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Schauder estimates for nonlocal kinetic equations and applications ☆

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ABSTRACT

In this paper we develop a new method based on Littlewood-Paley's decomposition and heat kernel estimates in integral form, to establish Schauder's estimate for the following degenerate nonlocal equation in $\mathbb{R}^{2d}$ with Hölder coefficients:

$$\partial_t u = \mathscr{L}^{(\alpha)}_{\kappa;\mathrm{v}} u + b \cdot \nabla u + f,\ u_0 = 0,$$

where $u = u(t, x, \mathrm{v})$ and $\mathscr{L}^{(\alpha)}_{\kappa;\mathrm{v}}$ is a nonlocal α-stable-like operator with $\alpha \in (1, 2)$ and kernel function κ, which acts on the variable v. As an application, we show the strong well-posedness to the following degenerate stochastic differential equation with Hölder drift b:

$$\mathrm{d}Z_t = b(t, Z_t)\mathrm{d}t + (0, \sigma(t, Z_t)\mathrm{d}L^{(\alpha)}_t), \quad Z_0 = (x, \mathrm{v}) \in \mathbb{R}^{2d},$$

where $L^{(\alpha)}_t$ is a d-dimensional rotationally invariant and symmetric α-stable process with $\alpha \in (1, 2)$, and $b : \mathbb{R}_+ \times \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ is a (γ, β)-order Hölder continuous function in (x, v) with $\gamma \in \big(\frac{2+\alpha}{2(1+\alpha)}, 1\big)$ and $\beta \in \big(1 - \frac{\alpha}{2}, 1\big)$, $\sigma : \mathbb{R}_+ \times \mathbb{R}^{2d} \to \mathbb{R}^d \otimes \mathbb{R}^d$ is a Lipschitz function. Moreover, we also show that for almost all ω, the following random transport equation has a unique C^1_b-solution:

$$\partial_t u(t, x, \omega) + (b(t, x) + L^{(\alpha)}_t(\omega)) \cdot \nabla_x u(t, x, \omega) = 0, \quad u(0, x) = \varphi(x),$$

where $\varphi \in C^1_b(\mathbb{R}^d)$ and $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ is a bounded continuous function of (t, x) and γ-order Hölder continuous in x uniformly in t with $\gamma \in \big(\frac{2+\alpha}{2(1+\alpha)}, 1\big)$.

RÉSUMÉ

Dans ce traité, nous développons une nouvelle méthode basée sur la décomposition de Littlewood-Paley et les estimées du noyau de la chaleur sous la forme intégrale,

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qui établit alors l'estimation de Schauder pour l'équation non-locale et dégénérée suivante dans $\mathbb{R}^{2d}$ avec les coefficients Hölder :

$$\partial_t u = \mathscr{L}^{(\alpha)}_{\kappa;\mathrm{v}} u + b \cdot \nabla u + f,\ u_0 = 0,$$

où $u = u(t, x, \mathrm{v})$, $\mathscr{L}^{(\alpha)}_{\kappa;\mathrm{v}}$ est un opérateur du type α-stable non-local avec $\alpha \in (1, 2)$, et κ est un noyau, qui agit sur la variable v. Comme une application, nous montrons que l'équation différentielle stochastique dégénérée suivante avec la dérive de Hölder b est fortement bien posée :

$$\mathrm{d}Z_t = b(t, Z_t)\mathrm{d}t + (0, \sigma(t, Z_t)\mathrm{d}L^{(\alpha)}_t),\quad Z_0 = (x, \mathrm{v}) \in \mathbb{R}^{2d},$$

où $L^{(\alpha)}_t$ est un processus d-dimensionnel invariant en rotation, symétrique et α-stable avec $\alpha \in (1, 2)$, et aussi $b : \mathbb{R}_+ \times \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ est une fonction continue de l'ordre Hölder (γ, β) en (x, v) avec $\gamma \in \big(\frac{2+\alpha}{2(1+\alpha)}, 1\big)$ et $\beta \in \big(1-\frac{\alpha}{2}, 1\big)$, $\sigma : \mathbb{R}_+ \times \mathbb{R}^{2d} \to \mathbb{R}^d \otimes \mathbb{R}^d$ est une fonction lipschitzienne. De plus, nous montrons également que pour presque tous ω, l'équation de transport aléatoire suivante possède une unique C^1_b-solution :

$$\partial_t u(t, x, \omega) + (b(t, x) + L^{(\alpha)}_t(\omega)) \cdot \nabla_x u(t, x, \omega) = 0,\quad u(0, x) = \varphi(x),$$

où $\varphi \in C^1_b(\mathbb{R}^d)$ et $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ est une fonction continue bornée de (t, x) ainsi continue à l'ordre Hölder γ en x et uniformément en t avec $\gamma \in \big(\frac{2+\alpha}{2(1+\alpha)}, 1\big)$.

Contents

1. Introduction

Let $T > 0$. Consider the following backward transport equation (a first order PDE):

$$\partial_s u + b \cdot \nabla_x u + f = 0,\quad u(T, x) = \varphi(x), \tag{1.1}$$

where $b(s, x), f(s, x) : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\varphi(x) : \mathbb{R}^d \to \mathbb{R}^d$ are measurable functions. It is a classical fact that if b, f and φ are C^1_b-functions in x uniformly with respect to s, then the above equation is well-posed, and the unique solution is in fact given by

$$u(s, x) = \varphi(X_{s,T}(x)) + \int_s^T f(t, X_{s,t}(x))\mathrm{d}t,$$

where for each $x \in \mathbb{R}^d$, $X_{s,t}(x)$ solves the following ordinary differential equation (abbreviated as ODE):

$$\dot{X}_{s,t}(x) = b(t, X_{s,t}(x)), \quad X_{s,s}(x) = x, \quad t \geq s. \tag{1.2}$$

From this observation and by studying transport equation (1.1), DiPerna and Lions [17] developed a well-posed theory for ODE (1.2) when $b \in \mathbb{W}^{1,1}_{loc}$ (the first order Sobolev space) has bounded divergence (see also [1] for the investigation of ODE (1.2) with BV-vector field b). The corresponding results for SDEs are referred to [21], [50], [19] and [53]. It should be noticed that if b is only Hölder continuous, then PDE (1.1) would be ill-posed (see [22] for counterexamples). On the other hand, when b is Hölder continuous, under some random perturbations, it was shown in [22] that the following transport equation (a stochastic PDE) is well-posed:

$$\mathrm{d}u + (b \cdot \nabla_x u)\mathrm{d}s + u \circ \mathrm{d}W_s = 0, \quad u(T, x) = \varphi(x), \tag{1.3}$$

where $\circ$ stands for the Stratonovich integral, and W is a standard d-dimensional Brownian motion on some probability space. In this paper we shall show a new regularization mechanism for PDE (1.1) with Hölder drift b under some discontinuous Lévy noise perturbations by establishing the Schauder estimate for some degenerate nonlocal equations.

More precisely, we are concerned with the following *degenerate* nonlocal equation in $\mathbb{R}^{2d}$:

$$\partial_t u = \mathscr{L}^{(\alpha)}_{\kappa;\mathrm{v}} u + b \cdot \nabla u + f, \ u_0 = 0, \tag{1.4}$$

where $u = u(t, x, \mathrm{v})$ and $\mathscr{L}^{(\alpha)}_{\kappa;\mathrm{v}}$ is an α-stable-like operator acting on the variable v with the form:

$$\mathscr{L}^{(\alpha)}_{\kappa;\mathrm{v}} u(x, \mathrm{v}) := \int_{\mathbb{R}^d} (u(x, \mathrm{v} + w) + u(x, \mathrm{v} - w) - 2u(x, \mathrm{v})) \frac{\kappa(t, x, \mathrm{v}, w)}{|w|^{d+\alpha}} \mathrm{d}w, \tag{1.5}$$

where $\alpha \in (0, 2)$ and $\kappa(t, x, \mathrm{v}, w)$ is symmetric in w and bounded from above and below, and $b(t, x, \mathrm{v})$ takes the form

$$b(t, x, \mathrm{v}) = (b^{(1)}(t, x, \mathrm{v}), b^{(2)}(t, x, \mathrm{v})) : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d. \tag{1.6}$$

Note that a typical example of equation (1.4) is the following nonlocal kinetic equation:

$$\partial_t u + \mathrm{v} \cdot \nabla_x u = \mathscr{L}^{(\alpha)}_{\kappa;\mathrm{v}} u + f, \tag{1.7}$$

which naturally occurs in the study of linearized spatial inhomeogeneous Boltzmann equation with non-cutoff kernel (cf. [46], [12]). For $\lambda > 0$, we introduce the dilation operator $\mathbb{T}_\lambda : (t, x, \mathrm{v}) \mapsto (\lambda^\alpha t, \lambda^{1+\alpha} x, \lambda \mathrm{v})$ and define

$$u_\lambda := u \circ \mathbb{T}_\lambda, \quad \kappa_\lambda(\cdot, w) := \kappa(\cdot, \lambda w) \circ \mathbb{T}_\lambda, \ f_\lambda := \lambda^\alpha f \circ \mathbb{T}_\lambda.$$

By scaling, it is easy to see that $(u_\lambda, \kappa_\lambda, f_\lambda)$ still satisfies equation (1.7). Such a multi-scale feature leads us to consider the anisotropic Hölder spaces throughout this paper (see also [39] for more discussion). The first goal of this paper is to establish the following Schauder's apriori estimate for equation (1.4) under some natural Hölder and hypoellipticity assumptions on κ and b (see $(\mathbf{H}^{\alpha,\vartheta}_{\beta,\gamma})$ below for precise statement):

$$\|u\|_{L^\infty([0,T];\mathbf{C}^{(\alpha+\gamma)/(1+\alpha)}_x \cap \mathbf{C}^{\alpha+\beta}_\mathrm{v})} \leq C\|f\|_{L^\infty([0,T];\mathbf{C}^{\gamma/(1+\alpha)}_x \cap \mathbf{C}^\beta_\mathrm{v})}, \tag{1.8}$$

where $\alpha \in (1,2)$, $\beta \in (0,1)$, $\gamma \in [\beta, 1+\alpha)$, and $\mathbf{C}_x^{\gamma/(1+\alpha)}$ and $\mathbf{C}_{\rm v}^\beta$ stand for the Hölder spaces in x and v, respectively. In particular, if $\gamma \in (1, 1+\alpha)$, then

$$\|\nabla u\|_{L^\infty([0,T]\times\mathbb{R}^d\times\mathbb{R}^d)} \leq C\|f\|_{L^\infty([0,T];\mathbf{C}_x^{\gamma/(1+\alpha)}\cap\mathbf{C}_{\rm v}^\beta)},$$

which is crucial for the application in SDEs with Hölder coefficients. Moreover, if $b(t,x,{\rm v}) = ({\rm v}, 0)$, then (1.8) holds for any $\alpha \in (0,2)$. Notice that the increasing of the regularity of u in x and v are different due to the multi-scale property and the loss of the regularity in degenerate component x.

In the theory of PDEs, Schauder's estimates play a basic role in constructing the classical solution for quasilinear PDEs. Nowadays, there are many ways to prove such an estimate for heat equations (see [24], [33], [35]). In recent years, Schauder's estimate for nonlocal equations are also drawn great interests (see [4], [2], [18], [29], [55], etc.). However, most of the works are concentrated on the non-degenerate case. In the degenerate case, Lunardi [38] showed Schauder's estimate in anisotropic Hölder spaces for linear degenerate Kolmogorov's equations. Here it is natural to use the anisotropic Hölder spaces due to the feature of multiple scales in different directions. Later, in [37] and [40], the authors established Schauder's estimate for hypoelliptic Kolmogorov equations with partial nonlinear smooth drifts (corresponding to $b^{(1)}(t,x,{\rm v}) = {\rm v}$ in (1.6)). For general variable coefficient b, to the best of our knowledge, the authors in [6] first establish the sharp Schauder estimate for degenerate nonlinear Kolmogorov equations under some weak Hörmander's conditions, which in our case, corresponds to (1.8) with $\alpha = 2$ and $\gamma = \beta$. Their method is based on complex forward parametrix expansions. We mention that the L^p-maximal regularity for degenerate nonlocal Kolmogorov's equations with constant coefficients was also obtained in [12,11], [27].

In a very recent work [31], under some ellipticity and Hölder assumptions on κ (see Definition 1.1 and Assumption 1.5 of [31]), Imbert and Silvestre showed the following Schauder's interior estimate for kinetic equation (1.7):

$$\|u\|_{\mathbf{C}_\ell^{\alpha+\beta}(Q_{1/2})} \leq C\Big(\|u\|_{\mathbf{C}_\ell^{(1+\alpha)\beta/\alpha}(Q_1)} + \|f\|_{\mathbf{C}_\ell^{\beta}(Q_1)}\Big), \quad \beta \in \Big(0, \tfrac{\alpha(1\wedge\alpha)}{1+\alpha}\Big], \tag{1.9}$$

where $Q_r := (-r^\alpha, 0] \times B_{r^{1+\alpha}} \times B_r$ and $\mathbf{C}_\ell^\beta(Q_r)$ is some locally kinetic Hölder space of $(t,x,{\rm v})$ defined in [31, Section 2]. Note the following embedding:

$$\|u\|_{\mathbf{C}_x^{(\alpha+\beta)/(1+\alpha)}(Q_r)\cap\mathbf{C}_{\rm v}^{\alpha+\beta}(Q_r)} \leq C\|u\|_{\mathbf{C}_\ell^{\alpha+\beta}(Q_r)}.$$

Thus (1.8) coincides with (1.9) locally except the assumptions in [31] are weaker than ours. However, our result allows $\beta \in (0,1)$ and $\gamma > 1$, which in particular solves the questions in Section 1.2 of [31]. As potential applications, by suitable localization technique, we plan to use the global Schauder estimate (1.8) and the standard bootstrap argument to derive higher regularity estimates for the non-cutoff Boltzmann equation in a future work (cf. [26] and [30]).

To establish Schauder's estimate (1.8), we develop a completely new method, that is based on Littlewood-Paley's decomposition and heat kernel estimates of some integral forms. Roughly speaking, when we consider the usual heat equation, in terms of the Besov characterization of Hölder spaces, the key point is the following integral form estimate of the heat kernel (see Lemma 3.1 below): for any $\beta \geq 0$ and some constant $C = C(d,\beta) > 0$,

$$\int_0^t \left(\int_{\mathbb{R}^d} |x|^\beta |\mathcal{R}_j p_s(x)| {\rm d}x\right) {\rm d}s \leq C 2^{-2j-\beta j}, \quad \forall t \geq 0, \quad j \in \mathbb{N},$$

where $\mathcal{R}_j$ is the usual block operator in Littlewood-Paley's decomposition, and $p_s(x)$ is the Gaussian heat kernel. Unlike the usual method by firstly showing Schauder's estimate for constant coefficient equations, then freezing it for variable coefficient equations, we directly do it by Duhamel's formula for variable coefficient equations (see Theorem 3.2 below), which looks simpler. Moreover, the advantage of our method is that it provides more flexibility to borrow the spatial regularity of coefficients to compensate the time singularity when we use it to treat degenerate equations, which allows us to obtain the sharp Schauder estimate (1.8).

Another goal of this paper is to use the Schauder estimate (1.8) to show the strong well-posedness as well as the C^1-stochastic diffeomorphism flow property to the following degenerate SDEs in $\mathbb{R}^{2d}$ driven by α-stable processes with Hölder drifts:

$$\mathrm{d}Z_t = b(t, Z_t)\mathrm{d}t + (0, \mathrm{d}L^{(\alpha)}_t), \quad Z_0 = (x, \mathrm{v}) \in \mathbb{R}^{2d},$$

where $L^{(\alpha)}_t$ is a symmetric and rotationally invariant α-stable process with $\alpha \in (1, 2)$, and $b : \mathbb{R}_+ \times \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ is a (γ, β)-order Hölder continuous function of $z = (x, \mathrm{v})$ with $\gamma \in \big(\frac{2+\alpha}{2(1+\alpha)}, 1\big)$ and $\beta \in \big(1 - \frac{\alpha}{2}, 1\big)$. In particular, we shall prove the well-posedness to the following random transport equation with Hölder coefficient:

$$\partial_t u(t, x, \omega) + (b(t, x) + L^{(\alpha)}_t(\omega)) \cdot \nabla_x u(t, x, \omega) = 0, \quad u(0, x) = \varphi(x), \tag{1.10}$$

where $\varphi \in C^1_b(\mathbb{R}^d)$ and $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ is a bounded continuous function and γ-order Hölder continuous in x uniformly in t with $\gamma \in \big(\frac{2+\alpha}{2(1+\alpha)}, 1\big)$. Compared with *stochastic* PDE (1.3) studied in [22], it is a little surprise that as a deterministic equation, PDE (1.10) would be ill-posed for each fixed ω, while in the probability sense, *random* PDE (1.10) could be well-posed for *almost all* ω (see Theorem 7.8 below). This phenomenon may be considered as a new regularization mechanism for PDE (1.1) under some additive noise perturbations on drift b.

In the nondegenerate Brownian diffusion case, there are numerous works devoted to the studies of strong and weak well-posedness for the SDEs with singular and even distributional drifts (see [36], [49], [20], [51], [54] and references therein). Meanwhile, in the nondegenerate and α-stable noise case, recently there are also several works (see [41], [10], [13], [8]) to study this problem, especially for the supercritical case $\alpha \in (0, 1)$, because in this case, from the view point of PDEs, the drift term plays a dominant role. On the other hand, in the degenerate Brownian diffusion case, Chaudru [5] showed a strong uniqueness result for SDEs with Hölder drifts (see also [47], [48]). More recently, Chaudru, Honoré and Menozii [7] used their Schauder's estimate [6] to establish the strong uniqueness for a chain of oscillators driven by Brownian motions. However, there are few works to study the degenerate SDEs with jumps and Hölder drifts, which is probably used to provide a probabilistic representation for Boltzman equations as in Constantin and Iyer's representation for Navier-Stokes equations [14] (see also [52]).

This paper is organized as follows: In Section 2, we recall the well-known anisotropic Besov and Hölder-Zygmund spaces for later use. In Section 3, we introduce the basic idea of using Littlewood-Paley's decomposition to establish Schauder's estimate for heat equations with variable coefficients. In Section 4, we prove several commutator estimates, which plays a crucial role in showing the Schauder estimate (1.8). In Section 5, we give the heat kernel estimate of integral form for nonlocal kinetic operators, which is the basic tool for proving Schauder's estimate. In Section 6, we prove the Schauder estimate (1.8) under some natural Hölder's assumptions on κ and b (see Theorem 6.3). In Section 7, we apply the Schauder estimate to the well-posedness of degenerate SDEs with Hölder drifts and also show the well-posedness of a random transport equation with Hölder drift. The key point is to establish the C^1-stochastic diffeomorphism flow property to the degenerate SDEs. Finally, in Section 8 we show the existence of smooth solutions for degenerate nonlocal equations with unbounded coefficients by a purely probabilistic argument, which has independent interest.

We conclude this introduction by introducing some conventions and notations used throughout this paper.

- We use $A \lesssim B$ to denote $A \le CB$ for some unimportant constant $C > 0$.
- We use $A \asymp B$ to denote $C^{-1}B \le A \le CB$ for some unimportant constant $C \ge 1$.
- For any $\varepsilon \in (0,1)$, we use $A \lesssim \varepsilon B + D$ to denote $A \le \varepsilon B + C_\varepsilon D$ for some constant $C_\varepsilon > 0$.
- For two operators $\mathscr{A}_1, \mathscr{A}_2$, we use $[\mathscr{A}_1, \mathscr{A}_2] := \mathscr{A}_1\mathscr{A}_2 - \mathscr{A}_2\mathscr{A}_1$ to denote their commutator.
- For a Banach space $\mathbb{B}$ and $T > 0$, we denote

$$\mathbb{L}^\infty_T(\mathbb{B}) := L^\infty([0,T];\mathbb{B}),\ \mathbb{L}^\infty_{loc}(\mathbb{B}) := \cap_{T>0}\mathbb{L}^\infty_T(\mathbb{B}),\ \mathbb{L}^\infty_T := L^\infty([0,T]\times\mathbb{R}^d).$$

- $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$, $\mathbb{R}_+ := [0,\infty)$, $a \vee b := \max(a,b)$, $a \wedge b := \min(a,b)$.

2. Anisotropic Besov and Hölder-Zygmund spaces

Let $N \in \mathbb{N}$. We first introduce the Hölder (and Hölder-Zygmund) spaces in $\mathbb{R}^N$. For $h \in \mathbb{R}^N$ and $f : \mathbb{R}^N \to \mathbb{R}$, the first order difference operator is defined by

$$\delta_h f(x) := f(x+h) - f(x).$$

For $\beta > 0$, let $\mathscr{C}^\beta$ be the usual β-order Hölder space consisting of all functions $f : \mathbb{R}^N \to \mathbb{R}$ with

$$\|f\|_{\mathscr{C}^\beta} := \|f\|_\infty + \cdots + \|\nabla^{[\beta]} f\|_\infty + [\nabla^{[\beta]} f]_{\mathscr{C}^{\beta-[\beta]}} < \infty,$$

where $[\beta]$ denotes the greatest integer less than β, and ∇^j stands for the j-order gradient, and

$$[f]_{\mathscr{C}^\gamma} := \sup_h \|\delta_h f\|_\infty/|h|^\gamma,\ \gamma \in [0,1).$$

The β-order Hölder-Zygmund space $\mathbf{C}^\beta$ is defined by

$$\|f\|_{\mathbf{C}^\beta} := \|f\|_\infty + [f]_{\mathbf{C}^\beta} < \infty,\ [f]_{\mathbf{C}^\beta} := \sup_h \|\delta_h^{[\beta]+1} f\|_\infty/|h|^\beta,$$

where for an integer m, $\delta_h^m := \delta_h \cdots \delta_h$ denotes the m-order difference operator. Notice that for $0 < \beta \notin \mathbb{N}$ and $m \in \mathbb{N}$ (cf. [44]),

$$\|f\|_{\mathscr{C}^\beta} \asymp \|f\|_{\mathbf{C}^\beta},\ \|f\|_{\mathbf{C}^m} \lesssim \|f\|_{\mathscr{C}^m}. \tag{2.1}$$

Let

$$\langle f, g\rangle := \int_{\mathbb{R}^N} f(x)g(x)\mathrm{d}x.$$

The adjoint operator of δ_h with respect to the above $\langle\cdot,\cdot\rangle$ is given by

$$\delta_h^* = -\delta_{-h} \Leftrightarrow \langle \delta_h f, g\rangle = \langle f, \delta_h^* g\rangle.$$

In particular, we have

$$\delta_h^*\delta_h f(x) = f(x+h) + f(x-h) - 2f(x), \tag{2.2}$$

and for any $f \in \mathscr{C}^2$,

$$\|\delta_h^*\delta_h f\|_\infty \le (2\|\nabla^2 f\|_\infty |h|^2) \wedge (4\|f\|_\infty). \tag{2.3}$$

Let $\mathscr{S}(\mathbb{R}^N)$ be the Schwartz space of all rapidly decreasing functions on $\mathbb{R}^N$, and $\mathscr{S}'(\mathbb{R}^N)$ the dual space of $\mathscr{S}(\mathbb{R}^N)$ called Schwartz generalized function (or tempered distribution) space. Given $f \in \mathscr{S}(\mathbb{R}^N)$, the Fourier transform $\hat{f}$ and inverse Fourier transform $\check{f}$ are defined by

$$\hat{f}(\xi) := (2\pi)^{-N/2} \int_{\mathbb{R}^N} \mathrm{e}^{-i\xi\cdot x} f(x)\mathrm{d}x, \quad \xi \in \mathbb{R}^N,$$

$$\check{f}(x) := (2\pi)^{-N/2} \int_{\mathbb{R}^N} \mathrm{e}^{i\xi\cdot x} f(\xi)\mathrm{d}\xi, \quad x \in \mathbb{R}^N.$$

Let $m = (m_1, \cdots, m_n) \in \mathbb{N}^n$ with $m_1 + \cdots + m_n = N$ and $a = (a_1, \cdots, a_n) \in [1, \infty)^n$ be fixed. We introduce the following distance in $\mathbb{R}^N$ by

$$|x - y|_a := \sum_{i=1}^n |x_i - y_i|^{1/a_i},\ x_i, y_i \in \mathbb{R}^{m_i}.$$

For $x = (x_1, \cdots, x_n)$, $t > 0$ and $s \in \mathbb{R}$, we denote

$$t^{sa}x := (t^{sa_1}x_1, \cdots, t^{sa_n}x_n) \in \mathbb{R}^N, \quad B_t^a := \Big\{x \in \mathbb{R}^N : |x|_a \le t\Big\}.$$

Clearly we have

$$|t^a x|_a = t|x|_a, \quad t > 0.$$

Let ϕ_0^a be a radial C^∞-function on $\mathbb{R}^N$ with

$$\phi_0^a(\xi) = 1 \ \text{ for } \ \xi \in B_1^a \ \text{ and } \ \phi_0^a(\xi) = 0 \ \text{ for } \ \xi \notin B_2^a.$$

For $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^N$ and $j \in \mathbb{N}$, define

$$\phi_j^a(\xi) := \phi_0^a(2^{-aj}\xi) - \phi_0^a(2^{-a(j-1)}\xi).$$

It is easy to see that for $j \in \mathbb{N}$, $\phi_j^a(\xi) = \phi_1^a(2^{-a(j-1)}\xi) \ge 0$ and

$$\mathrm{supp}\phi_j^a \subset B_{2^{j+1}}^a \setminus B_{2^{j-1}}^a, \quad \sum_{j=0}^k \phi_j^a(\xi) = \phi_0^a(2^{-ak}\xi) \to 1, \quad k \to \infty.$$

Definition 2.1 *(Anisotropic Besov and Hölder-Zygmund spaces).* For given $j \in \mathbb{N}_0$, the block operator $\mathcal{R}_j^a$ is defined on $\mathscr{S}'(\mathbb{R}^N)$ by

$$\mathcal{R}_j^a f(x) := (\phi_j^a \hat{f})\check{}(x) = \check{\phi}_j^a * f(x) = 2^{a\cdot m(j-1)} \int_{\mathbb{R}^N} \check{\phi}_1^a(2^{a(j-1)}y) f(x-y)\mathrm{d}y, \tag{2.4}$$

where $a \cdot m = a_1 m_1 + \cdots + a_n m_n$. For any $s \in \mathbb{R}$, the anisotropic Besov space $\mathbf{B}_{a,\infty}^s$ is defined by

$$\mathbf{B}_{a,\infty}^s := \Big\{f \in \mathscr{S}'(\mathbb{R}^N) : \|f\|_{\mathbf{B}_{a,\infty}^s} := \sup_{j\ge 0} \big(2^{sj}\|\mathcal{R}_j^a f\|_\infty\big) < \infty\Big\},$$

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and for $s \ge 0$, the anisotropic Hölder-Zygmund space $\mathbf{C}^s_a$ is defined by

$$\mathbf{C}^s_a := \Big\{ f \in \mathbb{R}^N \to \mathbb{R} : \|f\|_{\mathbf{C}^s_a} := \|f\|_\infty + [f]_{\mathbf{C}^s_a} < \infty \Big\},$$

where

$$[f]_{\mathbf{C}^s_a} := \sup_h \|\delta_h^{[s]+1} f\|_\infty / |h|_a^s.$$

In particular, if $a = (1, \cdots, 1)$, we shall drop the index a in $\mathbf{B}^s_{a,\infty}$, $\mathcal{R}^a_j$ and $\mathbf{C}^s_a$, and simply use $\mathbf{B}^s_\infty$, $\mathcal{R}_j$ and $\mathbf{C}^s$.

