Hierarchical Schrödinger type operators: the case of potentials with local singularities *

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Abstract

The goal of this paper is twofold. We prove that the operator $H = L + V$, a perturbation of the Taibleson-Vladimirov multiplier $L = \mathfrak{D}^{\alpha}$ by a potential $V(x) = b\,\|x\|^{-\alpha}$, $b \geq b_*$, is essentially self-adjoint and non-negative definite (the critical value b_* depends on α and will be specified later). While the operator H is non-negative definite the potential $V(x)$ may well take negative values, e.g. $b_* < 0$ for all $0 < \alpha < 1$. The equation $Hu = v$ admiits a Green function $g_H(x, y)$, the integral kernel of the operator H^{-1}. We obtain sharp lower- and upper bounds on the ratio of the functions $g_H(x, y)$ and $g_L(x, y)$. Examples illustrate our exposition.

Contents

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1 Introduction

The spectral theory of nested fractals similar to the Sierpinski gasket, i.e. the spectral theory of the corresponding Laplacians, is well understood. It has several important features: Cantor-like structure of the essential spectrum and, as result, the large number of spectral gaps, presence of infinite number of eigenvalues each of which has infinite multiplicity and compactly supported eigenstates, non-regularly varying heat kernels which contain oscilated in $\log t$ scale terms etc, see [21], [16] and [8].

The spectral properties mentioned above occure in the very precise form for the Taibleson-Vladimirov Laplacian $\mathfrak{D}^{\alpha}$, the operator of fractional derivative of order α. This operator can be introduced in several different forms (say, as L^2-multiplier in the p-adic analysis setting, see [41]) but we select the geometric approach [17], [36], [35], [4], [5], [6] and [7].

1.1 The Dyson hierarchical model

Let us fix an integer $p \geq 2$ and consider the family of partitions $\{\Pi_r : r \in \mathbb{Z}\}$ of the set $X = [0, +\infty)$ such that each Π_r consists of all p-adic intervals $I = [kp^r, (k+1)p^r)$. We call r the rank of the partition Π_r (respectively, the rank of the interval $I \in \Pi_r$). Each interval of rank r is the union of p disjoint intervals of rank $(r-1)$. Each point $x \in X$ belongs to a certain interval $I_r(x)$ of rank r, and the intersection of all p-adic intervals $I_r(x)$ is $\{x\}$.

The hierarchical distance $d(x, y)$ is defined as the length $|I|$ of the minimal p-adic interval I which contains x and y. Since any two points x and y belong to a certain p-adic interval, $d(x, y) < \infty$. Clearly $d(x, y) = 0$ if and only if $x = y$, $d(x, y) = d(y, x)$. Moreover, for arbitrary x, y and z holds the *ultrametric inequality* (which is stronger than the triangle inequality)

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}. \tag{1.1}$$

The ultrametric space (X, d) is *complete, separable* and *proper* metric space. In (X, d) the set of all open balls is countable and coincides with the set of all p-adic intervals. In particular, any two balls either do not intersect or one is a subset of another. Thus (X, d) is a totally disconnected separable topological space.The Borel σ-algebra generated by the ultrametric balls coincides with the Borel σ-algebra generated by the Eucledian balls.

The hierarchical Laplacian L is defined pointwise as

$$(Lf)(x) = \sum_{r=-\infty}^{+\infty} (1-\kappa)\kappa^{r-1} \left(f(x) - \frac{1}{|I_r(x)|} \int\limits_{I_r(x)} f dl \right). \tag{1.2}$$

The series in (1.2) diverges in general but it is finite and belongs to $L^2(0,\infty)$ for any f which has compact support and takes constant values on the p-adic intervals of a fixed rank r. The set of such functions we denote by $\mathcal{D}$.

