varepsilon(\phi_m)^2}{(\lambda_m-\lambda_0)^3}\ge C^{-2}\liminf_{\varepsilon\to0}\sum_{m=1}^\infty\frac{\mathrm{e}^{-2\lambda_m\varepsilon}\nu(\phi_m)^2}{(\lambda_m-\lambda_0)^3}=\infty,$$

which together with (3.25) and $\nu_\varepsilon(\phi_0)\to\nu(\phi_0)$ implies

$$
\begin{aligned}
\liminf_{\varepsilon\to0}I_\varepsilon&=\frac{1}{\{\mu(\phi_0)\nu(\phi_0)\}^2}\liminf_{\varepsilon\to0}\sum_{m=1}^\infty\frac{\{\nu_\varepsilon(\phi_0)\mu(\phi_m)+\mu(\phi_0)\nu_\varepsilon(\phi_m)\}^2}{(\lambda_m-\lambda_0)^3}\\
&\ge\frac{1}{\{\mu(\phi_0)\nu(\phi_0)\}^2}\liminf_{\varepsilon\to0}\frac{\frac12\{\mu(\phi_0)\nu_\varepsilon(\phi_m)\}^2-\|\phi_0\|_\infty^2\mu(\phi_m)^2}{(\lambda_m-\lambda_0)^3}=\infty.
\end{aligned}
$$

In conclusion, we have

$$\lim_{\varepsilon\to 0} I_\varepsilon = I. \tag{3.26}$$

This together with (3.23) for $\varepsilon = t^{-2}$ gives

$$\limsup_{t\to\infty} \big\{t^2 \mathbb{W}_2(\mu^\nu_{t,t^{-2}},\mu_0)^2\big\} \le I. \tag{3.27}$$

On the other hand, it is easy to see that

$$\|\mu^\nu_{t,\varepsilon} - \mu^\nu_t\|_{var} \le \frac{2\varepsilon}{t},\quad 0<\varepsilon<t,$$

so that

$$\mathbb{W}_2(\mu^\nu_t,\mu^\nu_{t,t^{-2}})^2 \le D^2\|\mu^\nu_{t,t^{-2}} - \mu^\nu_t\|_{var} \le 2D^2 t^{-3},\quad t>1. \tag{3.28}$$

Combining this with (3.27), we prove (3.18). □

4 Lower bound estimate and the finiteness of the limit

We will follow the idea of [1, 15], for which we need to modify $\tilde\mu^\nu_t$ as follows. For any $\beta>0$, consider

$$\tilde\mu^\nu_{t,\beta} := (1+\tilde\rho^\nu_{t,\beta})\mu_0,\quad \tilde\rho^\nu_{t,\beta} := P^0_{t^{-\beta}}\tilde\rho^\nu_t,\quad t>0.$$

According to Lemma 3.2, there exists $t_0>0$ such that

$$\tilde h^\nu_t := 1+\tilde\rho^\nu_t \ge \frac12,\quad \tilde h^\nu_{t,\beta} := 1+\tilde\rho^\nu_{t,\beta} \ge \frac12,\quad \beta>0, t\ge t_0. \tag{4.1}$$

Consequently, $\tilde\mu^\nu_{t,\beta}$ and $\tilde\mu^\nu_t$ are probability measures for any $\beta>0, t\ge t_0$.

Lemma 4.1. *For any $\beta>0$, there exists a constant $c>0$ such that $f_{t,\beta} := L_0^{-1}\tilde\rho^\nu_{t,\beta}$ satisfies*

$$\|f_{t,\beta}\|_\infty + \|L_0 f_{t,\beta}\|_\infty + \|\nabla f_{t,\beta}\|_\infty \le c t^{\frac{5\beta d}{4}-1},\quad t\ge 1.$$

