ROBUST ORLICZ SPACES: OBSERVATIONS AND CAVEATS

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Abstract. In this paper, we investigate two different constructions of robust Orlicz spaces as a generalisation of robust L^p-spaces. We show that a construction as norm closures of bounded continuous functions typically leads to spaces which are lattice-isomorphic to sublattices of a classical L^1-space, thus leading to dominated classes of contingent claims even for nondominated classes of probability measures. We further show that the mathematically very desirable property of σ-Dedekind completeness for norm closures of continuous functions ususally aready implies that the considered class of probability measures is dominated. Our second construction, which is top-down, is based on the consideration of the maximal domain of a worst-case Luxemburg norm. From an applied persepective, this approach can be justified by a uniform-boundedness-type result showing that, in typical situations, the worst-case Orlicz space agrees with the intersection of the corresponding individual Orlicz spaces.

Key words: Orlicz space, model uncertainty, nonlinear expectation, Dedekind completeness, Banach lattice, Choquet capacity

AMS 2010 Subject Classification: 46E30; 28A12; 46B42

1. Introduction

Since the beginning of this century, the simultaneous consideration of families of prior distributions instead of a single probability measure has become of fundamental importance for the risk assessment of financial positions. In this context, one often speaks of *model uncertainty* or *ambiguity*, where the uncertainty is modeled by a set $\mathfrak{P}$ of probability measures. Especially after the subprime mortgage crisis, the desire for mathematical models based on *nondominated* families of priors arose: no single probability measure can be chosen which determines whether an event is deemed certain or negligible. A model incorporating the phenomenon of mutually singular measures was found in a Brownian motion with uncertain volatility, the so-called G-Brownian motion, cf. Peng [31, 32]. To date, the latter is the most prominent example for a model consisting of a nondominated set of probability distributions, and an extensive strand of literature has formed around this model.

In order to maintain a certain degree of analytic tractability while still allowing for uncertainty in terms of nondominated priors, closures of the space C_b of bounded continuous functions under robust L^p-norms $\|\cdot\|_{L^p(\mathfrak{P})}$, for a nonempty set of priors $\mathfrak{P}$, have become a frequent choice for commodity spaces or spaces of contingent claims in the context of a G-Brownian motion, see, for instance, Beissner & Denis [3], Beissner & Riedel [4], Bion-Nadal & Kervarec [7], Denis et al. [11], or Denis & Kervarec [12]. One reason for this choice is certainly that, roughly speaking, all "nice" analytic properties of C_b carry over to the $\|\cdot\|_{L^p(\mathfrak{P})}$-closure. As a consequence, in the past decades, the analytic properties of these spaces have been studied extensively, see, e.g., Denis et al. [11] or Beissner & Denis [3], and a complete stochastic calculus has been developed based on these spaces, cf. Peng [33]. However, very little is known about their properties as Banach lattices.

On another note, there has been renewed interest in the role of *Orlicz spaces* in mathematical finance. They have, for instance, appeared as canonical model spaces for risk measures, premium principles, and utility maximisation problems; see Bellini et al. [5], Biagini & Černý [6], Delbaen & Owari [10], Gao et al. [15, 17, 18].

Date: July 17, 2020.

The second author gratefully acknowledges financial support of the German Research Foundation via CRC 1283.

The present manuscript lies at the intersection of these two strands of literature. One of its main goals is an order-theoretic study of robust spaces arising from closures of bounded continuous functions w.r.t. a robust Luxemburg norm introduced in equation (1) below, which we shall denote by $\mathcal{C}^\Phi(\mathfrak{P})$. These turn out to be separable in many typical situations. One of our main results, Theorem 13, states that, typically, $\mathcal{C}^\Phi(\mathfrak{P})$ is lattice-isomorphic to a sublattice of a classical L^1-space without changing the measurable space. As a consequence, its elements are still dominated by a single probability measure, and the $\mathfrak{P}$-quasi-sure order collapses to an almost sure order – even for nondominated sets $\mathfrak{P}$ of priors. In particular, Theorem 13 applies to the situation considered in the G-framework, showing that the robust closure $\mathcal{C}^\Phi(\mathfrak{P})$ not only inherits all nice analytic properties from C_b, but also its dominatedness. In the same spirit, we show that the robust closure of C_b is also too similar to the original space C_b in terms of *order completeness properties*, such as the existence of suprema for bounded countable families of contingent claims. Theorem 15 states that, in typical situations including the G-framework, σ-Dedekind completeness of $\mathcal{C}^\Phi(\mathfrak{P})$ already implies the dominatedness of the family of priors $\mathfrak{P}$ and even uniform integrability of the densities. We thereby qualify that what *prevents* nondominated models from being dominated is the lack of all order completeness properties of the Banach lattice $\mathcal{C}^\Phi(\mathfrak{P})$.
The ambient space in which we take the closure of C_b will be constructed as follows. We consider a fixed measurable space $(\Omega, \mathcal{F})$, a nonempty set of probability measures $\mathfrak{P}$ on $(\Omega, \mathcal{F})$, and a family $\Phi = (\phi_\mathbb{P})_{\mathbb{P}\in\mathfrak{P}}$ of Orlicz functions. As usual, we consider the quotient space $L^0(\mathfrak{P})$ of all real-valued measurable functions on $(\Omega, \mathcal{F})$ up to $\mathfrak{P}$-q.s. equality. On $L^0(\mathfrak{P})$, we may consider the robust Luxemburg norm

$$\|X\|_{L^\Phi(\mathfrak{P})} := \sup_{\mathbb{P}\in\mathfrak{P}} \|X\|_{L^{\phi_\mathbb{P}}(\mathbb{P})} \in [0,\infty], \quad \text{for } X \in L^0(\mathfrak{P}), \tag{1}$$

where $\|\cdot\|_{L^{\phi_\mathbb{P}}(\mathbb{P})}$ is the Luxemburg seminorm for $\phi_\mathbb{P}$ under the probability measure $\mathbb{P} \in \mathfrak{P}$. The robust Orlicz space $L^\Phi(\mathfrak{P})$ is then defined to be the space of all $X \in L^0(\mathfrak{P})$ with $\|X\|_{L^\Phi(\mathfrak{P})} < \infty$. Notice that these spaces arise naturally in the context of *variational preferences* as axiomatised by Maccheroni et al. [24]. These encompass prominent classes of preferences, such as multiple prior preferences introduced by Gilboa & Schmeidler [19] and multiplier preferences introduced by Hansen & Sargent [20] – see also [24, Section 4.2.1]. One of the most appealing qualities of variational preference relations is the handy separation of risk attitudes (measured by the prior-wise expected utility approach) and ambiguity or uncertainty attitudes (as expressed by the choice of $\mathfrak{P}$ and the penalisation γ). Aggregating expert opinions or the preferences of a cloud of variational preference agents, however, requires to consider more than one utility function. Thus, considering the preference relation on $L^0(\mathfrak{P})$ given by

$$X \trianglelefteq Y \quad :\iff \quad \inf_{\mathbb{P}\in\mathfrak{P}} \mathbb{E}_\mathbb{P}[u_\mathbb{P}(X)] - \gamma(\mathbb{P}) \le \inf_{\mathbb{P}\in\mathfrak{P}} \mathbb{E}_\mathbb{P}[u_\mathbb{P}(Y)] - \gamma(\mathbb{P}), \tag{2}$$

and following the arguments of Biagini & Černý [6], will naturally lead to robust Orlicz spaces as their canonical model space. Special cases of robust Orlicz spaces have been studied in the G-Framework by, e.g., Nutz & Soner [29] and Soner et al. [35], and for risk measures by Gao & Munari [16], Kupper & Svindland [22], Liebrich & Svindland [23], Owari [30], and Svindland [36].
A second approach to robust Orlicz spaces present in the literature, cf. Nutz [28] and Soner et al. [34], is given by the construction

$$\mathfrak{L}^\Phi(\mathfrak{P}) := \big\{X \in L^0(\mathfrak{P}) \,\big|\, \forall\, \mathbb{P} \in \mathfrak{P}\, \exists\, \alpha > 0 : \ \mathbb{E}_\mathbb{P}[\phi_\mathbb{P}(\alpha|X|)] < \infty\big\},$$

the "intersection" of the individual Orlicz spaces. In the situation of equation (2), this space collects minimal agreement among all agents under consideration, that is, they all can attach a well-defined utility to each of the objects in $\mathfrak{L}^\Phi(\mathfrak{P})$. In terms of modeling robustness, one may therefore be tempted to prefer $\mathfrak{L}^\Phi(\mathfrak{P})$ over $L^\Phi(\mathfrak{P})$, which depends on the modeling choice of the worst-case approach represented by the supremum over all priors in $\mathfrak{P}$. In Theorem 8, we show that, in many situations,

$$\mathfrak{L}^\Phi(\mathfrak{P}) = L^\Phi(\mathfrak{P}),$$

a uniform-boundedness type result which proves the equivalence of both constructions in terms of the extension of the resulting spaces in $L^0(\mathfrak{P})$.

Structure of the Paper: In Section 2, we start with a top-down construction of robust Orlicz spaces. We discuss basic prpoerties of the latter. We derive an equivalent conditions for a robust Orlicz space to coincide with a robust multiplicatively penalised L^1-space, cf. Theorem 4, and give sufficient conditions for the equality $L^\Phi(\mathfrak{P}) = \mathfrak{L}^\Phi(\mathfrak{P})$ to be valid, see Proposition 6 and Theorem 8. In Section 3, we consider a bottom-up approach to robust Orlicz spaces via norm closures of bounded continuous functions. We first derive a general condition ensuring the separability of $\mathcal{C}^\Phi(\mathfrak{P})$ (Lemma 12). Based on the latter, Theorem 13 states that, under very mild conditions, $\mathcal{C}^\Phi(\mathfrak{P})$ is lattice-isomorphic to a sublattice of $L^1(\mathbb{P}^*)$ for a suitable probability measure $\mathbb{P}^*$. Theorem 15, provides a set equivalent conditions for the σ-Dedekind completeness of $\mathcal{C}^\Phi(\mathfrak{P})$. In particular, we prove that the latter already implies the dominatedness of the set of priors $\mathfrak{P}$. We conclude by giving, in special yet relevant cases, an explicit description of the dual space of $\mathcal{C}^\Phi(\mathfrak{P})$, see Proposition 19. The proofs of Section 2 can be found in the Appendix A, and the proofs of Section 3 are contained in the Appendix B.