For $j \in \mathbb{N}_0$, by definition it is easy to see that

$$\mathcal{R}^a_j = \mathcal{R}^a_j \widetilde{\mathcal{R}}^a_j, \quad \text{where } \ \widetilde{\mathcal{R}}^a_j := \mathcal{R}^a_{j-1} + \mathcal{R}^a_j + \mathcal{R}^a_{j+1} \text{ with } \mathcal{R}^a_{-1} \equiv 0, \tag{2.5}$$

and $\mathcal{R}^a_j$ is symmetric in the sense that

$$\langle \mathcal{R}^a_j f, g\rangle = \langle f, \mathcal{R}^a_j g\rangle.$$

The cut-off low frequency operator S_k is defined by

$$S_k f := \sum_{j=0}^{k-1} \mathcal{R}^a_j f = 2^{a\cdot mk} \int_{\mathbb{R}^N} \check{\phi}^a_0(2^{ka}(x-y)) f(y) \mathrm{d}y \to f, \quad k \to \infty. \tag{2.6}$$

For $f, g \in \mathscr{S}'(\mathbb{R}^N)$, define

$$T_f g = \sum_{k \ge 2} S_{k-1} f \mathcal{R}^a_k g, \quad R(f,g) := \sum_{k\in\mathbb{N}} \sum_{i=-1,0,1} \mathcal{R}^a_k f \mathcal{R}^a_{k-i} g. \tag{2.7}$$

The Bony decomposition of fg is formally given by (cf. [3])

$$fg = T_f g + T_g f + R(f,g). \tag{2.8}$$

The key point of Bony's decomposition is

$$\mathcal{R}^a_j (S_{k-1} f \mathcal{R}^a_k g) = 0 \ \text{ for } \ |k-j| > 4. \tag{2.9}$$

Indeed, by Fourier's transform, we have

$$\big(\mathcal{R}^a_j (S_{k-1} f \mathcal{R}^a_k g)\big)\hat{} = \phi^a_j \cdot \Big((\phi_0(2^{a(2-k)} \cdot) \hat{f}) * (\phi^a_k \hat{g}) \Big).$$

Since the support of $(\phi_0(2^{a(2-k)}\cdot)\hat{f}) * (\phi^a_k \hat{g})$ is contained in $B^a_{2^{k+2}} \setminus B^a_{2^{k-2}}$, if $|k-j| > 4$, then

$$\phi^a_j \cdot \Big((\phi_0(2^{a(2-k)} \cdot) \hat{f}) * (\phi^a_k \hat{g}) \Big) = 0.$$

The following result gives the equivalence between $\mathbf{B}^s_{a,\infty}$ and $\mathbf{C}^s_a$ (cf. [45], [15]).

Theorem 2.2. *For any $s > 0$, it holds that*

$$\|f\|_{\mathbf{B}^s_{a,\infty}} \asymp \|f\|_{\mathbf{C}^s_a} \asymp \|f\|_{\mathbf{C}^{s/a_1}_{x_1}} + \cdots + \|f\|_{\mathbf{C}^{s/a_n}_{x_n}}, \tag{2.10}$$

where $\|f\|_{\mathbf{C}^{s/a_i}_{x_i}} := \sup_{x_j\in\mathbb{R}^{m_j}, j\neq i} \|f(x_1, \cdots, x_{i-1}, \cdot, x_{i+1}, \cdots, x_n)\|_{\mathbf{C}^{s/a_i}}$. *By convention we denote*

$$\mathbf{C}^s_a := \mathbf{B}^s_{a,\infty}, \quad s < 0.$$

We have the following interpolation inequality.

Corollary 2.3. *For any $0 \le s < r < t$, there is a constant $C > 0$ such that for any $\varepsilon \in (0, 1)$,*

$$\|f\|_{\mathbf{C}^r_a} \le C\|f\|^{(t-r)/(t-s)}_{\mathbf{C}^s_a} \|f\|^{(r-s)/(t-s)}_{\mathbf{C}^t_a} \le \varepsilon\|f\|_{\mathbf{C}^t_a} + C\varepsilon^{(s-r)/(t-r)}\|f\|_{\mathbf{C}^s_a}. \tag{2.11}$$

Proof. By (2.10) and the definition of $\mathbf{B}^r_{a,\infty}$, we have

$$\begin{aligned}\|f\|_{\mathbf{C}^r_a} \lesssim \|f\|_{\mathbf{B}^r_{a,\infty}} = \sup_{j\ge0} 2^{rj}\|\mathcal{R}^a_j f\|_\infty &\le \Big(\sup_{j\ge0} 2^{sj}\|\mathcal{R}^a_j f\|_\infty\Big)^{(t-r)/(t-s)} \Big(\sup_{j\ge0} 2^{tj}\|\mathcal{R}^a_j f\|_\infty\Big)^{(r-s)/(t-s)} \\ &= \|f\|^{(t-r)/(t-s)}_{\mathbf{B}^s_{a,\infty}} \|f\|^{(r-s)/(t-s)}_{\mathbf{B}^t_{a,\infty}} \lesssim \|f\|^{(t-r)/(t-s)}_{\mathbf{C}^s_a} \|f\|^{(r-s)/(t-s)}_{\mathbf{C}^t_a}.\end{aligned}$$

The desired interpolation inequality follows. □

3. Schauder's estimates for heat equations

Let $d \ge 1$. In this section we present the basic idea of proving Schauder's estimate for heat equations by Littlewood-Paley's theory. Let $(a^{ij}(t))$ be a measurable symmetric $d \times d$-matrix-valued function on $\mathbb{R}_+$ and satisfy that for some $c_0 \ge 1$,

$$c_0^{-1}|\xi|^2 \le a^{ij}(t)\xi_i\xi_j \le c_0|\xi|^2, \quad \xi \in \mathbb{R}^d, \ t \ge 0. \tag{3.1}$$

Here and below we use the usual summation convention: if an index appears in a product, then it will be summed automatically. Define for $0 \le s < t < \infty$ and $x \in \mathbb{R}^d$,

$$p_{s,t}(x) := \frac{\mathrm{e}^{-\langle A^{-1}_{s,t}x,x\rangle/2}}{\sqrt{2\pi \det(A_{s,t})}} = \frac{\mathrm{e}^{-\langle \bar{A}^{-1}_{s,t}x,x\rangle/(2(t-s))}}{\sqrt{2\pi(t-s)^d \det(\bar{A}_{s,t})}}, \tag{3.2}$$

where

$$A_{s,t} := \int_s^t a(r)\mathrm{d}r = (t-s)\int_0^1 a(s + (t-s)r)\mathrm{d}r =: (t-s)\bar{A}_{s,t}.$$

The following lemma is the key observation for Schauder's estimate of heat equation.

Lemma 3.1. *Under (3.1), for any $\beta \ge 0$, there is a constant $C = C(c_0, \beta, d) > 0$ such that for all $t \ge 0$ and $j \in \mathbb{N}$,*

$$\int_0^t \left(\int_{\mathbb{R}^d} |x|^\beta |\mathcal{R}_j p_{s,t}(x)|\mathrm{d}x\right)\mathrm{d}s \le C2^{-2j-\beta j}. \tag{3.3}$$

Proof. We first show that for any $m \in \mathbb{N}_0$ and $\beta \geq 0$, there is a constant $C = C(c_0, \beta, m, d) > 0$ such that for all $0 \leq s < t < \infty$ and $j \in \mathbb{N}$,

$$\int_{\mathbb{R}^d} |x|^\beta |\mathcal{R}_j p_{s,t}(x)| \mathrm{d}x \leq C 2^{-2jm} (t-s)^{-m} \Big(2^{-j} + (t-s)^{1/2}\Big)^\beta. \tag{3.4}$$

Recalling (2.4) and by the change of variables, we have

$$\int_{\mathbb{R}^d} |x|^\beta |\mathcal{R}_j p_{s,t}(x)| \mathrm{d}x = 2^{-jd-j\beta} \int_{\mathbb{R}^d} |x|^\beta \left| \int_{\mathbb{R}^d} p_{s,t}(2^{-j}(x-y)) \check{\phi}_1(y) \mathrm{d}y \right| \mathrm{d}x.$$

Since the support of ϕ_1 is contained in the annulus, by Fourier's transform we have

$$\int_{\mathbb{R}^d} p_{s,t}(2^{-j}(x-y)) \check{\phi}_1(y) \mathrm{d}y = \int_{\mathbb{R}^d} \Delta^m p_{s,t}(2^{-j}(x-\cdot))(y) \cdot \Delta^{-m} \check{\phi}_1(y) \mathrm{d}y, \ m \in \mathbb{N}_0,$$

where $\Delta^{-m}\check{\phi}_1 := (|\xi|^{-2m}\phi_1(\xi))^\vee$. Moreover, by (3.2) and elementary calculations, we have

$$2^{-j\beta-jd} \int_{\mathbb{R}^d} |x|^\beta |\Delta^m p_{s,t}(2^{-j}\cdot)(x)| \mathrm{d}x \leq C 2^{-2jm} (t-s)^{\beta/2-m}.$$

Hence,

$$\begin{aligned} \int_{\mathbb{R}^d} |x|^\beta |\mathcal{R}_j p_{s,t}(x)| \mathrm{d}x &\lesssim 2^{-jd-j\beta} \int_{\mathbb{R}^d} |x|^\beta |\Delta^m p_{s,t}(2^{-j}x)| \mathrm{d}x \int_{\mathbb{R}^d} |\Delta^{-m} \check{\phi}_1^a(y)| \mathrm{d}y \\ &\quad + 2^{-jd-j\beta} \int_{\mathbb{R}^d} |\Delta^m p_{s,t}(2^{-j}x)| \mathrm{d}x \int_{\mathbb{R}^d} |y|^\beta |\Delta^{-m} \check{\phi}_1^a(y)| \mathrm{d}y \\ &\lesssim 2^{-2jm} (t-s)^{\beta/2-m} + 2^{-j\beta-2jm} (t-s)^{-m}, \end{aligned}$$

which in turn gives (3.4).

Let $\mathscr{I}$ be the left hand side of (3.3). We make the following decomposition:

$$\mathscr{I} = \left(\int_{t-t\wedge 2^{-2j}}^{t} + \int_0^{t-t\wedge 2^{-2j}} \right) \left(\int_{\mathbb{R}^d} |x|^\beta |\mathcal{R}_j p_{s,t}(x)| \mathrm{d}x \right) \mathrm{d}s =: \mathscr{I}_1 + \mathscr{I}_2.$$

For $\mathscr{I}_1$, by (3.4) with $m = 0$, we have

$$\mathscr{I}_1 \lesssim \int_{t-t\wedge 2^{-2j}}^{t} \Big(2^{-j} + (t-s)^{-1/2}\Big)^\beta \mathrm{d}s = \int_0^{t\wedge 2^{-2j}} \Big(2^{-j} + s^{1/2}\Big)^\beta \mathrm{d}s \lesssim 2^{-2j-\beta j}.$$

For $\mathscr{I}_2$, by (3.4) with $m = 2$, we have

$$\mathscr{I}_2 \lesssim \int_0^{t-t\wedge 2^{-2j}} 2^{-4j} (t-s)^{-2} \Big(2^{-j} \vee (t-s)^{-1/2}\Big)^\beta \mathrm{d}s = 2^{-4j} \int_{t\wedge 2^{-2j}}^{t} s^{-2-\beta/2} \mathrm{d}s \lesssim 2^{-2j-\beta j}.$$

Combining the above two estimates, we obtain (3.3). □

Now we consider the following heat equation with variable coefficients:

$$\partial_t u = a^{ij}\partial_i\partial_j u + f, \quad u(0) = 0, \tag{3.5}$$

where $a : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ is a measurable symmetric matrix-valued function and satisfies

($\mathbf{H}^\beta_a$) For some $c_0 \geq 1$ and $\beta \in (0, 1)$, it holds that for all $t \geq 0$ and $x, y, \xi \in \mathbb{R}^d$,

$$c_0^{-1}|\xi|^2 \leq a^{ij}(t,x)\xi_i\xi_j \leq c_0|\xi|^2, \quad |a(t,x) - a(t,y)| \leq c_0|x-y|^\beta.$$

Below we use Lemma 3.1 to establish Schauder's estimate for heat equation (3.5).

Theorem 3.2. *Let $\beta \in (0, 1)$. Under ($\mathbf{H}^\beta_a$), there is a constant $C = C(c_0, \beta, d) > 0$ such that for any $T > 0$, and $u \in \mathbb{L}^\infty_T(\mathbf{B}^{2+\beta}_\infty)$ with $\partial_t u \in \mathbb{L}^\infty_T(\mathbf{B}^\beta_\infty)$ solving PDE (3.5),*

$$\|u\|_{\mathbb{L}^\infty_T(\mathbf{C}^{2+\beta})} \leq C\Big(\|f\|_{\mathbb{L}^\infty_T(\mathbf{C}^\beta)} + \|u\|_{\mathbb{L}^\infty_T}\Big).$$

Proof. Fix $x_0 \in \mathbb{R}^d$ and define

$$u_{x_0}(t,x) := u(t, x + x_0), \quad \tilde{a}_{x_0}(t,x) := a(t, x + x_0) - a(t, x_0).$$

It is easy to see that

$$\partial_t u_{x_0} = a^{ij}(t, x_0)\partial_i\partial_j u_{x_0} + \tilde{a}^{ij}_{x_0}\partial_i\partial_j u_{x_0} + f_{x_0}, \quad u_{x_0}(0) = 0.$$

Let $p^{x_0}_{s,t}$ be defined by (3.2) in terms of $a(t, x_0)$. For a space-time function f, define

$$P^{x_0}_{s,t} f(s,x) := \int_{\mathbb{R}^d} p^{x_0}_{s,t}(x-y) f(s,y)\mathrm{d}y.$$

By Duhamel's formula we have

$$u_{x_0}(t,x) = \int_0^t P^{x_0}_{s,t}\mathrm{tr}(\tilde{a}_{x_0} \cdot \nabla^2 u_{x_0})(s,x)\mathrm{d}s + \int_0^t P^{x_0}_{s,t} f_{x_0}(s,x)\mathrm{d}s =: I_1(t,x) + I_2(t,x).$$

Below, without loss of generality, we assume $x_0 = 0$ and drop the subscript and superscript x_0. First of all, for $I_1(t,x)$, by ($\mathbf{H}^\beta_a$) and Lemma 3.1, we have

$$\begin{aligned}|\mathcal{R}_j I_1(t,0)| &\leq \int_0^t |\mathcal{R}_j P_{s,t}\mathrm{tr}(\tilde{a} \cdot \nabla^2 u)(s,0)|\mathrm{d}s \lesssim \int_0^t \left(\int_{\mathbb{R}^d} |x|^\beta |\mathcal{R}_j p_{s,t}(x)|\mathrm{d}x\right)\mathrm{d}s \|\nabla^2 u\|_{\mathbb{L}^\infty_T} \\ &\lesssim 2^{-2j-\beta j}\|\nabla^2 u\|_{\mathbb{L}^\infty_T} \leq \varepsilon 2^{-2j-\beta j}\|u\|_{\mathbb{L}^\infty_T(\mathbf{B}^{2+\beta}_\infty)} + 2^{-2j-\beta j}\|u\|_{\mathbb{L}^\infty_T},\end{aligned}$$

where $\varepsilon > 0$ and the last inequality is due to the interpolation and Young's inequalities. For $I_2(t,x)$, by (2.5) and Lemma 3.1 again, we have

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$$
\begin{aligned}
|\mathcal{R}_j I_2(t,0)| &\le \int_0^t |\mathcal{R}_j P^{x_0}_{s,t} f(s,0)|\mathrm{d}s = \int_0^t \left| \int_{\mathbb{R}^d} \mathcal{R}_j p^{x_0}_{s,t}(y) f(s,y)\mathrm{d}y \right| \mathrm{d}s \\
&= \int_0^t \left| \int_{\mathbb{R}^d} \mathcal{R}_j \widetilde{\mathcal{R}}_j p^{x_0}_{s,t}(y) f(s,y)\mathrm{d}y \right| \mathrm{d}s = \int_0^t \left| \int_{\mathbb{R}^d} \widetilde{\mathcal{R}}_j p^{x_0}_{s,t}(y) \mathcal{R}_j f(s,y)\mathrm{d}y \right| \mathrm{d}s \\
&\le \int_0^t \left(\int_{\mathbb{R}^d} |\widetilde{\mathcal{R}}_j p_{s,t}(x)|\mathrm{d}x \right) \mathrm{d}s \|\mathcal{R}_j f\|_{\mathbb{L}^\infty_T} \lesssim 2^{-2j-\beta j} \|f\|_{\mathbb{L}^\infty_T(\mathbf{B}^\beta_\infty)}.
\end{aligned}
$$

Combining the above estimates, we obtain that for any $\varepsilon \in (0,1)$ and $j \in \mathbb{N}$,

$$
2^{j(2+\beta)}|\mathcal{R}_j u(t,x_0)| = 2^{j(2+\beta)}|\mathcal{R}_j u_{x_0}(t,0)| \lesssim \varepsilon \|u\|_{\mathbb{L}^\infty_T(\mathbf{B}^{2+\beta}_\infty)} + \|u\|_{\mathbb{L}^\infty_T} + \|f\|_{\mathbb{L}^\infty_T(\mathbf{B}^\beta_\infty)}. \tag{3.6}
$$

Moreover, for $j = 0$, it is easy to see that

$$
|\mathcal{R}_j u(t,x_0)| \le \|u\|_{\mathbb{L}^\infty_T}.
$$

Thus by the definition of Besov space, we arrive at

$$
\|u\|_{\mathbb{L}^\infty_T(\mathbf{B}^{2+\beta}_\infty)} = \sup_{t\in[0,T]} \sup_{j\in\mathbb{N}_0} 2^{j(2+\beta)} \|\mathcal{R}_j u(t,\cdot)\|_\infty \le \varepsilon \|u\|_{\mathbb{L}^\infty_T(\mathbf{B}^{2+\beta}_\infty)} + C_\varepsilon \|u\|_{\mathbb{L}^\infty_T} + C\|f\|_{\mathbb{L}^\infty_T(\mathbf{B}^\beta_\infty)},
$$

which gives the desired estimate by choosing $\varepsilon = 1/2$ and Theorem 2.2. □

4. Commutator estimates

In the sequel, we shall only consider the following case of anisotropic Besov spaces:

$$
N = 2d,\ n = 2,\ m_1 = m_2 = d,\quad a = (1+\alpha, 1),\quad \text{where}\quad \alpha \in (0,2).
$$

For $h \in \mathbb{R}^d$ and $f(x,\mathrm{v}) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, we introduce

$$
\begin{aligned}
\delta_{h;1} f(x,\mathrm{v}) &:= \delta_h f(\cdot,\mathrm{v})(x), \quad \delta_{h;2} f(x,\mathrm{v}) := \delta_h f(x,\cdot)(\mathrm{v}), \\
\mathcal{R}^x_j f(x,\mathrm{v}) &:= \mathcal{R}_j f(\cdot,\mathrm{v})(x), \quad \mathcal{R}^{\mathrm{v}}_j f(x,\mathrm{v}) := \mathcal{R}_j f(x,\cdot)(\mathrm{v}),
\end{aligned}
$$

and for $\beta > 0$,

$$
\begin{aligned}
\|f\|_{\mathbf{C}^\beta_x} &:= \sup_{\mathrm{v}\in\mathbb{R}^d} \|f(\cdot,\mathrm{v})\|_{\mathbf{C}^\beta}, \quad \|f\|_{\mathbf{C}^\beta_{\mathrm{v}}} := \sup_{x\in\mathbb{R}^d} \|f(x,\cdot)\|_{\mathbf{C}^\beta}, \\
\|f\|_{\mathbf{B}^\beta_{x,\infty}} &:= \sup_{\mathrm{v}\in\mathbb{R}^d} \|f(\cdot,\mathrm{v})\|_{\mathbf{B}^\beta_\infty}, \quad \|f\|_{\mathbf{B}^\beta_{\mathrm{v},\infty}} := \sup_{x\in\mathbb{R}^d} \|f(x,\cdot)\|_{\mathbf{B}^\beta_\infty}.
\end{aligned}
$$

Moreover, for $\beta \in (0,1)$, we introduce the following semi-norm for later use:

$$
[\![f]\!]_{\mathbf{C}^{1+\beta}_a} := [f]_{\mathbf{C}^{(1+\beta)/(1+\alpha)}_x} + \|\nabla_{\mathrm{v}} f\|_{\mathbf{C}^\beta_{\mathrm{v}}}. \tag{4.1}
$$

For $\gamma, \beta \ge 0$, we define the mixed Hölder space

$$
\mathbf{C}^\gamma_x \mathbf{C}^\beta_{\mathrm{v}} := \left\{ f(x,\mathrm{v}) : \|f\|_{\mathbf{C}^\gamma_x \mathbf{C}^\beta_{\mathrm{v}}} := \|f\|_{\mathbf{C}^\gamma_x} + \|f\|_{\mathbf{C}^\beta_{\mathrm{v}}} + \sup_{h,h'} \|\delta^m_{h;1} \delta^n_{h';2} f\|_\infty / (|h|^\gamma |h'|^\beta) < \infty \right\},
$$

where $m \geq [\gamma] + 1$ and $n \geq [\beta] + 1$, and for $\gamma \in \mathbb{R}$ and $\beta \geq 0$,

$$\mathbf{B}^{\gamma}_{x,\infty}\mathbf{C}^{\beta}_{\mathrm{v}} := \left\{ f(x, \mathrm{v}) : \|f\|_{\mathbf{B}^{\gamma}_{x,\infty}\mathbf{C}^{\beta}_{\mathrm{v}}} := \sup_{j\in\mathbb{N}_0} 2^{\gamma j}\|\mathcal{R}^x_j f\|_{\mathbf{C}^{\beta}_{\mathrm{v}}} < \infty \right\}.$$

Notice that the definition of $\mathbf{C}^{\gamma}_x\mathbf{C}^{\beta}_{\mathrm{v}}$ does not depend on the choices of m and n (see [45]). In particular, by Theorem 2.2, we have for $\gamma \in \mathbb{R}$ and $\beta > 0$,

$$\sup_{j,\ell} 2^{\gamma j/(1+\alpha)}2^{\beta\ell}\|\mathcal{R}^x_j\mathcal{R}^a_\ell f\|_\infty \asymp \sup_j 2^{\gamma j/(1+\alpha)}\|\mathcal{R}^x_j f\|_{\mathbf{C}^{\beta}_a} \asymp \|f\|_{\mathbf{B}^{(\gamma+\beta)/(1+\alpha)}_{x,\infty}} + \|f\|_{\mathbf{B}^{\gamma/(1+\alpha)}_{x,\infty}\mathbf{C}^{\beta}_{\mathrm{v}}}, \tag{4.2}$$

and for $\gamma > 0$ and $\beta \geq 0$,

$$\mathbf{B}^{\gamma}_{x,\infty}\mathbf{C}^{\beta}_{\mathrm{v}} \asymp \mathbf{C}^{\gamma}_x\mathbf{C}^{\beta}_{\mathrm{v}}.$$

We list some easy properties for later use.

Lemma 4.1.

(i) *Let $\beta_1, \beta_2, \gamma_1, \gamma_2 \geq 0$, $\theta \in [0, 1]$ and*

$$\beta := \theta\beta_1 + (1-\theta)\beta_2, \quad \gamma := \theta\gamma_1 + (1-\theta)\gamma_2.$$

Then it holds that for some $C = C(d, \theta, \gamma_1, \gamma_2, \beta_1, \beta_2) > 0$,

$$\|f\|_{\mathbf{C}^{\gamma}_x\mathbf{C}^{\beta}_{\mathrm{v}}} \leq C\|f\|^{\theta}_{\mathbf{C}^{\gamma_1}_x\mathbf{C}^{\beta_1}_{\mathrm{v}}}\|f\|^{1-\theta}_{\mathbf{C}^{\gamma_2}_x\mathbf{C}^{\beta_2}_{\mathrm{v}}}. \tag{4.3}$$

(ii) *For all $j \in \mathbb{N}_0$, it holds that for some $C = C(d, \alpha) > 0$,*

$$\|\nabla_x\mathcal{R}^a_j f\|_\infty \leq C2^{(1+\alpha)j}\|\mathcal{R}^a_j f\|_\infty, \quad \|\nabla_{\mathrm{v}}\mathcal{R}^a_j f\|_\infty \leq C2^j\|\mathcal{R}^a_j f\|_\infty. \tag{4.4}$$

(iii) *For any $\beta \in (0, 2)$, it holds that for some $C = C(d, \alpha, \beta) > 0$,*

$$\|\mathcal{R}^a_j f\|_\infty \leq C2^{-\beta j}[f]_{\mathbf{C}^{\beta}_a}, \ j \geq 1. \tag{4.5}$$

(iv) *For any $\beta \in (0, 1 \wedge \alpha)$, it holds that for some $C = C(d, \alpha, \beta) > 0$,*

$$\|\nabla_{\mathrm{v}} f\|_{\mathbf{C}^{\beta}_a} \leq C[\![f]\!]_{\mathbf{C}^{1+\beta}_a}, \tag{4.6}$$

where $[\![f]\!]_{\mathbf{C}^{1+\beta}_a}$ is defined by (4.1).

Proof. (i) Let $m \geq [\gamma_1] + [\gamma_2] + 2$ and $n \geq [\beta_1] + [\beta_2] + 2$. The desired interpolation inequality follows by

$$\begin{aligned}\sup_{h,h'} \|\delta^m_{h;1}\delta^n_{h';2}f\|_\infty/(|h|^\gamma|h'|^\beta) &= \sup_{h,h'} \|\delta^m_{h;1}\delta^n_{h';2}f\|_\infty/(|h|^{\theta\gamma_1+(1-\theta)\gamma_2}|h'|^{\theta\beta_1+(1-\theta)\beta_2}) \\ &= \sup_{h,h'} \Big(\|\delta^m_{h;1}\delta^n_{h';2}f\|_\infty/(|h|^{\gamma_1}|h'|^{\beta_1})\Big)^\theta \Big(\|\delta^m_{h;1}\delta^n_{h';2}f\|_\infty/(|h|^{\gamma_2}|h'|^{\beta_2})\Big)^{1-\theta}.\end{aligned}$$

(ii) It is a direct consequence of definition (2.4).

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(iii) Noticing that for $j \geq 1$,

$$\mathcal{R}_j^a f(x) = \int_{\mathbb{R}^d} \check{\phi}_j^a(h) f(x+h) \mathrm{d}h \quad \text{and} \quad \int_{\mathbb{R}^d} \check{\phi}_j^a(h) \mathrm{d}h = \phi_j^a(0) = 0,$$

by (2.2) and $\check{\phi}_j^a(-h) = \check{\phi}_j^a(h)$, we have

$$\mathcal{R}_j^a f(x) = \frac{1}{2} \int_{\mathbb{R}^d} \check{\phi}_j^a(h) \delta_h^* \delta_h f(x) \mathrm{d}h.$$

Hence,

$$\|\mathcal{R}_j^a f\|_\infty \leq \frac{1}{2} \sup_h \|\delta_h^* \delta_h f\|_\infty / |h|_a^\beta \int_{\mathbb{R}^d} \check{\phi}_j^a(h) |h|_a^\beta \mathrm{d}h \lesssim 2^{-\beta j} [f]_{\mathbf{C}_a^\beta}.$$

(iv) By Theorem 2.2 and definition, we have

$$\begin{aligned} \|\nabla_{\mathrm{v}} f\|_{\mathbf{C}_a^\beta} &\lesssim \|\nabla_{\mathrm{v}} f\|_\infty + \sup_{j \in \mathbb{N}} 2^{\beta j} \|\mathcal{R}_j^a \nabla_{\mathrm{v}} f\|_\infty \overset{(4.4)}{\lesssim} \|\nabla_{\mathrm{v}} f\|_\infty + \sup_{j \in \mathbb{N}} 2^{(1+\beta)j} \|\mathcal{R}_j^a f\|_\infty \\ &\overset{(4.5)}{\lesssim} \|\nabla_{\mathrm{v}} f\|_\infty + [f]_{\mathbf{C}_a^{1+\beta}} \leq \|\nabla_{\mathrm{v}} f\|_\infty + [f]_{\mathbf{C}_x^{(1+\beta)/(1+\alpha)}} + [f]_{\mathbf{C}_{\mathrm{v}}^{1+\beta}} \lesssim [\![f]\!]_{\mathbf{C}_a^{1+\beta}}. \end{aligned}$$

The proof is complete. □

We now show several commutator estimates, which are extensions of [13, Lemma 2.3], and will play a key role in showing the Schauder estimate below.