Proof. By (2.6) and (3.8), we have

$$\begin{aligned} f_{t,\beta} &= -\sum_{m=1}^\infty \frac{\{\mu(\phi_0)\nu(\phi_m)+\nu(\phi_0)\mu(\phi_m)\}\mathrm{e}^{-(\lambda_m-\lambda_0)t^{-\beta}}}{t(\lambda_m-\lambda_0)^2\nu(\phi_0 P^0_t\phi_0^{-1})}\big(\phi_m\phi_0^{-1}\big),\\ L_0 f_{t,\beta} &= \sum_{m=1}^\infty \frac{\{\mu(\phi_0)\nu(\phi_m)+\nu(\phi_0)\mu(\phi_m)\}\mathrm{e}^{-(\lambda_m-\lambda_0)t^{-\beta}}}{t(\lambda_m-\lambda_0)\nu(\phi_0 P^0_t\phi_0^{-1})}\big(\phi_m\phi_0^{-1}\big). \end{aligned}$$

Combining these with (2.1), (2.12), (3.5), and

$$|\mu(\phi_0)\nu(\phi_m)+\nu(\phi_0)\mu(\phi_m)| \le \|\phi_0\|_\infty + \|\phi_m\|_\infty \le c_0 m,\quad m\ge 1$$

for some constant $c_0>0$, we find constants $c_1,c_2,c_3,c_4,c_5>0$ such that

$$\begin{aligned} &t\{\|f_{t,\beta}\|_\infty+\|L_0f_{t,\beta}\|_\infty\}\le c_1\sum_{m=1}^\infty \frac{\mathrm{e}^{-(\lambda_m-\lambda_0)t^{-\beta}}m^{\frac{3d+2}{2d}}}{\lambda_m-\lambda_0}\\ &\le c_2\sum_{m=1}^\infty \mathrm{e}^{-c_3m^{\frac{2}{d}}t^{-\beta}}m^{\frac{3d-2}{2d}}\le c_4\int_0^\infty \mathrm{e}^{-c_3s^{\frac{2}{d}}t^{-\beta}}s^{\frac{3d-2}{2d}}\mathrm{d}s\le c_5t^{\frac{\beta(5d-2)}{4}},\quad t\ge 1.\end{aligned}$$

Similarly, by (2.25) we find constants $c_1',c_2',c_3'>0$ such that

$$\begin{aligned} &t\|\nabla f_{t,\beta}\|_\infty\le c_1'\sum_{m=1}^\infty \frac{\mathrm{e}^{-(\lambda_m-\lambda_0)t^{-\beta}}m^{\frac{3d+4}{2d}}}{(\lambda_m-\lambda_0)^2}\\ &\le c_2'\sum_{m=1}^\infty \mathrm{e}^{-c_3m^{\frac{2}{d}}t^{-\beta}}m^{\frac{3d-4}{2d}}\le c_3't^{\frac{\beta(5d-4)}{4}},\quad t\ge 1.\end{aligned}$$

Then the proof is finished. □

Lemma 4.2. *For any $\beta\in(0,\frac{1}{20d}]$, there exits a constant $c>0$ such that*

$$t^2\mathbb{W}_2(\tilde\mu^\nu_{t,\beta},\mu_0)^2\ge \frac{1-ct^{-1}}{\{\mu(\phi_0)\nu(\phi_0)\}^2}\sum_{m=1}^\infty\frac{\{\mu(h\phi_0)\mu_0(\phi_m)+\mu(\phi_0)\nu(\phi_m)\}^2}{(\lambda_m-\lambda_0)^3}-ct^{-\frac14}.$$

Proof. To estimate $\mathbb{W}_2(\tilde\mu^\nu_{t,\beta},\mu_0)$ from below by using the argument in [1, 15], we take

$$\varphi_\theta^\varepsilon:=-\varepsilon\log P^0_{\frac{\varepsilon\theta}{2}}\mathrm{e}^{-\varepsilon^{-1}f_{t,\beta}},\quad \theta\in[0,1],\varepsilon>0.$$

