Derivative Estimates on Distributions of McKean-Vlasov SDEs *

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Abstract

By using the heat kernel parameter expansion with respect to the frozen SDEs, the intrinsic derivative is estimated for the law of Mckean-Vlasov SDEs with respect to the initial distribution. As an application, the total variation distance between the laws of two solutions is bounded by the Wasserstein distance for initial distributions. These extend some recent results proved for distribution-free noise by using the coupling method and Malliavin calculus.

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1 Introduction

Let $\mathscr{P}_2$ be the set of all probability measures on $\mathbb{R}^d$ with finite second moment, which is called the Wasserstein space under the metric

$$\mathbb{W}_2(\mu,\nu) := \inf_{\pi\in\mathscr{C}(\mu,\nu)} \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2\pi(\mathrm{d}x,\mathrm{d}y)\right)^{\frac{1}{2}}, \quad \mu,\nu\in\mathscr{P}_2,$$

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where $\mathscr{C}(\mu,\nu)$ is the set of all couplings of μ and ν. Consider the following distribution dependent SDE on $\mathbb{R}^d$:

$$\mathrm{d}X_t^\mu = b_t(X_t^\mu, \mathscr{L}_{X_t^\mu})\mathrm{d}t + \sigma_t(X_t^\mu, \mathscr{L}_{X_t^\mu})\mathrm{d}W_t, \quad \mathscr{L}_{X_0^\mu} = \mu \in \mathscr{P}_2, \tag{1.1}$$

where W_t is an m-dimensional Brownian motion on a complete filtration probability space $(\Omega, \{\mathscr{F}_t\}_{t\ge 0}, \mathbb{P})$, $\mathscr{L}_{X_t}$ is the law of X_t, and

$$b : \mathbb{R}_+ \times \mathbb{R}^d \times \mathscr{P}_2 \to \mathbb{R}^d, \quad \sigma : \mathbb{R}_+ \times \mathbb{R}^d \times \mathscr{P}_2 \to \mathbb{R}^d \otimes \mathbb{R}^m$$

are measurable. This type equations, known as Mckean-Vlasov or mean field SDEs, have been intensively investigated and applied, see for instance the monograph [3] and references therein.

To characterize the regularity of the law $\mathscr{L}_{X_t^\mu}$ with respect to the initial distribution μ, we investigate the derivative estimate of the functions

$$\mathscr{P}_2 \ni \mu \mapsto P_t f(\mu) := \mathbb{E} f(X_t^\mu), \quad f \in \mathscr{B}_b(\mathbb{R}^d), t > 0.$$

When the noise coefficient $\sigma_t(x,\mu)$ does not depend on μ, the Harnack inequality and derivative formula have been established in [13, 10] for $P_t f$ by using the coupling by change of measures and Malliavin calculus respectively. See also [2, 7, 8, 12] for extensions to distribution-path dependent SDEs/SPDEs, singular distribution dependent SDEs, and distribution dependent SDEs with jumps, where in [12] allows the noise to be also distribution dependent and establishes the gradient estimate on $P_t f(x) := (P_t f)(\delta_x)$ when the initial distribution is a Dirac measure. In this paper, we estimate the derivative of $P_t f(\mu)$ in μ by using the heat kernel parameter expansion with respect to the frozen SDE

$$\mathrm{d}X_t^{z,\mu} = b_t(z, \mu_t)\mathrm{d}t + \sigma_t(z, \mu_t)\mathrm{d}W_t \tag{1.2}$$

for fixed $(z,\mu) \in \mathbb{R}^d \times \mathscr{P}_2(\mathbb{R}^d)$, where $\mu_t := \mathscr{L}_{X_t^\mu}$. Since this SDE has constant coefficients, the solution has a Gaussian heat kernel which can be easily analyzed.

Before introducing the main result, we first recall the intrinsic derivative and L-derivative for functions on $\mathscr{P}_2$ which go back to [1] where the intrinsic derivative on the configurations space is introduced, see [11] for the link of different derivatives for measures.

Definition 1.1. Let $f : \mathscr{P}_2 \to \mathbb{R}$ and $g : \mathbb{R}^d \times \mathscr{P}_2 \to \mathbb{R}$.

(1) f is called intrinsically differentiable, if for any $\mu \in \mathscr{P}_2$,

$$L^2(\mathbb{R}^d \to \mathbb{R}^d; \mu) \ni \phi \mapsto D_\phi^L f(\mu) := \lim_{\varepsilon \downarrow 0} \frac{f(\mu \circ (\mathrm{Id} + \varepsilon\phi)^{-1}) - f(\mu)}{\varepsilon} \in \mathbb{R}$$

is a well defined bounded linear functional. In this case, the unique map

$$\mathscr{P}_2 \ni \mu \mapsto D^L f(\mu) \in L^2(\mathbb{R}^d \to \mathbb{R}^d; \mu)$$

such that $D^L_\phi f(\mu) = \langle \phi, D^L f(\mu)\rangle_{L^2(\mu)}$ holds for any $\mu \in \mathscr{P}_2$ and $\phi \in L^2(\mathbb{R}^d \to \mathbb{R}^d; \mu)$ is called the intrinsic derivative of f, and we denote

$$\|D^L f(\mu)\| := \|D^L f(\mu)(\cdot)\|_{L^2(\mu)}, \quad \mu \in \mathscr{P}_2.$$

If moreover

$$\lim_{\mu(|\phi|^2)\to 0} \frac{f(\mu\circ(\mathrm{Id}+\phi)^{-1}) - f(\mu) - D^L_\phi f(\mu)}{\sqrt{\mu(|\phi|^2)}} = 0, \quad \mu \in \mathscr{P}_2,$$

we call f L-differentiable, and in this case $D^L f$ is also called the L-derivative of f.

(2) We denote $f \in C^1(\mathscr{P}_2)$, if f is L-differentiable and its L-derivative has a version $D^L f(\mu)(x)$ jointly continuous in $(x,\mu) \in \mathbb{R}^d \times \mathscr{P}_2$.

(3) We denote $g \in C^{1,1}(\mathbb{R}^d \times \mathscr{P}_2)$, if $g(x,\cdot) \in C^1(\mathscr{P}_2)$ for $x \in \mathbb{R}^d$, $g(\cdot,\mu) \in C^1(\mathbb{R}^d)$ for $\mu \in \mathscr{P}_2$, $g(x,\mu), \nabla g(\cdot,\mu)(x)$ are jointly continuous in $(x,\mu) \in \mathbb{R}^d \times \mathscr{P}_2$, and $D^L g(x,\cdot)(\mu)(y)$ has a version jointly continuous in $(x,y,\mu) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathscr{P}_2$.

(4) A vector- or matrix-valued function is said in a class defined above, if so are its component functions.

To estimate the intrinsic derivative of $P_t f(\mu)$, we need the following condition. Let $|\cdot|$ and $\|\cdot\|$ denote the norm in $\mathbb{R}^d$ and the operator norm for linear operators repsectively.