Lemma 4.2. *(i) For any $\beta \in (0, \alpha \wedge 1)$ and $\gamma \in (-1-\beta, 0]$, there is a constant $C = C(d, \beta, \gamma) > 0$ such that for all $x, \mathrm{v} \in \mathbb{R}^d$ and $j \geq 5$,*

$$|[\mathcal{R}_j^a, \tilde{f}]g|(x, \mathrm{v}) \leq C 2^{-j(\gamma+1)} \Big(2^{-j\beta} + |x|^{\frac{\beta}{1+\alpha}} + |\mathrm{v}|^\beta \Big) [\![f]\!]_{\mathbf{C}_a^{1+\beta}} \|g\|_{\mathbf{C}_a^\gamma}, \tag{4.7}$$

where $[\![f]\!]_{\mathbf{C}_a^{1+\beta}}$ is defined by (4.1), and

$$\tilde{f}(x, \mathrm{v}) := f(x, \mathrm{v}) - f(0,0) - \mathrm{v} \cdot \nabla_{\mathrm{v}} f(0,0).$$

(ii) For any $\beta \in (0,1)$ and $\gamma \in (-\beta, 0]$, there is a constant $C = C(d, \beta, \gamma) > 0$ such that

$$\|[\mathcal{R}_j^x, f]g\|_\infty \leq C 2^{-j(\beta+\gamma)} [f]_{\mathbf{C}_x^\beta} \|g\|_{\mathbf{C}_x^\gamma}, \ j \geq 5. \tag{4.8}$$

Proof. We only prove (i) since (ii) is the same (see [13, Lemma 2.3]). We divide the proof into three steps.

(*Step 1.*) We first prove (4.7) for $\gamma = 0$. By definition we have

$$\begin{aligned} |\tilde{f}(\bar{x}, \bar{\mathrm{v}}) - \tilde{f}(x, \mathrm{v})| &= |f(\bar{x}, \bar{\mathrm{v}}) - f(x, \mathrm{v}) + (\mathrm{v} - \bar{\mathrm{v}}) \cdot \nabla_{\mathrm{v}} f(0,0)| \\ &\leq |x - \bar{x}|^{\frac{1+\beta}{1+\alpha}} [f]_{\mathbf{C}_x^{(1+\beta)/(1+\alpha)}} + |\mathrm{v} - \bar{\mathrm{v}}| \big(|x|^{\frac{\beta}{1+\alpha}} + |\mathrm{v}|^\beta + |\bar{\mathrm{v}}|^\beta \big) [\nabla_{\mathrm{v}} f]_{\mathbf{C}_a^\beta} \\ &\lesssim \Big(|x - \bar{x}|^{\frac{1+\beta}{1+\alpha}} + |\mathrm{v} - \bar{\mathrm{v}}| \big(|x|^{\frac{\beta}{1+\alpha}} + |\mathrm{v}|^\beta + |\bar{\mathrm{v}}|^\beta \big) \Big) [\![f]\!]_{\mathbf{C}_a^{1+\beta}}, \end{aligned} \tag{4.9}$$

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where the last step is due to (4.6). Notice that

$$[\mathcal{R}_j^a, \tilde{f}]g(x,\mathrm{v}) = \int_{\mathbb{R}^{2d}} \phi_j^a(x-\bar{x},\mathrm{v}-\bar{\mathrm{v}})\Big(f(\bar{x},\bar{\mathrm{v}}) - f(x,\mathrm{v})\Big)g(\bar{x},\bar{\mathrm{v}})\mathrm{d}\bar{x}\mathrm{d}\bar{\mathrm{v}}.$$

The estimate (4.7) for $\gamma = 0$ follows by (4.9).

(*Step 2.*) Next we consider the case $\gamma \in (-1-\beta, 0)$. By Bony's decomposition (2.8), we can write

$$[\mathcal{R}_j^a, \tilde{f}]g = [\mathcal{R}_j^a, T_{\tilde{f}}]g + \mathcal{R}_j^a(T_g\tilde{f}) - T_{\mathcal{R}_j^a g}\tilde{f} + \mathcal{R}_j^a R(\tilde{f}, g) - R(\tilde{f}, \mathcal{R}_j^a g). \tag{4.10}$$

We first treat $[\mathcal{R}_j^a, T_{\tilde{f}}]g$ that only contains the low frequency part of $\tilde{f}$. By (2.7) and (2.9), we have

$$[\mathcal{R}_j^a, T_{\tilde{f}}]g = \sum_{|k-j|\le 4}\Big(\mathcal{R}_j^a(S_{k-1}\tilde{f}\mathcal{R}_k^a g) - S_{k-1}\tilde{f}\mathcal{R}_j^a\mathcal{R}_k^a g\Big) = \sum_{|k-j|\le 4}[\mathcal{R}_j^a, S_{k-1}\tilde{f}]\mathcal{R}_k^a g. \tag{4.11}$$

Noting that by the definition (2.6) of S_{k-1},

$$S_{k-1}\tilde{f}(x,\mathrm{v}) = 2^{(2+\alpha)kd}\int_{\mathbb{R}^{2d}} \check{\phi}_0^a(2^{(1+\alpha)k}x', 2^k\mathrm{v}')\tilde{f}(x-x',\mathrm{v}-\mathrm{v}')\mathrm{d}x'\mathrm{d}\mathrm{v}',$$

using (4.9) and $\check{\phi}_0^a \in \mathscr{S}(\mathbb{R}^{2d})$, we have

$$\begin{aligned} &|S_{k-1}\tilde{f}(\bar{x},\bar{\mathrm{v}}) - S_{k-1}\tilde{f}(x,\mathrm{v})| \\ &\lesssim \Big(|x-\bar{x}|^{\frac{1+\beta}{1+\alpha}} + |\mathrm{v}-\bar{\mathrm{v}}|\big(2^{-k\beta} + |x|^{\frac{\beta}{1+\alpha}} + |\mathrm{v}|^\beta + |\bar{\mathrm{v}}|^\beta\big)\Big)[\![f]\!]_{\mathbf{C}_a^{1+\beta}}. \end{aligned} \tag{4.12}$$

On the other hand, noting that

$$[\mathcal{R}_j^a, S_{k-1}\tilde{f}]\mathcal{R}_k^a g(x,\mathrm{v}) = \int_{\mathbb{R}^{2d}} \check{\phi}_j^a(x-\bar{x},\mathrm{v}-\bar{\mathrm{v}})\Big(S_{k-1}\tilde{f}(\bar{x},\bar{\mathrm{v}}) - S_{k-1}\tilde{f}(x,\mathrm{v})\Big)\mathcal{R}_k^a g(\bar{x},\bar{\mathrm{v}})\mathrm{d}\bar{x}\mathrm{d}\bar{\mathrm{v}},$$

we have by (4.12) and (2.4),

$$|[\mathcal{R}_j^a, S_{k-1}\tilde{f}]\mathcal{R}_k^a g|(x,\mathrm{v}) \lesssim \Big(2^{-j(1+\beta)} + 2^{-j}\big(2^{-k\beta} + |x|^{\frac{\beta}{1+\alpha}} + |\mathrm{v}|^\beta\big)\Big)[\![f]\!]_{\mathbf{C}_a^{1+\beta}}\|\mathcal{R}_k^a g\|_\infty. \tag{4.13}$$

Thus, by (4.11) and (4.13), we get

$$\begin{aligned} |[\mathcal{R}_j^a, T_{\tilde{f}}]g|(x,\mathrm{v}) &\lesssim \sum_{|k-j|\le 4}\Big(2^{-j(1+\beta)} + 2^{-j}\big(2^{-k\beta} + |x|^{\frac{\beta}{1+\alpha}} + |\mathrm{v}|^\beta\big)\Big)[\![f]\!]_{\mathbf{C}_a^{1+\beta}}\|\mathcal{R}_k^a g\|_\infty \\ &\lesssim 2^{-j\gamma-j}\Big(2^{-j\beta} + |x|^{\frac{\beta}{1+\alpha}} + |\mathrm{v}|^\beta\Big)[\![f]\!]_{\mathbf{C}_a^{1+\beta}}\|g\|_{\mathbf{B}^\gamma_{a,\infty}}. \end{aligned}$$

(*Step 3.*) In this step we treat the remaining terms in (4.10). By (2.7) and (2.9), we have

$$\begin{aligned} \|\mathcal{R}_j^a(T_g\tilde{f})\|_\infty &= \Big\|\sum_{|k-j|\le 4}\mathcal{R}_j^a(S_{k-1}g\mathcal{R}_k^a\tilde{f})\Big\|_\infty \le \sum_{|k-j|\le 4}\|\mathcal{R}_j^a(S_{k-1}g\mathcal{R}_k^a\tilde{f})\|_\infty \\ &\le \sum_{|k-j|\le 4}\|S_{k-1}g\mathcal{R}_k^a\tilde{f}\|_\infty \lesssim \sum_{|k-j|\le 4}\sum_{m\le k-2}\|\mathcal{R}_m^a g\|_\infty\|\mathcal{R}_k^a\tilde{f}\|_\infty. \end{aligned}$$

Since $j \geq 5$ and $|k-j| \leq 4$, by (4.5), we further have

$$\|\mathcal{R}^a_j(T_g\tilde{f})\|_\infty \lesssim \|g\|_{\mathbf{B}^\gamma_{a,\infty}}[\tilde{f}]_{\mathbf{C}^{1+\beta}_a} \sum_{|k-j|\leq 4}\sum_{m\leq k-2} 2^{-m\gamma}2^{-k(1+\beta)} \lesssim \|g\|_{\mathbf{B}^\gamma_{a,\infty}}[f]_{\mathbf{C}^{1+\beta}_a}2^{-j(\gamma+1+\beta)},$$

where the last step is due to $\gamma < 0$ and $[\tilde{f}]_{\mathbf{C}^{1+\beta}_a} = [f]_{\mathbf{C}^{1+\beta}_a}$. Similarly,

$$\begin{aligned}\|T_{\mathcal{R}^a_j g}\tilde{f}\|_\infty &\leq \sum_{k\geq j-2}\|S_{k-1}\mathcal{R}^a_j g\mathcal{R}^a_k\tilde{f}\|_\infty \leq \sum_{k\geq j-2}\|S_{k-1}\mathcal{R}^a_j g\|_\infty\|\mathcal{R}^a_k\tilde{f}\|_\infty \\ &\leq \sum_{k\geq j-2} 2^{-k(1+\beta)}[\tilde{f}]_{\mathbf{C}^{1+\beta}_a}\|\mathcal{R}^a_j g\|_\infty \lesssim 2^{-j(\beta+1+\gamma)}[f]_{\mathbf{C}^{1+\beta}_a}\|g\|_{\mathbf{B}^\gamma_{a,\infty}}.\end{aligned}$$

Finally, since $1+\beta+\gamma > 0$, we have

$$\begin{aligned}\|\mathcal{R}^a_j R(\tilde{f},g)\|_\infty &\overset{(2.9)}{=} \Big\|\sum_{|i|\leq 1, k\geq j-4}\mathcal{R}^a_j(\mathcal{R}^a_k\tilde{f}\mathcal{R}^a_{k-i}g)\Big\|_\infty \lesssim \sum_{|i|\leq 1,k\geq j-4}\|\mathcal{R}^a_k\tilde{f}\|_\infty\|\mathcal{R}^a_{k-i}g\|_\infty \\ &\lesssim \sum_{k\geq j-4} 2^{-k(1+\beta+\gamma)}[\tilde{f}]_{\mathbf{C}^{1+\beta}_a}\|g\|_{\mathbf{B}^\gamma_{a,\infty}} \lesssim 2^{-j(1+\beta+\gamma)}[f]_{\mathbf{C}^{1+\beta}_a}\|g\|_{\mathbf{B}^\gamma_{a,\infty}},\end{aligned}$$

and

$$\|R(\tilde{f},\mathcal{R}^a_j g)\|_\infty = \Big\|\sum_{|i|\leq 1,|k-j|\leq 1}\mathcal{R}^a_{k-i}\tilde{f}\mathcal{R}^a_k\mathcal{R}^a_j g\Big\|_\infty \lesssim [f]_{\mathbf{C}^{1+\beta}_a}\|g\|_{\mathbf{B}^\gamma_{a,\infty}}2^{-j(1+\beta+\gamma)}.$$

Combining the above calculations, we complete the proof. □

The following lemma extends (4.8) to more general case.

Lemma 4.3. *For any $0 \leq \beta \leq \gamma < 1$ and $\eta \in (-\gamma, 0]$, there is a constant $C = C(\beta,\gamma,\eta) > 0$ such that*

$$\|[\mathcal{R}^x_j, f]g\|_{\mathbf{C}^\beta_x} \leq C2^{(\beta-\gamma-\eta)j}[f]_{\mathbf{C}^\gamma_x}\|g\|_{\mathbf{C}^\eta_x}, \quad j \geq 5. \tag{4.14}$$

Proof. If $\ell \leq j+1$, then by (4.8),

$$\|\mathcal{R}^x_\ell[\mathcal{R}^x_j, f]g\|_\infty \leq \|[\mathcal{R}^x_j, f]g\|_\infty \lesssim 2^{-(\gamma+\eta)j}[f]_{\mathbf{C}^\gamma_x}\|g\|_{\mathbf{C}^\eta_x} \lesssim 2^{-\beta\ell}2^{(\beta-\gamma-\eta)j}[f]_{\mathbf{C}^\gamma_x}\|g\|_{\mathbf{C}^\eta_x}.$$

If $\ell > j+1$, since $\widehat{\mathcal{R}^x_\ell\mathcal{R}^x_j f} = \phi_\ell\phi_j\hat{f} \equiv 0$, we have

$$\mathcal{R}^x_\ell[\mathcal{R}^x_j, f]g = \mathcal{R}^x_\ell\mathcal{R}^x_j(fg) - \mathcal{R}^x_\ell(f\mathcal{R}^x_j g) = f\mathcal{R}^x_\ell\mathcal{R}^x_j g - \mathcal{R}^x_\ell(f\mathcal{R}^x_j g) = -[\mathcal{R}^x_\ell, f]\mathcal{R}^x_j g.$$

Thus by (4.8) again and $\beta \leq \gamma$, we have for $\ell > j+1$,

$$\|\mathcal{R}^x_\ell[\mathcal{R}^x_j, f]g\|_\infty = \|[\mathcal{R}^x_\ell, f]\mathcal{R}^x_j g\|_\infty \lesssim 2^{-\gamma\ell}[f]_{\mathbf{C}^\gamma_x}\|\mathcal{R}^x_j g\|_\infty \lesssim 2^{-\beta\ell}2^{(\beta-\gamma-\eta)j}[f]_{\mathbf{C}^\gamma_x}\|g\|_{\mathbf{C}^\eta_x}.$$

Hence,

$$\|[\mathcal{R}^x_j, f]g\|_{\mathbf{C}^\beta_x} \lesssim \sup_{\ell\in\mathbb{N}_0} 2^{\beta\ell}\|\mathcal{R}^x_\ell[\mathcal{R}^x_j, f]g\|_\infty \lesssim 2^{(\beta-\gamma-\eta)j}[f]_{\mathbf{C}^\gamma_x}\|g\|_{\mathbf{C}^\eta_x}.$$

The proof is complete. □

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Lemma 4.4. *Let $\beta, \gamma_2 \in (0,1]$, $\theta \in (0,1)$ and $\gamma, \gamma_1 \in (0, 1+\alpha)$. Under the conditions*

$$\gamma \vee \gamma_2 \le \gamma_1, \quad \theta\gamma_2 \le \gamma + \beta \le (1-\theta)\gamma_1 + \theta\gamma_2, \quad \beta \le \theta\gamma_2, \tag{4.15}$$

there is a constant $C > 0$ such that for all $j \ge 5$,

$$\|[\mathcal{R}^x_j, f]g\|_{\mathbf{C}^\beta_a} \le 2^{-\frac{\gamma}{1+\alpha}j}\Big([f]_{\mathbf{C}^{\gamma_1/(1+\alpha)}_x} + [f]_{\mathbf{C}^{\gamma_2}_{\mathrm{v}}}\Big)\Big(\|g\|_{\mathbf{C}^{(\gamma+\beta-(1-\theta)\gamma_1-\theta\gamma_2)/(1+\alpha)}_x} + \|g\|_{\mathbf{C}^{(\gamma-\gamma_1)/(1+\alpha)}_x \mathbf{C}^\beta_{\mathrm{v}}}\Big). \tag{4.16}$$

Proof. First of all, by applying (4.14) with $(\frac{\beta}{1+\alpha}, \frac{\gamma_1}{1+\alpha}, \frac{\beta+\gamma-\gamma_1}{1+\alpha})$ in place of (β, γ, η), we have

$$\|[\mathcal{R}^x_j, f]g\|_{\mathbf{C}^{\beta/(1+\alpha)}_x} \lesssim 2^{-\frac{\gamma}{1+\alpha}j}[f]_{\mathbf{C}^{\gamma_1/(1+\alpha)}_x}\|g\|_{\mathbf{C}^{(\beta+\gamma-\gamma_1)/(1+\alpha)}_x}.$$

Thus, by definition it suffices to prove

$$\|[\mathcal{R}^x_j, f]g\|_{\mathbf{C}^\beta_{\mathrm{v}}} \lesssim \sup_{\ell\in\mathbb{N}_0} 2^{\ell\beta}\|\mathcal{R}^{\mathrm{v}}_\ell[\mathcal{R}^x_j, f]g\|_\infty \lesssim \text{RHS of (4.16)}. \tag{4.17}$$

(Case: $\ell \le \frac{j}{1+\alpha}$). Since $\gamma_2 < \gamma_1$ and $\gamma + \beta < (1-\theta)\gamma_1 + \theta\gamma_2 \le \gamma_1$, by (4.8), we have

$$\begin{aligned}\|\mathcal{R}^{\mathrm{v}}_\ell[\mathcal{R}^x_j, f]g\|_\infty &\lesssim \|[\mathcal{R}^x_j, f]g\|_\infty \lesssim 2^{-\frac{\gamma+\beta}{1+\alpha}j}[f]_{\mathbf{C}^{\gamma_1/(1+\alpha)}_x}\|g\|_{\mathbf{C}^{(\gamma+\beta-\gamma_1)/(1+\alpha)}_x} \\ &\lesssim 2^{-\ell\beta}2^{-\frac{\gamma}{1+\alpha}j}[f]_{\mathbf{C}^{\gamma_1/(1+\alpha)}_x}\|g\|_{\mathbf{C}^{(\gamma+\beta-(1-\theta)\gamma_1-\theta\gamma_2)/(1+\alpha)}_x}.\end{aligned}$$

(Case: $\ell > \frac{j}{1+\alpha}$). Notice that

$$\mathcal{R}^{\mathrm{v}}_\ell[\mathcal{R}^x_j, f]g = [\mathcal{R}^x_j, f]\mathcal{R}^{\mathrm{v}}_\ell g + [\mathcal{R}^{\mathrm{v}}_\ell, [\mathcal{R}^x_j, f]]g =: I_1 + I_2.$$

For I_1, since $\gamma < \gamma_1$, by (4.8), we have

$$|I_1| \lesssim 2^{-\frac{\gamma}{1+\alpha}j}[f]_{\mathbf{C}^{\gamma_1/(1+\alpha)}_x}\|\mathcal{R}^{\mathrm{v}}_\ell g\|_{\mathbf{C}^{(\gamma-\gamma_1)/(1+\alpha)}_x} \lesssim 2^{-\frac{\gamma}{1+\alpha}j}2^{-\beta\ell}[f]_{\mathbf{C}^{\gamma_1/(1+\alpha)}_x}\|g\|_{\mathbf{C}^{(\gamma-\gamma_1)/(1+\alpha)}_x\mathbf{C}^\beta_{\mathrm{v}}}.$$

For I_2, by definition, (4.15) and (4.8), we have

$$\begin{aligned}|I_2| &= \left|\int_{\mathbb{R}^d} \check\phi_\ell(\bar{\mathrm{v}})[\mathcal{R}^x_j, \delta_{\bar{\mathrm{v}};2} f(\cdot, \mathrm{v})]g(\cdot, \mathrm{v}-\bar{\mathrm{v}})\mathrm{d}\bar{\mathrm{v}}\right| \\ &\lesssim 2^{-\frac{\gamma+\beta-\theta\gamma_2}{1+\alpha}j}\int_{\mathbb{R}^d} \check\phi_\ell(\bar{\mathrm{v}})[\delta_{\bar{\mathrm{v}};2} f]_{\mathbf{C}^{(1-\theta)\gamma_1/(1+\alpha)}_x}\|g\|_{\mathbf{C}^{(\gamma+\beta-(1-\theta)\gamma_1-\theta\gamma_2)/(1+\alpha)}_x}\mathrm{d}\bar{\mathrm{v}} \\ &\lesssim 2^{-\frac{\gamma+\beta-\theta\gamma_2}{1+\alpha}j}\left(\int_{\mathbb{R}^d} \check\phi_\ell(\bar{\mathrm{v}})|\bar{\mathrm{v}}|^{\theta\gamma_2}\mathrm{d}\bar{\mathrm{v}}\right)[f]_{\mathbf{C}^{(1-\theta)\gamma_1/(1+\alpha)}_x\mathbf{C}^{\theta\gamma_2}_{\mathrm{v}}}\|g\|_{\mathbf{C}^{(\gamma+\beta-(1-\theta)\gamma_1-\theta\gamma_2)/(1+\alpha)}_x} \\ &\lesssim 2^{-\frac{\gamma+\beta-\theta\gamma_2}{1+\alpha}j}2^{-\theta\gamma_2\ell}[f]_{\mathbf{C}^{(1-\theta)\gamma_1/(1+\alpha)}_x\mathbf{C}^{\theta\gamma_2}_{\mathrm{v}}}\|g\|_{\mathbf{C}^{(\gamma+\beta-(1-\theta)\gamma_1-\theta\gamma_2)/(1+\alpha)}_x} \\ &\lesssim 2^{-\frac{\gamma}{1+\alpha}j}2^{-\beta\ell}\Big([f]_{\mathbf{C}^{\gamma_1/(1+\alpha)}_x} + [f]_{\mathbf{C}^{\gamma_2}_{\mathrm{v}}}\Big)\|g\|_{\mathbf{C}^{(\gamma+\beta-(1-\theta)\gamma_1-\theta\gamma_2)/(1+\alpha)}_x}.\end{aligned}$$

Hence, for all $\ell \in \mathbb{N}_0$ and $j \ge 5$,

$$\|\mathcal{R}^{\mathrm{v}}_\ell[\mathcal{R}^x_j, f]g\|_\infty \lesssim 2^{-\frac{\gamma}{1+\alpha}j}2^{-\beta\ell}\Big([f]_{\mathbf{C}^{\gamma_1/(1+\alpha)}_x} + [f]_{\mathbf{C}^{\gamma_2}_{\mathrm{v}}}\Big)\Big(\|g\|_{\mathbf{C}^{(\gamma+\beta-(1-\theta)\gamma_1-\theta\gamma_2)/(1+\alpha)}_x} + \|g\|_{\mathbf{C}^{(\gamma-\gamma_1)/(1+\alpha)}_x\mathbf{C}^\beta_{\mathrm{v}}}\Big),$$

which gives (4.17). The proof is complete. □

The following corollary will be used to improve the regularity of u along the degenerate direction x in the proof of Theorem 6.3.

Corollary 4.5. *Let $\beta \in ((1-\alpha)\vee 0, 1)$, $\gamma \in [\beta, 1+\alpha)$ and $\gamma_1 \in (1, 1+\alpha) \cap [\gamma, \infty)$.*

(i) *For any $\varepsilon \in (0,1)$, there are $\theta > 0$ close to zero and $C_\varepsilon > 0$ such that for all $j \geq 5$,*

$$\|[\mathcal{R}^x_j, b\cdot\nabla_x]u\|_{\mathbf{C}^{\theta\beta}_a} \leq 2^{-\frac{(1-\theta)\gamma}{1+\alpha}j}\Big([b]_{\mathbf{C}^{\gamma_1/(1+\alpha)}_x} + \|\nabla_{\mathrm{v}} b\|_\infty\Big)\Big(\varepsilon\|u\|_{\mathbf{C}^{(\alpha+(1-\theta)\gamma+\theta\beta)/(1+\alpha)}_x} + C_\varepsilon\|u\|_{\mathbf{C}^{\alpha+\beta}_{\mathrm{v}}}\Big).$$

(ii) *For any $\theta \in (0, (\alpha+\beta-1)/\beta]$, there is a constant $C > 0$ such that for all $j \geq 5$,*

$$\|[\mathcal{R}^x_j, b\cdot\nabla_{\mathrm{v}}]u\|_{\mathbf{C}^{\theta\beta}_a} \leq C2^{-\frac{(1-\theta)\gamma}{1+\alpha}j}\Big([b]_{\mathbf{C}^{\gamma/(1+\alpha)}_x} + [b]_{\mathbf{C}^{\beta}_{\mathrm{v}}}\Big)\|u\|_{\mathbf{C}^{\alpha+\beta}_{\mathrm{v}}},$$

$$\|[\mathcal{R}^x_j, \mathscr{L}^{(\alpha)}_{\kappa;\mathrm{v}}]u\|_{\mathbf{C}^{\theta\beta}_a} \leq C2^{-\frac{(1-\theta)\gamma}{1+\alpha}j}\Big([\kappa]_{\mathbf{C}^{\gamma/(1+\alpha)}_x} + [\kappa]_{\mathbf{C}^{\beta}_{\mathrm{v}}}\Big)\|u\|_{\mathbf{C}^{\alpha+\beta}_{\mathrm{v}}},$$

where $\mathscr{L}^{(\alpha)}_{\kappa;\mathrm{v}}$ is defined by (1.5).