We have $\varphi_0^\varepsilon=f_{t,\beta}$, $\|\varphi_\theta^\varepsilon\|_\infty\le\|f_{t,\beta}\|_\infty$, and by [15, Lemma 2.9], there exists a constant $c_1>0$ such that for any $\varepsilon\in(0,1)$,

$$\begin{aligned}&\varphi_1^\varepsilon(y)-\varphi_0^\varepsilon(x)\le\frac12\big\{\rho(x,y)^2+\varepsilon\|(L_0f_{t,\beta})^+\|_\infty+c_1\sqrt{\varepsilon}\|\nabla f_{t,\beta}\|_\infty^2\big\},\quad x,y\in M,\\ &\int_M(\varphi_0^\varepsilon-\varphi_1^\varepsilon)\mathrm{d}\mu_0\le\frac12\int_M|\nabla f_{t,\beta}|^2\mathrm{d}\mu_0+c\varepsilon^{-1}\|\nabla f_{t,\beta}\|_\infty^4.\end{aligned}$$

Therefore, by the Kantorovich dual formula, $\phi_0^\varepsilon=f_{t,\beta}$ and the integration by parts formula

$$\int_M f_{t,\beta}\tilde\rho^\nu_{t,\beta}\mathrm{d}\mu_0=\int_M f_{t,\beta}L_0f_{t,\beta}\mathrm{d}\mu_0=-\int_M|\nabla f_{t,\beta}|^2\mathrm{d}\mu_0,$$

we find a constant $c>0$ such that

$$\begin{aligned}&c\big(\varepsilon\|L_0f_{t,\beta}\|_\infty+\varepsilon^{\frac12}\|\nabla f_{t,\beta}\|_\infty^2\big)+\frac12\mathbb{W}_2(\tilde\mu^\nu_{t,\beta},\mu_0)^2\ge\int_M\varphi_1^\varepsilon\mathrm{d}\mu_0-\int_M\varphi_0^\varepsilon\mathrm{d}\tilde\mu^\nu_{t,\beta}\\ &=\int_M(\varphi_1^\varepsilon-\varphi_0^\varepsilon)\mathrm{d}\mu_0-\int_M f_{t,\beta}\tilde\rho^\nu_{t,\beta}\mathrm{d}\mu_0=\int_M(\varphi_1^\varepsilon-\varphi_0^\varepsilon)\mathrm{d}\mu_0-\int_M f_{t,\beta}L_0f_{t,\beta}\mathrm{d}\mu_0\\ &\ge\frac12\int_M|\nabla f_{t,\beta}|^2\mathrm{d}\mu_0-c\varepsilon^{-1}\|\nabla f_{t,\beta}\|_\infty^4.\end{aligned}\tag{4.2}$$

Taking $\varepsilon = t^{-\frac{3}{2}}$ and applying Lemma 4.1, when $\beta \le \frac{1}{20d}$ we find a constant $c' > 0$ such that

$$t^2\mathbb{W}_2(\tilde\mu^\nu_{t,\beta},\mu_0)^2 \ge t^2\mu_0(|\nabla f_{t,\beta}|^2) - c' t^{-\frac{1}{4}},\quad t\ge t_0. \tag{4.3}$$

Combining this with (3.5) and (4.3), we complete the proof. $\square$

Lemma 4.3. *There exist constants $c, t_0 > 0$ such that for any $\nu = h\mu \in \mathscr{P}_0$ with $h \in L^2(\mu)$, $\tilde\mu^\nu_{t,\beta}, \tilde\mu^\nu_t \in \mathscr{P}_0$ for $t \ge t_0$ and*

$$t\mathbb{W}_2(\tilde\mu^\nu_{t,\beta},\tilde\mu^\nu_t) \le c\|h\|_{L^2(\mu)} t^{-\beta},\quad t\ge t_0.$$

Proof. $\tilde\mu^\nu_{t,\beta}, \tilde\mu^\nu_t \in \mathscr{P}_0$ for large t is implied by Lemma 3.2. Next, by (4.1), we have

$$\mathscr{M}(\tilde h^\nu_t, \tilde h^\nu_{t,\beta}) \ge \frac{1}{2},$$

so that [15, Lemma 2.3] implies

$$\mathbb{W}_2(\tilde\mu^\nu_{t,\beta},\tilde\mu^\nu_t)^2 \le \int_M \frac{|\nabla L_0^{-1}(\tilde h^\nu_t - \tilde h^\nu_{t,\beta})|^2}{\mathscr{M}(\tilde h^\nu_t, \tilde h^\nu_{t,\beta})}\mathrm{d}\mu_0 \le 2\mu_0(|\nabla L_0^{-1}(\tilde\rho^\nu_t - \tilde\rho^\nu_{t,\beta})|^2). \tag{4.4}$$

