

HÖLDER REGULARITY AND GRADIENT ESTIMATES FOR SDES DRIVEN BY CYLINDRICAL α -STABLE PROCESSES

ZHEN-QING CHEN, ZIMO HAO AND XICHENG ZHANG

ABSTRACT. We establish Hölder regularity and gradient estimates for the transition semigroup of the solutions to the following SDE:

$$dX_t = \sigma(t, X_{t-})dZ_t + b(t, X_t)dt, \quad X_0 = x \in \mathbb{R}^d,$$

where $(Z_t)_{t \geq 0}$ is a d -dimensional cylindrical α -stable process with $\alpha \in (0, 2)$, $\sigma(t, x) : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d$ is bounded measurable, uniformly nondegenerate and Lipschitz continuous in x uniformly in t , and $b(t, x) : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is bounded β -Hölder continuous in x uniformly in t with $\beta \in [0, 1]$ satisfying $\alpha + \beta > 1$. Moreover, we also show the existence and regularity of the distributional density of $X(t, x)$. Our proof is based on Littlewood-Paley's theory.

1. Introduction

Consider the following stochastic differential equation (SDE) in \mathbb{R}^d :

$$dX_{s,t} = \sigma(t, X_{s,t})dW_t + b(t, X_{s,t})dt, \quad t \geq s \quad \text{with } X_{s,s} = x \in \mathbb{R}^d, \quad (1.1)$$

where W is a d -dimensional standard Brownian motion, $b(t, x) : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a bounded measurable function, and $\sigma(t, x) : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d$ is a $d \times d$ matrix-valued measurable function. Suppose that σ satisfies the following uniformly elliptic condition: there is some $c_0 \geq 1$ so that

$$c_0^{-1}|\xi| \leq |\sigma(t, x)\xi| \leq c_0|\xi| \quad \text{for every } t \geq 0 \text{ and } x, \xi \in \mathbb{R}^d. \quad (\mathbf{H}^\sigma)$$

Under (\mathbf{H}^σ) and $\lim_{|x-y| \rightarrow 0} \sup_{t \geq 0} |\sigma(t, x) - \sigma(t, y)| = 0$, it is well known that for each starting point $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, SDE (1.1) admits a unique weak solution $X_{s,t}(x)$ (cf. [14]). If, in addition, $\sigma(t, x)$ is Hölder continuous in x uniformly in t , then (cf. [19]) for each $\varphi \in C_b^2(\mathbb{R}^d)$,

$$P_{s,t}\varphi(x) := \mathbb{E}[\varphi(X_{s,t}(x))]$$

is C^2 in x , and uniquely solves the following backward Kolmogorov equation:

$$\left(\partial_s + \frac{1}{2} \sigma^{ik}(s, \cdot) \sigma^{jk}(s, \cdot) \partial_i \partial_j + b^i(s, \cdot) \partial_i \right) P_{s,t}\varphi = 0, \quad s \leq t.$$

Here and below we use the usual Einstein convention: if an index appears twice in a product, then it will be summed automatically.

A natural question is what kind regularity does $P_{s,t}\varphi$ have for solutions $X_{s,t}$ to SDEs driven by cylindrical stable processes, that is, for SDE (1.1) with Brownian motion W replaced by a d -dimensional cylindrical α -stable process $Z =$

Keywords: Hölder regularity, Gradient estimate, Littlewood-Paley's decomposition, Heat kernel, Cylindrical Lévy process.

Research of Z.-Q. Chen is partially supported by Simons Foundation grant 520542 and a Victor Klee Faculty Fellowship at UW. Research of X. Zhang is partially supported by NNSFC grant of China (No. 11731009) and the DFG through the CRC 1283 "Taming uncertainty and profiting from randomness and low regularity in analysis, stochastics and their applications".

$(Z_t^1, \dots, Z_t^d)_{t \geq 0}$? Here $\alpha \in (0, 2)$ and Z^1, \dots, Z^d are independent 1-dimensional α -stable process with Lévy measure $dz/|z|^{1+\alpha}$. In this paper we shall consider the following SDE driven by cylindrical α -stable process Z :

$$dX_{s,t} = \sigma(t, X_{s,t-})dZ_t + b(t, X_{s,t})dt, \quad t \geq s \quad \text{with } X_{s,s} = x \in \mathbb{R}^d. \quad (1.2)$$

Notice that the (time-dependent) generator for the solution $X_{s,t}$ of this SDE is

$$\mathcal{L}_t^{\sigma,b} u(x) := \sum_{i=1}^d \text{p.v.} \int_{\mathbb{R}} \left(u(x + \sigma^i(t, x)z_i) - u(x) \right) \frac{dz_i}{|z_i|^{1+\alpha}} + b^i(t, x) \partial_i u(x), \quad (1.3)$$

where p.v. stands for the Cauchy principle value. It is well known that if σ and b are Lipschitz continuous in x and uniformly in t , then there is a unique strong solution $X_{s,t}(x)$ to SDE (1.2). When $b = 0$, Bass and Chen [2] established the weak well-posedness for SDE (1.2) under the assumption that $\sigma(t, x) = \sigma(x)$ is bounded, continuous and non-degenerate for each $x \in \mathbb{R}^d$. When σ is Lipschitz continuous in x uniformly in t and satisfies (\mathbf{H}^σ) , b is β -order Hölder continuous with $\alpha + \beta > 1$, Chen, Zhang and Zhao [9] recently showed the strong well-posedness for SDE (1.2) (see also [7]). Very recently, assuming that $\alpha \in (0, 1)$, $b = 0$ and $\sigma(t, x) = \sigma(x)$ is Lipschitz continuous satisfying (\mathbf{H}^σ) , Kulczycki, Ryznar and Sztonyk [13] showed the following Hölder estimate for the semigroup P_t associated with SDE (1.2): for any $\gamma \in (0, \alpha)$ and $T > 0$, there is a positive constant $C = C(d, \alpha, c_0, \|\nabla\sigma\|_\infty, \gamma, T)$ so that for all $t \in (0, T]$,

$$\|P_t \varphi\|_{\mathbf{C}^\gamma} \leq C t^{-\gamma/\alpha} \|\varphi\|_\infty,$$

where $P_t \varphi(x) := \mathbb{E}[\varphi(X_t(x))]$ and \mathbf{C}^γ is the space of bounded γ -Hölder continuous functions on \mathbb{R}^d . It was not known until now if the above Hölder estimate holds when $\alpha \in [1, 2)$, nor was gradient estimate for P_t for any $\alpha \in (0, 2)$. These properties will be addressed in this paper under a more general setting. When Z is the rotationally invariant α -stable process and b, σ are Lipschitz continuous, the gradient estimate for P_t was obtained in [16] using subordination technique.

The aim of this paper is to establish the following regularity estimates for the transition semigroup $\{P_{s,t}; t \geq s\}$ of the unique solution $\{X_{s,t}(x); t \geq s\}$ to SDE (1.2):

$$P_{s,t} \varphi(x) := \mathbb{E}[\varphi(X_{s,t}(x))].$$

Note that $\{P_{s,t}; t \geq s\}$ is the time-inhomogeneous semigroup generated by the non-local operator $\mathcal{L}_t^{\sigma,b}$ of (1.3).

Theorem 1.1. *Let $\alpha \in (0, 2)$ and $\beta \in [0, 1]$ with $\alpha + \beta > 1$. Assume (\mathbf{H}^σ) , $\|\nabla\sigma\|_\infty \leq c_1$ for some $c_1 > 0$, and one of the following conditions holds:*

- (i) $b = 0$ and $\beta = 1$; (ii) $\alpha \in (1/2, 2)$ and $\sup_{t \geq 0} \|b(t, \cdot)\|_{\mathbf{C}^\beta} \leq c_2$.

Let $\gamma \in [0, \alpha + \alpha \wedge \beta)$ and $\eta \in (-((\alpha + \beta - 1) \wedge 1), \gamma]$. For any $T > 0$, there exists a constant $C = C(d, c_0, c_1, c_2, \alpha, \beta, \gamma, \eta, T) > 0$ such that for all $0 \leq s < t \leq T$,

$$\|P_{s,t} \varphi\|_{\mathbf{B}_{\infty,\infty}^\eta} \leq C(t-s)^{(\eta-\gamma)/\alpha} \|\varphi\|_{\mathbf{B}_{\infty,\infty}^\eta}, \quad (1.4)$$

where $\mathbf{B}_{p,q}^\eta$ is the usual Besov space. In particular, we have the following assertions:

(A) Let $\varphi \in \cup_{\eta < (\alpha + \beta - 1) \wedge 1} \mathbf{B}_{\infty,\infty}^{-\eta}$. For any $0 \leq s < t$, $P_{s,t} \varphi \in \cap_{\gamma < \alpha + \alpha \wedge \beta} \mathbf{B}_{\infty,\infty}^\gamma$ and the following backward Kolmogorov equation is satisfied: for all $x \in \mathbb{R}^d$,

$$P_{t_0,t} \varphi(x) = P_{t_1,t} \varphi(x) + \int_{t_0}^{t_1} \mathcal{L}_s^{\sigma,b} P_{s,t} \varphi(x) ds, \quad 0 \leq t_0 < t_1 < t. \quad (1.5)$$

(B) For $\alpha \in (1/2, 2)$, the following gradient estimate holds: for $0 \leq s < t \leq T$,

$$\|\nabla P_{s,t}\varphi\|_\infty \leq C(t-s)^{-1/\alpha}\|\varphi\|_\infty. \quad (1.6)$$

(C) For each $s < t$, the random variable $X_{s,t}(x)$ admits a density $p_{s,t}(x, \cdot)$ with

$$p_{s,t}(x, \cdot) \in \cap_{\eta < (\alpha + \beta - 1) \wedge 1} \mathbf{B}_{1,1}^\eta. \quad (1.7)$$

We would like to emphasize that for SDE (1.2) driven by cylindrical α -stable process Z , since the support of the Lévy measure of Z is concentrated along the coordinate axes, it seems to be quite difficult to obtain quantitative regularity results as stated in Theorem 1.1 by using methods from [13] and [16]. A new approach is needed to study regularity theory for SDEs driven by cylindrical stable processes. A key ingredient in our approach is the use of Littlewood-Paley's theory.

Remark 1.2. (i) The precise definition of the Besov space $\mathbf{B}_{p,q}^s$ is given in Definition 2.3 below. It is known that for non-integer $s > 0$, $\|f\|_{\mathbf{B}_{\infty,\infty}^s} \asymp \|f\|_{\mathbf{C}^s}$. Hence (1.4) in particular yields that, under condition either (i) or (ii) of Theorem 1.1, for every $\alpha \in (0, 2)$, $T > 0$, and non-integer $\gamma \in (0, \alpha + \alpha \wedge \beta)$, there is a constant $C = C(d, c_0, c_1, c_2, \alpha, \beta, \gamma, T)$ so that for all $0 \leq s < t \leq T$,

$$\|P_{s,t}\varphi\|_{\mathbf{C}^\gamma} \leq C(t-s)^{-\gamma/\alpha}\|\varphi\|_\infty.$$

This significantly extends the main results of [13] where α is restricted to be in $(0, 1)$, $\sigma(t, x) = \sigma(x)$, $b(t, x) \equiv 0$ and $\gamma \in (0, \alpha)$.

