Convergence in Wasserstein Distance for Empirical Measures of Dirichlet Diffusion Processes on Manifolds*

Feng-Yu Wang[a),b)]

[a)] Center for Applied Mathematics, Tianjin University, Tianjin 300072, China

[b)] Department of Mathematics, Swansea University, Bay Campus, Swansea, SA1 8EN, United Kingdom

July 1, 2020

Abstract

Let M be a d-dimensional connected compact Riemannian manifold with boundary ∂M, let $V \in C^2(M)$ such that $\mu(\mathrm{d}x) := \mathrm{e}^{V(x)}\mathrm{d}x$ is a probability measure, and let X_t be the diffusion process generated by $L := \Delta + \nabla V$ with $\tau := \inf\{t \geq 0 : X_t \in \partial M\}$. Consider the empirical measure $\mu_t := \frac{1}{t}\int_0^t \delta_{X_s}\mathrm{d}s$ under the condition $t < \tau$ for the diffusion process. If $d \leq 3$, then for any initial distribution not fully supported on ∂M,

$$c\sum_{m=1}^\infty \frac{2}{(\lambda_m - \lambda_0)^2} \leq \liminf_{t\to\infty} \inf_{T\geq t} \Big\{t\mathbb{E}\big[\mathbb{W}_2(\mu_t,\mu_0)^2\big|T<\tau\big]\Big\}$$

$$\leq \limsup_{t\to\infty} \sup_{T\geq t} \Big\{t\mathbb{E}\big[\mathbb{W}_2(\mu_t,\mu_0)^2\big|T<\tau\big]\Big\} \leq \sum_{m=1}^\infty \frac{2}{(\lambda_m - \lambda_0)^2}$$

holds for some constant $c \in (0,1]$ with $c = 1$ when ∂M is convex, where $\mu_0 := \phi_0^2\mu$ for the first Dirichet eigenfunction ϕ_0 of L, $\{\lambda_m\}_{m\geq 0}$ are the Dirichlet eigenvalues of $-L$ listed in the increasing order counting multiplicities, and the upper bound is finite if and only if $d \leq 3$. When $d = 4$, $\sup_{T\geq t}\mathbb{E}\big[\mathbb{W}_2(\mu_t,\mu_0)^2\big|T<\tau\big]$ decays in the order $t^{-1}\log t$, while for $d \geq 5$ it behaves like $t^{-\frac{2}{d-2}}$, as $t \to \infty$.

AMS subject Classification: 60D05, 58J65.
Keywords: Conditional empirical measure, Dirichlet diffusion process, Wasserstein distance, eigenvalues, eigenfunctions.

*Supported in part by NNSFC (11771326, 11831014, 11921001), and DFG through the CRC ?Taming uncertainty and profiting from randomness and low regularity in analysis, stochastics and their applications?.

1 Introduction

Let M be a d-dimensional connected complete Riemannian manifold with a smooth boundary ∂M. Let $V \in C^2(M)$ such that $\mu(\mathrm{d}x) = \mathrm{e}^{V(x)}\mathrm{d}x$ is a probability measure on M, where $\mathrm{d}x$ is the Riemannian volume measure. Let X_t be the diffusion process generated by $L := \Delta + \nabla V$ with hitting time

$$\tau := \inf\{t \geq 0 : X_t \in \partial M\}.$$

Denote by $\mathscr{P}$ the set of all probability measures on M, and let $\mathbb{E}^\nu$ be the expectation taken for the diffusion process with initial distribution $\nu \in \mathscr{P}$. We consider the empirical measure

$$\mu_t := \frac{1}{t}\int_0^t \delta_{X_s}\mathrm{d}s, \quad t > 0$$

under the condition that $t < \tau$. Since $\tau = 0$ when $X_0 \in \partial M$, to ensure $\mathbb{P}^\nu(\tau > t) > 0$, where $\mathbb{P}^\nu$ is the probability taken for the diffusion process with initial distribution ν, we only consider

$$\nu \in \mathscr{P}_0 := \big\{\nu \in \mathscr{P} :\ \nu(M^\circ) > 0\big\}, \quad M^\circ := M \setminus \partial M.$$

Let $\mu_0 = \phi_0^2\mu$, where ϕ_0 is the first Dirichlet eigenfunction. We investigate the convergence rate of $\mathbb{E}^\nu[\mathbb{W}_2(\mu_t, \mu_0)^2|t < \tau]$ as $t \to \infty$, where $\mathbb{W}_2$ is the L^2-Wasserstein distance induced by the Riemannian metric ρ. In general, for any $p \geq 1$,

$$\mathbb{W}_p(\mu_1, \mu_2) := \inf_{\pi\in\mathscr{C}(\mu_1,\mu_2)}\left(\int_{M\times M}\rho(x,y)^p\pi(\mathrm{d}x,\mathrm{d}y)\right)^{\frac{1}{p}}, \quad \mu_1, \mu_2 \in \mathscr{P},$$

where $\mathscr{C}(\mu_1, \mu_2)$ is the set of all probability measures on $M \times M$ with marginal distributions μ_1 and μ_2, and $\rho(x, y)$ is the Riemannian distance between x and y, i.e. the length of the shortest curve on M linking x and y.

Recently, the convergence rate under $\mathbb{W}_2$ has been characterized in [21] for the empirical measures of the L-diffusion processes without boundary (i.e. $\partial M = \emptyset$) or with a reflecting boundary. Moreover, the convergence of $\mathbb{W}_2(\mu_t^\nu, \mu_0)$ for the conditional empirical measure

$$\mu_t^\nu := \mathbb{E}^\nu(\mu_t|t < \tau), \quad t > 0$$

is investigated in [20]. Comparing with $\mathbb{E}^\nu[\mathbb{W}_2(\mu_t, \mu_0)^2|t < \tau]$, in μ_t^ν the conditional expectation inside the Wasserstein distance. According to [20], $\mathbb{W}_2(\mu_t^\nu, \mu_0)^2$ behaves as t^{-2}, whereas the following result says that $\mathbb{E}[\mathbb{W}_2(\mu_t, \mu_0)^2|t < \tau]$ decays at a slower rate, which coincides with the rate of $\mathbb{E}[\mathbb{W}_2(\hat\mu_t, \mu)^2]$ given by [21, Theorems 1.1, 1.2], where $\hat\mu_t$ is the empirical measure of the reflecting diffusion process generated by L.

Theorem 1.1. *Let $\{\lambda_m\}_{m\geq 0}$ be the Dirichlet eigenvalues of $-L$ listed in the increasing order counting multiplicities. Then for any $\nu \in \mathscr{P}_0$, the following assertions hold.*

(1) *In general,*

$$\limsup_{t\to\infty}\Big\{t\sup_{T\geq t}\mathbb{E}^\nu\big[\mathbb{W}_2(\mu_t, \mu_0)^2\big|T < \tau\big]\Big\} \leq \sum_{m=1}^\infty \frac{2}{(\lambda_m - \lambda_0)^2}, \tag{1.1}$$

and there exists a constant $c>0$ *such that*

$$\liminf_{t\to\infty}\Big\{t\inf_{T\ge t}\mathbb{E}^\nu\big[\mathbb{W}_2(\mu_t,\mu_0)^2\big|T<\tau\big]\Big\}\ge c\sum_{m=1}^\infty\frac{2}{(\lambda_m-\lambda_0)^2}. \tag{1.2}$$

If ∂M *is convex, then* (1.2) *holds for* $c=1$ *so that*

$$\lim_{t\to\infty}\Big\{t\mathbb{E}^\nu\big[\mathbb{W}_2(\mu_t,\mu_0)^2\big|T<\tau\big]\Big\}=\sum_{m=1}^\infty\frac{2}{(\lambda_m-\lambda_0)^2}\ \text{uniformly in } T\ge t.$$

(2) *When* $d=4$, *there exists a constant* $c>0$ *such that*

$$\sup_{T\ge t}\mathbb{E}^\nu\big[\mathbb{W}_2(\mu_t,\mu_0)^2\big|T<\tau\big]\le ct^{-1}\log t,\quad t\ge 2. \tag{1.3}$$

(3) *When* $d\ge 5$, *there exist a constant* $c>1$ *such that*

$$c^{-1}t^{-\frac{2}{d-2}}\le\mathbb{E}^\nu\big[\mathbb{W}_1(\mu_t,\mu_0)^2\big|T<\tau\big]\le\mathbb{E}^\nu\big[\mathbb{W}_2(\mu_t,\mu_0)^2\big|T<\tau\big]\le ct^{-\frac{2}{d-2}},\quad T\ge t\ge 2.$$

Let X_t^0 be the diffusion process generated by $L_0:=L+2\nabla\log\phi_0$ in M°. It is well known that for any initial distribution supported on M°, the law of $\{X_s^0:s\in[0,t]\}$ is the weak limit of the conditional distribution of $\{X_s:s\in[0,t]\}$ given $T<\tau$ as $T\to\infty$. Therefore, the following is a direct consequence of Theorem 1.1.

Corollary 1.2. *Let* $\mu_t^0=\frac{1}{t}\int_0^t\delta_{X_s^0}\mathrm{d}s$. *Let* $\nu\in\mathscr{P}_0$ *with* $\nu(M^\circ)=1$.

(1) *In general,*

$$\limsup_{t\to\infty}\Big\{t\mathbb{E}^\nu\big[\mathbb{W}_2(\mu_t^0,\mu_0)^2\big]\Big\}\le\sum_{m=1}^\infty\frac{2}{(\lambda_m-\lambda_0)^2},$$

and there exists a constant $c>0$ *such that*

$$\liminf_{t\to\infty}\Big\{t\inf_{T\ge t}\big[\mathbb{W}_2(\mu_t^0,\mu_0)^2\big]\Big\}\ge c\sum_{m=1}^\infty\frac{2}{(\lambda_m-\lambda_0)^2}.$$

If ∂M *is convex, then*

$$\lim_{t\to\infty}\Big\{t\mathbb{E}^\nu\big[\mathbb{W}_2(\mu_t,\mu_0)^2\big]\Big\}=\sum_{m=1}^\infty\frac{2}{(\lambda_m-\lambda_0)^2}.$$

(2) *When* $d=4$, *there exists a constant* $c>0$ *such that*

$$\mathbb{E}^\nu\big[\mathbb{W}_2(\mu_t^0,\mu_0)^2\big]\le ct^{-1}\log t,\quad t\ge 2.$$

(3) *When* $d\ge 5$, *there exists a constant* $c>1$ *such that*

$$c^{-1}t^{-\frac{2}{d-2}}\le\mathbb{E}^\nu\big[\mathbb{W}_2(\mu_t^0,\mu_0)^2\big]\le ct^{-\frac{2}{d-2}},\quad t\ge 2.$$

In the next section, we first recall some facts on the Dirichlet semigroup and the diffusion semigroup P_t^0 generated by $L_0:=L+2\nabla\log\phi_0$, then establish the Bismut derivative formula for P_t^0 which will be used to estimate the lower bound of $\mathbb{E}^\nu[\mathbb{W}_2(\mu_t,\mu_0)^2|t<\tau)$. With these preparations, we prove Propositions 3.1 and 4.1 in Sections 3 and 4 respectively, which imply Theorem 1.1.

2 Some preparations

As in [21], we first recall some well known facts on the Dirichlet semigroup, see for instances [5, 6, 12, 19]. Let $\{\phi_m\}_{m\geq 0}$ be the eigenbasis of the Dirichlet operator L in $L^2(\mu)$, with Dirichlet eigenvalues $\{\lambda_m\}_{m\geq 0}$ of $-L$ listed in the increasing order counting multiplicities. Then $\lambda_0 > 0$ and

$$\|\phi_m\|_\infty \leq \alpha_0\sqrt{m},\quad \alpha_0^{-1} m^{\frac{2}{d}} \leq \lambda_m - \lambda_0 \leq \alpha_0 m^{\frac{2}{d}},\quad m \geq 1 \tag{2.1}$$

holds for some constant $\alpha_0 > 1$. Let ρ_∂ be the Riemannian distance function to the boundary ∂M. Then $\phi_0^{-1}\rho_\partial$ is bounded such that

$$\|\phi_0^{-1}\|_{L^p(\mu_0)} < \infty,\quad p \in [1,3). \tag{2.2}$$

The Dirichlet heat kernel has the representation

$$p_t^D(x,y) = \sum_{m=0}^\infty \mathrm{e}^{-\lambda_m t}\phi_m(x)\phi_m(y),\quad t>0, x,y \in M. \tag{2.3}$$

Let $\mathbb{E}^x$ denote the expectation for the L-diffusion process starting at point x. Then Dirichlet diffusion semigroup generated by L is given by

$$\begin{aligned} P_t^D f(x) &:= \mathbb{E}^x[f(X_t)1_{\{t<\tau\}}] = \int_M p_t^D(x,y)f(y)\mu(\mathrm{d}y) \\ &= \sum_{m=0}^\infty \mathrm{e}^{-\lambda_m t}\mu(\phi_m f)\phi_m(x),\quad t>0, f\in L^2(\mu). \end{aligned} \tag{2.4}$$

Consequently,

$$\lim_{t\to\infty}\left\{\mathrm{e}^{\lambda_0 t}\mathbb{P}^\nu(t<\tau)\right\} = \lim_{t\to\infty}\left\{\mathrm{e}^{\lambda_0 t}\nu(P_t^D 1)\right\} = \mu(\phi_0)\nu(\phi_0),\quad \nu\in\mathscr{P}_0. \tag{2.5}$$

Moreover, there exists a constant $c>0$ such that

$$\|P_t^D\|_{L^p(\mu)\to L^q(\mu)} := \sup_{\mu(|f|^p)\leq 1}\|P_t^D f\|_{L^q(\mu)} \leq c\mathrm{e}^{-\lambda_0 t}(1\wedge t)^{-\frac{d(q-p)}{2pq}},\quad t>0, q\geq p\geq 1. \tag{2.6}$$

On the other hand, let $L_0 = L + 2\nabla\log\phi_0$. Noting that $L_0 f = \phi_0^{-1}L(f\phi_0) + \lambda_0 f$, L_0 is a self-adjoint operator in $L^2(\mu_0)$ and the associated semigroup $P_t^0 := \mathrm{e}^{tL_0}$ satisfies

$$P_t^0 f = \mathrm{e}^{\lambda_0 t}\phi_0^{-1}P_t^D(f\phi_0),\quad f\in L^2(\mu_0),\quad t\geq 0. \tag{2.7}$$

So, $\{\phi_0^{-1}\phi_m\}_{m\geq 0}$ is an eigenbasis of L_0 in $L^2(\mu_0)$ with

$$L_0(\phi_m\phi_0^{-1}) = -(\lambda_m-\lambda_0)\phi_m\phi_0^{-1},\quad P_t^0(\phi_m\phi_0^{-1}) = \mathrm{e}^{-(\lambda_m-\lambda_0)t}\phi_m\phi_0^{-1},\quad m\geq 0, t\geq 0. \tag{2.8}$$