To estimate the upper bound in this inequality, we first observe that by (3.7) and (3.8), when $\nu = h\mu$ we have

$$\begin{aligned} L_0^{-1}(\tilde\rho^\nu_{t,\beta} - \tilde\rho^\nu_t) &= L_0^{-1}(P^0_{t^{-\beta}}\tilde\rho^\nu_t - \tilde\rho^\nu_t) = \int_0^{t^{-\beta}} P^0_r\tilde\rho^\nu_t \mathrm{d}r\\ &= \frac{1}{t\nu(\phi_0 P^0_t\phi_0^{-1})}\int_0^{t^{-\beta}} (-L_0)^{-1}(P^0_r - \mu_0)g\,\mathrm{d}r, \end{aligned} \tag{4.5}$$

where

$$g := \mu(\phi_0)h\phi_0^{-1} + \nu(\phi_0)\phi_0^{-1}.$$

Since $\|h\|_{L^2(\mu)} \ge \mu(h) = 1$,

$$\|g\|_{L^2(\mu_0)} \le \|\phi_0\|_\infty(1+\|h\|_{L^2(\mu)}) \le 2\|\phi_0\|_\infty\|h\|_{L^2(\mu)}. \tag{4.6}$$

By (2.10), (4.6) and the fact that $(-L_0)^{-\frac{1}{2}} = c\int_0^\infty P^0_{s^2}\mathrm{d}s$ for some constant $c>0$, we find a constants $c_1, c_2 > 0$ such that

$$\begin{aligned} \big\|\nabla L_0^{-1}(P^0_r - \mu_0)g\|_{L^2(\mu_0)} &= \big\|L_0^{-\frac{1}{2}}(P^0_r-\mu_0)g\|_{L^2(\mu_0)} \le \int_0^\infty \|(P^0_{r+s^2}-\mu_0)g\|_{L^2(\mu_0)}\mathrm{d}s\\ &\le c_1\|h\|_{L^2(\mu)}\int_1^\infty \mathrm{e}^{-(\lambda_1-\lambda_0)(s^2+r)}\mathrm{d}s \le c_2\|h\|_{L^2(\mu)},\quad r\in[0,1]. \end{aligned}$$

Therefore, by (3.5) and (4.5), we find constants $c>0$ such that

$$\|\nabla L_0^{-1}(\tilde\rho^\nu_{t,\beta} - \tilde\rho^\nu_t)\|_{L^2(\mu_0)} \le \frac{c'}{t}\int_0^{t^{-\beta}} \big\|\nabla L_0^{-1}(P^0_r-\mu_0)g\|_{L^2(\mu_0)}\mathrm{d}r \le ct^{-(1+\beta)}\|h\|_{L^2(\mu)},\quad t\ge t_0.$$

Combining this with (4.4) we finish the proof. $\square$

We are now ready to prove the following result.

Proposition 4.4. *For any $\nu \in \mathscr{P}_0$,*

$$\liminf_{t\to\infty} \left\{t^2 \mathbb{W}_2(\mu_t^\nu, \mu_0)^2\right\} \ge I > 0, \tag{4.7}$$

and $I < \infty$ provided either $d \le 5$, or $d \ge 6$ but $\nu = h\mu$ with $\mu(h^p) \wedge \mu_0(|h\phi_0^{-1}|^q) < \infty$ for some $p > \frac{2d}{d+6}$ and $q > \frac{2(d+2)}{d+6}$.