(H) For any $t \geq 0$, $b_t, \sigma_t \in C^{1,1}(\mathbb{R}^d \times \mathscr{P}_2)$, and there exists an increasing function $K : [0,\infty) \to [0,\infty)$ such that for any $t \geq 0, x, y \in \mathbb{R}^d$ and $\mu \in \mathscr{P}_2(\mathbb{R}^d)$,

$$K_t^{-1}\mathrm{Id} \leq (\sigma_t\sigma_t^*)(x,\mu) \leq K_t \mathrm{Id},$$

$$\begin{aligned}&|b_t(x,\mu)| + \|\nabla b_t(\cdot,\mu)(x)\| + \|D^L\{b_t(x,\cdot)\}(\mu)\| \\ &\quad + \|\nabla\{\sigma_t(\cdot,\mu)\}(x)\|^2 + \|D^L\{\sigma_t(x,\cdot)\}(\mu)\|^2 \leq K_t,\end{aligned}$$

$$\begin{aligned}&\|D^L\{b_t(x,\cdot)\}(\mu) - D^L\{b_t(y,\cdot)\}(\mu)\| + \|D^L\{\sigma_t(x,\cdot)\}(\mu) - D^L\{\sigma_t(y,\cdot)\}(\mu)\| \\ &\leq K_t|x-y|^2.\end{aligned}$$

It is well known that SDE (1.1) is well-posed under the assumption **(H)**, so that $P_t f$ is well defined on $\mathscr{P}_2$ for any $t \geq 0$ and $f \in \mathscr{B}_b(\mathbb{R}^d)$. In general, for any $s \geq 0$ and $X^\mu_{s,s} \in L^2(\Omega \to \mathbb{R}^d, \mathscr{F}_s, \mathbb{P})$ with $\mathscr{L}_{X^\mu_{s,s}} = \mu$, let $X^\mu_{s,t}$ be the unique solution of (1.1) for $t \geq s$:

$$\mathrm{d}X^\mu_{s,t} = b_t(X^\mu_{s,t}, \mathscr{L}_{X^\mu_{s,t}})\mathrm{d}t + \sigma_t(X^\mu_{s,t}, \mathscr{L}_{X^\mu_{s,t}})\mathrm{d}W_t, \quad t \geq s, \mathscr{L}_{X^\mu_{s,s}} = \mu \in \mathscr{P}_2. \tag{1.3}$$

We denote $P^*_{s,t}\mu = \mathscr{L}_{X^\mu_{s,t}}$ and investigate the regularity of

$$P_{s,t}f(\mu) := \mathbb{E}f(X^\mu_{s,t}) = \int_{\mathbb{R}^d} f\mathrm{d}(P^*_{s,t}\mu), \quad f \in \mathscr{B}_b(\mathbb{R}^d).$$

By the uniqueness, we have the flow property

$$P_{s,t}^* = P_{r,t}^* P_{s,r}^*, \quad 0 \le s \le r \le t.$$

However, due to the distribution dependence, $P_{s,t}$ is no-longer a semigroup, i.e. in general $P_{s,t} \neq P_{r,t}P_{s,r}$ and

$$P_t f(\mu) \neq \int_{\mathbb{R}^d} P_t f(x)\mu(\mathrm{d}x),$$

so that the regularity of $P_t f(\mu)$ in $\mu \in \mathscr{P}_2$ can not be deduced from that of $P_t f(x) := P_t f(\delta_x)$ for $x \in \mathbb{R}^d$, see for instance [13] for details.

We now state the main result of the paper as follows.

Theorem 1.1. *Assume* **(H)**. *Then for any $t > s$ and $f \in \mathscr{B}_b(\mathbb{R}^d)$, $P_{s,t}f$ is L-differentiable, and there exists an increasing function $C : [0,\infty) \to (0,\infty)$ such that*

$$\|D^L P_{s,t} f(\mu)\| \le \frac{C_t \|f\|_\infty}{\sqrt{t-s}}, \quad t > s \ge 0, f \in \mathscr{B}_b(\mathbb{R}^d). \tag{1.4}$$

Consequently, for any $t > s \ge 0, \mu, \nu \in \mathscr{P}_2$,

$$\|P_{s,t}^*\mu - P_{s,t}^*\nu\|_{var} := \sup_{\|f\|_\infty \le 1} |P_{s,t}f(\mu) - P_{s,t}f(\nu)| \le \frac{C_t\|f\|_\infty}{\sqrt{t-s}} \mathbb{W}_2(\mu,\nu). \tag{1.5}$$

Remark 1.1. We may also apply Malliavin calculus to establish a derivative formula for $D^L P_{s,t} f(\mu)$ as in [12], where the usual derivative in initial points (rather than in initial distributions) are studied. However, in this way we need stronger conditions on the coefficients, i.e. $b_t(x,\mu)$ and $\sigma_t(x,\mu)$ also have bounded second order derivatives in x. Let us explain this in more details.

Firstly, under **(H)**, the Malliavin matrix

$$M_{s,t} := \big\{ \langle D(X_{s,t}^\mu)_i, D(X_{s,t}^\mu)_j \rangle_{\mathbb{H}} \big\}_{1 \le i \le j}$$

is invertible with $\mathbb{E}\|M_{s,t}^{-1}\|^2 < \infty$ for $t > s \ge 0$, where D is the Malliavin gradient, $\mathbb{H}$ is the Cameron-Martin space in Malliavin calculus, and $(X_{s,t}^\mu)_i$ is the i-th component of $X_{s,t}^\mu$.

Next, for any $\phi \in L^2(\mathbb{R}^d \to \mathbb{R}^d; \mu)$, let $v_{s,t}^\phi = D_\phi^L X_{s,t}^\mu$, which exists in $L^2(\mathbb{P})$ and satisfies

$$\mathbb{E}|v_{s,t}^\phi|^2 \le c(t)\mu(|\phi|^2)$$

for some constant $c(t) > 0$, see [10, Proposition 3.2].

Then for any $f \in C_b^1(\mathbb{R}^d)$, by the chain rule and the integration by parts formula for the Malliavin gradient D, we have

$$D_\phi^L P_{s,t} f(\mu) = \mathbb{E}\langle \nabla f(X_{s,t}^\mu), v_{s,t}^\phi \rangle = \sum_{i=1}^d \mathbb{E}[\partial_i f(X_{s,t}^\mu)(v_{s,t}^\phi)_i]$$

$$
\begin{aligned}
&= \sum_{i,j,k=1}^{d} \mathbb{E}\big[\partial_i f(X_{s,t}^{\mu})(M_{s,t})_{ij}(M_{s,t}^{-1})_{jk}(v_{s,t}^{\phi})_k\big] \\
&= \sum_{i,j,k=1}^{d} \mathbb{E}\big[\langle Df(X_{s,t}^{\mu}), D(X_{s,t}^{\mu})_j\rangle_{\mathbb{H}}(M_{s,t}^{-1})_{jk}(v_{s,t}^{\phi})_k\big] \\
&= \sum_{i,j,k=1}^{d} \mathbb{E}\big[f(X_{s,t}^{\mu})D^*\big\{(M_{s,t}^{-1})_{jk}(v_{s,t}^{\phi})_k D(X_{s,t}^{\mu})_j\big\}\big],
\end{aligned}
$$

where D^* is the Malliavin divergence. To make the above calculations meaningful, we need to verify that $(M_{s,t}^{-1})_{jk}(v_{s,t}^{\phi})_k D(X_{s,t}^{\mu})_j$ belongs to the domain of D^*, for which the second order derivatives of coefficients will be involved. For instance, as shown in [10, Proposition 3.2] that $v_{s,t}^{\phi}$ solves an SDE involving in the first order derivatives of b and σ, making Malliavin derivative to this SDE we see that $Dv_{s,t}^{\phi}$ solves an SDE containing the second order derivatives of coefficients.