Notation: For a set $S \neq \emptyset$ and a function $f\colon S \to (-\infty, \infty]$, the EFFECTIVE DOMAIN of f will be denoted by $\mathrm{dom}(f) := \{s \in S \mid f(s) < \infty\}$.

Throughout, we consider a measurable space $(\Omega, \mathcal{F})$ and a nonempty set $\mathfrak{P}$ of probability measures $\mathbb{P}$ on $(\Omega, \mathcal{F})$. The latter give rise to an equivalence relation on the real vector space $\mathcal{L}^0(\Omega, \mathcal{F})$ of all real-valued random variables on $(\Omega, \mathcal{F})$. More precisely, we define

$$f \sim g \quad :\iff \quad \forall \mathbb{P} \in \mathfrak{P}: \; \mathbb{P}(f = g) = 1.$$

The quotient space $L^0(\mathfrak{P}) := \mathcal{L}^0(\Omega, \mathcal{F})/\sim$ is the space of all real-valued random variables on $(\Omega, \mathcal{F})$ up to $\mathfrak{P}$-q.s. equality. The elements $f\colon \Omega \to \mathbb{R}$ in the equivalence class $X \in L^0(\mathfrak{P})$ are called REPRESENTATIVES, and are denoted by $f \in X$. For X and Y in $L^0(\mathfrak{P})$, we set

$$X \preceq Y \quad :\iff \quad \forall f \in X \, \forall g \in Y \, \forall \mathbb{P} \in \mathfrak{P}: \; \mathbb{P}(f \leq g) = 1,$$

a well-defined notion of a vector space order on $L^0(\mathfrak{P})$. We refer to $\preceq$ as the $\mathfrak{P}$-Q.S. ORDER on $L^0(\mathfrak{P})$. Notice that $(L^0(\mathfrak{P}), \preceq)$ is a vector lattice. In fact, for $X, Y \in L^0(\mathfrak{P})$ and representatives $f \in X, g \in Y$, the minimum $X \wedge Y$ is the equivalence class generated by $f \wedge g$, whereas the maximum $X \vee Y$ is the equivalence class generated by $f \vee g$. We denote the vector sublattice of all bounded real-valued random variables up to $\mathfrak{P}$-q.s. equality by $L^\infty(\mathfrak{P})$. The latter is a Banach lattice, when endowed with the norm

$$\|X\|_{L^\infty(\mathfrak{P})} := \inf\big\{m > 0 \,\big|\, \forall f \in \mathcal{X}: \inf_{\mathbb{P} \in \mathfrak{P}} \mathbb{P}(|f| \leq m) = 1\big\}, \quad X \in L^\infty(\mathfrak{P}).$$

As usual, $\mathbf{ca}$ denotes the space of all signed measures with finite total variation. We denote by $\mathbf{ca}_+$ or $\mathbf{ca}_+^1$ the subset of all finite measures or probability measures, respectively. For $\mu \in \mathbf{ca}$, let $|\mu|$ denote the total variation measure of μ. Given two nonempty sets $\mathfrak{Q}, \mathfrak{R} \subset \mathbf{ca}$, we write $\mathfrak{Q} \ll \mathfrak{R}$ if $\sup_{\mu \in \mathfrak{Q}} |\mu|(N) = 0$ for all events $N \in \mathcal{F}$ with $\sup_{\nu \in \mathfrak{R}} |\nu|(N) = 0$. We write, $\mathfrak{Q} \approx \mathfrak{R}$ if $\mathfrak{Q} \ll \mathfrak{R}$ and $\mathfrak{R} \ll \mathfrak{Q}$. For singletons $\mathfrak{Q} = \{\mu\}$, we use the notation $\mu \ll \mathfrak{R}$, $\mathfrak{R} \ll \mu$, and $\mathfrak{R} \approx \mu$. Finally, $\mathbf{ca}(\mathfrak{P}) := \{\mu \in \mathbf{ca} \,|\, \mu \ll \mathfrak{P}\}$ denotes the space of all countably additive signed measures, which are absolutely continuous with respect to $\mathfrak{P}$. The subsets $\mathbf{ca}_+(\mathfrak{P})$ and $\mathbf{ca}_+^1(\mathfrak{P})$ are defiend in an analogous way. For all $\mu \in \mathbf{ca}$, $X \in L^0(\mathfrak{P})_+$, and $f, g \in X$, $\int f \, d\mu$ and $\int g \, d\mu$ are well-defined and satisfy

$$\int f \, d\mu = \int g \, d\mu.$$

We shall therefore henceforth write

$$\mu X := \int X \, d\mu := \int f \, d\mu, \quad \text{for } f \in X,$$

if $X \in L^0(\mathfrak{P})_+$ or it contains a $|\mu|$-integrable representative.

2. Robust Orlicz spaces and penalised versions of robust L^p-spaces

In this section, we introduce the main objects of this manuscript, robust versions of Orlicz spaces, and investigate their basic properties. For the theory of classical Orlicz spaces, we refer to [13, Chapter 2]. An *Orlicz function* is a lower semicontinuous, nondecreasing, and convex function $\phi\colon [0,\infty) \to [0,\infty]$ with $\phi(0) = 0$ such that there are $x_0, x_1 > 0$ such that $\phi(x_0) > 0$ and $\phi(x_1) < \infty$.[1] Throughout this section, we consider a general measurable space $(\Omega, \mathcal{F})$, a nonempty set of probability measures $\mathfrak{P}$, a family $\Phi = (\phi_{\mathbb{P}})_{\mathbb{P} \in \mathfrak{P}}$ of Orlicz functions, and define

$$\phi_{\text{Max}}(x) := \sup_{\mathbb{P} \in \mathfrak{P}} \phi_{\mathbb{P}}(x), \quad \text{for all } x \in [0,\infty). \tag{3}$$

Notice that, by definition, ϕ_{Max} is a lower semicontinuous, nondecreasing, and convex function $[0,\infty) \to [0,\infty]$ with $\phi_{\text{Max}}(0) = 0$. However, in general, ϕ_{Max} is not an Orlicz function, since $\phi_{\text{Max}}(x_0) \in [0,\infty)$ for some $x_0 \in (0,\infty)$ cannot be guaranteed.

2.1. Robust Orlicz spaces as Banach lattices.

Definition 1. For $X \in L^0(\mathfrak{P})$, the (Φ-)Luxemburg norm is defined via

$$\|X\|_{L^\Phi(\mathfrak{P})} := \inf\left\{\lambda > 0 \,\middle|\, \sup_{\mathbb{P} \in \mathfrak{P}} \mathbb{E}_{\mathbb{P}}\left[\phi_{\mathbb{P}}(\lambda^{-1}|X|)\right] \le 1\right\} = \sup_{\mathbb{P} \in \mathfrak{P}} \|X\|_{L^{\phi_{\mathbb{P}}}(\mathbb{P})} \in [0,\infty].$$ [2]

We define by $L^\Phi(\mathfrak{P}) := \operatorname{dom}(\|\cdot\|_{L^\Phi(\mathfrak{P})})$ the (Φ-)robust Orlicz space.

Example 2. Let $(\Omega, \mathcal{F}, \mathfrak{P})$ be a probability prior space and $\phi\colon [0,\infty) \to [0,\infty]$ be an Orlicz function.

(1) For an arbitrary function $\gamma\colon \mathfrak{P} \to [0,\infty)$, consider

$$\phi_{\mathbb{P}}(x) := \frac{\phi(x)}{1 + \gamma(\mathbb{P})}, \quad \text{for } x \ge 0.$$

This leads to an *additively penalised robust Orlicz space* with Luxemburg norm

$$\|X\|_{L^\Phi(\mathfrak{P})} = \inf\left\{\lambda > 0 \,\middle|\, \sup_{\mathbb{P} \in \mathfrak{P}} \mathbb{E}_{\mathbb{P}}\left[\phi(\lambda^{-1}|X|)\right] - \gamma(\mathbb{P}) \le 1\right\}, \quad \text{for } X \in L^0(\mathfrak{P}).$$

For $\phi(x) = \infty \cdot \mathbf{1}_{(1,\infty)}$, the Luxemburg norm is, independently of γ, given by

$$\|X\|_{L^\Phi(\mathfrak{P})} = \sup_{\mathbb{P} \in \mathfrak{P}} \|X\|_{L^\infty(\mathbb{P})}, \quad \text{for } X \in L^0(\mathfrak{P}).$$

Introducing the, up to a sign, convex risk measure

$$\rho(X) := \sup_{\mathbb{P} \in \mathfrak{P}} \mathbb{E}_{\mathbb{P}}[X] - \gamma(\mathbb{P}) \in [0,\infty], \quad \text{for } X \in L^0(\mathfrak{P})_+,$$

the robust Luxemburg norm can be expressed as

$$\|X\|_{L^\Phi(\mathfrak{P})} = \inf\left\{\lambda > 0 \,\middle|\, \rho\left(\phi(\lambda^{-1}|X|)\right) \le 1\right\}, \quad \text{for } X \in L^0(\mathfrak{P}).$$

(2) For $\theta\colon \mathfrak{P} \to (0,\infty)$ with $\sup_{\mathbb{P} \in \mathfrak{P}} \theta(\mathbb{P}) < \infty$, we consider

$$\phi_{\mathbb{P}}(x) := \phi\big(\theta(\mathbb{P})x\big) \quad \text{for } \mathbb{P} \in \mathfrak{P} \text{ and } x \ge 0.$$

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[1] This definition precludes triviality of ϕ, i.e. the cases $\phi \equiv 0$ and $\phi = \infty \cdot 1_{(0,\infty)}$.

[2] Defining $\|\cdot\|_{L^{\phi_{\mathbb{P}}}(\mathbb{P})}$ in the usual way, we obtain a seminorm on $L^\Phi(\mathfrak{P})$, not a norm as on the classical Orlicz space $L^{\phi_{\mathbb{P}}}(\mathbb{P})$.