Proof. Let $\beta \in (0, \frac{1}{20d}]$. By (3.19), Lemma 4.2 and Lemma 4.3, there exist constants $c, t_0 > 0$ such that for $\nu = h\mu \in \mathscr{P}_0$ and $t \ge t_0$,

$$\begin{aligned}
&t\mathbb{W}_2(\mu_t^\nu, \tilde\mu_t^\nu) \le c\|h\|_{L^2(\mu)} t^{-\beta t},\\
&t\mathbb{W}_2(\tilde\mu_{t,\beta}^\nu, \mu_0) \ge \left(\{(1-ct^{-1})I - ct^{-\frac14})^+\right\}^{\frac12},\\
&t\mathbb{W}_2(\mu_t^\nu, \tilde\mu_t^\nu) \le ct\mathrm{e}^{-(\lambda_1-\lambda_0)t/2}\|h\|_{L^2(\mu)}^{\frac12}.
\end{aligned}$$

Then

$$t\mathbb{W}_2(\mu_t^\nu, \mu_0) \ge \left(\{(1-ct^{-1})I - ct^{-\frac14})^+\right\}^{\frac12} - c\|h\|_{L^2(\mu)} t^{-\beta t} - ct\mathrm{e}^{-(\lambda_1-\lambda_0)t/2}\|h\|_{L^2(\mu)}^{\frac12},\quad t \ge t_0. \tag{4.8}$$

In general, let $\mu_{t,\varepsilon}^\nu = \mu_{t-\varepsilon}^{\nu_\varepsilon}$ be in the proof of Proposition 3.4. Applying (4.8) to $\mu_{t,t^{-2}}^\nu$ replacing μ_y^ν and using (3.22), (3.26), we obtain

$$\liminf_{t\to\infty} \left\{t\mathbb{W}_2(\mu_{t,t^{-2}}^\nu, \mu_0)\right\} \ge \sqrt{I},$$

which together with (3.28) proves (4.7).

It remains to prove $I > 0$ and $I < \infty$ the under given conditions, where due to (3.25), $I < \infty$ is equivalent to

$$I' := \sum_{m=1}^\infty \frac{\nu(\phi_m)^2}{(\lambda_m - \lambda_0)^3} < \infty. \tag{4.9}$$

Below we first prove $I > 0$ then shown $I' < \infty$ under the given conditions.

(a) $I > 0$. If this is not true, then

$$\mu(h\phi_0)\mu(\phi_m) = -\mu(\phi_0)\mu(h\phi_m),\quad m \ge 1.$$

Combining this with the representation in $L^2(\mu)$

$$f = \sum_{m=0}^\infty \mu(f\phi_m)\phi_m,\quad f \in L^2(\mu),$$

where the equation holds point-wisely if $f \in C_b(M)$ by the continuity, we obtain

$$\mu(\phi_0)\nu(f) = \sum_{m=0}^\infty \mu(f\phi_m)\mu(\phi_0)\nu(\phi_m) = 2\mu(f\phi_0)\nu(\phi_0)\mu(\phi_0) - \sum_{m=0}^\infty \mu(f\phi_m)\mu(\phi_m)\nu(\phi_0)$$

$$= 2\mu(f\phi_0)\nu(\phi_0)\mu(\phi_0) - \nu(\phi_0)\mu(f), \quad f \in C_b(M).$$

Consequently,

$$0 \le \mu(\phi_0)\frac{\mathrm{d}\nu}{\mathrm{d}\mu} = 2\phi_0\nu(\phi_0)\mu(\phi_0) - \nu(\phi_0),$$

which is however impossible since the upper bound is negative in a neighborhood of ∂M, because $\nu(M^\circ) > 0$ implies $\nu(\phi_0) > 0$ for $\phi_0 > 0$ in M°, and ϕ_0 is continuous with $\phi_0|_{\partial M} = 0$. Therefore, we must have $I > 0$.