Consequently,

$$P_t^0 f = \sum_{m=0}^\infty \mu_0(f\phi_m\phi_0^{-1})\mathrm{e}^{-(\lambda_m-\lambda_0)t}\phi_m\phi_0^{-1},\quad f\in L^2(\mu_0), \tag{2.9}$$

and the heat kernel of P_t^0 with respect to μ_0 is given by

$$p_t^0(x,y) = \sum_{m=0}^{\infty} (\phi_m\phi_0^{-1})(x)(\phi_m\phi_0^{-1})(y)\mathrm{e}^{-(\lambda_m-\lambda_0)t}, \quad x,y \in M, t>0. \tag{2.10}$$

By the intrinsic ultracontractivity, see for instance [13], there exists a constant $\alpha_1 \geq 1$ such that

$$\|P_t^0 - \mu_0\|_{L^1(\mu_0)\to L^\infty(\mu_0)} := \sup_{\mu_0(|f|)\leq 1} \|P_t^0 f - \mu_0(f)\|_\infty \leq \frac{\alpha_1 \mathrm{e}^{-(\lambda_1-\lambda_0)t}}{(1\wedge t)^{\frac{d+2}{2}}}, \quad t>0. \tag{2.11}$$

Combining this with the semigroup property and the contraction of P_t^0 in $L^p(\mu)$ for any $p \geq 1$, we find a constant $\alpha_2 \geq 1$ such that

$$\|P_t^0 - \mu_0\|_{L^p(\mu_0)} := \sup_{\mu_0(|f|^p)\leq 1} \|P_t^0 f - \mu_0(f)\|_{L^p(\mu_0)} \leq \alpha_2 \mathrm{e}^{-(\lambda_1-\lambda_0)t}, \quad t\geq 0, p\geq 1. \tag{2.12}$$

By the interpolation theorem, (2.11) and (2.12) yield that for some constant $\alpha_3 > 0$,

$$\|P_t^0 - \mu_0\|_{L^p(\mu_0)\to L^q(\mu_0)} \leq \alpha_3 \mathrm{e}^{-(\lambda_1-\lambda_0)t}\{1\wedge t\}^{-\frac{(d+2)(q-p)}{2pq}}, \quad t>0, \infty\geq q>p\geq 1. \tag{2.13}$$

By this and (2.8), there exists a constant $\alpha_4 > 0$ such that

$$\|\phi_m\phi_0^{-1}\|_\infty \leq \alpha_4 m^{\frac{d+2}{2d}}, \quad m\geq 1. \tag{2.14}$$

In the remainder of this section, we establish the Bismut derivative formula for P_t^0, which is not included by existing results due to the singularity of $\nabla \log \phi_0$ in L_0. Let X_t^0 be the diffusion process generated by L_0, which solves the following Itô SDE on M°, see [8]:

$$\mathrm{d}^I X_t^0 = \nabla(V + 2\log\phi_0)(X_t^0)\mathrm{d}t + \sqrt{2}U_t \mathrm{d}B_t, \tag{2.15}$$

where B_t is the d-dimensional Brownian motion, and $U_t \in O_{X_t^0}(M)$ is the horizontal lift of X_t^0 to the frame bundle $O(M)$. Let Ric and Hess be the Ricci curvature and the Hessian tensor on M respectively. Then the Bakry-Emery curvature of L_0 is given by

$$\mathrm{Ric}_{L_0} := \mathrm{Ric} - \mathrm{Hess}_{V+2\log\phi_0}.$$

Let $\mathrm{Ric}_{L^0}^{\#}(U_t) \in \mathbb{R}^d \otimes \mathbb{R}^d$ be defined by

$$\langle \mathrm{Ric}_{L^0}^{\#}(U_t)a, b\rangle_{\mathbb{R}^d} = \mathrm{Ric}_{L_0}(U_t a, U_t b), \quad a,b\in\mathbb{R}^d.$$

We consider the following ODE on $\mathbb{R}^d \otimes \mathbb{R}^d$:

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_t = -\mathrm{Ric}_{L^0}^{\#}(U_t)Q_t, \quad Q_0 = I, \tag{2.16}$$

where I is the identity matrix.

Lemma 2.1. *For any $\varepsilon > 0$, there exist constants $\delta_1, \delta_2 > 0$ such that*

$$\mathbb{E}^x\Big[\mathrm{e}^{\delta_1 \int_0^t \{\phi_0(X_s)\}^{-2}\mathrm{d}s}\Big] \le \delta_2 \phi_0^{-\varepsilon}(x)\mathrm{e}^{\delta_2 t}, \quad t \ge 0, x \in M^\circ. \tag{2.17}$$

Consequently,

(1) *For any $\varepsilon > 0$ and $p > 1$, there exists a constant $\kappa > 0$ such that*

$$|\nabla P_t f(x)|^2 \le \kappa \phi_0(x)^{-\varepsilon} \mathrm{e}^{\kappa t}\{P_t|\nabla f|^{2p}(x)\}^{\frac{1}{p}}, \quad f \in C_b^1(M).$$

(2) *For any $\varepsilon > 0$ and $p \ge 1$, there exists a constant $\kappa > 0$ such that for any stopping time τ',*

$$\mathbb{E}^x[\|Q_{t\wedge\tau'}\|^p] \le \kappa\phi_0(x)^{-\varepsilon}\mathrm{e}^{\kappa t}, \quad t \ge 0.$$

Proof. Since $L\phi_0 = -\lambda_0\phi_0$, $\phi_0 > 0$ in M°, $\|\phi_0\|_\infty < \infty$ and $|\nabla\phi_0|$ is strictly positive in a neighborhood of ∂M, we find a constant $c_1, c_2 > 0$ such that

$$L_0 \log \phi_0^{-1} = -\phi_0^{-1}L\phi_0 + \phi_0^{-2}|\nabla\phi_0|^2 - 2\phi_0^{-2}|\nabla\phi_0|^2 \le c_1 - c_2\phi_0^{-2}.$$

So, by (2.15) and Itô's formula, we obtain

$$\mathrm{d}\log\phi_0^{-1}(X_t^0) \le \{c_1 - c_2\phi_0^{-2}(X_t^0)\}\mathrm{d}t + \sqrt{2}\langle\nabla\log\phi_0^{-1}(X_t^0), U_t\mathrm{d}B_t\rangle.$$

This implies

$$\mathbb{E}^x\int_0^t [\phi_0^{-2}(X_s^0)]\mathrm{d}s \le ct + c\log(1+\phi_0^{-1})(x), \quad t \ge 0 \tag{2.18}$$

for some constant $c > 0$, and for any constant $\delta > 0$,

$$\begin{aligned}
&\mathbb{E}^x\Big[\mathrm{e}^{\delta c_2\int_0^t \phi_0^{-2}(X_s^0)\}\mathrm{d}s}\Big] \le \mathbb{E}^x\Big[\mathrm{e}^{\delta\log\phi_0^{-1}(x)+\delta\log\phi_0(X_t^0)+c_1\delta t-\delta\sqrt{2}\int_0^t\langle\nabla\log\phi_0(X_s^0),U_s\mathrm{d}B_s\rangle}\Big]\\
&\le \mathrm{e}^{c_1\delta t}\phi_0^{-\delta}(x)\|\phi_0\|_\infty^\delta\big(\mathbb{E}^x[\mathrm{e}^{4\delta^2\int_0^t|\nabla\log\phi_0|^2(X_s^0)\mathrm{d}s}]\big)^{\frac{1}{2}}.
\end{aligned}$$

Let $c_3 = 4\|\nabla\phi_0\|_\infty^2$, and take $\delta \in (0, c_2/c_3]$, we derive

$$\mathbb{E}^x\Big[\mathrm{e}^{\delta c_2\int_0^t \phi_0^{-2}(X_s^0)\}\mathrm{d}s}\Big] \le \mathrm{e}^{2c_1\delta t}\phi_0^{-2\delta}(x), \quad \delta \in (0, c_2/c_3].$$

This implies (2.17). Below we prove assertions (1) and (2) respectively.

Since $V \in C_b^2(M)$ and $\phi_0 \in C_b^2(M)$ with $\phi_0 > 0$ in M°, there exists a constant $\alpha_1 > 0$ such that

$$\mathrm{Ric}_{L_0}(U,U) \ge -\alpha_1\phi_0^{-1}(x)|U|^2, \quad x \in M^\circ, U \in T_xM. \tag{2.19}$$

By (2.15), (2.19), and the formulas of Itô and Bochner, for fixed $t > 0$ this implies

$$\begin{aligned}
&\mathrm{d}|\nabla P_{t-s}^0 f|^2(X_s^0)\\
&= \big\{L_0|\nabla P_{t-s}^0 f|^2(X_s^0) - 2\langle\nabla P_{t-s}^0 f, \nabla L_0 P_{t-s}^0 f\rangle\big\}\mathrm{d}s + \sqrt{2}\langle\nabla|\nabla P_{t-s}^0 f|^2(X_s^0), U_s\mathrm{d}B_s\rangle
\end{aligned}$$

$$\geq 2\mathrm{Ric}_{L^0}(\nabla P^0_{t-s}f, \nabla P^0_{t-s}f)(X^0_s)\mathrm{d}s + \sqrt{2}\langle \nabla |\nabla P^0_{t-s}f|^2(X^0_s), U_s \mathrm{d}B_s\rangle$$
$$\geq -2\alpha_1\{\phi_0^{-1}|\nabla P^0_{t-s}f)|^2\}(X^0_s)\mathrm{d}s + \sqrt{2}\langle \nabla |\nabla P^0_{t-s}f|^2(X_s), U_s \mathrm{d}B_s\rangle \mathrm{d}s.$$

Then

$$|\nabla P_t f(x)|^2 = \mathbb{E}^x|\nabla P_t f|^2(X^0_0) \leq \mathbb{E}^x\big[|\nabla f|^2(X^0_t)\mathrm{e}^{2\int_0^t 2\alpha_1 \phi^{-1}(X^0_u)\mathrm{d}u}\big]$$
$$\leq \big\{\mathbb{E}^x \mathrm{e}^{\frac{2\alpha_1 p}{p-1}\int_0^t \phi^{-1}(X^0_u)\mathrm{d}u}\big]\big\}^{\frac{p-1}{p}}\{P_t|\nabla f|^{2p}(x)\}^{\frac{1}{p}}.$$

Combining this with (2.17), we prove (1).

Next, by (2.16) and (2.19), we obtain

$$\|Q_{t\wedge\tau'}\| \leq \mathrm{e}^{\alpha_1\int_0^t \phi^{-1}(X^0_s)\mathrm{d}s}, \quad t \geq 0.$$

This together with (2.17) implies (2).

□

Lemma 2.2. *For any $t > 0$ and $\gamma \in C^1([0,t])$ with $\gamma(0) = 0$ and $\gamma(t) = 1$, we have*

$$\nabla P^0_t f(x) = \mathbb{E}^x\bigg[f(X^0_t)\int_0^t \gamma'(s)Q^*_s \mathrm{d}B_s\bigg], \quad x \in M^\circ, f \in \mathscr{B}_b(M^\circ). \tag{2.20}$$

Consequently, for any $\varepsilon > 0$ and $p > 1$, here exists a constant $c > 0$ such that

$$|\nabla P^0_t f| \leq \frac{c\phi_0^{-\varepsilon}}{\sqrt{1\wedge t}}(P^0_t|f|^p)^{\frac{1}{p}}, \quad t > 0, f \in \mathscr{B}_b(M^\circ). \tag{2.21}$$

Proof. Since (2.21) follows from (2.20) with $\gamma(s) := \frac{t-s}{t}$ and Lemma 2.1(2), it suffices to prove the Bismut formula (2.20). By an approximation argument, we only need to prove for $f \in C^1_b(M)$. The proof is standard by Elworthy-Li's martingale argument [7], see also [15]. By $\|\nabla f\|_\infty < \infty$ and Lemma 2.1(1) for $\varepsilon = \frac{1}{4}$, we find a constant $c_1 > 0$ such that

$$|\nabla P^0_s f|(x) \leq c_1\phi_0^{-1/4}(x), \quad s \in [0,t], x \in M^\circ. \tag{2.22}$$

Next, since $L\phi_0 = -\lambda_0\phi_0$ implies $L_0\phi_0^{-1} = \lambda_0\phi_0^{-1}$, by Itô's formula we obtain

$$\mathbb{E}^x[\phi_0^{-1}(X^0_{t\wedge\tau_n})] \leq \phi_0^{-1}(x)\mathrm{e}^{\lambda_0 t}, \quad t \geq 0, n \geq 1, \tag{2.23}$$

where $\tau_n := \inf\{t \geq 0 : \phi_0(X^0_s) \leq \frac{1}{n}\} \uparrow \infty$ as $n \uparrow \infty$ by noting that the process X^0_t is non-explosive in M°.

Moreover, by Itô's formula, for any $a \in \mathbb{R}^d$, we have

$$\mathrm{d}\langle \nabla P^0_{t-s}f(X^0_s), U_sQ_s a\rangle = \sqrt{2}\,\mathrm{Hess}_{P_{t-s}f}(U_s\mathrm{d}B_s, U_sQ_s a)(X^0_s),$$
$$\mathrm{d}P_{t-s}f(X^0_s) = \sqrt{2}\,\langle \nabla P^0_{t-s}f(X^0_s), U_s\mathrm{d}B_s\rangle, \quad s \in [0,t].$$

Due to the integration by part formula, this and $\gamma(0)=0$ imply

$$
\begin{aligned}
&-\frac{1}{\sqrt{2}}\mathbb{E}^x\bigg[f(X^0_{t\wedge\tau_n})\int_0^{t\wedge\tau_n}\gamma'(s)\langle Q_s a,\mathrm{d}B_s\rangle\bigg]\\
&=\mathbb{E}\bigg[\int_0^{t\wedge\tau_n}\langle\nabla P^0_{t-s}f(X^0_s),U_sQ_sa\rangle\mathrm{d}(1-\gamma)(s)\bigg]\\
&=\mathbb{E}\big[(1-\gamma)(t\wedge\tau_n)\langle\nabla P^0_{t-t\wedge\tau_n}f(X^0_{t\wedge\tau_n}),Q_{t\wedge\tau_n}a\rangle\big]-\langle\nabla P_tf(x),U_0a\rangle\\
&\quad-\mathbb{E}\bigg[\int_0^{t\wedge\tau_n}(1-\gamma)(s)\mathrm{d}\langle\nabla P^0_{t-s}f(X^0_s),U_sQ_sa\rangle\bigg]\\
&=\mathbb{E}\big[(1-\gamma)(t\wedge\tau_n)\langle\nabla P^0_{t-t\wedge\tau_n}f(X^0_{t\wedge\tau_n}),Q_{t\wedge\tau_n}a\rangle\big]-\langle\nabla P_tf(x),U_0a\rangle,\quad n\ge1.
\end{aligned}
\tag{2.24}
$$

Since γ is bounded with $\gamma(t)=1$ such that $(1-\gamma)(t\wedge\tau_n)\to0$ as $n\to\infty$, and (2.22), (2.23) and Lemma 2.1(2) imply

$$\sup_{n\ge1}\mathbb{E}^x\big[\langle\nabla P^0_{t-t\wedge\tau_n}f(X^0_{t\wedge\tau_n}),Q_{t\wedge\tau_n}a\rangle^2\big]\le c_1\sup_{n\ge1}\big(\mathbb{E}[\phi_0^{-1}(X^0_{t\wedge\tau_n})]\big)^{\frac12}\big(\mathbb{E}^x\|Q_{t\wedge\tau_n}\|^4\big)^{\frac12}<\infty,$$

by the dominated convergence theorem, we may take $n\to\infty$ in (2.24) to derive (2.20). □

3 Upper bound estimates

In this section we prove the following result which includes upper bound estimates in Theorem 1.1.

