Hardy's Inequality and Green Function on Metric Measure Spaces

Jun Cao, Alexander Grigor'yan and Liguang Liu

Abstract

We prove an abstract form of Hardy's inequality for local and non-local regular Dirichlet forms on metric measure spaces, using the Green operator of the Dirichlet form in question. Under additional assumptions such as the volume doubling, the reverse volume doubling, and certain natural estimates of the Green function, we obtain the "classical" form of Hardy's inequality containing distance to a reference point or set.

Contents

2010 *Mathematics Subject Classification*. Primary: 31E05; Secondary: 34B27; 32W30.

Key words and phrases. Hardy's inequality, Green function, heat kernel, Dirichlet form, metric measure space.

J. Cao was supported by the NNSF of China (# 11871254) and the China Scholarship Council (# 201708330186); A. Grigor'yan was supported by SFB1283 of the German Research Foundation; L. Liu was supported by the National Natural Science Foundation of China (# 11771446).

1 Introduction

The classical Hardy inequality was first proved by Hardy [35] in order to find an elementary proof of a double series inequality of Hilbert. A modern form of the Hardy inequality in $\mathbb{R}^n$, $n > 2$, is as follows:

$$\frac{(n-2)^2}{4}\int_{\mathbb{R}^n}\frac{|f(x)|^2}{|x|^2}\,dx \le \int_{\mathbb{R}^n}|\nabla f(x)|^2\,dx \quad \text{for all } f \in C_c^1(\mathbb{R}^n), \tag{1.1}$$

where $C_c^1(\mathbb{R}^n)$ denotes the class of continuously differentiable functions on $\mathbb{R}^n$ with compact support (cf. [36]). Hardy's inequality has found numerous applications in various areas of mathematics such as partial differential equations, geometric analysis, probability theory etc. We refer the reader to the monographs [1, 15, 44, 49] and the references therein for more information about Hardy's inequality in Euclidean spaces and related historical reviews.

Generalizations of (1.1) to Riemannian manifolds can be found in [9, 14, 24, 43]. Let M be a Riemannian manifold, Δ be the Laplace-Beltrami operator on M, and μ be the Riemannian measure. Then, for any smooth positive function ϕ on M satisfying the equation $-\Delta\phi + \Phi\phi = 0$ for some smooth function Φ, the following version of the Hardy inequality is true:

$$\int_M \frac{-\Delta\phi}{\phi} f^2\,d\mu \le \int_M |\nabla f|^2\,d\mu \quad \text{for all } f \in C_c^2(M). \tag{1.2}$$

The following short proof of (1.2) was given in [23, Section 4.4] and [24, p. 258]. Consider the weighted manifold $(M, \tilde{\mu})$ with $d\tilde{\mu} = \phi^2\,d\mu$. An easy calculation shows that the weighted Laplacian

$$\Delta_{\tilde{\mu}} u := \phi^{-2}\,\mathrm{div}(\phi^2 \nabla u)$$

satisfies the following identities: the product rule

$$-\phi\,\Delta_{\tilde{\mu}}(\phi^{-1} f) = -\Delta f + \frac{\Delta\phi}{\phi} f$$

and the Green formula

$$-\int_M u\Delta_{\tilde{\mu}} u\,d\tilde{\mu} = \int_M |\nabla u|^2 d\tilde{\mu} \ge 0 \quad \text{for all } u \in C_c^2(M).$$

Multiplying the former identity with $f\phi^{-2}$, integrating it against $d\tilde{\mu}$ and applying the latter identity with $u = \phi^{-1} f$, we obtain (1.2) (see also [30]).

Note that (1.2) is sharp in the sense that it recovers the sharp Hardy inequality (1.1) when $M = \mathbb{R}^n$, $n > 2$, because, for the function $\phi(x) = |x|^{-\frac{n-2}{2}}$, we have

$$\frac{-\Delta\phi(x)}{\phi(x)} = \frac{(n-2)^2}{4}\frac{1}{|x|^2}.$$

Motivated by (1.1) and (1.2), the main aim of this paper is to establish Hardy's inequality on general metric measure spaces (M, d, μ), including manifolds and fractal spaces (see [2, 3, 22, 42, 52]). In such a general setting, we replace the energy integral $\int_M |\nabla f|^2 \, d\mu$ in (1.2) by a Dirichlet form $(\mathcal{E}, \mathcal{F})$ and use instead of $-\Delta$ its inverse – the Green operator G. Hence, for a certain class of positive functions h on M, (1.2) transforms to

$$\int_M \frac{h}{Gh} f^2 \, d\mu \leq \mathcal{E}(f, f) \;\text{ for all } f \in \mathcal{F}. \tag{1.3}$$

The Hardy inequality in the form (1.3) is proved in this paper in Theorem 3.1 for strongly local regular Dirichlet forms and in Theorem 4.5 – for general (non-local) regular Dirichlet forms (in the latter case for a somewhat smaller class of functions h).

Given a Radon measure ν on M, one can ask under what condition the following even more general form of Hardy's inequality is valid:

$$\int_M f^2 \, d\nu \leq \mathcal{E}(f, f). \tag{1.4}$$

This question was studied in [19, 6, 50] where the answer was given in terms of a certain testing inequality expressed via the Dirichlet form and the measure ν. Our versions of Hardy's inequality are much more explicit and do not follow from the results of [19, 6, 50].

Assume further that the metric measure space (M, d, μ) satisfies the volume doubling condition (**VD**), the reverse volume doubling condition (**RVD**), and that the Dirichlet form $(\mathcal{E}, \mathcal{F})$ admits the Green function $G(x, y)$ satisfying a certain estimate $(\mathbf{G})_\beta$ where β is a positive parameter that is called the *walk dimension* (see Section 2 for the definitions of these conditions). Under these hypotheses we establish in Theorem 5.6 the "classical form" of Hardy's inequality: for any $x_o \in M$ and all $f \in \mathcal{F}$

$$\int_M \frac{f(x)^2}{d(x_o, x)^\beta} \, d\mu(x) \leq C\mathcal{E}(f, f). \tag{1.5}$$

Note that $\mathbb{R}^n$ satisfies the hypotheses of Theorem 5.6 provided $n > 2$ and $\beta = 2$. Theorem 5.6 applies also on many fractals spaces where the estimates of the Green functions with $\beta > 2$ are available. Let us emphasize that in Theorem 5.6 the Dirichlet form does not have to be local.

Recall that $\mathbb{R}^n$ with $n > 2$ admits also a weighted Hardy inequality (see [45, p.657, (7)], [7, Corollary 4] or [16, Theorem 13]):

$$\frac{(n-\sigma-2)^2}{4} \int_{\mathbb{R}^n} \frac{f(x)^2}{|x|^{\sigma+2}} dx \leq \int_{\mathbb{R}^n} \frac{|\nabla f(x)|^2}{|x|^\sigma} dx \tag{1.6}$$

for any $\sigma \in [0, n-2)$. We establish an analogue of (1.6) for strongly local Dirichlet forms, under the hypotheses **(VD)**, **(RVD)** and $(\mathbf{G})_2$. Our weighted Hardy inequality is stated in Theorem 7.5 and has the form

$$\int_M \frac{f(x)^2}{d(x, x_o)^2} w(x)\, d\mu(x) \le C \int_M w\, d\Gamma(f, f), \tag{1.7}$$

where $\Gamma(f, f)$ is the energy measure of f and w is a certain admissible function that is determined by a distance function to a certain closed null set Σ in M (see Definition 7.3). In particular, in the case of a singleton $\Sigma = \{x_o\}$ we obtain the following generalization of (1.6):

$$\int_M \frac{f(x)^2}{d(x, x_o)^{\sigma+2}} d\mu(x) \le C \int_M \frac{1}{d(x, x_o)^{\sigma}}\, d\Gamma(f, f)$$

(see Proposition 7.6).

This part of our work is most technically involved since it requires investigation of a weighted Dirichlet form $(\mathcal{E}^{(w)}, \mathcal{F}^{(w)})$, where $\mathcal{E}^{(w)}(f, f)$ is defined by the right hand side of (1.7), and establishing the estimate $(\mathbf{G})_2$ for the Green function of $(\mathcal{E}^{(w)}, \mathcal{F}^{(w)})$ (Theorem 7.5). For the latter we use the following two highly nontrivial results:

▷ the equivalence

$$(\mathbf{G})_\beta \Leftrightarrow (\mathbf{UE})_\beta + (\mathbf{NLE})_\beta, \tag{1.8}$$

where $(\mathbf{UE})_\beta$ and $(\mathbf{NLE})_\beta$ denote certain upper and lower bounds of the heat kernel of $(\mathcal{E}, \mathcal{F})$ (see Theorem 6.1);

▷ the stability of $(\mathbf{UE})_2 + (\mathbf{NLE})_2$ under certain non-uniform change of weight ([30], [55, Theorem 1.0.1]; the latter works only in the case $\beta = 2$).

Our weighted Hardy inequality (1.7) seems to be entirely new in the setting of Dirichlet forms.

This paper is organized as follows.

In Section 2 we describe our basic setup: define the aforementioned conditions **(VD)** and **(RVD)**, recall some basic facts about Dirichlet forms, introduce the Green operator G and define the condition $(\mathbf{G})_\beta$.

In Section 3 we prove the Hardy inequality for strongly local regular Dirichlet forms (Theorem 3.1).

In Section 4 we prove the Hardy inequality for general (non-local) regular Dirichlet forms (Theorem 4.5).

In Section 5 we apply Theorem 4.5 to specific settings. In particular, we obtain the discrete Hardy inequality on $\mathbb{Z}^n$ (Theorem 5.1). Under the assumptions **(VD)**, **(RVD)**, $(\mathbf{G})_\beta$, we deduce from Theorem 4.5 the explicit form (1.5) of the Hardy inequality (Theorem 5.6). Besides, in a similar setting, we prove the fractional Hardy inequality for the subordinated Dirichlet form (Theorem 5.9).

In Section 6 we prove the equivalence (1.8) for strongly local Dirichlet forms (Theorem 6.1). This equivalence is interesting on its own merit, but we need it for the proof of the weighted Hardy inequality in Section 7 (Theorem 7.5). Previously (1.8) was known in the setting of random walks on graphs – see [31]. Different ways of characterization of the heat kernel upper and lower estimates have been considered in a large number of papers; see for example, [4, 31, 32, 22, 26, 27, 29, 28] and references therein. In particular, it was proved in [27] that $(\mathbf{UE})_\beta$ and $(\mathbf{NLE})_\beta$ are equivalent to certain estimates of the restricted Green functions G^B in balls B provided G^B are jointly continuous off the diagonal. However, we do not apply this result since the proof of the joint continuity of G^B would have required at least as much work as a direct proof of (1.8).

The main ingredients of the proof of Theorem 6.1 are the mean exit time estimate $(\mathbf{E})_\beta$ and the elliptic Harnack inequality (**H**). Our strategy for the proof of (**H**) is based on the argument in [27, Lemma 8.2], but the crucial point here is to gain upper and lower bounds for a positive harmonic function via an integral of the Green function with respect to a certain Riesz measure (see the proof of Proposition 6.8).

In Section 7, we prove the weighted Hardy inequality (1.7) (Theorem 7.5) and give explicit examples of the weight w when the set Σ is a single point or an affine space in $\mathbb{R}^n$, or the boundary of a bounded convex set (Propositions 7.6, 7.7, 7.9).

Notation 1.1. Throughout the paper we use the following notation.

For any $p \in [1, \infty]$ and any open set $\Omega \subset M$, denote as usual by $L^p(\Omega, \mu)$ or $L^p(\Omega)$ the real-valued Lebesgue space in Ω. In case $\Omega = M$ we write $L^p = L^p(M, \mu)$. We use $(\cdot, \cdot)$ to denote the inner product in L^2. Set

$$L^p_{\text{loc}} = \{f : f \in L^p(\Omega) \text{ for any precompact open set } \Omega \subset M\}.$$

For any set $E \subset M$, $\overline{E}$ is the closure of E, and $E^c = M \setminus E$.

For any function $f : M \to \mathbb{R}$, its support supp f is the complement of the largest open set where $f = 0$ μ-a.e..

For any open set $\Omega \subset M$, $C(\Omega)$ is the space of all continuous functions on Ω with sup-norm, and $C_c(\Omega)$ is the subspace of $C(\Omega)$ consisting of functions with compact supports. In case $\Omega = M$ we write $C = C(M)$ and $C_c = C_c(M)$.

The letters C and c are used to denote positive constants that are independent of the variables in question, but may vary at each occurrence.

The relation $u \lesssim v$ (resp., $u \gtrsim v$) between functions u and v means that $u \leq Cv$ (resp., $u \geq Cv$) for a positive constant C and for a specified range of the variables. We write $u \simeq v$ if $u \gtrsim v \gtrsim u$.

2 Basic setup

2.1 Metric measure space

Let (M, d) be a locally compact separable metric space. Assume that all the metric balls

$$B(x, r) = \{y \in M : d(y, x) < r\}$$

in M are precompact. Let μ be a Radon measure on M with full support. Such a triple (M, d, μ) will be referred to a *metric measure space*. For convenience of notation, for any $x, y \in M$ and $r > 0$, we write

$$V(x, r) = \mu(B(x, r)) \quad \text{and} \quad V(x, y) = \mu(B(x, d(x, y))).$$

The metric measure space (M, d, μ) is said to satisfy the *volume doubling condition* (**VD**) if there exists $C_D \in (1, \infty)$ such that

$$V(x, 2r) \leq C_D V(x, r) \text{ for all } x \in M \text{ and } r > 0.$$

Note that the volume doubling condition is equivalent to

$$\frac{V(x,R)}{V(x,r)} \leq C\left(\frac{R}{r}\right)^{\alpha_+} \quad \text{for all } x \in M \text{ and } 0 < r \leq R, \tag{2.1}$$

where C is a positive constant and $\alpha_+ = \log_2 C_D$. The exponent α_+ is called the *upper volume dimension* of (M, d, μ).

We say that (M, d, μ) satisfies the *reverse volume doubling condition* (**RVD**) if there exists $c > 0$ such that

$$\frac{V(x,R)}{V(x,r)} \geq c\left(\frac{R}{r}\right)^{\alpha_-} \quad \text{for all } x \in M \text{ and } 0 < r \leq R. \tag{2.2}$$

The exponent α_- is called the *lower volume dimension* of (M, d, μ).

It is known that if (M, d) is connected and diam $M = \infty$ (or, equivalently, if $\mu(M) = \infty$; see [13, Proposition 2.1]) then

$$(\mathbf{VD}) \Rightarrow (\mathbf{RVD})$$

with $\alpha_- = \log_2(1 + C_D^{-2})$ (see [13, Proposition 2.2], [21, Theorem 1.1], [28, Corollary 5.3], [28, Proposition 5.2]). Clearly, if both (**VD**) and (**RVD**) are satisfied then $0 < \alpha_- \leq \alpha_+$. If (M, d, μ) satisfies (**RVD**) then $\mu(\{x\}) = 0$ for any $x \in M$, so that (M, d, μ) is non-atomic.

The conditions (**VD**) and (**RVD**) are known to hold on many families of metric measure spaces. For example, (**VD**) and (**RVD**) are satisfied for the Euclidean spaces $\mathbb{R}^n$, convex unbounded domains in $\mathbb{R}^n$, Riemannian manifolds of non-negative Ricci curvature, nilpotent Lie groups, and on many fractal-like spaces (see [2, 3, 12, 13, 22, 27, 28, 31, 34, 41, 52, 57]).

2.2 Dirichlet forms

Let (M, d, μ) be a metric measure space and $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on L^2, that is, $\mathcal{E}$ is a symmetric, non-negative definite, closed, Markovian bilinear form in L^2 with the domain $\mathcal{F}$ that is a dense subspace of L^2 (see [20]). The domain $\mathcal{F}$ is a Hilbert space with the following norm:

$$\|u\|_{\mathcal{F}}^2 = \mathcal{E}(u,u) + \|u\|_{L^2}^2.$$

The Dirichlet form $(\mathcal{E}, \mathcal{F})$ is called *regular* if $\mathcal{F} \cap C_c$ is dense both in $\mathcal{F}$ (with respect to the norm $\|\cdot\|_{\mathcal{F}}$) and in C_c (with respect to the supremum norm).

Definition 2.1. For any open set $\Omega \subset M$ and a set $A \Subset \Omega$, a *cutoff function* ϕ of the pair (A, Ω) is any function $\phi \in \mathcal{F} \cap C_c(\Omega)$ such that $0 \leq \phi \leq 1$ in M and $\phi = 1$ in an open neighborhood of $\overline{A}$.

It is known that if $(\mathcal{E}, \mathcal{F})$ is regular then, for any open set $\Omega \subset M$ and any $A \Subset \Omega$, there exists always a cutoff function of (A, Ω) (see [20, p.27]).

A Dirichlet form $(\mathcal{E}, \mathcal{F})$ is called *strongly local* if $\mathcal{E}(u, v) = 0$ for any two functions $u, v \in \mathcal{F}$ with compact supports such that $u =$ const in some open neighborhood of supp v.

Any Dirichlet form $(\mathcal{E}, \mathcal{F})$ has the generator – a non-negative definite self-adjoint operator $\mathcal{L}$ in L^2 such that dom $(\mathcal{L}) \subset \mathcal{F}$ and

$$\mathcal{E}(u,v) = (\mathcal{L}u, v) \quad \text{for all } u \in \operatorname{dom}(\mathcal{L}),\ v \in \mathcal{F}.$$

For any $t \geq 0$ set $P_t = e^{-t\mathcal{L}}$ so that P_t is a bounded, self-adjoint, positivity preserving operator in L^2. The family $\{P_t\}_{t\geq 0}$ is called the *heat semigroup* of $(\mathcal{E}, \mathcal{F})$. If P_t for $t > 0$ has an integral kernel then the latter is called the heat kernel and is denoted by $p_t(x,y)$ so that for all $f \in L^2$ and $t > 0$

$$P_t f(x) = \int_M p_t(x,y) f(y)\, d\mu(y) \quad \text{for } \mu\text{-a.a. } x \in M.$$

Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form in $L^2(M,\mu)$. For any non-empty open set $\Omega \subset M$, define $\mathcal{F}(\Omega)$ as the closure of $\mathcal{F} \cap C_c(\Omega)$ in $\mathcal{F}$. Then $\mathcal{F}(\Omega)$ is dense in $L^2(\Omega)$ and $(\mathcal{E},\mathcal{F}(\Omega))$ is a regular Dirichlet from in $L^2(\Omega)$, that is called the restriction of $(\mathcal{E},\mathcal{F})$ to Ω. Denote by $\mathcal{L}^\Omega$ the generator of $(\mathcal{E},\mathcal{F}(\Omega))$ and by $\{P_t^\Omega\}$ the corresponding heat semigroup. It is known that, for any $0 \le f \in L^2(\Omega)$ and $t \ge 0$,

$$P_t^\Omega f \le P_t f.$$

Set also

$$\lambda_{\min}(\Omega) = \inf \operatorname{spec} \mathcal{L}^\Omega.$$

It is known that

$$\lambda_{\min}(\Omega) = \inf_{u\in\mathcal{F}(\Omega)\setminus\{0\}} \frac{\mathcal{E}(u,u)}{\|u\|_{L^2}^2} = \inf_{u\in\mathcal{F}\cap C_c(\Omega)\setminus\{0\}} \frac{\mathcal{E}(u,u)}{\|u\|_{L^2}^2} \tag{2.3}$$

2.3 Green function

The positivity preserving property of the heat semigroups allows to extend $P_t f$ from $f \in L^2$ to all non-negative measurable functions u on M (of course, the value $+\infty$ for $P_t f$ is allowed in this case). It is easy to verify that the semigroup property $P_{t+s} f = P_t(P_s f)$ holds also in this extended setting.

Define the *Green operator* Gf for all non-negative measurable functions f on M by

$$Gf = \int_0^\infty P_t f\, dt.$$

Of course, the value $+\infty$ is allowed for Gf.

A function $G(x,y)$ on $M \times M$ is called the *Green function* (or the Green kernel) if it takes values in $[0,+\infty]$, is jointly measurable, non-negative, and satisfies for any non-negative f the identity

$$Gf(x) = \int_M G(x,y) f(y)\, d\mu(y) \;\; \text{for } \mu\text{-a.a. } x \in M.$$

For instance, if the heat semigroup $\{P_t\}$ has the heat kernel $p_t(x,y)$ then

$$G(x,y) = \int_0^\infty p_t(x,y)\, dt$$

(although the integral here may diverge). Note that the Green function is always symmetric in x, y which follows from the symmetry of P_t.

Let Ω be a non-empty open subset of M. Denote by P_t^Ω the heat semigroup of $(\mathcal{E},\mathcal{F}(\Omega))$ and by G^Ω – the Green operator. It is known that, for any non-negative f,

$$0 \le P_t^\Omega f \le P_t f,$$

whence also

$$0 \le G^\Omega f \le Gf.$$

Remark 2.2. Assume that $\lambda_{\min}(\Omega) > 0$. Then the operator $\mathcal{L}^\Omega$ has a bounded inverse in $L^2(\Omega)$, and $(\mathcal{L}^\Omega)^{-1} = G^\Omega|_{L^2(\Omega)}$. In this case G^Ω has the following property (see [27, Lemma 5.1]): for any $f \in L^2(\Omega)$, we have $G^\Omega f \in \mathcal{F}(\Omega)$ and

$$\mathcal{E}(G^\Omega f, \phi) = (f, \phi) \quad \text{for all } \phi \in \mathcal{F}(\Omega). \tag{2.4}$$

The following two-sided estimates for the Green function $G(x, y)$ are fundamental for us to derive Hardy's inequalities.

Definition 2.3. Given $\beta > 0$, we say that condition $(\mathbf{G})_\beta$ is satisfied if the Green function $G(x, y)$ exists, is jointly continuous in $M \times M \setminus \text{diag}$, and

$$G(x, y) \simeq \frac{d(x, y)^\beta}{V(x, y)} \text{ for all distinct } x, y \in M. \tag{2.5}$$

Note that the estimate (2.5) can be obtained from certain heat kernel bounds as follows.