- (ii) Note that $\mathcal{L}_s^{\sigma,b}u(x)$ is pointwisely well defined for any $u \in \mathbf{C}^\gamma$ with $\gamma > \max\{\alpha, 1\}$. Under either condition (i) or (ii) of Theorem 1.1, $\alpha + \alpha \wedge \beta > 1$. Thus $\mathcal{L}_s^{\sigma,b}P_{s,t}\varphi$ in (1.5) is pointwisely well defined once it is established that $P_{s,t}\varphi \in \cap_{\gamma < \alpha + \alpha \wedge \beta} \mathbf{B}_{\infty,\infty}^\gamma$.
- (iii) In the condition (ii) of Theorem 1.1, due to some moment estimate, α is required to be greater than $1/2$ (see Remark 4.7 below). This restriction also appears in the Schauder estimate of nonlocal PDEs in [4]. For variable coefficient σ , since we can only improve the regularity to $\alpha + \alpha \wedge 1$ even without drift b , we have to require $\alpha > 1/2$ for gradient estimate also. An open problem is to drop the restriction $\alpha > 1/2$.
- (iv) We note that when b and σ are time-independent, by a purely probabilistic method, Debussche and Fournier [10] obtained the existence and low regularity of the densities for SDE (1.2) under some Hölder assumptions on σ and b . Compared with their results, for $\alpha \in [1, 2)$, in the time independent case, the property (1.7) is covered by [10, Theorem 1.1]. However, for $\alpha \in (1/2, 1)$, the regularity (1.7) is better than [10, Theorem 1.1]. The method in [10] does not seem to be applicable to the time-dependent case and more general SDEs driven by Poisson random measures. Our method is mostly analytic combined by some probabilistic argument and has more flexibility.
- (v) We point out that resolvent Hölder regularity can be established under a much weaker assumption on σ , at least in the time-independent and driftless situation. Suppose that $\sigma(x)$ is continuous and satisfies condition (\mathbf{H}^σ) . Then by [2] for each $x \in \mathbb{R}^d$, there is a unique weak solution to $dX_t = \sigma(X_{t-})dZ_t$ with $X_0 = x$, where Z is a cylindrical stable process on \mathbb{R}^d . Denote the law of X with $X_0 = x$ by \mathbb{P}_x and its mathematical expectation

by \mathbb{E}_x . It follows from [3, Proposition 2.1] and a scaling argument, that there are constants $c_3 \geq c_4 > 0$ so that for every $x_0 \in \mathbb{R}^d$ and $r > 0$,

$$\mathbb{E}_x [\tau_{B(x_0, r)}] \leq c_3 r^\alpha \quad \text{for } x \in B(x_0, r) \quad (1.8)$$

and

$$\mathbb{E}_x [\tau_{B(x_0, r)}] \geq c_4 r^\alpha \quad \text{for } x \in B(x_0, r/2), \quad (1.9)$$

where $B(x_0, r) := \{x : |x - x_0| < r\}$ and $\tau_{B(x_0, r)} = \inf\{t > 0 : X_t \notin B(x_0, r)\}$. Assume further that $\sigma(x)$ is uniformly continuous on \mathbb{R}^d , we know from [3, Theorem, 2.9] that there is a constant $\gamma \in (0, 1)$ that depends on the modulo of continuity of $\sigma(x)$, the constant c_0 in (\mathbf{H}^σ) , d and α so that every bounded function that is harmonic in $B(x_0, r)$ is γ -Hölder continuous on $B(x_0, r/2)$. This together with (1.8)-(1.9) and the proof of [5, Proposition 2.4] yields that there is some constant $\gamma_1 \in (0, 1)$ depending on the modulo of continuity of $\sigma(x)$, the constant c_0 in (\mathbf{H}^σ) , d and α so that for every bounded function φ on \mathbb{R}^d and $\lambda > 0$,

$$|R_\lambda \varphi(x) - R_\lambda \varphi(y)| \leq C|x - y|^{\gamma_1} \|\varphi\|_\infty \quad \text{for } |x - y| \leq 1.$$

Here $R_\lambda \varphi(x) := \mathbb{E}_x \int_0^\infty e^{-\lambda t} \varphi(X_t) dt$ is the λ -resolvent of φ .

This paper is organized as follows: In Section 2, we introduce some basic estimates for later use. In Section 3, we present the estimates of Littlewood-Paley's type for the heat kernel of nonlocal operators with constant coefficients. In Section 4, we show the regularity estimates for general nonlocal equations by freezing coefficients. In Section 5, we apply the a priori estimate obtained in Section 4 to show our main results. We use $:=$ as a way of definition. Throughout this paper we shall use the following conventions and notations:

- The letter C denotes a constant, whose value may change in different places.
- We use $A \lesssim B$ to denote $A \leq CB$ for some unimportant constant $C > 0$.
- $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ := [0, \infty)$, $a \vee b := \max(a, b)$, $a \wedge b := \min(a, b)$.
- $\nabla_x := \partial_x := (\partial_{x_1}, \dots, \partial_{x_d})$, $\partial_i := \partial_{x_i} := \partial/\partial x_i$.
- For $x \in \mathbb{R}^d$ and $r > 0$, we denote $B(x_0, r) := \{x \in \mathbb{R}^d : |x - x_0| < r\}$ and $B_r := B(0, r)$.
- For $p \in [1, \infty]$, we use $\|\cdot\|_p$ to denote the usual norm in $L^p(\mathbb{R}^d, dx)$.

2. Preliminaries

Let $\phi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a bounded Lipschitz function with

$$\|\nabla^\ell \phi\|_\infty \leq \kappa \leq 1/2, \quad \ell = 0, 1. \quad (2.1)$$

We shall use the following definitions: for a C^1 -function $f : \mathbb{R}^d \rightarrow \mathbb{R}$,

$$\begin{cases} \Gamma_\phi(x) := x + \phi(x), & \mathcal{D}_0^\phi f(x) := f(x + \phi(x)) - f(x + \phi(0)), \\ \mathcal{D}_1^\phi f(x) := f(x + \phi(x)) - f(x + \phi(0)) - (\phi(x) - \phi(0)) \cdot \nabla f(x). \end{cases} \quad (2.2)$$

The following lemma is elementary (cf. [18, Lemma 2.1]).

Lemma 2.1. *Under (2.1), it holds that for any $x, y \in \mathbb{R}^d$,*

$$\frac{1}{2}|x - y| \leq |\Gamma_\phi(x) - \Gamma_\phi(y)| \leq 2|x - y|. \quad (2.3)$$

Moreover, there is a constant $C = C(d) > 0$ such that for all $x \in \mathbb{R}^d$,

$$|\det(\nabla \Gamma_\phi(x)) - 1 - \operatorname{div} \phi(x)| \leq C \|\nabla \phi\|_\infty^2 \leq C \kappa^2. \quad (2.4)$$

For $\beta > 0$, let \mathbf{C}^β be the space of β -order Hölder continuous functions on \mathbb{R}^d with norm

$$\|f\|_{\mathbf{C}^\beta} := \|f\|_\infty + \cdots + \|\nabla^{[\beta]}f\|_\infty + [\nabla^{[\beta]}f]_{\mathbf{C}^{\beta-[\beta]}} < \infty,$$

where $[\beta]$ denotes the largest integer not exceeding β , and ∇^j stands for the j -order gradient, and for $\gamma \in [0, 1]$,

$$[f]_{\mathbf{C}^\gamma} := \sup_h \|f(\cdot + h) - f(\cdot)\|_\infty / |h|^\gamma.$$

We use the following convention: By \mathbf{C}^0 we denote the space of bounded measurable functions. For two functions $f, g : \mathbb{R}^d \rightarrow \mathbb{R}$, let

$$\langle f, g \rangle := \int_{\mathbb{R}^d} f(x)g(x)dx,$$

whenever it is well defined. The following lemma will play a crucial role in the proof of Theorem 1.1.

Lemma 2.2. *Assume that $\phi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfies (2.1). For any $\theta \in [0, 1]$, there exists a constant $C = C(d, \theta) > 0$ such that for any $f \in L^\infty(\mathbb{R}^d)$ and $g \in L^1(\mathbb{R}^d)$ with $\nabla g \in L^1(\mathbb{R}^d)$,*

$$|\langle \mathcal{D}_0^\phi f, g \rangle| \leq C\kappa^\theta \|f\|_\infty (\mu_0(|g|) + \mu_\theta(|\nabla g|)^\theta \mu_\theta(|g|)^{1-\theta}), \quad (2.5)$$

and for any $f \in L^\infty(\mathbb{R}^d)$ and $g \in L^1(\mathbb{R}^d)$ with $\nabla g, \nabla^2 g \in L^1(\mathbb{R}^d)$,

$$|\langle \mathcal{D}_1^\phi f, g \rangle| \leq C\kappa^{1+\theta} \|f\|_{\mathbf{C}^\theta} \left(\sum_{j=0}^1 \mu_j(|\nabla^j g|) + \mu_{1+\theta}(|\nabla^2 g|)^\theta \mu_{1+\theta}(|\nabla g|)^{1-\theta} \right), \quad (2.6)$$

where $\mu_\theta(dx) := (|x| \wedge 1)^\theta dx$ and $\mu_\theta(f) := \int_{\mathbb{R}^d} f(x)\mu_\theta(dx)$.

Proof. We first assume that

$$\phi(0) = 0. \quad (2.7)$$

Step 1. Under (2.7), by a change of variable, we have

$$\begin{aligned} |\langle \mathcal{D}_0^\phi f, g \rangle| &= \left| \int_{\mathbb{R}^d} (f(\Gamma_\phi(x)) - f(x))g(x)dx \right| \\ &= \left| \int_{\mathbb{R}^d} f(x) \left(g(\Gamma_\phi^{-1}(x)) \det(\nabla \Gamma_\phi^{-1}(x)) - g(x) \right) dx \right| \\ &\leq \|f\|_\infty \left(\int_{\mathbb{R}^d} |g(\Gamma_\phi^{-1}(x))| |\det(\nabla \Gamma_\phi^{-1}(x)) - 1| dx \right. \\ &\quad \left. + \int_{\mathbb{R}^d} |g(\Gamma_\phi^{-1}(x)) - g(x)| dx \right). \end{aligned} \quad (2.8)$$

Since

$$\det(\nabla \Gamma_\phi^{-1}(x)) = (\det \nabla \Gamma_\phi)^{-1} \circ \Gamma_\phi^{-1}(x), \quad (2.9)$$

by (2.4) and (2.1) we have

$$\begin{aligned} &\int_{\mathbb{R}^d} |g(\Gamma_\phi^{-1}(x))| \cdot |\det(\nabla \Gamma_\phi^{-1}(x)) - 1| dx \\ &= \int_{\mathbb{R}^d} |g(x)| \cdot |\det(\nabla \Gamma_\phi(x)) - 1| dx \lesssim \kappa \int_{\mathbb{R}^d} |g(x)| dx = \kappa \mu_0(|g|). \end{aligned} \quad (2.10)$$

On the other hand, noting that

$$|g(\Gamma_\phi(x)) - g(x)| \leq |\phi(x)| \int_0^1 |\nabla g|(x + s\phi(x)) ds = |\phi(x)| \int_0^1 |\nabla g|(\Gamma_{s\phi}(x)) ds,$$

and since $\phi(0) = 0$,

$$|\phi(x)| = |\phi(x) - \phi(0)| \leq (\|\nabla\phi\|_\infty |x|) \wedge \|\phi\|_\infty \leq \kappa(|x| \wedge 1),$$

we have by the change of variable again, for any $\theta \in [0, 1]$,

$$\begin{aligned} \int_{\mathbb{R}^d} |g(\Gamma_\phi^{-1}(x)) - g(x)| dx &= \int_{\mathbb{R}^d} |g(x) - g(\Gamma_\phi(x))| \det(\nabla\Gamma_\phi(x)) dx \\ &= \int_{\mathbb{R}^d} |g(x) - g(\Gamma_\phi(x))|^\theta |g(x) - g(\Gamma_\phi(x))|^{1-\theta} \det(\nabla\Gamma_\phi(x)) dx \\ &\stackrel{(2.3)}{\lesssim} \int_{\mathbb{R}^d} \left(|\phi(x)| \int_0^1 |\nabla g|(\Gamma_{s\phi}(x)) ds \right)^\theta \left(|g(\Gamma_\phi(x))| + |g(x)| \right)^{1-\theta} dx \\ &\leq \kappa^\theta \int_{\mathbb{R}^d} \left(\int_0^1 |\nabla g|(\Gamma_{s\phi}(x)) ds \right)^\theta \left(|g(\Gamma_\phi(x))| + |g(x)| \right)^{1-\theta} (|x| \wedge 1)^\theta dx \\ &\leq \kappa^\theta \left(\int_0^1 \mu_\theta(|\nabla g| \circ \Gamma_{s\phi}) ds \right)^\theta \left(\mu_\theta(|g| \circ \Gamma_\phi + |g|) \right)^{1-\theta}, \end{aligned}$$

where the last step is due to Hölder's inequality with respect to μ_θ . Moreover,

$$\mu_\theta(|\nabla^j g| \circ \Gamma_{s\phi}) = \int_{\mathbb{R}^d} |\nabla^j g(x)| \cdot \left(|\Gamma_{s\phi}^{-1}(x)| \wedge 1 \right)^\theta \det(\nabla\Gamma_{s\phi}^{-1}(x)) dx \stackrel{(2.3)}{\lesssim} \mu_\theta(|\nabla^j g|),$$

where $j = 0, 1$ and $s \in [0, 1]$. Hence,

$$\int_{\mathbb{R}^d} |g(\Gamma_\phi^{-1}(x)) - g(x)| dx \lesssim \kappa^\theta \mu_\theta(|\nabla g|)^\theta \mu_\theta(|g|)^{1-\theta}, \quad (2.11)$$

which together with (2.8) and (2.10) yields the desired estimate (2.5).