Proposition 3.1. *Let $\nu\in\mathscr{P}_0$.*

(1) (1.1) *holds.*

(2) *When $d=4$, there exists a constant $c>0$ such that* (1.3) *holds.*

(3) *When $d\ge5$, there exists a constant $c>0$ such that*

$$\sup_{T\ge t}\mathbb{E}^\nu\big[\mathbb{W}_2(\mu_t,\mu_0)^2\big|T<\tau\big]\le ct^{-\frac{2}{d-2}},\quad t\ge2.$$

The main tool in the study of the upper bound estimate is the following inequality due to [1], see also [21, Lemma 2.3]: for any probability density $g\in L^2(\mu_0)$,

$$\mathbb{W}_2(g\mu_0,\mu_0)^2\le\int_M\frac{|\nabla L_0(g-1)|^2}{\mathscr{M}(g,1)}\mathrm{d}\mu_0,\tag{3.1}$$

where $\mathscr{M}(a,b):=\frac{a-b}{\log a-\log b}1_{\{a\wedge b>0\}}$. To apply this inequality, as in [21], we first modify μ_t by $\mu_{t,r}:=\mu_tP^0_r$ for some $r>0$, where for a probability measure ν on M°, νP^0_r is the law of the L_0-diffusion process X^0_r with initial distribution ν. Obviously, by (2.10) we have

$$
\begin{aligned}
&\rho_{t,r}:=\frac{\mathrm{d}\mu_{t,r}}{\mathrm{d}\mu_0}=\frac1t\int_0^tp^0_r(X_s,\cdot)\mathrm{d}s=1+\sum_{m=1}^\infty\mathrm{e}^{-(\lambda_m-\lambda_0)r}\psi_m(t)\phi_m\phi_0^{-1},\\
&\psi_m(t):=\frac1t\int_0^t\{\phi_m\phi_0^{-1}\}(X_s)\mathrm{d}s,
\end{aligned}
\tag{3.2}
$$

which are well-defined on the event $\{t < \tau\}$.

Lemma 3.2. *If $d \le 3$ and $\nu = h\mu$ with $h\phi_0^{-1} \in L^p(\mu_0)$ for some $p > \frac{d+2}{2}$, then there exists a constant $c > 0$ such that*

$$\sup_{T\ge t}\Big|t\mathbb{E}^\nu\big[\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^2)\big|T<\tau\big] - 2\sum_{m=1}^\infty \frac{\mathrm{e}^{-2(\lambda_m-\lambda_0)r}}{(\lambda_m-\lambda_0)^2}\Big|$$
$$\le ct^{-1}\big(r^{-\frac{(d-2)^+}{2}} + 1_{\{d=2\}}\log r^{-1}\big),\quad r\in(0,1], t\ge 1.$$

Proof. By the Markov property, (2.7) and (2.4), we have

$$\begin{aligned}(3.3)\qquad &\mathbb{E}^x[f(X_s)1_{\{T<\tau\}}] = \mathbb{E}^x\big[1_{\{s<\tau\}}f(X_s)\mathbb{E}^{X_s}1_{\{T-s<\tau\}}\big]\\ &= P_s^D\{fP_{T-s}^D 1\}(x) = \mathrm{e}^{-\lambda_0 T}\big(\phi_0 P_s^0\{fP_{T-s}^0\phi_0^{-1}\}\big)(x),\quad s<T.\end{aligned}$$

By the same reason, and noting that $\mathbb{E}^\nu = \int_M \mathbb{E}^x\nu(\mathrm{d}x)$, we derive

$$\begin{aligned}&\mathbb{E}^\nu[f(X_{s_1})f(X_{s_2}))1_{\{T<\tau\}}] = \int_M \mathbb{E}^x\big[1_{\{s_1<\tau\}}f(X_{s_1})\mathbb{E}^{X_{s_1}}\{f(X_{s_2-s_1})1_{\{T-s_1<\tau\}}\}\big]\nu(\mathrm{d}x)\\ &= \mathrm{e}^{-\lambda_0 T}\nu\big(\phi_0 P_{s_1}^0[fP_{s_2-s_1}^0\{fP_{T-s_2}^0\phi_0^{-1}\}]\big),\quad s_1<s_2<T.\end{aligned}$$

In particular, the formula with $f=1$ yields

$$\mathbb{P}^\nu(T<\tau) = \mathrm{e}^{-\lambda_0 T}\nu(\phi_0 P_T^0\phi_0^{-1}).$$

Combining these with (3.2), (2.8), $\mathbb{E}^\nu(\xi|T<\tau) := \frac{\mathbb{E}^\nu[\xi 1_{\{T<\tau\}}]}{\mathbb{P}^\nu(T<\tau)}$ for an integrable random variable ξ, and the symmetry of P_t^0 in $L^2(\mu_0)$, for $\nu = h\mu$ we obtain

$$\begin{aligned}(3.4)\qquad &t\mathbb{E}^\nu\big[\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^2)\big|T<\tau\big] = \sum_{m=1}^\infty \frac{t\mathbb{E}^\nu[\psi_m(t)^2|T<\tau]}{\mathrm{e}^{2(\lambda_m-\lambda_0)r}(\lambda_m-\lambda_0)}\\ &= \sum_{m=1}^\infty \frac{2\int_0^t\mathrm{d}s_1\int_{s_1}^t \mathbb{E}^\nu\big[1_{\{T<\tau\}}(\phi_m\phi_0^{-1})(X_{s_1})(\phi_m\phi_0^{-1})(X_{s_2})\big]\mathrm{d}s_2}{t\mathrm{e}^{2(\lambda_m-\lambda_0)r}(\lambda_m-\lambda_0)\nu(\phi_0P_T^0\phi_0^{-1})}\\ &= \sum_{m=1}^\infty \frac{2\int_0^t\mathrm{d}s_1\int_{s_1}^t \nu\big(\phi_0^{-1}P_{s_1}^0\{\phi_m\phi_0^{-1}P_{s_2-s_1}^0[\phi_m\phi_0^{-1}P_{T-s_2}^0\phi_0^{-1}]\}\big)\mathrm{d}s_2}{t\mathrm{e}^{2(\lambda_m-\lambda_0)r}(\lambda_m-\lambda_0)\nu(\phi_0P_T^0\phi_0^{-1})}\\ &= \sum_{m=1}^\infty \frac{2\int_0^t\mathrm{d}s_1\int_{s_1}^t \mu_0\big(\{P_{s_1}^0(h\phi_0^{-1})\}\phi_m\phi_0^{-1}P_{s_2-s_1}^0[\phi_m\phi_0^{-1}P_{T-s_2}^0\phi_0^{-1}]\big)\mathrm{d}s_2}{t\mathrm{e}^{2(\lambda_m-\lambda_0)r}(\lambda_m-\lambda_0)\mu_0(\phi_0^{-1}P_T^0(h\phi_0^{-1}))}.\end{aligned}$$

By (2.13), $\|\phi_0^{-1}\|_{L^2(\mu_0)} = 1$ and $\|h\phi_0^{-1}\|_{L^1(\mu_0)} = \mu(h\phi_0) \le \|\phi_0\|_\infty < \infty$, we find a constant $c_1 > 0$ such that

$$\begin{aligned}(3.5)\qquad &\big|\mu_0(\phi_0^{-1}P_T^0(h\phi_0^{-1})) - \mu(\phi_0)\nu(\phi_0)\big| \le \|\phi_0^{-1}(P_T^0-\mu_0)(h\phi_0^{-1})\|_{L^1(\mu_0)}\\ &\le \|P_T^0-\mu_0\|_{L^1(\mu_0)\to L^2(\mu_0)}\|h\phi_0^{-1}\|_{L^1(\mu_0)} \le c_1\mathrm{e}^{-(\lambda_1-\lambda_0)T},\quad T\ge 1.\end{aligned}$$

On the other hand, write

$$\mu_0\big(\{P^0_{s_1}(h\phi_0^{-1})\}\phi_m\phi_0^{-1}P^0_{s_2-s_1}[\phi_m\phi_0^{-1}P^0_{T-s_2}\phi_0^{-1}]\big) \\ = \nu(\phi_0)\mu(\phi_0)\mathrm{e}^{-(\lambda_m-\lambda_0)(s_2-s_1)} + J_1(s_1,s_2)+J_2(s_1,s_2)+J_3(s_1,s_2), \tag{3.6}$$

where, due to (2.8),

$$\begin{aligned} J_1(s_1,s_2) &:= \mu_0\big(\{P^0_{s_1}(h\phi_0^{-1})-\mu(h\phi_0)\}\phi_m\phi_0^{-1}P^0_{s_2-s_1}[\phi_m\phi_0^{-1}(P^0_{T-s_2}\phi_0^{-1}-\mu(\phi_0))]\big),\\ J_2(s_1,s_2) &:= \mu(\phi_0)\mathrm{e}^{-(\lambda_m-\lambda_0)(s_2-s_1)}\mu_0\big(\{P^0_{s_1}(h\phi_0^{-1})-\mu(h\phi_0)\}\{\phi_m\phi_0^{-1}\}^2\big),\\ J_3(s_1,s_2) &:= \mu(h\phi_0)\mathrm{e}^{-(\lambda_m-\lambda_0)(s_2-s_1)}\mu_0\big(\{\phi_m\phi_0^{-1}\}^2\{P^0_{T-s_2}\phi_0^{-1}]-\mu(\phi_0)\}\big). \end{aligned}$$

By (3.4), (3.5) and (3.6), we find a constant $\kappa>0$ such that

$$\sup_{T\ge t}\Big|t\mathbb{E}^\nu\big[\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^2)\big|T<\tau\big]-2\sum_{m=1}^\infty\frac{\mathrm{e}^{-2(\lambda_m-\lambda_0)r}}{(\lambda_m-\lambda_0)^2}\Big| \\ \le\frac{\kappa}{t}\sum_{m=1}^\infty\bigg(\frac{\mathrm{e}^{-2(\lambda_m-\lambda_0)r}}{(\lambda_m-\lambda_0)^2}+\frac{\mathrm{e}^{-2(\lambda_m-\lambda_0)r}}{\lambda_m-\lambda_0}\int_0^t\mathrm{d}s_1\int_{s_1}^t|J_1+J_2+J_3|(s_2,s_2)\mathrm{d}s_2\bigg),\quad t\ge1. \tag{3.7}$$

Since $\|h\phi_0^{-1}\|_{L^p(\mu_0)}<\infty$, $\|\phi_0^{-1}\|_{L^\theta(\mu_0)}<\infty$ for $\theta<3$ due to (2.2), $\|\phi_m\phi_0^{-1}\|_{L^2(\mu_0)}=1$, by (2.13), for any $\theta\in(\frac52,3)$, we find constants $c_1,c_2>0$ such that

$$|J_1(s_1,s_2)|\le c_1\|P^0_{s_1}-\mu_0\|_{L^p(\mu_0)\to L^\infty(\mu_0)}\|P^0_{T-s_2}-\mu_0\|_{L^\theta(\mu_0)\to L^\infty(\mu_0)} \\ \le c_2\mathrm{e}^{-(\lambda_1-\lambda_0)(s_1+T-s_2)}(1\wedge s_1)^{-\frac{d+2}{2p}}\{1\wedge(T-s_2)\}^{-\frac{d+2}{2\theta}}, \tag{3.8}$$

and

$$|(J_2+J_3)(s_1,s_2)| \\ \le c_1\mathrm{e}^{-(\lambda_m-\lambda_0)(s_2-s_1)}\big(\|P^0_{s_1}-\mu_0\|_{p\to\infty}+\|P^0_{T-s_2}-\mu_0\|_{L^\theta(\mu_0)\to L^\infty(\mu_0)}\big) \\ \le c_2\mathrm{e}^{-(\lambda_m-\lambda_0)(s_2-s_1)}\big(\{1\wedge s_1\}^{-\frac{d+2}{2p}}\mathrm{e}^{-(\lambda_1-\lambda_0)s_1}+\{1\wedge(T-s_2)\}^{-\frac{d+2}{2\theta}}\mathrm{e}^{-(\lambda_1-\lambda_0)(t-s_2)}\big). \tag{3.9}$$

Since $q>\frac52$ and $p>\frac{d+2}{2}$ imply $\frac{d+2}{2q}\vee\frac{d+2}{2p}<1$ for $d\le3$, by (3.8) and (3.9), we find a constant $c>0$ such that

$$\int_0^t\mathrm{d}s_1\int_{s_1}^t|J_1+J_2+J_3|(s_1,s_2)\mathrm{d}s_2\le\frac{c}{t},\quad T\ge t\ge1, m\ge1.$$

Combining this with (3.7) and (2.1), we find constants $c_3,c_4,c_5,c_6>0$ such that

$$\sup_{T\ge t}\Big|t\mathbb{E}^\nu[\big[\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^2)\big|T<\tau\big]-\sum_{m=0}^\infty\frac{\mathrm{e}^{-(\lambda_m-\lambda_0)r}}{(\lambda_m-\lambda_0)^2}\Big| \\ \le\frac{c_3}{t}\sum_{m=1}^\infty\frac{\mathrm{e}^{-2(\lambda_m-\lambda_0)r}}{\lambda_m-\lambda_0}\le\frac{c_4}{t}\int_1^\infty s^{-\frac2d}\mathrm{e}^{-c_5s^{\frac2d}r}\mathrm{d}s\le c_6t^{-1}\big(r^{-\frac{(d-2)^+}{2}}+1_{\{d=2\}}\log r^{-1}\big),\quad t\ge1.$$

□

Lemma 3.3. *There exists a constant $c > 0$ such that for any $t > 0$ and nonnegative random variable $\xi \in \sigma(X_s : s \le t)$,*

$$\sup_{T\ge t} \mathbb{E}^\nu[\xi|T < \tau] \le c\mathbb{E}^\nu[\xi|t < \tau], \quad t \ge 1, \nu \in \mathscr{P}_0.$$

Proof. By the Markov property, (2.6) for $p = q = \infty$ and (2.5), we find constants $c_1, c_2 > 0$ such that

$$\begin{aligned}
&\mathbb{E}^\nu[\xi 1_{\{T<\tau\}}] = \mathbb{E}^\nu[\xi 1_{\{t<\tau\}} P^D_{T-t}1(X_t)] \le c_1 \mathrm{e}^{-\lambda_0(T-t)}\mathbb{E}^\nu[\xi 1_{\{t<\tau\}}],\\
&\mathbb{P}^\nu(T < \tau) \ge c_2 \mathbb{P}^\nu(t < \tau)\mathrm{e}^{-(T-t)\lambda_0}, \quad T \ge t \ge 1.
\end{aligned}$$

Then

$$\mathbb{E}^\nu[\xi|T < \tau] = \frac{\mathbb{E}^\nu[\xi 1_{\{T<\tau\}}]}{\mathbb{P}^\nu(T < \tau} \le \frac{c_1\mathbb{E}^\nu[\xi 1_{\{t<\tau\}}]}{c_2\mathbb{P}^\nu(t < \tau)} = \frac{c_1}{c_2}\mathbb{E}^\nu[\xi|t < \tau].$$