Lemma 2.4. *Assume that (M, d, μ) satisfies* (**VD**) *and that the heat kernel of $(\mathcal{E}, \mathcal{F})$ exists and satisfies the following estimates, for any $t > 0$ and μ-a.a. $x, y \in M$:*

$$p_t(x, y) \lesssim \frac{1}{V(x, t^{1/\beta})} \wedge \frac{1}{V(x, y)} \tag{2.6}$$

and

$$p_t(x, y) \gtrsim \frac{1}{V(x, t^{1/\beta})} \quad \text{if } t \geq d(x, y)^\beta. \tag{2.7}$$

Then the Green function satisfies the estimate

$$G(x, y) \simeq \int_{d(x,y)}^\infty \frac{r^{\beta-1}}{V(x, r)} dr, \tag{2.8}$$

for μ-a.a. $x, y \in M$. Moreover, if in addition (**RVD**) *holds with $\alpha_- > \beta$ then the Green function satisfies (2.5) for μ-a.a. $x, y \in M$.*

Proof. Set for simplicity $\rho = d(x, y)$. It follows from (2.7) that

$$G(x, y) \gtrsim \int_{\rho^\beta}^\infty \frac{dt}{V(x, t^{1/\beta})} = \int_\rho^\infty \frac{\beta r^{\beta-1} dr}{V(x, r)},$$

which proves the lower bound in (2.8). It follows from (2.6) that

$$\begin{aligned} G(x, y) &\lesssim \int_{\rho^\beta}^\infty \frac{dt}{V(x, t^{1/\beta})} + \int_0^{\rho^\beta} \frac{dt}{V(x, \rho)} \\ &= \int_\rho^\infty \frac{\beta r^{\beta-1} dr}{V(x, r)} + \frac{\rho^\beta}{V(x, \rho)}. \end{aligned}$$

It remains to observe that, by (**VD**),

$$\int_\rho^\infty \frac{r^{\beta-1} dr}{V(x, r)} \geq \int_\rho^{2\rho} \frac{r^{\beta-1} dr}{V(x, r)} \geq \frac{\rho^\beta}{V(x, 2\rho)} \gtrsim \frac{\rho^\beta}{V(x, \rho)}, \tag{2.9}$$

whence the upper bound in (2.8) follows.

If (**RVD**) is satisfied with $\alpha_- > \beta$ then

$$\begin{aligned}\int_\rho^\infty \frac{r^{\beta-1}}{V(x,r)}dr &= \frac{\rho^\beta}{V(x,\rho)}\int_\rho^\infty \frac{V(x,\rho)\,r^\beta}{V(x,r)\rho^\beta}\frac{dr}{r}\\ &\lesssim \frac{\rho^\beta}{V(x,\rho)}\int_\rho^\infty \left(\frac{\rho}{r}\right)^{\alpha_-}\left(\frac{r}{\rho}\right)^{\beta}\frac{dr}{r}\\ &= \frac{\rho^\beta}{V(x,\rho)}\int_1^\infty s^{-(\alpha_- -\beta)}\frac{ds}{s}\\ &\lesssim \frac{\rho^\beta}{V(x,\rho)},\end{aligned}$$

which together with (2.9) implies that

$$G(x,y) \simeq \int_\rho^\infty \frac{r^{\beta-1}}{V(x,r)}dr \simeq \frac{\rho^\beta}{V(x,\rho)}.$$

□

Example 2.5. Assume that the heat kernel $p_t(x,y)$ on (M,d,μ) exists and satisfies the following sub-Gaussian estimate: for all $t > 0$ and μ-a.a. $x, y \in M$,

$$p_t(x,y) \asymp \frac{C}{V(x,t^{1/\beta})}\exp\left\{-c\left(\frac{d(x,y)}{t^{1/\beta}}\right)^{\frac{\beta}{\beta-1}}\right\}, \tag{2.10}$$

where $\beta > 1$ is the *walk dimension* and the symbol $\asymp$ means that both inequalities with $\leq$ and $\geq$ are satisfied but with different values of positive constants C and c. For example, (2.10) is satisfied with $\beta = 2$ on any Riemannian manifold of non-negative Ricci curvature (see [46]) as well as with $\beta > 2$ on many fractal spaces (see [2, 3, 22, 25, 42]).

Clearly, (2.10) implies both (2.6) and (2.7). Indeed, (2.6) and (2.7) are trivial in the case $t \geq d(x,y)^\beta$, while in the case $t < d(x,y)^\beta$ we have, setting $r = d(x,y)$,

$$V(x,r)\,p_t(x,y) \lesssim \frac{V(x,r)}{V(x,t^{1/\beta})}\exp\left(-c\left(\frac{r}{t^{1/\beta}}\right)^{\frac{\beta}{\beta-1}}\right) \lesssim \left(\frac{r}{t^{1/\beta}}\right)^{\alpha_+}\exp\left(-c\left(\frac{r}{t^{1/\beta}}\right)^{\frac{\beta}{\beta-1}}\right) \leq C$$

so that

$$p_t(x,y) \leq \frac{C}{V(x,r)},$$

which proves (2.6).

Example 2.6. For certain jump processes on fractal spaces the heat kernel satisfies the following stable-like estimate

$$p_t(x,y) \simeq \frac{1}{V(x,t^{1/\beta})} \wedge \frac{t}{V(x,y)\,d(x,y)^\beta} \tag{2.11}$$

(see [11]). For example, if

$$V(x,r) \simeq r^\alpha$$

then (2.11) becomes

$$p_t(x,y) \simeq \frac{1}{t^{\alpha/\beta}} \wedge \frac{t}{d(x,y)^{\alpha+\beta}} \simeq \frac{1}{t^{\alpha/\beta}}\left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}.$$

This estimate is satisfied with $\alpha = n$ for a symmetric stable process in $\mathbb{R}^n$ of index β.

If $t \geq d(x,y)^\beta$ then (2.11) becomes

$$p_t(x,y) \simeq \frac{1}{V(x,t^{1/\beta})},$$

while in the case $t < d(x,y)^\beta$ inequality (2.11) implies

$$p_t(x,y) \simeq \frac{t}{V(x,y)\,d(x,y)^\beta} \leq \frac{1}{V(x,y)}.$$

Hence, in the both cases the estimates (2.6) and (2.7) are satisfied, and by Lemma 2.4 the Green function satisfies (2.5).

3 Hardy's inequality for strongly local regular Dirichlet forms

In the setting of strongly local regular Dirichlet forms, in order to prove an abstract version of Hardy's inequality, we adopt the method of change of measure explained in Introduction. The following theorem is the main result of this section.

Theorem 3.1. *Let $(\mathcal{E},\mathcal{F})$ be a strongly local regular Dirichlet form in $L^2(M,\mu)$. Assume that $\lambda_{\min}(\Omega) > 0$ for all precompact open sets $\Omega \subset M$. Let h be a non-negative measurable function on M such that*

$$G(h \wedge a) \in L^\infty_{\mathrm{loc}} \tag{3.1}$$

for any positive constant a. Then, for any $f \in \mathcal{F}$,

$$\int_M \frac{h}{Gh} f^2\, d\mu \leq \mathcal{E}(f,f). \tag{3.2}$$

If h and Gh vanish simultaneously at some points then at these points we set $\frac{h}{Gh} = 0$.

Before the proof, let us recall some necessary notions from the theory of strongly local Dirichlet forms. According to [20, Section 3.2] (see also [10, Section 4.3]), for any $u \in \mathcal{F} \cap L^\infty$, there exists a unique positive Radon measure $\Gamma(u,u)$ on M such that

$$\int_M f\, d\Gamma(u,u) = \mathcal{E}(uf,u) - \frac{1}{2}\mathcal{E}(u^2,f) \quad \text{for all } f \in \mathcal{F} \cap C_c.$$

This measure $\Gamma(u,u)$ is called the *energy measure* of u. For any $u, v \in \mathcal{F} \cap L^\infty$, define a signed energy measure $\Gamma(u,v)$ by

$$\int_M f\, d\Gamma(u,v) = \frac{1}{2}\Big(\mathcal{E}(uf,v) + \mathcal{E}(u,vf) - \mathcal{E}(uv,f)\Big) \quad \text{for all } f \in \mathcal{F} \cap C_c.$$

Note that $\Gamma(u,v)$ is symmetric and bilinear, and it can be extended to all $u, v \in \mathcal{F}$. It is known that

$$\mathcal{E}(u,v) = \int_M d\Gamma(u,v) \quad \text{for all } u,v \in \mathcal{F} \tag{3.3}$$

(see [8], [20, Lemma 3.2.3]).

Let $\mathcal{F}_{\mathrm{loc}}$ be the space of all μ-measurable functions u on M satisfying the following property: for every precompact open subset $\Omega \subset M$ there exists a function $u' \in \mathcal{F}$ such that $u = u'$ μ-a.e. on Ω. The locality of $(\mathcal{E}, \mathcal{F})$ allows to extend $\mathcal{E}(u, v)$ to all $u \in \mathcal{F}_{\mathrm{loc}}$ and $v \in \mathcal{F}_c$, where $\mathcal{F}_c$ denotes a subspace of $\mathcal{F}$ consisting of functions with compact support. Indeed, there exists $u' \in \mathcal{F}$ such that $u = u'$ in a neighborhood of $\operatorname{supp} v$, and $\mathcal{E}(u', v)$ is obviously independent of the choice of u', so that we set $\mathcal{E}(u, v) := \mathcal{E}(u', v)$. It follows that the identity (3.3) holds also for $u \in \mathcal{F}_{\mathrm{loc}}$ and $v \in \mathcal{F}_c$.

It is known that the space $\mathcal{F} \cap L^\infty$ is closed under multiplication of functions (see [20, Theorem 1.4.2(ii)]). This implies that also $\mathcal{F}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$ is closed under multiplication[1].

For strongly local Dirichlet forms, $\Gamma(u, v)$ can be extended to all $u, v \in \mathcal{F}_{\mathrm{loc}}$ (see [10, Theorem 4.3.11], [53, p.189]). Moreover, $\Gamma(u, v)$ satisfies the following Leibniz product rule

$$d\Gamma(uv, w) = u\, d\Gamma(v, w) + v\, d\Gamma(u, w) \quad \text{for all } u, v \in \mathcal{F}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}} \text{ and } w \in \mathcal{F}_{\mathrm{loc}} \tag{3.4}$$

(see [53, p.190]).

The following lemma is a key ingredient for the proof of Theorem 3.1.

Lemma 3.2. *Let $(\mathcal{E}, \mathcal{F})$ be a strongly local regular Dirichlet form on $L^2(M, \mu)$. If ϕ is a positive measurable function on M such that*

$$\text{both } \phi \text{ and } \phi^{-1} \text{ belong to } \mathcal{F}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}, \tag{3.5}$$

then

$$\mathcal{E}(f, f) - \mathcal{E}(\phi, \phi^{-1} f^2) = \int_M \phi^2\, d\Gamma(\phi^{-1} f, \phi^{-1} f) \geq 0 \quad \text{for all } \ f \in \mathcal{F}_c \cap L^\infty. \tag{3.6}$$

Consequently, we have

$$\mathcal{E}(\phi, \phi^{-1} f^2) \leq \mathcal{E}(f, f) \ \text{ for all } \ f \in \mathcal{F}_c \cap L^\infty. \tag{3.7}$$

Remark 3.3. If in addition to (3.5) $\phi \in \operatorname{dom}(\mathcal{L})$ then

$$\mathcal{E}(\phi, \phi^{-1} f^2) = \left(\mathcal{L}\phi, \phi^{-1} f^2\right) = \int_M \frac{\mathcal{L}\phi}{\phi} f^2\, d\mu.$$

Hence, (3.7) becomes

$$\int_M \frac{\mathcal{L}\phi}{\phi} f^2\, d\mu \leq \mathcal{E}(f, f),$$

which coincides with (1.2) when (M, d, μ) is a Riemannian manifold and $\mathcal{L} = -\Delta$.

Proof. Since $\phi^{-1} \in \mathcal{F}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$ and the both functions f and f^2 lie in $\mathcal{F}_c \cap L^\infty$, we obtain that

$$\phi^{-1} f \ \text{ and } \phi^{-1} f^2 \in \mathcal{F}_c \cap L^\infty \tag{3.8}$$

(indeed, both $\phi^{-1} f$ and $\phi^{-1} f^2$ belong to $\mathcal{F}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$ and have compact supports). By (3.3) we have

$$\mathcal{E}(f, f) - \mathcal{E}(\phi, \phi^{-1} f^2) = \int_M d\Gamma(f, f) - \int_M d\Gamma(\phi, \phi^{-1} f^2).$$

---

[1]Indeed, if $f, g \in \mathcal{F}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$ then, for any precompact open set Ω, there exist $f', g' \in \mathcal{F}$ such that $f = f'$ and $g = g'$ in Ω. Both f' and g' can be chosen to be bounded on M because otherwise f' can be replaced by $(f' \wedge C) \vee (-C)$ for any $C > \|f\|_{L^\infty(\Omega)}$, and the same is valid for g'. Hence, $f'g' \in \mathcal{F} \cap L^\infty$. Since $fg = f'g'$ in Ω, we conclude that $fg \in \mathcal{F}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$.

Applying the Leibniz rule (3.4), we obtain

$$\begin{aligned} d\Gamma(f,f) - d\Gamma(\phi, \phi^{-1} f^2) &= d\Gamma(\left(\phi^{-1} f\right)\phi, f) - d\Gamma(\phi, \left(\phi^{-1} f\right) f) \\ &= \left(\phi^{-1} f\, d\Gamma(\phi, f) + \phi\, d\Gamma(\phi^{-1} f, f)\right) - \left(\phi^{-1} f\, d\Gamma(\phi, f) + f\, d\Gamma(\phi, \phi^{-1} f)\right) \\ &= \phi\, d\Gamma(\phi^{-1} f, \phi\phi^{-1} f) - f\, d\Gamma(\phi, \phi^{-1} f) \\ &= \left(\phi^2\, d\Gamma(\phi^{-1} f, \phi^{-1} f) + f\, d\Gamma(\phi^{-1} f, \phi)\right) - f\, d\Gamma(\phi, \phi^{-1} f) \\ &= \phi^2\, d\Gamma(\phi^{-1} f, \phi^{-1} f), \end{aligned}$$

whence it follows that

$$\mathcal{E}(f,f) - \mathcal{E}(\phi, \phi^{-1} f^2) = \int_M \phi^2\, d\Gamma(\phi^{-1} f, \phi^{-1} f) \geq 0.$$

This proves (3.6) and, hence, (3.7). □

Proof of Theorem 3.1. It suffices to prove (3.2) for all $f \in \mathcal{F} \cap C_c$ since for any $f \in \mathcal{F}$ there exists a sequence $\{f_n\}$ from $\mathcal{F} \cap C_c$ converging to f in $\mathcal{F}$. Applying (3.2) to each f_n, passing to the limit as $n \to \infty$ and using Fatou's lemma in the left hand side, we obtain (3.2) for f.

Hence, we assume further that $f \in \mathcal{F} \cap C_c$. Let Ω be a precompact open subset of M containing supp f so that $f \in \mathcal{F}(\Omega)$. Let a, ε be positive constants. Set

$$h_a = h \wedge a$$

and consider in Ω the function

$$\phi = G^\Omega h_a + \varepsilon.$$

By (3.1), ϕ is bounded in Ω. Since $\lambda_{\min}(\Omega) > 0$ and $h_a \in L^2(\Omega)$, we have $G^\Omega h_a \in \mathcal{F}(\Omega)$ and, hence, $\phi \in \mathcal{F}_{\text{loc}}(\Omega)$. Since $\phi \geq \varepsilon$, it follows that $\phi^{-1} \in \mathcal{F}_{\text{loc}} \cap L^\infty(\Omega)$ (indeed, $\phi^{-1} = F \circ \phi$ where $F(t) := \varepsilon^{-1} \wedge t^{-1}$ is Lipschitz; see [20, Theorem 1.4.2(v)]). Therefore, ϕ satisfies the hypotheses of Lemma 3.2 in Ω, and we conclude that

$$\mathcal{E}(\phi, \phi^{-1} f^2) \leq \mathcal{E}(f,f).$$

By (3.8) we have $\phi^{-1} f^2 \in \mathcal{F}_c(\Omega)$, and by (2.4) and the strong locality

$$\begin{aligned} \mathcal{E}(\phi, \phi^{-1} f^2) &= \mathcal{E}(G^\Omega h_a + \varepsilon, \phi^{-1} f^2) = \mathcal{E}(G^\Omega h_a, \phi^{-1} f^2) \\ &= \left(h_a, \phi^{-1} f^2\right) = \int_\Omega \frac{h_a}{G^\Omega h_a + \varepsilon} f^2 d\mu \\ &\geq \int_\Omega \frac{h_a}{Gh + \varepsilon} f^2 d\mu \end{aligned}$$

whence

$$\int_\Omega \frac{h_a}{Gh + \varepsilon} f^2 d\mu \leq \mathcal{E}(f,f).$$

Letting $a \to \infty$, $\varepsilon \to 0$, and $\Omega \to M$, we obtain (3.2). □

A non-negative measurable function u on M is called *excessive* if $P_t u \leq u$ for all $t \geq 0$. It follows that $P_t u \leq P_s u$ for all $t \geq s \geq 0$.

Corollary 3.4. *Let $(\mathcal{E},\mathcal{F})$ be a strongly local regular Dirichlet form on $L^2(M,\mu)$. Assume that $\lambda_{\min}(\Omega) > 0$ for all precompact open sets $\Omega \subset M$. Let $u \in L^\infty_{\mathrm{loc}}$ be a positive excessive function on M. Then, for any $f \in \mathcal{F}$,*

$$-\int_M f^2 \partial_t \log(P_t u)\, d\mu \le \mathcal{E}(f,f). \tag{3.9}$$

Proof. Fix $t > 0$ and set

$$h = -\partial_t P_t u$$

so that h is a non-negative measurable function on M. We have

$$\begin{aligned} Gh &= \int_0^\infty P_s h\, ds = -\int_0^\infty P_s\left(\partial_t P_t u\right) ds \\ &= -\int_0^\infty \partial_t\left(P_{t+s}u\right) ds = -\int_0^\infty \partial_s\left(P_{t+s}u\right) ds \\ &= -\int_t^\infty \partial_s\left(P_s u\right) ds \le P_t u. \end{aligned}$$

Hence,

$$Gh \le P_t u \le u$$

which implies that $Gh \in L^\infty_{\mathrm{loc}}$. By Theorem 3.1 we conclude that

$$\int_M \frac{h}{Gh} f^2 d\mu \le \mathcal{E}(f,f).$$

Observing that

$$\frac{h}{Gh} \ge \frac{-\partial_t P_t u}{P_t u} = -\partial_t \log(P_t u),$$

we obtain (3.9). □

4 Hardy's inequality for regular Dirichlet forms

In this section, we prove an analogue of Theorem 3.1 for general (non-local) regular Dirichlet forms. The main result is Theorem 4.5 below.

4.1 Extended Dirichlet forms

Given a regular Dirichlet form $(\mathcal{E},\mathcal{F})$ on L^2, denote by $\mathcal{F}_e$ the family of all μ-measurable functions u on M such that u is finite μ-a.e. on M and there exists an sequence $\{u_n\} \subset \mathcal{F}$ such that

$$\lim_{n\to\infty} u_n = u \quad \mu\text{-a.e. on } M \quad \text{and} \quad \lim_{n,m\to\infty} \mathcal{E}(u_n - u_m, u_n - u_m) = 0.$$

For any $u \in \mathcal{F}_e$, the limit

$$\mathcal{E}(u,u) = \lim_{n\to\infty} \mathcal{E}(u_n,u_n)$$

exists and does not depend on the choice of the sequence $\{u_n\}$ ([20, Theorem 1.5.2(i)]), Moreover, by [20, Theorem 1.5.2(iii)],

$$\mathcal{F} = \mathcal{F}_e \cap L^2.$$

The pair $(\mathcal{E}, \mathcal{F}_e)$ is called an *extended Dirichlet form.*

Recall that by [20, Theorem 1.4.2(ii)] the space $\mathcal{F} \cap L^\infty$ is closed under multiplication of functions. The following lemma extends this property to $\mathcal{F}_e$.

Lemma 4.1. *Assume that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on the metric measure space (M, d, μ). Then, for any $u \in \mathcal{F}_e \cap L^\infty_{\mathrm{loc}}$ and any $\psi \in \mathcal{F}_c \cap L^\infty$, we have*

$$u\psi \in \mathcal{F} \cap L^\infty. \tag{4.1}$$

Consequently,

$$\mathcal{F}_e \cap L^\infty_{\mathrm{loc}} \subset \mathcal{F}_{\mathrm{loc}}. \tag{4.2}$$

Proof. Let us first show that (4.1) implies (4.2). Indeed, given a function $u \in \mathcal{F}_e \cap L^\infty_{\mathrm{loc}}$ and a precompact open subset $\Omega \subset M$, we need to find a function $g \in \mathcal{F}$ such that $u = g$ μ-a.e. on Ω. Let ψ be a cutoff function of Ω in M. By (4.1) we have $g := u\psi \in \mathcal{F}$. Since $g = u$ in Ω, we obtain (4.2).