Step 2. Under (2.7), as above, we have

$$\langle \mathcal{D}_1^\phi f, g \rangle = \int_{\mathbb{R}^d} \left(f(\Gamma_\phi(x)) - f(x) - \phi(x) \cdot \nabla f(x) \right) g(x) dx = \langle f, \bar{\mathcal{D}}_1^\phi g \rangle,$$

where

$$\begin{aligned} \bar{\mathcal{D}}_1^\phi g(x) &:= g(\Gamma_\phi^{-1}(x)) \det(\nabla\Gamma_\phi^{-1}(x)) - g(x) + (g \cdot \operatorname{div}\phi)(x) + (\phi \cdot \nabla g)(x) \\ &= g(\Gamma_\phi^{-1}(x)) \left(\det(\nabla\Gamma_\phi^{-1}(x)) (1 + (\operatorname{div}\phi)(\Gamma_\phi^{-1}(x))) - 1 \right) \\ &\quad - (g \cdot \operatorname{div}\phi)(\Gamma_\phi^{-1}(x)) \det(\nabla\Gamma_\phi^{-1}(x)) + (g \cdot \operatorname{div}\phi)(x) \\ &\quad + g(\Gamma_\phi^{-1}(x)) - g(x) + \phi(x) \cdot \nabla g(x) \\ &=: \mathcal{G}_1 g(x) + \mathcal{G}_2 g(x) + \mathcal{G}_3 g(x). \end{aligned}$$

In particular,

$$\langle \mathcal{D}_1^\phi f, g \rangle = \langle f, \mathcal{G}_1 g \rangle + \langle f, \mathcal{G}_2 g \rangle + \langle f, \mathcal{G}_3 g \rangle.$$

Notice that by (2.9) and (2.4),

$$\begin{aligned} &|\det(\nabla\Gamma_\phi^{-1}(x)) (1 + (\operatorname{div}\phi)(\Gamma_\phi^{-1}(x))) - 1| \\ &= \frac{|\det(\nabla\Gamma_\phi) - 1 - \operatorname{div}\phi| \circ \Gamma_\phi^{-1}(x)}{\det \nabla\Gamma_\phi \circ \Gamma_\phi^{-1}(x)} \leq C\kappa^2. \end{aligned}$$

Hence,

$$\langle f, \mathcal{G}_1 g \rangle \lesssim \kappa^2 \|f\|_\infty \int_{\mathbb{R}^d} |g \circ \Gamma_\phi^{-1}(x)| dx \lesssim \kappa^2 \|f\|_\infty \int_{\mathbb{R}^d} |g(x)| dx. \quad (2.12)$$

Moreover, by the change of variable again, we have

$$\begin{aligned} |\langle f, \mathcal{G}_2 g \rangle| &= \left| \int_{\mathbb{R}^d} g(x) \cdot \operatorname{div} \phi(x) \cdot (f(\Gamma_\phi(x)) - f(x)) dx \right| \\ &\leq \|\operatorname{div} \phi\|_\infty \|\phi\|_\infty^\theta \|f\|_{\mathbf{C}^\theta} \int_{\mathbb{R}^d} |g(x)| dx \leq \kappa^{1+\theta} \|f\|_{\mathbf{C}^\theta} \mu_0(|g|). \end{aligned} \quad (2.13)$$

For $\mathcal{G}_3 g(x)$, due to $\Gamma_\phi^{-1}(x) = x - \phi \circ \Gamma_\phi^{-1}(x)$, we have

$$\begin{aligned} \mathcal{G}_3 g(x) &= \phi(x) \cdot \nabla g(x) - \phi \circ \Gamma_\phi^{-1}(x) \cdot \int_0^1 \nabla g(x - s\phi \circ \Gamma_\phi^{-1}(x)) ds \\ &= (\phi(x) - \phi \circ \Gamma_\phi^{-1}(x)) \cdot \int_0^1 \nabla g(x - s\phi \circ \Gamma_\phi^{-1}(x)) ds \\ &\quad + \phi(x) \cdot \int_0^1 (\nabla g(x) - \nabla g(x - s\phi \circ \Gamma_\phi^{-1}(x))) ds \\ &=: \mathcal{G}_{31} g(x) + \mathcal{G}_{32} g(x). \end{aligned}$$

For $\mathcal{G}_{31} g(x)$, since

$$|\phi(x) - \phi \circ \Gamma_\phi^{-1}(x)| \leq \|\nabla \phi\|_\infty |\phi \circ \Gamma_\phi^{-1}(x)| \leq \kappa^2 (|x| \wedge 1),$$

we have

$$\begin{aligned} |\langle f, \mathcal{G}_{31} g \rangle| &\lesssim \kappa^2 \|f\|_\infty \int_0^1 \int_{\mathbb{R}^d} (|x| \wedge 1) |\nabla g|(x - s\phi \circ \Gamma_\phi^{-1}(x)) dx ds \\ &\lesssim \kappa^2 \|f\|_\infty \int_0^1 \int_{\mathbb{R}^d} (|\Gamma_\phi(x)| \wedge 1) |\nabla g|(\Gamma_\phi(x) - s\phi(x)) dx ds \\ &\lesssim \kappa^2 \|f\|_\infty \int_0^1 \int_{\mathbb{R}^d} (|x| \wedge 1) |\nabla g|(\Gamma_{(1-s)\phi}(x)) dx ds \lesssim \kappa^2 \|f\|_\infty \mu_1(|\nabla g|). \end{aligned}$$

For $\mathcal{G}_{32} g(x)$, it is similar to (2.11) that for any $\theta \in [0, 1]$,

$$|\langle f, \mathcal{G}_{32} g \rangle| \lesssim \kappa^{1+\theta} \|f\|_\infty \mu_{1+\theta}(|\nabla^2 g|)^\theta \mu_{1+\theta}(|\nabla g|)^{1-\theta}. \quad (2.14)$$

Combining (2.12)-(2.14), we obtain (2.6) under (2.7).

Step 3. In the general case, without assuming (2.7), if we define

$$\bar{f}(x) := f(x + \phi(0)), \quad \bar{\phi}(x) := \phi(x) - \phi(0),$$

then

$$\mathcal{D}_0^\phi f(x) = \mathcal{D}_0^{\bar{\phi}} \bar{f}(x), \quad \mathcal{D}_1^\phi f(x) = \mathcal{D}_1^{\bar{\phi}} \bar{f}(x) + \bar{\phi}(x) \cdot \nabla(\bar{f} - f)(x).$$

For (2.5), it follows by (i). For (2.6), by (ii), it remains to make the following estimate:

$$\begin{aligned} |\langle \bar{\phi} \cdot \nabla(\bar{f} - f), g \rangle| &= \left| \int_{\mathbb{R}^d} (\operatorname{div} \bar{\phi}(x) g(x) + \bar{\phi}(x) \cdot \nabla g(x)) (\bar{f} - f)(x) dx \right| \\ &\leq \int_{\mathbb{R}^d} (\kappa |g(x)| + \kappa (|x| \wedge 1) |\nabla g(x)|) |\phi(0)|^\theta \cdot [f]_{\mathbf{C}^\theta} dx \\ &\leq \kappa^{1+\theta} (\mu_0(|g|) + \mu_1(|\nabla g|)) [f]_{\mathbf{C}^\theta}. \end{aligned}$$

The proof is complete. \square

Let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz space of all rapidly decreasing functions on \mathbb{R}^d , and $\mathcal{S}'(\mathbb{R}^d)$ the dual space of $\mathcal{S}(\mathbb{R}^d)$ called Schwartz generalized function (or tempered distribution) space. Given $f \in \mathcal{S}(\mathbb{R}^d)$, its Fourier transform \hat{f} and inverse Fourier transform \check{f} are defined by

$$\hat{f}(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i\xi \cdot x} f(x) dx, \quad \check{f}(x) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i\xi \cdot x} f(\xi) d\xi.$$

Let ϕ_0 be a radial C^∞ -function on \mathbb{R}^d with

$$\phi_0(\xi) = 1 \text{ for } |\xi| \leq 1 \text{ and } \phi_0(\xi) = 0 \text{ for } |\xi| \geq 2.$$

Define for $j \in \mathbb{N}$,

$$\phi_j(\xi) := \phi_0(2^{-j}\xi) - \phi_0(2^{1-j}\xi).$$

It is easy to see that for $j \in \mathbb{N}$, $\phi_j(\xi) = \phi_1(2^{1-j}\xi) \geq 0$ and

$$\text{supp}\phi_j \subset B_{2^{j+1}} \setminus B_{2^{j-1}}, \quad \sum_{j=0}^k \phi_j(\xi) = \phi_0(2^{-k}\xi) \rightarrow 1, \quad k \rightarrow \infty.$$

Definition 2.3. For given $j \in \mathbb{N}_0$, the block operator \mathcal{R}_j is defined on $\mathcal{S}'(\mathbb{R}^d)$ by

$$\mathcal{R}_j f(x) := (\phi_j \hat{f})^\vee(x) = \check{\phi}_j * f(x) = 2^{d(j-1)} \int_{\mathbb{R}^d} \check{\phi}_1(2^{j-1}y) f(x-y) dy. \quad (2.15)$$

For any $s \in \mathbb{R}$ and $p, q \in [1, \infty]$, the Besov space $\mathbf{B}_{p,q}^s$ is defined by

$$\mathbf{B}_{p,q}^s := \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{\mathbf{B}_{p,q}^s} := \left(\sum_{j \in \mathbb{N}_0} 2^{sqj} \|\mathcal{R}_j f\|_p^q \right)^{1/q} < \infty \right\}.$$

Remark 2.4. It is well known that for $0 < s \notin \mathbb{N}$ (cf. [15]):

$$\|f\|_{\mathbf{B}_{\infty,\infty}^s} \asymp \|f\|_{C^s}. \quad (2.16)$$

Moreover, let $\bar{\mathcal{R}}_j := \mathcal{R}_{j-1} + \mathcal{R}_j + \mathcal{R}_{j+1}$ with convention $\mathcal{R}_{-1} = 0$. Since $\phi_{j-1} + \phi_j + \phi_{j+1} = 1$ on $B_{2^{j+1}} \setminus B_{2^{j-1}}$, we have

$$(\phi_{j-1} + \phi_j + \phi_{j+1})\phi_j = \phi_j.$$

Consequently,

$$\mathcal{R}_j \bar{\mathcal{R}}_j = \bar{\mathcal{R}}_j \mathcal{R}_j = \mathcal{R}_j. \quad (2.17)$$

The following commutator estimate is proven in [9] (see also [1]).

Lemma 2.5. Let $p, p_1, p_2 \in [1, \infty]$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. For any $\beta \in (0, 1)$ and $\gamma \in (-\beta, 0]$, there is a constant $C = C(d, \beta, \gamma) > 0$ such that for all $j \in \mathbb{N}$,

$$\|[\mathcal{R}_j, f]g\|_p \leq C 2^{-j(\beta+\gamma)} \|f\|_{\mathbf{B}_{p_2,\infty}^\beta} \|g\|_{\mathbf{B}_{p_1,\infty}^\gamma}, \quad j \in \mathbb{N}_0, \quad (2.18)$$

where $[\mathcal{R}_j, f]g := \mathcal{R}_j(fg) - f\mathcal{R}_jg$.

We also need the following Gronwall inequality of Volterra type (cf. [17, Lemma 2.2]).

Lemma 2.6. Let $f \in L^1_{loc}(\mathbb{R}_+; \mathbb{R}_+)$ and $T > 0$. Suppose that for some $\gamma, \beta \in [0, 1)$ and $C_1, C_2 > 0$,

$$f(t) \leq C_1 t^{-\beta} + C_2 \int_0^t (t-s)^{-\gamma} f(s) ds, \quad t \in (0, T].$$

Then there is a constant $C_3 = C_3(C_2, T, \gamma, \beta) > 0$ such that

$$f(t) \leq C_3 C_1 t^{-\beta}, \quad t \in (0, T].$$

3. Gradient estimates for heat kernel of nonlocal operator with constant coefficient

Fix $\alpha \in (0, 2)$. Let $(Z_t)_{t \geq 0}$ be an α -stable process with Lévy measure

$$\nu(A) = \int_0^\infty \frac{dr}{r^{1+\alpha}} \int_{\mathbb{S}^{d-1}} \mathbf{1}_A(r\omega) \pi(d\omega), \quad A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}), \quad (3.1)$$

where π is a finite measure over the unit sphere \mathbb{S}^{d-1} . Note that the α -stable process Z has the scaling property

$$(\lambda^{-1/\alpha} Z_{\lambda t})_{t \geq 0} \stackrel{(d)}{=} (Z_t)_{t \geq 0}, \quad \forall \lambda > 0, \quad (3.2)$$

and for any $\gamma > \alpha > \beta \geq 0$.

$$\int_{|z| < 1} |z|^\gamma \nu(dz) + \int_{|z| > 1} |z|^\beta \nu(dz) < \infty. \quad (3.3)$$

Let $N(dt, dz)$ be the associated Poisson random measure, that is,

$$N([0, t] \times A) := \sum_{s \in (0, t]} \mathbf{1}_A(\Delta Z_s), \quad A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}), t > 0,$$

where $\Delta Z_s := Z_s - Z_{s-}$ is the jump size of Z at time s . Let $\phi(t, z) : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a measurable function with

$$|\phi(t, z)| \leq C|z|, \quad (t, z) \in \mathbb{R}_+ \times \mathbb{R}^d.$$

We consider the following time-inhomogenous Lévy process: for $0 \leq s \leq t < \infty$,

$$X_{s,t}^\phi := \begin{cases} \int_s^t \int_{\mathbb{R}^d} \phi(r, z) N(dr, dz), & \alpha \in (0, 1), \\ \int_s^t \int_{|z| \leq 1} \phi(r, z) \tilde{N}(dr, dz) + \int_s^t \int_{|z| > 1} \phi(r, z) N(dr, dz), & \alpha = 1, \\ \int_s^t \int_{\mathbb{R}^d} \phi(r, z) \tilde{N}(dr, dz), & \alpha \in (1, 2), \end{cases}$$

where $\tilde{N}(dr, dz) := N(dr, dz) - dr\nu(dz)$ is the compensated Poisson random measure. Define

$$P_{s,t}^\phi f(x) := \mathbb{E}f(x + X_{s,t}^\phi), \quad f \in C_b^2(\mathbb{R}^d). \quad (3.4)$$