The operator L admits a complete system of compactly supported eigenfunctions. Indeed, let I be a p-adic interval of rank r, and $I_1, I_2, ..., I_p$ be its p-adic subintervals of rank $r-1$. Let us consider p functions

$$\psi_{I_i} = \frac{1_{I_i}}{|I_i|} - \frac{1_I}{|I|}.$$

Each function ψ_{I_i} belongs to $\mathcal{D}$ and satisfies

$$L\psi_{I_i} = \kappa^{r-1}\psi_{I_i}.$$

When I runs over the set of all p-adic intervals the set of eigenfunctions ψ_{I_i} is complete in $L^2(0,\infty)$. In particular, L is a essentially self-adjoint operator with pure point spectrum

$$Spec(L) = \{0\} \cup \{\kappa^r : r \in \mathbb{Z}\}.$$

Clearly each eigenvalue κ^{r-1} has infinite multiplicity. In particular, the spectrum of L coincides with its essential part.

We shell see below that writing $\kappa = p^{-\alpha}$ the operator L can be identified with the Taibleson-Vladimirov operator $\mathfrak{D}^\alpha$, the operator of fractional derivative of order α.

According to [5] the operator $L : \mathcal{D} \to L^2(0,\infty)$ can be represented as a hypersingular integral operator

$$Lf(x) = \int_0^\infty (f(x) - f(y))\, J(x,y)dy,$$

$$J(x,y) = \frac{\kappa^{-1} - 1}{1 - \kappa p^{-1}} \cdot \frac{1}{\mathrm{d}(x,y)^{1+\alpha}}.$$

There are already several publications on the spectrum of the hierarchical Laplacian acting on a general ultrametric measure space (X, d, m) [2], [1], [36], [35], [4], [5], [6], [7]. Accordingly, the hierarchical Schrödinger operator was studied in [18], [36], [37], [38], [13], [31], [32], [33] (the hierarchical lattice of Dyson) and in [43], [42], [27] (the field of p-adic numbers).

By the general theory developed in [4], [5] and [6], any hierarchical Laplacian L acts in $L^2(X,m)$, is essentially self-adjoint and can be represented as a hypersingular integral operator

$$Lf(x) = \int_X (f(x) - f(y))J(x,y)dm(y). \tag{1.3}$$

The operator L has a pure point spectrum, its Markovian semigroup $(e^{-tL})_{t>0}$ admits with respect to m a continuous transition density $p(t,x,y)$. In terms of certain (intrinsically related to L) ultrametric $d_*(x,y)$ the functions $J(x,y)$ and $p(t,x,y)$ can be represented in the form

$$J(x,y) = \int_0^{1/d_*(x,y)} N(x,\tau)d\tau, \tag{1.4}$$

$$p(t,x,y)=t\int_{0}^{1/d_{*}(x,y)}N(x,\tau)\exp(-t\tau)d\tau. \tag{1.5}$$

The function $N(x,\tau)$ is called the *spectral function* and will be specified later.

1.2 Outline

Let us describe the main body of the paper. In Section 2 we introduce the notion of homogeneous hierarchical Laplacian L and list its basic properties e.g. the spectrum of the operator L is pure point, all eigenvalues of L have infinite multiplicity and compactly supported eigenfunctions, the heat kernel $p(t,x,y)$ exists and is a continuous function having certain asymptotic properties etc. For the basic facts related to the ultrametric analysis of heat kernels listed here we refere to [5], [6].

As a special example we consider the case $X=\mathbb{Q}_p$, the ring of p-adic numbers endowed with its standard ultrametric $d(x,y)=\|x-y\|_p$ and the normed Haar measure m. The hierarchical Laplacian L in our example coincides with the Taibleson-Vladimirov operator $\mathfrak{D}^{\alpha}$, the operator of fractional derivative of order α, see [41], [43], and [27]. The most complete sourse for the basic definitions and facts related to the p-adic analysis is [26] and [40].