□

Lemma 3.4. *Let $d \le 3$ and denote $\nu_0 = \frac{\phi_0}{\mu(\phi_0)}\mu$. For any $\varepsilon \in (\frac{d}{4} \vee \frac{d^2}{2d+4}, 1) \ne \emptyset$, there exists a constant $c > 0$ such that*

$$\sup_{T\ge t} \mathbb{E}^{\nu_0}\big[|\rho_{t,r}(y) - 1|^2\big|T < \tau\big] \le c\phi_0^{-2}(y)t^{-1}r^{-\varepsilon}, \quad t \ge 1, r \in (0, 1], y \in M^\circ.$$

Proof. By Lemma 3.3, it suffices to prove for $T = t$ replacing $T \ge t$. For fixed $y \in M^\circ$, let $f = p^0_r(\cdot, y) - 1$. We have

$$\rho_{t,r}(y) - 1 = \frac{1}{t}\int_0^t f(X_s)\mathrm{d}s.$$

Then

$$\mathbb{E}^{\nu_0}\big[|\rho_{t,r}(y) - 1|^2 1_{\{t<\tau\}}\big] = \frac{2}{t^2}\int_0^t \mathrm{d}s_1 \int_{s_1}^t \mathbb{E}^{\nu_0}\big[1_{\{t<\tau\}} f(X_{s_1})f(X_{s_2})\big]\mathrm{d}s_2. \tag{3.10}$$

By (3.3), $\mu_0(f) = 0$, and the symmetry of P^0_t in $L^2(\mu_0)$, we obtain

$$\begin{aligned}
I &:= \mathrm{e}^{\lambda_0 t}\mathbb{E}^{\nu_0}\big[1_{\{t<\tau\}} f(X_{s_1})f(X_{s_2})\big] = \mu(\phi_0)^{-1}\mu_0\big(P^0_{s_1}\{fP^0_{s_2-s_1}(fP^0_{t-s_2}\phi_0^{-1})\}\big)\\
&= \mu(\phi_0)^{-1}\mu_0\big(fP^0_{s_2-s_1}(fP^0_{t-s_2}\phi_0^{-1})\big) = \mu(\phi_0)^{-1}\mu_0\big(\{fP^0_{t-s_2}\phi_0^{-1}\}P^0_{s_2-s_1}f\big)\\
&= \mu(\phi_0)^{-1}\mu_0\big(\{fP^0_{t-s_2}\phi_0^{-1}\}\{P^0_{s_2-s_1} - \mu_0\}f\big).
\end{aligned} \tag{3.11}$$

Taking $q \in (\frac{5}{2}, 3)$ so that $\varepsilon_1 := \frac{d+2}{2q} < 1$ for $d \le 3$ and $\|\phi_0^{-1}\|_{L^q(\mu_0)} < \infty$ due to (2.2), for any $p \in (1, 2]$ we deduce from this and (2.13) that

$$\begin{aligned}
\mu(\phi_0)I &\le \|f\|_{L^p(\mu_0)}\|P^0_{t-s_2}\phi_0^{-1}\|_{L^\infty(\mu_0)}\|(P^0_{s_2-s_1} - \mu_0)f\|_{L^{\frac{p}{p-1}}(\mu_0)}\\
&\le \|f\|_{L^p(\mu_0)}\|P^0_{t-s_2}\|_{L^q(\mu_0)\to L^\infty(\mu_0)}\|\phi_0^{-1}\|_{L^q(\mu_0)}\|P^0_{s_2-s_1} - \mu_0\|_{L^2(\mu_0)\to L^{\frac{p}{p-1}}(\mu_0)}\|f\|_{L^2(\mu_0)}\\
&\le c_1\|f\|_{L^p(\mu_0)}\|f\|_{L^2(\mu_0)}\{1 \wedge (t - s_2)\}^{-\varepsilon_1}\{1 \wedge (s_2 - s_1)\}^{-\frac{(d+2)(2-p)}{2p}}\mathrm{e}^{-(\lambda_1-\lambda_0)(s_2-s_1)}
\end{aligned} \tag{3.12}$$

holds for some constants $c_1>0$. Since $f=p^0_r(\cdot,y)-1$ and $\inf\phi_0^{-1}>0$, by (2.6) and (2.7), we find constants $\beta_1,\beta_2>0$ such that

$$\begin{aligned}&\|f\|_{L^p(\mu_0)}\le 1+\|p^0_r(\cdot,y)\|_{L^p(\mu_0)}\le 1+\mathrm{e}^{r\lambda_0}\phi_0^{-1}(y)\|\phi_0^{-1}p^D_r(\cdot,y)\|_{L^p(\mu_0)}\\&\le 1+\beta_1\phi_0^{-1}(y)\|\phi_0\|_\infty^{\frac{2-p}{p}}\|p^D_r(\cdot,y)\|_{L^p(\mu)}\le \beta_2\phi_0^{-1}(y)r^{-\frac{d(p-1)}{2p}},\quad r\in(0,1],p\in[1,2].\end{aligned}$$

Combining this with (3.12) we find a constant $c_2>0$ such that

$$I\le c_2\phi_0^{-2}(y)r^{-\frac{d(p-1)}{2p}-\frac{d}{4}}\{1\wedge(t-s_2)\}^{-\varepsilon_1}\{1\wedge(s_2-s_1)\}^{-\frac{(d+2)(2-p)}{2p}}\mathrm{e}^{-(\lambda_1-\lambda_0)(s_2-s_1)},\quad p\in(1,2].$$

Taking $p>p_0:=1\vee\frac{2(d+2)}{d+6}$ such that

$$\varepsilon_2:=\frac{(d+2)(2-p)}{4p}\le\frac{5(2-p)}{4p}<1,$$

we arrive at

$$I\le c_2\phi^{-2}(y)r^{-\frac{d(p-1)}{2p}-\frac{d}{4}}\{1\wedge(t-s_2)\}^{-\varepsilon_1}\{1\wedge(s_2-s_1)\}^{-\varepsilon_2}\mathrm{e}^{-(\lambda_1-\lambda_0)(s_2-s_1)}$$

for some constants $\varepsilon_1,\varepsilon_2\in(0,1)$. Combining this with (3.10), we obtain

$$\mathbb{E}^{\nu_0}\big[|\rho_{t,r}(y)-1|^2\big|t<\tau\big]\le c\phi_0^{-2}(y)t^{-1}r^{-\frac{d(p-1)}{2p}-\frac{d}{4}},\quad t\ge1.$$

Noting that

$$\lim_{p\downarrow p_0}\Big\{\frac{d(p-1)}{2p}+\frac{d}{4}\Big\}=\frac{d}{4}\vee\frac{d^2}{2d+4}<1\text{ for }d\le3,$$

for any $\varepsilon\in(\frac{d}{4}\vee\frac{d^2}{2d+4},1)$, there exists $p>p_0$ such that $\frac{d}{4}\vee\frac{d^2}{2d+4}\le\varepsilon$. Therefore, the proof is finished. □

Lemma 3.5. *Let $d\le3$ and denote $\psi_m(t)=\frac{1}{t}\int_0^t(\phi_m\phi_0^{-1})(X_s)\mathrm{d}s$. Then there exists a constant $c>0$ such that for any $p\in[1,2]$,*

$$\sup_{T\ge t}\mathbb{E}^{\nu_0}\big[|\psi_m(t)|^{2p}\big|t<\tau\big]\le cm^{\frac{p(d+4)-d-8}{2d}}t^{-p},\quad t\ge1,m\ge1,r\in(0,1).$$

Proof. By Lemma 3.3, it suffices to prove for $T=t$ replacing $T\ge t$. By Hölder's inequality, we have

$$\mathbb{E}^{\nu_0}\big[|\psi_m(t)|^{2p}|T<\tau\big]\le\big\{\mathbb{E}^{\nu_0}\big[|\psi_m(t)|^2|T<\tau\big]\big\}^{2-p}\big\{\mathbb{E}^{\nu_0}\big[|\psi_m(t)|^4|T<\tau\big]\big\}^{p-1}.$$

Combining this with (2.5), it suffices to find a constant $c>0$ such that

$$\mathbb{E}^{\nu_0}\big[|\psi_m(t)|^2 1_{\{t<\tau\}}\big]\le\frac{c\mathrm{e}^{-\lambda_0 t}}{tm^{\frac{2}{d}}},\quad t\ge1,r\in(0,1),\tag{3.13}$$

$$\mathbb{E}^{\nu_0}\big[|\psi_m(t)|^4 1_{\{t<\tau\}}\big]\le c\sqrt{m}\,\mathrm{e}^{-\lambda_0 t}t^{-2},\quad t\ge1,r\in(0,1).\tag{3.14}$$

(a) Proof of (3.13). Let $\hat\phi_m=\phi_m\phi_0^{-1}$. We have

$$\mathbb{E}^{\nu_0}\big[|\psi_m(t)|^2 1_{\{t<\tau\}}\big]=\frac{2}{t^2}\int_0^t \mathrm{d}s_1\int_{s_1}^t \mathbb{E}^{\nu_0}\big[1_{\{t<\tau\}}\hat\phi_m(X_{s_1})\hat\phi_m(X_{s_2})\big]\mathrm{d}s_2. \tag{3.15}$$

By (2.8), (3.3), $\mu_0(|\hat\phi_m|^2)=1$, and the symmetry of P_t^0 in $L^2(\mu_0)$, we find a constant $c_1>0$ such that

$$\begin{aligned}
&\mathrm{e}^{\lambda_0 t}\mathbb{E}^{\nu_0}\big[1_{\{T<\tau\}}\hat\phi_m(X_{s_1})\hat\phi_m(X_{s_2})\big]=\nu_0\big(\phi_0P_{s_1}^0\{\hat\phi_mP_{s_2-s_1}^0(\hat\phi_mP_{t-s_2}^0\phi_0^{-1})\}\big)\\
&=\frac{1}{\mu(\phi_0)}\mu_0\big(\hat\phi_mP_{s_2-s_1}^0(\hat\phi_mP_{t-s_2}^0\phi_0^{-1})\big)=\frac{\mathrm{e}^{-(\lambda_m-\lambda_0)(s_2-s_1)}}{\mu(\phi_0)}\mu_0\big(|\hat\phi_m|^2P_{t-s_2}^0\phi_0^{-1})\big)\\
&\le c_1\mathrm{e}^{-(\lambda_m-\lambda_0)(s_2-s_1)}\|P_{t-s_2}\|_{L^p(\mu_0)\to\infty(\mu_0)}\|\phi_0^{-1}\|_{L^p(\mu_0)},\quad p>1.
\end{aligned}$$

Since $d\le 3$, we may take $p\in(1,3)$ such that $\varepsilon:=\frac{d+2}{2q}<1$ and $\|\phi_0^{-1}\|_{L^p(\mu_0)}<\infty$ due to (2.2), so that this and (2.13) imply

$$\mathrm{e}^{\lambda_0 t}\mathbb{E}^{\nu_0}\big[1_{\{t<\tau\}}\hat\phi_m(X_{s_1})\hat\phi_m(X_{s_2})\big]\le c_2\mathrm{e}^{-(\lambda_m-\lambda_0)(s_2-s_1)}\{1\wedge(t-s_2)\}^{-\varepsilon}$$

for some constant $c_3>0$. Therefore, (3.13) follows from (3.15) and (2.1).

(b) Proof of (3.14). For any $s>0$ we have

$$\begin{aligned}
&s^4\mathbb{E}^{\nu_0}\big[|\psi_m(s)|^4 1_{\{s<\tau\}}\big]\\
&=24\int_0^s\mathrm{d}s_2\int_{s_1}^s\mathrm{d}s_2\int_{s_2}^s\mathrm{d}s_3\int_{s_3}^s\mathbb{E}^{\nu_0}\big[1_{\{s<\tau\}}\hat\phi_m(X_{s_1})\hat\phi_m(X_{s_2})\hat\phi_m(X_{s_3})\hat\phi_m(X_{s_4})\big]\mathrm{d}s_4\\
&=24\int_0^s\mathrm{d}s_2\int_{s_1}^s\mathrm{d}s_2\int_{s_2}^s\mathrm{d}s_3\int_{s_3}^s\mathbb{E}^{\nu_0}\big[1_{\{s_3<\tau\}}\hat\phi_m(X_{s_1})\hat\phi_m(X_{s_2})g_s(s_3,s_4)\big]\mathrm{d}s_4,
\end{aligned}\tag{3.16}$$

where due to (3.3) and the Markov property,

$$\begin{aligned}
g_s(s_3,s_4)&:=\mathbb{E}^{\nu_0}\big[1_{\{s<\tau\}}\hat\phi_m(X_{s_3})\hat\phi_m(X_{s_4})\big|X_r:r\le s_3\big]\\
&=\hat\phi_m(X_{s_3})\mathbb{E}^{X_{s_3}}\big[1_{\{s-s_3<\tau\}}\hat\phi_m(X_{s_4-s_3})\big]\\
&=\mathrm{e}^{-\lambda_0(s-s_3)}\big\{\hat\phi_m\phi_0P_{s_4-s_3}^0(\hat\phi_mP_{s-s_4}^0\phi_0^{-1})\big\}(X_{s_3}),\quad 0<s_3<s_4\le s.
\end{aligned}\tag{3.17}$$

So, by Fubini's theorem and Schwarz's inequality, we obtain

$$\begin{aligned}
I(s)&:=s^4\mathrm{e}^{\lambda_0 s}\mathbb{E}^{\nu_0}\big[|\psi_m(s)|^4 1_{\{s<\tau\}}\big]\\
&=12\mathrm{e}^{\lambda_0 s}\int_0^s\mathrm{d}r_1\int_{r_1}^s\mathbb{E}^{\nu_0}\bigg[1_{\{r_1<\tau\}}g_s(r_1,r_2)\bigg|\int_0^{r_1}\hat\phi_m(X_r)\mathrm{d}r\bigg|^2\bigg]\mathrm{d}r_2\\
&\le 12\sup_{r\in[0,s]}\sqrt{I(r)}\int_0^s\mathrm{d}r_1\int_{r_1}^s\Big\{\mathrm{e}^{2\lambda_0 s-\lambda_0 r_1}\mathbb{E}^{\nu_0}\big[1_{\{r_1<\tau\}}g_s(r_1,r_2)^2\big]\Big\}^{\frac12}\mathrm{d}r_2.
\end{aligned}$$

Consequently,

$$I(t)\le\sup_{s\in[0,t]}I(s)\le\bigg(12\sup_{s\in[0,t]}\int_0^s\mathrm{d}r_1\int_{r_1}^s\Big\{\mathrm{e}^{\lambda_0(2s-r_1)}\mathbb{E}^{\nu_0}\big[1_{\{r_1<\tau\}}g_s(r_1,r_2)^2\big]\Big\}^{\frac12}\mathrm{d}r_2\bigg)^2. \tag{3.18}$$