By Itô's formula, one has

$$\partial_t P_{s,t}^\phi f = \mathcal{L}_{\phi_t}^{(\alpha)} P_{s,t}^\phi f, \quad (3.5)$$

where $\phi_t(z) := \phi(t, z)$ and

$$\mathcal{L}_{\phi_t}^{(\alpha)} f(x) := \int_{\mathbb{R}^d} \left(f(x + \phi(t, z)) - f(x) - \phi^{(\alpha)}(t, z) \cdot \nabla f(x) \right) \nu(dz),$$

and

$$\phi^{(\alpha)}(t, z) := \left(\mathbf{1}_{\alpha \in (1, 2)} + \mathbf{1}_{\alpha = 1} \mathbf{1}_{|z| \leq 1} \right) \phi(t, z). \quad (3.6)$$

Lemma 3.1. (Duhamel's formula) Let $\varphi \in \mathbf{C}^0$ and $f \in L_{loc}^\infty(\mathbb{R}_+; \mathbf{C}^0)$. Define

$$u(t, x) := P_{0,t}^\phi \varphi(x) + \int_0^t P_{s,t}^\phi f(s, x) ds.$$

Then $u \in L_{loc}^\infty(\mathbb{R}_+; \mathbf{C}^0)$ uniquely solves $\partial_t u = \mathcal{L}_{\phi_t}^{(\alpha)} u + f$ in the distributional sense, that is, for any $\psi \in C_c^2(\mathbb{R}^d)$ and $t > 0$,

$$\langle u(t), \psi \rangle = \langle \varphi, \psi \rangle + \int_0^t \langle u(s), \mathcal{L}_{-\phi_s}^{(\alpha)} \psi \rangle ds + \int_0^t \langle f(s), \psi \rangle ds. \quad (3.7)$$

Proof. Recall that \mathbf{C}^0 stands for the space of bounded measurable functions. Clearly, $u \in L_{loc}^\infty(\mathbb{R}_+; \mathbf{C}^0)$. Let $(\rho_\varepsilon)_{\varepsilon \in (0,1)}$ be a family of mollifiers in \mathbb{R}^d with support in B_ε . Define

$$u_\varepsilon(t) := u(t) * \rho_\varepsilon, \quad \varphi_\varepsilon := \varphi * \rho_\varepsilon, \quad f_\varepsilon(t) := f(t) * \rho_\varepsilon.$$

Clearly,

$$u_\varepsilon(t, x) = P_{0,t}^\phi \varphi_\varepsilon(x) + \int_0^t P_{s,t}^\phi f_\varepsilon(s, x) ds.$$

By (3.5) and the integration by parts, one sees that

$$u_\varepsilon(t, x) = \varphi_\varepsilon(x) + \int_0^t \mathcal{L}_{\phi_s}^{(\alpha)} u_\varepsilon(s, x) ds + \int_0^t f_\varepsilon(s, x) ds. \quad (3.8)$$

Hence, for any $\psi \in C_c^2(\mathbb{R}^d)$,

$$\langle u_\varepsilon(t), \psi \rangle = \langle \varphi_\varepsilon, \psi \rangle + \int_0^t \langle u_\varepsilon(s), \mathcal{L}_{-\phi_s}^{(\alpha)} \psi \rangle ds + \int_0^t \langle f_\varepsilon(s), \psi \rangle ds.$$

Taking $\varepsilon \rightarrow 0$, we obtain (3.7). On the other hand, if we take $\psi(\cdot) = \rho_\varepsilon(x - \cdot)$ in (3.7), then we get (3.8) and the uniqueness follows. \square

Below we always make the following assumptions:

($\mathbf{H}^{\phi, \nu}$) ϕ satisfies the following nondegeneracy conditions

$$0 < c_0^\phi := \inf_{\omega_0 \in \mathbb{S}^{d-1}} \inf_{t, \lambda > 0} \int_{\mathbb{S}^{d-1}} \frac{|\omega_0 \cdot \phi(t, \lambda \omega)|^2}{\lambda^2} \pi(d\omega), \quad \sup_{t, z} \frac{|\phi(t, z)|}{|z|} =: c_1^\phi < \infty,$$

and

$$\mathbf{1}_{\alpha=1} \int_{R_1 < |z| < R_2} \phi(t, z) \nu(dz) = 0, \quad 0 < R_1 < R_2. \quad (3.9)$$

Notice that for all $0 \leq s < t < \infty$,

$$c_0^\phi = c_0^{\phi_{s,t}} \quad \text{and} \quad c_1^\phi = c_1^{\phi_{s,t}}, \quad (3.10)$$

and by (3.2) and (3.9),

$$X_{s,t}^\phi \stackrel{(d)}{=} (t-s)^{-1/\alpha} X_{0,1}^{\phi_{s,t}}, \quad (3.11)$$

where

$$\phi_{s,t}(r, z) := (t-s)^{1/\alpha} \phi(s+r(t-s), (t-s)^{-1/\alpha} z).$$

The following lemma can be proved as in [8] (see also [4]). For the readers' convenience, we provide a detailed proof here.

Lemma 3.2. *Under ($\mathbf{H}^{\phi, \nu}$), for each $0 \leq s < t < \infty$, $X_{s,t}$ admits a C^∞ -smooth density $p_{s,t}^\phi(x)$ which satisfies that for any $\beta \in [0, \alpha)$ and $n \in \mathbb{N}_0$,*

$$\int_{\mathbb{R}^d} |x|^\beta |\nabla^n p_{s,t}^\phi(x)| dx \leq C(t-s)^{\frac{\beta-n}{\alpha}}, \quad (3.12)$$

where $C = C(c_0^\phi, c_1^\phi, n, d, \alpha, \beta) > 0$. Moreover, for each $0 \leq s < t < \infty$ and $x \in \mathbb{R}^d$,

$$p_{s,t}^\phi(x) = (t-s)^{-d/\alpha} p_{0,1}^{\phi_{s,t}}((t-s)^{-1/\alpha} x). \quad (3.13)$$

Proof. First of all, by (3.11) one sees that (3.13) holds. Thus by (3.10), it suffices to prove (3.12) for $s = 0$ and $t = 1$. We only consider the case $\alpha \in (0, 1)$ and write

$$X_{0,1}^\phi = \int_0^1 \int_{|z|<1} \phi(r, z) N(dr, dz) + \int_0^1 \int_{|z|\geq 1} \phi(r, z) N(dr, dz) =: \tilde{X}_{0,1}^\phi + \bar{X}_{0,1}^\phi.$$

Note that the characteristic function of $\tilde{X}_{0,1}^\phi$ is given by

$$\mathbb{E} e^{i\xi \cdot \tilde{X}_{0,1}^\phi} = e^{\psi_1(\xi)},$$

where for $\delta \in (0, \infty]$,

$$\psi_\delta(\xi) := \int_0^1 \int_{|z|<\delta} \left(e^{i\xi \cdot \phi(t, z)} - 1 \right) \nu(dz) dt.$$

We claim that there is a constant $c_2 > 0$ such that for all $\xi \in \mathbb{R}^d$,

$$\operatorname{Re} \psi_1(\xi) \leq -c_2 \left(|\xi|^2 \wedge |\xi|^\alpha \right). \quad (3.14)$$

Indeed, by a change of variable, we have

$$\begin{aligned} \operatorname{Re} \psi_\infty(\xi) &= \int_0^1 \int_0^\infty \frac{1}{r^{1+\alpha}} \int_{\mathbb{S}^{d-1}} \left(\cos(\xi \cdot \phi(t, r\omega)) - 1 \right) \pi(d\omega) dr dt \\ &= |\xi|^\alpha \int_0^1 \int_0^\infty \frac{1}{r^{1+\alpha}} \int_{\mathbb{S}^{d-1}} \left(\cos(\xi \cdot \phi(t, r\omega/|\xi|)) - 1 \right) \pi(d\omega) dr dt \\ &\leq |\xi|^\alpha \int_0^1 \int_0^\delta \frac{1}{r^{1+\alpha}} \int_{\mathbb{S}^{d-1}} \left(\cos(\xi \cdot \phi(t, r\omega/|\xi|)) - 1 \right) \pi(d\omega) dr dt. \end{aligned}$$

Note that

$$\lim_{r \rightarrow 0} \frac{1 - \cos(r)}{r^2} = \frac{1}{2}. \quad (3.15)$$

By $(\mathbf{H}^{\phi, \nu})$, since $|\xi \cdot \phi(t, r\omega/|\xi|)| \leq c_1^\phi r$, we can choose δ small enough so that

$$\begin{aligned} \operatorname{Re} \psi_\infty(\xi) &\leq -\frac{|\xi|^\alpha}{3} \int_0^1 \int_0^\delta \frac{1}{r^{1+\alpha}} \int_{\mathbb{S}^{d-1}} |\xi \cdot \phi(t, r\omega/|\xi|)|^2 \pi(d\omega) dr dt \\ &\leq -c_0^\phi \frac{|\xi|^\alpha}{3} \int_0^\delta \frac{r^2}{r^{1+\alpha}} dr, \end{aligned}$$

and therefore, there are constants $c_3 > 0$ and $M > 0$ such that for all $|\xi| > M$,

$$\begin{aligned} \operatorname{Re} \psi_1(\xi) &= \operatorname{Re} \psi_\infty(\xi) + |\psi_\infty(\xi) - \psi_1(\xi)| \\ &\leq -c_0^\phi \frac{|\xi|^\alpha}{3} \int_0^\delta \frac{r^2}{r^{1+\alpha}} dr + 2 \int_{|z|>1} \nu(dz) \leq -c_3 |\xi|^\alpha. \end{aligned}$$

On the other hand, by (3.15) and $(\mathbf{H}^{\phi, \nu})$, for δ small enough, we also have for $|\xi| \leq M$,

$$\begin{aligned} \operatorname{Re} \psi_1(\xi) &\leq \int_0^1 \int_{|z|\leq\delta} \left(\cos(\xi \cdot \phi(t, z)) - 1 \right) \nu(dz) dt \\ &\leq -c_4 \int_0^1 \int_{|z|\leq\delta} |\xi \cdot \phi(t, z)|^2 \nu(dz) dt \\ &= -c_4 |\xi|^2 \int_0^1 \int_0^\delta \frac{1}{r^{1+\alpha}} \int_{\mathbb{S}^{d-1}} |\bar{\xi} \cdot \phi(t, r\omega)|^2 \pi(d\omega) dr dt \\ &\leq -c_4 c_0^\phi |\xi|^2 \int_0^\delta \frac{r^2}{r^{1+\alpha}} dr, \end{aligned}$$

where $\bar{\xi} := \xi/|\xi|$. This proves the claim (3.14).

By (3.14), it is easy to see that $e^{\psi_1(\xi)} \in \mathcal{S}(\mathbb{R}^d)$ is a Schwartz function. Thus the random variable $\tilde{X}_{0,1}^\phi$ has a C^∞ -smooth density $\rho \in \mathcal{S}(\mathbb{R}^d)$. Noting that for $\beta < \alpha$,

$$\mathbb{E}|\bar{X}_{0,1}^\phi|^\beta \leq C(c_1^\phi, \beta, \alpha, d) < \infty, \quad p_{0,1}^\phi(x) = \mathbb{E}\rho(x + \bar{X}_{0,1}^\phi),$$

we have

$$\begin{aligned} \int_{\mathbb{R}^d} |x|^\beta |\nabla^n p_{0,1}^\phi(x)| dx &\leq \mathbb{E} \int_{\mathbb{R}^d} |x|^\beta |\nabla^n \rho|(x + \bar{X}_{0,1}^\phi) dx \\ &\lesssim \int_{\mathbb{R}^d} |x|^\beta |\nabla^n \rho|(x) dx + \mathbb{E}|\bar{X}_{0,1}^\phi|^\beta \int_{\mathbb{R}^d} |\nabla^n \rho|(x) dx < \infty. \end{aligned}$$

The desired estimate (3.12) for $s = 0$ and $t = 1$ follows. \square

The following is a key lemma, which is similar to [11, Lemma 3.1].

Lemma 3.3. *Assume $(\mathbf{H}^{\phi, \nu})$ and let $p_{s,t}(x) := p_{s,t}^\phi(x)$ be as in Lemma 3.2.*

(i) *For any $n \in \mathbb{N}_0$, there is a constant $C > 0$ such that for all $0 \leq s < t < \infty$,*

$$\|\nabla^n \mathcal{R}_0 p_{s,t}\|_1 \leq C. \quad (3.16)$$

(ii) *For any $n \in \mathbb{N}_0$, $\vartheta \geq 0$ and $\beta \in [0, \alpha)$, there is a constant $C > 0$ such that for all $0 \leq s < t < \infty$ and $j \in \mathbb{N}$,*

$$m_\beta(|\nabla^n \mathcal{R}_j p_{s,t}|) \leq C 2^{(n-\vartheta)j} (t-s)^{-\vartheta/\alpha} ((t-s)^{\beta/\alpha} + 2^{-j\beta}), \quad (3.17)$$

where $m_\beta(dx) := |x|^\beta dx$ and $m_\beta(\varphi) := \int_{\mathbb{R}^d} \varphi(x) m_\beta(dx)$.