In the next sections we consider the Schrödinger type operator $H=\mathfrak{D}^{\alpha}+V$ with potential $V\in L^1_{loc}$ having local singularity, e.g. $V(x)=b\,\|x\|_p^{-\alpha}$, $0<\alpha<1$. The main aim here is to prove that the the symmetric operator H defined via quadratic forms on $\mathcal{D}$, the set of locally constant compactly supported functions, is semibounded and whence admits a self-adjoint extension. Under certain condition on V we will prove that H is indeed an essentially self-adjoint operator.

We also prove several results about the negative part of the spectrum of H. For instance, if $V\in L^p$ for some $p>1/\alpha$, then the operator H has essential spectrum equals to the spectrum of $\mathfrak{D}^{\alpha}$. In particular, if H has any negative spectrum, then it consists of a sequence of negative eigenvalues of finite multiplicity. If this sequence is infinite then it converges to zero.

In the concluding section we consider the operator $H=\mathfrak{D}^{\alpha}+b\,\|x\|_p^{-\alpha}$ assuming that $0<\alpha<1$ and $b\geq b_*$, the critical value which will be specified later. We prove that the equation $Hu=v$ admits a fundamenthal solution $g_H(x,y)$ (the Green function of the operator H). The function $g_H(x,y)$ is continuous and takes finite values off the diagonal. Let $g_{\mathfrak{D}^{\alpha}}(x,y)$ be the Green function of the operator $\mathfrak{D}^{\alpha}$. The main result of this section is the following statement: for any $b\geq b_*$ there exists $\frac{\alpha-1}{2}\leq\beta<\alpha$ such that

$$\frac{g_H(x,y)}{g_{\mathfrak{D}^{\alpha}}(x,y)}\asymp\left(\frac{\|x\|_p}{\|y\|_p}\wedge\frac{\|y\|_p}{\|x\|_p}\right)^{\beta},$$

where the sign $\asymp$ means that the ratio of the left- and right hand sides is bounded from below and above by positive constants. This result must be compared with the Green function estimates for Schrödinger operators on complete Riemanian manifolds, see [22].

2 Preliminaries

2.1 Homogeneous ultrametric space

Let (X, d) be a locally compact and separable ultrametric space. Recall that a metric d is called a *ultrametric* if it satisfies the ultrametric inequality

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}, \tag{2.1}$$

that is stronger than the usual triangle inequality. The basic consequence of the ultrametric property is that each open ball is a closed set. Moreover, each point x of a ball B can be regarded as its center, any two balls A and B either do not intersect or one is a subset of another etc. In particular, the ultrametric space (X, d) is totally disconnected, see [6] and references therein. In this paper we assume that the ultrametric space (X, d) is *proper*, that is, each closed ball is a compact set.

To any ultrametric space (X, d) one can associate in a standard fashion a tree $\mathcal{T}$. The vertices of the tree are metric balls, the boundary $\partial\mathcal{T}$ can be identified with the one-point compactification $X \cup \{\varpi\}$ of X. We refere to [6] for a treatment of the association between an ultrametric space and the tree of its metric balls.

Definition 2.1 *An ultrametric measure space (X, d, m) is called* homogeneous *if the group of its isometries acts transitively and preserves the measure.*

The following remarkable result is due to M. Del Muto and A. Figà-Talamanca [15, Section 2].

Theorem 2.2 *Any homogeneous ultrametric measure space (X, d, m) can be identified with certain locally compact Abelian group $\mathfrak{G}$ equipped with a translation invariant ultrametric $\mathfrak{d}$ and the Haar measure $\mathfrak{m}$.*

For example, the set $X = [0, +\infty[$ equipped with the ultrametric structure generated by p-adic intervals can be identified with $\mathbb{Q}_p$, the ring of p-adic numbers.

The identification in Theorem 2.2 is not unique. One possible way to define such identification is to choose the sequence $a = \{a_n\}$ of forward degrees associated with the tree of balls $\mathcal{T}$. This sequence is two-sided if X is non-compact and perfect (has no isolated points), it is one-sided if X is compact and perfect, or if X is discrete. In the 1st case we identify X with Ω_a, the ring of a-adic numbers, in the 2nd case with $\Delta_a \subset \Omega_a$, the ring of a-adic integers, and in the 3rd case with the discrete group $[\Omega_a : \Delta_a]$. We refere the reader to the monograph [23] for the comprehensive treatment of special groups Ω_a, Δ_a and $[\Omega_a : \Delta_a]$.