This leads to a *multiplicatively penalised robust Orlicz space* with Luxemburg norm

$$\|X\|_{L^\Phi(\mathfrak{P})} = \sup_{\mathbb{P}\in\mathfrak{P}} \theta(\mathbb{P})\|X\|_{L^\phi(\mathbb{P})}, \quad \text{for } X \in L^0(\mathfrak{P}).$$

For $p \in [1,\infty)$ and $\phi = x^p$, we obtain the weighted robust L^p-norm

$$\|X\|_{L^\phi(\mathfrak{P})} = \sup_{\mathbb{P}\in\mathfrak{P}} \theta(\mathbb{P})\|X\|_{L^p(\mathbb{P})}, \quad \text{for } X \in L^0(\mathfrak{P}),$$

and, for $\phi(x) = \infty \cdot 1_{(1,\infty)}$, the Luxemburg norm is given by

$$\|X\|_{L^\Phi(\mathfrak{P})} = \sup_{\mathbb{P}\in\mathfrak{P}} \theta(\mathbb{P})\|X\|_{L^\infty(\mathbb{P})}, \quad \text{for } X \in L^0(\mathfrak{P}).$$

The resulting spaces will be referred to as *weighted robust L^p-spaces*, for $1 \le p \le \infty$.

As in the classical case, robust Orlicz spaces are Banach lattices.

Proposition 3. $\big(L^\Phi(\mathfrak{P}), \|\cdot\|_{L^\Phi(\mathfrak{P})}\big)$ *is a Banach lattice, and* $L^\Phi(\mathfrak{P}) \subset L^0(\mathfrak{P})$ *is an ideal.*

A robust Orlicz space may be reduced to a robust L^1-space if and only if it contains all bounded random variables.

Theorem 4. *The following statements are equivalent:*

(1) $L^\infty(\mathfrak{P}) \subset L^\Phi(\mathfrak{P})$,
(2) ϕ_{Max} *is an Orlicz function, i.e., there exists some* $x_0 \in (0,\infty)$ *with* $\phi_{\mathrm{Max}}(x_0) \in [0,\infty)$,
(3) *There exists a nonempty set* $\mathfrak{Q}$ *of probability measures with* $\mathfrak{P} \subset \mathfrak{Q}$ *and a weight function* $\theta\colon \mathfrak{Q} \to (0,\infty)$ *with* $\sup_{\mathbb{Q}\in\mathfrak{Q}} \theta(\mathbb{Q}) < \infty$ *such that* $\|\cdot\|_{L^\Phi(\mathfrak{P})} = \sup_{\mathbb{Q}\in\mathfrak{Q}} \theta(\mathbb{Q})\|\cdot\|_{L^1(\mathbb{Q})}$.

In this case, $L^\Phi(\mathfrak{P})$ *is a weighted robust* L^1*-space, and there is a constant* $\kappa > 0$ *such that*

$$\|X\|_{L^\Phi(\mathfrak{P})} \le \kappa \sup_{\mathbb{P}\in\mathfrak{P}} \|X\|_{L^\infty(\mathbb{P})}, \quad \textit{for } X \in L^\infty(\mathfrak{P}).$$

Example 5.

(1) We consider the setup of Example 2. Let $\theta\colon \mathfrak{P} \to (0,\infty)$ with $c := \sup_{\mathbb{P}\in\mathfrak{P}} \theta(\mathbb{P}) < \infty$, $\gamma\colon \mathfrak{P} \to [0,\infty)$, and ϕ be a joint Orlicz function. Let

$$\phi_{\mathbb{P}}(x) := \frac{\phi\big(\theta(\mathbb{P})x\big)}{1+\gamma(\mathbb{P})}, \quad \text{for } x \ge 0,$$

corresponding to the case of a *doubly penalised robust Orlicz space*. Then, for $x_0 \in (0,\infty)$ with $cx_0 \in \mathrm{dom}(\phi)$,

$$\phi_{\mathrm{Max}}(x_0) = \sup_{\mathbb{P}\in\mathfrak{P}} \phi_{\mathbb{P}}(x_0) = \sup_{\mathbb{P}\in\mathfrak{P}} \frac{\phi\big(\theta(\mathbb{P})x_0\big)}{1+\gamma(\mathbb{P})} \le \phi(cx_0) < \infty.$$

By Proposition 4, we obtain that $L^\Phi(\mathfrak{P})$ is a weighted robust L^1-space.

(2) For each fixed probability measure $\mathbb{P}^*$ on $(\Omega,\mathcal{F})$, Proposition 4 shows that the classical space $L^\infty(\mathbb{P}^*)$ is a robust L^1-space, although this result could, of course, also be obtained in a more direct manner. Let $\mathfrak{P}$ be the set of all probability measures $\mathbb{P}$ on $(\Omega,\mathcal{F})$ that are absolutely continuous with respect to $\mathbb{P}^*$. Consider $\phi_{\mathbb{P}}(x) = x$ for all $x \ge 0$ and $\mathbb{P} \in \mathfrak{P}$, leading a robust L^1-space over $\mathfrak{P}$. Then,

$$\|X\|_{\mathfrak{P},\Phi} = \|X\|_{L^\infty(\mathbb{P}^*)}, \quad \text{for } X \in L^0(\mathfrak{P}) = L^0(\mathbb{P}^*).$$

(3) Let $\mathbb{P}^*$ be a probability measure on $(\Omega,\mathcal{F})$, and consider a convex monetary risk measure $\rho\colon L^\infty(\mathbb{P}^*) \to \mathbb{R}$, which enjoys the Fatou property, and satisfies $\rho(0) = 0$. The dual representation, up to a sign,

$$\rho(X) = \sup_{Z\in\mathrm{dom}(\rho^*)} \mathbb{E}[ZX] - \rho^*(Z), \quad \text{for } X \in L^\infty(\mathbb{P}^*),$$

is a well-known consequence, where ρ^* is the convex conjugate of ρ. In the situation of Example 2 (1), set

$$\mathfrak{P} := \{Z\mathrm{d}\mathbb{P}^* \,|\, Z \in \mathrm{dom}(\rho^*)\},$$
$$\gamma(Z\mathrm{d}\mathbb{P}^*) := \rho^*(Z), \quad \text{for } Z \in \mathrm{dom}(\rho^*),$$
$$\phi_{\mathbb{P}}(x) := x, \quad \text{for } x \geq 0 \text{ and } \mathbb{P} \in \mathfrak{P}.$$

Then $L^\Phi(\mathfrak{P})$ contains $L^\infty(\mathfrak{P})$ as a sublattice. In general, we have $\mathfrak{P} \ll \mathbb{P}^*$, but $\mathfrak{P} \approx \mathbb{P}^*$ may fail without further conditions on ρ. We can always define the "projection"

$$\widehat{\rho}(Y) := \rho(J^{-1}(Y)), \quad \text{for } Y \in L^\infty(\mathfrak{P}),$$

though, where $J\colon L^\infty(\mathbb{P}^*) \to L^\infty(\mathfrak{P})$ is the natural projection. In that case, $L^\Phi(\mathfrak{P})$ serves as the maximal sensible domain of definition of $\widehat{\rho}$. For a detailed discussion of such spaces for a substantially wider class of risk measures, we refer to [23]. See also [22, 30, 36].

2.2. **An alternative path to robust Orlicz spaces.** In this section, we focus on a way to translate the concept of Orlicz spaces to a robust setting *without* using a robust Luxemburg norm and the worst-case approach represented by the supremum over all models $\mathbb{P} \in \mathfrak{P}$. One may indeed wonder if this modelling assumption is actually necessary to produce the largest commodity space on which the analytic behaviour of utility can be captured well with respect to any model considered in the uncertainty profile. An alternative would be provided by the space

$$\mathfrak{L}^\Phi(\mathfrak{P}) := \{X \in L^0(\mathfrak{P}) \,|\, \forall\, \mathbb{P} \in \mathfrak{P}\, \exists\, \alpha > 0 :\ \mathbb{E}_{\mathbb{P}}[\phi_{\mathbb{P}}(\alpha|X|)] < \infty\}.$$

One can show that $\mathfrak{L}^\Phi(\mathfrak{P})$ is a vector sublattice of $L^0(\mathfrak{P})$. Moreover, independent of Φ, $L^\Phi(\mathfrak{P}) \subset \mathfrak{L}^\Phi(\mathfrak{P})$ holds *a priori*. A special case of this space has, e.g., been studied in [28] and [34].
The next proposition shows that, frequently, $L^\Phi(\mathfrak{P}) = \mathfrak{L}^\Phi(\mathfrak{P})$ if $\mathfrak{P}$ is assumed to be countably convex.

Proposition 6. *Suppose that $\mathfrak{P}$ is countably convex, and assume that there exist constants $(c_{\mathbb{P}})_{\mathbb{P}\in\mathfrak{P}} \subset (0,\infty)$ such that*

$$\phi_{\mathrm{Max}}(x) \leq \phi_{\mathbb{P}}(c_{\mathbb{P}}x) \quad \text{for all } x \geq 0 \text{ and } \mathbb{P} \in \mathfrak{P}. \tag{4}$$

Then, $\mathfrak{L}^\Phi(\mathfrak{P}) = L^\Phi(\mathfrak{P})$.

Example 7. Without Condition (4), the assertion of Proposition 6 fails to hold. As an example, consider the case, where $\Omega = \mathbb{R}$ endowed with the Borel σ-algebra $\mathcal{F}$, and $\mathfrak{P}$ is given by the set of all probability measures $\mathbb{P}$, which are absolutely continuous w.r.t. $\mathbb{P}^* := \mathcal{N}(0,1)$ with bounded density $\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{P}^*} \geq \frac{1}{2}$. We set $\mathfrak{P}_n$ to be the set of all $\mathbb{P} \in \mathfrak{P}$ with $n \leq \left\|\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{P}^*}\right\|_{L^\infty(\mathbb{P}^*)} < n+1$, and

$$\phi_{\mathbb{P}}(x) := x^n, \quad \text{for } x \geq 0,\ n \in \mathbb{N}, \text{ and } \mathbb{P} \in \mathfrak{P}_n.$$

Then,

$$\{X \in L^0(\mathbb{P}^*) \,|\, \forall\, n \in \mathbb{N} :\ \mathbb{E}_{\mathbb{P}^*}[|X|^n] < \infty\} \subset \mathfrak{L}^\Phi(\mathfrak{P}).$$

If $U\colon \Omega \to \mathbb{R}$ is the identity, i.e., $U \sim \mathcal{N}(0,1)$ under $\mathbb{P}^*$, then $U \in \mathfrak{L}^\Phi(\mathfrak{P})$, but Stirling's formula implies that, for all $\alpha > 0$,

$$\sup_{\mathbb{P}\in\mathfrak{P}} \mathbb{E}_{\mathbb{P}}[\phi_{\mathbb{P}}(\alpha|U|)] \geq \sup_{n\in\mathbb{N}} \tfrac{1}{2}\mathbb{E}_{\mathbb{P}^*}[\alpha^n|U|^n] = \infty,$$

and $U \notin L^\Phi(\mathfrak{P})$ follows.