Now let us prove (4.1). We use the following result [20, (1.3.18) and (1.4.8)]: for any Borel measurable function f on M,

$$f \in \mathcal{F} \Leftrightarrow f \in L^2 \text{ and } \lim_{\tau\to\infty} \mathcal{E}^{(\tau)}(f, f) < \infty, \tag{4.3}$$

where

$$\mathcal{E}^{(\tau)}(f, f) = \frac{\tau}{2} \int_{M\times M} (f(x) - f(y))^2 \, d\sigma_\tau(x, y) + \tau \int_M f^2 s_\tau \, d\mu, \tag{4.4}$$

for some positive symmetric Radon measure $\sigma_\tau(\cdot, \cdot)$ on $M \times M$ satisfying $\sigma_\tau(M, E) \le \mu(E)$ for any Borel measurable set E, and s_τ is a function such that $0 \le s_\tau \le 1$ on M. It is also known that $\mathcal{E}^{(\tau)}(f, f)$ is non-decreasing as $\tau \to \infty$ so that the limit in (4.3) always exists, finite or infinite. Moreover, by [20, Theorem 1.5.2(i)&(ii)] if $f \in \mathcal{F}_e$ then

$$\lim_{\tau\to\infty} \mathcal{E}^{(\tau)}(f, f) = \mathcal{E}(f, f) < \infty.$$

Let $u \in \mathcal{F}_e \cap L^\infty_{\mathrm{loc}}$ and $\psi \in \mathcal{F}_c \cap L^\infty$. Without loss of generality we can assume that u and ψ are Borel measurable. Clearly, we have $u\psi \in L^\infty \cap L^2$ so that, by (4.3), in order to prove that $u\psi \in \mathcal{F}$, it suffices to verify that

$$\lim_{\tau\to\infty} \mathcal{E}^{(\tau)}(u\psi, u\psi) < \infty.$$

Without loss of generality, we can assume that $\|\psi\|_{L^\infty} = 1$. The set $\{x \in M : |\psi(x)| > 1\}$ is a Borel set of μ-measure zero. Modifying ψ on this set by setting $\psi = 0$ we can assume without loss of generality that

$$|\psi(x)| \le 1 \text{ for all } x \in M.$$

Let Ω be a precompact open set containing $\operatorname{supp}\psi$. Similarly, we can assume that $\psi(x) = 0$ for all $x \in \Omega^c$.

Without loss of generality, we can also assume that $\|u\|_{L^\infty(\Omega)} = 1$. Modifying u on the Borel null set $\{x \in \Omega : u(x) > 1\}$, we can assume that

$$|u(x)| \le 1 \text{ for all } x \in \Omega.$$

Let us verify that, for all $x, y \in M$,

$$|u(x)\psi(x) - u(y)\psi(y)| \leq |\psi(x) - \psi(y)| + |u(x) - u(y)|\,. \tag{4.5}$$

Indeed, if $x, y \in \Omega$ then

$$\begin{aligned}|u(x)\psi(x) - u(y)\psi(y)| &\leq |u(x)|\,|\psi(x) - \psi(y)| + |\psi(y)|\,|u(x) - u(y)| \\ &\leq |\psi(x) - \psi(y)| + |u(x) - u(y)|\,.\end{aligned}$$

If $x \in \Omega^c$ and $y \in \Omega$ then $\psi(x) = 0$ and

$$|u(x)\psi(x) - u(y)\psi(y)| = |u(y)|\,|\psi(y)| = |u(y)|\,|\psi(x) - \psi(y)| \leq |\psi(x) - \psi(y)|\,,$$

and if $x, y \in \Omega^c$ then $|u(x)\psi(x) - u(y)\psi(y)| = 0$.

It follows from (4.4) and (4.5) that

$$\begin{aligned}\int_{M\times M} (u\psi(x) - u\psi(y))^2 \, d\sigma_\tau(x, y) \leq & \int_{M\times M} (\psi(x) - \psi(y))^2 \, d\sigma_\tau(x, y) \\ & + \int_{M\times M} (u(x) - u(y))^2 \, d\sigma_\tau(x, y).\end{aligned}$$

Since $|u\psi| \leq |u|$, we have also

$$\int_M (u\psi)^2 \, s_\tau \, d\mu \leq \int_M u^2 s_\tau \, d\mu.$$

It follows that

$$\mathcal{E}^{(\tau)}(u\psi, u\psi) \leq \mathcal{E}^{(\tau)}(\psi, \psi) + \mathcal{E}^{(\tau)}(u, u)$$

and, hence,

$$\lim_{\tau\to\infty} \mathcal{E}^{(\tau)}(u\psi, u\psi) \leq \lim_{\tau\to\infty} \mathcal{E}^{(\tau)}(\psi, \psi) + \lim_{\tau\to\infty} \mathcal{E}^{(\tau)}(u, u) < \infty,$$

which finishes the proof. □

4.2 Transience of Dirichlet forms

According to [20, Section 1.5], a Dirichlet form $(\mathcal{E}, \mathcal{F})$ is called *transient* if there exists a bounded μ-measurable function g that is strictly positive μ-a.e. on M and such that

$$\int_M |u| g \, d\mu \leq \sqrt{\mathcal{E}(u, u)} \quad \text{for all } u \in \mathcal{F}.$$

By [20, Lemma 1.5.5], if $(\mathcal{E}, \mathcal{F})$ is transient then $\mathcal{E}(u, v)$ is an inner product in $\mathcal{F}_e$ and $\mathcal{F}_e$ with this inner product is a Hilbert space. By [20, Theorem 1.5.4], if $(\mathcal{E}, \mathcal{F})$ is transient, then, for any non-negative μ-measurable function f on M satisfying

$$\int_M f G f \, d\mu < \infty,$$

we have that $Gf \in \mathcal{F}_e$ and

$$\mathcal{E}(Gf, \phi) = \int_M f\phi \, d\mu \quad \text{for all } \phi \in \mathcal{F}_e. \tag{4.6}$$

As it follows from [20, Lemma 1.5.1], in order to show that $(\mathcal{E}, \mathcal{F})$ is transient, it suffices to find a μ-a.e. strictly positive function $g \in L^1$ such that

$$Gg(x) < \infty \quad \text{for } \mu\text{-a.a. } x \in M. \tag{4.7}$$

Lemma 4.2. *If the Green function $G(x,y)$ exists and belongs to $L^1_{\mathrm{loc}}(M\times M)$ then $(\mathcal{E},\mathcal{F})$ is transient.*

Proof. It suffices to construct a strictly positive function $g\in L^1$ such that

$$Gg\in L^1_{\mathrm{loc}},$$

which will imply (4.7). Observe first that if A and B are precompact subsets of M then

$$\int_B G1_A d\mu=\int_B\left(\int_A G(x,y)\,d\mu(y)\right)d\mu(x)=\|G\|_{L^1(B\times A)}<\infty. \tag{4.8}$$

Fix a point $x_o\in M$, set $B_k=B(x_o,2^k)$,

$$A_0=B_0,\quad A_k=B_k\setminus B_{k-1}\text{ for }\ k\geq 1,$$

and define g by

$$g=\sum_{k=0}^{\infty}c_k 1_{A_k},$$

where $\{c_k\}_{k=0}^{\infty}$ is sequence of positive reals yet to be determined. Clearly, $g>0$ on M. By (4.8) we have, for all indices k,n,

$$\int_{B_n}G1_{A_k}d\mu=\|G\|_{L^1(B_n\times A_k)}$$

and, hence,

$$\int_{B_n}Ggd\mu=\sum_{k=0}^{\infty}c_k\|G\|_{L^1(B_n\times A_k)}. \tag{4.9}$$

Choose c_k for all $k=0,1,\ldots$ so that

$$c_k\|G\|_{L^1(B_k\times A_k)}\leq 2^{-k}.$$

Then the series in (4.9) converges for any n, whence $Gg\in L^1_{\mathrm{loc}}$ follows. □

Corollary 4.3. *If (M,d,μ) satisfies* (**VD**) *and, for some $\beta>0$.*

$$G(x,y)\lesssim\frac{d(x,y)^\beta}{V(x,y)}\ \text{ for }\mu\text{-a.a. }\ x,y\in M,$$

then $(\mathcal{E},\mathcal{F})$ is transient. In particular, (**VD**) + $(\mathbf{G})_\beta$ *imply the transience.*

Proof. Indeed, by (**VD**), we have, for any ball $B(x,R)\subset M$,

$$\begin{aligned}\int_{B(x,R)}\frac{d(x,y)^\beta}{V(x,y)}d\mu(y)&=\sum_{j=0}^{\infty}\int_{B\left(x,2^{-j}R\right)\setminus B\left(x,2^{-(j+1)}R\right)}\frac{d(x,y)^\beta}{V(x,y)}d\mu(y)\\&\leq\sum_{j=0}^{\infty}(2^{-j}R)^\beta\frac{V(x,2^{-j}R)}{V(x,2^{-(j+1)}R)}\\&\leq C_D\sum_{j=0}^{\infty}(2^{-j}R)^\beta\end{aligned}$$

$$\simeq R^{\beta}. \tag{4.10}$$

Now, for any ball $B(x_o, R)$, we obtain, using (4.10),

$$\begin{aligned}\int_{B(x_o,R)}\int_{B(x_o,R)} G(x,y)\,d\mu(y)\,d\mu(x) &\lesssim \int_{B(x_o,R)}\left(\int_{B(x_o,R)} \frac{d(x,y)^{\beta}}{V(x,y)}\,d\mu(y)\right)d\mu(x)\\ &\le \int_{B(x_o,R)}\left(\int_{B(x,2R)} \frac{d(x,y)^{\beta}}{V(x,y)}\,d\mu(y)\right)d\mu(x)\\ &\lesssim \int_{B(x_o,R)} R^{\beta}\,d\mu(x) < \infty,\end{aligned}$$

which implies $G \in L^1_{\text{loc}}(M \times M)$. Hence, $(\mathcal{E},\mathcal{F})$ is transient by Lemma 4.2. □

4.3 Admissible functions and Hardy's inequality

Definition 4.4. Let G be the Green operator of a Dirichlet form. A positive μ-measurable function h on M is called (μ, G)-admissible if it satisfies the following three conditions:

(i) $Gh \in L^{\infty}_{\text{loc}}$;

(ii) $(Gh)^{-1} \in L^{\infty}_{\text{loc}}$;

(iii) $\int_M hGh\,d\mu < \infty$.

The next theorem is our main result about Hardy's inequality for general regular Dirichlet forms.

Theorem 4.5. *Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form on (M, d, μ) and G be its Green operator. If h is a (μ, G)-admissible function on M, then the following Hardy inequality holds:*

$$\int_M \frac{h}{Gh} f^2 d\mu \le \mathcal{E}(f,f) \quad \text{for all } f \in \mathcal{F}. \tag{4.11}$$

Remark 4.6. If $(\mathcal{E},\mathcal{F})$ is strongly local then Theorem 3.1 gives the same Hardy inequality (4.11) under a weaker hypothesis (3.1) instead of (μ, G)-admissibility.

Proof. Due to the regularity of the Dirichlet form $(\mathcal{E},\mathcal{F})$, it suffices to show (4.11) for all $f \in \mathcal{F} \cap C_c$ (see the proof of Theorem 3.1).

Let us first verify that if a (μ, G)-admissible function h exists then $(\mathcal{E},\mathcal{F})$ is transient. Indeed, it suffices to construct a positive function $g \in L^1$ such that $g \le h$ (then (4.7) is satisfied by $Gh \in L^{\infty}_{\text{loc}}$). Indeed, define a sequence $\{A_k\}_{k=0}^{\infty}$ of subsets of M as in Lemma 4.2, choose positive c_k so that

$$c_k \mu(A_k) \le 2^{-k},$$

and set

$$g(x) = \min\{c_k, h(x)\} \quad \text{if } x \in A_k.$$

Clearly, $0 < g \le h$ and

$$\int_{A_k} g\,d\mu \le c_k \mu(A_k) \le 2^{-k}$$

whence $g \in L^1$ follows.

By [20, Theorem 1.5.4], the condition (iii) of Definition 4.4 and the transience of $(\mathcal{E}, \mathcal{F})$ imply that

$$w := Gh \in \mathcal{F}_e. \tag{4.12}$$

The condition (i) of Definition 4.4, that is, $w \in L^\infty_{\text{loc}}$, and (4.12) imply by Lemma 4.1 that

$$w \in \mathcal{F}_{\text{loc}}.$$

By condition (ii) of Definition 4.4, for any ball $B \subset M$ there is $\varepsilon > 0$ such that $w \geq \varepsilon$ in B. By using [20, Theorem 1.4.2(v)], we conclude that $w^{-1} \in \mathcal{F}_{\text{loc}}$ (indeed, we have $w^{-1} = F \circ w$, where $F(t) := \varepsilon^{-1} \wedge t^{-1}$ is a Lipschitz function). Hence, $w^{-1} \in \mathcal{F}_{\text{loc}} \cap L^\infty_{\text{loc}}$. It follows that, for any $f \in \mathcal{F} \cap C_c$,

$$w^{-1} f^2 \in \mathcal{F} \subset \mathcal{F}_e. \tag{4.13}$$

By the transience of $(\mathcal{E}, \mathcal{F})$ and (4.6), we obtain

$$\int_M \frac{h}{Gh} f^2 \, d\mu = \int_M h \left(w^{-1} f^2 \right) d\mu = \mathcal{E}(Gh, w^{-1} f^2) = \mathcal{E}(w, w^{-1} f^2).$$

Hence, the proof of (4.11) amounts to verifying that

$$\mathcal{E}(w, w^{-1} f^2) \leq \mathcal{E}(f, f) \quad \text{for all } f \in \mathcal{F} \cap C_c. \tag{4.14}$$

According to [20, Lemma 4.5.4, Theorem 4.5.2] and [20, Theorem 7.2.1], a regular Dirichlet form $\mathcal{E}(u, v)$ admits a Beurling-Deny and LeJan decomposition:

$$\mathcal{E}(u, v) = \mathcal{E}^{(c)}(u, v) + \int_{M \times M} (\tilde{u}(x) - \tilde{u}(y))(\tilde{v}(x) - \tilde{v}(y)) \, dJ(x, y) + \int_M \tilde{u}(x) \tilde{v}(x) \, dk(x), \tag{4.15}$$

for all $u, v \in \mathcal{F}_e$, where $\mathcal{E}^{(c)}$ is a strongly local symmetric form with domain $\mathcal{F}_e$, $\tilde{u}$ and $\tilde{v}$ denote quasi continuous versions of u and v, J is a symmetric positive Radon measure on $M \times M \setminus \text{diag}$ (the jumping measure) and k is a positive Radon measure on M (the killing measure).

Let now w be a quasi continuous version of Gh. Then $w^{-1} f^2$ and $w^{-1} f$ are also quasi continuous. By (4.12), (4.13) and (4.15), we have

$$\begin{aligned} \mathcal{E}(w, w^{-1} f^2) = \mathcal{E}^{(c)}(w, w^{-1} f^2) & \\ + \int_{(M \times M) \setminus \text{diag}} & (w(x) - w(y)) \left(w(x)^{-1} f(x)^2 - w(y)^{-1} f(y)^2 \right) dJ(x, y) \\ + \int_M & w(x) w(x)^{-1} f(x)^2 \, dk(x). \end{aligned} \tag{4.16}$$

By $f \in \mathcal{F} \cap C_c$ and (4.15), we have

$$\mathcal{E}(f, f) = \mathcal{E}^{(c)}(f, f) + \int_{(M \times M) \setminus \text{diag}} (f(x) - f(y))^2 \, dJ(x, y) + \int_M f(x)^2 dk(x). \tag{4.17}$$

In order to prove (4.14), we compare the corresponding terms in the right hand sides of (4.16) and (4.17). Clearly, the third terms in the the right hand sides of (4.16) and (4.17) are equal to each other. Since that both w and w^{-1} are in $\mathcal{F}_{\rm loc} \cap L^\infty_{\rm loc}$, the argument in Lemma 3.2 shows that

$$\mathcal{E}^{(c)}(w, w^{-1}f^2) \le \mathcal{E}^{(c)}(f, f).$$

Finally, in order to compare the middle terms, observe that, for all $x, y \in M$,

$$\begin{aligned}
&(w(x) - w(y))\Big(w(x)^{-1} f(x)^2 - w(y)^{-1} f(y)^2\Big) \\
&\quad = f(x)^2 + f(y)^2 - w(x)w(y)^{-1} f(y)^2 - w(y)w(x)^{-1} f(x)^2 \\
&\quad = \Big(f(x) - f(y)\Big)^2 + 2 f(x) f(y) - w(x)w(y)\Big(w(y)^{-1} f(y)\Big)^2 - w(y)w(x)\Big(w(x)^{-1} f(x)\Big)^2 \\
&\quad = \Big(f(x) - f(y)\Big)^2 + w(x)w(y)\Big[2w(x)^{-1} f(x)w(y)^{-1} f(y) - \Big(w(y)^{-1} f(y)\Big)^2 - \Big(w(x)^{-1} f(x)\Big)^2\Big] \\
&\quad = \Big(f(x) - f(y)\Big)^2 - w(x)w(y)\Big(w(x)^{-1} f(x) - w(y)^{-1} f(y)\Big)^2 \\
&\quad \le \Big(f(x) - f(y)\Big)^2.
\end{aligned}$$

This proves (4.14) and, hence, (4.11). □

Remark 4.7. As we see from the proof, the positivity of the function h was used only in the first part in order to prove that $(\mathcal{E}, \mathcal{F})$ is transient. If it is known a priori that $(\mathcal{E}, \mathcal{F})$ is transient then we can allow h to be non-negative provided all the conditions (i)-(iii) of Definition 4.4 are satisfied.

We conclude this section with the following corollary.

Corollary 4.8. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on (M, d, μ), and $\mathcal{L}$ be its generator. If a positive function $\phi \in \operatorname{dom}(\mathcal{L})$ satisfies $\phi, \phi^{-1} \in L^\infty_{\rm loc}$ and $\int_M \phi\, \mathcal{L}\phi\, d\mu < \infty$, then*

$$\int_M \frac{\mathcal{L}\phi}{\phi} f^2\, d\mu \le \mathcal{E}(f, f) \quad \text{for all } f \in \mathcal{F}. \tag{4.18}$$

Proof. Indeed, applying Theorem 4.5 with $h = \mathcal{L}\phi$ and observing that $\phi = Gh$, we obtain (4.18) from (4.11). □

5 Some "classical" versions of Hardy's inequality

In this section, we apply Theorem 4.5 to obtain various versions of Hardy's inequality on metric measure spaces, which are generalizations of classical/discrete/fractional Hardy's inequality.

5.1 Discrete Hardy's inequality

We show here how Theorem 4.5 yields a discrete Hardy's inequality in $\mathbb{Z}^n$, where $n \in \mathbb{N}$. For any $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$, we set

$$\|k\| = |k_1| + \cdots + |k_n|$$

and define the graph structure in $\mathbb{Z}^n$ as follows: for $k, m \in \mathbb{Z}^n$ we say that k and m are neighbors and write $k \sim m$ if $\|k - m\| = 1$.

Define for all $s \geq 1$ the function

$$\omega(s) = \sum_{i=1}^{\infty} \binom{4i}{2i} \frac{1}{2^{4i-1}(4i-1)} \frac{1}{s^{2i}} = \frac{1}{4s^2} + \frac{5}{64s^4} + \frac{21}{512s^6} + \cdots .$$

Denote

$$\Gamma = \{k \in \mathbb{Z}^n : k_i = 0 \text{ for some } i = 1, ..., n\} .$$

Theorem 5.1. *For any function* $f : \mathbb{Z}^n \to \mathbb{R}$ *such that* $f \in l^2(\mathbb{Z}^n)$ *and* $f|_\Gamma = 0$, *the following discrete Hardy inequality holds:*

$$2n \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \omega(||k||) f(k)^2 \leq \sum_{\{k,m \in \mathbb{Z}^n : m \sim k\}} |f(m) - f(k)|^2. \tag{5.1}$$

Since

$$\omega(l) \geq \frac{1}{4l^2},$$

the inequality (5.1) implies

$$\frac{n}{2} \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \frac{f(k)^2}{||k||^2} \leq \sum_{\{k,m \in \mathbb{Z}^n : m \sim k\}} |f(m) - f(k)|^2. \tag{5.2}$$

If $n = 1$ and a function $f : \mathbb{Z} \to \mathbb{R}$ vanishes for $k \leq 0$, we obtain from (5.1)

$$\sum_{k=1}^{\infty} \omega(k) f(k)^2 \leq \sum_{k=1}^{\infty} (f(k) - f(k-1))^2 . \tag{5.3}$$

This inequality was proved in [39, 40] and shown there to be optimal. Of course, (5.3) implies the classical discrete Hardy inequality

$$\frac{1}{4} \sum_{k=1}^{\infty} \frac{f(k)^2}{k^2} \leq \sum_{k=1}^{\infty} (f(k) - f(k-1))^2 ,$$

where the constant $1/4$ is the best possible (see [36, p. 239]).

Proof of Theorem 5.1. Define the distance on $\mathbb{Z}^n$ by $d(k,m) = ||k-m||$ and let μ be the degree measure, that is, $\mu(k) = 2n$ for all $k \in \mathbb{Z}^n$. The Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $\mathbb{Z}^n$ is given by

$$\mathcal{E}(f,f) = \frac{1}{2} \sum_{\{k.m \in \mathbb{Z}^n : m \sim k\}} |f(m) - f(k)|^2,$$

where $\mathcal{F} = l^2(\mathbb{Z}^n)$. The discrete Laplacian Δ is defined on all functions $f : \mathbb{Z}^n \to \mathbb{R}$ by

$$\Delta f(k) = \frac{1}{2n} \sum_{m \sim k} (f(m) - f(k)), \quad k \in \mathbb{Z}^n.$$

It is known that the generator $\mathcal{L}$ of $(\mathcal{E}, \mathcal{F})$ coincides with $-\Delta|_{l^2}$ (see [38]).