2.2 Homogeneous hierarchical Laplacian

Let (X, d, m) be a non-compact homogeneous ultrametric measure space. Let $\mathcal{B}$ be the set of all open balls, $\mathcal{B}(x) \subset \mathcal{B}$ the set of balls centred at x, and $C : \mathcal{B} \to (0, \infty)$ a function satisfying the following conditions:

(i) $C(A) = C(B)$ for any two balls A and B of the same diameter,

(ii) $\lambda(B) := \sum_{T\in\mathcal{B}:\, B\subseteq T} C(T) < \infty$ for all $B \in \mathcal{B}$,

(iii) $\sup_{B\in\mathcal{B}(x)} \lambda(B) = \infty$ for any non-isolated x.

The class of functions $C(B)$ satisfying these conditions is reach enough, e.g. one can choose

$$C(B) = (1/m(B))^{\alpha} - (1/m(B'))^{\alpha}$$

for any two closest neighboring balls $B \subset B'$. In this case $\lambda(B) = (1/m(B))^{\alpha}$.

Let $\mathcal{D}$ be the set of all locally constant compactly supported functions. *The homogeneous hierarchical Laplacian* L is defined (pointwise) as

$$Lf(x) := \sum_{B\in\mathcal{B}(x)} C(B) \left(f(x) - \frac{1}{m(B)} \int_B f dm \right). \tag{2.2}$$

The operator $L : \mathcal{D} \to L^2(X, m)$ is symmetric and admits a complete system of eigenfunctions

$$f_B = \frac{\mathbf{1}_B}{m(B)} - \frac{\mathbf{1}_{B'}}{m(B')}, \tag{2.3}$$

where the couple $B \subset B'$ runs over all nearest neighboring balls in $\mathcal{B}$. We have

$$Lf_B(x) = \lambda(B')f_B(x). \tag{2.4}$$

Since the system $\{f_B : B \in \mathcal{B}\}$ is complete, we conclude that $L : \mathcal{D} \to L^2(X, m)$ is an essentially self-adjoint operator.

The intrinsic ultrametric $d_*(x, y)$ associated wth L is defined as follows

$$d_*(x, y) := \begin{cases} 0 & \text{when} \quad x = y \\ 1/\lambda(x \curlywedge y) & \text{when} \quad x \neq y \end{cases}, \tag{2.5}$$

where $x \curlywedge y$ is the minimal ball containing both x and y. In particular, for any non-singletone ball B we have

$$\lambda(B) = \frac{1}{\operatorname{diam}_*(B)}. \tag{2.6}$$

The spectral function $\tau \to N(\tau)$, see equation (1.4), is defined as the left-continuous step-function having jumps at the points $\lambda(B)$, and taking values

$$N(\lambda(B)) = 1/m(B).$$

The volume function $V(r)$ is defined by setting $V(r) = m(B)$ where the ball B has d_*-radius r. It is easy to see that

$$N(\tau) = 1/V(1/\tau). \tag{2.7}$$

The Markovian semigroup $P_t = e^{-tL}, t > 0$, admits a continuous density $p(t, x, y)$ w.r.t. m, we call it *the heat kernel.* The function $p(t, x, y)$ can be represented in the form (1.5).