The remainder of the paper is organized as follows. In Section 2, we formulate $P_{s,t}f(\mu)$ using classical SDEs with parameter μ and the parameter expansion of heat kernels with respect to the frozen SDE (1.2), and estimate the L-derivative for functions of $P_{s,t}^*\mu$. With these preparations, we prove Theorem 1.1 in Section 3.

2 Preparations

We first represent $P_{s,t}f(\mu)$ by using a Markov semigroup $P_{s,t}^{\mu}$ with parameter μ, then introduce the heat kernel expansion of $P_{s,t}^{\mu}$ with respect to the frozen SDEs. Since the frozen SDE has explicit Gaussian heat kernel, this enables us to calculate the intrinsic derivative of $P_t f(\mu)$ with respect to μ.

2.1 A representation of $P_{s,t}$

For any $s \geq 0, x \in \mathbb{R}^d$ and $\mu \in \mathscr{P}_2$, consider the decoupled SDE

$$
\mathrm{d}X_{s,t}^{x,\mu} = b_t(X_{s,t}^{x,\mu}, P_{s,t}^*\mu)\mathrm{d}t + \sigma_t(X_{s,t}^{x,\mu}, P_{s,t}^*\mu)\mathrm{d}W_t, \quad X_{s,s}^{x,\mu} = x, t \geq s. \tag{2.1}
$$

In this SDE, the measure variable $P_{s,t}^*\mu$ is fixed, so that it reduces to the classical time inhomogeneous SDE. Let $P_{s,t}^{\mu}$ be the associated Markov semigroup, i.e.

$$
P_{s,t}^{\mu}f(x) = \mathbb{E}f(X_{s,t}^{x,\mu}), \quad t \geq s, f \in \mathscr{B}_b(\mathbb{R}^d), x \in \mathbb{R}^d.
$$

Since $X_{s,t}^{\mu}$ solves (2.1) with the random initial value $X_{s,s}^{\mu}$ replacing x, and since $\mathscr{L}_{X_{s,s}^{\mu}} = \mu$, by the standard Markov property of solutions to (2.1), we have

$$
P_{s,t}f(\mu) := \mathbb{E}f(X_{s,t}^{\mu}) = \int_{\mathbb{R}^d} P_{s,t}^{\mu}f(x)\mu(\mathrm{d}x), \quad t \geq s, f \in \mathscr{B}_b(\mathbb{R}^d), \mu \in \mathscr{P}_2. \tag{2.2}
$$

Since for any $g \in C_b^1(\mathbb{R}^d)$ the function $\mu \mapsto \mu(g) := \int_{\mathbb{R}^d} g \mathrm{d}\mu$ is L-differentiable with $D^L\mu(g) = \nabla g$, we first study the derivative of $P_{s,t}^\mu f(x)$ in x.

Lemma 2.1. *Assume* **(H)**. *Then for any $f \in \mathscr{B}_b(\mathbb{R}^d)$ and $t > s \geq 0$, we have $P_{s,t}^\mu f \in C^1(\mathbb{R}^d)$ such that $(\nabla P_{s,t}^\mu f)(x)$ is continuous in $(x,\mu) \in \mathbb{R}^d \times \mathscr{P}_2$, and*

$$\|\nabla P_{s,t}^\mu f\|_\infty \leq \frac{CK_t\|f\|_\infty}{\sqrt{t-s}} \mathrm{e}^{CK_t}, \quad t > s, f \in \mathscr{B}_b(\mathbb{R}^d), \mu \in \mathscr{P}_2 \tag{2.3}$$

holds for some constant $C > 0$.

Proof. Since **(H)** implies that $P_{s,t}^*\mu$ is Lipschitz continuous in $\mu \in \mathscr{P}_2$, see for instance [13], the desired assertions follow from **(H)** and the Bismut formula

$$\nabla_v P_{s,t}^\mu f(x) = \mathbb{E}\bigg[\frac{f(X_{s,t}^{x,\mu})}{t-s} \int_s^t \big\langle \{\sigma_r(\sigma_r\sigma_r^*)^{-1}\}(X_{s,r}^{x,\mu}, P_{s,r}^*\mu) v_{s,r}^{x,\mu}, \mathrm{d}W_r \big\rangle\bigg], \quad v \in \mathbb{R}^d \tag{2.4}$$

for $f \in \mathscr{B}_b(\mathbb{R}^d)$, where $v_{s,t}^{x,\mu} := \frac{\mathrm{d}}{\mathrm{d}\varepsilon} X_{s,t}^{x+\varepsilon v,\mu}|_{\varepsilon=0}$ solves the linear SDE

$$\mathrm{d}v_{s,t}^{x,\mu} = \{\nabla_{v_{s,t}^{x,\mu}} b_t(\cdot, P_{s,t}^*\mu)\}(X_{s,t}^{x,\mu})\mathrm{d}t + \{\nabla_{v_{s,t}^{x,\mu}} \sigma_t(\cdot, P_{s,t}^*\mu)\}(X_{s,t}^{x,\mu})\mathrm{d}W_t, \quad t \geq s, v_{s,s}^{x,\mu} = v. \tag{2.5}$$

By **(H)**, $v_{s,t}^{x,\mu}$ is continuous in $(x,\mu) \in \mathbb{R}^d \times \mathscr{P}_2$ and

$$\mathbb{E}|v_{s,t}^{x,\mu}|^2 \leq |v|^2 \mathrm{e}^{CK_t}, \quad t \geq s, v \in \mathbb{R}^d$$

holds for some constant $C > 0$, so that (2.4) implies that $(\nabla P_{s,t}^\mu f)(x)$ is continuous in $(x,\mu) \in \mathbb{R}^d \times \mathscr{P}_2$ and satisfies (2.3).

To prove (2.4), for fixed $t > s$, take

$$h_u = \int_s^u \{\sigma_r^*(\sigma_r\sigma_r^*)^{-1}\}(X_{s,r}^{x,\mu}, P_{s,r}^*\mu) v_{s,r}^{x,\mu} \mathrm{d}r, \quad u \in [s,t].$$