Consider the subset

$$\Omega = \mathbb{Z}^n \setminus \Gamma$$

and set

$$\mathcal{F}(\Omega) = \{f \in \mathcal{F} : f|_{\Gamma} = 0\}$$

so that $(\mathcal{E}, \mathcal{F}(\Omega))$ be the restriction of $(\mathcal{E}, \mathcal{F})$ to Ω. For any $N \in \mathbb{N}$, consider the following function on $\mathbb{Z}^n$:

$$\phi_N(k) = \begin{cases} \|k\|^{\frac{1}{2}} = (|k_1| + \cdots + |k_n|)^{\frac{1}{2}} & \text{if } 0 \le \|k\| \le N \\ N^{\frac{1}{2}} & \text{if } \|k\| > N. \end{cases}$$

Clearly, if $\|k\| > N$ then

$$\Delta\phi_N(k) = 0.$$

For any $k \in \Omega$ with $0 < \|k\| \le N-1$, there exist n vertices $m \sim k$ satisfying $\phi_N(m) = (\|k\|+1)^{\frac{1}{2}}$, and another n vertices $m \sim k$ satisfying $\phi_N(m) = (\|k\| - 1)^{\frac{1}{2}}$, which implies that

$$\begin{aligned} -\frac{\Delta\phi_N(k)}{\phi_N(k)} &= \frac{1}{2n} \sum_{m\sim k} \frac{\phi_N(k) - \phi_N(m)}{\phi_N(k)} \\ &= \frac{2\|k\|^{\frac{1}{2}} - (\|k\| + 1)^{\frac{1}{2}} - (\|k\| - 1)^{\frac{1}{2}}}{2\|k\|^{\frac{1}{2}}} \\ &= \frac{1}{2}\left(2 - \left(1 + \frac{1}{\|k\|}\right)^{\frac{1}{2}} - \left(1 - \frac{1}{\|k\|}\right)^{\frac{1}{2}}\right). \end{aligned}$$

Using the Taylor expansions of the functions $t \mapsto (1+t)^{\frac{1}{2}}$ and $t \mapsto (1-t)^{\frac{1}{2}}$ that converge in $[-1, 1]$, we obtain

$$\begin{aligned} &2 - (1+t)^{\frac{1}{2}} - (1-t)^{\frac{1}{2}} \\ &\quad = 2 - \sum_{j=0}^{\infty} \frac{\frac{1}{2}(\frac{1}{2} - 1)\cdots(\frac{1}{2} - j + 1)}{j!} t^j - \sum_{j=0}^{\infty} \frac{\frac{1}{2}(\frac{1}{2} - 1)\cdots(\frac{1}{2} - j + 1)}{j!} (-t)^j \\ &\quad = -2 \sum_{i=1}^{\infty} \frac{\frac{1}{2}(\frac{1}{2} - 1)\cdots(\frac{1}{2} - 2i + 1)}{(2i)!} t^{2i} \\ &\quad = 2 \sum_{i=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (4i - 3)}{2^{2i}(2i)!} t^{2i} \\ &\quad = \sum_{i=1}^{\infty} \binom{4i}{2i} \frac{t^{2i}}{2^{4i-1}(4i - 1)} = \omega\left(\frac{1}{t}\right). \end{aligned}$$

It follows that

$$-\frac{\Delta\phi_N(k)}{\phi_N(k)} = \frac{1}{2}\omega(\|k\|) \quad \text{for all } k \in \Omega \text{ with } 0 < \|k\| \le N - 1.$$

If $k \in \Omega$ and $\|k\| = N$, then there exist n vertices $m \sim k$ satisfying $\phi_N(m) = (\|k\| - 1)^{\frac{1}{2}} = (N - 1)^{\frac{1}{2}}$, and another n vertices $m \sim k$ satisfying $\phi_N(m) = N^{\frac{1}{2}}$, which implies that

$$-\frac{\Delta\phi_N(k)}{\phi_N(k)} = \frac{1}{2}\left(\frac{N^{\frac{1}{2}} - (N-1)^{\frac{1}{2}}}{N^{\frac{1}{2}}}\right).$$

Hence, we obtain that, for all $k \in \Omega$,

$$-\frac{\Delta\phi_N(k)}{\phi_N(k)} = \eta_N(k) := \frac{1}{2}\begin{cases} \omega(\|k\|) & \text{if } 0 < \|k\| \leq N-1 \\ \frac{N^{\frac{1}{2}}-(N-1)^{\frac{1}{2}}}{N^{\frac{1}{2}}} & \text{if } \|k\| = N \\ 0 & \text{if } \|k\| \geq N. \end{cases} \tag{5.4}$$

Set

$$h_N = \phi_N \eta_N$$

so that

$$-\Delta\phi_N = h_N \text{ in } \Omega. \tag{5.5}$$

Note that $\phi_N \geq 0$ and $h_N \geq 0$ in Ω. In particular, the function ϕ_N is non-negative and superharmonic in Ω (let us mention that outside Ω it may happen that $-\Delta\phi_N < 0$, for example, $-\Delta\phi_N(0) < 0$). Since ϕ_N is non-constant, it follows that $(\mathcal{E}, \mathcal{F}(\Omega))$ is transient. In particular, the Green function G^Ω exists. It follows from (5.5) by the comparison principle that

$$\phi_N \geq G^\Omega h_N \text{ in } \Omega. \tag{5.6}$$

It is easy to see that the function $h = h_N$ satisfies in Ω all the conditions (i)-(iii) of Definition 4.4. Indeed, (i) holds by (5.6), (ii) holds because $G^\Omega h_N > 0$ by the strong minimum principle for superharmonic functions on graphs, and (iii) holds because h_N has a finite support.

By Remark 4.7, we can apply Theorem 4.5 with $h = h_N$ and conclude that, for all $f \in \mathcal{F}(\Omega)$,

$$\int_\Omega \frac{h_N}{G^\Omega h_N} f^2\, d\mu \leq \mathcal{E}(f,f). \tag{5.7}$$

The left-hand side here can be estimates by (5.6) and (5.4) as follows:

$$\int_\Omega \frac{h_N}{G^\Omega h_N} f^2\, d\mu \geq \int_\Omega \frac{h_N}{\phi_N} f^2\, d\mu = \int_\Omega \eta_N f^2\, d\mu \geq \sum_{0<\|k\|<N} \frac{1}{2}\omega(\|k\|)\, f(k)^2 2n.$$

Combining with (5.7) and letting $N \to \infty$, we obtain (5.1). □

5.2 Hardy's inequality and distance function

In this subsection we obtain an explicit form of Hardy's inequality under the hypotheses **(VD)**, **(RVD)** and $(\mathbf{G})_\beta$. For that, we construct explicitly (μ, G)-admissible functions that can be used in Theorem 4.5. The main result is stated in Theorem 5.6 below.

Let us begin with the following Selberg-type integral formula on (M, d, μ).

Lemma 5.2. *Assume that (M, d, μ) satisfies* **(VD)** *and* **(RVD)** *with lower volume dimension α_-. If β and ε are positive reals such that $\beta + \varepsilon < \alpha_-$, then the following estimate*

$$\int_M \frac{d(x,z)^\beta}{V(x,z)} \frac{d(z,y)^\varepsilon}{V(z,y)}\, d\mu(z) \simeq \frac{d(x,y)^{\beta+\varepsilon}}{V(x,y)} \tag{5.8}$$

holds uniformly for all distinct $x, y \in M$.

Proof. By condition (**RVD**), there exists a large constant $K > 2$ such that for all $x \in M$ and $R > 0$,

$$\frac{V(x, KR)}{V(x, R)} \geq 2. \tag{5.9}$$

Set $r = d(x, y)$. In order to prove the lower bound in (5.8), observe first that

$$d(x,z) < \frac{r}{2} \Rightarrow d(y,z) \simeq r,$$

whence

$$\begin{aligned}\int_M \frac{d(x,z)^\beta}{V(x,z)} \frac{d(z,y)^\varepsilon}{V(z,y)} d\mu(z) &\geq \int_{B(x,r/2)\setminus B(x,r/2K)} \frac{d(x,z)^\beta}{V(x,z)} \frac{d(y,z)^\varepsilon}{V(y,z)} d\mu(z) \\ &\simeq \frac{r^{\beta+\varepsilon}}{V(x,r)^2} \left(V(x, r/2) - V(x, r/2K)\right).\end{aligned}$$

Using further (5.9), we obtain

$$\int_M \frac{d(x,z)^\beta}{V(x,z)} \frac{d(z,y)^\varepsilon}{V(z,y)} d\mu(z) \gtrsim \frac{r^{\beta+\varepsilon}}{V(x,r)^2} \frac{1}{2} V(x, r/2) \simeq \frac{r^{\beta+\varepsilon}}{V(x,r)}.$$

Before we prove the upper bound in (5.8), observe that, by (4.10), for any $\sigma > 0$,

$$\int_{B(x,R)} \frac{d(x,z)^\sigma}{V(x,z)} d\mu(z) \lesssim R^\sigma. \tag{5.10}$$

Let us prove that, for any $0 < \theta < \alpha_-$,

$$\int_{B(x,R)^c} \frac{d(x,z)^\theta}{V(x,z)^2} d\mu(z) \lesssim \frac{R^\theta}{V(x,R)} \tag{5.11}$$

uniformly in $x \in M$ and $R > 0$. Indeed, applying (2.2) and $\theta < \alpha_-$, we obtain

$$\begin{aligned}\int_{B(x,R)^c} \frac{d(x,z)^\theta}{V(x,z)^2} d\mu(z) &\leq \sum_{j=0}^{\infty} \int_{B\left(x,2^{j+1}R\right)\setminus B\left(x,2^jR\right)} \frac{d(x,z)^\theta}{V(x,z)^2} d\mu(z) \\ &\lesssim \sum_{j=0}^{\infty} \frac{(2^{j+1}R)^\theta}{V(x,2^jR)} \\ &\lesssim \frac{R^\theta}{V(x,R)} \sum_{j=0}^{\infty} 2^{j\theta} \frac{V(x,R)}{V(x,2^jR)} \\ &\lesssim \frac{R^\theta}{V(x,R)} \sum_{j=0}^{\infty} 2^{j(\theta-\alpha_-)} \\ &\simeq \frac{R^\theta}{V(x,R)},\end{aligned}$$

which proves (5.11).

Now, we use (5.10) and (5.11) to verify the upper bound in (5.8). Using (5.2) and (5.10), we obtain

$$\int_{B(x,r/2)} \frac{d(x,z)^\beta}{V(x,z)} \frac{d(z,y)^\varepsilon}{V(z,y)} d\mu(z) \simeq \frac{r^\varepsilon}{V(x,r)} \int_{B(x,r/2)} \frac{d(x,z)^\beta}{V(x,z)} d\mu(z) \lesssim \frac{r^{\beta+\varepsilon}}{V(x,r)}. \tag{5.12}$$

Similarly, if $r/2 \le d(z,x) < 2r$, then

$$d(z,y) \le d(z,x) + d(x,y) < 3r \quad \text{and} \quad V(x,z) \simeq V(x,r),$$

which, together with (5.10) implies

$$\int_{B(x,2r)\setminus B(x,r/2)} \frac{d(x,z)^\beta}{V(x,z)} \frac{d(z,y)^\varepsilon}{V(z,y)}\, d\mu(z) \lesssim \frac{r^\beta}{V(x,r)} \int_{B(y,3r)} \frac{d(z,y)^\varepsilon}{V(z,y)}\, d\mu(z) \lesssim \frac{r^{\beta+\varepsilon}}{V(x,r)}. \tag{5.13}$$

For any $z \in M$ satisfying $d(z,x) \ge 2r$, we have by (**VD**) that

$$d(z,y) \simeq d(x,z) \ \text{ and } \ V(z,y) \simeq V(x,z),$$

which yields by (5.11) and $\beta + \varepsilon < \alpha_-$ that

$$\int_{B(x,2r)^c} \frac{d(x,z)^\beta}{V(x,z)} \frac{d(z,y)^\varepsilon}{V(z,y)}\, d\mu(z) \simeq \int_{B(x,2r)^c} \frac{d(x,z)^{\beta+\varepsilon}}{V(x,z)^2}\, d\mu(z) \lesssim \frac{r^{\beta+\varepsilon}}{V(x,r)}. \tag{5.14}$$

Adding up (5.12), (5.13) and (5.14), we conclude that

$$\int_M \frac{d(x,z)^\beta}{V(x,z)} \frac{d(z,y)^\varepsilon}{V(z,y)}\, d\mu(z) \lesssim \frac{r^{\beta+\varepsilon}}{V(x,r)},$$

which finishes the proof of (5.8). □

Remark 5.3. The Selberg integral formula [51, p. 118, (6)] in $\mathbb{R}^n$ says that, for all distinct $x, y \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} |x-z|^{-a_1} |z-y|^{-a_2}\, dz = C_{n,a_1,a_2} |x-y|^{n-a_1-a_2}, \tag{5.15}$$

for any are positive reals a_1, a_2 satisfying $a_1 + a_2 > n$, where

$$C_{n,a_1,a_2} = \pi^{\frac{n}{2}} \frac{\Gamma(\frac{n-a_1}{2})\Gamma(\frac{n-a_2}{2})\Gamma(\frac{a_1+a_2-n}{2})}{\Gamma(\frac{a_1}{2})\Gamma(\frac{a_2}{2})\Gamma(\frac{2n-a_1-a_2}{2})}.$$

The inequalities (5.8) can be regarded as a mild generalization of the identity (5.15).

We use Lemma 5.2 in order to construct a function h that is admissible in the sense of Definition 4.4.

Lemma 5.4. *Assume that* (M, d, μ) *satisfies* (**VD**) *and* (**RVD**) *with lower volume dimension* α_-. *Let* β *and* ε *be positive reals such that* $\beta + \varepsilon < \alpha_-$ *and let the Green function* $G(x,y)$ *satisfy* $(\mathbf{G})_\beta$. *Fix an arbitrary point* $x_o \in M$, *a real* $\rho > 0$ *and define*

$$h(x) = \begin{cases} \frac{\rho^\varepsilon}{V(x_o,\rho)} & \text{if } d(x_o,x) < \rho \\ \frac{d(x_o,x)^\varepsilon}{V(x_o,x)} & \text{if } d(x_o,x) \ge \rho. \end{cases} \tag{5.16}$$

Then, the Green potential of h satisfies

$$\inf_{B(x_o,R)} Gh > 0 \qquad \forall \ R > 0 \tag{5.17}$$

and

$$Gh(x) \le C \begin{cases} \frac{\rho^{\beta+\varepsilon}}{V(x_o,\rho)} & \text{if } d(x_o,x) < 2\rho \\ \frac{d(x_o,x)^{\beta+\varepsilon}}{V(x_o,x)} & \text{if } d(x_o,x) \ge 2\rho \end{cases} \tag{5.18}$$

where C is a positive constant independent of ρ, x, x_o.

Proof. The inequality (5.17) follows from $\inf_{B(x_o,R)} h > 0$ and

$$\inf_{x,y\in B(x_o,R)} G(x,y) \gtrsim \inf_{x,y\in B(x_o,R)} \frac{d(x,y)^\beta}{V(x,y)} \gtrsim \frac{R^\beta}{V(x_o,R)} > 0. \tag{5.19}$$

Indeed, setting $r = d(x,y)$, we obtain

$$\begin{aligned}\frac{R^\beta}{V(x_o,R)} \Big/ \frac{r^\beta}{V(x,r)} &= \frac{V(x,r)}{V(x_o,R)}\left(\frac{R}{r}\right)^\beta \lesssim \frac{V(x,r)}{V(x,R)}\left(\frac{R}{r}\right)^\beta \\ &\lesssim \left(\frac{r}{R}\right)^{\alpha_-}\left(\frac{R}{r}\right)^\beta = \left(\frac{r}{R}\right)^{\alpha_--\beta} \lesssim 1,\end{aligned}$$

which proves (5.19) and, hence, (5.17).

In order to prove (5.18), we apply $(\mathbf{G})_\beta$, (5.16) and split the integral in the definition of Gh into two parts as follows:

$$\begin{aligned} Gh(x) &\simeq \int_M \frac{d(x,y)^\beta}{V(x,y)} h(y)\, d\mu(y) \\ &\simeq \int_{B(x_o,\rho)} \frac{d(x,y)^\beta}{V(x,y)} \frac{\rho^\varepsilon}{V(x_o,\rho)}\, d\mu(y) + \int_{B(x_o,\rho)^c} \frac{d(x,y)^\beta}{V(x,y)} \frac{d(x_o,y)^\varepsilon}{V(x_o,y)}\, d\mu(y) \\ &=: I_1 + I_2. \end{aligned} \tag{5.20}$$

Set $r = d(x_o,x)$. We estimate I_1 and I_2 in (5.20) by considering two cases: $r \ge 2\rho$ and $r < 2\rho$.

Case $r \ge 2\rho$. If $y \in B(x_o,\rho)$ then

$$d(x,y) \le d(x_o,x) + d(x_o,y) < r + \rho < 2r$$

and

$$d(x,y) \ge d(x_o,x) - d(x_o,y) > r - \rho > r/2$$

so that

$$d(x,y) \simeq r \text{ and } V(x,y) \simeq V(x_o,r).$$

It follows that

$$I_1 \simeq \int_{B(x_o,\rho)} \frac{r^\beta}{V(x_o,r)} \frac{\rho^\varepsilon}{V(x_o,\rho)} d\mu(y) \simeq \frac{r^\beta \rho^\varepsilon}{V(x_o,r)} \lesssim \frac{r^{\beta+\varepsilon}}{V(x_o,r)}.$$

By Lemma 5.2, we have

$$I_2 \lesssim \frac{r^{\beta+\varepsilon}}{V(x_o,r)}.$$

Combining the last two estimates and (5.20), we obtain

$$Gh(x) \lesssim \frac{r^{\beta+\varepsilon}}{V(x_o,r)} \text{ provided } r \ge 2\rho.$$

Case $r < 2\rho$. In this case, applying (5.10) gives

$$I_1 \simeq \frac{\rho^\varepsilon}{V(x_o,\rho)} \int_{B(x_o,\rho)} \frac{d(x,y)^\beta}{V(x,y)}\, d\mu(y) \lesssim \frac{\rho^{\beta+\varepsilon}}{V(x_o,\rho)}.$$

By (**VD**) and (5.10) we obtain

$$\int_{B(x_o,4\rho)\setminus B(x_o,\rho)} \frac{d(x,y)^\beta}{V(x,y)} \frac{d(x_o,y)^\varepsilon}{V(x_o,y)}\, d\mu(y) \simeq \frac{\rho^\varepsilon}{V(x_o,\rho)} \int_{B(x_o,4\rho)\setminus B(x_o,\rho)} \frac{d(x,y)^\beta}{V(x,y)}\, d\mu(y)$$
$$\lesssim \frac{\rho^{\beta+\varepsilon}}{V(x_o,\rho)}. \tag{5.21}$$

Finally, if $y \in B(x_o, 4\rho)^c$ then $r < \frac{1}{2} d(x_o, y)$ and

$$d(x,y) \le d(x_o,y) + d(x_o,x) < 2d(x_o,y)$$

and

$$d(x,y) \ge d(x_o,y) - d(x_o,x) > \frac{1}{2} d(x_o,y),$$

whence

$$V(x,y) \simeq V(x_o,y).$$

Using also (5.11), we obtain

$$\int_{B(x_o,4\rho)^c} \frac{d(x,y)^\beta}{V(x,y)} \frac{d(x_o,y)^\varepsilon}{V(x_o,y)}\, d\mu(y) \simeq \int_{B(x_o,4\rho)^c} \frac{d(x_o,y)^{\beta+\varepsilon}}{V(x_o,y)^2}\, d\mu(y) \lesssim \frac{\rho^{\beta+\varepsilon}}{V(x_o,\rho)}. \tag{5.22}$$

Combining (5.21) and (5.22) yields

$$I_2 \lesssim \frac{\rho^{\beta+\varepsilon}}{V(x_o,\rho)}.$$

Substituting the estimates of I_1 and I_2 into (5.20), we obtain

$$Gh(x) \lesssim \frac{\rho^{\beta+\varepsilon}}{V(x_o,\rho)} \text{ provided } r < 2\rho,$$

which finishes the proof of (5.18). □

Corollary 5.5. *Under the hypotheses of Lemma 5.4, assume that $\beta + 2\varepsilon < \alpha_-$. Then the function h in (5.16) is (μ, G)-admissible.*

Proof. The hypotheses (5.17) and (5.18) imply that h satisfies the conditions (i) and (ii) of Definition 4.4. Let us verify the remaining condition (iii). By (5.18), (5.10), (5.11) and $\beta + 2\varepsilon < \alpha_-$, we obtain

$$\int_M h\, Gh\, d\mu = \left(\int_{B(x_o,\rho)} + \int_{B(x_o,2\rho)\setminus B(x_o,\rho)} + \int_{B(x_o,2\rho)^c} \right) h\, Gh\, d\mu$$
$$\lesssim \int_{B(x_o,\rho)} \frac{\rho^\varepsilon}{V(x_o,\rho)} \frac{\rho^{\beta+\varepsilon}}{V(x_o,\rho)}\, d\mu(x)$$
$$+ \int_{B(x_o,2\rho)\setminus B(x_o,\rho)} \frac{d(x_o,x)^\varepsilon}{V(x_o,x)} \frac{\rho^{\beta+\varepsilon}}{V(x_o,\rho)}\, d\mu(x)$$
$$+ \int_{B(x_o,2\rho)^c} \frac{d(x_o,x)^\varepsilon}{V(x_o,x)} \frac{d(x_o,x)^{\beta+\varepsilon}}{V(x_o,x)}\, d\mu(x)$$
$$\lesssim \frac{\rho^{\beta+2\varepsilon}}{V(x_o,\rho)} < \infty,$$

which finishes the proof. □

Applying Theorem 4.5 with the admissible function h as in (5.16), we derive Hardy's inequality (1.5).

Theorem 5.6. *Assume that (M, d, μ) satisfies* (**VD**) *and* (**RVD**) *with lower volume dimension α_-. Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on M that satisfies $(\mathbf{G})_\beta$ with $0 < \beta < \alpha_-$. Then, for all $x_o \in M$ and $f \in \mathcal{F}$,*

$$\int_M \frac{f(x)^2}{d(x_o, x)^\beta}\, d\mu(x) \lesssim \mathcal{E}(f, f).$$

Proof. Choose a number ε such that $0 < 2\varepsilon < \alpha_- - \beta$. For this ε and $\rho \in (0, \infty)$, we define the function h as in (5.16) and adopt all other notation from Lemma 5.4. By Corollary 5.5, h is (μ, G)-admissible. By Theorem 4.5 we conclude that, for all $f \in \mathcal{F}$,

$$\int_M f^2 \frac{h}{Gh}\, d\mu \le \mathcal{E}(f, f). \tag{5.23}$$

Applying (5.16) and (5.18), we obtain

$$\int_M f^2 \frac{h}{Gh}\, d\mu \ge \int_{B(x_o, 2\rho)^c} f^2 \frac{h}{Gh}\, d\mu \gtrsim \int_{B(x_o, 2\rho)^c} \frac{f(x)^2}{d(x_o, x)^\beta}\, d\mu(x),$$

where the implicit constant in the last step is independent of x_o and ρ. Substituting into (5.23), we obtain

$$\int_{B(x_o, 2\rho)^c} \frac{f(x)^2}{d(x_o, x)^\beta}\, d\mu(x) \lesssim \mathcal{E}(f, f),$$

where the implicit constant is independent of x_o and ρ. Letting $\rho \to 0$ yields

$$\int_M \frac{f(x)^2}{d(x_o, x)^\beta}\, d\mu(x) \lesssim \mathcal{E}(f, f),$$

which concludes the proof. □

Remark 5.7. If $(\mathcal{E}, \mathcal{F})$ is strongly local then the proof of Theorem 5.6 simplifies as in this case we apply Theorem 3.1 instead of Theorem 4.5 and, hence, do not need Corollary 5.5.