Proof. (i) Since $\gamma_1 > 1$, one can choose $\theta > 0$ small enough such that

$$1 + \frac{(1-\theta)\gamma - \gamma_1}{1+\alpha} \leqslant \frac{(\alpha + (1-\theta)\gamma + \theta\beta)\alpha}{(1+\alpha)(\alpha+\theta\beta)}. \tag{4.18}$$

By Lemma 4.4 with $(\theta\beta, (1-\theta)\gamma, \gamma_1, 1, \theta\beta)$ in place of $(\beta, \gamma, \gamma_1, \gamma_2, \theta)$, we have

$$\begin{aligned}\|[\mathcal{R}^x_j, b\cdot\nabla_x]u\|_{\mathbf{C}^{\theta\beta}_a} &\lesssim 2^{-\frac{(1-\theta)\gamma}{1+\alpha}j}\Big([b]_{\mathbf{C}^{\gamma_1/(1+\alpha)}_x} + [b]_{\mathbf{C}^{1}_{\mathrm{v}}}\Big)\\ &\quad\times\Big(\|\nabla_x u\|_{\mathbf{C}^{((1-\theta)\gamma-(1-\theta\beta)\gamma_1)/(1+\alpha)}_x} + \|\nabla_x u\|_{\mathbf{C}^{((1-\theta)\gamma-\gamma_1)/(1+\alpha)}_x\mathbf{C}^{\theta\beta}_{\mathrm{v}}}\Big)\\ &\lesssim 2^{-\frac{(1-\theta)\gamma}{1+\alpha}j}\Big([b]_{\mathbf{C}^{\gamma_1/(1+\alpha)}_x} + \|\nabla_{\mathrm{v}} b\|_\infty\Big)\\ &\quad\times\Big(\|u\|_{\mathbf{C}^{1+((1-\theta)\gamma-(1-\theta\beta)\gamma_1)/(1+\alpha)}_x} + \|u\|_{\mathbf{C}^{1+((1-\theta)\gamma-\gamma_1)/(1+\alpha)}_x\mathbf{C}^{\theta\beta}_{\mathrm{v}}}\Big).\end{aligned}$$

By (4.18), (4.3) and Young's inequality, for any $\varepsilon \in (0,1)$, we have

$$\|u\|_{\mathbf{C}^{1+((1-\theta)\gamma-\gamma_1)/(1+\alpha)}_x\mathbf{C}^{\theta\beta}_{\mathrm{v}}} \lesssim \|u\|_{\mathbf{C}^{(\alpha+(1-\theta)\gamma+\theta\beta)\alpha/((1+\alpha)(\alpha+\theta\beta))}_x\mathbf{C}^{\theta\beta}_{\mathrm{v}}} \lesssim \varepsilon\|u\|_{\mathbf{C}^{(\alpha+(1-\theta)\gamma+\theta\beta)/(1+\alpha)}_x} + \|u\|_{\mathbf{C}^{\alpha+\theta\beta}_{\mathrm{v}}},$$

and

$$\|u\|_{\mathbf{C}^{1+((1-\theta)\gamma-(1-\theta\beta)\gamma_1)/(1+\alpha)}_x} \leq \varepsilon\|u\|_{\mathbf{C}^{(\alpha+(1-\theta)\gamma+\theta\beta)/(1+\alpha)}_x} + \|u\|_\infty.$$

Substituting these two estimates into the above estimate, we obtain the desired estimate.
(ii) By Lemma 4.4 with $(\theta\beta, (1-\theta)\gamma, \gamma, \beta, \theta)$ in place of $(\beta, \gamma, \gamma_1, \gamma_2, \theta)$, we have

$$\begin{aligned}\|[\mathcal{R}^x_j, b\cdot\nabla_{\mathrm{v}}]u\|_{\mathbf{C}^{\theta\beta}_a} &\lesssim 2^{-\frac{(1-\theta)\gamma}{1+\alpha}j}\Big([b]_{\mathbf{C}^{\gamma/(1+\alpha)}_x} + [b]_{\mathbf{C}^{\beta}_{\mathrm{v}}}\Big)\Big(\|\nabla_{\mathrm{v}} u\|_{\mathbf{C}^0_x} + \|\nabla_{\mathrm{v}} u\|_{\mathbf{C}^{-\theta\gamma/(1+\alpha)}_x\mathbf{C}^{\theta\beta}_{\mathrm{v}}}\Big)\\ &\lesssim 2^{-\frac{(1-\theta)\gamma}{1+\alpha}j}\Big([b]_{\mathbf{C}^{\gamma/(1+\alpha)}_x} + [b]_{\mathbf{C}^{\beta}_{\mathrm{v}}}\Big)\|u\|_{\mathbf{C}^{1+\theta\beta}_{\mathrm{v}}}\\ &\lesssim 2^{-\frac{(1-\theta)\gamma}{1+\alpha}j}\Big([b]_{\mathbf{C}^{\gamma/(1+\alpha)}_x} + [b]_{\mathbf{C}^{\beta}_{\mathrm{v}}}\Big)\|u\|_{\mathbf{C}^{\alpha+\beta}_{\mathrm{v}}},\end{aligned}$$

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where the last step is due to $1+\theta\beta\le\alpha+\beta$. Moreover, recalling (1.5) and (2.2), and noticing that

$$[\mathcal{R}^x_j,\mathscr{L}^{(\alpha)}_{\kappa;\mathrm{v}}]u(x,\mathrm{v})=\int_{\mathbb{R}^d}\Big([\mathcal{R}^x_j,\kappa(\cdot,\cdot,w)]\delta^*_{w;2}\delta_{w;2}u\Big)(x,\mathrm{v})\frac{\mathrm{d}w}{|w|^{d+\alpha}},$$

by Lemma 4.4 with $(\theta\beta,(1-\theta)\gamma,\gamma,\beta,\theta)$ in place of $(\beta,\gamma,\gamma_1,\gamma_2,\theta)$, we have

$$\begin{aligned}\|[\mathcal{R}^x_j,\mathscr{L}^{(\alpha)}_{\kappa;\mathrm{v}}]u\|_{\mathbf{C}^{\theta\beta}_a}&\lesssim\int_{\mathbb{R}^d}\big\|[\mathcal{R}^x_j,\kappa(\cdot,\cdot,w)]\delta^*_{w;2}\delta_{w;2}u\big\|_{\mathbf{C}^{\theta\beta}_a}\frac{\mathrm{d}w}{|w|^{d+\alpha}}\\&\lesssim 2^{-\frac{(1-\theta)\gamma}{1+\alpha}j}\Big([\kappa]_{\mathbf{C}^{\gamma/(1+\alpha)}_x}+[\kappa]_{\mathbf{C}^{\beta}_{\mathrm{v}}}\Big)\int_{\mathbb{R}^d}\|\delta^*_{w;2}\delta_{w;2}u\|_{\mathbf{C}^{\theta\beta}_{\mathrm{v}}}\frac{\mathrm{d}w}{|w|^{d+\alpha}}\\&\lesssim 2^{-\frac{(1-\theta)\gamma}{1+\alpha}j}\Big([\kappa]_{\mathbf{C}^{\gamma/(1+\alpha)}_x}+[\kappa]_{\mathbf{C}^{\beta}_{\mathrm{v}}}\Big)\|u\|_{\mathbf{C}^{\alpha+\theta\beta+\varepsilon}_{\mathrm{v}}}\int_{\mathbb{R}^d}\frac{(1\wedge|w|^{\alpha+\varepsilon})\mathrm{d}w}{|w|^{d+\alpha}}\\&\lesssim 2^{-\frac{(1-\theta)\gamma}{1+\alpha}j}\Big([\kappa]_{\mathbf{C}^{\gamma/(1+\alpha)}_x}+[\kappa]_{\mathbf{C}^{\beta}_{\mathrm{v}}}\Big)\|u\|_{\mathbf{C}^{\alpha+\beta}_{\mathrm{v}}},\end{aligned}$$

where $\varepsilon\in(0,(1-\theta)\beta)$. The proof is complete. □

5. Heat kernel estimates of nonlocal kinetic operators

In this section we consider the following nonlocal kinetic equation with constant coefficients:

$$\partial_t u=\mathscr{L}^{(\alpha)}_{\kappa;\mathrm{v}}u+U_t\mathrm{v}\cdot\nabla_x u+f=:\mathscr{K}u+f,\quad u(0)=0,$$

where $\kappa(t,w)$ and U_t are measurable functions and satisfy the following assumptions:

$$c_0^{-1}\le\kappa(t,w)\le c_0,\quad c_0\ge 1,\tag{5.1}$$

and

$$c_1:=\|U\|_\infty+\sup_{s<t}\Big((t-s)\|\Pi^{-1}_{s,t}\|\Big)<\infty,\ \text{ where }\ \Pi_{s,t}:=\int_s^t U_r\mathrm{d}r.\tag{5.2}$$

It is well known that under (5.1) and (5.2), there is a fundamental solution or heat kernel $p_{s,t}(x,\mathrm{v})$ to kinetic operator $\partial_t-\mathscr{K}$ so that (see [12, Lemma 2.5])

$$u(t,x,\mathrm{v})=\int_0^t P_{s,t}f(s,x,\mathrm{v})\mathrm{d}s:=\int_0^t\Big(\Gamma_{s,t}p_{s,t}*\Gamma_{s,t}f\Big)(s,x,\mathrm{v})\mathrm{d}s,\tag{5.3}$$

where operator $\Gamma_{s,t}$ is defined by

$$\Gamma_{s,t}f(x,\mathrm{v}):=f(x+\Pi_{s,t}\mathrm{v},\mathrm{v}).\tag{5.4}$$

Moreover, for any $\beta,\gamma\ge 0$ with $\beta+\gamma<\alpha$ and $n,m\in\mathbb{N}_0$, there is a constant $C>0$ such that

$$\int_{\mathbb{R}^{2d}}|x|^\beta|\mathrm{v}|^\gamma|\nabla^n_x\nabla^m_{\mathrm{v}}p_{s,t}(x,\mathrm{v})|\mathrm{d}x\mathrm{d}\mathrm{v}\le C(t-s)^{\frac{(\beta-n)(1+\alpha)+\gamma-m}{\alpha}},\quad\forall s<t.\tag{5.5}$$

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We now use (5.5) to show the following crucial lemma, which is an analogue of Lemma 3.1.

Lemma 5.1. *Under (5.1) and (5.2), for any $q > -1$ and $\beta, \gamma \geq 0$ with $\beta + \gamma < \alpha$, there is a constant $C > 0$ such that for all $j \in \mathbb{N}$ and $t > s \geq 0$,*

$$\int_0^t \int_{\mathbb{R}^{2d}} (t-s)^q |x|^\beta |\mathrm{v}|^\gamma \left|\mathcal{R}^a_j \Gamma_{s,t} p_{s,t}(x, \mathrm{v})\right| \mathrm{d}x\mathrm{d}\mathrm{v}\mathrm{d}s \leq C 2^{-((1+\alpha)\beta+\gamma+(q+1)\alpha)j}, \tag{5.6}$$

$$\int_0^t \int_{\mathbb{R}^{2d}} (t-s)^q |x|^\beta |\mathrm{v}|^\gamma \left|\mathcal{R}^x_j \Gamma_{s,t} p_{s,t}(x, \mathrm{v})\right| \mathrm{d}x\mathrm{d}\mathrm{v}\mathrm{d}s \leq C 2^{-(\beta+\frac{(q+1)\alpha+\gamma}{1+\alpha})j}. \tag{5.7}$$

Proof. We only prove the first one. The second one is similar. First of all, by the change of variables, we have

$$\mathscr{I}_{s,t} := \int_{\mathbb{R}^{2d}} |x|^\beta |\mathrm{v}|^\gamma \left|\mathcal{R}^a_j \Gamma_{s,t} p_{s,t}(x, \mathrm{v})\right| \mathrm{d}x\mathrm{d}\mathrm{v} = 2^{-((1+\alpha)(d+\beta)+d+\gamma)j} \int_{\mathbb{R}^{2d}} |x|^\beta |\mathrm{v}|^\gamma \times$$
$$\times \left| \int_{\mathbb{R}^{2d}} \check{\phi}^a_1(x-\bar{x}, \mathrm{v}-\bar{\mathrm{v}}) p_{s,t}(2^{-(1+\alpha)j}\bar{x} + \Pi_{s,t} 2^{-j}\bar{\mathrm{v}}, 2^{-j}\bar{\mathrm{v}}) \mathrm{d}\bar{x}\mathrm{d}\bar{\mathrm{v}} \right| \mathrm{d}x\mathrm{d}\mathrm{v}.$$

Let $\tilde{U}_r := U_{(t-s)r+s}$ and $\tilde{\kappa}_r := \kappa_{(t-s)r+s}$. By the scaling property of the heat kernel (see [12, (2.27)]), we have

$$p_{s,t}(x, \mathrm{v}) := p^{\kappa,U}_{s,t}(x, \mathrm{v}) = (t-s)^{-\frac{2d}{\alpha}-d} p^{\tilde{\kappa},\tilde{U}}_{0,1}((t-s)^{-\frac{1}{\alpha}-1}x, (t-s)^{-\frac{1}{\alpha}}\mathrm{v}). \tag{5.8}$$

Hence,

$$p^{\kappa,U}_{s,t}(2^{-(1+\alpha)j}\bar{x} + \Pi_{s,t}2^{-j}\bar{\mathrm{v}}, 2^{-j}\bar{\mathrm{v}}) = (t-s)^{-\frac{2d}{\alpha}-d} p^{\tilde{\kappa},\tilde{U}}_{0,1}(\hbar^{\alpha+1}\bar{x} + \hbar\theta_{s,t}\bar{\mathrm{v}}, \hbar\bar{\mathrm{v}}),$$

where

$$\hbar := (t-s)^{-\frac{1}{\alpha}} 2^{-j}, \quad \theta_{s,t} := \Pi_{s,t}/(t-s).$$

Since the support of ϕ^a_1 is contained in the annulus, by Fourier's transform,

$$(\widehat{\Delta^{-n}_{x,\mathrm{v}}\check{\phi}^a_1})(\xi, \eta) := (|\xi|^2 + |\eta|^2)^{-n} \phi^a_1(\xi, \eta) \in \mathscr{S}(\mathbb{R}^{2d}),$$

so that $\Delta^{-n}_{x,\mathrm{v}}\check{\phi}^a_1$ is a well-defined Schwartz function. Thus we have

$$\mathscr{U} := \int_{\mathbb{R}^{2d}} |x|^\beta |\mathrm{v}|^\gamma \left| \int_{\mathbb{R}^{2d}} \check{\phi}^a_1(x-\bar{x}, \mathrm{v}-\bar{\mathrm{v}}) p^{\tilde{\kappa},\tilde{U}}_{0,1}(\hbar^{\alpha+1}\bar{x} + \hbar\theta_{s,t}\bar{\mathrm{v}}, \hbar\bar{\mathrm{v}}) \mathrm{d}\bar{x}\mathrm{d}\bar{\mathrm{v}} \right| \mathrm{d}x\mathrm{d}\mathrm{v}$$
$$= \int_{\mathbb{R}^{2d}} |x|^\beta |\mathrm{v}|^\gamma \left| \int_{\mathbb{R}^{2d}} \Delta^{-n}_{x,\mathrm{v}}\check{\phi}^a_1(x-\bar{x}, \mathrm{v}-\bar{\mathrm{v}}) \Delta^n_{x,\mathrm{v}} p^{\tilde{\kappa},\tilde{U}}_{0,1}(\hbar^{\alpha+1}\bar{x} + \hbar\theta_{s,t}\bar{\mathrm{v}}, \hbar\bar{\mathrm{v}}) \mathrm{d}\bar{x}\mathrm{d}\bar{\mathrm{v}} \right| \mathrm{d}x\mathrm{d}\mathrm{v}$$
$$\leq \int_{\mathbb{R}^{2d}} |x|^\beta |\mathrm{v}|^\gamma |\Delta^{-n}_{x,\mathrm{v}}\check{\phi}^a_1(x, \mathrm{v})| \mathrm{d}x\mathrm{d}\mathrm{v} \int_{\mathbb{R}^{2d}} |\Delta^n_{x,\mathrm{v}} p^{\tilde{\kappa},\tilde{U}}_{0,1}(\hbar^{\alpha+1}\bar{x} + \hbar\theta_{s,t}\bar{\mathrm{v}}, \hbar\bar{\mathrm{v}})| \mathrm{d}\bar{x}\mathrm{d}\bar{\mathrm{v}}$$

$$+\int_{\mathbb{R}^{2d}}|\Delta^{-n}_{x,\mathrm{v}}\check{\phi}^a_1(x,\mathrm{v})|\mathrm{d}x\mathrm{dv}\int_{\mathbb{R}^{2d}}|\bar{x}|^\beta|\bar{\mathrm{v}}|^\gamma|\Delta^n_{x,\mathrm{v}}p^{\tilde{\kappa},\tilde{U}}_{0,1}(\hbar^{\alpha+1}\bar{x}+\hbar\theta_{s,t}\bar{\mathrm{v}},\hbar\bar{\mathrm{v}})|\mathrm{d}\bar{x}\mathrm{d}\bar{\mathrm{v}}.$$

By the chain rule, (5.5) and cumbersome calculations, we have

$$\int_{\mathbb{R}^{2d}}|\Delta^n_{x,\mathrm{v}}p^{\tilde{\kappa},\tilde{U}}_{0,1}(\hbar^{\alpha+1}x+\hbar\theta_{s,t}\mathrm{v},\hbar\mathrm{v})|\mathrm{d}x\mathrm{dv}\lesssim\hbar^{(\alpha+1)(n-d)-d}+\hbar^{n-(\alpha+2)d},$$

and

$$\int_{\mathbb{R}^{2d}}|x|^\beta|\mathrm{v}|^\gamma|\Delta^n_{x,\mathrm{v}}p^{\tilde{\kappa},\tilde{U}}_{0,1}(\hbar^{\alpha+1}x+\hbar\theta_{s,t}\mathrm{v},\hbar\mathrm{v})|\mathrm{d}x\mathrm{dv}\lesssim\Big(\hbar^{(\alpha+1)(n-d)-d}+\hbar^{n-(\alpha+2)d}\Big)\hbar^{-(\alpha+1)\beta-\gamma}.$$

Therefore,

$$\mathscr{U}\lesssim\Big(\hbar^{(\alpha+1)(n-d)-d}+\hbar^{n-(\alpha+2)d}\Big)\Big(1+\hbar^{-(\alpha+1)\beta-\gamma}\Big),$$

and

$$\begin{aligned}\mathscr{J}_{s,t}&\lesssim 2^{-((1+\alpha)\beta+\gamma)j}\hbar^{(\alpha+2)d}\Big(\hbar^{(\alpha+1)(n-d)-d}+\hbar^{n-(\alpha+2)d}\Big)\Big(1+\hbar^{-(\alpha+1)\beta-\gamma}\Big)\\&=\Big(\hbar^{(\alpha+1)n}+\hbar^n\Big)\Big(2^{-((1+\alpha)\beta+\gamma)j}+(t-s)^{\frac{(\alpha+1)\beta+\gamma}{\alpha}}\Big).\end{aligned}\tag{5.9}$$

Now we prove (5.6). Without loss of generality, assume $t>2^{-\alpha j}$. We denote the left hand side of (5.6) by $\mathscr{I}$, and make the following decomposition:

$$\mathscr{I}=\left(\int_{t-2^{-\alpha j}}^{t}+\int_0^{t-2^{-\alpha j}}\right)(t-s)^q\mathscr{J}_{s,t}\mathrm{d}s=:\mathscr{I}_1+\mathscr{I}_2.$$

For $\mathscr{I}_1$, using (5.9) with $n=0$, and by the change of variables, we have

$$\begin{aligned}\mathscr{I}_1&\lesssim\int_{t-2^{-\alpha j}}^{t}(t-s)^q\Big(2^{-((1+\alpha)\beta+\gamma)j}+(t-s)^{\frac{(\alpha+1)\beta+\gamma}{\alpha}}\Big)\mathrm{d}s\\&\lesssim\int_0^{2^{-\alpha j}}s^q\Big(2^{-((1+\alpha)\beta+\gamma)j}+s^{\frac{(\alpha+1)\beta+\gamma}{\alpha}}\Big)\mathrm{d}s\lesssim 2^{-((1+\alpha)\beta+\gamma+(q+1)\alpha)j}.\end{aligned}$$

For $\mathscr{I}_2$, choosing n large enough in (5.9) so that

$$1+q-\tfrac{n}{\alpha}+\tfrac{(\alpha+1)\beta}{\alpha}+\tfrac{\gamma}{\alpha}<0,$$

by similar calculations as above, we also have

$$\mathscr{I}_2\lesssim\int_{2^{-\alpha j}}^{t}s^q\Big((s^{-\frac{1}{\alpha}}2^{-j})^{(\alpha+1)n}+(s^{-\frac{1}{\alpha}}2^{-j})^n\Big)\Big(2^{-((1+\alpha)\beta+\gamma)j}+s^{\frac{(\alpha+1)\beta+\gamma}{\alpha}}\Big)\mathrm{d}s\lesssim 2^{-((1+\alpha)\beta+\gamma+(q+1)\alpha)j}.$$

Combining the above calculations, we obtain the desired estimate. □

6. Schauder's estimates for non-local degenerate equations

In this section we consider the following nonlocal degenerate equation in $\mathbb{R}^{2d}$:

$$\partial_t u = \mathscr{L}^{(\alpha)}_{\kappa;\mathrm{v}} u + b \cdot \nabla u - \lambda u + f, \ \lambda \geq 0, \tag{6.1}$$

where $\mathscr{L}^{(\alpha)}_{\kappa;\mathrm{v}}$ is defined by (1.5) and b is a measurable function with the form

$$b(t, x, \mathrm{v}) = (b^{(1)}(t, x, \mathrm{v}), b^{(2)}(t, x, \mathrm{v})).$$

Throughout this section we assume

($\mathbf{H}^{\alpha,\vartheta}_{\beta,\gamma}$) For some $c_0 \geq 1$ and $\vartheta, \beta \in (0, 1)$, it holds that for all $t \geq 0$ and $x, \mathrm{v}, w \in \mathbb{R}^d$,

$$c_0^{-1} \leq \kappa(t, x, \mathrm{v}, w) \leq c_0, \quad [\kappa(t, \cdot, w)]_{\mathbf{C}^\beta_{\mathrm{v}}} + [b^{(2)}(t, \cdot)]_{\mathbf{C}^\beta_{\mathrm{v}}} + [\![b^{(1)}(t, \cdot)]\!]_{\mathbf{C}^{1+\vartheta}_a} \leq c_0,$$

where $[\![\cdot]\!]_{\mathbf{C}^{1+\vartheta}_a}$ is defined by (4.1), and for some $\gamma \in [\beta, 1 + \alpha)$,

$$[\kappa(t, \cdot, w)]_{\mathbf{C}^{\gamma/(1+\alpha)}_x} + [b(t, \cdot)]_{\mathbf{C}^{\gamma/(1+\alpha)}_x} + |b(t, 0)| \leq c_0,$$

and for some closed and convex subset $\mathscr{E} \subset GL_d(\mathbb{R})$, where $GL_d(\mathbb{R})$ is the set of all invertible $d \times d$-matrices,

$$\nabla_{\mathrm{v}} b^{(1)}(t, x, \mathrm{v}) \in \mathscr{E}. \tag{6.2}$$

Definition 6.1 *(Classical solutions).* Let $\lambda \geq 0$. We call a bounded continuous function u defined on $\mathbb{R}_+ \times \mathbb{R}^{2d}$ a classical solution of PDE (6.1) if for some $\varepsilon \in (0, 1)$,

$$u \in C([0, \infty); \mathbf{C}^{(\alpha \vee 1)+\varepsilon}_{\mathrm{v}} \cap \mathbf{C}^{1+\varepsilon}_x),$$

and for all $t \geq 0$ and $x, \mathrm{v} \in \mathbb{R}^d$,

$$u(t, x, \mathrm{v}) = \int_0^t \Big(\mathscr{L}^{(\alpha)}_{\kappa;\mathrm{v}} u + b \cdot \nabla u - \lambda u + f \Big)(s, x, \mathrm{v}) \mathrm{d}s.$$

We have the following maximum principle for classical solutions.

Theorem 6.2 *(Maximum principle). Let $\lambda, T > 0$. For any classical solution u of PDE (6.1) in the sense of Definition 6.1, it holds that*

$$\|u\|_{\mathbb{L}^\infty_T} \leq (1 - \mathrm{e}^{-\lambda T}) \|f\|_{\mathbb{L}^\infty_T} / \lambda. \tag{6.3}$$

Proof. Let

$$\bar{u}(t, x, \mathrm{v}) := -u(t, x, \mathrm{v}) \mathrm{e}^{\lambda t} + \int_0^t \|f(s, \cdot, \cdot)\|_\infty \mathrm{e}^{\lambda s} \mathrm{d}s. \tag{6.4}$$

By (6.1), it is easy to see that for Lebesgue almost all $t > 0$,

$$\partial_t \bar{u} - \mathscr{L}^{(\alpha)}_{\kappa;\mathrm{v}} \bar{u} - b \cdot \nabla \bar{u} \geq 0.$$

Since $\underline{\lim}_{t\downarrow 0} \bar{u}(t, x, \mathrm{v}) = 0$, by [9, Theorem 6.1], we have

$$\bar{u}(t, x, \mathrm{v}) \geq 0.$$

Thus, by (6.4), we get

$$u(t, x, \mathrm{v}) \leq \mathrm{e}^{-\lambda t} \int_0^t \|f(s, \cdot, \cdot)\|_\infty \mathrm{e}^{\lambda s} \mathrm{d}s \leq (1 - \mathrm{e}^{-\lambda t}) \|f\|_{\mathbb{L}^\infty_t} / \lambda.$$

By symmetry, we obtain (6.3). □

The goal of this section is to prove the following Schauder's apriori estimate.

Theorem 6.3. *Let $\alpha \in (1, 2)$ and $\beta \in (0, 1)$, $\vartheta \in (0, \beta \wedge (\alpha - 1))$, $\gamma \in [\beta, 1 + \alpha)$. Under $(\mathbf{H}^{\alpha,\vartheta}_{\beta,\gamma})$, for any $T > 0$, there are $\theta = \theta(\alpha, \beta, \gamma, \vartheta) > 0$ small enough and $C > 0$ only depending on $T, d, c_0, \alpha, \beta, \vartheta, \gamma, \mathscr{E}$ such that for any $\lambda \geq 0$ and any classical solution u of (6.1),*

$$\|u\|_{\mathbb{L}^\infty_T(\mathbf{C}^{(\alpha+\gamma)/(1+\alpha)}_x \cap \mathbf{C}^{\alpha+\beta}_\mathrm{v})} + \|u\|_{\mathbb{L}^\infty_T(\mathbf{C}^{(1-\theta)\gamma/(1+\alpha)}_x \mathbf{C}^{\alpha+\theta\beta}_\mathrm{v})} \leq C \|f\|_{\mathbb{L}^\infty_T(\mathbf{C}^{\gamma/(1+\alpha)}_x \cap \mathbf{C}^\beta_\mathrm{v})}. \tag{6.5}$$

Remark 6.4. Although our result is stated for $\alpha \in (1, 2)$, it in fact also works for $\alpha = 2$. In this case, under $(\mathbf{H}^{\alpha,\beta}_{\beta,\beta})$, Chaudru, Honoré and Menozzi [6, Theorem 1] has proven (6.5) for $\gamma = \beta$. When $\gamma = \beta$, our assumption on $b^{(1)}$ is weaker since we only assume $(\mathbf{H}^{\alpha,\vartheta}_{\beta,\beta})$ for some $\vartheta \in (0, 1)$.

To prove this theorem we use the perturbation argument by freezing the coefficients along the characteristic curve as in [34]. We need the following well-known fact from ODE.