For $\lambda > 0$ the Markovian resolvent $G_\lambda = (\lambda + L)^{-1}$ admits a continuous strictly positive integral kernel $g(\lambda, x, y)$. The operator G_λ is well defined for $\lambda = 0$ (i.e. the Markovian semigroup $(P_t)_{t>0}$ is transient) if and only if for some (equivalently, for all) $x \in X$ the volume function $\tau \to 1/V(\tau)$ is integrable at ∞. The integral kernel $g(x, y) := g(0, x, y)$, called also the Green function, is of the form

$$g(x,y) = \int_r^{+\infty} \frac{d\tau}{V(\tau)}, \ r = d_*(x,y). \tag{2.8}$$

Under certain Tauberian conditions it takes the form

$$g(x,y) \asymp \frac{r}{V(r)}, \ r = d_*(x,y).[1] \tag{2.9}$$

2.3 Subordination

Let $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing homeomorphism. For any two nearest neighbouring balls $B \subset B'$ we define

$$C(B) = \Phi\left(1/m(B)\right) - \Phi\left(1/m(B')\right). \tag{2.10}$$

Let L_Φ be the corresponding hierarchical Laplacian.The following properties hold:

(i) $\lambda(B) = \Phi\left(1/m(B)\right)$. In particular, the Laplacians L_Φ and L_{Id} are related by the equation $L_\Phi = \Phi(L_{Id})$. [2]

(ii) $d_*(x,y) = 1/\Phi\left(1/m(x \curlywedge y)\right)$.

(iii) $V(r) \leq 1/\Phi^{-1}(1/r)$.

(iv) $V(r) \asymp 1/\Phi^{-1}(1/r)$ whenever both Φ and Φ^{-1} are doubling and $m(B') \leq cm(B)$ for some $c > 0$ and all nearest neighboring balls $B \subset B'$. In particular, in this case we have

$$p_\Phi(t,x,y) \asymp t \cdot \min\left\{\frac{1}{t}\Phi^{-1}\left(\frac{1}{t}\right), \frac{1}{m(x \curlywedge y)}\Phi\left(\frac{1}{m(x \curlywedge y)}\right)\right\}.$$

2.4 Multipliers

As a special case of the general construction consider $X = \mathbb{Q}_p$, the ring of p-adic numbers equipped with its standard ultrametric $\mathrm{d}(x,y) = \|x-y\|_p$. Notice that the ultrametric spaces $(\mathbb{Q}_p, \mathrm{d})$ and $([0,\infty), d)$ with non-eucledian d, as explained in the introduction, are isometrically isomorphic (the isometry can be established via identification of their trees of metric balls).

Let $\mathcal{F} : f \to \widehat{f}$ be the Fourier transform of the function f. It is known, see [40], [43], [27], that $\mathcal{F} : \mathcal{D} \to \mathcal{D}$ is a bijection.

[1]The sign $\asymp$ means that the ratio of the left- and right hand sides is bounded from below and above by positive constants.

[2]In the case $\Phi(\tau)$ is a Bernstein function the association $L_\Phi = \Phi(L_{Id})$ has been studied in the well-known Bochner's subordination theory [19].

Let $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing homeomorphism. The self-adjoint operator $\Phi(\mathfrak{D})$ we define as multiplier, that is,

$$\widehat{\Phi(\mathfrak{D})f}(\xi) = \Phi(\|\xi\|_p)\widehat{f}(\xi), \quad \xi \in \mathbb{Q}_p. \tag{2.11}$$

By [5, Theorem 3.1], $\Phi(\mathfrak{D})$ is a homogeneous hierarchical Laplacian. The eigenvalues $\lambda(B)$ of the operator $\Phi(\mathfrak{D})$ are numbers

$$\lambda(B) = \Phi\left(\frac{p}{m(B)}\right) = \Phi\left(\frac{p}{\operatorname{diam}(B)}\right). \tag{2.12}$$

Let $p_\Phi(t,x,y)$ be the heat kernel associated with the operator $\Phi(\mathfrak{D})$. Assuming that both Φ and Φ^{-1} are doubling we get the following relationship

$$p_\Phi(t,x,y) \asymp t \cdot \min\left\{\frac{1}{t}\Phi^{-1}\left(\frac{1}{t}\right), \frac{1}{\|x-y\|_p}\Phi\left(\frac{1}{\|x-y\|_p}\right)\right\}. \tag{2.13}$$