On the other hand, by the definition of ν_0, (3.3), (3.17) and that μ_0 is P_t^0-invariant, we obtain

$$
\begin{aligned}
&\mathbb{E}^{\nu_0}\big[1_{\{r_1<\tau\}}|g_s(r_1,r_2)|^2\big]\\
&\le \frac{\mathrm{e}^{-2\lambda_0(s-r_1)-\lambda_0 r_1}}{\mu(\phi_0)}\mu_0\big(P_{r_1}^0\{\phi_0^{-1}|\hat\phi_m\phi_0P_{r_2-r_1}^0(\hat\phi_mP_{s-r_2}^0\phi_0^{-1})|^2\}\big)\\
&= \frac{\mathrm{e}^{-\lambda_0(2s-r_1)}}{\mu(\phi_0)}\mu_0\big(\phi_0|\hat\phi_mP_{r_2-r_1}^0(\hat\phi_mP_{s-r_2}^0\phi_0^{-1})|^2\big)\\
&\le \frac{2\mathrm{e}^{-\lambda_0(2s-r_1)}}{\mu(\phi_0)}\mu_0\big(\phi_0\{|\hat\phi_m(P_{r_2-r_1}^0\hat\phi_m)\mu(\phi_0)|^2+|\hat\phi_mP_{r_2-r_1}^0(\hat\phi_m[P_{s-r_2}^0-\mu_0]\phi_0^{-1})|^2\}\big).
\end{aligned}
\tag{3.19}
$$

Then, by (3.17), (2.8), (3.3), $\mu_0(|\hat\phi_m|^2)=1$, and noting that μ_0 is P_t^0-invariant, we find a constant $c_1>0$ such that

$$
\begin{aligned}
&\mathbb{E}^{\nu_0}\big[1_{\{r_1<\tau\}}|g_s(r_1,r_2)|^2\big]\le 2\mathrm{e}^{-\lambda_0(2s-r_1)-2(\lambda_m-\lambda_0)(r_2-r_1)}\|\phi_0\|_\infty^2\\
&\quad+2\frac{\mathrm{e}^{-\lambda_0(2s-r_1)}\|\phi_m\|_\infty}{\mu(\phi_0)}\mu_0\big(|\hat\phi_m|\cdot\big|P_{r_2-r_1}^0(\hat\phi_m(P_{s-r_2}^0-\mu_0)\phi_0^{-1})\big|^2\big)\\
&\le c_1\mathrm{e}^{-\lambda_0(2s-r_1)}\Big\{\mathrm{e}^{-2(\lambda_m-\lambda_0)(r_2-r_1)}+\|\phi_m\|_\infty\|P_{r_2-r_1}^0(\hat\phi_m[P_{s-r_2}^0-\mu_0]\phi_0^{-1})\|_{L^4(\mu_0)}^2\Big\}.
\end{aligned}
$$

By (2.1), (2.13), $\|\hat\phi_m\|_{L^2(\mu_0)}=1$, $\|\phi_0^{-1}\|_{L^q(\mu_0)}<\infty$ and $\varepsilon:=\frac{d+2}{8}\vee\frac{d+2}{2q}<1$ for $q\in(\frac52,3)$ due to (2.2) and $d\le 3$, we find constants $c_2>0$ such that

$$
\begin{aligned}
&\|\phi_m\|_\infty\|P_{r_2-r_1}^0(\hat\phi_m[P_{s-r_2}^0-\mu_0]\phi_0^{-1})\|_{L^4(\mu_0)}^2\\
&\le \|\phi_m\|_\infty\|P_{r_2-r_1}^0\|_{L^2(\mu_0)\to L^4(\mu_0)}^2\|\hat\phi_m\|_{L^2(\mu_0)}^2\|(P_{s-r_2}^0-\mu_0)\phi_0^{-1}\|_{L^\infty(\mu_0)}^2\\
&\le \|\phi_m\|_\infty\|P_{r_2-r_1}^0\|_{L^2(\mu_0)\to L^4(\mu_0)}^2\|P_{s-r_2}^0-\mu_0\|_{L^q(\mu_0)\to L^\infty(\mu_0)}^2\|\phi_0^{-1}\|_{L^q(\mu_0)}^2\\
&\le c_2\sqrt{m}\,\mathrm{e}^{-2(\lambda_1-\lambda_0)(s-r_2)}\{1\wedge(r_2-r_1)\}^{-2\varepsilon}\{1\wedge(s-r_2)\}^{-2\varepsilon}.
\end{aligned}
$$

Therefore, there exist constants $c_3>0$ and $\varepsilon\in(0,1)$ such that

$$
\begin{aligned}
&\mathbb{E}^{\nu_0}\big[1_{\{r_1<\tau\}}|g_s(r_1,r_2)|^2\big]\le c_3\mathrm{e}^{-\lambda_0(2s-r_1)-(\lambda_m-\lambda_0)(r_2-r_1)}\\
&\qquad+c_3\sqrt{m}\,\mathrm{e}^{-\lambda_0(2s-r_1)-2(\lambda_1-\lambda_0)(s-r_2)}\{1\wedge(r_2-r_1)\}^{-2\varepsilon}\{1\wedge(t-r_2)\}^{-2\varepsilon}.
\end{aligned}
$$

Combining this with (3.18) and the definition of $I(t)$, we prove (3.14) for some constant $c>0$, and hence finish the proof. □

Lemma 3.6. *Let $d\le 3$. Then for any $p\in(1,\frac{3d+16}{5d+8}\wedge\frac{d+2}{d+1})\ne\emptyset$, there exists a constant $c>0$ such that*

$$\sup_{r>0,T\ge t}\mathbb{E}^{\nu_0}\big[\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^{2p})|T<\tau\big]\le ct^{-p},\quad t\ge 1.$$

Proof. By Lemma 3.3, it suffices to prove for $T=t$ replacing $T\ge t$. Let $p\in(1,\frac{3d+16}{5d+8}\wedge\frac{d+2}{d+1})$, where $p>1$ is equivalent to

$$\frac{p}{2p-1}<1, \tag{3.20}$$

while $p < \frac{3d+16}{5d+8} \wedge \frac{d+2}{d+1}$ implies

$$\frac{(d+2)(2p-2)}{4} + \frac{d(p-1)}{2} + \Big(\frac{p(d+4)+d}{4} - 2\Big)^+ < 1,$$

and hence there exists $\varepsilon \in (0,1)$ such that

$$\frac{(d+2)(2p-2+\varepsilon)}{4} + \frac{d(p-1)}{2} + \Big(\frac{p(d+4)+d}{4} - 2\Big)^+ < 1. \tag{3.21}$$

By (2.13), (2.21), $L_0^{-1} = -\int_0^\infty P_s^0 \mathrm{d}s$, and applying Hölder's inequality, we find a constant $c_1, c_2 > 0$ such that

$$\begin{aligned}
&\int_M \big|\nabla L_0^{-1}(\rho_{t,r}-1)\big|^{2p} \mathrm{d}\mu_0 \le \int_M \bigg(\int_0^\infty \big|\nabla P_s^0(\rho_{t,r}-1)\big| \mathrm{d}s\bigg)^{2p} \mathrm{d}\mu_0 \\
&\le c_1 \int_M \bigg(\int_0^\infty \frac{1}{\sqrt{s}} \big\{P_{\frac{s}{4}}^0 \big|P_{\frac{3s}{4}}^0(\rho_{t,r}-1)\big|^p\big\}^{\frac{1}{p}} \mathrm{d}s\bigg)^{2p} \phi_0^{-\varepsilon} \mathrm{d}\mu_0 \\
&\le c_1 \bigg(\int_0^\infty s^{-\frac{p}{2p-1}} \mathrm{e}^{-\frac{2p\theta s}{2p-1}} \mathrm{d}s\bigg)^{\frac{2p-1}{2p}} \int_0^\infty \mathrm{e}^{\theta s} \mu_0\big(\phi_0^{-\varepsilon}\big\{P_{\frac{s}{4}}^0 |P_{\frac{3s}{4}}^0(\rho_{t,r}-1)|^p\big\}^2\big) \mathrm{d}s, \quad \theta > 0.
\end{aligned} \tag{3.22}$$

Noting that $\frac{p}{2p-1} < 1$ due to (3.20), we obtain

$$\int_0^\infty s^{-\frac{p}{2p-1}} \mathrm{e}^{-\frac{2p\theta s}{2p-1}} \mathrm{d} < \infty, \quad \theta > 0. \tag{3.23}$$

Moreover, since $\|\phi_0^{-\varepsilon}\|_{L^{2\varepsilon^{-1}}(\mu_0)} = 1$, $\mu_0(\rho_{t,r}-1) = 0$, and P_t^0 is contractive in $L^p(\mu_0)$ for $p \ge 1$, by (2.13) and Hölder's inequality, we find a constant $c_2 > 0$ such that

$$\begin{aligned}
&\mu_0\big(\phi_0^{-\varepsilon}\big\{P_{\frac{s}{4}}^0 |P_{\frac{3s}{4}}^0(\rho_{t,r}-1)|^p\big\}^2\big) \le \big\|P_{\frac{s}{4}}^0 |P_{\frac{3s}{4}}^0(\rho_{t,r}-1)|^p\big\|_{L^{\frac{4}{2-\varepsilon}}(\mu_0)}^2 \|\phi_0^{-\varepsilon}\|_{L^{2\varepsilon^{-1}}(\mu_0)} \\
&\le \|P_{\frac{s}{4}}^0\|_{L^{\frac{4}{2-\varepsilon}}(\mu_0)}^2 \big\|(P_{\frac{s}{2}}^0 - \mu_0)(P_{\frac{s}{4}}^0 \rho_{t,r} - 1)|\big\|_{L^{\frac{4p}{2-\varepsilon}}(\mu_0)}^{2p} \\
&\le \|P_{\frac{s}{2}}^0 - \mu_0\|_{L^2(\mu_0) \to L^{\frac{4p}{2-\varepsilon}}(\mu_0)}^{2p} \|P_{\frac{s}{4}}^0 \rho_{t,r} - 1\|_{L^2(\mu_0)}^{2p} \\
&\le c_2 (1 \wedge s)^{-\frac{(d+2)(2p-2+\varepsilon)}{4}} \mathrm{e}^{-(\lambda_1 - \lambda_0)ps} \|P_{\frac{s}{4}}^0 \rho_{t,r} - 1\|_{L^2(\mu_0)}^{2p}.
\end{aligned}$$

Combining this with (3.23), we find a function $c : (0,\infty) \to (0,\infty)$ such that

$$\begin{aligned}
&\mathbb{E}^{\nu_0}\big[1_{\{t<\tau\}} \mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^{2p})\big] \\
&\le c(\theta) \int_0^\infty \mathrm{e}^{\theta s} (1 \wedge s)^{-\frac{(d+2)(2p-2+\varepsilon)}{4}} \mathrm{e}^{-(\lambda_1 - \lambda_0)ps} \mathbb{E}^{\nu_0}\big[1_{\{t<\tau\}} \|P_{\frac{s}{4}}^0 \rho_{t,r} - 1\|_{L^2(\mu_0)}^{2p}\big] \mathrm{d}s, \quad \theta > 0.
\end{aligned} \tag{3.24}$$

By (2.8), (3.2) and Hölder's inequality, we obtain

$$\|P_{\frac{s}{4}}^0 \rho_{t,r} - 1\|_{L^2(\mu_0)}^{2p} = \Big(\sum_{m=1}^\infty \mathrm{e}^{-(\lambda_m - \lambda_0)(2r + s/2)} |\psi_m(t)|^2\Big)^p$$

$$\leq \Big(\sum_{m=1}^\infty e^{-(\lambda_m-\lambda_0)(2r+s/2)}\Big)^{p-1}\sum_{m=1}^\infty e^{-(\lambda_m-\lambda_0)(2r+s/2)}|\psi_m(t)|^{2p}.$$

Noting that (2.1) implies

$$\sum_{m=1}^\infty e^{-(\lambda_m-\lambda_0)(2r+s/2)}\leq a_1\int_1^\infty e^{-\alpha_2(r+s/2)t^{\frac{2}{d}}}dt\leq \alpha_3(1\wedge s)^{-\frac{d}{2}}$$

for some constants $\alpha_1,\alpha_2,\alpha_3>0$, we derive

$$\mathbb{E}^{\nu_0}\big[\|P^0_{\frac{s}{4}}\rho_{t,r}-1\|^{2p}_{L^2(\mu_0)}\big|t<\tau\big]\leq c_3(1\wedge s)^{-\frac{d(p-1)}{2}}\sum_{m=1}^\infty e^{-(\lambda_m-\lambda_0)(2r+s/2)}\mathbb{E}^{\nu_0}\big[|\psi_m(t)|^{2p}\big|t<\tau\big]$$

for some constant $c_3>0$. Combining this with Lemma 3.5, (2.1), we find constants $c_4,c_5,c_6,c_7>0$ such that

$$\begin{aligned}&\mathbb{E}^{\nu_0}\big[\|P^0_{\frac{s}{4}}\rho_{t,r}-1\|^{2p}_{L^2(\mu_0)}\big|t<\tau\big]\leq c_4t^{-p}(1\wedge s)^{-\frac{d(p-1)}{2}}\int_1^\infty e^{-c_5su^{\frac{2}{d}}}u^{\frac{p(d+4)-d-8}{2d}}du\\&\leq c_6t^{-p}(1\wedge s)^{-\frac{d(p-1)}{2}}s^{2-\frac{p(d+4)+d}{4}}\int_s^\infty t^{\frac{p(d+4)+d}{4}-3}e^{-t}dt\\&\leq c_7t^{-p}(1\wedge s)^{-\frac{d(p-1)}{2}-(\frac{p(d+4)+d}{4}-2)^+}\log(2+s^{-1}),\end{aligned}$$

where the term $\log(2+s^{-1})$ comes when $\frac{p(d+4)+d}{4}-3=-1$. This together with (3.21) and (3.24) for $\theta\in(0,\lambda_1-\lambda_0)$ implies the desired estimate. □

Lemma 3.7. *Let $d\leq 3$. If $r_t=t^{-\alpha}$ for some $\alpha\in(1,\frac{4}{d}\wedge\frac{2d+4}{d^2})\neq\emptyset$, then $\rho_{t,r_t,r_t}:=(1-r_t)\rho_{t,r_t}+r_t$ satisfies*

$$\lim_{t\to\infty}\sup_{T\geq t}\mathbb{E}^{\nu_0}\big[\mu_0(|\mathscr{M}(\rho_{t,r_t,r_t},1)^{-1}-1|^q)\big|T<\tau\big]=0,\quad q\geq 1.$$

Proof. By Lemma 3.3, it suffices to prove for $T=t$ replacing $T\geq t$. By the same reason leading to (3.16) in [21], for any $\eta\in(0,1),y\in M$, we have

$$\mathbb{E}^{\nu_0}\big[|\mathscr{M}(\rho_{t,r_t,r_t}(y),1)^{-1}-1|^q\big|t<\tau\big]\leq\Big|\frac{1}{\sqrt{1-\eta}}-\frac{2}{2+\eta}\Big|^q+\mathbb{P}^{\nu_0}\big(|\rho_{t,r_t}(y)-1|>\eta\big).$$