(iii) *For any $n \in \mathbb{N}_0$, $\beta, \gamma \in [0, \alpha)$, there is a constant $C > 0$ such that for all nonnegative measurable $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $0 \leq s < t < \infty$, $j \in \mathbb{N}$,*

$$\int_0^t m_\beta(|\nabla^n \mathcal{R}_j p_{s,t}|) f(s) ds \leq C 2^{(n-\gamma-\beta)j} \int_0^t (t-s)^{-\frac{\gamma}{\alpha}} f(s) ds. \quad (3.18)$$

Proof. (i) Let $n \in \mathbb{N}_0$. By the definition of \mathcal{R}_0 , we have

$$\|\nabla^n \mathcal{R}_0 p_{s,t}\|_1 \lesssim \|\mathcal{R}_0 p_{s,t}\|_1 \leq \|\check{\phi}_0\|_1 \|p_{s,t}\|_1 = \|\check{\phi}_0\|_1 < \infty.$$

(ii) Since the support of ϕ_1 is contained in the annulus, for any $k \in \mathbb{N}_0$,

$$\Delta^{-k} \check{\phi}_1 := (|\xi|^{-2k} \phi_1(\xi))^\vee \in \mathcal{S}(\mathbb{R}^d).$$

Fix $j \in \mathbb{N}$. For simplicity of notation, we write

$$\tilde{p}_{0,1}(x) := p_{0,1}^{\phi_{s,t}}(x), \quad \check{h} := (t-s)^{-1/\alpha} 2^{-j}.$$

Thus by (3.13) and the change of variable, for any $k \in \mathbb{N}_0$, we have

$$\begin{aligned} \mathcal{R}_j p_{s,t}(x) &= (t-s)^{-d/\alpha} \int_{\mathbb{R}^d} \tilde{p}_{0,1}((t-s)^{-1/\alpha} 2^{-j} y) \check{\phi}_1(2^j x - y) dy \\ &= \check{h}^{d+k} 2^{jd} \int_{\mathbb{R}^d} (\Delta^k \tilde{p}_{0,1})(\check{h}y) (\Delta^{-k} \check{\phi}_1)(2^j x - y) dy. \end{aligned}$$

Therefore, for any $k \in \mathbb{N}_0$,

$$\begin{aligned} m_\beta(|\nabla^n \mathcal{R}_j p_{s,t}|) &= \int_{\mathbb{R}^d} |x|^\beta |\nabla^n \mathcal{R}_j p_{s,t}(x)| dx \\ &= \check{h}^{d+k} 2^{j(n-\beta)} \int_{\mathbb{R}^d} |x|^\beta \left| \int_{\mathbb{R}^d} (\Delta^k \tilde{p}_{0,1})(\check{h}y) (\nabla^n \Delta^{-k} \check{\phi}_1)(x-y) dy \right| dx \\ &\lesssim \check{h}^{d+k} 2^{j(n-\beta)} \int_{\mathbb{R}^d} |x|^\beta |\Delta^k \tilde{p}_{0,1}(\check{h}y)| dy \int_{\mathbb{R}^d} |\nabla^n \Delta^{-k} \check{\phi}_1(x)| dx \end{aligned}$$

$$\begin{aligned}
& + \hbar^{d+k} 2^{j(n-\beta)} \int_{\mathbb{R}^d} |\Delta^k \tilde{p}_{0,1}(\hbar y)| dy \int_{\mathbb{R}^d} |x|^\beta |\nabla^n \Delta^{-k} \check{\phi}_1(x)| dx \\
& \stackrel{(3.12)}{\lesssim} \hbar^k 2^{j(n-\beta)} (\hbar^{-\beta} + 1),
\end{aligned}$$

which in turn gives (3.17) by simple interpolation.

(iii) Let \mathcal{I} be the left hand side of (3.18). Without loss of generality, we assume $t > 2^{-\alpha j}$. We make the following decomposition:

$$\mathcal{I} = \left(\int_{(t-2^{-\alpha j}) \vee 0}^t + \int_0^{(t-2^{-\alpha j}) \vee 0} \right) \mu_\beta \left(|\nabla^n \mathcal{R}_j p_{s,t}| \right) f(s) ds =: \mathcal{I}_1 + \mathcal{I}_2.$$

For \mathcal{I}_1 , by (3.17) with $\vartheta = \gamma$, we have

$$\begin{aligned}
\mathcal{I}_1 & \lesssim 2^{(n-\gamma)j} \int_{(t-2^{-\alpha j}) \vee 0}^t (t-s)^{-\frac{\gamma}{\alpha}} \left(2^{-j} + (t-s)^{\frac{1}{\alpha}} \right)^\beta f(s) ds \\
& \leq 2^{(n-\gamma-\beta)j} \int_0^t (t-s)^{-\frac{\gamma}{\alpha}} f(s) ds.
\end{aligned}$$

For \mathcal{I}_2 , by (3.17) with $\vartheta = \gamma + \beta$, we have

$$\begin{aligned}
\mathcal{I}_2 & \lesssim 2^{(n-\gamma-\beta)j} \int_0^{(t-2^{-\alpha j}) \vee 0} (t-s)^{-\frac{\gamma+\beta}{\alpha}} \left(2^{-j} + (t-s)^{\frac{1}{\alpha}} \right)^\beta f(s) ds \\
& \leq 2^{(n-\gamma-\beta)j} \int_0^t (t-s)^{-\frac{\gamma}{\alpha}} f(s) ds.
\end{aligned}$$

Combining the above two estimates, we obtain (3.18) for $j \in \mathbb{N}$. \square

4. Regularity estimate for nonlocal equations

In this section we fix $\alpha \in (0, 2)$ and consider the following time-dependent non-local operator:

$$\mathcal{L}_{\phi_t}^{(\alpha)} u(x) := \int_{\mathbb{R}^d} \left(u(x + \phi(t, x, z)) - u(x) - \phi^{(\alpha)}(t, x, z) \cdot \nabla u(x) \right) \nu(dz),$$

where ν takes the form (3.1), $\phi(t, x, z) : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a measurable function, and

$$\phi^{(\alpha)}(t, x, z) := \left(\mathbf{1}_{\alpha \in (1,2)} + \mathbf{1}_{\alpha=1} \mathbf{1}_{|z| \leq 1} \right) \phi(t, x, z).$$

Recall that π is the finite measure on \mathbb{S}^{d-1} associated with the Lévy measure ν in (3.1). We assume for some $c_0, c_1 > 0$,

$$|\phi(t, x, z)| \leq c_0 |z|, \quad |\nabla_x \phi(t, x, z)| \leq c_1 |z|, \quad (4.1)$$

and

$$\inf_{\omega_0 \in \mathbb{S}^{d-1}} \inf_{x \in \mathbb{R}^d} \inf_{t, \lambda > 0} \int_{\mathbb{S}^{d-1}} \frac{|\omega_0 \cdot \phi(t, x, \lambda \omega)|^2}{\lambda^2} \pi(d\omega) =: c_2 > 0, \quad (4.2)$$

and

$$\mathbf{1}_{\{\alpha=1\}} \int_{R_1 < |z| < R_2} \phi(t, x, z) \nu(dz) = 0 \quad \text{for any } 0 < R_1 < R_2. \quad (4.3)$$

Clearly, $\mathcal{L}_{\phi_t}^{(\alpha)} u(x)$ is well defined pointwisely if $u \in \mathbf{C}^\gamma$ for some $\gamma > \alpha$. Let $b : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a measurable function and satisfy that for some $\beta \in [0, 1]$,

$$|b(t, x)| \leq c_3, \quad |b(t, x) - b(t, y)| \leq c_3 |x - y|^\beta. \quad (4.4)$$

The following parameter set will be used for stating the dependence of a constant.

$$\Theta := (d, \alpha, c_0, c_1, c_2, c_3, \beta).$$

Fix $s \geq 0$. Consider the following nonlocal equation:

$$\partial_t u_s = \mathcal{L}_{\phi_t}^{(\alpha)} u_s + \mathbf{1}_{\{\alpha > 1/2\}} b \cdot \nabla u_s \quad \text{for } t \geq s \quad \text{with } u_s(s, x) = \varphi(x). \quad (4.5)$$

In order to introduce the classical solution of (4.5), we define

$$\mathbf{C}^{\gamma-} := \cap_{\gamma' < \gamma} \mathbf{C}^{\gamma'}.$$

Definition 4.1. Fix $s \geq 0$ and $\gamma > \alpha \vee 1$. For $\varphi \in \mathbf{C}^{\gamma-}$, we call a function $u_s \in C([s, \infty); \mathbf{C}^{\gamma-})$ a classical solution of nonlocal-PDE (4.5) with initial value φ at time s if for all $t \geq s$ and $x \in \mathbb{R}^d$,

$$u_s(t, x) = \varphi(x) + \int_s^t \left(\mathcal{L}_{\phi_r}^{(\alpha)} + \mathbf{1}_{\{\alpha > 1/2\}} b \cdot \nabla \right) u_s(r, x) dr.$$

By the proof of [6, Theorem 6.1], maximum principle holds for classical solutions of (4.5) and so classical solution to (4.5) is unique. Fix $\gamma > \alpha \vee 1$ and suppose $u_s(t, x)$ is the classical solution to (4.5). To explicitly reflect its dependence on its initial value φ at time s , we write

$$Q_{s,t}\varphi(x) := u_s(t, x).$$

It follows from the uniqueness of classical solution to (4.5) that for any $s < r < t$,

$$Q_{s,t}\varphi = Q_{r,t}Q_{s,r}\varphi. \quad (4.6)$$

We first establish the following a priori regularity estimate.

Theorem 4.2. Let $\alpha \in (0, 2)$, $\beta \in [0, 1]$ with $\alpha + \beta > 1$ and $\gamma \in [0, \alpha + \alpha \wedge \beta)$. Under conditions (4.1)-(4.4), for any $T > 0$ and $\eta \in (-((\alpha + \beta - 1) \wedge 1), \gamma]$, there is a constant $C = C(T, \Theta, \gamma, \eta) > 0$ such that for any $0 \leq s < t \leq T$, $\varphi \in \mathbf{C}^2$ and any classical solution $Q_{s,t}\varphi(x) = u_s(t, x)$ of the nonlocal-PDE (4.5),

$$\|Q_{s,t}\varphi\|_{\mathbf{B}_{\infty,\infty}^{\gamma}} \leq C(t-s)^{\frac{\eta-\gamma}{\alpha}} \|\varphi\|_{\mathbf{B}_{\infty,\infty}^{\eta}}. \quad (4.7)$$

To show (4.7), we use the freezing coefficient argument. Without loss of generality, we assume $s = 0$ and write $u(t, x) = Q_{0,t}\varphi(x)$. Fix $y \in \mathbb{R}^d$ and let \mathcal{S}^y be the set of all solutions θ^y of the following ODE:

$$\dot{\theta}^y(t) = -b(t, \theta^y(t)), \quad \theta^y(0) = y.$$

It is well known that for any $t > 0$ (e.g. [11, Lemma 6.5]),

$$\cup_{y \in \mathbb{R}^d} \cup_{\theta^y \in \mathcal{S}^y} \{\theta^y(t)\} = \mathbb{R}^d. \quad (4.8)$$

Define

$$u^y(t, x) := u(t, x + \theta^y(t)), \quad \phi^y(t, x, z) := \phi^y(t, x, z) := \phi(t, x + \theta^y(t), z)$$

and

$$b^y(t, x) := b(t, x + \theta^y(t)) - b(t, \theta^y(t)).$$

It is easy to see that

$$\partial_t u^y + \mathcal{L}_{\phi_t^y}^{(\alpha)} u^y + \mathbf{1}_{\{\alpha > 1/2\}} b^y \cdot \nabla u^y = 0, \quad u^y(0, x) = \varphi^y(x) := \varphi(x + y).$$

In the following, for notional simplicity we drop the superscript y from u, φ, b . With this notation, u satisfies

$$\partial_t u + \mathcal{L}_{\phi_t}^{(\alpha)} u + \mathbf{1}_{\{\alpha > 1/2\}} b \cdot \nabla u = 0, \quad u(0) = \varphi,$$

and b satisfies

$$|b(t, x)| \leq 2c_3(|x|^\beta \wedge 1). \quad (4.9)$$

Next we freeze ϕ at point 0. Define $\psi(t, z) := \phi(t, 0, z)$ and

$$\mathcal{A}u := \mathcal{L}_{\phi_t}^{(\alpha)}u - \mathcal{L}_{\psi_t}^{(\alpha)}u.$$

Then we have

$$\partial_t u + \mathcal{L}_{\psi_t}^{(\alpha)}u + \mathcal{A}u + \mathbf{1}_{\{\alpha > 1/2\}} b \cdot \nabla u = 0, \quad u(0) = \varphi. \quad (4.10)$$

Let $P_{s,t}^\psi$ be defined by (3.4) in terms of $\phi = \psi$ and $p_{s,t}^\psi$ the corresponding heat kernel, that is,

$$P_{s,t}^\psi f(x) = \int_{\mathbb{R}^d} p_{s,t}^\psi(x-y) f(y) dy.$$

Since u is a classical solution of (4.10), by Lemma 3.1,

$$u(t, x) = P_{0,t}^\psi \varphi^y(x) + \int_0^t P_{s,t}^\psi(\mathcal{A}u)(s, x) ds + \mathbf{1}_{\{\alpha > 1/2\}} \int_0^t P_{s,t}^\psi(b \cdot \nabla u)(s, x) ds.$$

For $j \in \mathbb{N}_0$, acting on both sides of the above equation by \mathcal{R}_j , we obtain

$$\begin{aligned} \mathcal{R}_j u(t, 0) &= \mathcal{R}_j P_{0,t}^\psi \varphi^y(0) + \int_0^t \mathcal{R}_j P_{s,t}^\psi(\mathcal{A}u)(s, 0) ds \\ &\quad + \mathbf{1}_{\{\alpha > 1/2\}} \int_0^t \mathcal{R}_j P_{s,t}^\psi(b \cdot \nabla u)(s, 0) ds. \end{aligned} \quad (4.11)$$

Lemma 4.3. *For any $T > 0$ and $\eta \leq \gamma$, there is a constant $C = C(T, \gamma, \eta, \Theta) > 0$ such that for all $j \in \mathbb{N}_0$, $t \in (0, T]$ and $y \in \mathbb{R}^d$,*

$$2^{\gamma j} |\mathcal{R}_j P_{0,t}^\psi \varphi^y(0)| \leq C t^{\frac{\eta-\gamma}{\alpha}} \|\varphi\|_{\mathbf{B}_{\infty,\infty}^\eta},$$

where $\varphi^y(x) := \varphi(x+y)$.