As an example of application, we apply Theorem 5.6 to deduce the following estimate of $\lambda_{\min}(\Omega)$.

Corollary 5.8. *Under the assumptions of Theorem 5.6, for any nonempty open bounded $\Omega \subset M$, we have*

$$\lambda_{\min}(\Omega) \gtrsim \left(\operatorname{diam}(\Omega)\right)^{-\beta}. \tag{5.24}$$

Proof. Set $D = \operatorname{diam} \Omega$, fix a point $x_o \in \Omega$ and let $u \in \mathcal{F} \cap C_c(\Omega)$. We have $\operatorname{supp} u \subset \Omega$ and

$$\|u\|_{L^2}^2 = \int_{B(x_o, D)} |u(x)|^2\, d\mu(x) \le \int_M \left(\frac{D}{d(x, x_o)}\right)^\beta |u(x)|^2\, d\mu(x).$$

By Theorem 5.6, we have

$$\int_M \frac{u(x)^2}{d(x_o, x)^\beta}\, d\mu(x) \lesssim \mathcal{E}(u, u).$$

Combining the last two inequalities yields

$$\|u\|_{L^2}^2 \lesssim D^\beta \mathcal{E}(u, u),$$

which implies (5.24) by (2.3). □

5.3 Subordinated Green function and fractional Hardy's inequality

For any $\delta \in (0,1)$ the operator $\mathcal{L}^\delta$ generates the *subordinated* heat semigroup $\left\{e^{-t\mathcal{L}^\delta}\right\}_{t\geq 0}$ and the associated Dirichlet form $(\mathcal{E}^{(\delta)}, \mathcal{F}^{(\delta)})$. It is well-known that

$$e^{-t\mathcal{L}^\delta} = \int_0^\infty \eta_t^{(\delta)}(s) e^{-s\mathcal{L}}\, ds \quad \text{for all } t \geq 0,$$

where $\left\{\eta_t^{(\delta)}(s)\right\}_{t\geq 0}$ is a family of non-negative continuous functions on $[0,\infty)$ that is called a *subordinator* (see [59], [22, Section 5.4]). Moreover, if $(\mathcal{E},\mathcal{F})$ is regular, then $(\mathcal{E}^{(\delta)}, \mathcal{F}^{(\delta)})$ is also regular (see [47, Proposition 3.1]). If $\{e^{-t\mathcal{L}}\}_{t\geq 0}$ has the heat kernel $p_t(x,y)$ then $\{e^{-t\mathcal{L}^\delta}\}_{t\geq 0}$ has the heat kernel

$$p_t^{(\delta)}(x,y) = \int_0^\infty \eta_t^{(\delta)}(s) p_s(x,y)\, ds \quad \text{for all } x,y \in M.$$

Using the identity

$$\int_0^\infty \eta_t^{(\delta)}(s)\, dt = \frac{s^{\delta-1}}{\Gamma(s)} \quad \text{for all } s > 0 \tag{5.25}$$

(see [47, (6)]), we obtain the following expression for the subordinated Green function $G^{(\delta)}$:

$$G^{(\delta)}(x,y) = \int_0^\infty p_t^{(\delta)}(x,y)\, dt = \int_0^\infty \int_0^\infty \eta_t^{(\delta)}(s) p_s(x,y)\, ds\, dt = c_\delta \int_0^\infty s^{\delta-1} p_s(x,y)\, ds. \tag{5.26}$$

Theorem 5.9. *Assume that (M,d,μ) satisfies* (**VD**) *and* (**RVD**) *with lower volume dimension α_-. Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form on M. Assume that the heat kernel of $(\mathcal{E},\mathcal{F})$ exists and satisfies (2.6) and (2.7) for some $\beta \in (0,\alpha_-)$. Then, for any $\delta \in (0,1)$, the subordinated Green kernel $G^{(\delta)}$ satisfies*

$$G^{(\delta)}(x,y) \simeq \frac{d(x,y)^{\delta\beta}}{V(x,y)} \quad \text{for distinct } x,y \in M. \tag{$\mathbf{G}^{(\delta)})_\beta$$

Consequently, there exists a constant $C > 0$ such that, for all $f \in \mathcal{F}^{(\delta)}$,

$$\int_M \frac{f(x)^2}{d(x_o,x)^{\beta\delta}}\, d\mu(x) \leq C\mathcal{E}^{(\delta)}(f,f). \tag{5.27}$$

Proof. The inequality (5.27) follows directly from Theorem 5.6 and $(\mathbf{G}^{(\delta)})_\beta$. Let us verify that the subordinated Green kernel $G^{(\delta)}$ satisfies $(\mathbf{G}^{(\delta)})_\beta$. By (5.26), (2.7) and (**VD**), we obtain the lower bound of $G^{(\delta)}$:

$$G^{(\delta)}(x,y) \geq c_\delta \int_{d(x,y)^\beta}^{2d(x,y)^\beta} s^{\delta-1} p_s(x,y)\, ds \gtrsim \int_{d(x,y)^\beta}^{2d(x,y)^\beta} \frac{s^{\delta-1}}{V(x,s^{1/\beta})}\, ds \simeq \frac{d(x,y)^{\delta\beta}}{V(x,y)}.$$

Recall that, by Lemma 2.4, (2.6) and (2.7) imply $(\mathbf{G})_\beta$. Applying (5.26), (2.6) and $(\mathbf{G})_\beta$, we obtain the upper bound of $G^{(\delta)}$:

$$G^{(\delta)}(x,y) = c_\delta \left(\int_0^{d(x,y)^\beta} + \int_{d(x,y)^\beta}^\infty\right) s^{\delta-1} p_s(x,y)\, ds$$

$$\lesssim \int_0^{d(x,y)^\beta} \frac{s^{\delta-1}}{V(x,y)}\,ds + d(x,y)^{\beta(\delta-1)} \int_{d(x,y)^\beta}^\infty p_s(x,y)\,ds$$
$$\lesssim \frac{d(x,y)^{\delta\beta}}{V(x,y)} + d(x,y)^{\beta(\delta-1)} G(x,y)$$
$$\simeq \frac{d(x,y)^{\delta\beta}}{V(x,y)},$$

which finishes the proof. □

In $\mathbb{R}^n$ the following fractional version of Hardy's inequality is known:

$$c_{n,\delta} \int_{\mathbb{R}^n} \frac{f(x)^2}{|x|^{2\delta}}\,dx \le \int_{\mathbb{R}^n} \left|(-\Delta)^{\frac{\delta}{2}} f(x)\right|^2 dx \quad \text{for all } f \in C_0^\infty(\mathbb{R}^n), \tag{5.28}$$

where the constant $c_{n,d\delta} := \left(\frac{2^\delta \Gamma(\frac{n+2\delta}{4})}{\Gamma(\frac{n-2\delta}{4})}\right)^2$ is best possible (see [5, p. 1873, Corollary 1]). Consider in $\mathbb{R}^n$ ($n \ge 3$) the Dirichlet form $(\mathcal{E}, \mathcal{F})$ where

$$\mathcal{E}(f,f) = \int_{\mathbb{R}^n} |\nabla f|^2\,dx \tag{5.29}$$

and

$$f \in \mathcal{F} = W^{1,2} = \{f \in L^2(\mathbb{R}^n) : \nabla f \in L^2(\mathbb{R}^n)\}. \tag{5.30}$$

The generator of $(\mathcal{E}, \mathcal{F})$ is the Laplacian $-\Delta = -\sum_{j=1}^n \partial_{x_j}^2$, the heat kernel $\{p_t\}_{t>0}$ of the heat semigroup $\{e^{t\Delta}\}_{t\ge 0}$ is the Gauss-Weierstrass function

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right), \tag{5.31}$$

and the Green function is given by

$$G(x,y) = \int_0^\infty p_t(x,y)\,dt = \frac{\Gamma(\frac{n-2}{2})}{4\pi^{n/2}} |x-y|^{2-n}. \tag{5.32}$$

For the subordinated Dirichlet form $(\mathcal{E}^{(\delta)}, \mathcal{F}^{(\delta)})$ we have

$$\mathcal{F}^{(\delta)} = \left\{ f \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+2\delta}}\,dx\,dy < \infty \right\}$$

and

$$\mathcal{E}^{(\delta)}(f,f) = \left((-\Delta)^\delta f, f\right) = \int_{\mathbb{R}^n} \left|(-\Delta)^{\frac{\delta}{2}} f(x)\right|^2 dx \quad \text{for } f \in \mathcal{F}^{(\delta)}.$$

(see [22, Theorem 5.2]). Hence, by Theorem 5.9 with $\beta = 2$ we obtain (5.28) with *some* constant $c_{n,\delta} > 0$.

Let us show how Theorem 4.5 yields (5.28) with the sharp constant. An exact computation shows that

$$G^{(\delta)}(x,y) = \frac{\Gamma(\frac{n-2\delta}{2})}{4^\delta \pi^{n/2} \Gamma(\delta)} |x-y|^{2\delta-n}.$$

(see [51, p. 117]). By Theorem 4.5, we have the Hardy inequality (4.11), where we choose the admissible function h to be

$$h_r(x) = \begin{cases} r^{\epsilon-n}, & |x| \le r \\ |x|^{\epsilon-n}, & |x| > r, \end{cases}$$

where $r > 0$ and $0 < \epsilon < n - 2\delta$. Now let in (4.11) $r \to 0$. By the Selberg integral formula (see (5.15) or [51, p. 118, (6)]), we obtain

$$\lim_{r\to 0} \frac{h_r(x)}{G^{(\delta)}h_r(x)} = \frac{|x|^{\epsilon-n}}{\frac{\Gamma(\frac{n-2\delta}{2})}{4^\delta \pi^{n/2}\Gamma(\delta)} \int_{\mathbb{R}^n} |x-y|^{2\delta-n}|y|^{\epsilon-n}\,dy} = \frac{2^{2\delta}\Gamma(\frac{2\delta+\epsilon}{2})\Gamma(\frac{n-\epsilon}{2})}{\Gamma(\frac{\epsilon}{2})\Gamma(\frac{n-2\delta-\epsilon}{2})} \frac{1}{|x|^{2\delta}}.$$

Taking here $\epsilon = \frac{n-2\delta}{2}$, we obtain

$$\lim_{r\to 0} \frac{h_r(x)}{G^{(\delta)}h_r(x)} = \left(\frac{2^\delta \Gamma(\frac{n+2\delta}{4})}{\Gamma(\frac{n-2\delta}{4})}\right)^2 \frac{1}{|x|^{2\delta}} = \frac{c_{n,\delta}}{|x|^{2\delta}},$$

which implies (5.28).

6 Green functions and heat kernels

The main goal of this section is to show the equivalence between the Green function estimate $(\mathbf{G})_\beta$ and the upper/lower bound of the heat kernel. This equivalence will be used in Section 7 in order to obtain a weighted Hardy inequality.

The following theorem is the main result of this section.

Theorem 6.1. *Assume that $(\mathcal{E}, \mathcal{F})$ is a strongly local regular Dirichlet form on the metric measure space (M, d, μ) that satisfies* (**VD**) *and* (**RVD**) *with lower volume dimension α_-. Then, for any $0 < \beta < \alpha_-$, the following two statements are equivalent:*

(i) the Green function $G(x, y)$ exists, is jointly continuous off-diagonal, and satisfies $(\mathbf{G})_\beta$;

(ii) the heat kernel $p_t(x, y)$ exists, is Hölder continuous in $x, y \in M$, and satisfies the following upper bound estimate

$$p_t(x,y) \le \frac{C}{V(x, t^{1/\beta})} \exp\left\{-c\left(\frac{d(x,y)}{t^{1/\beta}}\right)^{\frac{\beta}{\beta-1}}\right\} \qquad (\mathbf{UE})_\beta$$

as well as the near-diagonal lower bound estimate

$$p_t(x,y) \ge \frac{C^{-1}}{V(x, t^{1/\beta})} \quad \text{when } d(x,y) < \epsilon t^{1/\beta} \qquad (\mathbf{NLE})_\beta$$

for all $x, y \in M$ and all $t \in (0, \infty)$, where C and c, ϵ are positive constants.

Combining Theorems 6.1 and 5.9, we have the following fractional version of Hardy's inequality for strongly local Dirichlet forms.

Corollary 6.2. *Assume that (M, d, μ) satisfies* (**VD**) *and* (**RVD**) *with lower volume dimension α_-. Let $(\mathcal{E}, \mathcal{F})$ be a strongly local regular Dirichlet form on M and satisfies $(\mathbf{G})_\beta$ for some $\beta \in (0, \alpha_-)$. Then, given any $\delta \in (0, 1)$, the subordinated Green kernel $G^{(\delta)}$ satisfies $(\mathbf{G}^{(\delta)})_\beta$. Moreover, there exists a constant $C > 0$ such that for all $f \in \mathcal{F}^{(\delta)}$,*

$$\int_M \frac{f(x)^2}{d(x_o, x)^{\beta\delta}}\, d\mu(x) \le C\mathcal{E}^{(\delta)}(f, f).$$

6.1 Overview of the proof of Theorem 6.1

The detailed proof of Theorem 6.1 is presented in the subsections below. Here we give an overview of the proof. In Section 6.2 we prove the implication (ii) $\Rightarrow$ (i). The estimates and the continuity of the Green functions follow from similar properties of the heat kernel upon integration in time.

The proof of the implication (i) $\Rightarrow$ (ii) is much more involved. For that we need the following definitions.

Definition 6.3. Let $\Omega \subset M$ be an open subset. A function $u \in \mathcal{F}$ is said to be *harmonic* in Ω if

$$\mathcal{E}(u,\phi) = 0 \quad \text{for all } \phi \in \mathcal{F}(\Omega).$$

A function $u \in \mathcal{F}$ is said to be *superharmonic* (resp. *subharmonic*) in Ω if

$$\mathcal{E}(u,\phi) \geq 0 \quad (\text{ resp. } \mathcal{E}(u,\phi) \leq 0) \text{ for all } 0 \leq \phi \in \mathcal{F}(\Omega).$$

Definition 6.4. We say that the *elliptic Harnack inequality* (**H**) holds if there exist constants $C \in (1,\infty)$ and $\delta \in (0,1)$ such that, for any ball $B \subset M$ and for any function $u \in \mathcal{F}$ that is harmonic and non-negative in B,

$$\operatorname*{esssup}_{x\in\delta B} u(x) \leq C \operatorname*{essinf}_{x\in\delta B} u(x).$$

Definition 6.5. We say that the mean exit time estimate $(\mathbf{E})_\beta$ holds if there exist constants $C \in (1,\infty)$ and $\delta \in (0,1)$ such that, for any ball $B \subset M$ of radius $r > 0$, the restricted Green operator G^B exists and satisfies

$$\operatorname*{esssup}_{x\in B} G^B 1(x) \leq Cr^\beta$$

and

$$\operatorname*{essinf}_{x\in\delta B} G^B 1(x) \geq C^{-1} r^\beta.$$

It is known that $(\mathbf{UE})_\beta + (\mathbf{NLE})_\beta \Leftrightarrow (\mathbf{E})_\beta + (\mathbf{H})$ (see [32, Theorem 7.4]). We show in Sections 6.4 and 6.5 that $(\mathbf{G})_\beta \Rightarrow (\mathbf{E})_\beta$ and $(\mathbf{G})_\beta \Rightarrow (\mathbf{H})$, thus yielding (i) $\Rightarrow$ (ii).

6.2 Proof of $(\mathbf{UE})_\beta + (\mathbf{NLE})_\beta \Rightarrow (\mathbf{G})_\beta$

Proof of Theorem 6.1 (ii) $\Rightarrow$ (i). By [32, Theorem 7.4], the heat kernel is Hölder continuous in $x, y \in M$. The Green function can be then defined pointwise by the identity

$$G(x,y) = \int_0^\infty p_t(x,y)\,dt. \tag{6.1}$$

The estimates (2.5) of the Green function have been already proved in Lemma 2.4 (see also Example 2.5).

Let us now prove the continuity of $G(x,y)$ off-diagonal. By (6.1) we have

$$|G(x,y) - G(x_o,y)| \leq \int_0^\infty |p_t(x,y) - p_t(x_o,y)|\,dt. \tag{6.2}$$

Next, we will use the following elementary estimate: for all $0 \le a < 1$, $R > 0$ and $x \in M$,

$$\int_0^\infty t^{-a} \left(\frac{1}{V(x, t^{1/\beta})} \wedge \frac{1}{V(x, R)} \right) dt \lesssim \frac{R^{(1-a)\beta}}{V(x, R)}. \tag{6.3}$$

Indeed, using (**RVD**) and $a\beta + \alpha_- > \beta$, we obtain

$$\begin{aligned}\int_{R^\beta}^\infty t^{-a} \frac{1}{V(x, t^{1/\beta})} dt &= \frac{1}{V(x, R)} \int_{R^\beta}^\infty t^{-a} \frac{V(x, R)}{V(x, t^{1/\beta})} dt \\ &\lesssim \frac{1}{V(x, R)} \int_{R^\beta}^\infty t^{-a} \left(\frac{R}{t^{1/\beta}} \right)^{\alpha_-} dt \\ &\simeq \frac{1}{V(x, R)} \int_0^1 \left(\frac{s}{R} \right)^{a\beta} s^{\alpha_-} \beta R^\beta s^{-(\beta+1)} ds \\ &\simeq \frac{\beta R^{(1-a)\beta}}{V(x, R)} \int_0^1 s^{a\beta + \alpha_- - \beta - 1} ds \simeq \frac{R^{(1-a)\beta}}{V(x, R)}.\end{aligned}$$

By $a < 1$ we have also

$$\int_0^{R^\beta} t^{-a} \frac{1}{V(x, R)} dt \simeq \frac{R^{(1-a)\beta}}{V(x, R)},$$

whence (6.3) follows.

For any $x \in M$ and positive t, R, consider the cylinder

$$D((t, x), R) = B(x, R) \times (t - R^\beta, t].$$

It was proved in [4, Corollary 4.2] that $(\mathbf{UE})_\beta + (\mathbf{NLE})_\beta$ imply the following property: there exist $\theta, \delta \in (0, 1)$ such that, for any continuous caloric function u in $D((t, x_o), R)$ and for all $x \in B(x_o, \delta R)$

$$|u(t, x) - u(t, x_o)| \lesssim \left(\frac{d(x, x_o)}{R} \right)^\theta \underset{(s,z) \in D((t,x_o),R)}{\operatorname{osc}} u(s, z).$$

Fix $y \in M$ so that $u(t, x) = p_t(x, y)$ is a non-negative continuous caloric function on $M \times (0, \infty)$. Fix also distinct points $x, x_o \in M$ and set $r = d(x, x_o)$. For any $t > T := 2(r/\delta)^\beta$, if we take $R = (t/2)^{1/\beta}$ (this implies that $d(x, x_o) < \delta R$), then the function u is caloric in the cylinder $D((t, x_o), R)$, which implies that

$$|p_t(x, y) - p_t(x_o, y)| \lesssim \left(\frac{r}{R} \right)^\theta \sup_{t/2 \le s \le t} \sup_{z \in B(x_o, R)} p_s(y, z). \tag{6.4}$$

For $s \in [t/2, t]$ we have by $(UE)_\beta$

$$p_s(y, z) \lesssim \frac{1}{V(y, t^{1/\beta})} \exp\left(-c \left(\frac{d(y, z)}{t^{1/\beta}} \right)^{\frac{\beta}{\beta-1}} \right).$$

Since

$$d(y, z) \ge d(y, x_o) - d(x_o, z) \ge d(y, x_o) - R = d(y, x_o) - (t/2)^{1/\beta},$$

it follows that

$$p_s(y, z) \lesssim \frac{1}{V(y, t^{1/\beta})} \exp\left(-c \left(\frac{d(y, x_o)}{t^{1/\beta}} \right)^{\frac{\beta}{\beta-1}} \right) \lesssim \frac{1}{V(y, t^{1/\beta})} \wedge \frac{1}{V(y, x_o)}$$

(see Example 2.5). Substituting into (6.4), we conclude that, for all $t > T := 2\,(r/\delta)^{\beta}$,

$$|p_t\,(x,y) - p_t\,(x_o,y)| \lesssim \left(\frac{r}{t^{1/\beta}}\right)^{\theta}\left(\frac{1}{V\,(y,t^{1/\beta})} \wedge \frac{1}{V\,(y,x_o)}\right).$$

Applying (6.3) with $a = \theta/\beta$ we obtain

$$\int_T^\infty |p_t\,(x,y) - p_t\,(x_o,y)|\,dt \lesssim \int_T^\infty \left(\frac{r}{t^{1/\beta}}\right)^{\theta}\left(\frac{1}{V\,(y,t^{1/\beta})} \wedge \frac{1}{V\,(y,x_o)}\right)dt \lesssim r^{\theta}\frac{d\,(x_o,y)^{\beta-\theta}}{V\,(x_o,y)}.$$

Similarly, we obtain

$$\int_0^T p_t\,(x,y)\,dt \le \int_0^T \left(\frac{T}{t}\right)^{\theta/\beta} p_t\,(x,y)\,dt \lesssim \int_0^\infty \left(\frac{r^\beta}{t}\right)^{\theta/\beta}\left(\frac{1}{V\,(y,t^{1/\beta})} \wedge \frac{1}{V\,(y,x)}\right)dt \lesssim r^{\theta}\frac{d\,(x,y)^{\beta-\theta}}{V\,(x,y)}$$

and

$$\int_0^T p_t\,(x_o,y)\,dt \lesssim r^{\theta}\frac{d\,(x_o,y)^{\beta-\theta}}{V\,(x_o,y)}.$$