Combining this with Lemma 3.4 we find constants $c>0$ and $\varepsilon\in(0,\alpha^{-1})$ such that

$$\mathbb{E}^{\nu_0}\big[|\mathscr{M}(\rho_{t,r_t,r_t}(y),1)^{-1}-1|^q\big|t<\tau\big]\leq\Big|\frac{1}{\sqrt{1-\eta}}-\frac{2}{2+\eta}\Big|^q+c\eta^{-1}\phi_0(y)^{-2}t^{-1+\alpha\varepsilon}.$$

Since $\mu_0(\phi_0^{-2})=1$, we obtain

$$\mathbb{E}^{\nu_0}\big[\mu_0(|\mathscr{M}(\rho_{t,r_t,r_t},1)^{-1}-1|^q)\big|t<\tau\big]\leq\Big|\frac{1}{\sqrt{1-\eta}}-\frac{2}{2+\eta}\Big|^q+c\eta^{-1}t^{-1+\alpha\varepsilon},\quad\eta\in(0,1),t\geq 1.$$

Noting that $\alpha\varepsilon<1$, by letting first $t\to\infty$ then $\eta\to 0$, we finish the proof. □

Lemma 3.8. *Let $\mu_{t,r,r} = (1+\rho_{t,r,r})\mu_0$, where $\rho_{t,r,r} := (1-r)\rho_{t,r} + r, r \in (0,1]$. Assume that $\nu = h\mu$ with $h\phi_0^{-1} \in L^p(\mu_0)$ for some $p > 1$. Then there exists a constant $c > 0$ such that*

$$\sup_{T\geq t} \mathbb{E}^\nu\big[\mathbb{W}_2(\mu_{t,r,r}, \mu_t)^2\big|T < \tau\big] \leq cr, \quad t > 0, r \in (0,1].$$

Proof. By Lemma 3.3, it suffices to prove for $T = t$ replacing $T \geq t$. Firstly, it is easy to see that

$$\mathbb{W}_2(\mu_{t,r,r}, \mu_{t,r})^2 \leq D^2\|\mu_{t,r,r} - \mu_{t,r}\|_{var} = D^2\mu_0(|\rho_{t,r,r} - \rho_{t,r}|) \leq 2D^2 r, \quad r \in (0,1]. \tag{3.25}$$

Next, by the definition of $\mu_{t,r}$, we have

$$\pi(\mathrm{d}x, \mathrm{d}y) := \mu_t(\mathrm{d}x)P_r^0(x, \mathrm{d}y) \in \mathscr{C}(\mu_t, \mu_{t,r}),$$

where $P_r^0(x,\cdot)$ is the distribution of X_r^0 starting at x. So,

$$\mathbb{W}_2(\mu_t, \mu_{t,r})^2 \leq \int_M \mathbb{E}^x[\rho(x, X_r^0)^2]\mu_t(\mathrm{d}x). \tag{3.26}$$

Moreover, by Itô's formula and $L_0 = L + 2\nabla \log \phi_0$, we find a constant $c_1 > 0$ such that

$$\mathrm{d}\rho(x, X_r^0)^2 = L_0\rho(x,\cdot)^2(X_r^0)\mathrm{d}r + \mathrm{d}M_r \leq \big\{c_1 + c_1\phi_0^{-1}(X_r^0)\big\}\mathrm{d}r + \mathrm{d}M_r$$

holds for some martingale M_r. Combining this with (2.18), and noting that $\log(1+\phi_0^{-1}) \geq \log(1+\|\phi_0\|_\infty^{-1}) > 0$, we find a constant $c_2 > 0$ such that

$$\begin{aligned}\mathbb{W}_2(\mu_t, \mu_{t,r})^2 &\leq c_1 r + c_1\int_M \left(\mathbb{E}^x\int_0^r \phi_0^{-1}(X_s^0)\mathrm{d}s\right)\mu_t(\mathrm{d}x)\\ &\leq c_2 r\mu_t(\log(1+\phi_0^{-1})) = \frac{c_2 r}{t}\int_0^t \log\{1+\phi_0^{-1}(X_s)\}\mathrm{d}s, \quad r \in (0,1].\end{aligned}$$

Combining this with (3.25), (3.3), $\|P_t^0\|_{L^p(\mu_0)} = 1$ for $t \geq 0$ and $p \geq 1$, and noting that

$$\inf_{t\geq 0} \mu_0(h\phi_0^{-1}P_t^0\phi_0^{-1}) > 0,$$

we find constants $c_3, c_4 > 0$ such that

$$\begin{aligned}\mathbb{E}^\nu[\mathbb{W}_2(\mu_{t,r,r}, \mu_t)^2|t < \tau] &= \frac{\mathbb{E}^\nu[1_{\{t<\tau\}}\mathbb{W}_2(\mu_{t,r,r}, \mu_t)^2]}{\mathbb{P}^\nu(t<\tau)}\\ &\leq \frac{c_3 r}{t\mu_0(h\phi_0^{-1}P_t^0\phi_0^{-1})}\int_0^t \mu_0(h\phi_0^{-1}P_s^0\log\{1+\phi_0^{-1}\})\mathrm{d}s\\ &\leq c_3 r\|h\phi_0^{-1}\|_{L^p(\mu_0)}\|\log(1+\phi_0^{-1})\|_{L^{\frac{p}{p-1}}(\mu_0)} \leq c_4 r, \quad r \in (0,1].\end{aligned} \tag{3.27}$$

Combining this with (3.25) we finish the proof. □

We are now ready to prove the main result in this section.

Proof of Proposition 3.1(1). Since the upper bound is infinite for $d \geq 4$, it suffices to consider $d \leq 3$.

(a) We first assume that $\nu = h\mu$ with $h \leq C\phi_0$ for some constant $C > 0$. In this case, by (2.5) and $\mathbb{E}^\nu = \int_M \mathbb{E}^x \nu(\mathrm{d}x)$, there exists a constant $c_0 > 0$ such that

$$\mathbb{E}^\nu(\cdot|t < \tau) \leq c_0 \mathbb{E}^{\nu_0}(\cdot|t < \tau), \quad t \geq 1. \tag{3.28}$$

Let $\mu_{t,r_t,r_t} = \{(1 - r_t)\rho_{t,r_t} + r_t\}\mu_0$ with $r_t = t^{-\alpha}$ for some $\alpha \in (1, \frac{4}{d} \wedge \frac{2d+4}{d^2})$. By Lemma 3.8 and the triangle inequality of $\mathbb{W}_2$, there exists a constant $c_1 > 0$ such that for any $t \geq 1$,

$$\mathbb{E}^\nu\big[\mathbb{W}_2(\mu_t, \mu_0)^2\big|t < \tau\big] \leq (1 + \varepsilon)\mathbb{E}^\nu\big[\mathbb{W}_2(\mu_{t,r_t,r_t}, \mu_0)^2\big|t < \tau\big] + c_1(1 + \varepsilon^{-1})t^{-\alpha}, \quad \varepsilon > 0. \tag{3.29}$$

On the other hand, by (3.1), (3.28), Lemmas 3.2, 3.6 and 3.7, there exists $p > 1$ such that

$$\begin{aligned}
&\limsup_{t\to\infty} t\mathbb{E}^\nu\big[\mathbb{W}_2(\mu_{t,r_t,r_t}, \mu_0)^2\big|t < \tau\big] \leq \limsup_{t\to\infty} t\mathbb{E}^\nu\bigg[\int_M \frac{|\nabla L_0^{-1}(\rho_{t,r_t} - 1)|^2}{\mathscr{M}(\rho_{t,r_t,r_t}, 1)}\mathrm{d}\mu_0\bigg|t < \tau\bigg] \\
&\leq \limsup_{t\to\infty} t\Big\{\mathbb{E}^\nu\big[\mu_0(|\nabla L_0^{-1}(\rho_{t,r_t} - 1)|^2)\mathrm{d}\mu_0\big|t < \tau\big] \\
&\quad + \big(\mathbb{E}^\nu\big[\mu_0(|\nabla L_0^{-1}(\rho_{t,r_t} - 1)|^{2p})\mathrm{d}\mu_0\big|t < \tau\big]\big)^{\frac{1}{p}}\big(\mathbb{E}^\nu\big[\mu_0(|\mathscr{M}(\rho_{t,r_t,r_t}, 1)^{-1} - 1|^{\frac{p}{p-1}})\big|t < \tau\big]\big)^{\frac{p-1}{p}}\Big\} \\
&= \limsup_{t\to\infty} t\mathbb{E}^\nu\big[\mu_0(|\nabla L_0^{-1}(\rho_{t,r_t} - 1)|^2)\mathrm{d}\mu_0\big|t < \tau\big] \leq \sum_{m=1}^\infty \frac{2}{(\lambda_m - \lambda_0)^2}.
\end{aligned}$$

Combining this with (3.29) where $\alpha > 1$, we prove (1.1).

(b) In general, for any $t \geq 2$ and $\varepsilon \in (0, 1)$, we consider

$$\mu_t^\varepsilon := \frac{1}{t - \varepsilon}\int_\varepsilon^t \delta_{X_s}\mathrm{d}s.$$

Letting D be the diameter of D, we find a constant $c_1 > 0$ such that

$$\mathbb{W}_2(\mu_t^\varepsilon, \mu_t)^2 \leq D^2\|\mu_t - \mu_t^\varepsilon\|_{var} \leq c_1\varepsilon t^{-1}, \quad t \geq 2, \varepsilon \in (0, 1). \tag{3.30}$$

On the other hand, by the Markov property we obtain

$$\begin{aligned}
&\mathbb{E}^\nu\big[1_{\{t<\tau\}}\mathbb{W}_2(\mu_t^\varepsilon, \mu_0)^2\big] = \mathbb{E}^\nu\big[1_{\{\varepsilon<\tau\}}\mathbb{E}^{X_\varepsilon}(1_{\{t-\varepsilon<\tau\}}\mathbb{W}_2(\mu_{t-\varepsilon}, \mu_0)^2)\big] \\
&= \mathbb{P}^\nu(\varepsilon < \tau)\mathbb{E}^{\nu_\varepsilon}\big[1_{\{t-\varepsilon<\tau\}}\mathbb{W}_2(\mu_{t-\varepsilon}, \mu_0)^2\big] \\
&= \mathbb{P}^{\nu_\varepsilon}(t - \varepsilon < \tau)\mathbb{P}^\nu(\varepsilon < \tau)\mathbb{E}^{\nu_\varepsilon}\big[\mathbb{W}_2(\mu_{t-\varepsilon}, \mu_0)^2\big|t - \varepsilon < \tau\big],
\end{aligned}$$

where $\nu_\varepsilon = h_\varepsilon\mu$ with

$$h_\varepsilon(y) := \frac{1}{\mathbb{P}^\nu(\varepsilon < \tau)}\int_M p_\varepsilon^D(x, y)\nu(\mathrm{d}x) \leq c(\varepsilon, \nu)\phi_0(y)$$

for some constant $c(\varepsilon, \nu) > 0$. Moreover, by (2.3), (2.5) and $\nu_\varepsilon = h_\varepsilon\mu$, we have

$$\lim_{t\to\infty} \frac{\mathbb{P}^{\nu_\varepsilon}(t - \varepsilon < \tau)\mathbb{P}^\nu(\varepsilon < \tau)}{\mathbb{P}^\nu(t < \tau)} = 1.$$

So, (a) implies

$$\begin{aligned}
&\limsup_{t\to\infty}\Big\{t\mathbb{E}^\nu\big[\mathbb{W}_2(\mu_t^\varepsilon,\mu_0)^2\big|t<\tau\big]\Big\}\\
&=\limsup_{t\to\infty}\frac{\mathbb{P}^{\nu_\varepsilon}(t-\varepsilon<\tau)\mathbb{P}^\nu(\varepsilon<\tau)}{\mathbb{P}^\nu(t<\tau)}\Big\{t\mathbb{E}^{\nu_\varepsilon}\big[\mathbb{W}_2(\mu_{t-\varepsilon},\mu_0)^2\big|t-\varepsilon<\tau\big]\Big\}\\
&\le\sum_{m=1}^\infty\frac{2}{(\lambda_m-\lambda_0)^2}.
\end{aligned}$$

Combining this with (3.30), we arrive at

$$\begin{aligned}
&\limsup_{t\to\infty}\Big\{t\mathbb{E}^\nu\big[\mathbb{W}_2(\mu_t,\mu_0)^2\big|t<\tau\big]\Big\}\\
&\le(1+\varepsilon^{\frac12})\limsup_{t\to\infty}\Big\{t\mathbb{E}^\nu\big[\mathbb{W}_2(\mu_t^\varepsilon,\mu_0)^2\big|t<\tau\big]\Big\}+c_1\varepsilon(1+\varepsilon^{-\frac12})\\
&\le(1+\varepsilon^{\frac12})\sum_{m=1}^\infty\frac{2}{(\lambda_m-\lambda_0)^2}+c_1\varepsilon(1+\varepsilon^{-\frac12}),\quad\varepsilon\in(0,1).
\end{aligned}$$

By letting $\varepsilon\to0$, we derive (1.1). □

Proof of Proposition 3.1(2)-(3). Let $d\ge4$. By (3.30), it suffices to prove the desired estimates for μ_t^1 replacing μ_t. Therefore, we may and do assume $\nu=h\mu$ with $\|h\phi_0^{-1}\|_\infty<\infty$. Since

$$\lim_{p\downarrow p_0}\Big\{\frac d2+\frac{(d+2)(p-1)}{2p}-2\Big\}=\frac{2(d-4)}{3},$$

by Lemma 3.2(1), for any $k>\frac{2(d-4)}{3}$, there exist constants $c_1,c_2>0$ such that

$$\begin{aligned}
t\mathbb{E}^\nu\big[\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^2\big|T<\tau\big]&\le c_1\sum_{m=1}^\infty\frac{\mathrm{e}^{-2(\lambda_m-\lambda_0)r}}{(\lambda_m-\lambda_0)^2}+c_1t^{-1}r^{-k}\\
&\le c_2\big\{1+1_{\{d=4\}}\log r^{-1}+t^{-1}r^{-k}\big\},\quad r\in(0,1),t\ge1,T\ge t.
\end{aligned}$$

Combining this with the following inequality due to [11, Theorem 2] for $p=2$:

$$\mathbb{W}_2(f\mu_0,\mu_0)^2\le4\mu_0(|\nabla L_0^{-1}(f-1)|^2),\quad f\mu_0\in\mathscr{P}_0,$$

we obtain

$$t\mathbb{E}^\nu\big[\mathbb{W}_2(\mu_{t,r,r},\mu_0)^2\big|T<\tau\big]\le c\big\{r^{-\frac{d-4}{2}}+1_{\{d=4\}}\log r^{-1}+t^{-1}r^{-k}\big\},\quad T\ge t\ge1,r\in(0,1).$$

By this and Lemma 3.8, we find a decreasing function $c:(\frac{2(d-4)}{3},\infty)\to(0,\infty)$ such that

$$\begin{aligned}
&\mathbb{E}^\nu\big[\mathbb{W}_2(\mu_t,\mu_0)^2\big|T<\tau\big]\le c(k)\big\{t^{-1}r^{-\frac{d-4}{2}}+t^{-1}1_{\{d=4\}}\log r^{-1}+t^{-2}r^{-k}+r\big\},\\
&T\ge t\ge1,r\in(0,1),k>\frac{2(d-4)}{3}.
\end{aligned}\tag{3.31}$$

(a) Let $d = 4$. We take $r = t^{-1}$ for $t > 1$, such that (3.31) implies (1.3) for some constant $c > 0$.