Proof. By definition and Hölder's inequality, we have for any $\eta \leq \gamma$,

$$\begin{aligned} |\mathcal{R}_j P_{0,t}^\psi \varphi^y(0)| &= \left| \int_{\mathbb{R}^d} \mathcal{R}_j p_{0,t}^\psi(-x) \varphi^y(x) dx \right| \stackrel{(2.17)}{=} \left| \int_{\mathbb{R}^d} \mathcal{R}_j p_{0,t}^\psi(-x) \bar{\mathcal{R}}_j \varphi^y(x) dx \right| \\ &\leq \|\mathcal{R}_j p_{0,t}^\psi\|_1 \|\bar{\mathcal{R}}_j \varphi\|_\infty \lesssim 2^{-\gamma j} t^{-\frac{\gamma-\eta}{\alpha}} \|\varphi\|_{\mathbf{B}_{\infty,\infty}^\eta}, \end{aligned}$$

where the last step is due to Lemma 3.3 and the definition of $\mathbf{B}_{\infty,\infty}^\eta$. \square

Lemma 4.4. *For any $T > 0$, $\gamma \in [0, \alpha)$ and $\varepsilon \in (0, \alpha - \gamma)$, there is a constant $C = C(T, \varepsilon, \gamma, \Theta) > 0$ such that for all $j \in \mathbb{N}_0$ and $t \in (0, T]$,*

$$2^{\gamma j} \int_0^t |\mathcal{R}_j P_{s,t}^\psi \mathcal{A}u(s, 0)| ds \leq \int_0^t (t-s)^{-\frac{\gamma+\varepsilon}{\alpha}} \|u(s)\|_{\mathbf{B}_{\infty,\infty}^{(\alpha-1)\vee 0+\varepsilon}} ds.$$

Proof. We only prove the estimate for $\alpha \in (1, 2)$. The case $\alpha \in (0, 1]$ is similar and easier. Since the time variable does not play any essential role in the following calculations, if there is no confusions, we shall drop the time variable for simplicity of notation. Let $\delta > 0$ be a fixed small number, which will be determined below. Since $\alpha \in (1, 2)$, by definition we can make the following decomposition:

$$\mathcal{A}u(x) = \int_{|z| \leq \delta} \mathcal{D}_1^{\phi(\cdot, z)} u(x) \nu(dz) + \int_{|z| > \delta} \mathcal{D}_1^{\phi(\cdot, z)} u(x) \nu(dz) =: \mathcal{A}_\delta u(x) + \bar{\mathcal{A}}_\delta u(x),$$

where $\mathcal{D}_1^{\phi(\cdot, z)} u(x)$ is defined by (see (2.2))

$$\mathcal{D}_1^{\phi(\cdot, z)} u(x) := u(x + \phi(x, z)) - u(x + \phi(0, z)) - (\phi(x, z) - \phi(0, z)) \cdot \nabla u(x).$$

We first treat $\mathcal{A}_\delta u$. Notice that by definition and Fubini's theorem,

$$\begin{aligned}\mathcal{R}_j P_{s,t}^\psi \mathcal{A}_\delta u(s, 0) &= \int_{\mathbb{R}^d} \mathcal{R}_j p_{s,t}^\psi(-x) \mathcal{A}_\delta u(s, x) dx \\ &= \int_{|z| \leq \delta} \int_{\mathbb{R}^d} \mathcal{R}_j p_{s,t}^\psi(-x) \mathcal{D}_1^{\phi(\cdot, z)} u(s, x) dx \nu(dz).\end{aligned}$$

By the assumption (4.1), one can choose δ small enough so that

$$\|\nabla^\ell \phi(s, \cdot, z)\|_\infty \leq C|z| \leq \frac{1}{2}, \quad |z| < \delta, \quad \ell = 0, 1.$$

In particular, the assumption (2.1) is satisfied. Let $\theta \in (\alpha - 1, 1)$. By (2.6) and Hölder's inequality, we have for any $j \in \mathbb{N}$,

$$\begin{aligned}\int_0^t |\mathcal{R}_j P_{s,t}^\psi \mathcal{A}_\delta u(s, 0)| ds &\leq \int_0^t \int_{|z| \leq \delta} \left| \int_{\mathbb{R}^d} \mathcal{R}_j p_{s,t}^\psi(-x) \mathcal{D}_1^{\phi(\cdot, z)} u(s, x) dx \right| \nu(dz) ds \\ &\lesssim \int_0^t \|u(s)\|_{\mathbf{C}^\theta} \left[\mu_{1+\theta} (|\nabla^2 \mathcal{R}_j p_{s,t}^\psi|)^\theta \mu_{1+\theta} (|\nabla \mathcal{R}_j p_{s,t}^\psi|)^{1-\theta} \right. \\ &\quad \left. + \mu_0 (|\mathcal{R}_j p_{s,t}^\psi|) + \mu_1 (|\nabla \mathcal{R}_j p_{s,t}^\psi|) \right] ds \\ &\lesssim \left(\int_0^t \mu_{1+\theta} (|\nabla^2 \mathcal{R}_j p_{s,t}^\psi|) \|u(s)\|_{\mathbf{C}^\theta} ds \right)^\theta \\ &\quad \times \left(\int_0^t \mu_{1+\theta} (|\nabla \mathcal{R}_j p_{s,t}^\psi|) \|u(s)\|_{\mathbf{C}^\theta} ds \right)^{1-\theta} \\ &\quad + \int_0^t \left[\mu_0 (|\mathcal{R}_j p_{s,t}^\psi|) + \mu_1 (|\nabla \mathcal{R}_j p_{s,t}^\psi|) \right] \|u(s)\|_{\mathbf{C}^\theta} ds \\ &=: I_1(t) + I_2(t).\end{aligned}$$

Let $\beta \in (0, \alpha)$. Since $\beta < \alpha < 1 + \theta$ and $\gamma + \varepsilon < \alpha$, and recalling

$$\mu_{1+\theta}(dx) = (|x| \wedge 1)^{1+\theta} dx \leq |x|^\beta dx = m_\beta(dx),$$

for $j \in \mathbb{N}$, by (3.18) and (2.16), we have

$$\begin{aligned}I_1(t) &\lesssim \left(\int_0^t m_\beta (|\nabla^2 \mathcal{R}_j p_{s,t}^\psi|) \|u(s)\|_{\mathbf{C}^\theta} ds \right)^\theta \left(\int_0^t m_\beta (|\nabla \mathcal{R}_j p_{s,t}^\psi|) \|u(s)\|_{\mathbf{C}^\theta} ds \right)^{1-\theta} \\ &\lesssim \left(2^{(2-\beta-\gamma-\varepsilon)j} \int_0^t (t-s)^{-\frac{\gamma+\varepsilon}{\alpha}} \|u(s)\|_{\mathbf{C}^\theta} ds \right)^\theta \\ &\quad \times \left(2^{(1-\beta-\gamma-\varepsilon)j} \int_0^t (t-s)^{-\frac{\gamma+\varepsilon}{\alpha}} \|u(s)\|_{\mathbf{C}^\theta} ds \right)^{1-\theta} \\ &\lesssim 2^{(1+\theta-\beta-\gamma-\varepsilon)j} \int_0^t (t-s)^{-\frac{\gamma+\varepsilon}{\alpha}} \|u(s)\|_{\mathbf{B}_{\infty, \infty}^\theta} ds,\end{aligned}$$

and also,

$$I_2(t) \lesssim 2^{-\gamma j} \int_0^t (t-s)^{-\frac{\gamma}{\alpha}} \|u(s)\|_{\mathbf{B}_{\infty, \infty}^\theta} ds.$$

For $j = 0$, by (3.16), we clearly have

$$\int_0^t |\mathcal{R}_0 P_{s,t}^\psi \mathcal{A}_\delta u(s, 0)| ds \lesssim \int_0^t \|u(s)\|_{\mathbf{B}_{\infty, \infty}^\theta} ds.$$

Thus, we obtain that for any $j \in \mathbb{N}_0$,

$$\int_0^t |\mathcal{R}_j P_{s,t}^\psi \mathcal{A}_\delta u(s, 0)| ds \lesssim 2^{(1+\theta-\beta-\varepsilon-\gamma)j} \int_0^t (t-s)^{-\frac{\gamma+\varepsilon}{\alpha}} \|u(s)\|_{\mathbf{B}_{\infty, \infty}^\theta} ds.$$

In particular, if we choose θ close to $\alpha - 1$ from above and β close to α from below so that

$$1 + \theta - \beta \leq \varepsilon, \quad \theta - (\alpha - 1) \leq \varepsilon,$$

then we get for any $j \in \mathbb{N}_0$,

$$2^{\gamma j} \int_0^t |\mathcal{R}_j P_{s,t}^\psi \mathcal{A}_\delta u(s, 0)| ds \lesssim \int_0^t (t-s)^{-\frac{\gamma+\varepsilon}{\alpha}} \|u(s)\|_{\mathbf{B}_{\infty,\infty}^{\alpha-1+\varepsilon}} ds. \quad (4.12)$$

Recall $m_\beta(dx) = |x|^\beta dx$. For $\mathcal{A}_\delta u$, letting $\bar{\phi}(x, z) := \phi(x, z) - \phi(0, z)$, by Fubini's theorem and the integration by parts, we have

$$\begin{aligned} |\mathcal{R}_j P_{s,t}^\psi \mathcal{A}_\delta u(0)| &= \left| \int_{\mathbb{R}^d} \mathcal{R}_j p_{s,t}^\psi(x) \mathcal{A}_\delta u(x) dx \right| \leq 2 \|u\|_\infty \int_{|z|>\delta} \int_{\mathbb{R}^d} |\mathcal{R}_j p_{s,t}^\psi(x)| dx \nu(dz) \\ &\quad + \left| \int_{|z|>\delta} \int_{\mathbb{R}^d} \left(\mathcal{R}_j p_{s,t}^\psi(x) \operatorname{div}_x \bar{\phi}(x, z) + \bar{\phi}(x, z) \cdot \nabla \mathcal{R}_j p_{s,t}^\psi(x) \right) u(x) dx \nu(dz) \right| \\ &\lesssim \|u\|_\infty m_0(|\mathcal{R}_j p_{s,t}^\psi|) + \|u\|_\infty m_1(|\nabla \mathcal{R}_j p_{s,t}^\psi|) \int_{|z|>\delta} |z| \nu(dz), \end{aligned}$$

where we have used that $|\operatorname{div}_x \bar{\phi}(x, z)| \lesssim |z|$ and $|\bar{\phi}(x, z)| \lesssim |x| \cdot |z|$. Thus by (3.18), we get for any $\gamma \in (0, \alpha)$,

$$\int_0^t |\mathcal{R}_j P_{s,t}^\psi \mathcal{A}_\delta u(s, 0)| ds \leq C 2^{-\gamma j} \int_0^t (t-s)^{-\frac{\gamma}{\alpha}} \|u(s)\|_\infty ds,$$

which together with (4.12) yields the desired estimate. \square

In order to obtain the gradient estimate, we need the following lemma to improve the regularity to higher order.

Lemma 4.5. *For any $T > 0$, $\gamma \in [0, \alpha + \alpha \wedge 1)$ and $\theta \in (\alpha, 2)$, there is a constant $C = C(T, \Theta, \gamma, \theta) > 0$ such that for all $j \in \mathbb{N}_0$ and $t \in (0, T)$,*

$$2^{\gamma j} \int_0^t |\mathcal{R}_j P_{s,t}^\psi \mathcal{A} u(s, 0)| ds \leq C \int_0^t (t-s)^{-\frac{\gamma'}{\alpha}} \|u(s)\|_{\mathbf{B}_{\infty,\infty}^\theta} ds,$$

where $\gamma' := \frac{\gamma}{2} \mathbf{1}_{\gamma \leq 2} + (\gamma - 1) \mathbf{1}_{\gamma > 2} < \alpha$.