The next theorem, which generalises [23, Proposition 4.2(ii)], shows that, for doubly penalised Orlicz spaces, cf. Example 5(1), the assumptions of countable convexity of the set $\mathfrak{P}$ can be further relaxed.

Theorem 8. *Suppose that $\mathfrak{P}$ is convex. Assume that Φ is doubly penalised with joint Orlicz function ϕ, multiplicative penalisation θ, and convex additive penalty function $\gamma\colon \mathfrak{P} \to [0,\infty)$ with countably* *convex lower level sets. Then,*

$$\mathfrak{L}^\Phi(\mathfrak{P}) = L^\Phi(\mathfrak{P}).$$

Remark 9. An example for an additive penalty function as demanded in Theorem 8 is given by the set $\mathfrak{P}$ of all probability measures in $\mathrm{dom}(\rho^*)$ for a convex monetary risk measure ρ with $\mathrm{dom}(\rho^*)\cap\mathbf{ca}^1_+ \neq \emptyset$. Assume that, in this situation, the multiplicative penalty is $\theta \equiv 1$. Then, there are two equally consistent ways to translate convergence in $L^\phi(\mathbb{P})$ to a robust setting given by the set $\mathfrak{P}$ of priors. One could either declare a net $(X_\alpha)_{\alpha\in I}$ to be convergent if it (i) converges in each space $L^\phi(\mathbb{P})$, for $\mathbb{P} \in \mathfrak{P}$, at equal or comparable speed to the same limit, or (ii) converges to the same limit in each space $L^\phi(\mathbb{P})$, for $\mathbb{P} \in \mathfrak{P}$. Convergence (i) is reflected by the norm $\|\cdot\|_{L^\Phi(\mathfrak{P})}$, and the equality of speeds may be relaxed by the additive penalty, whereas (ii) would be the natural choice of a topology on $\mathfrak{L}^\Phi(\mathfrak{P})$. Even though $\mathfrak{L}^\Phi(\mathfrak{P}) = L^\Phi(\mathfrak{P})$ holds, convergence (ii) might not be normable or even sequential. However, having both options at hand provides a degree of freedom to reflect different economic phenomena on an applied level.

3. Closures of continuous functions

By construction, Φ-robust Orlicz spaces are ideals in $L^0(\mathfrak{P})$ with respect to the $\mathfrak{P}$-q.s. order, and therefore particularly well-behaved with respect to order properties. Each Φ-robust Orlicz space is σ-Dedekind complete. Moreover, using arguments as in [16, Lemma 34], their (super) Dedekind completeness is equivalent to (super) Dedekind completeness of $L^0(\mathfrak{P})$ and (super) Dedekind completeness of $L^\infty(\mathfrak{P})$.[3] In conclusion, Φ-robust Orlicz spaces not only have the desirable Banach space property, but also behave well as vector lattices.

In contrast to the top-down construction of Φ-robust Orlicz spaces, one could also build a robsut space bottom-up, a path taken in, e.g., [3, 7, 11]. Starting with a space of *test random variables*, one could consider closing the test space in a larger ambient space with respect to the risk-uncertainty structure, leading to smaller spaces. The existing literature typically discusses (special cases of) these spaces as *Banach spaces* without further going into detail on their order-theoretic properties. This section therefore fills this gap, and explores their properties as Banach lattices. We shall observe that they tend to be not very tractable as vector lattices, and their well-behavedness with respect to order properties has strong consequences.

Assumption 10. Throughout this section, we assume that Ω is a Polish space, $\mathcal{F}$ is the Borel-σ-algebra on Ω, and $\mathfrak{P}$ is a nonempty set of probability measures. Moreover, there exists some $x_0 \in (0,\infty)$ such that $\phi_{\mathrm{Max}}(x_0) \in [0,\infty)$, or, equivalently, $L^\infty(\mathfrak{P}) \subset L^\Phi(\mathfrak{P})$.

Let C_b be the space of bounded continuous functions on Ω. By virtue of Assumption 10,

$$\iota : C_b \to L^\Phi(\mathfrak{P}), \quad f \mapsto [f].$$

is a well-defined, continuous, and injective lattice homeomorphism. We shall abuse notation and also refer to $\iota(C_b)$ as C_b, to the equivalence classes by capital letters though. Then, we consider

$$\mathcal{C}^\Phi(\mathfrak{P}) := \mathrm{cl}_{\|\cdot\|_{L^\Phi(\mathfrak{P})}}(C_b).$$

The space $\mathcal{C}^\Phi(\mathfrak{P})$ is a Banach lattice, when endowed with $\|\cdot\|_{L^\Phi(\mathfrak{P})}$ and the $\mathfrak{P}$-q.s. order.

Lemma 11. $\big(\mathcal{C}^\Phi(\mathfrak{P}), \preceq, \|\cdot\|_{L^\Phi(\mathfrak{P})}\big)$ *is a Banach lattice.*

We would like to point out that all results in this section apply to the spaces discussed in [3, 7, 11].

[3] For the definition of these notions, we refer to [2].

The following lemma provides a tightness criterion for the separability of $(\mathcal{C}^\Phi(\mathfrak{P}), \|\cdot\|_{L^\Phi(\mathfrak{P})})$. We decisively generalise [7, Proposition 2.6] to a degree, which is new in the existing literature to the best of our knowledge.

Lemma 12. *Assume that, for every $\varepsilon > 0$, there exists a compact set $K \subset \Omega$ with*

$$\|\mathbf{1}_{\Omega\setminus K}\|_{L^\Phi(\mathfrak{P})} < \varepsilon. \tag{5}$$

Then, $\mathcal{C}^\Phi(\mathfrak{P})$ is separable. In particular, condition (5) is met in either of the following situations:

(i) *Ω is compact.*
(ii) *$\mathfrak{P}$ is tight and* $\operatorname{dom}(\phi_{\mathrm{Max}}) = [0, \infty)$.

We emphasise that (ii) is treated as a rather mild condition in the literature, which is, in particular, satisfied in the G-Framework, see [33, Theorem 2.5].

The next theorem proves that the $\mathfrak{P}$-q.s. order on $\mathcal{C}^\Phi(\mathfrak{P})$ collapses to a $\mathbb{P}^*$-a.s. order under mild conditions, where $\mathbb{P}^* \ll \mathfrak{P}$ is an appropriately chosen probability measure. Recall that, for a signed measure μ such that each $X \in \mathcal{C}^\Phi(\mathfrak{P})$ is $|\mu|$-integrable, we define $\mu X := \int X\, d\mu$. We denote by $\mathcal{C}^\Phi(\mathfrak{P})^*$ the topological dual space of $\mathcal{C}^\Phi(\mathfrak{P})$ endowed with the operator norm $\|\cdot\|_{\mathcal{C}^\Phi(\mathfrak{P})^*}$. Furthermore, we set

$$\mathbf{ca}(\mathcal{C}^\Phi(\mathfrak{P})) := \mathbf{ca}(\mathfrak{P}) \cap \mathcal{C}^\Phi(\mathfrak{P})^*,$$

the set of signed measures absolutely continuous with respect to $\mathfrak{P}$ which define continuous linear functionals on $\mathcal{C}^\Phi(\mathfrak{P})$.

Theorem 13. *Suppose that $\mathcal{C}^\Phi(\mathfrak{P})$ is separable. Then, there exists a countable convex combination*

$$\mathbb{P}^* \in \mathbf{ca}\big(\mathcal{C}^\Phi(\mathfrak{P})\big)$$

of probability measures in $\mathfrak{P}$ such that the $\mathfrak{P}$-q.s. order and the $\mathbb{P}^*$-a.s. order coincide on $\mathcal{C}^\Phi(\mathfrak{P})$. Moreover, $\mathcal{C}^\Phi(\mathfrak{P})$ is lattice isomorphic to a sublattice $\mathcal{L} \subset L^1(\mathbb{P}^*)$. If, additionally, $\mathfrak{P}$ is countably convex, $\mathbb{P}^*$ can be chosen as an element of $\mathfrak{P}$.

Remark 14. The preceding theorem is akin to results of Nagel [26], see [25, Theorem 2.7.8]. However, these use Kakutani representation and isomorphisms between a multitude of Banach lattices, while our approach does not require a change of the underlying measurable space or topological structure. Moreover, it shows that the $\mathfrak{P}$-q.s. order collapses to a $\mathbb{P}^*$-a.s. order on every sublattice of $\mathcal{C}^\Phi(\mathfrak{P})$, that is, $\mathfrak{P}$ is dominated on each sublattice of $\mathcal{C}^\Phi(\mathfrak{P})$. Prominent sublattices of $\mathcal{C}^\Phi(\mathfrak{P})$ appearing in the literature are the $\|\cdot\|_{L^\Phi(\mathfrak{P})}$-closures of bounded Lipschitz functions, or bounded cylindrical Lipschitz functions, cf. [11, 21], respectively. The additional assumption of countable convexity is met if, e.g., $\mathfrak{P}$ is convex and weakly compact, such as in [3].

Another consequence of separability is that the only property which prevents $\mathfrak{P}$ being dominated is the mild order completeness property of σ-Dedekind completeness.