Lemma 6.5. *Let $N \geq 1$ and $b : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N$ be a time-dependent measurable vector field. Suppose that for each $t > 0$, $x \mapsto b(t, x)$ is continuous and for some $C > 0$ and all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^N$,*

$$|b(t, x)| \leq C(1 + |x|).$$

Then for each $x \in \mathbb{R}^N$, there is a global solution θ_t to the following ODE:

$$\dot{\theta}_t = b(t, \theta_t), \quad \theta_0 = x.$$

Moreover, if we denote by $\mathscr{S}_x := \{\theta_\cdot : \theta_0 = x\}$ the set of all solutions with starting point x, then for each $T > 0$,

$$\cup_{x\in\mathbb{R}^N} \cup_{\theta_\cdot \in \mathscr{S}_x} \{\theta_T\} = \mathbb{R}^N. \tag{6.6}$$

Proof. We only show (6.6). Fix $y \in \mathbb{R}^d$ and $T > 0$. Let $(\tilde{\theta}_t)_{t\in[0,T]}$ be the solution of ODE:

$$\dot{\tilde{\theta}}_t = -b(T - t, \tilde{\theta}_t), \ \tilde{\theta}_0 = y,$$

and $(\bar{\theta}_t)_{t\geq 0}$ solve the ODE

$$\dot{\bar{\theta}}_t = b(T + t, \bar{\theta}_t), \ \bar{\theta}_0 = y.$$

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Define

$$\theta_t := \tilde\theta_{T-t}\mathbf{1}_{t\le T} + \bar\theta_{t-T}\mathbf{1}_{t>T}.$$

It is easy to see that $\theta_T = y$ and $\theta_\cdot \in \mathscr{S}_x$ with $x = \tilde\theta_T$. □

Fix $(x_0, \mathrm{v}_0) \in \mathbb{R}^{2d}$. Let θ_t solve the following ODE in $\mathbb{R}^{2d}$:

$$\dot\theta_t = b(t,\theta_t),\ \ \theta_0 = (x_0, \mathrm{v}_0).$$

Define

$$\tilde u(t,x,\mathrm{v}) := u(t, x+\theta^{(1)}_t, \mathrm{v}+\theta^{(2)}_t),\quad \tilde f(t,x,\mathrm{v}) := f(t, x+\theta^{(1)}_t, \mathrm{v}+\theta^{(2)}_t),$$
$$\kappa_0(t,w) := \kappa(t,\theta_t,w),\quad \tilde\kappa(t,x,\mathrm{v},w) := \kappa(t, x+\theta^{(1)}_t, \mathrm{v}+\theta^{(2)}_t, w) - \kappa(t,\theta^{(1)}_t,\theta^{(2)}_t,w),$$

and

$$U_t := \nabla_{\mathrm{v}} b^{(1)}(t,\theta_t),\quad \tilde b(t,x,\mathrm{v}) := b(t, x+\theta^{(1)}_t, \mathrm{v}+\theta^{(2)}_t) - b(t,\theta_t) - (U_t\mathrm{v}, 0).$$

By (6.2), there is a constant $c_1 \ge 1$ only depending on $\mathscr{E}$ such that for all $0 \le s < t$,

$$|U_t| + (t-s)|\Pi^{-1}_{s,t}| \le c_1,\ \text{where } \Pi_{s,t} := \int_s^t U_r \mathrm{d}r. \tag{6.7}$$

It is easy to see that $\tilde u$ satisfies the following frozen equation:

$$\partial_t \tilde u = \mathscr{L}^{(\alpha)}_{\kappa_0;\mathrm{v}}\tilde u + U_t\mathrm{v}\cdot\nabla_x\tilde u - \lambda\tilde u + \mathscr{L}^{(\alpha)}_{\tilde\kappa;\mathrm{v}}\tilde u + \tilde b\cdot\nabla\tilde u + \tilde f,$$

where

$$\mathscr{L}^{(\alpha)}_{\kappa_0;\mathrm{v}} u(x,\mathrm{v}) := \int_{\mathbb{R}^d} \delta^{(2)}_w u(x,\mathrm{v})\kappa_0(t,w)\frac{\mathrm{d}w}{|w|^{d+\alpha}},\quad \delta^{(2)}_w := \delta^*_{w;2}\delta_{w;2}.$$

Below, without loss of generality, we drop the tilde over u, f, κ, b and assume $x_0 = \mathrm{v}_0 = 0$ and

$$|\kappa(t,x,\mathrm{v},w)| \le [\kappa(t,\cdot,w)]_{\mathbf{C}^{\gamma/(1+\alpha)}_x}|x|^{\frac{\gamma}{1+\alpha}} + [\kappa(t,\cdot,w)]_{\mathbf{C}^\beta_x}|\mathrm{v}|^\beta, \tag{6.8}$$

and

$$|b^{(1)}(t,x,\mathrm{v})| \le [\![b^{(1)}(t,\cdot)]\!]_{\mathbf{C}^{1+\vartheta}_a}\big(|x|^{\frac{1+\vartheta}{1+\alpha}} + |\mathrm{v}|^{1+\vartheta}\big), \tag{6.9}$$
$$|b^{(2)}(t,x,\mathrm{v})| \le [b^{(2)}(t,\cdot)]_{\mathbf{C}^{\gamma/(1+\alpha)}_x}|x|^{\frac{\gamma}{1+\alpha}} + [b^{(2)}(t,\cdot)]_{\mathbf{C}^\beta_{\mathrm{v}}}|\mathrm{v}|^\beta. \tag{6.10}$$

Let $p^{\kappa_0}_{s,t}(x,\mathrm{v})$ be the heat kernel of $\mathscr{L}^{(\alpha)}_{\kappa_0;\mathrm{v}} + U_t\mathrm{v}\cdot\nabla_x$. Define for $\lambda \ge 0$,

$$P^\lambda_{s,t}f(x,\mathrm{v}) := \Big(\Gamma_{s,t}p^\lambda_{s,t} * \Gamma_{s,t}f\Big)(x,\mathrm{v}),\quad p^\lambda_{s,t}(x,\mathrm{v}) := \mathrm{e}^{\lambda(s-t)}p^{\kappa_0}_{s,t}(x,\mathrm{v}), \tag{6.11}$$

where $\Gamma_{s,t}$ is defined by (5.4). By Duhamel's formula we have (see (5.3))

$$u(t,x,\mathrm{v}) = \int_0^t P^\lambda_{s,t}\mathscr{L}^{(\alpha)}_{\kappa;\mathrm{v}} u(s,x,\mathrm{v})\mathrm{d}s + \int_0^t P^\lambda_{s,t}(b\cdot\nabla u)(s,x,\mathrm{v})\mathrm{d}s + \int_0^t P^\lambda_{s,t} f(s,x,\mathrm{v})\mathrm{d}s. \tag{6.12}$$

We prepare the following lemmas.

Lemma 6.6. *Let $\alpha \in (0,2)$. Under $(\mathbf{H}^{\alpha,\vartheta}_{\beta,\gamma})$, for any $\varepsilon \in (0,1)$, there is a constant $C > 0$ such that for all $j \in \mathbb{N}$ and $\lambda \geq 0$, $t \in [0,T]$,*

$$\int_0^t |\mathcal{R}^a_j P^\lambda_{s,t}\mathscr{L}^{(\alpha)}_{\kappa;\mathrm{v}} u|(s,0,0)\mathrm{d}s \leq C2^{-(\alpha+\beta)j}\|u\|_{\mathbb{L}^\infty_T(\mathbf{C}^{\alpha+\varepsilon}_a)}, \tag{6.13}$$

$$\int_0^t |\mathcal{R}^x_j P^\lambda_{s,t}\mathscr{L}^{(\alpha)}_{\kappa;\mathrm{v}} u|(s,0,0)\mathrm{d}s \leq C2^{-\frac{\alpha+\gamma}{1+\alpha}j}\|u\|_{\mathbb{L}^\infty_T(\mathbf{C}^{(\gamma-\beta)/(1+\alpha)}_x\mathbf{C}^{\alpha+\varepsilon}_\mathrm{v})}. \tag{6.14}$$

Proof. (i) First of all, by (6.8), we have for $u \in \mathbf{C}^{\alpha+\varepsilon}_\mathrm{v}$,

$$\begin{aligned}\left|\mathscr{L}^{(\alpha)}_{\kappa;\mathrm{v}} u(x,\mathrm{v})\right| &= \left|\int_{\mathbb{R}^d} \delta^{(2)}_w u(x,\mathrm{v})\cdot\kappa(t,x,\mathrm{v},\mathrm{w})\frac{\mathrm{d}w}{|w|^{d+\alpha}}\right| \lesssim \left(|x|^{\frac{\gamma}{1+\alpha}} + |\mathrm{v}|^\beta\right)\int_{\mathbb{R}^d}\left|\delta^{(2)}_w u(x,\mathrm{v})\right|\frac{\mathrm{d}w}{|w|^{d+\alpha}}\\ &\lesssim \left(|x|^{\frac{\gamma}{1+\alpha}} + |\mathrm{v}|^\beta\right)\|u\|_{\mathbf{C}^{\alpha+\varepsilon}_\mathrm{v}}\int_{\mathbb{R}^d}(1\wedge|w|^{\alpha+\varepsilon})\frac{\mathrm{d}w}{|w|^{d+\alpha}} \lesssim \left(|x|^{\frac{\gamma}{1+\alpha}} + |\mathrm{v}|^\beta\right)\|u\|_{\mathbf{C}^{\alpha+\varepsilon}_\mathrm{v}}.\end{aligned}$$

Thus by definition (6.11), we have

$$\begin{aligned}\int_0^t |\mathcal{R}^a_j P^\lambda_{s,t}\mathscr{L}^{(\alpha)}_{\kappa;\mathrm{v}} u|(s,0,0)\mathrm{d}s &= \int_0^t \left|\int_{\mathbb{R}^{2d}} \mathcal{R}^a_j\Gamma_{s,t}p^\lambda_{s,t}(x,\mathrm{v})\cdot\left(\Gamma_{s,t}\mathscr{L}^{(\alpha)}_{\kappa;\mathrm{v}} u\right)(s,x,\mathrm{v})\mathrm{d}x\mathrm{d}\mathrm{v}\right|\mathrm{d}s\\ &\lesssim \|u\|_{\mathbb{L}^\infty_T(\mathbf{C}^{\alpha+\varepsilon}_\mathrm{v})}\int_0^t\int_{\mathbb{R}^{2d}}|\mathcal{R}^a_j\Gamma_{s,t}p^\lambda_{s,t}(x,\mathrm{v})|\left(|x+\Pi_{s,t}\mathrm{v}|^{\frac{\gamma}{1+\alpha}} + |\mathrm{v}|^\beta\right)\mathrm{d}x\mathrm{d}\mathrm{v}\mathrm{d}s,\end{aligned}$$

which in turn gives (6.13) by direct application of (5.6) and $\gamma \geq \beta$.

(ii) Notice that by (2.5) and $\mathcal{R}^x_j\Gamma_{s,t} = \Gamma_{s,t}\mathcal{R}^x_j$,

$$\begin{aligned}\int_0^t |\mathcal{R}^x_j P^\lambda_{s,t}\mathscr{L}^{(\alpha)}_{\kappa;\mathrm{v}} u|(s,0,0)\mathrm{d}s &= \int_0^t \left|\int_{\mathbb{R}^{2d}} \mathcal{R}^x_j\Gamma_{s,t}p^\lambda_{s,t}(x,\mathrm{v})\cdot\left(\Gamma_{s,t}\mathscr{L}^{(\alpha)}_{\kappa;\mathrm{v}} u\right)(s,x,\mathrm{v})\mathrm{d}x\mathrm{d}\mathrm{v}\right|\mathrm{d}s\\ &= \int_0^t \left|\int_{\mathbb{R}^{2d}} \widetilde{\mathcal{R}}^x_j\Gamma_{s,t}p^\lambda_{s,t}(x,\mathrm{v})\cdot\left(\Gamma_{s,t}\mathcal{R}^x_j\mathscr{L}^{(\alpha)}_{\kappa;\mathrm{v}} u\right)(s,x,\mathrm{v})\mathrm{d}x\mathrm{d}\mathrm{v}\right|\mathrm{d}s =: \mathscr{I}_1 + \mathscr{I}_2,\end{aligned}$$

where

$$\mathscr{I}_1 := \int_0^t \left|\int_{\mathbb{R}^{2d}} \widetilde{\mathcal{R}}^x_j\Gamma_{s,t}p^\lambda_{s,t}(x,\mathrm{v})\cdot\left(\Gamma_{s,t}\mathscr{L}^{(\alpha)}_{\kappa;\mathrm{v}}\mathcal{R}^x_j u\right)(s,x,\mathrm{v})\mathrm{d}x\mathrm{d}\mathrm{v}\right|\mathrm{d}s,$$

$$\mathscr{I}_2 := \int_0^t \left|\int_{\mathbb{R}^{2d}} \widetilde{\mathcal{R}}^x_j\Gamma_{s,t}p^\lambda_{s,t}(x,\mathrm{v})\cdot\left(\Gamma_{s,t}[\mathcal{R}^x_j, \mathscr{L}^{(\alpha)}_{\kappa;\mathrm{v}}]u\right)(s,x,\mathrm{v})\mathrm{d}x\mathrm{d}\mathrm{v}\right|\mathrm{d}s.$$

For $\mathscr{I}_1$, by the assumptions, we have

$$\left|\mathscr{L}^{(\alpha)}_{\kappa;\mathrm{v}}\mathcal{R}^x_j u(x,\mathrm{v})\right| = \left|\int_{\mathbb{R}^d} \delta^{(2)}_w \mathcal{R}^x_j u(x,\mathrm{v})\cdot\kappa(t,x,\mathrm{v},\mathrm{w})\frac{\mathrm{d}w}{|w|^{d+\alpha}}\right|$$
$$\lesssim \left(|x|^{\frac{\gamma}{1+\alpha}}+|\mathrm{v}|^\beta\right)\int_{\mathbb{R}^d}\left|\delta^{(2)}_w\mathcal{R}^x_j u(x,\mathrm{v})\right|\frac{\mathrm{d}w}{|w|^{d+\alpha}}$$
$$\lesssim \left(|x|^{\frac{\gamma}{1+\alpha}}+|\mathrm{v}|^\beta\right)\|\mathcal{R}^x_j u\|_{\mathbf{C}^{\alpha+\varepsilon}_{\mathrm{v}}},$$

and thus, by (5.7) and $\gamma \geq \beta$,

$$\mathscr{I}_1 \lesssim \|\mathcal{R}^x_j u\|_{\mathbb{L}^\infty_T(\mathbf{C}^{\alpha+\varepsilon}_{\mathrm{v}})}\int_0^t\int_{\mathbb{R}^{2d}}|\widetilde{\mathcal{R}}^x_j\Gamma_{s,t}p^\lambda_{s,t}|(x,\mathrm{v})\left(|x+\Pi_{s,t}\mathrm{v}|^{\frac{\gamma}{1+\alpha}}+|\mathrm{v}|^\beta\right)\mathrm{d}x\mathrm{d}\mathrm{v}\mathrm{d}s$$
$$\lesssim \|\mathcal{R}^x_j u\|_{\mathbb{L}^\infty_T(\mathbf{C}^{\alpha+\varepsilon}_{\mathrm{v}})}\left(2^{-\frac{\alpha+\gamma}{1+\alpha}j}+2^{-\frac{\alpha+\beta}{1+\alpha}j}\right)\lesssim 2^{-\frac{\alpha+\gamma}{1+\alpha}j}\|u\|_{\mathbb{L}^\infty_T(\mathbf{C}^{(\gamma-\beta)/(1+\alpha)}_x\mathbf{C}^{\alpha+\varepsilon}_{\mathrm{v}})}.$$

For $\mathscr{I}_2$, by definition, we have

$$\|[\mathcal{R}^x_j,\mathscr{L}^{(\alpha)}_{\kappa;\mathrm{v}}]u\|_\infty \lesssim \int_{\mathbb{R}^{2d}}\|\delta^{(2)}_w u\|_\infty\frac{\mathrm{d}w}{|w|^{d+\alpha}}\int_{\mathbb{R}^d}|\bar{x}|^{\frac{\gamma}{1+\alpha}}\check{\phi}_j(\bar{x})\mathrm{d}\bar{x}\lesssim\|u\|_{\mathbf{C}^{\alpha+\varepsilon}_{\mathrm{v}}}2^{-\frac{\gamma}{1+\alpha}j}.$$

Hence, by (5.7) again,

$$\mathscr{I}_2\lesssim\|u\|_{\mathbb{L}^\infty_T(\mathbf{C}^{\alpha+\varepsilon}_{\mathrm{v}})}2^{-\frac{\gamma}{1+\alpha}j}\int_0^t\int_{\mathbb{R}^{2d}}|\widetilde{\mathcal{R}}^x_j\Gamma_{s,t}p^\lambda_{s,t}|(x,\mathrm{v})\mathrm{d}x\mathrm{d}\mathrm{v}\mathrm{d}s\lesssim\|u\|_{\mathbb{L}^\infty_T(\mathbf{C}^{\alpha+\varepsilon}_{\mathrm{v}})}2^{-\frac{\alpha+\gamma}{1+\alpha}j}.$$

Combining the above calculations, we obtain (6.14). □

To treat the other terms in (6.12), we need the following lemma.

Lemma 6.7. *Let $c_1 \geq 1$ be the same as in (6.7). For $t \geq 0$ and $j \in \mathbb{N}$, define*

$$\Theta^t_j := \left\{\ell\in\mathbb{N}_0 : 2^\ell \leq 2^4c_1(2^j+t2^{(1+\alpha)j}),\ 2^j\leq 2^4c_1(2^\ell+t2^{(1+\alpha)\ell})\right\}.$$

(i) *Let $0 \leq s < t$ and $j \in \mathbb{N}$. For any $\ell \notin \Theta^{t-s}_j$, it holds that*

$$\langle\mathcal{R}^a_j f,\Gamma_{s,t}\mathcal{R}^a_\ell g\rangle = \int_{\mathbb{R}^{2d}}\mathcal{R}^a_j f(x,\mathrm{v})\cdot\Gamma_{s,t}\mathcal{R}^a_\ell g(x,\mathrm{v})\mathrm{d}x\mathrm{d}\mathrm{v} = 0. \tag{6.15}$$

(ii) *For any $\beta > 0$, there is a constant $C = C(c_1,\beta) > 0$ such that for all $j \in \mathbb{N}$ and $t \geq 0$,*

$$\sum_{\ell\in\Theta^t_j}2^{-\beta\ell}\leq C\left(2^{-j}+t2^{(\alpha-1)j}\right)^\beta,\quad \sum_{\ell\in\Theta^t_j}2^{\beta\ell}\leq C\left(2^j+t2^{(1+\alpha)j}\right)^\beta. \tag{6.16}$$

(iii) *For any $T > 0$, there exists a $j_0 = j_0(c_1,\alpha,T)\in\mathbb{N}$ such that for all $j \geq j_0$ and $t\in[0,T]$,*

$$\inf\left\{\ell : \ell\in\Theta^t_j\right\}\geq 5.$$

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Proof. (i) To prove (6.15), by Fourier's transform we have

$$\langle \mathcal{R}_j^a f, \Gamma_{s,t}\mathcal{R}_\ell^a g\rangle = \int_{\mathbb{R}^{2d}} \phi_j^a(\xi,\eta)\hat{f}(\xi,\eta)\phi_\ell^a(\xi,\eta-\Pi_{s,t}\xi)\hat{g}(\xi,\eta-\Pi_{s,t}\xi)\mathrm{d}\xi\mathrm{d}\eta.$$

Notice that

$$\mathrm{supp}\phi_j^a \subset \left\{(\xi,\eta) : 2^{j-1} \le |\xi|^{1/(1+\alpha)} + |\eta| \le 2^{j+1}\right\} =: \mathcal{I}_j.$$

Assuming $\langle \mathcal{R}_j^a\Gamma_{s,t}f, \Gamma_{s,t}\mathcal{R}_\ell^a g\rangle \neq 0$ for $j,\ell \in \mathbb{N}$, we must have

$$(\xi,\eta)\in\mathcal{I}_j \ \text{ and } \ (\xi,\eta-\Pi_{s,t}\xi)\in\mathcal{I}_\ell,$$

which implies that

$$|\xi| \le 2^{(1+\alpha)(j+1)}, \quad |\eta| \le 2^{j+1},$$

and

$$2^{\ell-1} \le |\xi|^{1/(1+\alpha)} + |\eta-\Pi_{s,t}\xi| \le 2\cdot 2^{j+1} + c_1(t-s)2^{(1+\alpha)(j+1)} \le 2^3c_1(2^j + (t-s)2^{(1+\alpha)j}).$$

By symmetry we also have

$$2^{j-1} \le 2^3c_1(2^\ell + (t-s)2^{(1+\alpha)\ell}).$$

If $\langle \mathcal{R}_j^a\Gamma_{s,t}f, \Gamma_{s,t}\mathcal{R}_0^a g\rangle \neq 0$ for $j\in\mathbb{N}$, we still have

$$2^{j-1} \le |\xi|^{1/(1+\alpha)} + |\eta-\Pi_{s,t}\xi| + |\Pi_{s,t}\xi| \le 2 + c_1(t-s)2^{1+\alpha}.$$

Combining the above calculations, one sees that for $\ell\notin\Theta_j^{t-s}$, (6.15) holds.

(ii) We only prove the first estimate in (6.16). If $\ell > j$, then $2^{-\ell}\le 2^{-j}$. If $\ell\le j$, then by the definition of Θ_j^t,

$$2^{-\ell} \le 2^4c_12^{-j}(1+(t-s)2^{\alpha\ell}) \le 2^4c_1(2^{-j}+(t-s)2^{(\alpha-1)j}) =: D,$$

which implies $\ell \ge -\ln D/\ln 2$. Thus, we have

$$\sum_{\ell\in\Theta_j^t} 2^{-\beta\ell} \le \sum_{\ell\ge -\ln D/\ln 2} 2^{-\beta\ell} \le (2D)^\beta/(1-2^{-\beta}) \lesssim (2^{-j}+(t-s)2^{(\alpha-1)j})^\beta.$$

(iii) By definition of Θ_j^t, it suffices to take $j_0 > \ln(2^4c_1(2^5+T2^{(1+\alpha)5}))/\ln 2$. □

Lemma 6.8. *Let $\alpha\in(1,2)$, $T>0$ and j_0 be as in (iii) of Lemma 6.7. Under $(\mathbf{H}^{\alpha,\vartheta}_{\beta,\gamma})$ with $\vartheta<\alpha-1$, there is a constant $C>0$ such that for all $j\ge j_0$, $\lambda\ge 0$ and $t\in[0,T]$,*

$$\int_0^t |\mathcal{R}_j^a P^\lambda_{s,t}(b\cdot\nabla u)|(s,0,0)\mathrm{d}s \le C2^{-(\alpha+\beta)j}\|u\|_{\mathbb{L}^\infty_T(\mathbf{C}_a^{\alpha+\beta-\vartheta})}, \tag{6.17}$$

$$\int_0^t |\mathcal{R}_j^x P^\lambda_{s,t}(b^{(1)}\cdot\nabla_x u)|(s,0,0)\mathrm{d}s \le C2^{-\frac{\alpha+\gamma}{1+\alpha}j}\|u\|_{\mathbb{L}^\infty_T(\mathbf{C}_x^{(\alpha+\gamma-\vartheta)/(1+\alpha)})}, \tag{6.18}$$

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$$\int_0^t |\mathcal{R}^x_j P^\lambda_{s,t}(b^{(2)}\cdot\nabla_{\rm v} u)|(s,0,0)\mathrm{d}s \le C2^{-\frac{\alpha+\gamma}{1+\alpha}j}\|\nabla_{\rm v} u\|_{\mathbb{L}^\infty_T(\mathbf{C}^{(\gamma-\beta)/(1+\alpha)}_x)}. \tag{6.19}$$

Proof. (i) Let $\Theta=\Theta^{t-s}_j$ be as in Lemma 6.7. By definition (6.11) and (6.15), we have

$$\begin{aligned}\int_0^t |\mathcal{R}^a_j P^\lambda_{s,t}(b\cdot\nabla u)|(s,0,0)\mathrm{d}s &= \int_0^t \left|\int_{\mathbb{R}^{2d}} \mathcal{R}^a_j \Gamma_{s,t} p^\lambda_{s,t}(x,{\rm v})\cdot\Gamma_{s,t}(b\cdot\nabla u)(s,x,{\rm v})\mathrm{d}x\mathrm{d}{\rm v}\right|\mathrm{d}s\\ &= \int_0^t \left|\sum_{\ell\in\Theta}\int_{\mathbb{R}^{2d}} \mathcal{R}^a_j\Gamma_{s,t}p^\lambda_{s,t}(x,{\rm v})\cdot\Gamma_{s,t}\mathcal{R}^a_\ell(b\cdot\nabla u)(s,x,{\rm v})\mathrm{d}x\mathrm{d}{\rm v}\right|\mathrm{d}s.\end{aligned} \tag{6.20}$$

Below we drop the time variable and write

$$\mathcal{R}^a_\ell(b\cdot\nabla u)(x,{\rm v}) = (b\cdot\nabla\mathcal{R}^a_\ell u)(x,{\rm v}) + [\mathcal{R}^a_\ell, b\cdot\nabla]u(x,{\rm v}) =: \mathscr{I}_1+\mathscr{I}_2.$$

For $\mathscr{I}_1$, by (6.9) and (4.4), we have

$$|(b^{(1)}\cdot\nabla_x\mathcal{R}^a_\ell u)(x,{\rm v})| \lesssim \Big(|x|^{\frac{1+\vartheta}{1+\alpha}}+|{\rm v}|^{1+\vartheta}\Big)2^{(1+\alpha)\ell}\|\mathcal{R}^a_\ell u\|_\infty \lesssim \Big(|x|^{\frac{1+\vartheta}{1+\alpha}}+|{\rm v}|^{1+\vartheta}\Big)2^{(1-\beta+\vartheta)\ell}\|u\|_{\mathbf{B}^{\alpha+\beta-\vartheta}_{a,\infty}},$$

and by (6.10) and (4.4),

$$|(b^{(2)}\cdot\nabla_{\rm v}\mathcal{R}^a_\ell u)(x,{\rm v})| \lesssim \Big(|x|^{\frac{\gamma}{1+\alpha}}+|{\rm v}|^{\beta}\Big)2^{\ell}\|\mathcal{R}^a_\ell u\|_\infty \lesssim \Big(|x|^{\frac{\gamma}{1+\alpha}}+|{\rm v}|^{\beta}\Big)2^{(1-\alpha-\beta+\vartheta)\ell}\|u\|_{\mathbf{B}^{\alpha+\beta-\vartheta}_{a,\infty}}.$$

Hence, by (2.10),

$$|\mathscr{I}_1| \lesssim \|u\|_{\mathbf{C}^{\alpha+\beta-\vartheta}_a}\Big((|x|^{\frac{1+\vartheta}{1+\alpha}}+|{\rm v}|^{1+\vartheta})2^{(1-\beta+\vartheta)\ell} + (|x|^{\frac{\gamma}{1+\alpha}}+|{\rm v}|^{\beta})2^{(1-\alpha-\beta+\vartheta)\ell}\Big). \tag{6.21}$$

For $\mathscr{I}_2$, due to $j\ge j_0$ and by (iii) of Lemma 6.7, we have $\ell\ge 5$ for $\ell\in\Theta^{t-s}_j$. Thus we can use (4.7) with $(\vartheta,\beta-1-\vartheta)$ in place of (β,γ) to derive that

$$\begin{aligned}\big|[\mathcal{R}^a_\ell, b^{(1)}\cdot\nabla_x]u(x,{\rm v})\big| &\lesssim 2^{-\ell(\beta-\vartheta)}\Big(2^{-\ell\vartheta}+|x|^{\frac{\vartheta}{1+\alpha}}+|{\rm v}|^{\vartheta}\Big)\|\nabla_x u\|_{\mathbf{C}^{\beta-1-\vartheta}_a}\\ &\lesssim \Big(2^{-\ell\beta}+2^{-\ell(\beta-\vartheta)}(|x|^{\frac{\vartheta}{1+\alpha}}+|{\rm v}|^{\vartheta})\Big)\|u\|_{\mathbf{C}^{\alpha+\beta-\vartheta}_a}.\end{aligned}$$

Moreover, by $\gamma\ge\beta$, $\alpha-\vartheta>1$ and the definition, we also have

$$\begin{aligned}\big|[\mathcal{R}^a_\ell, b^{(2)}\cdot\nabla_{\rm v}]u(x,{\rm v})\big| &= \left|\int_{\mathbb{R}^{2d}}\check\phi^a_\ell(x-\bar x,{\rm v}-\bar{\rm v})\Big(b^{(2)}(\bar x,\bar{\rm v})-b^{(2)}(x,{\rm v})\Big)\nabla_{\rm v}u(\bar x,\bar{\rm v})\mathrm{d}\bar x\mathrm{d}\bar{\rm v}\right|\\ &\lesssim \|\nabla_{\rm v}u\|_\infty\int_{\mathbb{R}^{2d}}|\check\phi^a_\ell(\bar x,\bar{\rm v})|\cdot\Big(|\bar x|^{\frac{\gamma}{1+\alpha}}+|\bar{\rm v}|^{\beta}\Big)\mathrm{d}\bar x\mathrm{d}\bar{\rm v} \lesssim 2^{-\ell\beta}\|u\|_{\mathbf{C}^{\alpha+\beta-\vartheta}_a}.\end{aligned}$$

Therefore,

$$|\mathscr{I}_2| \lesssim \|u\|_{\mathbf{C}^{\alpha+\beta-\vartheta}_a}\Big(2^{-\ell\beta}+2^{-\ell(\beta-\vartheta)}(|x|^{\frac{\vartheta}{1+\alpha}}+|{\rm v}|^{\vartheta})\Big). \tag{6.22}$$

Combining (6.20)-(6.22), and by (6.16) we get

$$\int_0^t |\mathcal{R}_j^a P^\lambda_{s,t}(b\cdot\nabla u)|(s,0,0)\mathrm{d}s \lesssim \|u\|_{\mathbb{L}^\infty_T(\mathbf{C}_a^{\alpha+\beta-\vartheta})}\int_0^t\int_{\mathbb{R}^{2d}}\left|\mathcal{R}^a_j\Gamma_{s,t}p^\lambda_{s,t}(x,\mathrm{v})\right|$$
$$\times\Big\{\Big(|x+\Pi_{s,t}\mathrm{v}|^{\frac{1+\vartheta}{1+\alpha}}+|\mathrm{v}|^{1+\vartheta}\Big)\Big(2^j+(t-s)2^{(1+\alpha)j}\Big)^{1-\beta+\vartheta}$$
$$+\Big(|x+\Pi_{s,t}\mathrm{v}|^{\frac{\gamma}{1+\alpha}}+|\mathrm{v}|^{\beta}\Big)\Big(2^{-j}+(t-s)2^{(\alpha-1)j}\Big)^{\alpha+\beta-\vartheta-1}$$
$$+\Big(2^{-j}+(t-s)2^{(\alpha-1)j}\Big)^{\beta-\vartheta}\Big(|x+\Pi_{s,t}\mathrm{v}|^{\frac{\vartheta}{1+\alpha}}+|\mathrm{v}|^{\vartheta}\Big)$$
$$+\Big(2^{-j}+(t-s)2^{(\alpha-1)j}\Big)^{\beta}\Big\}\mathrm{d}x\mathrm{d}\mathrm{v}\mathrm{d}s,$$

which in turn yields (6.17) by using (6.7) and (5.6) item by item, as well as $\gamma\geq\beta$.