Substituting the above three estimates into (6.2), we obtain

$$|G\,(x,y) - G\,(x_o,y)| \lesssim r^{\theta}\frac{d\,(x_o,y)^{\beta-\theta}}{V\,(x_o,y)} + r^{\theta}\frac{d\,(x,y)^{\beta-\theta}}{V\,(x,y)},$$

which proves the locally uniform Hölder continuity of $G\,(\cdot,y)$ in $M \setminus \{y\}$ with the Hölder exponent θ. Since $G\,(x,y)$ is symmetric, this implies a joint continuity of $G\,(x,y)$ in $(x,y) \in M \setminus \mathrm{diag}$. □

6.3 Existence of the restricted Green function

Lemma 6.6. *Let* (**VD**), (**RVD**) *and* $(\mathbf{G})_\beta$ *be satisfied with* $0 < \beta < \alpha_-$. *Then the following are true.*

(i) For any ball $B \subset M$, there exists a non-negative symmetric function $G^B(x,y)$ that is jointly measurable in $x,y \in B$ and satisfies

$$G^B f(x) = \int_B G^B(x,y) f(y)\,d\mu(y) \quad \text{for all } f \in L^2(B) \text{ and } \mu\text{-a.a. } x \in B. \tag{6.5}$$

(ii) There exist constants $\varepsilon \in (0,1)$ and $C > 0$ such that, for any ball B, the restricted Green function $G^B(x,y)$ satisfies

$$G^B(x,y) \le C\frac{d\,(x,y)^\beta}{V\,(x,y)} \quad \text{for } \mu\text{-a.a. } x,y \in B \tag{6.6}$$

and

$$G^B(x,y) \ge C^{-1}\frac{d\,(x,y)^\beta}{V\,(x,y)} \quad \text{for } \mu\text{-a.a. } x,y \in \varepsilon B. \tag{6.7}$$

Proof. By Corollary 5.8 we have, for any ball $B = B(x_o, R)$,

$$\lambda_{\min}(B) \gtrsim (\operatorname{diam}(B))^{-\beta} > 0.$$

By Remark 2.2, the operator $\mathcal{L}^B$ has a bounded inverse in $L^2(B)$, and the latter is exactly the restricted Green operator G^B. Besides, we have

$$0 \le G^B f \le Gf \quad \text{for all } 0 \le f \in L^2(B).$$

Let us now prove the existence of the integral kernel of G^B. For that, we will prove that, for any $0 < \delta < 1$, the operator $G - G^B$ acting from $L^2(\delta B)$ to $L^2(B)$, has an integral kernel. By [28, Lemma 3.3], for the existence of the integral kernel, it suffices to prove that

$$\left\| G - G^B \right\|_{L^2(\delta B) \to L^\infty(B)} < \infty,$$

that is,

$$\left\| Gf - G^B f \right\|_{L^\infty(B)} \lesssim \|f\|_{L^2} \quad \text{for any } 0 \le f \in L^2(\delta B). \tag{6.8}$$

The function $Gf - G^B f$ is harmonic in B. Due to $\lambda_{\min}(B) > 0$, we can apply the maximum principle for harmonic functions (see [27, Lemma 4.1]) and obtain, for any λ such that $\delta < \lambda < 1$,

$$\begin{aligned} 0 \le \operatorname*{esssup}_B \left(Gf - G^B f\right) &\le \operatorname*{esssup}_{B \setminus \lambda B} \left(Gf - G^B f\right) \\ &\le \operatorname*{esssup}_{x \in B \setminus (\lambda B)} Gf(x) \\ &\lesssim \sup_{x \in B \setminus (\lambda B)} \int_{\delta B} \frac{d(x,y)^\beta}{V(x,y)} f(y)\, d\mu(y). \end{aligned}$$

Since for all x, y in the above expression

$$(\lambda - \delta) R < d(x,y) < 2R,$$

it follows that

$$\frac{d(x,y)^\beta}{V(x,y)} \le \frac{(2R)^\beta}{V(x, (\lambda - \delta) R)} \lesssim (\lambda - \delta)^{\alpha_+} \frac{R^\beta}{V(x,R)} \lesssim (\lambda - \delta)^{\alpha_+} \frac{R^\beta}{V(x_o, R)}.$$

Therefore, we have

$$\left\| Gf - G^B f \right\|_{L^\infty(B)} \lesssim (\lambda - \delta)^{\alpha_+} \frac{R^\beta}{V(x_o, R)} \|f\|_{L^1} \tag{6.9}$$

whence (6.8) follows. Hence, the operator $G - G^B$ has an integral kernel, say $K^\delta(x,y)$ that is a non-negative jointly measurable function in $B \times \delta B$.

Clearly, the family $\{K^\delta\}_{\delta \in (0,1)}$ of kernels is consistent in the sense that, for all $0 < \delta' < \delta'' < 1$,

$$K^{\delta'}(x,y) = K^{\delta''}(x,y) \quad \text{for } \mu\text{-a.a. } x \in B \text{ and } y \in \delta' B.$$

Choose a sequence $\delta_k \nearrow 1$ and define in $B \times B$ the kernel

$$K(x,y) = K^{\delta_k}(x,y) \quad \text{for } \mu\text{-a.a. } x \in B \text{ and } y \in \delta_k B.$$

Finally, we define the Green function G^B by

$$G^B(x,y) = G(x,y) - K(x,y).$$

Similarly to the proof of [27, (5.8)], one shows that G^B satisfies (6.5).

Because the operator G^B is positivity preserving, it follows from [28, Lemma 3.2] that

$$G^B(x,y) \geq 0 \quad \text{for } \mu\text{-a.a. } x,y \in B.$$

Moreover, by the symmetry of $\mathcal{E}$, we have, for all $f,g \in \mathcal{F}(B)$,

$$(f, G^B g) = \mathcal{E}(G^B f, G^B g) = \mathcal{E}(G^B g, G^B f) = (g, G^B f),$$

which implies that

$$G^B(x,y) = G^B(y,x) \quad \text{for } \mu\text{-a.a. } x,y \in B.$$

By construction $G^B(x,y) \leq G(x,y)$ so that the upper bound (6.6) of $G^B(x,y)$ follows from $(\mathbf{G})_\beta$. In order to prove the lower bound (6.7) of $G^B(x,y)$, it suffices to verify that, for all $0 \leq f \in L^2(\varepsilon B)$,

$$\operatorname*{essinf}_{\varepsilon B} G^B f(x) \gtrsim \int_{\varepsilon B} \frac{d(x,y)^\beta}{V(x,y)} f(y)\, d\mu(y),$$

where $\varepsilon > 0$ is yet to be determined. Fix the parameters δ and λ from the previous part of the proof, for example, set $\delta = \frac{1}{2}$ and $\lambda = \frac{3}{4}$. Assuming that $\varepsilon < \frac{1}{2}$, we obtain by (6.9)

$$\left\| Gf - G^B f \right\|_{L^\infty(B)} \leq C \frac{R^\beta}{V(x_o, R)} \|f\|_{L^1}$$

so that, for μ-a.a. $x \in \varepsilon B$,

$$G^B f(x) \geq \int_{\varepsilon B} G(x,y) f(y)\, d\mu - C \frac{R^\beta}{V(x_o,R)} \int_{\varepsilon B} f\, d\mu. \tag{6.10}$$

Let us show that the second term in the right hand side of (6.10) is a small fraction of the first one. Since

$$G(x,y) \gtrsim \frac{d(x,y)^\beta}{V(x,y)},$$

so it suffices to verify that, for all $x,y \in \varepsilon B$,

$$\frac{R^\beta}{V(x_o,R)} \leq c(\varepsilon) \frac{d(x,y)^\beta}{V(x,y)}, \tag{6.11}$$

where $c(\varepsilon) \to 0$ as $\varepsilon \to 0$. Indeed, setting $r = d(x,y)$, we obtain

$$\frac{R^\beta}{V(x_o,R)} \Big/ \frac{r^\beta}{V(x,r)} = \frac{V(x,r)}{V(x_o,R)} \left(\frac{R}{r}\right)^\beta \leq \frac{V(x,r)}{V(x,R/2)} \left(\frac{R}{r}\right)^\beta \lesssim \left(\frac{r}{R}\right)^{\alpha_-} \left(\frac{R}{r}\right)^\beta = \left(\frac{R}{r}\right)^{\beta-\alpha_-} \lesssim \varepsilon^{\alpha_- - \beta}.$$

Since $\alpha_- > \beta$, this proves (6.11) with $c(\varepsilon) = C\varepsilon^{\alpha_- - \beta}$. It follows that

$$G^B f(x) \geq (1 - Cc(\varepsilon))\, Gf(x)$$

and, hence,

$$G^B(x,y) \geq (1 - Cc(\varepsilon))\, G(x,y) \quad \text{for } \mu\text{-a.a. } x,y \in \varepsilon B. \tag{6.12}$$

By choosing ε small enough we obtain (6.7). □

6.4 $(\mathbf{G})_\beta$ implies $(\mathbf{E})_\beta$

Proposition 6.7. *Let* (**VD**), (**RVD**) *and* $(\mathbf{G})_\beta$ *be satisfied and* $0 < \beta < \alpha_-$. *Then*

$$(\mathbf{G})_\beta \Rightarrow (\mathbf{E})_\beta.$$

Proof. Fix a ball $B = B(x_o, R) \subset M$. Then we obtain from (4.10)

$$\operatorname*{esssup}_B G^B 1 \le \operatorname*{esssup}_B G1_B \lesssim R^\beta.$$

Choose $\delta = \varepsilon$ where ε is the constant from (6.7). Then, for μ-a.a. $x \in \delta B$,

$$G^B 1(x) \ge \int_{\delta B} G^B(x,y)\, d\mu(y) \gtrsim \int_{\delta B} \frac{d(x,y)^\beta}{V(x,y)} d\mu(y).$$

Using (6.11), we conclude

$$G^B 1(x) \gtrsim \frac{R^\beta}{V(x_o,R)} V(x_o, \delta R) \gtrsim R^\beta,$$

which finishes the proof of $(\mathbf{E})_\beta$. □

6.5 $(\mathbf{G})_\beta$ implies (H)

Proposition 6.8. *Let* (**VD**), (**RVD**) *and* $(\mathbf{G})_\beta$ *be satisfied and* $0 < \beta < \alpha_-$. *Then*

$$(\mathbf{G})_\beta \Rightarrow (\mathbf{H}).$$

Proof. If the restricted Green functions G^B are continuous off-diagonal then this was proved in [27, Theorem 3.12 and Lemma 8.2]. Without the continuity of G^B, the key ingredient of the proof – [27, Lemma 6.2(ii)], breaks down[2]. To overcome this difficulty, we have developed here a new approach.

Let $u \in \mathcal{F}$ be non-negative and harmonic in a ball $B = B(x_o, R) \subset M$. We need to prove that

$$\operatorname*{esssup}_{\delta B} u \le C \operatorname*{essinf}_{\delta B} u \tag{6.13}$$

for some constants $C \in (1, \infty)$ and $\delta \in (0,1)$ independent of B. Without loss of generality, we can assume that $u \in L^\infty$ (see [32, p. 1280, Theorem 7.4] for how to remove this additional assumption). Also, by replacing u by u_+, we can assume without loss of generality that $u \ge 0$ on M.

By (6.12), there exists a small $\varepsilon \in (0, \frac{1}{4})$ so that for any ball B

$$\frac{1}{2} G(x,y) \le G^B(x,y) \le G(x,y) \qquad \text{for } \mu\text{-a.a. } x,y \in \varepsilon B. \tag{6.14}$$

Let us fix this ε and use in what follows. The further proof will be split into three steps.

Step 1. *Riesz measure and a reduced function.* Fix $B = B(x_o, R)$ and consider also the ball

$$B_1 = \frac{\varepsilon}{2} B.$$

By [27, Lemma 6.4], there exists the *reduced function* $\hat{u}$ of u with respect to $(\overline{B}_1, B)$ such that

- $\hat{u} \in \mathcal{F}(B)$;
- $\hat{u} = u$ in $\overline{B}_1$ and $0 \le \hat{u} \le u$ in M;
- $\hat{u}$ is harmonic in $B \setminus \overline{B}_1$ and superharmonic in B

(see Fig. 1).

[2] Note that a posteriori G^B is still continuous off-diagonal which follows from (**H**).

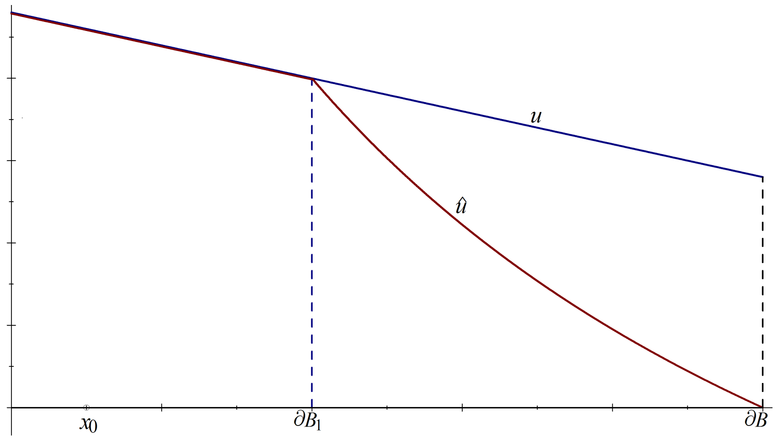

Figure 1: Functions u and $\hat{u}$.

By [27, Lemma 6.2(i)], there exists a regular non-negative Borel measure σ in B such that

$$\int_B \varphi\, d\sigma = \mathcal{E}(\hat{u}, \varphi) \quad \text{for all } \varphi \in \mathcal{F} \cap C_c(B). \tag{6.15}$$

The measure σ is called the *Riesz measure* of the superharmonic function $\hat{u}$. Moreover, the proof of [27, Lemma 6.2(i)] shows that σ does not charge any open set where $\hat{u}$ is harmonic. Since $\hat{u}$ is harmonic in the both sets B_1 and $B \setminus \overline{B}_1$, we obtain that $\operatorname{supp}\sigma \subset \partial B_1 =: S$. Consequently, the domain of integration in (6.15) can be reduced to S.

Step 2. Let Ω be an open neighborhood of $S = \partial B_1$, such that $\overline{\Omega} \subset B$, for example,

$$\Omega = (1+\tau)\, B_1 \setminus (1-\tau)\, \overline{B}_1$$

with a small $\tau \in \left(0, \frac{1}{2}\right)$. Consider also the ball

$$B_2 := \frac{1}{2} B_1 = \frac{\varepsilon}{4} B$$

so that $\overline{B}_2$ and $\overline{\Omega}$ are disjoint (see Fig. 2).

Fix a cutoff function ψ of (S, Ω). The aim of this step is to show that, for any function

$$0 \le \phi \in \mathcal{F} \cap C_c(B_2), \tag{6.16}$$

the following inequality holds:

$$\frac{1}{2}\mathcal{E}(\hat{u}, \psi G\phi) \le (u, \phi) \le \mathcal{E}(\hat{u}, \psi G\phi) \tag{6.17}$$

(see Fig. 3).

By Remark 2.2, both functions $G^B\phi$ and $(1-\psi)G^B\phi$ belong to $\mathcal{F}(B)$. Since $(1-\psi)G^B\phi$ vanishes in an open neighbourhood of S, we conclude by [27, Proposition A.3] that

$$(1-\psi)G^B\phi \in \mathcal{F}(B \setminus S).$$

Since $\hat{u}$ is harmonic $B \setminus S$ we have

$$\mathcal{E}(\hat{u}, (1-\psi)G^B\phi) = 0. \tag{6.18}$$

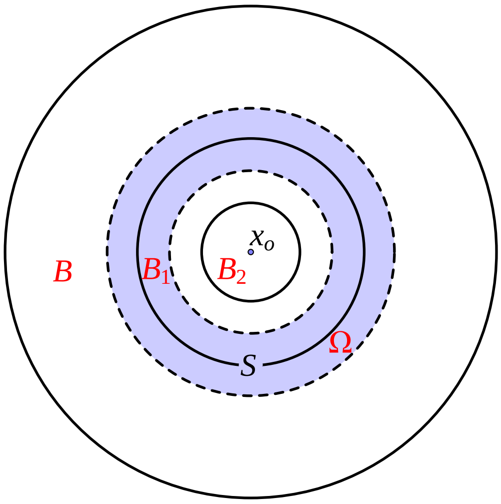

Figure 2: The sets B, B_1, B_2, Ω, S.

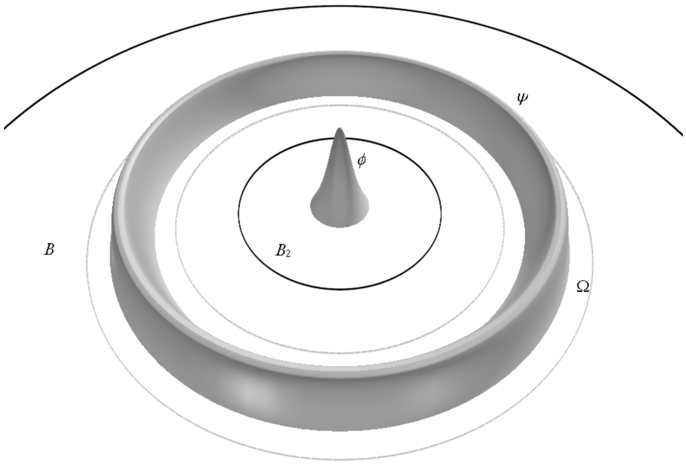

Figure 3: Functions ϕ and ψ

Since $u = \hat{u}$ in B_1, ϕ is supported in B_1, and $\hat{u} \in \mathcal{F}(B)$, we obtain, using Remark 2.2 and (6.18) that

$$\begin{aligned}(u, \phi) = (\hat{u}, \phi) &= \mathcal{E}(\hat{u}, G^B\phi) \\ &= \mathcal{E}(\hat{u}, \psi G^B\phi) + \mathcal{E}(\hat{u}, (1-\psi)G^B\phi) \\ &= \mathcal{E}(\hat{u}, \psi G^B\phi). \end{aligned} \tag{6.19}$$

By (6.14) we have

$$\frac{1}{2}G\phi \le G^B\phi \le G\phi \quad \mu\text{-a.a. in } \varepsilon B.$$

Since $\operatorname{supp}\psi \subset 2B_1 = \varepsilon B$, it follows that

$$\frac{1}{2}\psi G\phi \le \psi G^B\phi \le \psi G\phi \quad \mu\text{-a.a. in } B.$$

Since both functions $\psi G\phi$ and $\psi G^B\phi$ belong to $\mathcal{F}(B)$ and $\hat{u}$ is superharmonic in B, we obtain

$$\frac{1}{2}\mathcal{E}(\hat{u},\psi G\phi) \le \mathcal{E}(\hat{u},\psi G^B\phi) \le \mathcal{E}(\hat{u},\psi G\phi).$$

This inequality together with (6.19) yields (6.17).

Step 3. Now we can prove the Harnack inequality (6.13). As before, let ψ be a fixed cut-off function of (S,Ω) and ϕ be any function satisfying (6.16). Since $\operatorname{supp}\psi \cap \operatorname{supp}\phi = \emptyset$ and the Green function $G(x,y)$ is jointly continuous off-diagonal, the function $\psi(x)G(x,y)\phi(y)$ is jointly continuous in $(x,y) \in M \times M$. Clearly, we also have $\psi G\phi \in \mathcal{F} \cap C_c(B)$. Applying (6.15) with $\varphi = \psi G\phi$ and the Fubini theorem, we obtain

$$\begin{aligned}
\mathcal{E}(\hat{u},\psi G\phi) &= \int_S \psi(x)G\phi(x)\,d\sigma(x)\\
&= \int_S \psi(x)\left(\int_{B_2} G(x,y)\phi(y)\,d\mu(y)\right)d\sigma(x)\\
&= \int_{B_2}\left(\int_S \psi(x)\,G(x,y)\,d\sigma(x)\right)\phi(y)\,d\mu(y)\\
&= \int_{B_2}\left(\int_S G(x,y)\,d\sigma(x)\right)\phi(y)\,d\mu(y),
\end{aligned}$$

where in the last step we have used that $\psi = 1$ on S. Combining with (6.17), we obtain

$$\frac{1}{2}\int_{B_2}\left(\int_S G(x,y)\,d\sigma(x)\right)\phi(y)\,d\mu(y) \le (u,\phi) \le \int_{B_2}\left(\int_S G(x,y)\,d\sigma(x)\right)\phi(y)\,d\mu(y).$$

Since this is true for any non-negative $\phi \in \mathcal{F}\cap C_c(B_2)$ and $\mathcal{F}\cap C_c(B_2)$ is dense in $L^2(B_2)$, we conclude that

$$\frac{1}{2}\int_S G(x,y)\,d\sigma(x) \le u(y) \le \int_S G(x,y)\,d\sigma(x) \quad \text{for } \mu\text{-a.a. } y \in B_2.$$

Since $(\mathbf{G})_\beta$ implies

$$G(x,y) \simeq \frac{R^\beta}{V(x_o,R)} \ \text{ for all } x \in S \text{ and } y \in B_2,$$

we deduce that

$$u(y) \simeq \frac{R^\beta}{V(x_o,R)}\sigma(S) \quad \text{for } \mu\text{-a.a. } y \in B_2.$$

Hence, the Harnack inequality (6.13) holds with $\delta = \frac{1}{4}\varepsilon$. □

7 Weighted Hardy's inequality for strongly local Dirichlet forms

Let (M,d,μ) be a metric measure space and $(\mathcal{E},\mathcal{F})$ be a strongly local Dirichlet form on $L^2(M,\mu)$. The main aim of this section is to obtain a weighted version of Hardy's inequality for strongly local Dirichlet forms.

7.1 Intrinsic metric and weighted Dirichlet form

For all $x, y \in M$, define

$$d_i(x,y) = \sup\{u(x) - u(y) : \ u \in \mathcal{F} \cap C_c, \ d\Gamma(u,u) \le d\mu\}.$$

The function $d_i(x,y)$ is called the *intrinsic metric* of $(\mathcal{E},\mathcal{F})$. In general $d_i(x,y)$ is a pseudo-distance.