Theorem 15. *Suppose that $\mathcal{C}^\Phi(\mathfrak{P})$ is separable. Then, the following are equivalent:*

(1) *$\mathcal{C}^\Phi(\mathfrak{P})$ is super Dedekind complete.*
(2) *$\mathcal{C}^\Phi(\mathfrak{P})$ is σ-Dedekind complete.*
(3) $\mathcal{C}^\Phi(\mathfrak{P}) = \operatorname{cl}\big(L^\infty(\mathfrak{P})\big)$.
(4) *$\mathcal{C}^\Phi(\mathfrak{P}) \subset L^\Phi(\mathfrak{P})$ is an ideal.*

In this case, the probability measure $\mathbb{P}^*$ constructed in Theorem 13 satisfies $\mathbb{P}^* \approx \mathfrak{P}$. If, additionally, $\inf_{\mathbb{P}\in\mathfrak{P}} \phi_{\mathbb{P}}(x_0) \in (0, \infty)$ for some $x_0 \in (0, \infty)$, there is a probability measure $\mathbb{Q}^* \in \mathbf{ca}(\mathcal{C}^\Phi(\mathfrak{P}))$ such that $\mathbb{Q}^* \approx \mathfrak{P}$ and such that the set of densities $\left\{\frac{d\mathbb{P}}{d\mathbb{Q}^*} \,\middle|\, \mathbb{P} \in \mathfrak{P}\right\}$ is uniformly $\mathbb{Q}^*$-integrable.

We thus see that, in typical situations encountered in the literature, all order completeness properties agree, and their validity typically implies dominatedness of the underlying set of priors in a particularly strong form.

Separability is a desirable property from an analytic point of view. One may wonder what happens if one drops this assumption. For the next result, we emphasise that, if we view $\mathcal{C}^\Phi(\mathfrak{P})$ as a space of *measurable functions*, two properties should not be far-fetched: (i) σ-Dedekind completeness, (ii) many positive functionals which are integrals with respect to a measure are σ-order continuous. Recall that a bounded linear functional $\ell\colon \mathcal{X} \to \mathbb{R}$ on a Banach lattice $(\mathcal{X}, \preceq, \|\cdot\|)$ is σ-order continuous if for every non-increasing sequence $(x_n)_{n\in\mathbb{N}}$ with $\inf_{n\in\mathbb{N}} x_n = 0$, $\lim_{n\to\infty} |\ell(x_n)| = 0$.

Lemma 16. *Each σ-order continuous linear functional on $\mathcal{C}^\Phi(\mathfrak{P})$ is given by a unique signed measure in $\mathbf{ca}(\mathcal{C}^\Phi(\mathfrak{P}))$.*

Proposition 17. *Suppose that $\mathfrak{P}$ is equivalent to the set of positive σ-order continuous linear functionals on $\mathcal{C}^\Phi(\mathfrak{P})$. Then properties (2)–(4) in Theorem 15 are equivalent.*

Remark 18.

(1) The proof of Proposition 17 shows that (3) and (4) in Theorem 15 are always equivalent. If one of them holds, Lemma 16 proves that each measure $\mu \in \mathbf{ca}(\mathcal{C}^\Phi(\mathfrak{P}))$ is σ-order continuous. The additional assumption in Proposition 17 therefore only presents a restriction for property (2), the σ-Dedekind completeness of $\mathcal{C}^\Phi(\mathfrak{P})$.

(2) In general, not every measure $\mu \in \mathbf{ca} \cap \mathcal{C}^\Phi(\mathfrak{P})^*$ is σ-order continuous. As an example, consider $\Omega = [0,1]$ endowed with its σ-algebra $\mathcal{F}$ of Borel sets and set $\mathfrak{P}$ to be the set of all atomless probability measures. Consider the robust weighted L^1-space, where $\theta \equiv 1$. One shows that each $X \in \mathcal{C}^\Phi(\mathfrak{P})$ has a unique continuous representative f and satisfies $\|X\|_{L^\phi(\mathfrak{P})} = \|f\|_\infty$. Now consider the linear bounded functional $\ell(X) := f(1)$, $X \in \mathcal{C}^\Phi(\mathfrak{P})$, where $f \in X$ is continuous. Although it corresponds to the Dirac measure concentrated at 1, it neither lies in $\mathbf{ca}(\mathcal{C}^\Phi(\mathfrak{P}))$ nor is σ-order continuous on $\mathcal{C}^\Phi(\mathfrak{P})$.

(3) As is remarked after [9, Corollary 5.6], C_b does not admit any nontrivial σ-order continuous functional in our situation. One could therefore interpret Proposition 17 as a dichotomy: either $\mathcal{C}^\Phi(\mathfrak{P})$ behaves very much like the space of continuous functions, or it is an ideal of $L^\Phi(\mathfrak{P})$, which could be obtained more directly as the closure of $L^\infty(\mathfrak{P})$.

We conclude with a Riesz representation result for the dual of $\mathcal{C}^\Phi(\mathfrak{P})$, which extends [3, Proposition 4] to our setting.

Proposition 19. *Assume that $\mathfrak{P}$ is weakly compact and that* $\mathrm{dom}(\phi_{\mathrm{Max}}) = [0,\infty)$. *Then,*

$$\mathcal{C}^\Phi(\mathfrak{P})^* = \mathbf{ca}\big(\mathcal{C}^\Phi(\mathfrak{P})\big).$$

Appendix A. Proofs of Section 2

Proof of Proposition 3. The fact that $L^\Phi(\mathfrak{P})$ is an ideal of $L^0(\mathfrak{P})$ follows directly from the fact that each $\phi_\mathbb{P}$ is nondecreasing and convex and the fact that the supremum is subadditive. Hence, it is a vector lattice with respect to the $\mathfrak{P}$-q.s. order. In a similar way, it follows that $\|\cdot\|_{L^\Phi(\mathfrak{P})}$ defines a norm on $L^\Phi(\mathfrak{P})$. Let $(X_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. Notice that, since $\phi_\mathbb{P}$ is convex and nontrivial for all $\mathbb{P} \in \mathfrak{P}$, there exist $a_\mathbb{P} > 0$ and $b_\mathbb{P} \in \mathbb{R}$ such that

$$\phi_\mathbb{P}(x) \geq (a_\mathbb{P} x + b_\mathbb{P})^+ \quad \text{for all } x \in \mathbb{R}. \tag{6}$$

By possibly passing to a subsequence, we may assume that

$$\|X_n - X_{n+1}\|_{L^\Phi(\mathfrak{P})} < 4^{-n} \quad \text{for all } n \in \mathbb{N}.$$

For all $n \in \mathbb{N}$, let $\lambda_n > 0$ with $\|X_n - X_{n+1}\|_{L^\Phi(\mathfrak{P})} < \lambda_n \leq 4^{-n}$. In particular, $\lambda_n^{-1} 2^{-n} \geq 2^n$, i.e. we can fix $n_\mathbb{P} \in \mathbb{N}$ such that $a_\mathbb{P} \lambda_n^{-1} 2^{-n} + b_\mathbb{P} > 0$ holds for all $n \geq n_\mathbb{P}$. Markov's inequality together with equation (6) shows, for all $\mathbb{P}\in\mathfrak{P}$,

$$\begin{aligned}\sum_{n=n_{\mathbb{P}}}^{\infty}\mathbb{P}\big(|X_n-X_{n+1}|\geq 2^{-n}\big)&\leq\sum_{n=n_{\mathbb{P}}}^{\infty}\mathbb{P}\Big(\big(a_{\mathbb{P}}(\lambda_n^{-1}|X_n-X_{n+1}|)+b_{\mathbb{P}}\big)^+\geq\big(a_{\mathbb{P}}\lambda_n^{-1}2^{-n}+b_{\mathbb{P}}\big)^+\Big)\\&\leq\sum_{n=n_{\mathbb{P}}}^{\infty}(a_{\mathbb{P}}2^n+b_{\mathbb{P}})^{-1}\,\mathbb{E}_{\mathbb{P}}\left[\phi_{\mathbb{P}}\left(\lambda_n^{-1}|X_n-X_{n+1}|\right)\right]\\&\leq\sum_{n=n_{\mathbb{P}}}^{\infty}\frac{1}{a_{\mathbb{P}}2^n+b_{\mathbb{P}}}<\infty.\end{aligned}$$

Applying the Borel-Cantelli Lemma yields that

$$\inf_{\mathbb{P}\in\mathfrak{P}}\mathbb{P}\big(|X_n-X_{n+1}|\leq 2^{-n}\text{ eventually}\big)=1.$$

Hence, the event $\Omega^*:=\{\lim_{n\to\infty}X_n\text{ exists in }\mathbb{R}\}\in\mathcal{F}$ satisfies $\mathbb{P}(\Omega^*)=1$ for all $\mathbb{P}\in\mathfrak{P}$. We set X to be (the equivalence class in $L^0(\mathfrak{P})$ induced by) $\limsup_{n\to\infty}X_n$. Now, let $\mathbb{P}\in\mathfrak{P}$ and $\alpha>0$ be arbitrary. Choose $k\in\mathbb{N}$ such that $\sum_{i\geq k}\lambda_i\alpha\leq 1$. For $l>k$, we can estimate

$$\phi_{\mathbb{P}}(\alpha|X_{n_k}-X_{n_l}|)\leq\phi_{\mathbb{P}}\left(\sum_{i=k}^{l-1}\alpha|X_{n_{i+1}}-X_{n_i}|\right)\leq\sum_{i=k}^{l-1}\lambda_i\alpha\phi_{\mathbb{P}}\left(\lambda_i^{-1}|X_{n_{i+1}}-X_{n_i}|\right)\leq\sum_{i=k}^{\infty}\lambda_i\alpha.$$

Notice that the last bound is uniform in l and $\mathbb{P}$. Letting $l\to\infty$ and using lower semicontinuity of $\phi_{\mathbb{P}}$,

$$\phi_{\mathbb{P}}(\alpha|X_{n_k}-X|)\leq\sum_{i=k}^{\infty}\lambda_i\alpha.$$

This implies

$$\limsup_{k\to\infty}\sup_{\mathbb{P}\in\mathfrak{P}}\mathbb{E}_{\mathbb{P}}[\phi_{\mathbb{P}}\left(\alpha|X_{n_k}-X|\right)]\leq\lim_{k\to\infty}\sum_{i=k}^{\infty}\lambda_i\alpha=0.$$

As $\alpha>0$ was arbitrary, $X\in L^{\Phi}(\mathfrak{P})$ and $\lim_{k\to\infty}\|X_k-X\|_{L^{\Phi}(\mathfrak{P})}=0$ follow. □