(ii) As above by definition (6.11) and (2.5), we have

$$\int_0^t |\mathcal{R}_j^x P^\lambda_{s,t}(b^{(1)}\cdot\nabla_x u)|(s,0,0)\mathrm{d}s = \int_0^t\left|\int_{\mathbb{R}^{2d}}\mathcal{R}^x_j\Gamma_{s,t}p^\lambda_{s,t}(x,\mathrm{v})\cdot\Gamma_{s,t}(b^{(1)}\cdot\nabla_x u)(s,x,\mathrm{v})\mathrm{d}x\mathrm{d}\mathrm{v}\right|\mathrm{d}s$$
$$=\int_0^t\left|\int_{\mathbb{R}^{2d}}\widetilde{\mathcal{R}}^x_j\Gamma_{s,t}p^\lambda_{s,t}(x,\mathrm{v})\cdot\Gamma_{s,t}\mathcal{R}^x_j(b^{(1)}\cdot\nabla_x u)(s,x,\mathrm{v})\mathrm{d}x\mathrm{d}\mathrm{v}\right|\mathrm{d}s=\mathscr{I}_1+\mathscr{I}_2,$$

where we have used $\mathcal{R}^x_j\Gamma_{s,t}=\Gamma_{s,t}\mathcal{R}^x_j$, and

$$\mathscr{I}_1:=\int_0^t\left|\int_{\mathbb{R}^{2d}}\widetilde{\mathcal{R}}^x_j\Gamma_{s,t}p^\lambda_{s,t}(x,\mathrm{v})\cdot\Gamma_{s,t}(b^{(1)}\cdot\nabla_x\mathcal{R}^x_j u)(s,x,\mathrm{v})\mathrm{d}x\mathrm{d}\mathrm{v}\right|\mathrm{d}s,$$
$$\mathscr{I}_2:=\int_0^t\left|\int_{\mathbb{R}^{2d}}\widetilde{\mathcal{R}}^x_j\Gamma_{s,t}p^\lambda_{s,t}(x,\mathrm{v})\cdot\Gamma_{s,t}[\mathcal{R}^x_j,b^{(1)}\cdot\nabla_x]u(s,x,\mathrm{v})\mathrm{d}x\mathrm{d}\mathrm{v}\right|\mathrm{d}s.$$

For $\mathscr{I}_1$, noticing that by (6.9) and the definition of $\mathcal{R}^x_j$,

$$|b^{(1)}\cdot\nabla_x\mathcal{R}^x_j u|(x,\mathrm{v})\lesssim\Big(|x|^{\frac{1+\vartheta}{1+\alpha}}+|\mathrm{v}|^{1+\vartheta}\Big)2^j\|\mathcal{R}^x_j u\|_\infty,$$

we have

$$\mathscr{I}_1\lesssim 2^j\|\mathcal{R}^x_j u\|_{\mathbb{L}^\infty_T}\int_0^t\int_{\mathbb{R}^{2d}}|\widetilde{\mathcal{R}}^x_j\Gamma_{s,t}p^\lambda_{s,t}|(x,\mathrm{v})\Big(|x+\Pi_{s,t}\mathrm{v}|^{\frac{1+\vartheta}{1+\alpha}}+|\mathrm{v}|^{1+\vartheta}\Big)\mathrm{d}x\mathrm{d}\mathrm{v}\mathrm{d}s$$
$$\overset{(5.7)}{\lesssim}\|\mathcal{R}^x_j u\|_{\mathbb{L}^\infty_T}2^{-\frac{\vartheta}{1+\alpha}j}\lesssim\|u\|_{\mathbb{L}^\infty_T(\mathbf{B}^{(\alpha+\gamma-\vartheta)/(1+\alpha)}_{x,\infty})}2^{-\frac{\alpha+\gamma}{1+\alpha}j}.$$

For $\mathscr{I}_2$, by (4.8) with $(\frac{(1+\vartheta)\vee\gamma}{1+\alpha},\frac{-((1+\vartheta)\vee\gamma-\gamma)}{1+\alpha})$ in place of (β,γ) and (5.7), we have

$$\mathscr{I}_2\lesssim\int_0^t\int_{\mathbb{R}^{2d}}|\widetilde{\mathcal{R}}^x_j\Gamma_{s,t}p^\lambda_{s,t}|(x,\mathrm{v})\,\|[\mathcal{R}^x_j,b^{(1)}\cdot\nabla_x]u\|_\infty\mathrm{d}x\mathrm{d}\mathrm{v}\mathrm{d}s$$
$$\lesssim 2^{-\frac{\alpha+\gamma}{1+\alpha}j}\|\nabla_x u\|_{\mathbb{L}^\infty_T(\mathbf{C}_x^{-((1+\vartheta)\vee\gamma-\gamma)/(1+\alpha)})}\lesssim 2^{-\frac{\alpha+\gamma}{1+\alpha}j}\|u\|_{\mathbb{L}^\infty_T(\mathbf{C}_x^{(\alpha+\gamma-\vartheta)/(1+\alpha)})}.$$

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Combining the above two estimates and Theorem 2.2, we obtain (6.18).

(iii) As above, by definition we have

$$\int_0^t |\mathcal{R}_j^x P_{s,t}^\lambda (b^{(2)}\cdot\nabla_{\rm v} u)|(s,0,0)\mathrm{d}s = \int_0^t \left|\int_{\mathbb{R}^{2d}} \mathcal{R}_j^x \Gamma_{s,t} p_{s,t}^\lambda(x,\mathrm{v})\cdot \Gamma_{s,t}(b^{(2)}\cdot\nabla_{\rm v} u)(s,x,\mathrm{v})\mathrm{d}x\mathrm{d}\mathrm{v}\right|\mathrm{d}s$$

$$= \int_0^t \left|\int_{\mathbb{R}^{2d}} \widetilde{\mathcal{R}}_j^x \Gamma_{s,t} p_{s,t}^\lambda(x,\mathrm{v})\cdot \Gamma_{s,t}\mathcal{R}_j^x(b^{(2)}\cdot\nabla_{\rm v} u)(s,x,\mathrm{v})\mathrm{d}x\mathrm{d}\mathrm{v}\right|\mathrm{d}s = \mathscr{I}_1+\mathscr{I}_2,$$

where

$$\mathscr{I}_1 := \int_0^t \left|\int_{\mathbb{R}^{2d}} \widetilde{\mathcal{R}}_j^x \Gamma_{s,t} p_{s,t}^\lambda(x,\mathrm{v})\cdot \Gamma_{s,t}(b^{(2)}\cdot\nabla_{\rm v}\mathcal{R}_j^x u)(s,x,\mathrm{v})\mathrm{d}x\mathrm{d}\mathrm{v}\right|\mathrm{d}s,$$

$$\mathscr{I}_2 := \int_0^t \left|\int_{\mathbb{R}^{2d}} \widetilde{\mathcal{R}}_j^x \Gamma_{s,t} p_{s,t}^\lambda(x,\mathrm{v})\cdot \Gamma_{s,t}[\mathcal{R}_j^x, b^{(2)}\cdot\nabla_{\rm v}]u(s,x,\mathrm{v})\mathrm{d}x\mathrm{d}\mathrm{v}\right|\mathrm{d}s.$$

For $\mathscr{I}_1$, noticing that by (6.10),

$$|b^{(2)}\cdot\nabla_{\rm v}\mathcal{R}_j^x u|(x,\mathrm{v}) \lesssim \left(|x|^{\frac{\gamma}{1+\alpha}}+|\mathrm{v}|^\beta\right)\|\nabla_{\rm v}\mathcal{R}_j^x u\|_\infty,$$

we have

$$\mathscr{I}_1 \lesssim \|\nabla_{\rm v}\mathcal{R}_j^x u\|_{\mathbb{L}_T^\infty}\int_0^t\int_{\mathbb{R}^{2d}} |\widetilde{\mathcal{R}}_j^x \Gamma_{s,t} p_{s,t}^\lambda|(x,\mathrm{v})\left(|x+\Pi_{s,t}\mathrm{v}|^{\frac{\gamma}{1+\alpha}}+|\mathrm{v}|^\beta\right)\mathrm{d}x\mathrm{d}\mathrm{v}\mathrm{d}s$$

$$\overset{(5.7)}{\lesssim} \|\nabla_{\rm v}\mathcal{R}_j^x u\|_{\mathbb{L}_T^\infty}\left(2^{-\frac{\alpha+\gamma}{1+\alpha}j}+2^{-\frac{\alpha+\beta}{1+\alpha}j}\right) \lesssim 2^{-\frac{\alpha+\gamma}{1+\alpha}j}\|\nabla_{\rm v} u\|_{\mathbb{L}_T^\infty(\mathbf{B}_{x,\infty}^{(\gamma-\beta)/(1+\alpha)})}.$$

For $\mathscr{I}_2$, by the commutator estimate (4.8), we have

$$\mathscr{I}_2 \lesssim \int_0^t\int_{\mathbb{R}^{2d}} |\widetilde{\mathcal{R}}_j^x \Gamma_{s,t} p_{s,t}^\lambda|(x,\mathrm{v})\|[\mathcal{R}_j^x, b^{(2)}\cdot\nabla_{\rm v}]u\|_\infty \mathrm{d}x\mathrm{d}\mathrm{v}\mathrm{d}s \lesssim 2^{-\frac{\alpha+\gamma}{1+\alpha}j}\|\nabla_{\rm v} u\|_{\mathbb{L}_T^\infty}.$$

Combining the above calculations, we obtain (6.19). □

Lemma 6.9. *Let $\alpha\in(0,2)$. For any $\beta\in(0,1)$, there is a constant $C>0$ such that for all $j\geq 5$ and $\lambda\geq 0$, $t\in[0,T]$,*

$$\int_0^t |\mathcal{R}_j^a P_{s,t}^\lambda f|(s,0,0)\mathrm{d}s \leq C2^{-(\alpha+\beta)j}\|f\|_{\mathbb{L}_T^\infty(\mathbf{C}_a^\beta)}, \tag{6.23}$$

$$\int_0^t |\mathcal{R}_j^x P_{s,t}^\lambda f|(s,0,0)\mathrm{d}s \leq C2^{-\frac{\alpha+\gamma}{1+\alpha}j}\|f\|_{\mathbb{L}_T^\infty(\mathbf{C}_x^{\gamma/(1+\alpha)})}. \tag{6.24}$$

Proof. We only prove the first one. The second one is similar and easier by (5.7). Let $\Theta=\Theta_j^{t-s}$ be as in Lemma 6.7. By definition (6.11) and Lemma 6.7, we have

$$\begin{aligned}\int_0^t |\mathcal{R}_j^a P_{s,t}^\lambda f|(s,0,0)\mathrm{d}s &= \int_0^t \left|\int_{\mathbb{R}^{2d}} \mathcal{R}_j^a \Gamma_{s,t} p_{s,t}^\lambda(x,\mathrm{v})\cdot\Gamma_{s,t} f(s,x,\mathrm{v})\mathrm{d}x\mathrm{d}\mathrm{v}\right|\mathrm{d}s\\ &= \int_0^t \left|\sum_{\ell\in\Theta}\int_{\mathbb{R}^{2d}} \mathcal{R}_j^a \Gamma_{s,t} p_{s,t}^\lambda(x,\mathrm{v})\cdot\Gamma_{s,t}\mathcal{R}_\ell^a f(s,x,\mathrm{v})\mathrm{d}x\mathrm{d}\mathrm{v}\right|\mathrm{d}s\\ &\le \int_0^t \sum_{\ell\in\Theta}\|\mathcal{R}_\ell^a f(s)\|_\infty\left(\int_{\mathbb{R}^{2d}}|\mathcal{R}_j^a\Gamma_{s,t}p_{s,t}^\lambda(x,\mathrm{v})|\mathrm{d}x\mathrm{d}\mathrm{v}\right)\mathrm{d}s\\ &\le \|f\|_{\mathbb{L}^\infty_T(\mathbf{B}^\beta_{a,\infty})}\int_0^t\sum_{\ell\in\Theta}2^{-\ell\beta}\left(\int_{\mathbb{R}^{2d}}|\mathcal{R}_j^a\Gamma_{s,t}p_{s,t}^\lambda(x,\mathrm{v})|\mathrm{d}x\mathrm{d}\mathrm{v}\right)\mathrm{d}s\\ &\lesssim \|f\|_{\mathbb{L}^\infty_T(\mathbf{C}^\beta_a)}\int_0^t\left(2^{-j}+(t-s)2^{(\alpha-1)j}\right)^\beta\left(\int_{\mathbb{R}^{2d}}|\mathcal{R}_j^a\Gamma_{s,t}p_{s,t}^\lambda(x,\mathrm{v})|\mathrm{d}x\mathrm{d}\mathrm{v}\right)\mathrm{d}s,\end{aligned}$$

which gives (6.23) by the application of (5.6). □

Now we are in a position to give

Proof of Theorem 6.3. (i) We first show the following estimates:

$$\|u\|_{\mathbb{L}^\infty_T(\mathbf{C}^{\alpha+\beta}_a)}\le C\|f\|_{\mathbb{L}^\infty_T(\mathbf{C}^\beta_a)},\tag{6.25}$$

and for any $\varepsilon\in(0,1)$,

$$\|u\|_{\mathbb{L}^\infty_T(\mathbf{C}_x^{(\alpha+\gamma)/(1+\alpha)})}\le C\Big(\|f\|_{\mathbb{L}^\infty_T(\mathbf{C}_x^{\gamma/(1+\alpha)})}+\|u\|_{\mathbb{L}^\infty_T(\mathbf{C}_x^{(\gamma-\beta)/(1+\alpha)}\mathbf{C}_{\mathrm{v}}^{\alpha+\varepsilon})}\Big).\tag{6.26}$$

Since $\vartheta<\beta\wedge(\alpha-1)$, by Lemmas 6.6, 6.8 and 6.9, we have

$$|\mathcal{R}_j^a u(t,\theta_t)|\lesssim 2^{-(\alpha+\beta)j}\|u\|_{\mathbb{L}^\infty_T(\mathbf{C}_a^{\alpha+\beta-\vartheta})}+2^{-(\alpha+\beta)j}\|f\|_{\mathbb{L}^\infty_T(\mathbf{C}_a^\beta)},\ j\ge j_0,$$

where j_0 is the same as in (iii) of Lemma 6.7, and for any $\varepsilon\in(0,1)$,

$$|\mathcal{R}_j^x u(t,\theta_t)|\lesssim 2^{-\frac{\alpha+\gamma}{1+\alpha}j}\Big(\|u\|_{\mathbb{L}^\infty_T(\mathbf{C}_x^{(\alpha+\gamma-\vartheta)/(1+\alpha)})}+\|u\|_{\mathbb{L}^\infty_T(\mathbf{C}_x^{(\gamma-\beta)/(1+\alpha)}\mathbf{C}_{\mathrm{v}}^{\alpha+\varepsilon})}+\|f\|_{\mathbb{L}^\infty_T(\mathbf{C}_x^{\gamma/(1+\alpha)})}\Big),\ j\ge j_0.$$

Moreover, it is clear that

$$|\mathcal{R}_j^a u(t,\theta_t)|\le\|u\|_{\mathbb{L}^\infty_T},\ j\in\mathbb{N}_0.$$

By (6.6), Theorem 2.2 and (2.11), for any $\varepsilon'>0$, the above estimates lead to

$$\begin{aligned}\|u(t)\|_{\mathbf{C}_a^{\alpha+\beta}}&\lesssim\|u(t)\|_{\mathbf{B}^{\alpha+\beta}_{a,\infty}}=\sup_{j\in\mathbb{N}_0}2^{(\alpha+\beta)j}\|\mathcal{R}_j^a u(t)\|_\infty\\ &\lesssim\|u\|_{\mathbb{L}^\infty_T(\mathbf{C}_a^{\alpha+\beta-\vartheta})}+\|f\|_{\mathbb{L}^\infty_T(\mathbf{C}_a^\beta)}\lesssim\varepsilon'\|u\|_{\mathbb{L}^\infty_T(\mathbf{C}_a^{\alpha+\beta})}+\|u\|_{\mathbb{L}^\infty_T}+\|f\|_{\mathbb{L}^\infty_T(\mathbf{C}_a^\beta)},\end{aligned}$$

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and

$$\begin{aligned}\|u(t)\|_{\mathbf{C}_x^{(\alpha+\gamma)/(1+\alpha)}} &\lesssim \|u(t)\|_{\mathbf{B}_{x,\infty}^{(\alpha+\gamma)/(1+\alpha)}} = \sup_{j\in\mathbb{N}_0} 2^{\frac{\alpha+\gamma}{1+\alpha}j}\|\mathcal{R}_j^x u(t)\|_\infty \\ &\lesssim \|u\|_{\mathbb{L}_T^\infty(\mathbf{C}_x^{(\alpha+\gamma-\vartheta)/(1+\alpha)})} + \|u\|_{\mathbb{L}_T^\infty(\mathbf{C}_x^{(\gamma-\beta)/(1+\alpha)}\mathbf{C}_{\rm v}^{\alpha+\varepsilon})} + \|f\|_{\mathbb{L}_T^\infty(\mathbf{C}_x^{\gamma/(1+\alpha)})} \\ &\lesssim \varepsilon'\|u\|_{\mathbb{L}_T^\infty(\mathbf{C}_x^{(\alpha+\gamma)/(1+\alpha)})} + \|u\|_{\mathbb{L}_T^\infty(\mathbf{C}_x^{(\gamma-\beta)/(1+\alpha)}\mathbf{C}_{\rm v}^{\alpha+\varepsilon})} + \|f\|_{\mathbb{L}_T^\infty(\mathbf{C}_x^{\gamma/(1+\alpha)})},\end{aligned}$$

which in turn yield (6.25) and (6.26) by taking $\varepsilon' = 1/2$ and (6.3).

(ii) For $j \geq 1$, we have

$$\partial_t \mathcal{R}_j^x u = \mathscr{L}_{\kappa;{\rm v}}^{(\alpha)}\mathcal{R}_j^x u + b\cdot\nabla\mathcal{R}_j^x u + \mathcal{R}_j^x f + [\mathcal{R}_j^x, \mathscr{L}_{\kappa;{\rm v}}^{(\alpha)}]u + [\mathcal{R}_j^x, b\cdot\nabla]u.$$

For $\theta \in (0,1]$ being small enough, by Schauder's estimate (6.25) and Corollary 4.5 with $\gamma_1 = \gamma \vee (1+\vartheta)$, we have

$$\begin{aligned}\|\mathcal{R}_j^x u\|_{\mathbb{L}_T^\infty(\mathbf{C}_a^{\alpha+\theta\beta})} &\lesssim \|\mathcal{R}_j^x f\|_{\mathbb{L}_T^\infty(\mathbf{C}_a^{\theta\beta})} + \|[\mathcal{R}_j^x, \mathscr{L}_{\kappa;{\rm v}}^{(\alpha)}]u\|_{\mathbb{L}_T^\infty(\mathbf{C}_a^{\theta\beta})} + \|[\mathcal{R}_j^x, b\cdot\nabla]u\|_{\mathbb{L}_T^\infty(\mathbf{C}_a^{\theta\beta})} \\ &\lesssim \|\mathcal{R}_j^x f\|_{\mathbb{L}_T^\infty(\mathbf{C}_a^{\theta\beta})} + 2^{-\frac{(1-\theta)\gamma}{1+\alpha}j}\Big(\varepsilon\|u\|_{\mathbb{L}_T^\infty(\mathbf{C}_x^{(\alpha+(1-\theta)\gamma+\theta\beta)/(1+\alpha)})} + \|u\|_{\mathbb{L}_T^\infty(\mathbf{C}_{\rm v}^{\alpha+\beta})}\Big),\end{aligned} \tag{6.27}$$

where $j \geq 5$ and $\varepsilon \in (0,1)$. On the other hand, for $j = 0, \cdots, 5$,

$$\|\mathcal{R}_j^x u\|_{\mathbb{L}_T^\infty(\mathbf{C}_a^{\alpha+\theta\beta})} \lesssim \|\mathcal{R}_j^x u\|_{\mathbb{L}_T^\infty(\mathbf{C}_x^{(\alpha+\theta\beta)/(1+\alpha)})} + \|\mathcal{R}_j^x u\|_{\mathbb{L}_T^\infty(\mathbf{C}_{\rm v}^{\alpha+\theta\beta})} \lesssim \|u\|_{\mathbb{L}_T^\infty(\mathbf{C}_{\rm v}^{\alpha+\beta})},$$

and also,

$$\begin{aligned}\sup_{j\in\mathbb{N}_0} 2^{\frac{(1-\theta)\gamma}{1+\alpha}j}\|\mathcal{R}_j^x f\|_{\mathbb{L}_T^\infty(\mathbf{C}_a^{\theta\beta})} &\overset{(4.2)}{\lesssim} \|f\|_{\mathbb{L}_T^\infty(\mathbf{C}_x^{(\theta\beta+(1-\theta)\gamma)/(1+\alpha)})} + \|f\|_{\mathbb{L}_T^\infty(\mathbf{C}_x^{(1-\theta)\gamma/(1+\alpha)}\mathbf{C}_{\rm v}^{\theta\beta})} \\ &\overset{(4.3)}{\lesssim} \|f\|_{\mathbb{L}_T^\infty(\mathbf{C}_x^{\gamma/(1+\alpha)})} + \|f\|_{\mathbb{L}_T^\infty(\mathbf{C}_{\rm v}^{\beta})}.\end{aligned} \tag{6.28}$$

Hence, by (4.2), (6.27), (6.28) and (6.25), we obtain that for any $\varepsilon \in (0,1)$,

$$\begin{aligned}\|u\|_{\mathbb{L}_T^\infty(\mathbf{C}_x^{(\alpha+(1-\theta)\gamma+\theta\beta)/(1+\alpha)})} + \|u\|_{\mathbb{L}_T^\infty(\mathbf{C}_x^{(1-\theta)\gamma/(1+\alpha)}\mathbf{C}_{\rm v}^{\alpha+\theta\beta})} &\lesssim \sup_{j\in\mathbb{N}_0} 2^{\frac{(1-\theta)\gamma}{1+\alpha}j}\|\mathcal{R}_j^x u\|_{\mathbb{L}_T^\infty(\mathbf{C}_a^{\alpha+\theta\beta})} \\ \lesssim \|f\|_{\mathbb{L}_T^\infty(\mathbf{C}_x^{\gamma/(1+\alpha)})} + \|f\|_{\mathbb{L}_T^\infty(\mathbf{C}_{\rm v}^{\beta})} + \varepsilon\|u\|_{\mathbb{L}_T^\infty(\mathbf{C}_x^{(\alpha+(1-\theta)\gamma+\theta\beta)/(1+\alpha)})},\end{aligned}$$

which implies by taking $\varepsilon = \frac{1}{2}$,

$$\|u\|_{\mathbb{L}_T^\infty(\mathbf{C}_x^{(1-\theta)\gamma/(1+\alpha)}\mathbf{C}_{\rm v}^{\alpha+\theta\beta})} \lesssim \|f\|_{\mathbb{L}_T^\infty(\mathbf{C}_x^{\gamma/(1+\alpha)})} + \|f\|_{\mathbb{L}_T^\infty(\mathbf{C}_{\rm v}^{\beta})}.$$

Substituting this into (6.26) with θ being small enough, we obtain (6.5). □

Remark 6.10. The restriction of $\alpha \in (1,2)$ is only used in Lemma 6.8, which is caused by the moment problem due to $1+\vartheta < \alpha$. In particular, if $b^{(1)}(t,x,{\rm v}) = {\rm v} + b^{(1)}(t,x)$, then under the following restrictions:

$$\tfrac{1+\vartheta}{1+\alpha} < \alpha, \ \ \tfrac{\gamma}{1+\alpha} < \alpha, \ \ \alpha+\beta > 1, \tag{6.29}$$

which implies $\alpha > \frac{\sqrt{5}-1}{2}$, Theorem 6.3 still holds for $\alpha \in (\frac{\sqrt{5}-1}{2}, 1]$. Here we conjecture that the moment restriction is superfluous. At this moment we do not know how to drop it. Such a problem also appears in

[8]. Moreover, if $b(t,x,\mathrm{v}) = (-\mathrm{v}, 0)$, which corresponds to the kinetic equation (1.7), then Theorem 6.3 still holds for all $\alpha \in (0,2)$.

We have the following existence of classical solutions.