(b) $I' < \infty$ for $d \le 5$. By (2.6), (3.2), and $(-L_0)^{-\frac{1}{2}} = c\int_0^\infty P^0_{s^2}\mathrm{d}s$ for some constant $c > 0$, we obtain

$$\begin{aligned}\sqrt{I'} &= \Big\| \int_0^\infty (-L_0)^{-\frac{1}{2}}\{\psi_r^\nu - \nu(\phi_0)\}\mathrm{d}r\Big\|_{L^2(\mu_0)} \\ &\le c\int_0^\infty \mathrm{d}r \int_0^\infty \|(P^0_{s^2+r/2} - \mu_0)\psi^\nu_{r/2}\|_{L^2(\mu_0)}\mathrm{d}s.\end{aligned} \tag{4.10}$$

Noting that (3.2) and (2.8) imply $\|\psi^\nu_{r/2}\|_{L^1(\mu_0)} = \nu(\phi_0) < \infty$ and

$$\mu(\psi^\nu_{r/2}\phi_0) = \mathrm{e}^{\lambda_0 r/2}\int_{M\times M} p^D_{r/2}(x,y)\nu(\mathrm{d}x)\mu(\mathrm{d}y) \le \mathrm{e}^{\lambda_0 r/2},$$

by (2.4) and (2.5), we find a constant $c_1 > 0$ such that

$$\begin{aligned}&\|(P^0_{s^2+r/2} - \mu_0)\psi^\nu_{r/2}\|_{L^2(\mu_0)} \le \|\psi^\nu_{r/2}\|_{L^1(\mu_0)} + \mathrm{e}^{(s^2+r)\lambda_0}\|P^D_{s^2+r/2}\|_{L^1(\mu)\to L^2(\mu)} \\ &\le c_1(s^2+r)^{-\frac{d}{4}} \le c_1(s^2+r)^{-\frac{5}{4}}, \quad s^2 + r/2 \le 1, d \le 5,\end{aligned}$$

and due to (2.11)

$$\begin{aligned}&\|(P^0_{s^2+r/2} - \mu_0)\psi^\nu_{r/2}\|_{L^2(\mu_0)} \le \|P^0_{s^2+r/2} - \mu_0\|_{L^1(\mu_0)\to L^2(\mu_0)}\|\psi^\nu_{r/2}\|_{L^1(\mu_0)} \\ &\le c_2\mathrm{e}^{-(\lambda_1-\lambda_2)(s^2+r/2)}, \quad s^2 + r \ge 1\end{aligned}$$

holds for some constant $c_2 > 0$. Combining these with (4.10), we prove $I' < \infty$.

(c) $I' < \infty$ for $d \ge 6$ and $\nu = h\mu$ with $h \in L^p(\mu)$ for some $p > \frac{2d}{d+6}$. Since $\{\phi_m\phi_0^{-1}\}_{m\ge 0}$ is an orthonormal basis of $L^2(\mu_0)$ and $\mu_0(h\phi_0^{-1} - \mu(h\phi_0)) = 0$, we have

$$h\phi_0^{-1} - \mu(h\phi_0) = \sum_{m=1}^\infty \mu_0\big(\{h\phi_0^{-1} - \mu(h\phi_0)\}\phi_m\phi_0^{-1}\big)\phi_m\phi_0^{-1},$$

so that (2.6) and $\mu_0(\phi_m\phi_0^{-1}) = 0$ for $m \ge 1$ yield

$$\begin{aligned}(-L_0)^{-\frac{3}{2}}(h\phi_0^{-1} - \mu(h\phi_0)) &= \sum_{m=1}^\infty \frac{\mu_0\big(\{h\phi_0^{-1} - \mu(h\phi_0)\}\phi_m\phi_0^{-1}\big)}{(\lambda_m - \lambda_0)^{\frac{3}{2}}}\phi_m\phi_0^{-1} \\ &= \sum_{m=1}^\infty \frac{\mu(h\phi_m)}{(\lambda_m-\lambda_0)^{\frac{3}{2}}}\phi_m\phi_0^{-1}.\end{aligned}$$

Thus,

$$I' = \left\|(-L_0)^{-\frac{3}{2}}(h\phi_0^{-1} - \mu(h\phi_0))\right\|^2_{L^2(\mu_0)}. \tag{4.11}$$