Proof of Theorem 4. Suppose $L^{\infty}(\mathfrak{P})\subset L^{\Phi}(\mathfrak{P})$. Then, we can find some $\alpha>0$ such that

$$\sup_{\mathbb{P}\in\mathfrak{P}}\phi_{\mathbb{P}}(\alpha)=\sup_{\mathbb{P}\in\mathfrak{P}}\mathbb{E}_{\mathbb{P}}[\phi_{\mathbb{P}}(\alpha\mathbf{1}_{\Omega})]\leq 1.$$

Now let $\alpha>0$ with $\phi_{\mathrm{Max}}(\alpha)=\sup_{\mathbb{P}\in\mathfrak{P}}\phi_{\mathbb{P}}(\alpha)<\infty$. Since ϕ_{Max} is convex, we may w.l.o.g. assume that $\phi_{\mathrm{Max}}(\alpha)\leq 1$. For $\mathbb{P}\in\mathfrak{P}$ and $Z\in L^0(\mathbb{P})$, let

$$\|Z\|'_{\mathbb{P}}:=\sup\big\{\mathbb{E}_{\mathbb{P}}[|ZX|]\,\big|\,\|X\|_{L^{\phi_{\mathbb{P}}}(\mathbb{P})}=1\big\}.$$

Then, by [25, Theorem 2.6.9 & Corollary 2.6.6],[4]

$$\|X\|_{L^{\phi_{\mathbb{P}}}(\mathbb{P})}=\sup\big\{\mathbb{E}_{\mathbb{P}}[|ZX|]\,\big|\,Z\in L^0(\mathbb{P}),\ \|Z\|'_{\mathbb{P}}=1\big\}\quad\text{for all }\mathbb{P}\in\mathfrak{P}\text{ and }X\in L^{\phi_{\mathbb{P}}}(\mathbb{P}).\tag{7}$$

Since $\sup_{\mathbb{P}\in\mathfrak{P}}\phi_{\mathbb{P}}(\alpha)\leq 1$, $\|\mathbf{1}_{\Omega}\|_{L^{\phi_{\mathbb{P}}}(\mathbb{P})}\leq\alpha^{-1}$. Hence, for all $Z\in L^0(\mathbb{P})$ with $\|Z\|'_{\mathbb{P}}=1$,

$$\mathbb{E}_{\mathbb{P}}[|Z|]=\|\mathbf{1}_{\Omega}\|_{L^{\phi_{\mathbb{P}}}(\mathbb{P})}\mathbb{E}\big[\big|Z(\|\mathbf{1}_{\Omega}\|_{L^{\phi_{\mathbb{P}}}(\mathbb{P})})^{-1}\mathbf{1}_{\Omega})\big|\big]\leq\frac{1}{\alpha}.\tag{8}$$

For $\mathbb{P}\in\mathfrak{P}$, let $\mathfrak{Q}_{\mathbb{P}}$ denote the set of all probability measures $\mathbb{Q}\ll\mathbb{P}$ on $(\Omega,\mathcal{F})$ with

$$\tfrac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}=\tfrac{|Z|}{\mathbb{E}_{\mathbb{P}}[|Z|]}\quad\text{for some }Z\in L^0(\mathbb{P})\text{ with }\|Z\|'_{\mathbb{P}}=1.$$

[4] The cases $L^{\phi_{\mathbb{P}}}(\mathbb{P})\in\{L^1(\mathbb{P}),L^{\infty}(\mathbb{P})\}$ are not treated in this reference, but equation (7) is well known for them.

Note that by [13, (2.1.21)], $\mathbb{P}\in\mathfrak{Q}_{\mathbb{P}}$ holds for all $\mathbb{P}\in\mathfrak{P}$. Let $\mathfrak{Q}:=\{\mathbb{Q}\in\mathbf{ca}^1_+\mid\mathbb{Q}\in\mathfrak{Q}_{\mathbb{P}}\text{ for some }\mathbb{P}\in\mathfrak{P}\}$ and

$$\theta(\mathbb{Q}):=\sup_{\mathbb{P}\in\mathfrak{P}\colon\mathbb{Q}\in\mathfrak{Q}_{\mathbb{P}}}(\|\tfrac{d\mathbb{Q}}{d\mathbb{P}}\|'_{\mathbb{P}})^{-1},\quad\text{for }\mathbb{Q}\in\mathfrak{Q}.$$

Then, for $\mathbb{P}\in\mathfrak{P}$, $\mathbb{Q}\in\mathfrak{Q}_{\mathbb{P}}$, and $Z\in L^0(\mathbb{P})$ with $\|Z\|'_{\mathbb{P}}=1$ as well as $\frac{d\mathbb{Q}}{d\mathbb{P}}=\frac{|Z|}{\mathbb{E}_{\mathbb{P}}[|Z|]}$, (8) implies that

$$\left(\|\tfrac{d\mathbb{Q}}{d\mathbb{P}}\|'_{\mathbb{P}}\right)^{-1}=\mathbb{E}_{\mathbb{P}}[|Z|]\le\frac{1}{\alpha}.$$

Hence, $\sup_{\mathbb{Q}\in\mathfrak{Q}}\theta(\mathbb{Q})<\infty$, and, for $X\in L^0(\mathfrak{P})$,

$$\begin{aligned}\|X\|_{L^\Phi(\mathfrak{P})}&=\sup_{\mathbb{P}\in\mathfrak{P}}\|X\|_{L^{\phi_{\mathbb{P}}}(\mathbb{P})}=\sup_{\mathbb{P}\in\mathfrak{P}}\sup_{\mathbb{Q}\in\mathfrak{Q}_{\mathbb{P}}}\left(\left\|\frac{d\mathbb{Q}}{d\mathbb{P}}\right\|'_{\mathbb{P}}\right)^{-1}\mathbb{E}_{\mathbb{Q}}[|X|]\\&=\sup_{\mathbb{P}\in\mathfrak{P}\colon\mathbb{Q}\in\mathfrak{Q}_{\mathbb{P}}}\left(\|\tfrac{d\mathbb{Q}}{d\mathbb{P}}\|'_{\mathbb{P}}\right)^{-1}\sup_{\mathbb{Q}\in\mathfrak{Q}}\mathbb{E}_{\mathbb{Q}}[|X|]=\sup_{\mathbb{Q}\in\mathfrak{Q}}\theta(\mathbb{Q})\mathbb{E}_{\mathbb{Q}}[|X|].\end{aligned}$$

As $\|\mathbf{1}_A\|_{L^\Phi(\mathfrak{P})}=0$, for $A\in\mathcal{F}$, is equivalent to $\mathbf{1}_A=0$ in $L^0(\mathfrak{P})$, this representation also proves $\mathfrak{Q}\approx\mathfrak{P}$. At last, suppose that $L^\Phi(\mathfrak{P})$ reduces to a weighted robust L^1-space as in the assertion. From $\mathfrak{Q}\approx\mathfrak{P}$, we infer that the latter space contains $L^\infty(\mathfrak{Q})=L^\infty(\mathfrak{P})$. For the last statement, choose $\kappa:=\sup_{\mathbb{Q}\in\mathfrak{Q}}\theta(\mathbb{Q})$ or, equivalently, $\kappa:=\|\mathbf{1}_\Omega\|_{L^\Phi(\mathfrak{P})}$. □

Proof of Proposition 6. Due to Condition (4),

$$\|X\|_{L^{\phi_{\mathbb{P}}}(\mathbb{P})}\le\|X\|_{L^{\phi_{\mathrm{Max}}}(\mathbb{P})}\le c_{\mathbb{P}}\|X\|_{L^{\phi_{\mathbb{P}}}(\mathbb{P})}\quad\text{for all }\mathbb{P}\in\mathfrak{P}\text{ and }X\in L^0(\mathfrak{P}).$$

Therefore, we may concentrate on the case $\phi_{\mathbb{P}}=\phi_{\mathrm{Max}}=:\phi$ for all $\mathbb{P}\in\mathfrak{P}$. Let $X\in L^0(\mathfrak{P})\backslash L^\Phi(\mathfrak{P})$. Then, there exists a sequence $(\mathbb{P}_n)_{n\in\mathbb{N}}\subset\mathfrak{P}$ with

$$\|X\|_{L^\phi(\mathbb{P}_n)}\ge 2^n n\quad\text{for all }n\in\mathbb{N}.$$

Define $\mathbb{P}:=\sum_{n\in\mathbb{N}}2^{-n}\mathbb{P}_n$. Then, $\mathbb{P}\in\mathfrak{P}$, since $\mathfrak{P}$ is countably convex, and, for all $n\in\mathbb{N}$,

$$\|X\|_{L^\phi(\mathbb{P})}=2^{-n}\|X\|_{L^\phi(\mathbb{P}_n)}\ge n\to\infty\quad\text{as }n\to\infty,$$

which proves that $X\notin\mathfrak{L}^\Phi(\mathfrak{P})$. □

Proof of Theorem 8. We proceed similarly to the proof of [23, Proposition 4.2(ii)]. Let $X\in L^0(\mathfrak{P})\backslash L^\Phi(\mathfrak{P})$. Assume for contradiction there is $n\in\mathbb{N}$ and a constant $c>1$ such that, for all $\mathbb{P}\in\mathfrak{P}$,

$$\mathbb{E}_{\mathbb{P}}[\phi(\theta(\mathbb{P})2^{-n}|X|)]\le c(1+\gamma(\mathbb{P})).$$

Using the convexity of ϕ, we then obtain

$$\sup_{\mathbb{P}\in\mathfrak{P}}\tfrac{1}{1+\gamma(\mathbb{P})}\mathbb{E}_{\mathbb{P}}\left[\phi\left(\tfrac{\theta(\mathbb{P})}{c2^n}|X|\right)\right]\le 1,$$

which means $X\in L^\Phi(\mathfrak{P})$ and is thus contradictory. We must therefore be able to choose a sequence $(\mathbb{P}_n)_{n\in\mathbb{N}}\subset\mathfrak{P}$ with the property

$$\forall\,n\in\mathbb{N}:\ \mathbb{E}_{\mathbb{P}_n}[\phi(\theta(\mathbb{P}_n)2^{-n}|X|)]\ge 2^n(1+\gamma(\mathbb{P}_n)).$$