(b) When $d \geq 5$. Since

$$\lim_{k \downarrow \frac{2(d-4)}{3}} \left\{ 2 - \frac{2k}{d-2} \right\} = \frac{2d+4}{3(d-2)} > \frac{2}{d-2},$$

there exists $k > \frac{2(d-4)}{3}$ such that $2 - \frac{2k}{d-2} > \frac{2}{d-2}$. So, we may take $r = t^{-\frac{2}{d-2}}$ for $t > 1$ such that (3.31) implies the inequality in (3). □

4 Lower bound estimate

This section devotes to the proof of the following result, which together with Proposition 3.1 implies Theorem 1.1.

Proposition 4.1. *Let $\nu \in \mathscr{P}_0$. There exists a constant $c > 0$ such that (1.2) holds, and when ∂M is convex it holds for $c = 1$. Moreover, when $d \geq 5$, there exists a constant $c' > 0$ such that*

$$\inf_{T \geq t} \left\{ t \mathbb{E}[\mathbb{W}_2(\mu_t, \mu_0) | T < \tau] \right\} \geq c' t^{-\frac{2}{d-2}}, \quad t \geq 1. \tag{4.1}$$

To estimate the Wasserstein distance from below, we use the idea of [1] to construct a pair of functions in Kantorovich's dual formula, which leads to the following lemma.

Lemma 4.2. *There exists a constant $c > 0$ such that*

$$\mathbb{W}_2(\mu_{t,r}, \mu_0)^2 \geq \mu_0(|\nabla L_0^{-1}(\rho_{t,r} - 1)|^2) - c\|\rho_{t,r} - 1\|_\infty^{\frac{7}{3}}(1 + \|\rho_{t,r} - 1\|_\infty^{\frac{1}{3}}), \quad t, r > 0.$$

Proof. Let $f = L_0^{-1}(\rho_{t,r} - 1)$, and take

$$\varphi_\theta^\varepsilon = -\varepsilon \log P^0_{\frac{\varepsilon\theta}{2}} \mathrm{e}^{-\varepsilon^{-1} f}, \quad \theta \in [0,1], \varepsilon > 0.$$

We have $\varphi_0 = f$ and by [21, Lemma 2.9],

$$\begin{aligned} &\varphi_1^\varepsilon(y) - f(x) \leq \frac{1}{2}\left\{ \rho(x,y)^2 + \varepsilon \|(L_0 f)^+\|_\infty + c_1 \varepsilon^{\frac{1}{2}} \|\nabla f\|_\infty^2 \right\}, \\ &\mu_0(f - \varphi_1^\varepsilon) \leq \frac{1}{2}\mu_0(|\nabla f|^2) + c_1 \varepsilon^{-1} \|\nabla f\|_\infty^4. \end{aligned}$$

Since $L_0 f = \rho_{t,r} - 1$, this and the integration by parts formula imply

$$\begin{aligned} &\frac{1}{2}\mathbb{W}_2(\mu_{t,r}, \mu_0)^2 + \varepsilon\|\rho_{t,r} - 1\|_\infty + c_1 \varepsilon^{\frac{1}{2}} \|\nabla f\|_\infty^2 \geq \mu_0(\varphi_1^\varepsilon) - \mu_{t,r}(f) \\ &= \mu_0(\varphi_1^\varepsilon - f) - \mu_0(f L_0 f) \geq \frac{1}{2}\mu_0(|\nabla L_0^{-1} f|^2) - c_1 \varepsilon^{-1} \|\nabla f\|_\infty^4, \quad \varepsilon > 0. \end{aligned} \tag{4.2}$$

Next, by Lemma 2.1(1) for $p = \infty$ and (2.12), we find constants $c_2, c_3, c_4 > 0$ such that

$$\begin{aligned}
\|\nabla f\|_\infty &= \|\nabla L_0^{-1}(\rho_{t,r}-1)\|_\infty \le \int_0^\infty \|\nabla P_s^0(\rho_{t,r}-1)\|_\infty \mathrm{d}s \\
&\le c_2 \int_0^\infty (1+s^{-\frac{1}{2}})\|P_{s/2}^0(\rho_{t,r}-1)\|_\infty \mathrm{d}s \\
&\le c_3\|\rho_{t.r}-1\|_\infty \int_0^\infty (1+s^{-\frac{1}{2}})\mathrm{e}^{-(\lambda_1-\lambda_0)s/2}\mathrm{d}s \le c_4\|\rho_{t.r}-1\|_\infty.
\end{aligned}$$

Combining this with (4.2) we find a constant $c_5 > 0$ such that

$$\mathbb{W}_2(\mu_{t,r},\mu_0)^2 \ge \mu_0(|\nabla L_0^{-1}f|^2) - c_5\big\{\varepsilon\|\rho_{t,r}-1\|_\infty + \varepsilon^{\frac{1}{2}}\|\rho_{t,r}-1\|_\infty^2 + \varepsilon^{-1}\|\rho_{t,r}-1\|_\infty^4\big\}, \quad \varepsilon > 0.$$

By taking $\varepsilon = \|\rho_{t,r}-1\|_\infty^{\frac{4}{3}}$ we finish the proof. □

By Lemma 4.2, to derive a sharp lower bound of $\mathbb{W}_2(\mu_{t,r},\mu_0)^2$, we need to estimate $\|\rho_{t,r}-1\|_\infty$ and $\mathbb{E}^\nu\big[\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^2)\big|T<\tau\big]$, which are included in the following three lemmas.

Lemma 4.3. *For any $r > 0$ and $\nu = h\mu$ with $\|h\phi_0^{-1}\|_\infty < \infty$, there exists a constant $c(r) > 0$ such that*

$$\sup_{T\ge t}\mathbb{E}^\nu\big[\|\rho_{t,r}-1\|_\infty^4\big|T<\tau\big] \le c(r)t^{-2}, \quad t \ge 1.$$

Proof. By Lemma 3.3 and (3.28), it suffices to prove for $\nu = \nu_0$ and $T = t$ replacing $T \ge t$, i.e. for a constant $c(r) > 0$ we have

$$\mathbb{E}^{\nu_0}\big[\|\rho_{t,r}-1\|_\infty^4\big|t<\tau\big] \le c(r)t^{-2}, \quad t \ge 1. \tag{4.3}$$

By (3.19), (2.8), (2.12), and $\|\phi_0^{-1}\|_{L^2(\mu_0)} = 1$, we find a constant $c_1 > 0$ such that

$$\begin{aligned}
&\mathbb{E}^{\nu_0}[1_{\{r_1<\tau\}}|g_s(r_1,r_2)|^2] \\
&\le c_1\mathrm{e}^{-\lambda_0(2s-\lambda_1)}\|\hat{\phi}_m\|_\infty^4\big\{\mathrm{e}^{-(\lambda_m-\lambda_0)(r_2-r_1)} + \mathrm{e}^{-(\lambda_1-\lambda_0)(s-r_2)}\big\}, \quad s > r_2 > r_1 > 0.
\end{aligned}$$

By (3.18) and $\mathbb{P}^{\nu_0}(t<\tau) \ge c_0\mathrm{e}^{-\lambda_0 t}$ for some constant $c_0 > 0$ and all $t \ge 1$, this implies

$$\mathbb{E}^{\nu_0}[|\psi_m(t)|^4|t<\tau] := \frac{\mathbb{E}^{\nu_0}[|\psi_m(t)|^4 1_{\{t<\tau\}}]}{P^{\nu_0}(t<\tau)} \le c_2\|\hat{\phi}_m\|_\infty^4 t^{-2}, \quad m \ge 1, t > 1$$

for some constant $c_2 > 0$. Combining with (3.2) gives

$$\begin{aligned}
&\mathbb{E}^{\nu_0}\big[\|\rho_{t,r}-1\|_\infty^4\big|t<\tau\big] \\
&\le \bigg(\sum_{m=1}^\infty \mathrm{e}^{-(\lambda_m-\lambda_0)r}\|\hat{\phi}_m\|_\infty^{\frac{4}{3}}\bigg)^3 \sum_{m=1}^\infty \mathrm{e}^{-(\lambda_m-\lambda_0)r}e^{\lambda_0 t}\mathbb{E}^{\nu_0}[1_{\{r_1<\tau\}}|\psi_m(t)|^4] \\
&\le \bigg(\sum_{m=1}^\infty \mathrm{e}^{-(\lambda_m-\lambda_0)r}\|\hat{\phi}_m\|_\infty^{\frac{4}{3}}\bigg)^3 c_2 t^{-2}\sum_{m=1}^\infty \mathrm{e}^{-(\lambda_m-\lambda_0)r}\|\hat{\phi}_m\|_\infty^4.
\end{aligned}$$

By (2.1) and (2.14), this implies (4.3) for some constant $c(r) > 0$. □

Lemma 4.4. *Let $\nu = h\mu$ with $\|h\phi_0^{-1}\|_\infty < \infty$. Then for any $r > 0$ there exists a constant $c(r) > 0$ such that*

$$\sup_{T\geq t}\Big|t\mathbb{E}^\nu\big[\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^2)\big|T<\tau\big] - 2\sum_{m=1}^\infty \frac{\mathrm{e}^{-2(\lambda_m-\lambda_0)r}}{(\lambda_m-\lambda_0)^2}\Big| \leq \frac{c(r)}{t}, t\geq 1.$$

Proof. Let $\{J_i : i = 1,2,3\}$ be in (3.6). By (2.12), (2.14), and $\|\hat\phi_m\|_{L^2(\mu_0)} = 1$, we find a constant $c_1 > 0$ such that for any $T \geq t \geq s_2 \geq s_1 > 0$,

$$\begin{aligned}
|J_1(s_1,s_2)| &\leq \|h\phi_0^{-1}\|_\infty \|P^0_{s_1} - \mu_0\|_{L^\infty(\mu_0)}\|\phi_m\phi_0^{-1}|^2_\infty \|P^0_{T-s_2} - \mu_0\|_{L^1(\mu_0)}\|\phi_0^{-1}\|_{L^1(\mu_0)}\\
&\leq c_1\|\phi_m\phi_0^{-1}\|^2_\infty \mathrm{e}^{-(\lambda_1-\lambda_0)(t+s_1-s_2)},\\
|J_2(s_1,s_2)| &\leq \|\phi_0\|_\infty \mathrm{e}^{-(\lambda_m-\lambda_0)(s_2-s_1)}\|h\phi_0^{-1}\|_\infty \|P^0_{s_1} - \mu_0\|_{L^\infty(\mu_0)}\\
&\leq c_1\mathrm{e}^{-(\lambda_1-\lambda_0)s_2},\\
|J_3(s_1,s_2)| &\leq \|\phi_0\|_\infty \mathrm{e}^{-(\lambda_m-\lambda_0)(s_2-s_1)}\|\phi_m\phi_0^{-1}\|^2_\infty \|P^0_{T-s_2} - \mu_0\|_{L^1(\mu_0)}\|\phi_0^{-1}\|_{L^1(\mu_0)}\\
&\leq c_1\|\phi_m\phi_0^{-1}\|^2_\infty \mathrm{e}^{-(\lambda_1-\lambda_0)(t-s_1)}.
\end{aligned}$$

Substituting these into (3.7) and applying (2.1) and (2.14), we find a constant $c(r) > 0$ such that the desired estimate holds. □

Lemma 4.5. *Let $\nu = h\mu$ with $\|h\phi_0^{-1}\|_\infty < \infty$. Then for any $r > 0$ and $p \geq 2$, there exists a constant $c(r,p) > 0$ such that*

$$\||\nabla L_0^{-1}(\rho_{t,r}-1)|^{2p}\|_{L^{2p}(\mu_0)} \leq c(r,p), \quad t > 0.$$

Proof. Since $\rho_{t,r} = \frac{1}{t}\int_0^t p_r^0(X_s,\cdot)\mathrm{d}s$, we have $\mu_0(\rho_{t,r}) = 1$ and $\|\rho_{t,r}\|_\infty \leq \|p_r^0\|_\infty < \infty$. Then by (2.12) and $\|\phi_0^{-1}\|_{L^2(\mu_0)} = 1$, we find a constant $c_1(r) > 0$ such that

$$\begin{aligned}
&\mu_0\big(\phi_0^{-1}\{P^0_{\frac{s}{4}}|P^0_{\frac{3s}{4}}(\rho_{t,r}-1)|^p\}^2\big) \leq \|\phi_0^{-1}\|_{L^2(\mu_0)}\|(P^0_{\frac{3s}{4}} - \mu_0)\rho_{t,r}\|^{2p}_{L^{4p}(\mu_0)}\\
&\leq \|P^0_{\frac{3s}{4}} - \mu_0\|^{2p}_{L^{4p}(\mu_0)}\|\rho_{t,r}\|^{2p}_\infty \leq c_1(r)\mathrm{e}^{-3(\lambda_1-\lambda_0)s}.
\end{aligned}$$

Combining this with (3.22) for $\varepsilon = 1$ and $\theta \in (0, \frac{1}{\lambda_1-\lambda_0})$, we finish the proof. □

Finally, since $\mu_{t,r} = \mu_t P_r^0$, to derive a lower bound of $\mathbb{W}_2(\mu_t,\mu_0)$ from that of $\mathbb{W}_2(\mu_{t,r},\mu_0)$, we present the following result.

Lemma 4.6. *There exist two constants $K_1, K_2 > 0$ such that for any probability measures μ_1, μ_2 on M°,*

$$\mathbb{W}_2(\mu_1P_t^0, \mu_2P_t^0) \leq K_1\mathrm{e}^{K_2t}\mathbb{W}_2(\mu_1,\mu_2), \quad t \geq 0. \tag{4.4}$$

When ∂M is convex, this estimate holds for $K_1 = 1$.

Proof. When ∂M is convex, by [20, Lemma 2.16], there exists a constant K such that

$$\mathrm{Ric} - \mathrm{Hess}_{V+2\log\phi_0} \geq -K,$$

so that the desired estimate holds for $K_1 = 1$ and $K_2 = K$, see [14].

In general, following the line of [18], we make the boundary from non-convex to convex by using a conformal change of metric. Let N be the inward normal unit vector field of ∂M. Then the second fundamental form of ∂M is a two-tensor on the tangent space of ∂M defined by

$$\mathbb{I}(X,Y) := -\langle \nabla_X N, Y\rangle, \quad X, Y \in T\partial M.$$

Since M is compact, we find a function $f \in C_b^\infty(M)$ such that $f \geq 1, N \parallel \nabla f$ on ∂M, and $N \log f|_{\partial M} + \mathbb{I}(u,u) \geq 0$ holds on ∂M for any $u \in T\partial M$ with $|u| = 1$. By [18, Lemma 2.1] or [19, Theorem 1.2.5], ∂M is convex under the metric

$$\langle \cdot, \cdot \rangle' = f^{-2}\langle \cdot, \cdot \rangle.$$

Let Δ', ∇' and Hess' be the Laplacian, gradient and Hessian induced by the new metric $\langle \cdot, \cdot \rangle'$. We have $\nabla' = f^2\nabla$ and (see (2.2) in [16])

$$L_0 = f^{-2}\Delta' + f^{-2}\nabla'\{V + 2\log\phi_0 + (d-2)f^{-1}\}.$$

Then the L_0-diffusion process X_t^0 with X_0^0 having distribution μ_1 can be constructed by solving the following Itô SDE on M° with metric $\langle \cdot, \cdot \rangle'$ (see [2])

$$\mathrm{d}^I X_t^0 = \big\{f^{-2}\nabla'(V + 2\log\phi_0 + (d-2)f^{-1})\big\}(X_t^0)\mathrm{d}t + \sqrt{2}f^{-1}(X_t^0)U_t\mathrm{d}B_t, \tag{4.5}$$

where B_t is the d-dimensional Brownian motion, and U_t is the horizontal lift of X_t^0 to the frame bundle $O'(M)$ with respect to the metric $\langle \cdot, \cdot \rangle'$.