Then the Malliavin derivative $w_r^{x,\mu} := D_h X_{s,r}^{x,\mu}$ along h solves the SDE

$$\begin{aligned}
\mathrm{d}w_{s,r}^{x,\mu} &= \Big[\{\nabla_{w_{s,r}^{x,\mu}} b_r(\cdot, P_{s,r}^*\mu)\}(X_{s,r}^{x,\mu}) + \sigma_r(X_{s,r}^{x,\mu}, P_{s,r}^*\mu)h_r'\Big]\mathrm{d}r + \{\nabla_{w_{s,r}^{x,\mu}} \sigma_r(\cdot, P_{s,r}^*\mu)\}(X_{s,r}^{x,\mu})\mathrm{d}W_r \\
&= \Big[\{\nabla_{w_{s,r}^{x,\mu}} b_r(\cdot, P_{s,r}^*\mu)\}(X_{s,r}^{x,\mu}) + v_{s,r}^{x,\mu}\Big]\mathrm{d}r + \{\nabla_{w_{s,r}^{x,\mu}} \sigma_r(\cdot, P_{s,r}^*\mu)\}(X_{s,r}^{x,\mu})\mathrm{d}W_r, \quad r \in [s,t], w_{s,s}^{x,\mu} = 0,
\end{aligned}$$

see for instance [10, Proposition 3.5]. It is easy to see from (2.5) that $\bar{v}_r := (r-s)v_{s,r}^{x,\mu}$ solves the same equation. By the uniqueness we obtain $(t-s)v_{s,t}^{x,\mu} = D_h X_{s,t}^{x,\mu}$, so that the chain rule and the integration by parts formula yield

$$\begin{aligned}
&\nabla_v P_{s,t}^\mu f(x) = \mathbb{E}\langle \nabla f(X_{s,t}^{x,\mu}), v_{s,t}^{x,\mu}\rangle = \frac{1}{t-s}\mathbb{E}\langle \nabla f(X_{s,t}^{x,\mu}), D_h X_{s,t}^{x,\mu}\rangle \\
&= \frac{1}{t-s}\mathbb{E} D_h\{f(X_{s,t}^{x,\mu})\} = \mathbb{E}\bigg[\frac{f(X_{s,t}^{x,\mu})}{t-s} \int_s^t \big\langle \{\sigma_r^*(\sigma_r\sigma_r^*)^{-1}\}(X_{s,r}^{x,\mu}, P_{s,r}^*\mu) v_{s,r}^{x,\mu}, \mathrm{d}W_r \big\rangle\bigg].
\end{aligned}$$

$\square$

Combining (2.2) with Lemma 2.1, we have the following result.

Lemma 2.2. *Assume* **(H)**. *Let $t > s$ and $f \in \mathscr{B}_b(\mathbb{R}^d)$. If for any $x \in \mathbb{R}^d$, the function $\mu \mapsto P^{\mu}_{s,t} f(x)$ is L-differentiable with*

$$\sup_{x\in\mathbb{R}^d} \big\|D^L\{P^{\cdot}_{s,t} f(x)\}(\mu)\big\| < \infty, \tag{2.6}$$

then $P_{s,t} f(\mu)$ is L-differentiable in μ with

$$D^L P_{s,t} f(\mu) = \nabla P^{\mu}_{s,t} f + \int_{\mathbb{R}^d} D^L\{P^{\cdot}_{s,t} f(x)\}(\mu)\mu(\mathrm{d}x). \tag{2.7}$$

Consequently, there exists a constant $C > 0$ such that for any $f \in \mathscr{B}_b(\mathbb{R}^d)$ and $\mu \in \mathscr{P}_2$,

$$\big\|D^L P_{s,t} f(\mu)\big\| \le \frac{CK_t\|f\|_\infty}{\sqrt{t-s}}\mathrm{e}^{CK_t} + \sup_{x\in\mathbb{R}^d} \big\|D^L\{P^{\cdot}_{s,t} f(x)\}(\mu)\big\|, \quad t > s \ge 0. \tag{2.8}$$

Proof. Obviously, (2.8) is implied by (2.3) and (2.7). So, we only need to prove that $P_{s,t} f(\mu)$ is L-differentiable and satisfies (2.7).

(1) We first prove that $P_{s,t} f(\mu)$ is intrinsically differentiable and satisfies (2.7). For any $g \in C^1_b(\mathbb{R}^d)$, the function $\mu \mapsto \mu(g) := \int_{\mathbb{R}^d} g\mathrm{d}\mu$ is L-differentiable with $D^L\mu(g) = \nabla g$. So, for any $\mu \in \mathscr{P}_2$, the function

$$\mathscr{P}_2 \ni \nu \mapsto P^{\mu}_{s,t} f(\nu) := \int_{\mathbb{R}^d} P^{\mu}_{s,t} f\mathrm{d}\nu \tag{2.9}$$

is L-differentiable with $D^L(P^{\mu}_{s,t} f)(\nu) = \nabla P^{\mu}_{s,t} f,\ \nu \in \mathscr{P}_2$. Combining this with (2.2), (2.3) and (2.6), and using the dominated convergence theorem, we conclude that the map

$$L^2(\mathbb{R}^d \to \mathbb{R}^d; \mu) \ni \phi \mapsto D^L_\phi P_{s,t} f(\mu) = \langle \nabla P^{\mu}_{s,t} f, \phi\rangle_{L^2(\mu)} + \int_{\mathbb{R}^d} D^L_\phi\{P^{\cdot}_{s,t} f(x)\}(\mu)\mu(\mathrm{d}x)$$

is a bounded linear functional, so that by definition, $P_{s,t} f(\mu)$ is intrinsically differentiable in $\mu \in \mathscr{P}_2$, and the formula (2.7) holds true.

(2) By (2.7), for any $\phi \in L^2(\mathbb{R}^d \to \mathbb{R}^d; \mu)$, we have

$$\begin{aligned}
&P_{s,t} f(\mu\circ(\mathrm{Id}+\phi)^{-1}) - P_{s,t} f(\mu) - D^L_\phi P_{s,t} f(\mu)\\
&= \int_{\mathbb{R}^d} \big\{P^{\mu\circ(\mathrm{Id}+\phi)^{-1}}_{s,t} f(x+\phi(x)) - P^{\mu\circ(\mathrm{Id}+\phi)^{-1}}_{s,t} f(x) - \langle \nabla P^{\mu\circ(\mathrm{Id}+\phi)^{-1}}_{s,t} f(x), \phi(x)\rangle\big\}\mu(\mathrm{d}x)\\
&\quad + \int_{\mathbb{R}^d} \big\{P^{\mu\circ(\mathrm{Id}+\phi)^{-1}}_{s,t} f(x) - P^{\mu}_{s,t} f(x) - D^L_\phi[P^{\cdot}_{s,t} f(x)](\mu)\big\}\mu(\mathrm{d}x)\\
&\quad + \int_{\mathbb{R}^d} \langle \nabla P^{\mu\circ(\mathrm{Id}+\phi)^{-1}}_{s,t} f(x) - \nabla P^{\mu}_{s,t} f(x), \phi(x)\rangle\mu(\mathrm{d}x).
\end{aligned}$$

Combining this with Lemma 2.1, (2.6), and the L-differentiability of $P^{\mu}_{s,t} f(x)$ in μ, we may apply the dominated convergence theorem to derive

$$\lim_{\|\phi\|_{L^2(\mu)}\downarrow 0} \frac{|P_{s,t} f(\mu\circ(\mathrm{Id}+\phi)^{-1}) - P_{s,t} f(\mu) - D^L_\phi P_{s,t} f(\mu)|}{\|\phi\|_{L^2(\mu)}} = 0,$$

that is, $P_{s,t} f(\mu)$ is L-differentiable. □

According to Lemma 2.2, to estimate $\|D^L P_{s,t}f(\mu)\|$, it remains to investigate the L-derivative of $P^\mu_{s,t}f(x)$ in μ. To this end, we let $p^\mu_{s,t}(x,y)$ be the heat kernel of $P^\mu_{s,t}$ for $t>s$, which exists and is differentiable in x and y under conditions **(H)**. We have

$$P^\mu_{s,t}f(x)=\int_{\mathbb{R}^d}p^\mu_{s,t}(x,y)f(y)\mathrm{d}y,\quad f\in\mathscr{B}_b(\mathbb{R}^d),t>s,x\in\mathbb{R}^d. \tag{2.10}$$

So, to investigate the L-derivative of $P^\mu_{s,t}f(x)$, we need to study that of $p^\mu_{s,t}(x,y)$, for which we will use the heat kernel parameter expansion.