Fix $\mathbb{P}^*\in\mathfrak{P}$ and consider the measures

$$\mathbb{Q}_n:=\tfrac{\gamma(\mathbb{P}_n)}{1+\gamma(\mathbb{P}_n)}\mathbb{P}^*+\tfrac{1}{1+\gamma(\mathbb{P}_n)}\mathbb{P}_n,\quad\text{for }n\in\mathbb{N}.$$

By convexity of γ, $\gamma(\mathbb{Q}_n)\le\gamma(\mathbb{P}^*)+1$, $n\in\mathbb{N}$. Using countable convexity of the lower level sets of γ,

$$\mathbb{Q}:=\sum_{n=1}^\infty 2^{-n}\mathbb{Q}_n\in\mathfrak{P}$$

satisfies $\gamma(\mathbb{Q}) \leq \gamma(\mathbb{P}^*) + 1$. For $\alpha > 0$ arbitrary, set $I := \{n \in \mathbb{N} \mid \theta(\mathbb{Q})\alpha \geq \theta(\mathbb{P}_n)2^{-n}\}$. We estimate

$$\mathbb{E}_{\mathbb{Q}}[\phi(\alpha\theta(\mathbb{Q})|X|)] \geq \sum_{n=1}^{\infty} 2^{-n}\mathbb{E}_{\mathbb{Q}_n}[\phi(\theta(\mathbb{Q})\alpha|X|)] \geq \sum_{n=1}^{\infty} \frac{\mathbb{E}_{\mathbb{P}_n}[\phi(\theta(\mathbb{Q})\alpha|X|)]}{2^n(1+\gamma(\mathbb{P}_n))}$$
$$\geq \sum_{n\in I} \frac{\mathbb{E}_{\mathbb{P}_n}[\phi(\theta(\mathbb{P}_n)2^{-n}|X|)]}{2^n(1+\gamma(\mathbb{P}_n))} \geq \sum_{n\in I} \frac{2^n(1+\gamma(\mathbb{P}_n))}{2^n(1+\gamma(\mathbb{P}_n))} = \infty.$$

This proves $X \notin \mathfrak{L}^{\Phi}(\mathfrak{P})$. □

Appendix B. Proofs of Section 3

Proof of Lemma 11. $\big(C_b, \preceq\big)$ is a sublattice of $\big(L^{\Phi}(\mathfrak{P}), \preceq\big)$. By [25, Proposition 1.2.3(ii)], the closure $(\mathcal{C}^{\Phi}(\mathfrak{P}), \preceq)$ is a sublattice as well. As $\|\cdot\|_{L^{\Phi}(\mathfrak{P})}$ is a lattice norm on $\mathcal{C}^{\Phi}(\mathfrak{P})$, $\big(\mathcal{C}^{\Phi}(\mathfrak{P}), \|\cdot\|_{L^{\Phi}(\mathfrak{P})}\big)$ is a Banach lattice by construction. □

Proof of Lemma 12. By Theorem 4, there exists some constant $\kappa > 0$ such that

$$\|X\|_{L^{\Phi}(\mathfrak{P})} \leq \kappa \|X\|_{L^{\infty}(\mathfrak{P})}, \quad \text{for all } X \in C_b(\Omega).$$

Let d be a metric consistent with the topology on Ω, and $(\omega_n)_{n\in\mathbb{N}}$ dense in Ω. For $m, n \in \mathbb{N}$ and $\omega \in \Omega$, let $X_{m,n}(\omega) := d(\omega, \omega_n) \wedge m$. Then, the algebra $\mathcal{A} \subset C_b(\Omega)$ generated by $(X_{m,n})_{m,n\in\mathbb{N}}$ over the rational numbers is still of countable cardinality. Let $X \in C_b(\Omega)$, $\varepsilon > 0$, and $K \subset \Omega$ compact with

$$\|\mathbf{1}_{\Omega\setminus K}\|_{L^{\Phi}(\mathfrak{P})} < \frac{\varepsilon}{2(1+2\|X\|_{\infty})}.$$

By the Stone-Weierstrass Theorem, there exists some $X_0 \in \mathcal{A}$ with $\|X_0\|_{\infty} \leq 1 + \|X\|_{\infty}$ and

$$\big\|(X - X_0)\mathbf{1}_K\big\|_{L^{\infty}(\mathfrak{P})} < \frac{\varepsilon}{2\kappa}.$$

Hence,

$$\|X - X_0\|_{L^{\Phi}(\mathfrak{P})} \leq \big\|(X - X_0)\mathbf{1}_K\big\|_{L^{\Phi}(\mathfrak{P})} + \big\|(X - X_0)\mathbf{1}_{\Omega\setminus K}\big\|_{L^{\Phi}(\mathfrak{P})}$$
$$\leq \kappa\big\|(X - X_0)\mathbf{1}_K\big\|_{L^{\infty}(\mathfrak{P})} + \big(\|X\|_{\infty} + \|X_0\|_{\infty}\big)\|\mathbf{1}_{\Omega\setminus K}\|_{L^{\Phi}(\mathfrak{P})} < \varepsilon.$$

Now, (i) trivially implies (5). Under condition (ii), let $0 < \delta < \varepsilon$ and $K \subset \Omega$ compact with

$$\phi_{\mathrm{Max}}(\delta^{-1}) \sup_{\mathbb{P}\in\mathfrak{P}} \mathbb{P}(\Omega\setminus K) \leq 1.$$

Then,

$$\sup_{\mathbb{P}\in\mathfrak{P}} \mathbb{E}_{\mathbb{P}}[\phi_{\mathbb{P}}(\delta^{-1}\mathbf{1}_K)] \leq \sup_{\mathbb{P}\in\mathfrak{P}} \mathbb{E}_{\mathbb{P}}[\phi_{\mathrm{Max}}(\delta^{-1}\mathbf{1}_K)] = \phi_{\mathrm{Max}}(\delta^{-1}) \sup_{\mathbb{P}\in\mathfrak{P}} \mathbb{P}(\Omega\setminus K) \leq 1.$$

This entails $\|\mathbf{1}_{\Omega\setminus K}\|_{L^{\Phi}(\mathfrak{P})} \leq \delta < \varepsilon$. □

Proof of Theorem 13. By the Banach-Alaoglu Theorem, separability of $\mathcal{C}^{\Phi}(\mathfrak{P})$ implies that the unit ball $\mathfrak{B} := \{\ell \in \mathcal{C}^{\Phi}(\mathfrak{P})^* \mid \|\ell\|_{\mathcal{C}^{\Phi}(\mathfrak{P})^*} \leq 1\}$ endowed with the weak* topology is compact and metrisable. As such, it is separable. Assumption 10 and Theorem 4(iii) show that, for each $\mathbb{P} \in \mathfrak{P}$, a multiple of this measure lies in $\mathcal{B}$. We thus define the subset

$$\mathfrak{R} := \{c\mathbb{P} \mid \mathbb{P} \in \mathfrak{P},\, c > 0,\, \|c\mathbb{P}\|_{\mathcal{C}^{\Phi}(\mathfrak{P})^*} = 1\} \subset \mathfrak{B}.$$

For all $0 \neq X \in \mathcal{C}^{\Phi}(\mathfrak{P})_+$ we can find $\mu \in \mathfrak{R}$ such that $\mu X > 0$. As $\mathfrak{R}$ is separable with respect to the relative weak* topology, we may choose a dense sequence $(c_k\mathbb{P}_k)_{k\in\mathbb{N}} \subset \mathfrak{R}$ and conclude that, for $X \in \mathcal{C}^{\Phi}(\mathfrak{P})_+$, $\sup_{k\in\mathbb{N}} c_k\mathbb{E}_{\mathbb{P}_k}[X] > 0$ holds if and only if $X \neq 0$. Consider the measure

$$\mu^* := \sum_{k=1}^{\infty} 2^{-k} c_k \mathbb{P}_k \in \mathbf{ca}(\mathcal{C}^{\Phi}(\mathfrak{P})),$$

and $\mathbb{P}^* := (\mu^*(\Omega))^{-1}\mu^*$, which is a countable convex combination of $(\mathbb{P}_k)_{k\in\mathbb{N}}$. By construction, the functional $\mathbb{E}_{\mathbb{P}^*}[\cdot] \in \mathcal{C}^\Phi(\mathfrak{P})^*$ is strictly positive. Hence, for $X, Y \in \mathcal{C}^\Phi(\mathfrak{P})$, $X \preceq Y$ if and only if $\mathbb{E}_{\mathbb{P}^*}[(Y-X)^-] = 0$, which immediately proves that both orders coincide on $\mathcal{C}^\Phi(\mathfrak{P})$. $\mathbb{P}^* \ll \mathfrak{P}$ allows us to consider the natural projection $J\colon \mathcal{C}^\Phi(\mathfrak{P}) \to L^1(\mathbb{P}^*)$, and we immediately obtain that $\mathcal{C}^\Phi(\mathfrak{P})$ is lattice isomorphic to the sublattice $J\big(\mathcal{C}^\Phi(\mathfrak{P})\big) \subset L^1(\mathbb{P}^*)$. □

Proof of Lemma 16. We first remark that each sequence $(X_n)_{n\in\mathbb{N}} \subset \mathcal{C}^\Phi(\mathfrak{P})_+$ possessing representatives $(f_n)_{n\in\mathbb{N}}$ such that $f_n \downarrow 0$ holds pointwise, yields $\inf_{n\in\mathbb{N}} X_n$ in $\mathcal{C}^\Phi(\mathfrak{P})$. Consider now the vector lattice

$$C_b \subset \mathcal{L} := \{f \in \mathcal{L}^0(\Omega, \mathcal{F}) \mid [f] \in \mathcal{C}^\Phi(\mathfrak{P})\}.$$

The linear functional

$$\widetilde{\ell}\colon \mathcal{L} \to \mathbb{R}, \quad f \mapsto \ell([f]),$$

satisfies $\widetilde{\ell}(f_n) \downarrow 0$ for all sequences $(f_n)_{n\in\mathbb{N}} \subset \mathcal{L}$ such that $f_n \downarrow 0$ pointwise. By [1, Lemma 4.65(3)], $\sigma(C_b) = \mathcal{F}$. Hence, [8, Theorem 7.8.1] provides a unique finite measure μ on $(\Omega, \mathcal{F})$ such that

$$\widetilde{\ell}(f) = \int f\,d\mu \quad \text{for all } f \in \mathcal{L}.$$

Moreover, for all $X \in \mathcal{C}^\Phi(\mathfrak{P})$ and all $f, g \in X$ we have

$$\int f\,d\mu = \widetilde{\ell}(f) = \ell(X) = \widetilde{\ell}(g) = \int g\,d\mu.$$

In particular, considering that $\mathbf{1}_N \in \mathcal{L}$ for all $N \in \mathcal{F}$ satisfying $\sup_{\mathbb{P}\in\mathfrak{P}} \mathbb{P}(N) = 0$, $\mu \in \mathbf{ca}(\mathfrak{P})$ follows. □

For the sake of clarity, we give the proof of Proposition 17 in advance of Theorem 15.