Noting that $\mu_0((h\phi_0^{-1} - \mu(h\phi_0)) = 0$ and $(-L_0)^{-\frac{3}{2}} = c\int_0^\infty P^0_{t^{\frac{2}{3}}}\mathrm{d}t$ for some constant $c > 0$, combining this with (2.4), (2.5), (2.11), $\|h\phi_0^{-1}\|_{L^1(\mu_0)} < \infty$, and $\|h\|_{L^p(\mu_0)} < \infty$ for some $p \in (\frac{2(d+2)}{d+8}, 2)$ as we have assumed, we find constants $c_1, c_2 > 0$ such that

$$\begin{aligned}
&\|(-L_0)^{-\frac{3}{2}}(h\phi_0^{-1} - \mu(h\phi_0))\|_{L^2(\mu_0)} \le \int_0^\infty \|(P^0_{t^{\frac{2}{3}}} - \mu_0)\{h\phi_0^{-1}\}\|_{L^2(\mu_0)}\mathrm{d}t\\
&\le \|h\phi_0^{-1}\|_{L^1(\mu_0)}\int_1^\infty \|P^0_{t^{\frac{2}{3}}} - \mu_0\|_{L^1(\mu_0)\to L^2(\mu_0)}\mathrm{d}t + \int_0^1 \mathrm{e}^{\lambda_0 t^{\frac{3}{2}}}\|\phi_0^{-1}P^D_{t^{\frac{3}{2}}}\{h - \nu(\phi_0)\phi_0\}\|_{L^2(\mu_0)}\mathrm{d}t\\
&= \|h\phi_0^{-1}\|_{L^1(\mu_0)}\int_1^\infty \|P^0_{t^{\frac{2}{3}}} - \mu_0\|_{L^1(\mu_0)\to L^2(\mu_0)}\mathrm{d}t + \int_0^1 \mathrm{e}^{\lambda_0 t^{\frac{3}{2}}}\|P^D_{t^{\frac{3}{2}}}\{h - \nu(\phi_0)\phi_0\}\|_{L^2(\mu)}\mathrm{d}t\\
&\le c_1\int_1^\infty \mathrm{e}^{-(\lambda_1-\lambda_0)t}\mathrm{d}t + c_1\int_0^1 \|P^D_{t^{\frac{3}{2}}}\|_{L^p(\mu)\to L^2(\mu)}\mathrm{d}t \le \frac{c_1}{\lambda_1 - \lambda_0} + c_2\int_0^1 t^{-\frac{d(2-p)}{6p}}\mathrm{d}t < \infty,
\end{aligned}$$

since $p > \frac{2d}{d+6}$ implies $\frac{d(2-p)}{6p} < 1$. Combining this with (4.11) we prove (4.9).

(d) $I' < \infty$ for $d \ge 6$ and $\nu = h\mu$ with $h\phi_0^{-1} \in L^q(\mu_0)$ for some $q > \frac{2(d+2)}{d+8}$. By (2.11) we find constants $c_1, c_2 > 0$ such that

$$\begin{aligned}
&\|(-L_0)^{-\frac{3}{2}}(h\phi_0^{-1} - \mu(h\phi_0))\|_{L^2(\mu_0)} \le \int_0^\infty \|(P^0_{t^{\frac{2}{3}}} - \mu_0)\{h\phi_0^{-1}\}\|_{L^2(\mu_0)}\mathrm{d}t\\
&\le \int_0^\infty \|P^0_{t^{\frac{2}{3}}} - \mu_0\|_{L^q(\mu_0)\to L^2(\mu_0)}\|h\phi_0^{-1}\|_{L^q(\mu_0)}\mathrm{d}t\\
&\le c_1\int_0^\infty \{1\wedge t\}^{-\frac{(d+2)(2-q)}{6q}}\mathrm{e}^{-(\lambda_1-\lambda_0)t^{\frac{2}{3}}}\mathrm{d}t < \infty
\end{aligned}$$

since $q > \frac{2(d+2)}{d+8}$ implies $\frac{(d+2)(2-q)}{6q} < 1$.

□

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