2.2 Parameter expansion for $p^\mu_{s,t}$

Since heat kernel $p^\mu_{s,t}$ is less explicit, we make use of its parameter expansion with respect to the heat kernel of the Gaussian process

$$X^{x,\mu,z}_{s,r,t}=x+\int_r^t b_u(z,P^*_{s,u}\mu)\mathrm{d}u+\int_r^t\sigma_u(z,P^*_{s,u}\mu)\mathrm{d}W_u,\quad t\ge r\ge s\ge 0,x\in\mathbb{R}^d$$

for fixed $z\in\mathbb{R}^d$ and $\mu\in\mathscr{P}_2$. For any $t\ge r\ge s\ge 0$, let

$$\begin{aligned}m^{\mu,z}_{s,r,t}&:=\int_r^t b_u(z,P^*_{s,u}\mu)\mathrm{d}u,\quad m^{\mu,z}_{s,t}:=m^{\mu,z}_{s,s,t},\\ a^{\mu,z}_{s,r,t}&:=\int_r^t(\sigma_u\sigma_u^*)(z,P^*_{s,u}\mu)\mathrm{d}u,\quad a^{\mu,z}_{s,t}:=a^{\mu,z}_{s,s,t}.\end{aligned} \tag{2.11}$$

By **(H)**, we have

$$|m^{\mu,z}_{s,r,t}|+|a^{\mu,z}_{s,r,t}|\le(t-r)K_t,\quad t\ge r\ge s\ge 0. \tag{2.12}$$

Obviously, the law of $X^{x,\mu,z}_{s,r,t}$ is the d-dimensional normal distribution entered at $x+m^{\mu,z}_{s,r,t}$ with covariance matrix $a^{\mu,z}_{s,r,t}$, i.e. the distribution density function is

$$p^{\mu,z}_{s,r,t}(x,y)=\frac{\exp[-\frac{1}{2}\langle(a^{\mu,z}_{s,r,t})^{-1}(y-x-m^{\mu,z}_{s,r,t}),y-x-m^{\mu,z}_{s,r,t}\rangle]}{(2\pi)^{\frac{d}{2}}(\det\{a^{\mu,z}_{s,r,t}\})^{\frac{1}{2}}},\quad y\in\mathbb{R}^d,t>r\ge s. \tag{2.13}$$

When $r=s$, we simply denote $p^{\mu,z}_{s,t}=p^{\mu,z}_{s,s,t}$, so that

$$p^{\mu,z}_{s,t}(x,y)=\frac{\exp[-\frac{1}{2}\langle(a^{\mu,z}_{s,t})^{-1}(y-x-m^{\mu,z}_{s,t}),y-x-m^{\mu,z}_{s,t}\rangle]}{(2\pi)^{\frac{d}{2}}(\det\{a^{\mu,z}_{s,t}\})^{\frac{1}{2}}},\quad y\in\mathbb{R}^d,t>s. \tag{2.14}$$

For any $0\le s\le r<t$ and $y,z\in\mathbb{R}^d$, let

$$\begin{aligned}H^\mu_{s,r,t}(y,z):=&\big\langle b_r(z,P^*_{s,r}\mu)-b_r(y,P^*_{s,r}\mu),\nabla p^{\mu,z}_{s,r,t}(\cdot,z)(y)\big\rangle\\ &+\frac{1}{2}\mathrm{tr}\left[\big\{(\sigma_r\sigma_r^*)(z,P^*_{s,r}\mu)-(\sigma_r\sigma_r^*)(y,P^*_{s,r}\mu)\big\}\nabla^2p^{\mu,z}_{s,r,t}(\cdot,z)(y)\right].\end{aligned} \tag{2.15}$$

By the parameter expansion, see for instance [9, Lemma 3.1], we have

$$p^{\mu}_{s,t}(x,z) = p^{\mu,z}_{s,t}(x,z) + \sum_{m=1}^{\infty} \int_s^t \mathrm{d}r \int_{\mathbb{R}^d} H^{\mu,m}_{s,r,t}(y,z) p^{\mu,z}_{s,r}(x,y)\mathrm{d}y, \tag{2.16}$$

where $H^{\mu,m}_{s,r,t}$ for $m \in \mathbb{N}$ are defined by

$$\begin{aligned} &H^{\mu,1}_{s,r,t} := H^{\mu}_{s,r,t}, \\ &H^{\mu,m}_{s,r,t}(y,z) := \int_r^t \mathrm{d}u \int_{\mathbb{R}^d} H^{\mu,m-1}_{s,u,t}(z',z) H^{\mu}_{s,r,u}(y,z')\mathrm{d}z',\ m \geq 2. \end{aligned} \tag{2.17}$$

Combining (2.16) with (2.11), (2.13) and (2.14), to estimate $D^L P^{\mu}_{s,t} f$, it suffices to study the L-derivative of $b_r(y, P^*_{u_1,u_2}\mu)$ and $(\sigma_r\sigma_r^*)(y, P^*_{u_1,u_2}\mu)$ in μ for $r \geq 0$ and $u_2 \geq u_1 \geq 0$. So, we present the following lemma.

Lemma 2.3. *Assume* **(H)** *and let $t > s \geq 0$. Then for any $F \in C^1(\mathscr{P}_2)$ with bounded $\|D^L F\|$, $F(P^*_{s,t}\mu)$ is L-differentiable in μ such that*

$$\|D^L F(P^*_{s,t}\cdot)(\mu)\| \leq \|D^L F\|_\infty \mathrm{e}^{4K_t(t-s)}. \tag{2.18}$$

Consequently, for any $r \geq 0, t \geq s \geq 0$ and $y \in \mathbb{R}^d$, $b_r(y, P^*_{s,t}\mu)$ and $(\sigma_r\sigma_r^*)(y, P^*_{s,t}\mu)$ are L-differentiable in μ, and

$$\max\Big\{\|D^L b_r(y, P^*_{s,t}\cdot)(\mu)\|, \|D^L(\sigma_r\sigma_r^*)(y, P^*_{s,t}\cdot)(\mu)\|\Big\} \leq K_r \mathrm{e}^{4K_t(t-s)}, \quad \mu \in \mathscr{P}_2.$$