Proof of Proposition 17.
(2) implies (3): Let $A \subset \Omega$ closed. Then, there exists a sequence $(X_n)_{n\in\mathbb{N}} \subset C_b(\Omega)$ with $X_n \downarrow \mathbf{1}_A$ pointwise. By monotone convergence,

$$\lim_{n\to\infty} \mu X_n = \mu \mathbf{1}_A \tag{9}$$

follows for all $\mu \in \mathbf{ca}_+$. By σ-Dedekind completeness of $\mathcal{C}^\Phi(\mathfrak{P})$, there is a maximal lower bound $V \in \mathcal{C}^\Phi(\mathfrak{P})$ of $\{X_n \mid n \in \mathbb{N}\}$. $0 \preceq V \preceq \mathbf{1}_A$ holds in $L^\Phi(\mathfrak{P})$ *a priori*. Moreover, for each nonnegative σ-order continuous functional μ on $\mathcal{C}^\Phi(\mathfrak{P})$,

$$\lim_{n\to\infty} \mu X_n = \mu V. \tag{10}$$

Combining the global condition of the proposition with equations (9) and (10) now implies that $\mathbf{1}_A = V \in \mathcal{C}^\Phi(\mathfrak{P})$. At last, consider the π-system $\Pi := \{A \subset \Omega \mid A \text{ closed}\}$, which is a subset of

$$\Lambda := \{A \in \mathcal{F} \mid \mathbf{1}_A \in \mathcal{C}^\Phi(\mathfrak{P})\}.$$

The latter is a λ-system, which can be shown similar to the conjunction of the global condition of the proposition, and equations (9) and (10). Since $\mathcal{C}^\Phi(\mathfrak{P})$ is σ-Dedekind complete, the latter is a λ-system. By Dynkin's lemma, it follows that $\Lambda = \mathcal{F}$.
$\mathcal{C}^\Phi(\mathfrak{P})$ contains all representatives of simple functions. As $\mathcal{C}^\Phi(\mathfrak{P})$ is closed, Theorem 4 implies $L^\infty(\mathfrak{P}) \subset \mathcal{C}^\Phi(\mathfrak{P})$. The latter space is closed, hence $\mathrm{cl}(L^\infty(\mathfrak{P})) \subset \mathcal{C}^\Phi(\mathfrak{P})$.
(3) always implies (4): $L^\infty(\mathfrak{P})$ is an ideal in $L^\Phi(\mathfrak{P})$, and norm closures of ideals in Banach lattices remain ideals ([25, Proposition 1.2.3(iii)]).
(4) implies (2): Suppose that $\mathcal{C}^\Phi(\mathfrak{P})$ is an ideal in $L^\Phi(\mathfrak{P})$ with respect to the $\mathfrak{P}$-q.s. order. Let $\mathcal{D} \subset \mathcal{C}^\Phi(\mathfrak{P})$ be order bounded from above and countable. Since $L^\Phi(\mathfrak{P})$ is σ-Dedekind complete, $U := \sup \mathcal{D}$ exists in $L^\Phi(\mathfrak{P})$. Let $Y \in \mathcal{C}^\Phi(\mathfrak{P})$ be any upper bound of $\mathcal{D}$ and $X \in \mathcal{D}$. Then $X \preceq U \preceq Y$. As $\mathcal{C}^\Phi(\mathfrak{P})$ is an ideal, $U \in \mathcal{C}^\Phi(\mathfrak{P})$, and we have proved that $\mathcal{C}^\Phi(\mathfrak{P})$ is σ-Dedekind complete. □

Proof of Theorem 15. Theorem 13 provides a strictly positive linear functional in the present situation. Hence, (1) is equivalent to (2) by [27, Lemma A.3].

Now suppose that (1) holds. $\mathcal{C}^\Phi(\mathfrak{P})$ is thus a separable and σ-Dedekind complete Banach lattice. As such it cannot contain a sublattice isomorphic to ℓ^∞, the space of all bounded sequences with real values. By [25, Corollary 2.4.3(x)], $\|\cdot\|_{L^\Phi(\mathfrak{P})}$ is order continuous on $\mathcal{C}^\Phi(\mathfrak{P})$. By [25, Theorem 2.4.2(vii)], each continuous linear functional on $\mathcal{C}^\Phi(\mathfrak{P})$ is order continuous. We have seen in the proof of Theorem 13 that $\mathbf{ca}(\mathcal{C}^\Phi(\mathfrak{P}))_+ \approx \mathfrak{P}$. Proposition 17 shows that (3) holds. Finally, the proof of Proposition 17 shows that (3) implies (4), which in turn implies (2).

For the rest of the proof, we thus assume that the equivalent conditions (1)–(4) hold. Let $\mathbb{P}^* \in \mathbf{ca}\big(\mathcal{C}^\Phi(\mathfrak{P})\big)$ as in Theorem 13. For all $A \in \mathcal{F}$, $\mathbf{1}_A \in \mathcal{C}^\Phi(\mathfrak{P})$ is implied by (3), and it follows that $\mathfrak{P} \approx \mathbb{P}^*$. Finally, assume that $\beta := \inf_{\mathbb{P}\in\mathfrak{P}} \phi_\mathbb{P}(x_0) \in (0,\infty]$ for some $x_0 \in (0,\infty)$. In a first step, we prove that $\mathfrak{P}$ is a bounded subset of $\mathcal{C}^\Phi(\mathfrak{P})^*$. To this effect, note first that $L^\Phi(\mathfrak{P}) = L^\infty(\mathfrak{P})$ if $\beta = \infty$, and we can assume $\beta < \infty$ w.l.o.g. Second, note that, for all $\mathbb{P} \in \mathfrak{P}$, $\phi_\mathbb{P}(x_0) \geq \beta > 0$. Hence, by [13, (2.1.21)],

$$\sup_{\mathbb{P}\in\mathfrak{P}} \mathbb{E}_\mathbb{P}[|X|] \leq \frac{x_0(1+\beta)}{\beta}\|X\|_{L^\Phi(\mathfrak{P})} \quad \text{for all } X \in L^\Phi(\mathfrak{P}),$$

which proves the claim. Now, order continuity of the norm on $\mathcal{C}^\Phi(\mathfrak{P})$ shows that the coherent monetary risk measure $\rho(X) := \sup_{\mathbb{P}\in\mathfrak{P}} \mathbb{E}_\mathbb{P}[X]$, $X \in L^\infty(\mathfrak{P}) = L^\infty(\mathbb{P}^*)$, is continuous from above. The existence of a probability measure $\mathbb{Q}^*$ such that $\mathbb{Q}^* \approx \mathfrak{P}$, $\mathbb{E}_{\mathbb{Q}^*}[X] \leq \sup_{\mathbb{P}\in\mathfrak{P}} \mathbb{E}_\mathbb{P}[X]$ for all $X \in L^\infty(\mathfrak{P})$, and the set of densities $\{\frac{d\mathbb{P}}{d\mathbb{Q}^*} \mid \mathbb{P} \in \mathfrak{P}\}$ is uniformly $\mathbb{Q}^*$-integrable, is a well-known consequence, see, e.g., [14, Corollary 4.38]. By density of $L^\infty(\mathfrak{P})$ in $\mathcal{C}^\Phi(\mathfrak{P})$, we also have $\mathbb{Q}^* \in \mathbf{ca}(\mathcal{C}^\Phi(\mathfrak{P}))$. □

Proof of Proposition 19. The inclusion $\mathbf{ca}(\mathcal{C}^\Phi(\mathfrak{P})) \subset \mathcal{C}^\Phi(\mathfrak{P})^*$ holds by definition. For the converse inclusion, it suffices to show that every positive bounded linear functional ℓ can be represented by some finite measure μ. To this end, let $\ell \in \mathcal{C}^\Phi(\mathfrak{P})^*$ be positive. Moreover, let $(X_n)_{n\in\mathbb{N}} \subset C_b(\Omega)$ with $X_n \downarrow 0$ as $n \to \infty$ and $\alpha > 0$. Then, $\phi_{\text{Max}}(\alpha X_n) \in C_b(\Omega)$ for all $n \in \mathbb{N}$ with $\phi_{\text{Max}}(\alpha X_n) \downarrow 0$ as $n \to \infty$. Since $\mathfrak{P}$ is weakly compact, [11, Corollary 33] implies

$$\lim_{n\to\infty} \sup_{\mathbb{P}\in\mathfrak{P}} \mathbb{E}_\mathbb{P}[\phi_\mathbb{P}(\alpha|X_n|)] = 0 \quad \text{as } n \to \infty.$$

This suffices to conclude $\lim_{n\to\infty} \|X_n\|_{L^\Phi(\mathfrak{P})} = 0$. Hence, we can apply the Daniell-Stone Theorem to the lattice $\mathcal{L} := \{f \in \mathcal{L}^0(\Omega, \mathcal{F}) \mid [f] \in C_b\}$ and the functional $\widetilde{\ell}(f) := \ell([f])$ in a similar fashion to Lemma 16 to obtain $\ell = \mu$ for a unique measure $\mu \in \mathbf{ca}(\mathfrak{P})_+$ such that

$$\ell(X) = \mu X, \quad \text{for all } X \in C_b(\Omega).$$

By standard arguments, one can extend this identity to all of $\mathcal{C}^\Phi(\mathfrak{P})$ and prove $\mu \in \mathbf{ca}(\mathcal{C}^\Phi(\mathfrak{P}))$ by density of $C_b \subset \mathcal{C}^\Phi(\mathfrak{P})$. To this end, one uses the observation from the proof of Proposition 3 that every norm convergent sequence contains a $\mathfrak{P}$-q.s. convergent subsequence. □

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