Theorem 6.11. *Let $\alpha \in (1,2)$ and $\beta \in (0,1), \vartheta \in (0, \beta \wedge (\alpha-1))$, $\gamma \in (1, 1+\alpha)$. Under $(\mathbf{H}^{\alpha,\vartheta}_{\beta,\gamma})$, for any $f \in \mathbb{L}^\infty_{loc}(\mathbf{C}^{\gamma/(1+\alpha)}_x \cap \mathbf{C}^\beta_{\mathrm{v}})$, there is a unique classical solution u in the sense of Definition 6.1 such that for some $\theta = \theta(\alpha,\beta,\gamma,\vartheta) > 0$ small enough, for any $T > 0$ and some $C > 0$ being independent of $\lambda > 0$,*

$$\|u\|_{\mathbb{L}^\infty_T(\mathbf{C}^{(\alpha+\gamma)/(1+\alpha)}_x \cap \mathbf{C}^{\alpha+\beta}_{\mathrm{v}})} + \|u\|_{\mathbb{L}^\infty_T(\mathbf{C}^{(1-\theta)\gamma/(1+\alpha)}_x \mathbf{C}^{\alpha+\theta\beta}_{\mathrm{v}})} \leq C\|f\|_{\mathbb{L}^\infty_T(\mathbf{C}^{\gamma/(1+\alpha)}_x \cap \mathbf{C}^\beta_{\mathrm{v}})}, \ \|u\|_{\mathbb{L}^\infty_T} \leq \lambda^{-1}\|f\|_{\mathbb{L}^\infty_T}. \tag{6.30}$$

Proof. Let $(\rho_n)_{n\in\mathbb{N}}$ and $(\rho'_n)_{n\in\mathbb{N}}$ be the usual mollifiers in $\mathbb{R}^{3d}$ and $\mathbb{R}^{2d}$ respectively. Define

$$\kappa_n(t,x,\mathrm{v},w) := \kappa(t,\cdot) * \rho_n(x,\mathrm{v},w), \ n \in \mathbb{N},$$

and

$$b_n(t,x,\mathrm{v}) := b(t,\cdot) * \rho'_n(x,\mathrm{v}), \quad f_n(t,x,\mathrm{v}) := f(t,\cdot) * \rho'_n(x,\mathrm{v}), \ n \in \mathbb{N}.$$

Fix $\lambda, T > 0$. By Theorem 8.2 in appendix, there is a unique smooth $u_n \in C([0,T]; \mathscr{C}^2(\mathbb{R}^{2d}))$ solving the following PDE:

$$\partial_t u_n = \mathscr{L}^{(\alpha)}_{\kappa_n;\mathrm{v}} u_n + b_n \cdot \nabla u_n - \lambda u_n + f_n, \quad u_n(0) = 0. \tag{6.31}$$

Under $(\mathbf{H}^{\alpha,\vartheta}_{\beta,\gamma})$, it is easy to see that κ_n and b_n satisfy $(\mathbf{H}^{\alpha,\vartheta}_{\beta,\gamma})$ uniformly. By Theorem 6.3, there is a constant $C > 0$ such that for all $n \in \mathbb{N}$,

$$\|u_n\|_{\mathbb{L}^\infty_T(\mathbf{C}^{(\alpha+\gamma)/(1+\alpha)}_x \cap \mathbf{C}^{\alpha+\beta}_{\mathrm{v}})} + \|u_n\|_{\mathbb{L}^\infty_T(\mathbf{C}^{(1-\theta)\gamma/(1+\alpha)}_x \mathbf{C}^{\alpha+\theta\beta}_{\mathrm{v}})} \leq C\|f_n\|_{\mathbb{L}^\infty_T(\mathbf{C}^{\gamma/(1+\alpha)}_x \cap \mathbf{C}^\beta_{\mathrm{v}})} \leq C\|f\|_{\mathbb{L}^\infty_T(\mathbf{C}^{\gamma/(1+\alpha)}_x \cap \mathbf{C}^\beta_{\mathrm{v}})}. \tag{6.32}$$

In particular, since $\alpha \in (1,2)$ and $\gamma \in (1, 1+\vartheta)$, we also have for some $\varepsilon > 0$,

$$\|\nabla u_n\|_{\mathbb{L}^\infty_T(\mathbf{C}^\varepsilon)} \leq C\|u_n\|_{\mathbb{L}^\infty_T(\mathbf{C}^{(\alpha+\gamma)/(1+\alpha)}_x \cap \mathbf{C}^{\alpha+\beta}_{\mathrm{v}})} \leq C\|f\|_{\mathbb{L}^\infty_T(\mathbf{C}^{\gamma/(1+\alpha)}_x \cap \mathbf{C}^\beta_{\mathrm{v}})}.$$

Hence, from approximation equation (6.31) and the above uniform estimates, one sees that

$$\sup_n \|\partial_t u_n \cdot \mathbf{1}_{|x|+|\mathrm{v}|\leq m}\|_{\mathbb{L}^\infty_T} \leq C_m, \quad m \in \mathbb{N}.$$

Thus by Ascolli-Arzela's theorem and a standard diagonalization argument, there are subsequence n_k and continuous function $u : [0,T] \times \mathbb{R}^{2d} \to \mathbb{R}$ such that for each $m \in \mathbb{N}$,

$$\lim_{k\to\infty} \sup_{t\in[0,T],|x|+|\mathrm{v}|\leq m} |u_{n_k}(t,x,\mathrm{v}) - u(t,x,\mathrm{v})| = 0.$$

Moreover, we also have

$$u \in \mathbb{L}^\infty_T(\mathbf{C}^{(\alpha+\gamma)/(1+\alpha)}_x \cap \mathbf{C}^{\alpha+\beta}_{\mathrm{v}}).$$

In fact, by (2.10) and Fatou's lemma, we have

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$$\begin{aligned}\|u(t)\|_{\mathbf{C}_{\mathrm{v}}^{\alpha+\beta}} &\lesssim \sup_{j\geq 0} 2^{(\alpha+\beta)j}\|\mathcal{R}_j^{\mathrm{v}}u(t)\|_\infty \lesssim \sup_{j\geq 0} 2^{(\alpha+\beta)j} \varliminf_{n\to\infty} \|\mathcal{R}_j^{\mathrm{v}}u_n(t)\|_\infty \\ &\lesssim \varliminf_{n\to\infty} \|u_n(t)\|_{\mathbf{C}_{\mathrm{v}}^{\alpha+\beta}} \overset{(6.32)}{\lesssim} \|f\|_{\mathbb{L}_T^\infty(\mathbf{C}_x^{\gamma/(1+\alpha)}\cap\mathbf{C}_{\mathrm{v}}^\beta)}.\end{aligned}$$

Noticing the following interpolation inequality (see [33, Theorem 3.2.1]),

$$\|\nabla f\|_\infty \leq C\|f\|_{\mathbf{C}^{1+\varepsilon}}^{1/(1+\varepsilon)}\|f\|_\infty^{\varepsilon/(1+\varepsilon)},$$

we further have for each $m\in\mathbb{N}$,

$$\lim_{k\to\infty} \sup_{t\in[0,T],|x|+|\mathrm{v}|\leq m} |\nabla u_{n_k}(t,x,\mathrm{v}) - \nabla u(t,x,\mathrm{v})| = 0.$$

By taking limits for equation (6.31), we obtain that u satisfies (6.1) in the sense of Definition 6.1. By (6.32) and (6.3), we complete the proof. □

Remark 6.12. If we do not assume $\gamma>1$ in Theorem 6.11, then under $(\mathbf{H}^{\alpha,\vartheta}_{\beta,\gamma})$, for any $f\in\mathbb{L}_T^\infty(\mathbf{C}_a^\beta)$, we can show the existence of $u\in\mathbb{L}_T^\infty(\mathbf{C}_a^{\alpha+\beta})$ solving PDE (6.1) in the distributional sense since $b\cdot\nabla u$ is a well-defined distribution under the above regularity.

7. Degenerate SDEs with Hölder drifts

7.1. Pathwise uniqueness of SDEs with multiplicative Lévy noises

Let $L_t^{(\alpha)}$ be a symmetric and rotationally invariant α-stable process with $\alpha\in(1,2)$ on some probability space $(\Omega,\mathscr{F},\mathbb{P})$, so that whose generator is given by the fractional Laplacian $\Delta^{\alpha/2}$. In this section we consider the following degenerate SDE with jumps in $\mathbb{R}^{2d}$:

$$\mathrm{d}Z_{s,t} = b(t,Z_{s,t})\mathrm{d}t + (0,\sigma(t,Z_{s,t})\mathrm{d}L_t^{(\alpha)}),\quad Z_{s,s}=z\in\mathbb{R}^{2d},\ t\geq s\geq 0, \tag{7.1}$$

where $\sigma:\mathbb{R}_+\times\mathbb{R}^{2d}\to\mathbb{R}^d\otimes\mathbb{R}^d$ and $b:\mathbb{R}_+\times\mathbb{R}^{2d}\to\mathbb{R}^{2d}$ are measurable functions satisfying

$(\widetilde{\mathbf{H}}^{\alpha,\vartheta}_{\beta,\gamma})$ σ is Lipschitz continuous in x uniformly in t, and for some $c_0\geq 1$ and all $t\geq 0$,

$$c_0^{-1}|\xi|\leq|\sigma(t,z)\xi|\leq c_0|\xi|,\quad \xi\in\mathbb{R}^d,\ z\in\mathbb{R}^{2d},$$

and for some $\vartheta,\beta\in(0,1)$ and $\gamma\in(1,1+\alpha)$,

$$|b(t,0)| + [b(t,\cdot)]_{\mathbf{C}_x^{\gamma/(1+\alpha)}} + [b^{(2)}(t,\cdot)]_{\mathbf{C}_{\mathrm{v}}^\beta} + [\![b^{(1)}(t,\cdot)]\!]_{\mathbf{C}_a^{1+\vartheta}} \leq c_0.$$

Moreover, (6.2) holds.

Let $N(\mathrm{d}t,\mathrm{d}w)$ be the Poisson random measure associated with $L^{(\alpha)}$, i.e.,

$$N((0,t]\times\Gamma) := \sum_{0<s\leq t} 1_\Gamma(L_s^{(\alpha)} - L_{s-}^{(\alpha)}),\quad t>0,\ \Gamma\in\mathscr{B}(\mathbb{R}^d\setminus\{0\}).$$

Let $\tilde N(\mathrm{d}t,\mathrm{d}w) := N(\mathrm{d}t,\mathrm{d}w) - \mathrm{d}t\mathrm{d}w/|w|^{d+\alpha}$ be the compensated Poisson random measure. By the Lévy-Itô decomposition, we can write for each $t>0$,

$$L_t^{(\alpha)} = \int_0^t \int_{\mathbb{R}^d} w\tilde{N}(\mathrm{d}s, \mathrm{d}w).$$

Thus, if we let $Z_{s,t} = (X_{s,t}, V_{s,t})$, then SDE (7.1) can be written as

$$\begin{cases} \mathrm{d}X_{s,t} = b^{(1)}(t, Z_{s,t})\mathrm{d}t, \quad (X_{s,s}, V_{s,s}) = (x, \mathrm{v}), \\ \mathrm{d}V_{s,t} = b^{(2)}(t, Z_{s,t})\mathrm{d}t + \int_{\mathbb{R}^d} \sigma(t, Z_{s,t})w\tilde{N}(\mathrm{d}t, \mathrm{d}w). \end{cases}$$

In particular, the generator of this SDE is given by $\mathscr{L}^{(\alpha)}_{\sigma;\mathrm{v}} + b \cdot \nabla$ with

$$\begin{aligned} \mathscr{L}^{(\alpha)}_{\sigma;\mathrm{v}} f(x, \mathrm{v}) &= \mathrm{p.v.} \int_{\mathbb{R}^d} \Big(f(x, \mathrm{v} + \sigma(t, z)w) - f(x, \mathrm{v})\Big) \frac{\mathrm{d}w}{|w|^{d+\alpha}} \\ &= \mathrm{p.v.} \int_{\mathbb{R}^d} \Big(f(x, \mathrm{v} + w) - f(x, \mathrm{v})\Big) \kappa(t, z, w) \frac{\mathrm{d}w}{|w|^{d+\alpha}}, \end{aligned}$$

where $z = (x, \mathrm{v})$ and

$$\kappa(t, z, w) := \det(\sigma^{-1}(t, z))|w|^{d+\alpha}/|\sigma^{-1}(t, z)w|^{d+\alpha}.$$

Under $(\widetilde{\mathbf{H}}^{\alpha,\vartheta}_{\beta,\gamma})$, it is easy to see that for some $c_0 > 1$,

$$c_0^{-1} \leq \kappa(t, z, w) \leq c_0, \quad |\kappa(t, z, w) - \kappa(t, z', w)| \leq c_0|z - z'|.$$

We have the following result.

Theorem 7.1. *Let $\alpha \in (1, 2)$, $\beta \in (1 - \frac{\alpha}{2}, 1)$, $\vartheta \in (0, \beta \wedge (\alpha - 1))$ and $\gamma \in (1 + \frac{\alpha}{2}, 1 + \alpha)$. Under $(\widetilde{\mathbf{H}}^{\alpha,\vartheta}_{\beta,\gamma})$, for each $s \geq 0$ and $z \in \mathbb{R}^{2d}$, there exists a unique strong solution $(Z_{s,t})_{t\geq s}$ to SDE (7.1).*

Proof. Since the coefficients are continuous and linear growth, the existence of a solution is well known (cf. [32]). By Yamada-Watanabe's theorem (cf. [28]), it suffices to show the pathwise uniqueness. Without loss of generality, we assume $s = 0$ and simply write

$$Z_t := Z_{0,t}.$$

Since b is unbounded, to construct Zvonkin's transformation (cf. [56]), we need to cutoff b. For $m \in \mathbb{N}$, let χ_m be a smooth cutoff function in $\mathbb{R}^{2d}$ with

$$\chi_m(z) = 1, \ |z| \leq m, \ \chi_m(z) = 0, \ |z| > m + 1.$$

Fix $T > 0$ and $m \in \mathbb{N}$. Consider the following backward equation:

$$\partial_s \mathbf{u}^m_\lambda + \mathscr{L}^{(\alpha)}_{\sigma;\mathrm{v}} \mathbf{u}^m_\lambda - \lambda \mathbf{u}^m_\lambda + b \cdot \nabla \mathbf{u}^m_\lambda + b\chi_m = 0, \ \mathbf{u}^m_\lambda(T, \cdot) = 0. \tag{7.2}$$

Under $(\widetilde{\mathbf{H}}^{\alpha,\vartheta}_{\beta,\gamma})$, by Theorem 6.11, there is a unique classical solution $\mathbf{u}^m_\lambda$ to the above equation with regularity: for some $C_m \geq 1$, $\theta > 0$ small enough and all $\lambda \geq 1$,

$$\|\mathbf{u}^m_\lambda\|_{\mathbb{L}^\infty_T(\mathbf{C}_x^{(\alpha+\gamma)/(1+\alpha)} \cap \mathbf{C}_{\mathrm{v}}^{\alpha+\beta})} + \|\mathbf{u}^m_\lambda\|_{\mathbb{L}^\infty_T(\mathbf{C}_x^{(1-\theta)\gamma/(1+\alpha)} \mathbf{C}_{\mathrm{v}}^{\alpha+\theta\beta})} \leq C_m, \quad \|\mathbf{u}^m_\lambda\|_{\mathbb{L}^\infty_T} \leq C_m \lambda^{-1}.$$

Since $\gamma \in (1+\frac{\alpha}{2}, 1+\alpha)$ and $\beta \in (1-\frac{\alpha}{2}, 1)$, by interpolation inequalities (2.11) and (4.3), there is an $\varepsilon_0 > 0$ small enough such that

$$\|\mathbf{u}^m_\lambda\|_{\mathbb{L}^\infty_T(\mathbf{C}^{1+\varepsilon_0}_x \mathbf{C}^{\alpha/2+\varepsilon_0}_{\rm v})} + \|\mathbf{u}^m_\lambda\|_{\mathbb{L}^\infty_T(\mathbf{C}^{1+\alpha/2+\varepsilon_0}_{\rm v})} \le c_\lambda,$$

where $c_\lambda \to 0$ as $\lambda \to \infty$. In particular, for $\lambda \ge 1$ large enough,

$$\|\nabla \mathbf{u}^m_\lambda\|_{\mathbb{L}^\infty_T(\mathbf{C}^{\alpha/2+\varepsilon_0}_{\rm v})} \le \tfrac{1}{2}. \tag{7.3}$$

Define

$$\Phi^m_t(z) := z + \mathbf{u}^m_\lambda(t,z).$$

By (7.3), one sees that

$$z \mapsto \Phi^m_t(z) \text{ is a diffeomorphism on } \mathbb{R}^{2d}$$

and

$$\|\nabla \Phi^m_t\|_\infty + \|\nabla(\Phi^m_t)^{-1}\|_\infty \le 2.$$

Moreover, by (7.2) we have

$$\partial_s \Phi^m + \mathscr{L}^{(\alpha)}_{\sigma;{\rm v}} \Phi^m + b\cdot\nabla\Phi^m = b(1-\chi_m) + \lambda \mathbf{u}^m_\lambda. \tag{7.4}$$

Let Z_t and Z'_t be two solutions of SDE (7.1) defined on the same probability space with the same starting point z. Define a stopping time

$$\tau_m := \inf\Big\{t > 0 : |Z_t| \wedge |Z'_t| \ge m+1 \text{ or } |\Delta L^{(\alpha)}_t| > m\Big\},$$

and let

$$g^m_s(z,w) := \Phi^m_s(z + (0, \sigma(s,z)w)) - \Phi^m_s(z).$$

By Itô's formula and (7.4), one sees that

$$\begin{aligned}\Phi^m_{t\wedge\tau_m}(Z_{t\wedge\tau_m}) - \Phi^m_0(z) &= \int_0^{t\wedge\tau_m}\int_{\mathbb{R}^d} \Big(\Phi^m_s(Z_{s-} + (0,\sigma(s,Z_{s-})w)) - \Phi^m_s(Z_{s-})\Big)\tilde N({\rm d}s,{\rm d}w)\\ &\quad + \int_0^{t\wedge\tau_m} \Big(\partial_s\Phi^m + b\cdot\nabla\Phi^m + \mathscr{L}^{(\alpha)}_{\sigma;{\rm v}}\Phi^m\Big)(s,Z_s){\rm d}s\\ &= \int_0^{t\wedge\tau_m}\int_{B_m} g^m_s(Z_{s-},w)\tilde N({\rm d}s,{\rm d}w) + \lambda\int_0^{t\wedge\tau_m} \mathbf{u}^m_\lambda(s,Z_s){\rm d}s,\end{aligned}$$

where we have used that $(b(1-\chi_m))(s,Z_s) = 0$ for $s < \tau_m$ and

$$\int_0^{t\wedge\tau_m}\int_{B^c_m} g^m_s(Z_{s-},w)\tilde N({\rm d}s,{\rm d}w) = \sum_{0<s\le t\wedge\tau_m} g^m_s(Z_{s-},\Delta L^{(\alpha)}_s)\cdot \mathbf{1}_{|\Delta L^{(\alpha)}_s|>m} = 0.$$

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Noticing that

$$\nabla_z g_s^m(z,w) = (\nabla\Phi_s^m)(z+(0,\sigma(s,z)w))(\mathbb{I}_{2d\times 2d}+(0,\nabla\sigma(s,z)w)) - \nabla\Phi_s^m(z),$$

by (7.3) we have

$$\|\nabla_z g_s^m(\cdot,w)\|_\infty \le 2(\|\sigma\|_\infty|w|)^{\alpha/2+\varepsilon_0} + 2\|\nabla_z\sigma\|_\infty|w|.$$

Hence, by the isometry formula of stochastic integrals,

$$\begin{aligned}\mathbb{E}|Z_{t\wedge\tau_m}-Z'_{t\wedge\tau_m}|^2 &\le 4\mathbb{E}|\Phi^m_{t\wedge\tau_m}(Z_{t\wedge\tau_m})-\Phi^m_{t\wedge\tau_m}(Z'_{t\wedge\tau_m})|^2\\ &\lesssim \mathbb{E}\int_0^{t\wedge\tau_m}\int_{B_m}|g_s^m(Z_{s-},w)-g_s^m(Z'_{s-},w)|^2\frac{\mathrm{d}w}{|w|^{d+\alpha}}\mathrm{d}s\\ &\quad+\lambda\mathbb{E}\int_0^{t\wedge\tau_m}|\mathbf{u}_\lambda^m(s,Z_s)-\mathbf{u}_\lambda^m(s,Z'_s)|^2\mathrm{d}s\\ &\lesssim \mathbb{E}\int_0^{t\wedge\tau_m}|Z_s-Z'_s|^2\mathrm{d}s\left(\int_{B_m}\frac{(|w|^{\alpha+2\varepsilon_0}+|w|^2)\mathrm{d}w}{|w|^{d+\alpha}}+\lambda\right),\end{aligned}$$

which yields by Gronwall's inequality

$$Z_{t\wedge\tau_m} = Z'_{t\wedge\tau_m},\quad t\ge 0.$$

Finally, letting $m\to\infty$, we obtain the pathwise uniqueness. □

Remark 7.2. By suitable localization technique, we can directly construct the solution by Zvonkin's transformation without using Yamada-Watanabe's theorem.

7.2. C^1-stochastic diffeomorphism flows for SDEs with additive Lévy noise

In this subsection we consider the C^1-stochastic diffeomorphism flows property for SDE (7.1) with additive Lévy noises. We introduce the following spaces: For a Fréchet space $\mathbb{F}$ and time interval I, define

$$C(I;\mathbb{F}) := \{f: I\to\mathbb{F} \text{ is continuous}\},\quad D(I;\mathbb{F}) := \{f: I\to\mathbb{F} \text{ is càdlàg}\}.$$

For $k\in\mathbb{N}_0$, let $\mathbb{C}^k$ be the Fréchet space of all k-order continuous differentiable functions with Fréchet metric:

$$d(f,g) := \sum_{j=0}^k\sum_{n\in\mathbb{N}} 2^{-n}\left(1\wedge\sup_{|x|<n}|\nabla^j f(x)-\nabla^j g(x)|\right).$$

We have

Theorem 7.3. *Let $\alpha\in(1,2)$, $\gamma\in(1+\frac{\alpha}{2},1+\alpha)$ and $\beta\in(1-\frac{\alpha}{2},1)$. Assume $\sigma\equiv 1$ and*

$$b(t,x,\mathrm{v}) = (\mathrm{v}+b^{(1)}(t,x), b^{(2)}(t,x,\mathrm{v})), \tag{7.5}$$

with

$$b^{(1)} \in \mathbb{L}^\infty_{loc}(\mathbf{C}^{\gamma/(1+\alpha)}_x), \quad b^{(2)} \in \mathbb{L}^\infty_{loc}(\mathbf{C}^{\gamma/(1+\alpha)}_x \cap \mathbf{C}^\beta_{\rm v}).$$

Then the unique strong solution $\{Z_{s,t}(z), t > s \geq 0, z \in \mathbb{R}^{2d}\}$ of SDE (7.1) forms a C^1-stochastic diffeomorphism flow. More precisely, there is a null set $\mathcal{N}$ such that for all $\omega \notin \mathcal{N}$,

(i) *For all $0 \leq s < r < t$, it holds that*

$$Z_{s,t}(z,\omega) = Z_{r,t}(Z_{s,r}(z,\omega),\omega), \quad \forall z \in \mathbb{R}^{2d},$$

and

$$z \mapsto Z_{s,t}(z,\omega) \text{ is a } C^1\text{-diffeomorphism on } \mathbb{R}^{2d}.$$

(ii) *$t \mapsto Z_{s,t}(\cdot,\omega) \in D([s,\infty);\mathbb{C}^1)$ and $s \mapsto Z_{s,t}(\cdot,\omega) \in D([0,t];\mathbb{C}^1)$.*

Proof. Fix $T, \lambda > 0$. Consider the following backward equation:

$$\partial_s \mathbf{u}_\lambda + \Delta^{\alpha/2}_{\rm v} \mathbf{u}_\lambda - \lambda \mathbf{u}_\lambda + b \cdot \nabla \mathbf{u}_\lambda + b = 0, \ \mathbf{u}_\lambda(T,\cdot) = 0,$$

where $\Delta^{\alpha/2}_{\rm v}$ is the fractional Laplacian acting on the variable v. Under the assumptions of the theorem, by Theorem 6.11, there is a unique classical solution $\mathbf{u}^m_\lambda$ to the above equation with the following regularity: for some $C \geq 1$, $\theta > 0$ small enough and all $\lambda \geq 1$,

$$\|\mathbf{u}_\lambda\|_{\mathbb{L}^\infty_T(\mathbf{C}^{(\alpha+\gamma)/(1+\alpha)}_x \cap \mathbf{C}^{\alpha+\beta}_{\rm v})} + \|\mathbf{u}_\lambda\|_{\mathbb{L}^\infty_T(\mathbf{C}^{(1-\theta)\gamma/(1+\alpha)}_x \mathbf{C}^{\alpha+\theta\beta}_{\rm v})} \leq C, \quad \|\mathbf{u}_\lambda\|_{\mathbb{L}^\infty_T} \leq C\lambda^{-1}.$$

As in Theorem 7.1, by (2.11) and (4.3), there are $\lambda \geq 1$ large enough and $\varepsilon_0 > 0$ such that

$$\|\nabla \mathbf{u}_\lambda(t,\cdot,\cdot)\|_{\mathbf{C}^{\alpha/2+\varepsilon_0}_{\rm v}} \leq \tfrac{1}{2}, \ \|\mathbf{u}_\lambda\|_{\mathbb{L}^\infty_T(\mathbf{C}^{1+\varepsilon_0}_x \mathbf{C}^{\alpha/2+\varepsilon_0}_{\rm v})} + \|\mathbf{u}_\lambda\|_{\mathbb{L}^\infty_T(\mathbf{C}^{1+\alpha/2+\varepsilon_0}_{\rm v})} < \infty. \tag{7.6}$$

Define

$$\Phi_t(z) := z + \mathbf{u}_\lambda(t,z).$$

By (7.6), one sees that

$$z \mapsto \Phi_t(z) \text{ is a diffeomorphism on } \mathbb{R}^{2d}$$

and

$$\|\nabla \Phi_t\|_\infty + \|\nabla \Phi^{-1}_t\|_\infty \leq 2.$$

Let

$$\widetilde{Z}_t := \Phi_t(Z_t).$$

By Itô's formula, one sees that

$$\begin{aligned}\widetilde{Z}_t = \Phi_t(Z_t) &= \Phi_0(z) + \int_0^t \Big(\partial_s\Phi + b\cdot\nabla\Phi + \Delta_{\rm v}^{\alpha/2}\Phi\Big)(s, Z_s)\mathrm{d}s \\ &\quad + \int_0^t\int_{\mathbb{R}^d} \Big(\Phi_s(Z_{s-} + (0,w)) - \Phi_s(Z_{s-})\Big)\tilde{N}(\mathrm{d}s,\mathrm{d}w) \\ &= \tilde{z} + \int_0^t \widetilde{b}(s,\widetilde{Z}_s)\mathrm{d}s + \int_0^t\int_{\mathbb{R}^d} \widetilde{g}_s(\widetilde{Z}_{s-},w)\tilde{N}(\mathrm{d}s,\mathrm{d}w), \end{aligned} \tag{7.7}$$

where $\tilde{z} := \Phi_0(z)$ and

$$\widetilde{b}(t,z) := \lambda\mathbf{u}_\lambda(t,\Phi_t^{-1}(z)), \quad \widetilde{g}_t(z,w) := \Phi_t(\Phi_t^{-1}(z) + (0,w)) - z.$$

In particular, $\{Z_{s,t}, t \geq 0\}$ solves SDE (7.1) if and only if $\{\widetilde{Z}_{s,t}, t \geq 0\}$ solves SDE (7.7) (see [10, Lemma 3.4]).

Claim: There are $\varepsilon_1, \varepsilon_2 \in (0,\varepsilon_0)$ and constant $C > 0$ such that for all $t \in [0,T]$ and $w \in \mathbb{R}^d$,

$$\|\nabla_z\widetilde{g}_t(\cdot,w)\|_{\mathbf{C}^{\varepsilon_1}} \leq C(|w|^{\alpha/2+\varepsilon_2} \wedge 1), \quad \|\nabla_z\widetilde{b}\|_{\mathbb{L}^\infty_T(\mathbf{C}^{\varepsilon_1})} < \infty.$$

Indeed, noticing that

$$\begin{aligned}\nabla_z\widetilde{g}_t(z,w) &= (\nabla\Phi_t)(\Phi_t^{-1}(z) + (0,w))\cdot\nabla\Phi_t^{-1}(z) - \mathbb{I} \\ &= \Big((\nabla\Phi_t)(\Phi_t^{-1}(z) + (0,w)) - \nabla\Phi_t\circ\Phi_t^{-1}(z)\Big)\cdot\nabla\Phi_t^{-1}(z),\end{aligned}$$

by (7.6) we have

$$\|\nabla_z\widetilde{g}_t(\cdot,w)\|_\infty \leq \|\nabla\Phi_t\|_{\mathbf{C}_{\rm v}^{\alpha/2+\varepsilon_0}}(|w|^{\alpha/2+\varepsilon_0} \wedge 1)\|\nabla\Phi_t^{-1}\|_\infty \lesssim 1 \wedge |w|^{\alpha/2+\varepsilon_0},$$

and by definition,

$$[\nabla_z\widetilde{g}_t(\cdot,w)]_{\mathbf{C}^{\varepsilon_0}} \leq [\nabla\Phi_t(\Phi_t^{-1}(\cdot) + (0,w))]_{\mathbf{C}^{\varepsilon_0}}\|\nabla\Phi_t^{-1}\|_\infty + \|\nabla\Phi_t\|_\infty[\nabla\Phi_t^{-1}]_{\mathbf{C}^{\varepsilon_0}} \lesssim 1.$$

Thus we obtain the first claim by standard interpolation technique. The second one is easy by definition and (7.6).