Proof. Still we only consider $\alpha \in (1, 2)$. Let $\theta \in (\alpha, 2)$. Noting that

$$\begin{aligned} &|u(x + \phi(x, z)) - u(x + \phi(0, z)) - (\phi(x, z) - \phi(0, z)) \cdot \nabla u(x)| \\ &\leq |\phi(x, z) - \phi(0, z)| \int_0^1 |\nabla u(x + s\phi(x, z) + (1-s)\phi(0, z)) - \nabla u(x)| ds \\ &\stackrel{(4.1)}{\lesssim} |x| |z| \left((|z|^{\theta-1} \|\nabla u\|_{\mathbf{C}^{\theta-1}}) \wedge \|\nabla u\|_\infty \right) \lesssim |x| (|z|^\theta \wedge |z|) \|u\|_{\mathbf{C}^\theta}, \end{aligned}$$

by (2.16), we have

$$|\mathcal{A} u(x)| \lesssim |x| \cdot \|u\|_{\mathbf{C}^\theta} \int_{\mathbb{R}^d} (|z|^\theta \wedge |z|) \nu(dz) \lesssim |x| \cdot \|u\|_{\mathbf{B}_{\infty,\infty}^\theta}.$$

Hence,

$$\begin{aligned} \int_0^t |\mathcal{R}_j P_{s,t}^\psi \mathcal{A} u(s, 0)| ds &\leq \int_0^t \int_{\mathbb{R}^d} |\mathcal{R}_j p_{s,t}^\psi(-x) \mathcal{A} u(s, x)| dx ds \\ &\lesssim \int_0^t m_1(|\mathcal{R}_j p_{s,t}^\psi|) \|u(s)\|_{\mathbf{B}_{\infty,\infty}^\theta} ds, \end{aligned}$$

which implies by (3.18) that for any $\bar{\gamma} \in [0, \alpha]$,

$$2^{(\bar{\gamma} + \bar{\gamma} \wedge 1)j} \int_0^t |\mathcal{R}_j P_{s,t}^\psi \mathcal{A}u(s, 0)| ds \leq C \int_0^t (t-s)^{-\frac{\bar{\gamma}}{\alpha}} \|u(s)\|_{\mathbf{B}_{\infty, \infty}^\theta} ds.$$

Here $m_1(dx) := |x|dx$. The proof is completed by the change of $\gamma = \bar{\gamma} + \bar{\gamma} \wedge 1$. \square

Next comes to treat the drift term.

Lemma 4.6. *Let $\alpha \in (1/2, 2)$, $\beta \in [0, \alpha \wedge 1)$ with $\alpha + \beta > 1$. Under (4.4), for any $T > 0$, $\gamma \in [0, \alpha]$ and $\varepsilon \in (0, (\alpha - \gamma) \wedge \beta)$, there is a constant $C = C(T, \Theta, \gamma, \varepsilon) > 0$ such that for all $j \in \mathbb{N}_0$ and $t \in (0, T]$,*

$$2^{\gamma j} \int_0^t |\mathcal{R}_j P_{s,t}^\psi (b \cdot \nabla u)(s, 0)| ds \leq C \int_0^t (t-s)^{-\frac{\gamma + \varepsilon}{\alpha}} \|u(s)\|_{\mathbf{B}_{\infty, \infty}^{1-\beta+\varepsilon}} ds, \quad (4.13)$$

and for any $\gamma \in [0, \alpha + \beta)$,

$$2^{\gamma j} \int_0^t |\mathcal{R}_j P_{s,t}^\psi (b \cdot \nabla u)(s, 0)| ds \leq C \int_0^t (t-s)^{-\frac{(\gamma - \beta) \vee 0}{\alpha}} \|\nabla u(s)\|_{\infty} ds. \quad (4.14)$$

Proof. For $j \in \mathbb{N}_0$, by definition and (2.17), we have

$$\begin{aligned} \mathcal{R}_j P_{s,t}^\psi (b \cdot \nabla u)(0) &= \int_{\mathbb{R}^d} \mathcal{R}_j p_{s,t}^\psi(-x) (b \cdot \nabla u)(x) dx = \int_{\mathbb{R}^d} \bar{\mathcal{R}}_j p_{s,t}^\psi(-x) \mathcal{R}_j (b \cdot \nabla u)(x) dx \\ &= \int_{\mathbb{R}^d} \bar{\mathcal{R}}_j p_{s,t}^\psi(-x) (b \cdot \nabla \mathcal{R}_j u)(x) dx + \int_{\mathbb{R}^d} \bar{\mathcal{R}}_j p_{s,t}^\psi(-x) [\mathcal{R}_j, b \cdot \nabla] u(x) dx. \end{aligned}$$

By (4.9) and (2.18), we have for any $\theta \in (1 - \beta, 1)$,

$$\begin{aligned} |\mathcal{R}_j P_{s,t}^\psi (b \cdot \nabla u)(0)| &\lesssim \|\nabla \mathcal{R}_j u\|_{\infty} m_{\beta}(|\bar{\mathcal{R}}_j p_{s,t}^\psi|) + \|[\mathcal{R}_j, b \cdot \nabla] u\|_{\infty} \|\bar{\mathcal{R}}_j p_{s,t}^\psi\|_1 \\ &\lesssim 2^{(1-\theta)j} \|u\|_{\mathbf{C}^\theta} m_{\beta}(|\bar{\mathcal{R}}_j p_{s,t}^\psi|) + 2^{(1-\theta-\beta)j} \|u\|_{\mathbf{C}^\theta} \|\bar{\mathcal{R}}_j p_{s,t}^\psi\|_1, \end{aligned}$$

where $m_{\beta}(dx) = |x|^{\beta} dx$. Since $\gamma + \varepsilon < \alpha$, by (3.18) and (2.16), we obtain that for any $\theta \in (1 - \beta, 1)$,

$$\int_0^t |\mathcal{R}_j P_{s,t}^\psi (b \cdot \nabla u)(s, 0)| ds \leq C 2^{(1-\theta-\beta-\gamma-\varepsilon)j} \int_0^t (t-s)^{-\frac{\gamma+\varepsilon}{\alpha}} \|u(s)\|_{\mathbf{B}_{\infty, \infty}^\theta} ds,$$

which implies (4.13) by choosing $\theta = 1 - \beta + \varepsilon$. Moreover, we also have

$$\begin{aligned} \int_0^t |\mathcal{R}_j P_{s,t}^\psi (b \cdot \nabla u)(s, 0)| ds &\leq \int_0^t \int_{\mathbb{R}^d} |\mathcal{R}_j p_{s,t}^\psi(-x) (b \cdot \nabla u)(s, x)| dx ds \\ &\stackrel{(4.9)}{\lesssim} \int_0^t m_{\beta}(|\mathcal{R}_j p_{s,t}^\psi|) \|\nabla u(s)\|_{\infty} ds \stackrel{(3.18)}{\lesssim} 2^{-(\bar{\gamma} + \beta)j} \int_0^t (t-s)^{-\frac{\bar{\gamma}}{\alpha}} \|\nabla u(s)\|_{\infty} ds, \end{aligned}$$

where $\bar{\gamma} \in [0, \alpha]$. Thus we obtain (4.14) by letting $\gamma = \bar{\gamma} + \beta$. \square

Remark 4.7. The reason of requiring $\alpha \in (1/2, 2)$ in Lemma 4.6 is due to $\beta < \alpha \wedge 1$ and $\alpha + \beta > 1$. Here we require $\beta < \alpha$ because of the moment estimate in (3.12). It is an open problem whether this restrict $\alpha > 1/2$ can be dropped.

Now we are in a position to give

Proof of Theorem 4.2. We divide the proof into two steps.

Step 1. In this step we prove (4.7) for $\gamma \in [0, \alpha]$. Let

$$\delta := (\alpha - 1) \vee (1 - \beta), \quad \eta \in (\delta - \alpha, \gamma].$$

By (4.11) and Lemmas 4.3, 4.4 and (4.13), for any $\varepsilon \in (0, (\alpha - \gamma) \wedge \beta)$, we have

$$\|u(t)\|_{\mathbf{B}_{\infty, \infty}^{\gamma}} = \sup_{j \geq 0} 2^{\gamma j} \|\mathcal{R}_j u(t)\|_{\infty} \stackrel{(4.8)}{=} \sup_{j \geq 0} 2^{\gamma j} \sup_y |\mathcal{R}_j u^y(t, 0)|$$

$$\lesssim t^{\frac{\eta-\gamma}{\alpha}} \|\varphi\|_{\mathbf{B}_{\infty,\infty}^{\eta}} + \int_0^t (t-s)^{-\frac{\gamma+\varepsilon}{\alpha}} \|u(s)\|_{\mathbf{B}_{\infty,\infty}^{\delta+\varepsilon}} ds. \quad (4.15)$$

Since $\delta < \alpha$ and $\eta > \delta - \alpha$, one can choose $\varepsilon \in (0, (\alpha - \gamma) \wedge \beta)$ small enough and η' so that

$$\delta + 2\varepsilon < \alpha, \quad \eta' \in (\delta + \varepsilon - \alpha, (\delta + \varepsilon) \wedge \eta].$$

Thus by (4.15) with $\gamma = \delta + \varepsilon$ and $\eta = \eta'$, we have

$$\|u(t)\|_{\mathbf{B}_{\infty,\infty}^{\delta+\varepsilon}} \lesssim t^{\frac{\eta'-\delta-\varepsilon}{\alpha}} \|\varphi\|_{\mathbf{B}_{\infty,\infty}^{\eta'}} + \int_0^t (t-s)^{-\frac{\delta+2\varepsilon}{\alpha}} \|u(s)\|_{\mathbf{B}_{\infty,\infty}^{\delta+\varepsilon}} ds,$$

which implies by Gronwall's inequality (see Lemma 2.6) that for all $t \in (0, T]$,

$$\|u(t)\|_{\mathbf{B}_{\infty,\infty}^{\delta+\varepsilon}} \lesssim t^{\frac{\eta'-\delta-\varepsilon}{\alpha}} \|\varphi\|_{\mathbf{B}_{\infty,\infty}^{\eta'}} \lesssim t^{\frac{\eta'-\delta-\varepsilon}{\alpha}} \|\varphi\|_{\mathbf{B}_{\infty,\infty}^{\eta}}. \quad (4.16)$$

Now substituting (4.16) into (4.15), we obtain that for all $t \in (0, T]$,

$$\begin{aligned} \|u(t)\|_{\mathbf{B}_{\infty,\infty}^{\gamma}} &\lesssim \|\varphi\|_{\mathbf{B}_{\infty,\infty}^{\eta}} \left(t^{\frac{\eta-\gamma}{\alpha}} + \int_0^t (t-s)^{-\frac{\gamma+\varepsilon}{\alpha}} s^{\frac{\eta'-\delta-\varepsilon}{\alpha}} ds \right) \\ &\lesssim t^{\frac{\eta-\gamma}{\alpha}} \|\varphi\|_{\mathbf{B}_{\infty,\infty}^{\eta}} \Rightarrow \|Q_{s,t}\varphi\|_{\mathbf{B}_{\infty,\infty}^{\gamma}} \leq C(t-s)^{\frac{\eta-\gamma}{\alpha}} \|\varphi\|_{\mathbf{B}_{\infty,\infty}^{\eta}}. \end{aligned} \quad (4.17)$$

Thus we obtain (4.7) for any $\gamma \in [0, \alpha)$ since $\delta - \alpha = -((\alpha + \beta - 1) \wedge 1)$.