Proof. It suffices to prove the first assertion. We first prove the intrinsic differentiability. Let $\mu \in \mathscr{P}_2$ and $\phi \in L^2(\mathbb{R}^d \to \mathbb{R}^d; \mu)$. Since $\mathscr{L}_{X^{\mu}_{s,s}} = \mu$ implies

$$\mathscr{L}_{X^{\mu}_{s,s} + \varepsilon\phi(X^{\mu}_{s,s})} = \mu \circ (\mathrm{Id} + \varepsilon\phi)^{-1}, \quad \varepsilon \geq 0,$$

we have $\mathscr{L}_{X^{\varepsilon}_{s,t}} = P^*_{s,t}(\mu \circ (\mathrm{Id} + \varepsilon\phi)^{-1})$ for $X^{\varepsilon}_{s,t}$ solving (1.3) with initial value $X^{\varepsilon}_{s,s} = X^{\mu}_{s,s} + \varepsilon\phi(X^{\mu}_{s,s})$. By [10, Proposition 3.1] for $\eta = \phi(X^{\mu}_0)$ and [10, (4.21)] for time s replacing 0, for any $\delta \geq 0$,

$$v^{\phi,\delta}_{s,t} := D^L_\phi X^{\delta}_{s,t} = \lim_{\varepsilon \downarrow 0} \frac{X^{\delta+\varepsilon}_{s,t} - X^{\delta}_{s,t}}{\varepsilon}, \quad t \geq s$$

exists in $L^2(\Omega \to C([s,T]; \mathbb{R}^d); \mathbb{P})$ for any $T > 0$, and solves the linear SDEs:

$$\begin{aligned} \mathrm{d}v^{\phi,\delta}_{s,t} = &\Big[\nabla_{v^{\phi,\delta}_{s,t}} b_t(X^{\delta}_{s,t}, \mathscr{L}_{X^{\delta}_{s,t}}) + \mathbb{E}\big\{\langle D^L b_t(z,\cdot)(\mathscr{L}_{X^{\delta}_{s,t}})(X^{\delta}_{s,t}), v^{\phi,\delta}_{s,t}\rangle\big\}\big|_{z=X^{\delta}_{s,t}}\Big]\mathrm{d}t \\ &+ \Big[\nabla_{v^{\phi,\delta}_{s,t}} \sigma_t(X^{\delta}_{s,t}, \mathscr{L}_{X^{\delta}_{s,t}}) + \mathbb{E}\big\{D^L \sigma_t(z,\cdot)(\mathscr{L}_{X^{\delta}_{s,t}})(X^{\delta}_{s,t}) v^{\phi,\delta}_{s,t}\big\}\big|_{z=X^{\delta}_{s,t}}\Big]\mathrm{d}W_t, \\ v^{\phi,\delta}_{s,s} = &\ \phi(X_0), \quad t \geq s. \end{aligned} \tag{2.19}$$

Fromm **(H)** we see that $v^{\phi,\varepsilon}_{s,t}$ is continuous in ε and

$$\mathbb{E}|v^{\phi,\delta}_{s,t}|^2 \leq \mu(|\phi|^2)\mathrm{e}^{8(t-s)K_t}, \quad t \geq s, \phi \in L^2(\mathbb{R}^d \times \mathbb{R}^d; \mu). \tag{2.20}$$

By the chain rule, see for instance [10, Proposition 3.1], we have

$$D^L_\phi F(P^*_{s,t}\cdot)(\mu) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} F(\mathscr{L}_{X^\varepsilon_{s,t}})\Big|_{\varepsilon=0} = \mathbb{E}\langle (D^L F)(P^*_{s,t}\mu)(X^\mu_{s,t}), v^{\phi,0}_{s,t}\rangle. \tag{2.21}$$

Combining this with **(H)** and (2.20), we obtain

$$\begin{aligned}|D^L_\phi F(P^*_{s,t}\cdot)(\mu)| &\le \|(D^L F)(P^*_{s,t}\mu)\| \sqrt{\mathbb{E}|v^{\phi,0}_{s,t}|^2}\\ &\le \|\phi\|_{L^2(\mu)} \|D^L F\|_\infty \mathrm{e}^{4(t-s)K_t}, \quad \phi \in L^2(\mathbb{R}^d \to \mathbb{R}^d; \mu).\end{aligned}$$

Therefore, $F(P^*_{s,t}\mu)$ is intrinsically differentiable in μ such that (2.18) holds.

It remains to verify the L-differentiability. By the chain rule and (2.21), we obtain

$$\begin{aligned}&F(P^*_{s,t}\mu \circ (\mathrm{Id}+\phi)^{-1}) - F(P^*_{s,t}\mu) - D^L_\phi F(P^*_{s,t}\cdot)(\mu) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\varepsilon} F(\mathscr{L}_{X^\varepsilon_{s,t}})\mathrm{d}\varepsilon - D^L_\phi F(P^*_{s,t}\cdot)(\mu)\\ &= \int_0^1 \big\{\mathbb{E}\langle (D^L F)(P^*_{s,t}\mu \circ (\mathrm{Id}+\varepsilon\phi)^{-1})(X^\varepsilon_{s,t}), v^{\phi,\varepsilon}_{s,t}\rangle - \mathbb{E}\langle (D^L F)(P^*_{s,t}\mu)(X^\mu_{s,t}), v^{\phi,0}_{s,t}\rangle\big\}\mathrm{d}\varepsilon.\end{aligned}$$

Combining this with $F \in C^1(\mathscr{P}_2)$ with bounded $\|D^L F\|$, the continuity of $v^{\phi,\varepsilon}_{s,t}$ in ε, (2.20), and that $X^\varepsilon_{s,t} \to X^\mu_{s,t}$ when $\|\phi\|_{L^2(\mu)} \to 0$, by the dominated theorem we prove

$$\lim_{\|\phi\|_{L^2(\mu)}\downarrow 0} \frac{|F(P^*_{s,t}\mu \circ (\mathrm{Id}+\phi)^{-1}) - F(P^*_{s,t}\mu) - D^L_\phi F(P^*_{s,t}\cdot)(\mu)|}{\|\phi\|_{L^2(\mu)}} = 0,$$

thus, $F(P^*_{s,t}\mu)$ is L-differentiable in μ.

□

3 Proof of Theorem 1.1

According to Lemma 2.2, (2.10) and (2.16), to estimate $\|D^L P_{s,t} f(\mu)\|$, it suffices to handle the derivative of $p^\mu_{s,t}$ and $H^{\mu,m}_{s,r,t}$ in μ. To this end, for fixed $T > 0$, we introduce the Gaussian heat kernel

$$h_T(s,y) = \frac{\exp[-\frac{|y|^2}{8sK_T}]}{(8\pi s K_T)^{\frac{d}{2}}}, \quad y \in \mathbb{R}^d, s > 0, \tag{3.1}$$

which satisfies the Chapman-Kolmogorov equation

$$\int_{\mathbb{R}^d} h_T(s_1, y-z) h_T(s_2, z)\mathrm{d}z = h_T(s_1+s_2, y), \quad s_1, s_2 > 0, y \in \mathbb{R}^d. \tag{3.2}$$

By **(H)**, there exists a constant $K_1(T)$, which increases in T, such that

$$p^{\mu,z}_{s,r,t}(y,z) \le K_1(T) h_T(t-r, y-z)\mathrm{e}^{-\frac{|y-z|^2}{8(t-r)K_T}}, \quad y, z \in \mathbb{R}^d, 0 \le s \le r < t \le T, \mu \in \mathscr{P}_2.$$

Consequently, there exists a constant $K_2(T)$, which increases in T, such that

$$\begin{aligned} &p^{\mu,z}_{s,r,t}(y,z)\Big(1+\frac{|y-z|^2}{t-r}+\frac{|y-z|}{(t-r)^{\frac{1}{2}}}\Big)\\ &\le K_2(T)h_T(t-r,y-z),\quad y,z\in\mathbb{R}^d, 0\le s\le r<t\le T,\mu\in\mathscr{P}_2. \end{aligned}\tag{3.3}$$