Let us introduce the following hypotheses $(H1)$-$(H3)$ that will be used in what follows.

(H1) For any $u \in \mathcal{F}$, the energy measure $\Gamma(u,u)$ is absolutely continuous with respect to μ.

(H2) The intrinsic metric d_i coincides with the original metric d.

(H3) The metric space (M,d) is complete.

It is known that, under these assumptions, the metric space (M,d) is geodesic. Besides, for any non-empty subset E of M, the function $f(x) = d(x,E)$ belongs to $\mathcal{F}_{\rm loc}$ and satisfies $d\Gamma(f,f) \le d\mu$ (see [37]).

For example, $(H1)$-$(H3)$ are satisfied if M is a geodesically complete Riemannian manifold, d is the geodesic distance, μ is the Riemannian measure, and $(\mathcal{E},\mathcal{F})$ is given by the Dirichlet integral

$$\mathcal{E}(f,f) = \int_M |\nabla f|^2 \, d\mu,$$

where $f \in W^{1,2}(M)$.

Let $w : \ M \to (0,\infty]$ be a continuous, locally integrable function, where "continuous" in this context means that w is continuous on $\{w < \infty\}$ and lower semi-continuous on M. Define a weighted bilinear form $\mathcal{E}^{(w)}$ by

$$\mathcal{E}^{(w)}(u,v) = \int_M w \, d\Gamma(u,v) \ \text{ for all } u, v \in \mathcal{F} \cap C_c$$

and set

$$C^{(w)} = \left\{u \in \mathcal{F} \cap C_c : \ \mathcal{E}^{(w)}(u,u) < \infty\right\}.$$

We will use the following result from [55, Corollary 6.1.6].

Proposition 7.1. *Let* $(\mathcal{E},\mathcal{F})$ *satisfy* $(H1)$-$(H3)$ *and let* $w : \ M \to (0,\infty]$ *be a continuous, locally integrable function. Define*

$$d\mu_w = w d\mu.$$

Then the symmetric bilinear form $(\mathcal{E}^{(w)}, C^{(w)})$ *is closable and its closure* $(\mathcal{E}^{(w)}, \mathcal{F}^{(w)})$ *is a strongly local regular Dirichlet form on* $L^2(M,\mu_w)$ *that also satisfies* $(H1)$-$(H3)$*.*

7.2 Admissible weights and the weighted Hardy inequality

Motivated by [30, 55], we introduce the following definitions. Given a set $\Sigma \subset M$ and $\rho \in (0,1]$, define for any $x_o \in \Sigma$ and $s \ge 0$ the set

$$\Sigma_\rho(x_o,s) := \{x \in M : \ d(x,x_o) \le s \text{ and } d(x,\Sigma) \ge \rho s\}.$$

Set also

$$\widehat{\Sigma}_\rho(x_o, r) = \bigcup_{0 \le s \le r} \Sigma_\rho(x_o, s).$$

For example, if $\Sigma = \{x_o\}$ then $\Sigma_\rho(x_o, s)$ is the annulus $\overline{B}(x_o, s) \setminus B(x_o, \rho s)$, and $\widehat{\Sigma}_\rho(x_o, r)$ coincides with the closed ball $\overline{B}(x_o, r)$.

Definition 7.2. Let Σ be a non-empty subset of M. Fix $\rho \in (0, 1)$. The set Σ is called *ρ-accessible* if the following conditions are satisfied:

- ▷ Σ is closed and $\mu(\Sigma) = 0$;
- ▷ there exists $\rho' \in (\rho, 1]$ such that, for any $x_o \in \Sigma$ and $s \in (0, \infty)$, the set $\Sigma_{\rho'}(x_o, s)$ is nonempty;
- ▷ for any $x_o \in \Sigma$ and $r \in (0, \infty)$, the set $\widehat{\Sigma}_\rho(x_o, r)$ is path connected.

For example, if (M, d) is a non-compact complete geodesic space and $\Sigma = \{x_0\}$ then all these conditions are satisfied so that a singleton is ρ-accessible for any $\rho \in (0, 1)$. Other examples of ρ-accessible sets include closed subsets of a hyperplane in the Euclidean space and the boundaries of uniform and Reifenberg domains (see [55, p. 5] or [56, p. 163]).

Definition 7.3. A function $w : M \to (0, \infty]$ is called admissible if there exist a set $\Sigma \subset M$ and a function $a : [0, \infty) \to (0, \infty]$ such that

$$w(x) = a(d(x, \Sigma)) \ \text{ for all } x \in M,$$

and the following conditions are satisfied:

(i) the set Σ is ρ-accessible for some $\rho \in (0, 1)$;

(ii) the function a is continuous, non-increasing, $a(r) < \infty$ for $r > 0$, and there exists a constant $c \in (0, 1)$ such that, for any $r > 0$,

$$a(2r) \ge c\, a(r); \tag{7.1}$$

(iii) there exists a positive constant C such that, for any $x_o \in \Sigma$ and any $r > 0$,

$$\mu_w(B(x_o, r)) \le C a(r) \mu(B(x_o, r)), \tag{7.2}$$

where $d\mu_w = w\, d\mu$.

It follows that any admissible function w is continuous and locally integrable with respect to μ.

For example, the function $a(r) = r^{-\sigma}$ satisfies (ii) for any $\sigma > 0$. If $\mu(B(x_o, r)) \simeq r^\alpha$ for all $r > 0$ and $x_o \in \Sigma$ then $a(r) = r^{-\sigma}$ satisfies (iii) if and only if $0 < \sigma < \alpha$ (cf. [30, Sec. 4.3] and Proposition 7.6 below).

The notion of an admissible weight function was used in [55] to prove the following result.

Theorem 7.4. *[55, Thm 1.0.1, Prop. 4.2.2] If a strongly local Dirichlet form $(\mathcal{E},\mathcal{F})$ in (M,d,μ) satisfies $(H1)$-$(H3)$ as well as the uniform parabolic Harnack inequality, and w is an admissible weight on M then the weighted Dirichlet form $(\mathcal{E}^{(w)},\mathcal{F}^{(w)})$ in (M,d,μ_w) also satisfies the uniform parabolic Harnack inequality.*

Our main result in this Section is the following weighted Hardy inequality for admissible weights w.

Theorem 7.5. *Assume that (M,d,μ) satisfies* (**VD**) *and* (**RVD**) *with the lower volume dimension $\alpha_- > 2$. Let $(\mathcal{E},\mathcal{F})$ be a strongly local regular Dirichlet form on $L^2(M,\mu)$ that satisfies $(H1)$-$(H3)$ as well as $(\mathbf{G})_2$. Let w be an admissible function on M as in Definition 7.3. Define*

$$d\mu_w = w d\mu$$

and assume additionally that (M,d,μ_w) satisfies (**RVD**) *with the lower volume dimension $\alpha_-^{(w)} > 2$. Then the Green function of $(\mathcal{E}^{(w)},\mathcal{F}^{(w)})$ satisfies $(\mathbf{G})_2$ and, for all $x_o \in M$ and $f \in \mathcal{F} \cap C_c$, the following weighted Hardy inequality holds:*

$$\int_M \frac{f(x)^2}{d(x,x_o)^2} w(x)\, d\mu(x) \lesssim \int_M w\, d\Gamma(f,f). \tag{7.3}$$

Proof. By Theorem 6.1, the hypothesis $(\mathbf{G})_2$ implies that the heat kernel p_t of $(\mathcal{E},\mathcal{F})$ satisfies $(\mathbf{UE})_2$ and $(\mathbf{NLE})_2$. Further, by [4, Theorems 3.1 & 3.2] (see also [54]), the conditions $(\mathbf{UE})_2$ and $(\mathbf{NLE})_2$ are equivalent to the parabolic Harnack inequality for $(\mathcal{E},\mathcal{F})$.

Since w admissible, we conclude by Theorem 7.4, that the parabolic Harnack inequality for $(\mathcal{E},\mathcal{F})$ implies the parabolic Harnack inequality for $(\mathcal{E}^{(w)},\mathcal{F}^{(w)})$. Hence, the heat kernel $p_t^{(w)}$ of $(\mathcal{E}^{(w)},\mathcal{F}^{(w)})$ also satisfies the Gaussian estimates $(\mathbf{UE})_2$ and $(\mathbf{NLE})_2$, with measure μ_w instead of μ.

It was proved in [55, Proposition 4.2.2] that the measure μ_w satisfies (**VD**) (which is a consequence of (7.2)). By hypothesis μ_w satisfies also (**RVD**) with $\alpha_-^{(w)} > 2$. Applying Lemma 2.4 in the space (M,d,μ_w) we obtain that the Green function of $(\mathcal{E}^{(w)},\mathcal{F}^{(w)})$ satisfies $(\mathbf{G})_2$ with respect to the measure μ_w. By Theorem 5.6, we obtain that, for all $f \in \mathcal{F}^{(w)}$,

$$\int_M \frac{f(x)^2}{d(x_o,x)^\beta}\, d\mu(x) \lesssim \mathcal{E}^{(w)}(f,f). \tag{7.4}$$

Let us now prove (7.3) for all $f \in \mathcal{F} \cap C_c$. If the right hand side of (7.3) is ∞, then (7.3) is trivially satisfied. If the right hand side of (7.3) is finite then $f \in C^{(w)} \subset \mathcal{F}^{(w)}$ and

$$\int_M w\, d\Gamma(f,f) = \mathcal{E}^{(w)}(f,f),$$

so that (7.3) follows from (7.4). □

7.3 The 1^{st} example: Σ is a singleton

Here we apply Theorem 7.5 to get an explicit version of the weighted Hardy inequality in the case when Σ is a singleton.

Proposition 7.6. *Assume that* (M, d, μ) *satisfies* (**VD**) *and* (**RVD**) *with lower volume dimension* $\alpha_- \in (2, \infty)$. *Let* $(\mathcal{E}, \mathcal{F})$ *be a strongly local regular Dirichlet form on* $L^2(M, \mu)$ *that satisfies* $(H1)$-$(H3)$ *and admits the Green function* $G(x, y)$ *satisfying* $(\mathbf{G})_2$. *Fix some* $0 \le \sigma < \alpha_- - 2$. *Then the following inequality holds for all* $x_o \in M$ *and* $f \in \mathcal{F} \cap C_c$*:*

$$\int_M \frac{f(x)^2}{d(x_o, x)^{\sigma+2}}\, d\mu(x) \lesssim \int_M \frac{1}{d(x_o, x)^{\sigma}}\, d\Gamma(f, f) \tag{7.5}$$

Proof. We will apply Theorem 7.5 with $\Sigma = \{x_o\}$ and the weight

$$w(x) = d(x, x_o)^{-\sigma} \text{ for all } x \in M. \tag{7.6}$$

We need to verify that the weight w is admissible and that μ_w satisfies (**VD**) and (**RVD**) with $\alpha_-^{(w)} > 2$.

The conditions (i) and (ii) of Definition 7.3 are obviously satisfied with $a(r) = r^{-\sigma}$. In order to prove the condition (iii) of Definition 7.3 as well as (**VD**) and (**RVD**) for μ_w, we will use the following estimate

$$\mu_w(B(x, r)) \simeq [r + d(x, x_o)]^{-\sigma} \mu(B(x, r)), \tag{7.7}$$

that holds for all $x \in M$ and $r > 0$. Clearly, (7.7) with $x = x_o$ implies (iii). Next, it follows from (7.7) and the conditions (**VD**) and (**RVD**) for μ that, for any $\lambda > 1$, $x \in M$ and $r > 0$,

$$\lambda^{\alpha_- - \sigma} \lesssim \frac{\mu_w(B(x, \lambda r))}{\mu_w(B(x, r))} \lesssim \lambda^{\alpha_+},$$

which implies that μ_w satisfies (**VD**) and (**RVD**) with the upper volume dimension α_+ and the lower volume dimension $\alpha_- - \sigma > 2$. Hence, applying the inequality (7.3) of Theorem 7.5, we obtain (7.5).

Now let us prove (7.7), assuming $0 \le \sigma < \alpha_-$. For any $y \in B(x, r)$, we have

$$d(y, x_o) \le d(y, x) + d(x, x_o) < r + d(x, x_o)$$

and, hence,

$$\mu_w(B(x, r)) = \int_{B(x,r)} d(x_o, y)^{-\sigma}\, d\mu(y) \ge [r + d(x, x_o)]^{-\sigma} \mu(B(x, r)). \tag{7.8}$$

On the other hand, we have

$$\begin{aligned}
\mu_w(B(x, r)) &= \int_{B(x,r)} d(x_o, y)^{-\sigma}\, d\mu(y) \\
&= \sigma \int_{B(x,r)} \left(\int_{d(x_o,y)}^{\infty} s^{-\sigma-1}\, ds \right) d\mu(y) \\
&= \sigma \int_0^{\infty} \left(\int_{B(x,r)\cap B(x_o,s)} d\mu(y) \right) s^{-\sigma-1}\, ds \\
&\le \sigma \int_{d(x,x_o)-r}^{\infty} \min\{\mu(B(x, r)), \mu(B(x_o, s))\}\, s^{-\sigma-1}\, ds \\
&\le \sigma \int_{d(x,x_o)-r}^{\frac{1}{2}[r+d(x,x_o)]} \mu(B(x_o, s)) s^{-\sigma-1}\, ds + \sigma \int_{\frac{1}{2}[r+d(x,x_o)]}^{\infty} \mu(B(x, r)) s^{-\sigma-1}\, ds.
\end{aligned} \tag{7.9}$$

Observe that

$$d(x,x_o) - r < s < \frac{1}{2}[r + d(x,x_o)] \;\; \Rightarrow \;\; d(x,x_o) < 3r$$

and, hence, by (**VD**) and (**RVD**) for μ,

$$\begin{aligned}\mu(B(x_o,s)) &= \frac{\mu(B(x_o,s))}{\mu(B(x_o,r+d(x,x_o)))}\mu(B(x_o,r+d(x,x_o))) \\ &\lesssim \left(\frac{s}{r+d(x,x_o)}\right)^{\alpha_-}\mu(B(x,7r)) \lesssim \left(\frac{s}{r+d(x,x_o)}\right)^{\alpha_-}\mu(B(x,r)).\end{aligned}$$

Substituting into (7.9) we obtain

$$\begin{aligned}\mu_w(B(x,r)) &\lesssim \mu(B(x,r))\int_0^{r+d(x,x_o)}\left(\frac{s}{r+d(x,x_o)}\right)^{\alpha_-} s^{-\sigma-1}\,ds + [r+d(x,x_o)]^{-\sigma}\mu(B(x,r)) \\ &\lesssim [r+d(x,x_o)]^{-\sigma}\mu(B(x,r)),\end{aligned}$$

which together with (7.8) yields (7.7). □

Denote by $Lip_c(\mathbb{R}^n)$ the class of Lipschitz functions in $\mathbb{R}^n$ with compact support. Since $Lip_c \subset W^{1,2}\cap C_c$, it follows from Proposition 7.6 that, for any $f \in Lip_c(\mathbb{R}^n)$ and $0 < \sigma < n-2$,

$$\int_{\mathbb{R}^n}\frac{f(x)^2}{|x|^{\sigma+2}}\,dx \lesssim \int_{\mathbb{R}^n}\frac{|\nabla f(x)|^2}{|x|^{\sigma}}dx, \tag{7.10}$$

which matches (1.6).

7.4 The 2^{nd} example: Σ is an subspace of $\mathbb{R}^n$

In this and the next subsection, we assume that $M = \mathbb{R}^n$ with $n \geq 3$, d is the Euclidean distance in $\mathbb{R}^n$, μ is the Lebesgue measure, and the Dirichlet form $(\mathcal{E},\mathcal{F})$ is given by (5.29)-(5.30), so that the conditions (**VD**), (**RVD**), $(\mathbf{G})_2$ are satisfied with $\alpha_- = \alpha_+ = n$.

Let Σ be an affine subspace of $\mathbb{R}^n$ of the codimension $k \in \{1,2,\dots,n-1\}$. Rotating and translating Σ, we can assume without loss of generality that $\Sigma = \mathbb{R}^{n-k}$. It is easy to observe that Σ is ρ-accessible for any $\rho \in (0,1)$. Any point $x \in \mathbb{R}^n$ can be written as

$$x = (x', x'') \in \mathbb{R}^k \times \mathbb{R}^{n-k} = \mathbb{R}^k \times \Sigma$$

so that $d(x,\Sigma) = |x'|$.

Proposition 7.7. *In the above setting, fix some $0 < \sigma < \min\{k, n-2\}$. Then the following inequality*

$$\int_{\mathbb{R}^n}\frac{f(x)^2}{|x|^2\,|x'|^{\sigma}}\,dx \lesssim \int_{\mathbb{R}^n}\frac{|\nabla f(x)|^2}{|x'|^{\sigma}}\,dx \tag{7.11}$$

holds for all $f \in Lip_c(\mathbb{R}^n)$.

Proof. We will apply Theorem 7.5 with the weight

$$w(x) = d(x,\Sigma)^{-\sigma} = |x'|^{-\sigma}$$

and the corresponding measure

$$d\mu_w(x) = |x'|^{-\sigma}\,dx.$$

Let us first prove that, for any $x \in \mathbb{R}^n$ and $r > 0$,

$$\mu_w(B(x,r)) \simeq r^n(r+|x|)^{-\sigma}. \tag{7.12}$$

Denote by B' and B'' the balls in $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$, respectively, and by μ' and μ'' – the Lebesgue measures in $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$. Since w depends only on x', we have

$$\mu_w = \mu'_w \times \mu''.$$

Since

$$B'(x', r/2) \times B''(x'', r/2) \subset B(x,r) \subset B'(x',r) \times B''(x'',r)$$

and by (7.7) with $x_o = 0$

$$\mu'_w(B'(x,r)) \simeq (r+|x|)^{-\sigma} r^k \text{ and } \mu''(B''(x'',r)) = \text{const}\, r^{n-k}$$

(where we use that $\sigma < k$), it follows that

$$\mu_w(B(x,r)) \simeq \mu'_w(B'(x,r))\,\mu''(x'',r) \simeq r^n(r+|x|)^{-\sigma}.$$

It follows from (7.12) that, for any $\lambda > 1$,

$$\lambda^{n-\sigma} \lesssim \frac{\mu_w(B(x,\lambda r))}{\mu_w(B(x,r))} \lesssim \lambda^n.$$

Hence, μ_w satisfies (**VD**) and (**RVD**) with $\alpha_+ = n$ and $\alpha_- = n-\sigma > 2$. By Theorem 7.5, the inquality (7.11) holds for all $f \in W^{1,2} \cap C_c(\mathbb{R}^n)$, in particular, for all $f \in Lip_c(\mathbb{R}^n)$. □

Remark 7.8. It follows from (7.12) and $|x|+|y| \simeq |x|+|x-y|$ that

$$\mu_w(B(x,|x-y|)) \simeq |x-y|^n(|x-y|+|x|)^{-\sigma} \simeq |x-y|^n(|x|+|y|)^{-\sigma}.$$

As it was shown in the proof of Theorem 7.5, the Green function $G_w(x,y)$ of the Dirichlet form $(\mathcal{E}^{(w)}, \mathcal{F}^{(w)})$ exists and satisfies $(\mathbf{G})_2$, which yields

$$G_w(x,y) \simeq \frac{|x-y|^2}{\mu_w(B(x,|x-y|))} \simeq |x-y|^{2-n}(|x|+|y|)^{-\sigma}.$$

7.5 The 3rd example: Σ is the boundary of a bounded convex domain

In this Subsection we apply Theorem 7.5 in order to prove the following statement.

Proposition 7.9. *Let $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) be a nonempty bounded convex domain. Fix $\sigma \in (0,1)$. Then the following inequality*

$$\int_{\mathbb{R}^n} \frac{f(x)^2}{|x|^2\, d(x,\partial\Omega)^\sigma}\, dx \lesssim \int_{\mathbb{R}^n} \frac{|\nabla f(x)|^2}{d(x,\partial\Omega)^\sigma}\, dx \tag{7.13}$$

holds for all $f \in Lip_c(\mathbb{R}^n)$.

In particular, (7.13) holds for any $f \in Lip(\overline{\Omega})$ with $f|_{\partial\Omega} = 0$ as this function extends to that in $Lip_c(\mathbb{R}^n)$ by setting $f = 0$ in $\overline{\Omega}^c$.

In the proof of Proposition 7.9 we use *signed distance function* $\delta(x)$ to $\partial\Omega$ that is defined by

$$\delta(x) = \begin{cases} -d(x, \partial\Omega) & \text{if } x \in \Omega, \\ d(x, \partial\Omega) & \text{if } x \in \mathbb{R}^n \setminus \Omega. \end{cases} \tag{7.14}$$

Note that δ is Lipschitz and, hence, differentiable almost everywhere on $\mathbb{R}^n$. It follows from [17, Theorem 5.1.5], that $|\nabla\delta| = 1$ for almost all $x \in \mathbb{R}^n$.