Let Y_0^0 be a random variable independent of B_t with distribution μ_2 such that

$$\mathbb{W}_2(\mu_1, \mu_2)^2 = \mathbb{E}[\rho(X_0^0, Y_0^0)^2]. \tag{4.6}$$

For any $x, y \in M^\circ$, let $P'_{x,y} : T_xM \to T_yM$ be the parallel transform along the minimal geodesic from x to y induced by the metric $\langle \cdot, \cdot \rangle'$, which is contained in M° by the convexity. Consider the coupling by parallel displacement

$$\mathrm{d}^I Y_t^0 = \big\{f^{-2}\nabla'(V + 2\log\phi_0 + (d-2)f^{-1})\big\}(Y_t^0)\mathrm{d}t + \sqrt{2}f^{-1}(Y_t^0)P'_{X_t^0,Y_t^0}U_t\mathrm{d}B_t. \tag{4.7}$$

As explained in [2, Section 3], we may assume that $(M^\circ, \langle \cdot, \cdot \rangle')$ does not have cut-locus such that $P'_{x,y}$ is a smooth map, which ensures the existence and uniqueness of Y_t^0. Since the distributions of X_0^0 and Y_0^0 are μ_1, μ_2 respectively, the law of (X_t^0, Y_t^0) is in the class $\mathscr{C}(\mu_1 P_t^0, \mu_2 P_t^0)$, so that

$$\mathbb{W}_2(\mu_1 P_t^0, \mu_2 P_t^0)^2 \leq \mathbb{E}[\rho(X_t^0, Y_t^0)^2], \quad t \geq 0. \tag{4.8}$$

Let $\rho'(x,y)$ be the Riemannian distance between x and y induced by $\langle \cdot, \cdot \rangle' := f^{-2}\langle \cdot, \cdot \rangle$. By $1 \leq f \in C_b^\infty(M)$ we have

$$\|f\|_\infty^{-1}\rho \leq \rho' \leq \rho. \tag{4.9}$$

Since except the term $f^{-2}\nabla'\log\phi_0$, all coefficients in the SDEs are in $C_b^\infty(M)$, by Itô's formula, there exists a constant K such that

$$\text{(4.10)} \qquad \mathrm{d}\rho'(X_t^0, Y_t^0)^2 \le \big\{K\rho'(X_t^0, Y_t^0)^2 + I\big\}\mathrm{d}t + \mathrm{d}M_t,$$

where M_t is a martingale and

$$I := \langle (f^{-2}\nabla'\log\phi_0)(\gamma_1), \dot\gamma_1\rangle' - \langle (f^{-2}\nabla'\log\phi_0)(\gamma_0), \dot\gamma_0\rangle'.$$

Let $\gamma : [0,1] \to M$ be the minimal geodesic from X_t^0 to Y_t^0 induced by the metric $\langle\cdot,\cdot\rangle'$, which is contained in M° by the convexity, we obtain

$$\begin{aligned} I &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\langle (f^{-2}\nabla'\log\phi_0)(\gamma_s), \dot\gamma_s\rangle'\mathrm{d}s \\ &= \int_0^1 \Big\{\frac{f^{-2}(\gamma_s)\mathrm{Hess}'_{\phi_0}(\dot\gamma_s,\dot\gamma_s) + \langle\nabla' f^{-2}(\gamma_s),\dot\gamma_s\rangle'\langle\nabla'\phi_0(\gamma_s),\dot\gamma_s\rangle'}{\phi_0(\gamma_s)} - \frac{\{\langle\nabla'\phi_0(\gamma_s),\dot\gamma_s\rangle'\}^2}{(f^2\phi_0^2)(\gamma_s)}\Big\}\mathrm{d}s \\ &\le \int_0^1 \Big\{(\phi_0^{-1}f^{-2})(\gamma_s)\mathrm{Hess}'_{\phi_0}(\dot\gamma_s,\dot\gamma_s) + \frac{f^2}{4}\big[\langle\nabla' f^{-2}(\gamma_s),\dot\gamma_s\rangle'\big]^2\Big\}\mathrm{d}s \le C\rho'(X_t^0,Y_t^0)^2 \end{aligned}$$

for some constant $C > 0$, where the last step is due to $\langle\dot\gamma_s,\dot\gamma_s\rangle' = \rho'(X_t^0,Y_t^0)^2$, $1 \le f \in C_b^\infty(M)$, and that by the proof of [20, Lemma 2.1] the convexity of ∂M under $\langle\cdot,\cdot\rangle'$ implies $\mathrm{Hess}'_{\phi_0} \le c\phi_0$ for some constant $c > 0$. This and (4.10) yield

$$\mathbb{E}[\rho'(X_t^0,Y_t^0)^2] \le \mathbb{E}[\rho'(X_0^0,Y_0^0)^2]\mathrm{e}^{(K+C)t}, \quad t \ge 0.$$

Combining this with (4.6) and (4.9), we prove (4.4) for some constant $K_1, K_2 > 0$. □

We are now ready to prove the main result in this section.

Proof of Proposition 4.1. (a) According to (3.30), it suffices to prove for $\nu = h\mu$ with $\|h\phi_0^{-1}\|_\infty < \infty$. Let $r > 0$ be fixed. By Lemma 4.2, we obtain

$$\text{(4.11)} \qquad \begin{aligned} t\mathbb{E}^\nu\big[\mathbb{W}_2(\mu_{t,r},\mu_0)^2\big|T<\tau\big] &\ge t\mathbb{E}^\nu\big[1_{\{\|\rho_{t,r}-1\|_\infty\le\varepsilon\}}\mathbb{W}_2(\mu_{t,r},\mu_0)^2\big|T<\tau\big] \\ &\ge t\mathbb{E}^\nu\big[1_{\{\|\rho_{t,r}-1\|_\infty\le\varepsilon\}}\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^2)\big|T<\tau\big] - c\varepsilon^2 \\ &\ge t\mathbb{E}^\nu\big[\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^2)\big|T<\tau\big] - c\varepsilon^2 \\ &\quad - t\mathbb{E}^\nu\big[1_{\{\|\rho_{t,r}-1\|_\infty>\varepsilon\}}\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^2)\big|T<\tau\big], \quad \varepsilon>0, T\ge t. \end{aligned}$$

By Lemma 4.3 and Lemma 4.5 with $p = 3$, we find some constants $c_1, c_2 > 0$ such that

$$\begin{aligned} &t\mathbb{E}^\nu\big[1_{\{\|\rho_{t,r}-1\|_\infty>\varepsilon\}}\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^2)\big|T<\tau\big] \le c_1 t\big\{\mathbb{P}^\nu\big(\|\rho_{t,r}-1\|_\infty>\varepsilon\big|T<\tau\big)\big\}^{\frac{2}{3}} \\ &\le c_1 t\varepsilon^{-\frac{8}{3}}\big\{\mathbb{E}^\nu\big(\|\rho_{t,r}-1\|_\infty^4\big|T<\tau\big)\big\}^{\frac{2}{3}} \le c_2\varepsilon^{-\frac{8}{3}}t^{-\frac{1}{3}}, \quad T\ge t. \end{aligned}$$

Combining this with (4.11) and Lemma 4.4, we find a constant $c_3 > 0$ such that

$$t\mathbb{E}^\nu\big[\mathbb{W}_2(\mu_{t,r},\mu_0)^2\big|T<\tau\big] \ge t\mathbb{E}^\nu\big[\mu_0(|\nabla L_0^{-1}(\rho_{t,r}-1)|^2)\big|T<\tau\big] - \varepsilon_t$$

$$\geq 2\sum_{m=1}^\infty \frac{\mathrm{e}^{-2(\lambda_m-\lambda_0)r}}{(\lambda_m-\lambda_0)^2} - \varepsilon_t - c_3 t^{-1},\quad T\geq t\geq 1,$$

where

$$\varepsilon_t := \inf_{\varepsilon>0}\{c\varepsilon^2 + c_2\varepsilon^{-\frac{8}{3}}t^{-\frac{1}{3}}\}\to 0 \text{ as } t\to\infty.$$

Therefore,

$$\liminf_{t\to\infty}\inf_{T\geq t}\Big\{tE^\nu\big[\mathbb{W}_2(\mu_{t,r},\mu_0)^2\big|T<\tau\big]\Big\}\geq 2\sum_{m=1}^\infty \frac{\mathrm{e}^{-2(\lambda_m-\lambda_0)r}}{(\lambda_m-\lambda_0)^2},\quad r>0.$$

Combining this with Lemma 4.6, we derive

$$\liminf_{t\to\infty}\inf_{T\geq t}\Big\{tE^\nu\big[\mathbb{W}_2(\mu_t,\mu_0)^2\big|T<\tau\big]\Big\}\geq 2K_1^{-1}\mathrm{e}^{-K_1 r}\sum_{m=1}^\infty \frac{\mathrm{e}^{-2(\lambda_m-\lambda_0)r}}{(\lambda_m-\lambda_0)^2},\quad r>0.$$

Letting $r\to 0$ we prove (1.2) for $c=K_1^{-1}$. By Lemma 4.6, we may take $c=1$ when ∂M is convex.

(b) The second assertion can be proved as in [21, Subsection 4.2]. For any $t\geq 1$ and $N\in\mathbb{N}$, let $\mu_N := \frac{1}{N}\sum_{i=1}^N \delta_{X_{t_i}}$, where $t_i := \frac{(i-1)t}{N}, 1\leq i\leq N$. [10, Proposition 4.2] (see also [9, Corollary 12.14]) implies

$$\mathbb{W}_1(\mu_N,\mu_0)^2\geq c_0 N^{-\frac{2}{d}},\quad N\in\mathbb{N}, t\geq 1 \tag{4.12}$$

for some constant $c_0>0$. Write

$$\mu_t = \frac{1}{N}\sum_{i=1}^N \frac{N}{t}\int_{t_i}^{t_{i+1}}\delta_{X_s}\mathrm{d}s.$$

By the convexity of $\mathbb{W}_2^2$, which follows from the Kantorovich dual formula, we have

$$\mathbb{W}_2(\mu_N,\mu_t)^2\leq \frac{1}{N}\sum_{i=1}^N\frac{N}{t}\int_{t_i}^{t_{i+1}}\mathbb{W}_2(\delta_{X_{t_i}},\delta_{X_s})^2\mathrm{d}s = \frac{1}{t}\sum_{i=1}^N\int_{t_i}^{t_{i+1}}\rho(X_{t_i},X_s)^2\mathrm{d}s \tag{4.13}$$

On the other hand, by the Markov property,

$$\mathbb{E}^\nu[\rho(X_{t_i},X_s)^2 1_{\{T<\tau\}}] = \mathbb{E}^\nu\big[1_{\{t_i<\tau\}}P^D_{s-t_i}\{\rho(X_{t_i},\cdot)^2 P^D_{T-s}1\}(X_{t_i})\big]. \tag{4.14}$$

Since $P_t^D 1\leq c_1\mathrm{e}^{-\lambda_0 t}$ for some constant $c_1>0$ and all $t\geq 0$, (2.7) implies

$$\begin{aligned} &P^D_{s-t_i}\{\rho(x,\cdot)^2 P^D_{T-s}1\}(x)\\ &\leq c_1\mathrm{e}^{-\lambda_0(T-s)}P^D_{s-t_i}\rho(x,\cdot)^2(x)\leq c_1\mathrm{e}^{-\lambda_0(T-s)}\phi_0(x)P^0_{s-t_i}\{\rho(x,\cdot)^2\phi_0^{-1}\}(x).\end{aligned} \tag{4.15}$$

It is easy to see that

$$L_0\{\rho(x,\cdot)^2\phi_0^{-1}\}\leq c_2\phi_0^{-2}$$

holds on M° for some constant $c_2>0$. So, by (2.18), we find a constant $c_3>0$ such that

$$P^0_{s-t_i}\{\rho(x,\cdot)^2\phi_0^{-1}\}(x)\le c_2\mathbb{E}^x\int_0^{s-t_i}\phi_0^{-2}(X_r)\mathrm{d}r\le c_3(s-t_i)\log(1+\phi_0^{-1}(x)).$$

Combining this with (4.14) and (4.15), and using $P_t^D1\le c_1\mathrm{e}^{-\lambda_0 t}$ observed above, we find a constant $c_5>0$ such that

$$\begin{aligned}&\mathbb{E}^\nu[\rho(X_{t_i},X_s)^2 1_{\{T<\tau\}}]\le c_4\mathrm{e}^{-\lambda_0 T}\nu(\log(1+\phi_0^{-1}))(s-t_i)\\ &\le c_4\|h\phi_0^{-1}\|_\infty\mu(\phi_0\log(1+\log\phi_0^{-1}))(s-t_i)\mathrm{e}^{-\lambda_0 T}\le c_5(s-t_i)\mathrm{e}^{-\lambda_0 T},\quad s\ge t_i.\end{aligned}$$

Since $\mathbb{P}^\nu(T<\tau)\ge c_0\mathrm{e}^{-\lambda_0 T}$ for some constant $c_0>0$ and all $T\ge1$, we find a constant $c>0$ such that

$$\mathbb{E}^\nu[\rho(X_{t_i},X_s)^2|T<\tau]\le c(s-t_i),\quad s\ge t_i.$$

Combining this with (4.12) and (4.13), we find a constant $c_6>0$ such that

$$\mathbb{E}^\nu[\mathbb{W}_1(\mu_t,\mu_0)^2|T<\tau]\ge\frac{c_1}{2}N^{-\frac{2}{d}}-c_6tN^{-1},\quad T\ge t.$$

Taking $N=\sup\{i\in\mathbb{N}:i\le\alpha t^{\frac{d}{d-2}}\}$ for some $\alpha>0$, we derive

$$t^{\frac{2}{d-2}}\inf_{T\ge t}\{\mathbb{E}^\nu[\mathbb{W}_1(\mu_0,\mu_t)^2|T<\tau]\}\ge\frac{c_2}{2\alpha^{\frac{2}{d}}}-\frac{2c'}{\alpha},\quad t\ge1.$$

Therefore,

$$t^{\frac{2}{d-2}}\inf_{T\ge t}\mathbb{E}^\nu[\mathbb{W}_1(\mu_0,\mu_t)^2|T<\tau]\ge\sup_{\alpha>0}\Big(\frac{c_2}{2\alpha^{\frac{2}{d}}}-\frac{2c'}{\alpha}\Big)>0,\quad t\ge1.$$

□

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