Lemma 3.1. *Assume* **(H)**. *There exists a constant $\bar K_T>0$ which increases in $T>0$, such that for any $0\le s\le r<t\le T, y,z\in\mathbb{R}^d$ and $m\ge 1$, $p^{\mu,z}_{s,r,t}(y,z)$ and $H^{\mu,m}_{s,r,t}$ are L-differentiable in $\mu\in\mathscr{P}_2$ satisfying*

$$\|D^L\{p^{\cdot,z}_{s,r,t}(y,z)\}(\mu)\|\le \bar K_T h_T(t-r,y-z),\tag{3.4}$$

$$|H^{\mu,m}_{s,r,t}(y,z)|\le \frac{\bar K_T^m(t-r)^{\frac{m}{2}-1}}{\Gamma(\frac{m}{2})}h_T(t-r,y-z),\quad m\ge 1,\tag{3.5}$$

$$\|D^L\{H^{\cdot,m}_{s,r,t}(y,z)\}(\mu)\|\le \frac{m\bar K_T^m(t-r)^{\frac{m}{2}-1}}{\Gamma(\frac{m}{2})}h_T(t-r,y-z),\quad m\ge 1.\tag{3.6}$$

Proof. By **(H)**, we have $|m^{\mu,z}_{s,r,t}|\le (t-r)K_T$, so that (3.3) yields

$$p^{\mu,z}_{s,r,t}(y,z)\Big(1+\frac{|y-z-m^{\mu,z}_{s,r,t}|^2}{t-r}+\frac{|y-z-m^{\mu,z}_{s,r,t}|}{(t-r)^{\frac{1}{2}}}\Big)\le C_1(T)h_T(t-r,y-z)\tag{3.7}$$

for some constant $C_1(T)>0$ increasing in T, and all $0\le s\le r<t\le T,\mu\in\mathscr{P}_2$ and $y,z\in\mathbb{R}^d$. Combining this with **(H)**, (2.13), (3.7) and applying Lemma 2.3, we prove the L-differentiability of $p^{\mu,z}_{s,r,t}(y,z)$ in $\mu\in\mathscr{P}_2$ and the estimate (3.4).

Next, by **(H)**, (2.13), (2.15) and (3.7), we find constants $C_2(T),C_3(T)>0$ increasing in $T>0$ such that for any $0\le s\le r<t\le T,\mu\in\mathscr{P}_2$ and $y,z\in\mathbb{R}^d$,

$$\begin{aligned} |H^{\mu}_{s,r.t}(y,z)|&\le C_2(T)p^{\mu,z}_{s,r,t}(y,z)|y-z|\Big(\frac{1}{t-r}+\frac{|y-z-m^{\mu,z}_{s,r,t}|^2}{(t-r)^2}+\frac{|y-z-m^{\mu,z}_{s,r,t}|}{t-r}\Big)\\ &\le C_3(T)(t-r)^{-\frac{1}{2}}h_T(t-r,y-z). \end{aligned}\tag{3.8}$$

Assume that for some $k\ge 1$ we have

$$|H^{\mu,k}_{s,r,t}(y,z)|\le C_3(T)^k(t-r)^{\frac{k}{2}-1}\Big(\prod_{i=1}^{k-1}\beta\Big(\frac{i}{2},\frac{1}{2}\Big)\Big)h_T(t-r,y-z).$$

Combining this with (2.17), (3.2), and (3.8), we derive

$$\begin{aligned} |H^{\mu,k+1}_{s,r,t}(y,z)|&\le \int_r^t \mathrm{d}u\int_{\mathbb{R}^d}|H^{\mu,k}_{s,u,t}(z',z)H^{\mu}_{s,r,u}(y,z')|\mathrm{d}z'\\ &\le C_3(T)^{k+1}h_T(t-r,y-z)\Big(\prod_{i=1}^{k-1}\beta\Big(\frac{i}{2},\frac{1}{2}\Big)\Big)\int_r^t(t-u)^{\frac{k}{2}-1}(u-r)^{-\frac{1}{2}}\mathrm{d}u \end{aligned}$$

$$= C_3(T)^{k+1}(t-r)^{\frac{k+1}{2}-1}h_T(t-r,y-z)\Big(\prod_{i=1}^{k}\beta\Big(\frac{i}{2},\frac{1}{2}\Big)\Big).$$

In conclusion, for any $m \geq 1$, we have

$$|H^{\mu,m}_{s,r,t}(y,z)| \leq C_3(T)^m(t-r)^{\frac{m}{2}-1}\Big(\prod_{i=1}^{m-1}\beta\Big(\frac{i}{2},\frac{1}{2}\Big)\Big)h_T(t-r,y-z),$$

which implies (3.5) for $\bar{K}_T = C_3(T)\Gamma(\frac{1}{2})$, since

$$\prod_{i=1}^{m-1}\beta\Big(\frac{i}{2},\frac{1}{2}\Big) = \prod_{i=1}^{m-1}\frac{\Gamma(\frac{i}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{i+1}{2})} = \frac{\Gamma(\frac{1}{2})^m}{\Gamma(\frac{m}{2})}. \tag{3.9}$$

Finally, by **(H)**, (2.14), (2.15), Lemma 2.3 and (3.8), we see that $H^{\mu,m}_{s,r,t}$ is L-differentiable in μ, and there exist constants $C_4(T), C_5(T) \geq C_3(T)$ increasing in $T > 0$ such that

$$\begin{aligned}
&\|D^L\{H^{\cdot}_{s,r,t}(y,z)\}(\mu)\| \\
&\leq C_4(T)p^{\mu,z}_{s,r,t}(y,z)|y-z|\Big(\frac{1}{t-r} + \frac{|y-z-m^{\mu,z}_{s,r,t}|^2}{(t-r)^2} + \frac{|y-z-m^{\mu,z}_{s,r,t}|}{t-r}\Big) \\
&\leq C_5(T)(t-r)^{-\frac{1}{2}}h_T(t-r,y-z).
\end{aligned} \tag{3.10}$$

Assume that for some $k \geq 1$ we have

$$\|D^L\{H^{\cdot,k}_{s,r,t}(y,z)\}(\mu)\| \leq kC_5(T)^k(t-r)^{\frac{k}{2}-1}\Big(\prod_{i=1}^{k-1}\beta\Big(\frac{i}{2},\frac{1}{2}\Big)\Big)h_T(t-r,y-z).$$

Combining this with (2.17), (3.2), and (3.10), we derive

$$\begin{aligned}
&\|D^L\{H^{\cdot,k+1}_{s,r,t}(y,z)\}(\mu)\| \\
&\leq \int_r^t \mathrm{d}u \int_{\mathbb{R}^d}\Big\{\|D^L\{H^{\cdot,k}_{s,u,t}(z',z)\}(\mu)\| \cdot |H^{\mu}_{s,r,u}(y,z')| \\
&\qquad\qquad + |H^{\mu,k}_{s,u,t}(z',z)| \cdot \|D^L\{H^{\cdot}_{s,r,u}(y,z')\}(\mu)\|\Big\}\mathrm{d}z' \\
&\leq (k+1)C_5(T)^{k+1}h_T(t-r,y-z)\Big(\prod_{i=1}^{k-1}\beta\Big(\frac{i}{2},\frac{1}{2}\Big)\Big)\int_r^t (t-u)^{\frac{k}{2}-1}(u-r)^{-\frac{1}{2}}\mathrm{d}u \\
&= (k+1)C_5(T)^{k+1}(t-r)^{\frac{k+1}{2}-1}h_T(t-r,y-z)\Big(\prod_{i=1}^{k}\beta\Big(\frac{i}{2},\frac{1}{2}\Big)\Big).
\end{aligned}$$

This together with (3.9) implies (3.6) for $\bar{K}_T = C_5(T)\Gamma(\frac{1}{2})$. $\square$

We are now ready to prove the main result.