Lemma 7.10. *Let $\Omega \subset R^n$ be a nonempty bounded convex domain. Then there exists a positive constant C such that, for any $x_o \in \mathbb{R}^n$ and $R, s \in (0, \infty)$,*

$$|\{x \in B(x_o, R) : d(x, \partial\Omega) < s\}| \le CR^{n-1} \min\{s, R\}. \tag{7.15}$$

Proof. Note that (7.15) holds trivially if $s \ge R$ or if $\{x \in B(x_o, R) : d(x, \partial\Omega) < s\} = \emptyset$. Hence, we assume in what follows that

$$0 < s < R \quad \text{and} \quad \{x \in B(x_o, R) : d(x, \partial\Omega) < s\} \neq \emptyset.$$

Define for all $t \in \mathbb{R}$ the set

$$\Omega_t = \{x \in \mathbb{R}^n : \delta(x) < t\},$$

where $\delta(x)$ is the function (7.14). We claim that Ω_t is a convex set for all $t \in \mathbb{R}$. Indeed, for $t < 0$ this was proved in [33, p. 17, the remark after Fig. 4]. Let us prove the convexity of Ω_t for $t > 0$. Note that, for $t > 0$, we have

$$\Omega_t = \left\{x \in \mathbb{R}^n : d\left(x, \overline{\Omega}\right) < t\right\}.$$

Fix two points $x, y \in \Omega_t$ and prove that the line segment $[x, y]$ is contained in Ω_t. Choose points $\tilde{x}, \tilde{y} \in \overline{\Omega}$ such that

$$|x - \tilde{x}| < t \quad \text{and} \quad |y - \tilde{y}| < t.$$

Any point $z \in [x, y]$ can be written as $z = \lambda x + (1 - \lambda)y$ for some $\lambda \in [0, 1]$. Since $\tilde{x}, \tilde{y} \in \overline{\Omega}$ and $\overline{\Omega}$ is convex, it follows that

$$\tilde{z} = \lambda\tilde{x} + (1 - \lambda)\tilde{y} \in \overline{\Omega}.$$

Since

$$\begin{aligned} |z - \tilde{z}| &= \left|\left(\lambda x + (1 - \lambda)y\right) - \left(\lambda\tilde{x} + (1 - \lambda)\tilde{y}\right)\right| \\ &\le \lambda\,|x - \tilde{x}| + (1 - \lambda)\,|y - \tilde{y}| < t, \end{aligned}$$

we conclude that $z \in \Omega_t$ and, hence, $[x, y] \subset \Omega_t$.

By the coarea formula (see [18, p. 112]), for any Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ and any Lebesgue measurable set $A \subset \mathbb{R}^n$,

$$\int_A |\nabla f|\, dx = \int_{\mathbb{R}} H^{n-1}(A \cap \{x \in \mathbb{R}^n : f(x) = t\})\, dt,$$

where H^{n-1} denotes the $(n-1)$-dimensional Hausdorff measure. Applying this formula with $f = \delta$ and using that $|\nabla\delta| = 1$ a.e., we obtain

$$|\{x \in B(x_o, R) : d(x, \partial\Omega) < s\}| = |\{x \in B(x_o, R) : |\delta(x)| < s\}|$$

$$= \int_{\{x\in B(x_o,R):\,|\delta(x)|<s\}} |\nabla\delta(x)|\,dx$$
$$= \int_{-s}^{s} H^{n-1}(\{x \in B(x_o,R) :\ \delta(x)=t\})\,dt. \tag{7.16}$$

Clearly, we have

$$\{x \in B(x_o,R) :\ \delta(x) = t\} = \partial\Omega_t \cap B(x_o,R) \subset \partial(\Omega_t \cap B(x_o,R))$$

and, hence,

$$H^{n-1}(\{x \in B(x_o,R) :\ \delta(x)=t\}) \le H^{n-1}(\partial(\Omega_t \cap B(x_o,R))). \tag{7.17}$$

(see Fig. 4 for the case $t \in (-\infty,0)$).

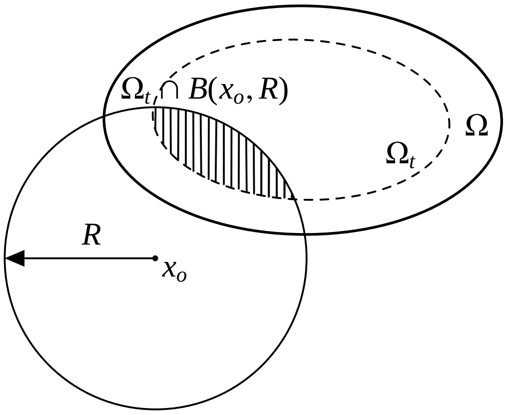

Figure 4: The sets $B(x_o,R)$, Ω and Ω_t for $t \in (-\infty,0)$

Next, we will use the following result of Xiao [58, Theorem 2.1]: for any $p \in (1,n)$ and any convex compact set $E \subset \mathbb{R}^n$, we have

$$H^{n-1}(\partial E) \le c_{p,n}\Big(\mathrm{cap}_p(E)\Big)^{\frac{n-1}{n-p}}, \tag{7.18}$$

where $c_{p,n}$ is a positive constant depending only on p and n, and cap_p is the variational p-capacity defined by

$$\mathrm{cap}_p(E) = \inf\left\{\int_{\mathbb{R}^n} |\nabla f|^p\,dx :\ f \in C_c^\infty(\mathbb{R}^n),\ \ f \ge 1 \text{ on } E\right\}.$$

Below we use the following two basic properties of cap_p (see [58]):

- ▷ *(Monotonicity)* if $E_1 \subset E_2$ are two compact subsets in $\mathbb{R}^n$ then $\mathrm{cap}_p(E_1) \le \mathrm{cap}_p(E_2)$;
- ▷ *(The ball capacity)* for any ball $B \subset \mathbb{R}^n$ of radius $r > 0$, we have $\mathrm{cap}_p(\overline{B}) \simeq r^{n-p}$.

It follows from (7.18) that

$$\begin{aligned} H^{n-1}(\partial(\Omega_t \cap B(x_o, R))) &= H^{n-1}\left(\partial\left(\overline{\Omega_t \cap B(x_o, R)}\right)\right) \\ &\lesssim \left(\mathrm{cap}_p\left(\overline{\Omega_t \cap B(x_o, R)}\right)\right)^{\frac{n-1}{n-p}} \\ &\lesssim \left(\mathrm{cap}_p(\overline{B(x_o, R)})\right)^{\frac{n-1}{n-p}} \simeq R^{n-1}. \end{aligned} \tag{7.19}$$

Combining (7.16), (7.17) and (7.19), we obtain

$$|\{x \in B(x_o, R) : \ d(x, \partial\Omega) < s\}| \lesssim R^{n-1}s,$$

which was to be proved. □

Lemma 7.11. *Let $\Omega \subset R^n$ be a nonempty bounded convex domain. Then, for any $\sigma \in (0, 1)$, the weight function*

$$w(x) = d(x, \partial\Omega)^{-\sigma} \quad \textit{for all } x \in \mathbb{R}^n$$

satisfies the relation

$$\mu_w(B(x_o, r)) \simeq r^n(r + d(x_o, \partial\Omega))^{-\sigma} \tag{7.20}$$

uniformly in $x_o \in \mathbb{R}^n$ and $r > 0$.

Proof. Obviously, for any $y \in B(x_o, r)$, we have

$$d(y, \partial\Omega) \leq d(y, x_0) + d(x_o, \partial\Omega) < r + d(x_o, \partial\Omega),$$

whence leading to

$$\mu_w(B(x_o, r)) = \int_{B(x_o,r)} d(y, \partial\Omega)^{-\sigma}\, dy \geq [r + d(x_o, \partial\Omega)]^{-\sigma}|B(x_o, r)| \simeq r^n(r + d(x_o, \partial\Omega))^{-\sigma}.$$

In order to prove the matching upper bound of $\mu_w(B(x_o, r))$, we consider the following two cases.

Case 1: let $d(x_o, \partial\Omega) \geq 2r$. In this case, for any $y \in B(x_o, r)$, we have

$$d(y, \partial\Omega) \geq d(x_o, \partial\Omega) - d(x_o, y) > d(x_o, \partial\Omega)/2,$$

which implies

$$\mu_w(B(x_o, r)) = \int_{B(x_o,r)} d(y, \partial\Omega)^{-\sigma}\, dy \lesssim r^n d(x_o, \partial\Omega)^{-\sigma} \simeq r^n(r + d(x_o, \partial\Omega))^{-\sigma}.$$

Case 2: let $d(x_o, \partial\Omega) < 2r$. By the Fubini theorem and Lemma 7.10, we obtain

$$\begin{aligned} \mu_w(B(x_o, r)) &= \int_{B(x_o,r)} d(y, \partial\Omega)^{-\sigma}\, dy \simeq \int_{B(x_o,r)} \left(\int_{d(y,\partial\Omega)}^{\infty} s^{-\sigma-1}\, ds\right) dy \\ &\simeq \int_0^{\infty} \left(\int_{\{y\in B(x_o,r):\, d(y,\partial\Omega)<s\}} dy\right) s^{-\sigma-1}\, ds \\ &\lesssim r^{n-1} \int_0^{\infty} \min\{r, s\} s^{-\sigma-1}\, ds \simeq r^{n-\sigma} \simeq r^n(r + d(x_o, \partial\Omega))^{-\sigma}, \end{aligned}$$

which finishes the proof. □

Now can prove Proposition 7.9.

Proof of Proposition 7.9. Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain. By [48, Remark 2.4(c)], any bounded convex domain is a John domain. Moreover, by [48, Examples and Remarks 2.13(c)] that any convex John domain is a uniform domain. Hence, Ω is a uniform domain. By [55, p. 5] or [56, p.163], the boundary $\Sigma = \partial\Omega$ is ρ-accessible for some $\rho \in (0, 1)$. It follows from the estimate (7.20) of Lemma 7.11 that the weight $w(x) = d(x, \partial\Omega)^{-\sigma}$ is admissible as in Definition 7.3 with $a(r) = r^{-\sigma}$.

The estimate (7.20) of Lemma 7.11 implies also that, for any ball $B(x_o, r)$ with $x_o \in \mathbb{R}^n$ and $r > 0$ and for any $\lambda \in (1, \infty)$,

$$\lambda^{n-\sigma} \lesssim \frac{\mu_w(B(x_o, \lambda r))}{\mu_w(B(x_o, r))} \lesssim \lambda^n.$$

Hence, the metric measure space $(\mathbb{R}^n, |\cdot|, \mu_w)$ (with the Euclidean distance $|\cdot|$) satisfies (**VD**) and (**RVD**) with the upper volume dimension n and the lower volume dimension $n - \sigma > 2$. By Theorem 7.5 we obtain the Hardy inequality (7.13). □

Remark 7.12. Theorem 7.5 also says that the Green function $G_w(x, y)$ of $(\mathcal{E}^{(w)}, \mathcal{F}^{(w)})$ satisfies $(\mathbf{G})_2$. Using Lemma 7.11 and that

$$|x - y| + d(x, \partial\Omega) \simeq |x - y| + d(y, \partial\Omega) \simeq |x - y| + d(x, \partial\Omega) + d(y, \partial\Omega),$$

we obtain the following explicit estimate of the weighted Green function:

$$G_w(x, y) \simeq \frac{|x - y|^2}{\mu_w(B(x, |x - y|))} \simeq |x - y|^{2-n} \Big(|x - y| + d(x, \partial\Omega) + d(y, \partial\Omega) \Big)^{\sigma}.$$

Acknowledgement. The authors would like to thank Dr. Meng Yang for valuable discussions on the theory of of Dirichlet forms.

References

[1] A. A. Balinsky, W. D. Evans, and R. T. Lewis. *The analysis and geometry of Hardy's inequality*. Universitext. Springer, Cham, 2015. [2]

[2] M. T. Barlow. Diffusions on fractals. In *Lectures on probability theory and statistics (Saint-Flour, 1995)*, volume 1690 of *Lecture Notes in Math.*, pages 1–121. Springer, Berlin, 1998. [3, 6, 9]

[3] M. T. Barlow and R. F. Bass. Brownian motion and harmonic analysis on Sierpinski carpets. *Canad. J. Math.*, 51(4):673–744, 1999. [3, 6, 9]

[4] M. T. Barlow, A. Grigor'yan, and T. Kumagai. On the equivalence of parabolic Harnack inequalities and heat kernel estimates. *J. Math. Soc. Japan*, 64(4):1091–1146, 2012. [4, 32, 42]

[5] W. Beckner. Pitt's inequality with sharp convolution estimates. *Proc. Amer. Math. Soc.*, 136(5): 1871–1885, 2008. [29]

[6] N. Belhadjrhouma and A. Ben Amor. Hardy's inequality in the scope of Dirichlet forms. *Forum Math.*, 24(4):751–767, 2012. [3]

[7] D. M. Bennett. An extension of Rellich's inequality. *Proc. Amer. Math. Soc.*, 106(4): 987?993, 1989. [3]

[8] A. Beurling and J. Deny. Espaces de Dirichlet. I. Le cas élémentaire. *Acta Math.*, 99:203–224, 1958. [10]

[9] G. Carron. Inégalités isopérimétriques de Faber-Krahn et conséquences. In *Actes de la Table Ronde de Géométrie Différentielle (Luminy, 1992)*, volume 1 of *Sémin. Congr.*, pages 205–232. Soc. Math. France, Paris, 1996. [2]

[10] Z.-Q. Chen and M. Fukushima. *Symmetric Markov processes, time change, and boundary theory*, volume 35 of *London Mathematical Society Monographs Series*. Princeton University Press, Princeton, NJ, 2012. [10, 11]

[11] Z.-Q. Chen, T. Kumagai, and J. Wang. Stability of heat kernel estimates for symmetric jump processes on metric measure spaces. *preprint*, 2016. [9]

[12] R. R. Coifman and G. Weiss. *Analyse harmonique non-commutative sur certains espaces homogènes*. Lecture Notes in Mathematics, Vol. 242. Springer-Verlag, Berlin-New York, 1971. Étude de certaines intégrales singulières. [6]

[13] T. Coulhon, G. Kerkyacharian, and P. Petrushev. Heat kernel generated frames in the setting of Dirichlet spaces. *J. Fourier Anal. Appl.*, 18(5):995–1066, 2012. [6]

[14] L. D'Ambrosio and S. Dipierro. Hardy inequalities on Riemannian manifolds and applications. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 31(3):449–475, 2014. [2]

[15] E. B. Davies. A review of Hardy inequalities. In *The Maz'ya anniversary collection, Vol. 2 (Rostock, 1998)*, volume 110 of *Oper. Theory Adv. Appl.*, pages 55–67. Birkhäuser, Basel, 1999. [2]

[16] E. B. Davies and A. M. Hinz. Explicit constants for Rellich inequalities in $L^p(\Omega)$. *Math. Z.*, 227(3):511?523, 1998. [3]

[17] D. E. Edmunds and W. D. Evans. *Hardy operators, function spaces and embeddings*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2004. [46]

[18] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*. Textbooks in Mathematics. CRC Press, Boca Raton, FL, revised edition, 2015. [46]

[19] P. J. Fitzsimmons. Hardy's inequality for Dirichlet forms. *J. Math. Anal. Appl.*, 250(2):548–560, 2000. [3]

[20] M. Fukushima, Y. Oshima, and M. Takeda. *Dirichlet forms and symmetric Markov processes*, volume 19 of *De Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1994. [6, 10, 11, 12, 13, 14, 15, 18]

[21] A. Grigor'yan. The heat equation on non-compact Riemannian manifolds. *Math. USSR Sb.*, 72:47–77, 1992. [6]

[22] A. Grigor'yan. Heat kernels and function theory on metric measure spaces. In *Heat kernels and analysis on manifolds, graphs, and metric spaces (Paris, 2002)*, volume 338 of *Contemp. Math.*, pages 143–172. Amer. Math. Soc., Providence, RI, 2003. [3, 4, 6, 9, 28, 29]

[23] A. Grigor'yan. Heat kernels on weighted manifolds and applications. volume 398 of *Contemp. Math.*, pages 93–191. Amer. Math. Soc., Providence, RI, 2006. [2]

[24] A. Grigor'yan. *Heat kernel and analysis on manifolds*, volume 47 of *AMS/IP Studies in Advanced Mathematics*. American Mathematical Society, Providence, RI; International Press, Boston, MA, 2009. [2]

[25] A. Grigor'yan. Heat kernels on metric measure spaces with regular volume growth In *Handbook of Geometric Analysis Vol.2* volume 13 of *Advanced Lectures in Math.*, pages 1–60. International Press, 2010. [9]

[26] A. Grigor'yan and J. Hu. Off-diagonal upper estimates for the heat kernel of the Dirichlet forms on metric spaces. *Invent. Math.*, 174(1):81–126, 2008. [4]

[27] A. Grigor'yan and J. Hu. Heat kernels and Green functions on metric measure spaces. *Canad. J. Math.*, 66(3):641–699, 2014. [4, 5, 6, 8, 34, 35, 36, 37]

[28] A. Grigor'yan and J. Hu. Upper bounds of heat kernels on doubling spaces. *Mosc. Math. J.*, 14(3):505–563, 641–642, 2014. [4, 6, 34, 35]

[29] A. Grigor'yan, J. Hu, and K.-S. Lau. Generalized capacity, Harnack inequality and heat kernels of Dirichlet forms on metric measure spaces. *J. Math. Soc. Japan*, 67(4):1485–1549, 2015. [4]

[30] A. Grigor'yan and L. Saloff-Coste. Stability results for Harnack inequalities. *Ann. Inst. Fourier (Grenoble)*, 55(3):825–890, 2005. [3, 4, 40, 41]

[31] A. Grigor'yan and A. Telcs. Sub-Gaussian estimates of heat kernels on infinite graphs. *Duke Math. J.*, 109(3):451–510, 2001. [4, 6]

[32] A. Grigor'yan and A. Telcs. Two-sided estimates of heat kernels on metric measure spaces. *Ann. Probab.*, 40(3):1212–1284, 2012. [4, 31, 36]

[33] M. Gromov. Sign and geometric meaning of curvature. *Rend. Sem. Mat. Fis. Milano*, 61:9–123 (1994), 1991. [46]

[34] Y. Han, D. Müller, and D. Yang. A theory of Besov and Triebel-Lizorkin spaces on metric measure spaces modeled on Carnot-Carathéodory spaces. *Abstr. Appl. Anal.*, pages Art. ID 893409, 250, 2008. [6]

[35] G. H. Hardy. Notes on some points in the integral calculus. *Messenger of Math.*, 57:12–16, 1928. [2]

[36] G. H. Hardy, J. E. Littlewood, and G. Pólya. *Inequalities*. Cambridge, at the University Press, 1952. 2d ed. [2, 20]

[37] F. Hirsch. Intrinsic metrics and Lipschitz functions. *J. Evolution Equations*, 3:11?25, 2003. [40]

[38] M. Keller. *Intrinsic metrics on graphs: a survey*. Mathematical technology of networks. Springer Proc. Math. Stat, 128: 81–119, 2015. [20]

[39] M. Keller, Y. Pinchover, and F. Pogorzelski. An improved discrete Hardy inequality. *Amer. Math. Monthly*, 125(4): 347–350, 2018. [20]

[40] M. Keller, Y. Pinchover, and F. Pogorzelski. Optimal Hardy inequalities for Schrödinger operators on graphs. *Comm. Math. Phys.*, 358(2): 767–790, 2018. [20]

[41] J. Kigami. *Analysis on fractals*, volume 143 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2001. [6]

[42] J. Kigami. Local Nash inequality and inhomogeneity of heat kernels. *Proc. London Math. Soc. (3)*, 89(2):525–544, 2004. [3, 9]

[43] I. Kombe and M. Özaydin. Improved Hardy and Rellich inequalities on Riemannian manifolds. *Trans. Amer. Math. Soc.*, 361(12):6191–6203, 2009. [2]

[44] A. Kufner, L. Maligranda, and L.-E. Persson. *The Hardy inequality*. Vydavatelský Servis, Plzevn, 2007. About its history and some related results. [2]

[45] R. T. Lewis. Singular elliptic operators of second order with purely discrete spectra. *Trans. Amer. Math. Soc.*, 271 (2): 653-666, 1982. [3]

[46] P. Li and S.-T. Yau. On the parabolic kernel of the Schrödinger operator. *Acta Math.*, 156:153–201, 1986. [9]

[47] I. McGillivray. A recurrence condition for some subordinated strongly local Dirichlet forms. *Forum Math.*, 9 (2): 229–246, 1997. [28]

[48] O. Martio and J. Sarvas. Injectivity theorems in plane and space. *Ann. Acad. Sci. Fenn. Ser. A I Math.* 4(2): 383-401, 1979. [49]

[49] B. Opic and A. Kufner. *Hardy-type inequalities*, volume 219 of *Pitman Research Notes in Mathematics Series*. Longman Scientific and Technical, Harlow, 1990. [2]

[50] M. Rao and H. Šikić. Potential-theoretic nature of Hardy's inequality for Dirichlet forms. *J. Math. Anal. Appl.*, 318(2):781–786, 2006. [3]

[51] E. M. Stein. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970. [24, 30]

[52] R. S. Strichartz and A. Teplyaev. Spectral analysis on infinite Sierpiński fractafolds. *J. Anal. Math.*, 116:255–297, 2012. [3, 6]

[53] K.-T. Sturm. Analysis on local Dirichlet spaces. I. Recurrence, conservativeness and L^p-Liouville properties. *J. Reine Angew. Math.*, 456:173–196, 1994. [11]

[54] K. T. Sturm. Analysis on local Dirichlet spaces. III. The parabolic Harnack inequality. *J. Math. Pures Appl. (9)*, 75(3):273–297, 1996. [42]

[55] S. Tasena. *Heat kernel analysis on weighted Dirichlet spaces*. ProQuest LLC, Ann Arbor, MI, 2011. Thesis (Ph.D.)–Cornell University. [4, 40, 41, 42, 49]

[56] S. Tasena, L. Saloff-Coste, and S. Dhompongsa. Poincaré inequality: from remote balls to all balls. *Nonlinear Anal.* 108: 161172, 2014. [41, 49]

[57] N. T. Varopoulos, L. Saloff-Coste, and T. Coulhon. *Analysis and geometry on groups*, volume 100 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1992. [6]

[58] J. Xiao. *P*-capacity vs surface-area. *Adv. Math.*, 308:1318–1336, 2017. [47]

[59] V. M. Zolotarev *One-dimensional stable distributions* Transl. Math. Monographs 65, Amer. Math. Soc., 1986 [28]

Jun Cao
Department of Applied Mathematics, Zhejiang University of Technology, Hangzhou 310023, People's Republic of China

E-mail: caojun1860@zjut.edu.cn

Alexander Grigor'yan

Department of Mathematics, University of Bielefeld, 33501 Bielefeld, Germany and Institute of Control Sciences of Russian Academy of Sciences, Moscow, Russia

E-mail: grigor@math.uni-bielefeld.de

Liguang Liu

School of Mathematics, Renmin University of China, Beijing 100872, People's Republic of China

E-mail: liuliguang@ruc.edu.cn