By the above claim and [23, Theorem 4.1], the unique solutions of SDE (7.7) and so SDE (7.1) define a C^1-stochastic diffeomorphism flow and (i), (ii) hold. See also [41,10] for more details. □

Remark 7.4. Although we here only consider the symmetric and rotationally invariant α-stable additive noise, by the same argument as used in [10], it is possible to consider more general additive Lévy noises, even cylindrical Lévy noises. On the other hand, by Remark 6.10, the α in Theorem 7.3 in fact can be less than 1, but with a lower bound $(\sqrt{17}-1)/4 \approx 0.78078$. Indeed, by restriction (6.29) and $\gamma \in (1+\frac{\alpha}{2}, 1+\alpha)$, we need to require

$$\alpha^2 + \alpha > 1 + \tfrac{\alpha}{2} \Rightarrow \alpha > (\sqrt{17}-1)/4.$$

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7.3. Application to random transport equations with Hölder coefficients

In this subsection we apply Theorem 7.3 to a random transport equation with Hölder coefficient. First of all, the following corollary is an easy consequence of Theorem 7.3.

Corollary 7.5. *Let $\alpha \in (1,2)$ and $\gamma \in (\frac{2+\alpha}{2(1+\alpha)}, 1)$. Assume that*

$$b^{(1)} : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d \in \mathbb{L}^\infty_{loc}(\mathbf{C}^\gamma).$$

Consider the following random ODE

$$\mathrm{d}Y_{s,t}(x,\omega)/\mathrm{d}t = b^{(1)}(t, Y_{s,t}(x,\omega)) + L^{(\alpha)}_t(\omega),\ \ t > s,\ \ Y_{s,s} = x. \tag{7.8}$$

For $\mathbb{P}$-almost all ω, there exists a family of solutions $\{Y_{s,t}(x,\omega), x \in \mathbb{R}^d, 0 \le s < t < \infty\}$ to the above random ODE so that

(i) *For each $s < r < t$, it holds that*

$$Y_{s,t}(x,\omega) = Y_{r,t}(Y_{s,r}(x,\omega),\omega), \quad x \in \mathbb{R}^d, \tag{7.9}$$

and

$$x \mapsto Y_{s,t}(x,\omega) \text{ is a } C^1\text{-diffeomorphism on } \mathbb{R}^d. \tag{7.10}$$

(ii) *For each $s \ge 0$, $t \mapsto Y_{s,t}(\cdot,\omega) \in C([s,\infty), \mathbb{C}^0) \cap D([s,\infty), \mathbb{C}^1)$.*

Proof. For $z = (x, \mathrm{v})$, by Theorem 7.3 let

$$Z_{s,t}(z) := (X_{s,t}(x,\mathrm{v}), L^{(\alpha)}_t - L^{(\alpha)}_s + \mathrm{v})$$

be the unique solution of degenerate SDE (7.1) with

$$\sigma \equiv 1, \quad b(t,x,\mathrm{v}) := (\mathrm{v} + b^{(1)}(t,x), 0).$$

Now we define

$$Y_{s,t}(x) := X_{s,t}(x, L^{(\alpha)}_s).$$

Clearly, it solves ODE (7.8). Now we check (i) and (ii) hold. By (i) of Theorem 7.3, we have

$$\begin{aligned}(Y_{s,t}(x), L^{(\alpha)}_t) &= (X_{s,t}(x, L^{(\alpha)}_s), L^{(\alpha)}_t) = Z_{s,t}(x, L^{(\alpha)}_s) = Z_{r,t} \circ Z_{s,r}(x, L^{(\alpha)}_s) \\ &= Z_{r,t}(X_{s,r}(x, L^{(\alpha)}_s), L^{(\alpha)}_r) = (X_{r,t}(X_{s,r}(x, L^{(\alpha)}_s), L^{(\alpha)}_r), L^{(\alpha)}_t) = (Y_{r,t} \circ Y_{s,r}(x), L^{(\alpha)}_t),\end{aligned}$$

which implies (7.9) and (7.10) by (i) of Theorem 7.3. Finally, (ii) follows by (ii) of Theorem 7.3 and equation (7.8). □

Remark 7.6. Here an open question is to show Davie's uniqueness [16] for the above random ODE, that is, for almost all ω, ODE (7.8) has a unique solution. See [42] for the study of random ODE $\mathrm{d}Y_t/\mathrm{d}t = b(t, Y_t + L^{(\alpha)}_t)$. We will study this problem in a future work.

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Finally, to apply the above result to random transport equations, we need the following real analysis result, which can be proven by the completely same method as in [43, p149, Theorem 7.21]. We omit the details.

Lemma 7.7. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Assume that for each point $t \in [a, b)$, the right derivative $f'_+ := \lim_{\varepsilon\downarrow 0}(f(t+\varepsilon) - f(t))/\varepsilon$ exists and $f'_+ \in L^1([a, b])$. Then f is absolutely continuous on $[a, b]$.*

Now we can state the following result.

Theorem 7.8. *Let $\alpha \in (1, 2)$. Assume $b \in C(\mathbb{R}_+ \times \mathbb{R}^d) \cap \mathbb{L}^\infty_{loc}(\mathbf{C}^\gamma)$ with $\gamma \in (\frac{2+\alpha}{2(1+\alpha)}, 1)$. For any $\varphi \in \mathscr{C}^1$ and almost all ω, there is a unique function $(t, x) \mapsto u(t, x, \omega) \in C(\mathbb{R}_+; \mathbb{C}^0) \cap D(\mathbb{R}_+; \mathbb{C}^1)$ so that for each $x \in \mathbb{R}^d$, $t \mapsto u(t, x, \omega)$ is absolutely continuous and*

$$\partial_t u(t, x, \omega) + (b(t, x) + L_t(\omega)) \cdot \nabla_x u(t, x, \omega) = 0, \quad u(0, x) = \varphi(x). \tag{7.11}$$

Proof. We only show the existence since the uniqueness is standard by the maximum principle (see [9, Theorem 6.1]). By Corollary 7.5, for each $x \in \mathbb{R}^d$, let $X_{s,t}(x, \omega)$ solve the following random ODE:

$$X_{s,t}(x, \omega) = x + \int_s^t \Big(b(r, X_{s,r}(x, \omega)) + L^{(\alpha)}_r(\omega)\Big)\mathrm{d}r, \quad 0 \le s \le t. \tag{7.12}$$

Define

$$u(t, x, \omega) := \varphi(X^{-1}_{0,t}(x, \omega)).$$

Clearly, by (ii) of Corollary 7.5, we have for almost all ω,

$$u(\cdot, \cdot, \omega) \in C(\mathbb{R}_+; \mathbb{C}^0) \cap D(\mathbb{R}_+; \mathbb{C}^1), \tag{7.13}$$

and by (i) of Corollary 7.5, for $\varepsilon > 0$,

$$u(t + \varepsilon, x) = \varphi(X^{-1}_{0,t} \circ X^{-1}_{t,t+\varepsilon}(x)) = u(t, X^{-1}_{t,t+\varepsilon}(x)).$$

Here and below we drop the ω. Hence,

$$\frac{u(t + \varepsilon, x) - u(t, x)}{\varepsilon} = \frac{X^{-1}_{t,t+\varepsilon}(x) - x}{\varepsilon} \int_0^1 (\nabla_x u)\Big(t, \theta X^{-1}_{t,t+\varepsilon}(x) + (1 - \theta)x\Big)\mathrm{d}\theta. \tag{7.14}$$

Since $\varepsilon \mapsto X_{t,t+\varepsilon}(\cdot) \in D(\mathbb{R}_+; \mathbb{C}^1)$ and $\nabla X^{-1}_{t,t+\varepsilon}(x) = (\nabla X_{t,t+\varepsilon})^{-1} \circ X^{-1}_{t,t+\varepsilon}(x)$, we have

$$\lim_{\varepsilon\downarrow 0} \sup_{|x|\le R} |\nabla X^{-1}_{t,t+\varepsilon}(x) - \mathbb{I}| = 0, \quad \forall R > 0. \tag{7.15}$$

Noticing that

$$X^{-1}_{t,t+\varepsilon}(x) - x = (x - X_{t,t+\varepsilon}(x)) \cdot \int_0^1 \nabla X^{-1}_{t,t+\varepsilon}(\theta x + (1 - \theta)X_{t,t+\varepsilon}(x))\mathrm{d}\theta,$$

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since $(t,x)\mapsto b(t,x)$ is continuous and $t\mapsto L^{(\alpha)}_t$ is right continuous, by (7.12) and (7.15), for each $(t,x)\in\mathbb{R}_+\times\mathbb{R}^d$, we have

$$\lim_{\varepsilon\downarrow 0}(X^{-1}_{t,t+\varepsilon}(x)-x)/\varepsilon=-b(t,x)-L^{(\alpha)}_t.$$

Therefore, by (7.14) and the continuity of $x\mapsto\nabla_x u(t,x)$,

$$\partial^+_t u(t,x):=\lim_{\varepsilon\downarrow 0}\frac{u(t+\varepsilon,x)-u(t,x)}{\varepsilon}=-(b(t,x)+L^{(\alpha)}_t)\cdot\nabla_x u(t,x),\ \forall(t,x)\in\mathbb{R}_+\times\mathbb{R}^d,$$

where $\partial^+_t u$ stands for the right derivative. Since $t\mapsto(b(t,x)+L^{(\alpha)}_t)\cdot\nabla_x u(t,x)$ is bounded, by Lemma 7.7, $t\mapsto u(t,x)$ is absolutely continuous. The proof is complete. □

Remark 7.9. If $\varphi\in L^\infty(\mathbb{R}^d)$ and $\mathrm{div} b\in L^1_{loc}(\mathbb{R}_+\times\mathbb{R}^d)$, then as in [22, Theorem 20], we can show that $u(t,x):=\varphi(X^{-1}_{0,t}(x))$ is the unique bounded weak solution of transport equation (7.11).

8. Appendix

In this section we use a probabilistic method to show the existence of a smooth solution to PDE (6.1) when the coefficients are smooth. We first recall the following result proved in [25].

Lemma 8.1. *Given $d_0\in\mathbb{N}$ and $c_0>1$, let $\kappa(x,z):\mathbb{R}^{d_0}\times B_1\to[c_0^{-1},c_0]$ be a smooth function with bounded derivatives of all orders. For any $\alpha\in(0,2)$, there is a measurable map $\Phi(x,z):\mathbb{R}^{d_0}\times B_1\to B_1$ such that for any nonnegative measurable function f,*

$$\int_{B_1} f\circ\Phi(x,z)\frac{\mathrm{d}z}{|z|^{d+\alpha}}=\int_{B_1} f(z)\kappa(x,z)\frac{\mathrm{d}z}{|z|^{d+\alpha}}.$$

Moreover, Φ enjoys the following properties:

(i) *$\Phi(x,0)=0$ and if $\kappa(x,-z)=\kappa(x,z)$, then $\Phi(x,-z)=-\Phi(x,z)$.*
(ii) *For all $i,j\in\mathbb{N}_0$, there is a $C_{ij}>0$ such that for all $x\in\mathbb{R}^{d_0}$ and $z\in B_1$,*

$$|\nabla^i_x\nabla^j_z\Phi(x,z)|\le C_{ij}|z|^{1-j},$$

where C_{ij} is a polynomial of $\|\nabla^m_x\nabla^n_z\kappa\|_\infty$, $m=1,\cdots,i,n=0,\cdots,j$.

Theorem 8.2. *Suppose that κ and b satisfy that for any $m\in\mathbb{N}$ and $t>0$,*

$$c_0^{-1}\le\kappa(t,x,\mathrm{v},w)\le c_0,\quad\|\nabla^m\kappa(t,\cdot)\|_\infty+\|\nabla^m b(t,\cdot)\|_\infty\le c_m.$$

Then there is a classical solution $u\in\cap_{m\in\mathbb{N}}C(\mathbb{R}_+;\mathscr{C}^m)$ to PDE (6.1).

Proof. We decompose the operator $\mathscr{L}^{(\alpha)}_{\kappa;\mathrm{v}}$ as two parts: small jumps $\widetilde{\mathscr{L}^{(\alpha)}_{\kappa;\mathrm{v}}}$ and large jumps $\overline{\mathscr{L}^{(\alpha)}_{\kappa;\mathrm{v}}}$,

$$\mathscr{L}^{(\alpha)}_{\kappa;\mathrm{v}}f=\left(\int_{B_1}+\int_{B_1^c}\right)\delta^{(2)}_w f(x,\mathrm{v})\kappa(t,x,\mathrm{v},w)\frac{\mathrm{d}w}{|w|^{d+\alpha}}=:\widetilde{\mathscr{L}^{(\alpha)}_{\kappa;\mathrm{v}}}f+\overline{\mathscr{L}^{(\alpha)}_{\kappa;\mathrm{v}}}f,$$

where

$$\delta_w^{(2)} f(x,\mathrm{v}) := f(x,\mathrm{v}+w) + f(x,\mathrm{v}-w) - 2f(x,\mathrm{v}).$$

By Lemma 8.1, there is a measurable map $g(t,x,\mathrm{v},w) : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \times B_1 \to B_1$ with

$$|\nabla^i_{x,\mathrm{v}} \nabla^j_w g(t,x,\mathrm{v},w)| \le C_{i,j} |w|^{1-j},$$

and so that

$$\widetilde{\mathscr{L}^{(\alpha)}_{\kappa;\mathrm{v}}} f(x,\mathrm{v}) = \int_{B_1} \delta^{(2)}_{g(t,x,\mathrm{v},w)} f(x,\mathrm{v}) \frac{\mathrm{d}w}{|w|^{d+\alpha}}.$$

Now we consider the following SDE:

$$\mathrm{d}Z_{s,t} = \int_{B_1} g(t, Z_{s,t-}, w)\tilde{N}(\mathrm{d}t, \mathrm{d}w) + b(t, Z_{s,t})\mathrm{d}t, \ \ Z_{s,s} = z.$$

Since the coefficients are smooth and have bounded derivatives of all orders greater than 1, it is well known that there is a unique solution $Z_{s,t}(z)$, which forms a C^∞-stochastic flows (cf. [23, Theorem 4.1]). More precisely, it holds that for any $s < r < t$ and $z \in \mathbb{R}^d$,

$$Z_{s,t}(z) = Z_{r,t} \circ Z_{s,r}(z), \quad a.s.,$$

and for any $j \in \mathbb{N}$ and $p \ge 1$,

$$\sup_{z\in\mathbb{R}^d} \sup_{0\le s<t\le T} \mathbb{E}|\nabla^j_z Z_{s,t}(z)|^p < \infty. \tag{8.1}$$

Moreover, let $f : \mathbb{R}_+ \times \mathbb{R}^{2d} \to \mathbb{R}$ be a measurable function with $\|\nabla^j_z f\|_\infty < \infty$ for each $j \in \mathbb{N}$. It is also well known that

$$(x,\mathrm{v}) \mapsto u(s,x,\mathrm{v}) := \int_s^T \mathrm{e}^{\lambda(s-t)} \mathbb{E} f(t, Z_{s,t}(x,\mathrm{v}))\mathrm{d}t \in \cap_m \mathscr{C}^m(\mathbb{R}^{2d})$$

solves the following PDE:

$$\partial_s u + \widetilde{\mathscr{L}^{(\alpha)}_{\kappa;\mathrm{v}}} u + b \cdot \nabla u - \lambda u = f, \quad u(T,\cdot) = 0.$$

From the representation, by the chain rule and (8.1), it is easy to see that for any $m \in \mathbb{N}$, there is a constant $C > 0$ such that for all $s \in [0,T]$ and $\lambda \ge 0$,

$$\|u(s)\|_{\mathscr{C}^m} \le C \int_s^T \mathrm{e}^{\lambda(s-t)} \|f(t,\cdot)\|_{\mathscr{C}^m} \mathrm{d}t \le C \int_s^T \|f(t,\cdot)\|_{\mathscr{C}^m} \mathrm{d}t, \tag{8.2}$$

and by the definition of $\overline{\mathscr{L}^{(\alpha)}_{\kappa;\mathrm{v}}} u$,

$$\|\overline{\mathscr{L}^{(\alpha)}_{\kappa;\mathrm{v}}} u\|_{\mathscr{C}^m} \le C\|u\|_{\mathscr{C}^m}.$$

Next we consider the following Picard's iteration: Fix $m \geq 2$. Let $u_0 \equiv 0$. For $n \in \mathbb{N}$, let u_n be the unique $\mathscr{C}^m$-valued solution of the following PDE:

$$\partial_s u_n + \widetilde{\mathscr{L}^{(\alpha)}_{\kappa;\mathrm{v}}} u_n + b \cdot \nabla u_n - \lambda u_n = f - \overline{\mathscr{L}^{(\alpha)}_{\kappa;\mathrm{v}}} u_{n-1}, \quad u_n(T, \cdot) = 0. \tag{8.3}$$

By (8.2) we have

$$\|u_n(s)\|_{\mathscr{C}^m} \leq C \int_s^T \|f(t) - \overline{\mathscr{L}^{(\alpha)}_{\kappa;\mathrm{v}}} u_{n-1}(t)\|_{\mathscr{C}^m} \mathrm{d}t \leq C \int_s^T \Big(\|f(t)\|_{\mathscr{C}^m} + \|u_{n-1}(t)\|_{\mathscr{C}^m} \Big) \mathrm{d}t,$$

which yields by Gronwall's inequality that for some $C > 0$,

$$\sup_n \sup_{s \in [0,T]} \|u_n(s)\|_{\mathscr{C}^m} \leq C \sup_{s \in [0,T]} \|f(s)\|_{\mathscr{C}^m}. \tag{8.4}$$

Similarly, we also have

$$\|u_n(s) - u_k(s)\|_{\mathscr{C}^m} \leq C \int_s^T \Big(\|u_{n-1}(t) - u_{k-1}(t)\|_{\mathscr{C}^m} \Big) \mathrm{d}t, \quad n, k \in \mathbb{N}.$$

By (8.4) and Fatou's lemma, we have

$$\limsup_{n,k \to \infty} \sup_{t \in [s,T]} \|u_n(t) - u_k(t)\|_{\mathscr{C}^m} \leq C \int_s^T \limsup_{n,k \to \infty} \Big(\|u_{n-1}(t) - u_{k-1}(t)\|_{\mathscr{C}^m} \Big) \mathrm{d}t,$$

which yields by Gronwall's inequality again

$$\limsup_{n,k \to \infty} \sup_{t \in [0,T]} \|u_n(t) - u_k(t)\|_{\mathscr{C}^m} = 0.$$

Therefore, there exists a $u \in C([0,T]; \mathscr{C}^m)$ so that

$$\lim_{n \to \infty} \sup_{t \in [0,T]} \|u_n(t) - u(t)\|_{\mathscr{C}^m} = 0.$$

By taking limits for equation (8.3), we obtain the existence of $\mathscr{C}^m$-valued solution to PDE (6.1). Since m is arbitrary, we complete the proof. □

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References

[1] L. Ambrosio, Transport equation and Cauchy problem for *BV* vector fields, Invent. Math. 158 (2) (2004) 227–260.
[2] J. Bae, M. Kassmann, Schauder estimates in generalized Hölder spaces, available at arXiv:1505.05498, 2015.
[3] H. Bahouri, J.-Y. Chemin, R. Danchin, Fourier Analysis and Nonlinear Partial Differential Equations, Grundlehren der Mathematischen Wissenschaften., vol. 343, Springer, Heidelberg, 2011.
[4] R.F. Bass, Regularity results for stable-like operators, J. Funct. Anal. 257 (8) (2009) 2693–2722.

[5] P.E. Chaudru de Raynal, Strong existence and uniqueness for stochastic differential equation with Hölder drift and degenerate noise, Ann. Inst. Henri Poincaré 53 (1) (2017) 259–286.
[6] P.E. Chaudru de Raynal, I. Honoré, S. Menozzi, Sharp Schauder estimates for some degenerate Kolmogorov equations, available at arXiv:1810.12227, 2018.
[7] P.E. Chaudru de Raynal, I. Honoré, S. Menozzi, Strong regularization by Brownian noise propagating through a weak Hörmander structure, available at arXiv:1810.12225, 2018.
[8] P.E. Chaudru de Raynal, S. Menozzi, E. Priola, Schauder estimates for drifted fractional operators in the supercritical case, available at arXiv:1902.02616, 2019.
[9] Z.-Q. Chen, E. Hu, L. Xie, X. Zhang, Heat kernels for non-symmetric diffusion operators with jumps, J. Differ. Equ. 263 (10) (2017) 6576–6634.
[10] Z.-Q. Chen, R. Song, X. Zhang, Stochastic flows for Lévy processes with Hölder drifts, Rev. Mat. Iberoam. 34 (4) (2018) 1755–1788.
[11] Z.-Q. Chen, X. Zhang, Propagation of regularity in L^p-spaces for Kolmogorov type hypoelliptic operators, available at arXiv:1706.02181, 2017.
[12] Z.-Q. Chen, X. Zhang, L^p-hypoelliptic regularity of nonlocal kinetic Fokker-Planck operators, J. Math. Pures Appl. 9 (116) (2018) 52–87.
[13] Z.-Q. Chen, X. Zhang, G. Zhao, Well-posedness of supercritical SDE driven by Lévy processes with irregular drifts, available at arXiv:1709.04632, 2017.
[14] P. Constatin, G. Iyer, A stochastic Lagrangian representation of the three-dimensional incompressible Navier-Stokes equations, Commun. Pure Appl. Math. 61 (2008) 330–345.
[15] S. Dachkovski, Anisotropic function spaces and related semi-linear hypoelliptic equations, Math. Nachr. 248/249 (2003) 40–61.
[16] A.M. Davie, Uniqueness of solutions of stochastic differential equations, Int. Math. Res. Not. 124 (24) (2007) 26 pages.
[17] R.J. Diperna, P.L. Lions, Ordinary differential equations, transport theory and Sobolev spaces, Invent. Math. 98 (3) (1989) 511–547.
[18] H. Dong, D. Kim, Schauder estimates for a class of non-local elliptic equations, Discrete Contin. Dyn. Syst. 33 (6) (2013) 2319–2347.
[19] S. Fang, D. Luo, A. Thalmaier, Stochastic differential equations with coefficients in Sobolev spaces, J. Funct. Anal. 259 (3) (2011) 1129–1168.
[20] E. Fedrizzi, F. Flandoli, Pathwise uniqueness and continuous dependence for SDEs with non-regular drift, Stochastics 83 (2011) 241–257.
[21] A. Figalli, Existence and uniqueness of martingale solutions for SDEs with rough or degenerate coefficients, J. Funct. Anal. 254 (1) (2008) 109–153.
[22] F. Flandoli, M. Gubinelli, E. Priola, Well-posedness of the transport equation by stochastic perturbation, Invent. Math. 180 (1) (2010) 1–53.
[23] T. Fujiwara, H. Kunita, Canonical SDE's based on semimartingales with spatial parameters. I. Stochastic flows of diffeomorphisms, Kyushu J. Math. 53 (2) (1999) 265–300.
[24] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Classics in Mathematics, Springer-Verlag, Berlin, 2001.
[25] Z. Hao, X. Peng, X. Zhang, Hörmander's hypoelliptic theorem for nonlocal operators, available at arXiv:1901.06621, 2019.
[26] C. Henderson, S. Snelson, C^∞-smoothing for weak solutions of the inhomogeneous Landau equation, available at arXiv: 1707.05710, 2017.
[27] L. Huang, S. Menozzi, E. Priola, L^p Estimates for degenerate non-local Kolmogorov operators, J. Math. Pures Appl. 121 (9) (2019) 162–215.
[28] N. Ikeda, S. Watanabe, Stochastic differential equations and diffusion processes, in: North-Holland Mathematical Library, vol. 24, 2nd edition, North-Holland Publishing Co./Kodansha, Ltd., Amsterdam/Tokyo, 1989.
[29] C. Imbert, T. Jin, R. Shvydkoy, Schauder estimates for an integro-differential equation with applications to a nonlocal Burgers equation, Ann. Fac. Sci. Toulouse Math. (6) 27 (4) (2018) 667–677.
[30] C. Imbert, C. Mouhot, A toy nonlinear model in kinetic theory, available at arXiv:1801.07891, 2018.
[31] C. Imbert, L. Silvestre, The Schauder estimate for kinetic integral equations, available at arXiv:1812.11870v2, 2018.
[32] J. Jacod, A.N. Shiryaev, Limit Theorems for Stochastic Processes, 2nd edition, Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), vol. 288, Springer-Verlag, Berlin, 2003.
[33] N.V. Krylov, Lectures on Elliptic and Parabolic Equations in Hölder Spaces, Graduate Studies in Mathematics, vol. 12, American Mathematical Society, Providence, RI, 1996.
[34] N.V. Krylov, E. Priola, Elliptic and parabolic second-order PDEs with growing coefficients, Commun. Partial Differ. Equ. 35 (3) (2010) 1–22.
[35] N.V. Krylov, E. Priola, Poisson stochastic process and basic Schauder and Sobolev estimates in the theory of parabolic equations, Arch. Ration. Mech. Anal. 225 (3) (2017) 1089–1126.
[36] N.V. Krylov, M. Röckner, Strong solutions of stochastic equations with singular time dependent drift, Probab. Theory Relat. Fields 131 (2) (2005) 154–196.
[37] L. Lorenzi, Schauder estimates for degenerate elliptic and parabolic problems with unbounded coefficients in $\mathbb{R}^N$, Differ. Integral Equ. 18 (5) (2005) 531–566.
[38] A. Lunardi, Schauder estimates for a class of degenerate elliptic and parabolic operators with unbounded coefficients in $\mathbb{R}^n$, Ann. Sc. Norm. Super. Pisa, Cl. Sci. (4) 24 (1) (1997) 133–164.
[39] L. Marino, Schauder estimates for degenerate stable Kolmogorov equations, available at arXiv:1910.03653, 2019.
[40] E. Priola, Global Schauder estimates for a class of degenerate Kolmogorov equations, Stud. Math. 194 (2) (2009) 117–153.
[41] E. Priola, Pathwise uniqueness for singular SDEs driven by stable processes, Osaka J. Math. 49 (2) (2012) 421–447.

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[42] E. Priola, Davie's type uniqueness for a class of SDEs with jumps, Ann. Inst. Henri Poincaré Probab. Stat. 54 (2) (2018) 694–725.
[43] W. Rudin, Real and Complex Analysis, 3rd edition, McGraw-Hill Book Co., New York, 1987.
[44] H. Triebel, Theory of Function Spaces, Monographs in Mathematics, vol. 78, Birkhäuser Verlag, Basel, 1983.
[45] H. Triebel, Theory of Function Spaces. II, Monographs in Mathematics, vol. 84, Birkhäuser Verlag, Basel, 1992.
[46] C. Villani, A review of mathematical topics in collisional kinetic theory, in: S. Friedlander, D. Serre (Eds.), Handbook of Fluid Mechanics, 2002.
[47] F.-Y. Wang, X. Zhang, Degenerate SDEs in Hilbert spaces with rough drifts, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 18 (4) (2015) 1550026.
[48] F.-Y. Wang, X. Zhang, Degenerate SDE with Hölder-Dini drift and non-Lipschitz coefficient, SIAM J. Math. Anal. 48 (3) (2016) 2189–2226.
[49] X. Zhang, Strong solutions of SDEs with singular drift and Sobolev diffusion coefficients, Stoch. Process. Appl. 115 (11) (2005) 1805–1818.
[50] X. Zhang, Stochastic flows of SDEs with irregular coefficients and stochastic transport equations, Bull. Sci. Math. France 134 (2010) 340–378.
[51] X. Zhang, Stochastic homeomorphism flows of SDEs with singular drifts and Sobolev diffusion coefficients, Electron. J. Probab. 16 (38) (2011) 1096–1116.
[52] X. Zhang, Stochastic Lagrangian particle approach to fractal Navier-Stokes equations, Commun. Math. Phys. 311 (2012) 133–155.
[53] X. Zhang, Well-posedness and large deviation for degenerate sdes with Sobolev coefficients, Rev. Mat. Iberoam. 29 (1) (2013) 25–52.
[54] X. Zhang, G. Zhao, Heat kernel and ergodicity of SDEs with distributional drifts, available at arXiv:1710.10537, 2017.
[55] X. Zhang, G. Zhao, Dirichlet problem for supercritical nonlocal operators, available at arXiv:1809.05712, 2018.
[56] A.K. Zvonkin, A transformation of the phase space of a diffusion process that removes the drift, Mat. Sb. (N.S.) 93 (135) (1974) 129–149, 152.

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