Proof of Theorem 1.1. By Lemma 3.1 with (2.16) and (3.2), $p^\mu_{s,t}(x,z)$ is L-differentiable in μ for $t>s$, and there exists a constant $\delta_T>0$ increasing in $T>0$ such that

$$\begin{aligned}
&\|D^L\{p^\cdot_{s,t}(x,z)\}(\mu)\| \le \bar K_T h_T(t-s,x-z)\\
&\quad+\sum_{m=1}^\infty \frac{(m+1)\bar K_T^{m+1}}{\Gamma(\frac{m}{2})}\int_s^t (t-s)^{\frac{m}{2}-1}\mathrm{d}r\int_{\mathbb{R}^d} h_T(t-r,y-z)h_T(r-s,x-y)\mathrm{d}y \qquad (3.11)\\
&\le \delta_T h_T(t-s,x-z).\quad 0\le s<t\le T, x,z\in\mathbb{R}^d,\mu\in\mathscr{P}_2.
\end{aligned}$$

This and (2.2) imply that $P^\mu_{s,t}f(x)$ is L-differentiable in μ such that

$$\|D^L\{P^\cdot_{s,t}f(x)\}(\mu)\|\le \|f\|_\infty\int_{\mathbb{R}^d}\|D^L\{p^\cdot_{s,t}(x,z)\}(\mu)\|\mathrm{d}z\le \delta_T\|f\|_\infty$$

holds for all $0\le s<t\le T$, $f\in\mathscr{B}_b(\mathbb{R}^d)$ and $\mu\in\mathscr{P}_2$. Combining this with Lemma 2.2, we prove that $P_{s,t}f(\mu)$ is L-differentiable in μ and (1.4) holds for some increasing $C:[0,\infty)\to(0,\infty)$. According to the proof of [10, Corollary 2.2(2)], we can show that (1.4) implies (1.5). We include below a simple proof for completeness.

Since $C_b^1(\mathbb{R}^d)$ is dense in $L^1(P^*_{s,t}\mu+P^*_{s,t}\nu)$, (1.5) is equivalent to

$$|P_{s,t}f(\mu)-P_{s,t}f(\nu)|\le \frac{C_t\|f\|_\infty}{\sqrt{t-s}}\mathbb{W}_2(\mu,\nu),\quad t>s, f\in C_b^1(\mathbb{R}^d),\mu,\nu\in\mathscr{P}_2. \qquad (3.12)$$

Let $f\in C_b^1(\mathbb{R}^d)$ be fixed. We first prove this inequality for μ,ν with compact supports. Let ξ,η be two bounded random variables such that $\mathscr{L}_\xi=\mu,\mathscr{L}_\eta=\nu$ and

$$\mathbb{E}|\xi-\eta|^2=\mathbb{W}_2(\mu,\nu)^2.$$

By Proposition 3.1 in [10] and (1.4), we obtain

$$\begin{aligned}
&|P_{s,t}f(\mu)-P_{s,t}f(\nu)|=\left|\int_0^1\frac{\mathrm{d}}{\mathrm{d}r}P_{s,t}f(\mathscr{L}_{r\xi+(1-r)\eta})\mathrm{d}r\right|\\
&\le\int_0^1\big|\mathbb{E}\langle D^LP_{s,t}f(\mathscr{L}_{r\xi+(1-r)\eta})(r\xi+(1-r)\eta),\xi-\eta\rangle\big|\mathrm{d}r\le\frac{C_t\|f\|_\infty}{\sqrt{t-s}}\mathbb{W}_2(\mu,\nu).
\end{aligned}$$

So, (3.12) holds.

Next, for any $\mu,\nu\in\mathscr{P}_2$, we choose $\{\mu_n,\nu_n\}_{n\ge1}\subset\mathscr{P}_2$ with compact supports such that

$$\lim_{n\to\infty}\big\{\mathbb{W}_2(\mu,\mu_n)+\mathbb{W}_2(\nu,\nu_n)\big\}=0.$$

Then by the last step,

$$|P_{s,t}f(\mu_n)-P_{s,t}f(\nu_n)|\le\frac{C_t\|f\|_\infty}{\sqrt{t-s}}\mathbb{W}_2(\mu_n,\nu_n),\quad n\ge1. \qquad (3.13)$$

If $P_{s,t}f(\gamma)$ is continuous in $\gamma \in \mathscr{P}_2$, then by letting $n \to \infty$ we obtain the desired estimate (3.12). To prove the continuity, for any $\gamma_1, \gamma_2 \in \mathscr{P}_2$, let ξ_1, ξ_2 be $\mathscr{F}_0$-measurable random variables such that $\mathscr{L}_{\xi_i} = \gamma_i, i = 1, 2$, and

$$\mathbb{W}_2(\gamma_1, \gamma_2)^2 = \mathbb{E}|\xi_1 - \xi_2|^2.$$

For any $\varepsilon \in [0,1]$, let $X_{s,t}^\varepsilon$ solve (1.3) with initial value $X_{s,s}^\varepsilon := \varepsilon\xi_1 + (1-\varepsilon)\xi_2$. By [10, Proposition 3.2 and (4.2)],

$$\nabla_{\xi_1-\xi_2} X_{s,t}^\varepsilon := \frac{\mathrm{d}}{\mathrm{d}\varepsilon} X_{s,t}^\varepsilon$$

exists in $L^2(\mathbb{P})$ with

$$\mathbb{E}|\nabla_{\xi_1-\xi_2} X_{s,t}^\varepsilon|^2 \le c(t)\mathbb{E}|\xi_1 - \xi_2|^2 = c(t)\mathbb{W}_2(\gamma_1, \gamma_2)^2$$

for some constant $c(t) > 0$. Then

$$\begin{aligned}|P_{s,t}f(\gamma_1) - P_{s,t}f(\gamma_2)| &= |\mathbb{E}f(X_{s,t}^1) - \mathbb{E}f(X_{s,t}^0)| = \left|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathbb{E}f(X_{s,t}^\varepsilon)\mathrm{d}\varepsilon\right| \\ &\le \int_0^1 \left|\mathbb{E}\langle \nabla f(X_{s,t}^\varepsilon), \nabla_{\xi_1-\xi_2} X_{s,t}^\varepsilon\rangle\right|\mathrm{d}s \le \sqrt{c(t)}\|\nabla f\|_\infty \mathbb{W}_2(\gamma_1, \gamma_2).\end{aligned}$$

Therefore, $P_{s,t}f(\gamma)$ is continuous in $\gamma \in \mathscr{P}_2$ and the proof is then finished. □

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