Step 2. In this step we improve the spatial regularity of $Q_{s,t}\varphi$ to order $\gamma \in [0, \alpha + \alpha \wedge \beta)$. We consider the case of $\alpha > 1/2$. The case of $\alpha \in (0, 1/2]$ and $b \equiv 0$ is easier. Let

$$\gamma \in (\alpha \vee 1, \alpha + \alpha \wedge \beta), \quad \eta \leq \gamma, \quad \theta \in (\alpha \vee 1, 2).$$

By Lemma 4.5 and (4.14), for γ' being as in Lemma 4.5, we have

$$\begin{aligned} \|u(t)\|_{\mathbf{B}_{\infty,\infty}^{\gamma}} &\lesssim t^{\frac{\eta-\gamma}{\alpha}} \|\varphi\|_{\mathbf{B}_{\infty,\infty}^{\eta}} + \int_0^t (t-s)^{-\frac{\gamma'}{\alpha}} \|u(s)\|_{\mathbf{B}_{\infty,\infty}^{\theta}} ds \\ &\quad + \int_0^t (t-s)^{-\frac{\gamma-\beta}{\alpha}} \|\nabla u(s)\|_{\infty} ds. \end{aligned} \quad (4.18)$$

In particular, for $\gamma'' := \gamma' \vee (\gamma - \beta) < \alpha$, we have

$$\|u(t)\|_{\mathbf{B}_{\infty,\infty}^{\gamma}} \lesssim t^{\frac{\eta-\gamma}{\alpha}} \|\varphi\|_{\mathbf{B}_{\infty,\infty}^{\eta}} + \int_0^t (t-s)^{-\frac{\gamma''}{\alpha}} \|u(s)\|_{\mathbf{B}_{\infty,\infty}^{\gamma}} ds,$$

which implies by Gronwall's inequality that for any $\eta \in (\gamma - \alpha, \gamma]$,

$$\|u(t)\|_{\mathbf{B}_{\infty,\infty}^{\gamma}} \lesssim t^{\frac{\eta-\gamma}{\alpha}} \|\varphi\|_{\mathbf{B}_{\infty,\infty}^{\eta}} \Rightarrow \|Q_{s,t}\varphi\|_{\mathbf{B}_{\infty,\infty}^{\gamma}} \lesssim (t-s)^{\frac{\eta-\gamma}{\alpha}} \|\varphi\|_{\mathbf{B}_{\infty,\infty}^{\eta}}. \quad (4.19)$$

For $\eta \in (\delta - \alpha, \alpha)$, by (4.6) and the estimate obtained in Step 1, we have

$$\begin{aligned} \|Q_{s,t}\varphi\|_{\mathbf{B}_{\infty,\infty}^{\gamma}} &= \|Q_{\frac{s+t}{2},t} Q_{s,\frac{s+t}{2}}\varphi\|_{\mathbf{B}_{\infty,\infty}^{\gamma}} \stackrel{(4.19)}{\lesssim} (t-s)^{\frac{\eta'-\gamma}{\alpha}} \|Q_{s,\frac{s+t}{2}}\varphi\|_{\mathbf{B}_{\infty,\infty}^{\eta'}} \\ &\stackrel{(4.17)}{\lesssim} (t-s)^{\frac{\eta-\gamma}{\alpha}} \|\varphi\|_{\mathbf{B}_{\infty,\infty}^{\eta}}, \end{aligned}$$

where $\eta' \in (\gamma - \alpha, \alpha) \subset (\gamma - \alpha, \gamma)$. The proof is completed by interpolation. \square

5. PROOF OF THEOREM 1.1

Now we give the proof of Theorem 1.1 under condition (ii). Case (i) is easier. Thus we assume $\alpha \in (1/2, 2)$. We divide the proof into three steps.

Step 1. Fix $T > 0$ and $\gamma \in (\alpha \vee 1, \alpha + \alpha \wedge 1)$. For any $\varphi \in \mathbf{C}^{\gamma-}$, we first show the existence of a classical solution $u^T \in C([0, T]; \mathbf{C}^{\gamma-})$ for the following backward nonlocal-PDE

$$\partial_s u^T + \mathcal{L}_s^{\sigma, b} u^T = 0, \quad u^T(T) = \varphi, \quad (5.1)$$

where $\mathcal{L}_s^{\sigma, b}$ is defined by (1.3). For $n \in \mathbb{N}$, define

$$\varphi_n(x) := \varphi * \rho_n(x), \quad b_n(t, x) := b(t, \cdot) * \rho_n(x), \quad \sigma_n(t, x) := \sigma(t, \cdot) * \rho_n(x),$$

where $(\rho_n)_{n \in \mathbb{N}}$ is a family of mollifiers in \mathbb{R}^d . It is easy to see that

$$\varphi_n \in C_b^\infty(\mathbb{R}^d), \quad \sigma_n, b_n \in L^\infty(\mathbb{R}_+; C_b^\infty(\mathbb{R}^d)).$$

It is well known that under these assumptions, for any $s \geq 0$ and $x \in \mathbb{R}^d$, the following SDE admits a unique strong solution $X_{s,t}^n(x)$:

$$dX_{s,t}^n = b_n(t, X_{s,t}^n)dt + \sigma_n(t, X_{s,t}^n)dZ_t, \quad X_{s,s}^n = x.$$

Moreover, $\{X_{s,t}^n(x), x \in \mathbb{R}^d, t \geq s \geq 0\}$ forms a C^∞ -stochastic flow, and

$$u_n^T(s, x) := \mathbb{E}\varphi_n(X_{s,T}^n(x)) \in C([0, T]; C_b^2(\mathbb{R}^d))$$

uniquely solves the following equation:

$$\partial_s u_n^T + \mathcal{L}_s^{\sigma_n, b_n} u_n^T = 0, \quad u_n^T(T) = \varphi_n. \quad (5.2)$$

Below we let

$$\phi_n(t, x, z) := \sigma_n(t, x)z, \quad \nu(dz) := \sum_{i=1}^d |z_i|^{-1-\alpha} \delta_0(dz_1) \cdots dz_i \cdots \delta_0(dz_d).$$

Under (\mathbf{H}^σ) and $\|\nabla\sigma\|_\infty < \infty$, it is easy to see that (4.1)-(4.4) hold uniformly for the above b_n, ϕ_n and ν . Thus for any $\gamma' \in (\alpha \vee 1, \gamma)$, one can use (4.7) to derive the following uniform estimate:

$$\sup_n \|u_n^T\|_{C([0, T]; \mathbf{C}^{\gamma'})} \leq C\|\varphi\|_{\mathbf{C}^{\gamma'}}.$$

By (5.2) and the above uniform estimate, one sees that for all $0 \leq t_0 < t_1 \leq T$,

$$\|u_n^T(t_1) - u_n^T(t_0)\|_\infty \leq \int_{t_0}^{t_1} \|\mathcal{L}_s^{\sigma_n, b_n} u_n^T(s)\|_\infty ds \leq C|t_1 - t_0|,$$

where $C > 0$ is independent of n . Now by Ascoli-Arzelà's lemma, there are function $u \in C([0, T]; \mathbf{C}^{\gamma-})$ and subsequence still denoted by n such that for any $T, R > 0$,

$$\lim_{n \rightarrow \infty} \|\nabla^j u_n^T - \nabla^j u^T\|_{C([0, T] \times B_R)} = 0, \quad j = 0, 1.$$

Taking $n \rightarrow \infty$ in (5.2), one finds that u is a classical solution of nonlocal-PDE (5.1) in the sense of Definition 4.1.

Step 2. Let $u^T \in C([0, T]; \mathbf{C}^{\gamma-})$ be the classical solution of nonlocal equation (5.1). Let $X_{s,t}(x)$ be the unique solution of SDE (1.2). By applying Itô's formula to $(t, x) \mapsto u^T(t, x)$, we obtain

$$u^T(T, X_{s,T}(x)) = u^T(s, x) + \int_s^T (\partial_t u^T + \mathcal{L}_t^{\sigma, b} u^T)(t, X_{s,t}(x))dt + \text{a martingale}.$$

Hence, by (5.1),

$$P_{s,T}\varphi(x) = \mathbb{E}\varphi(X_{s,T}(x)) = \mathbb{E}u^T(T, X_{s,T}(x)) = u^T(s, x).$$

The desired estimate (1.4) now follows by Theorem 4.2.

Step 3. For **(A)**, let $\varphi \in \mathbf{B}_{\infty,\infty}^{-\eta}$ for some $\eta < (\alpha + \beta - 1) \wedge 1$ and let $\varphi_\varepsilon := \varphi * \rho_\varepsilon$ be the mollifying approximation. Clearly, by (5.1) we have

$$\partial_s P_{s,t} \varphi_\varepsilon(x) + \mathcal{L}_s^{\sigma,b} P_{s,t} \varphi_\varepsilon(x) = 0.$$

In particular, for any $0 \leq t_0 < t_1 < t$ and $x \in \mathbb{R}^d$,

$$P_{t_0,t} \varphi_\varepsilon(x) = P_{t_1,t} \varphi_\varepsilon(x) + \int_{t_0}^{t_1} \mathcal{L}_s^{\sigma,b} P_{s,t} \varphi_\varepsilon(x) ds.$$

By (1.4) and taking limits $\varepsilon \rightarrow 0$, we obtain (1.5).

For **(B)**, since $\alpha > 1/2$ and $\alpha + \beta > 1$, one can choose $\gamma > 1$ and $\eta = 0$ in (1.4) so that

$$\|P_{s,t} \varphi\|_{\mathbf{C}^\gamma} \leq C(t-s)^{-\frac{\gamma}{\alpha}} \|\varphi\|_{\mathbf{B}_{\infty,\infty}^0} \leq C(t-s)^{-\frac{\gamma}{\alpha}} \|\varphi\|_\infty.$$

On the other hand, it is clear that

$$\|P_{s,t} \varphi\|_\infty \leq \|\varphi\|_\infty.$$

The desired gradient estimate now follows by interpolation theorem (see [12, p35, Theorem 3.2.1]).

For **(C)**, let $\eta \in (\delta - \alpha, 0)$. By (1.4), we have

$$\|P_{s,t} \varphi\|_\infty \leq C_{s,t} \|\varphi\|_{\mathbf{B}_{\infty,\infty}^\eta}.$$

From this, by Sobolev's embedding, one sees that

$$P_{s,t} \varphi(x) = \int_{\mathbb{R}^d} \varphi(y) p_{s,t}(x, y) dy, \quad p_{s,t}(x, \cdot) \in \mathbf{B}_{1,1}^{-\eta}.$$

Thus, we obtain the desired regularity.

REFERENCES

- [1] H. Bahouri, J.-Y. Chemin, and R. Danchin, *Fourier analysis and nonlinear partial differential equations*. Grundlehren der Mathematischen Wissenschaften, Vol.343. Springer, Heidelberg, 2011.
- [2] R. F. Bass and Z.-Q. Chen, Systems of equations driven by stable processes. *Probab. Theory Relat. Fields*, **134** (2006), 175–214.
- [3] R. F. Bass and Z.-Q. Chen, Regularity of harmonic functions for a class of singular stable-like processes. *Math. Z.*, **266** (2010), 489–503.
- [4] P.E. Chaudru de Raynal, S. Menozzi, and E. Priola, Schauder estimates for drifted fractional operators in the supercritical case. arXiv:1902.02616.
- [5] Z.-Q. Chen, D. A. Croydon, and T. Kumagai, Quenched invariance principles for random walks and elliptic diffusions in random media with boundary. *Ann. Probab.*, **43** (2015), 1594–1642.
- [6] Z.-Q. Chen, E. Hu, L. Xie, and X. Zhang, Heat kernels for non-symmetric diffusion operators with jumps. *J. Differential Equations*, **263** (2017), 6576–6634.
- [7] Z.-Q. Chen, R. Song, and X. Zhang, Stochastic flows for Lévy processes with Hölder drifts. *Rev. Mat. Iberoam.*, **34** (2018), 1755–1788.
- [8] Z.-Q. Chen and X. Zhang, L^p -maximal hypoelliptic regularity of nonlocal kinetic Fokker-Planck operators. *J. Math. Pures Appl. (9)*, **116** (2018), 52–87.
- [9] Z.-Q. Chen, X. Zhang, and G. Zhao, Well-posedness of supercritical SDE driven by Lévy processes with irregular drifts. arXiv:1709.04632.
- [10] A. Debussche and N. Fournier, Existence of densities for stable-like driven SDE's with Hölder continuous coefficients. *J. Funct. Anal.*, **264** (2013), 1757–1778.
- [11] Z. Hao, M. Wu, and X. Zhang, Schauder's estimate for nonlocal kinetic equations and its applications. arXiv:1903.09967.
- [12] N. V. Krylov, *Lectures on elliptic and parabolic equations in Hölder spaces*. Graduate Studies in Mathematics, Vol.12. American Mathematical Society, Providence, RI, 1996.
- [13] T. Kulczycki, M. Ryznar, and P. Sztonyk, Strong Feller property for SDEs driven by multiplicative cylindrical stable noise. arXiv:1811.05960.

- [14] D.W. Stroock and S. R. S. Varadhan, *Multidimensional diffusion processes*. Grundlehren der Mathematischen Wissenschaften, Vol.233. Springer-Verlag, Berlin-New York, 1979.
- [15] H. Triebel, *Theory of function spaces. II*. Monographs in Mathematics, Vol.84. Birkhäuser Verlag, Basel, 1992.
- [16] F.Y. Wang, L. Xu, and X. Zhang, Gradient estimates for SDEs driven by multiplicative Lévy noise. *J. Funct. Anal.*, **269** (2015), 3195-3219.
- [17] X. Zhang, Stochastic Volterra equations in Banach spaces and stochastic partial differential equation. *J. Funct. Anal.*, **258** (2010), 1361–1425.
- [18] X. Zhang, Degenerate irregular SDEs with jumps and application to integro-differential equations of Fokker-Planck type. *Electron. J. Probab.*, **18** (2013), no. 55, 25.
- [19] X. Zhang and G. Zhao, Heat kernel and ergodicity of SDEs with distributional drifts. arXiv:1710.10537.

ZHEN-QING CHEN: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE, WA 98195, USA, EMAIL: ZQCHEN@UW.EDU

XICHENG ZHANG: SCHOOL OF MATHEMATICS AND STATISTICS, WUHAN UNIVERSITY, WUHAN, HUBEI 430072, P.R.CHINA, EMAIL: XICHENGZHANG@GMAIL.COM

ZIMO HAO: SCHOOL OF MATHEMATICS AND STATISTICS, WUHAN UNIVERSITY, WUHAN, HUBEI 430072, P.R.CHINA, EMAIL: ZIMOHAO@WHU.EDU.CN