The Taibleson-Vladimirov operator $\mathfrak{D}^\alpha$ introduced in [40] and [43] is the multiplier corresponding to the function $\Phi(\tau) = \tau^\alpha$. On the set $\mathcal{D}$ it can be represented in the form

$$\mathfrak{D}^\alpha\psi(x) = -\frac{1}{\Gamma_p(-\alpha)}\int_{\mathbb{Q}_p}\frac{\psi(x)-\psi(y)}{\|x-y\|_p^{1+\alpha}}dm(y), \tag{2.14}$$

where $\Gamma_p(z) = (1-p^{z-1})(1-p^{-z})^{-1}$ is the p-adic Gamma-function [43, Sec.VIII.2]. The function $z \to \Gamma_p(z)$ is meromorphic in the complex plane $\mathbb{C}$ and satisfies the functional equation $\Gamma_p(z)\Gamma_p(1-z) = 1$.

By what we said above the heat kernel $p_\alpha(t,x,y)$, the transition density of the Markovian semigroup $(e^{-t\mathfrak{D}^\alpha})_{t>0}$, can be estimated as follows

$$p_\alpha(t,x,y) \asymp \frac{t}{(t^{1/\alpha}+\|x-y\|_p)^{1+\alpha}}, \tag{2.15}$$

In particular, the Markov semigroup $(e^{-t\mathfrak{D}^\alpha})_{t>0}$ is transient if and only if $\alpha < 1$. In the transient case the Green function $g_\alpha(x,y)$ can be computed explicitly

$$g_\alpha(x,y) = \frac{1}{\Gamma_p(\alpha)}\frac{1}{\|x-y\|_p^{1-\alpha}}. \tag{2.16}$$

For all facts listed above we refer the reader to [4], [5] and [6].

3 Schrödinger type operators

Let (X,d,m) be a homogeneous ultrametric measure space and L a homogeneous hierarchical Laplacian on it. In this section we embark on the study of Schrödinger type operators acting in $L^2(X,m)$. These are operators of the form

$$Hf(x) = Lf(x) + V(x)f(x). \tag{3.17}$$

Our goal is to find conditions such that one can associate with equation (3.17) a self-adjoint operator H.

3.1 The symbol of the hierarchical Laplacian

Identifying (X,d) with a locally compact Abelian group we can regard $-L$ as an isotropic Lévy generator. By (1.3), the operator L on $\mathcal{D}$ takes the form

$$Lf(x)=\int_X (f(x)-f(y))J(x-y)dm(y), \tag{3.18}$$

or equivalently, in terms of the Fourier transform,

$$\widehat{Lf}(\theta)=\widehat{L}(\theta)\cdot\widehat{f}(\theta),\ \theta\in\widehat{X}, \tag{3.19}$$

where $\widehat{X}$ is the dual Abelian group (e.g. $\widehat{\mathbb{Q}_p}$ can be identified with $\mathbb{Q}_p$) and

$$\widehat{L}(\theta)=\int_X [1-\mathrm{Re}\,\langle h,\theta\rangle]J(h)dm(h). \tag{3.20}$$

The function $\widehat{L}(\theta)\geq 0$, the symbol of symmetric Lévy generator $-L$, is a continuous *negative definite function* [11]. By [11, Proposition 7.15], the function $\sqrt{\widehat{L}(\theta)}$ is subadditive. By the subordination property [5, Theorem 3.1], the function $\widehat{L}(\theta)^2$ is the symbol of symmetric Lévy generator $-L^2$, so the function $\widehat{L}(\theta)=\sqrt{\widehat{L}(\theta)^2}$ is subadditive as well, i.e. it satisfies the triangle inequality

$$\widehat{L}(\theta_1+\theta_2)\leq\widehat{L}(\theta_1)+\widehat{L}(\theta_2). \tag{3.21}$$

Since $-L$ is an isotropic Lévy generator [5, Sec. 5.2] , more strong property holds true

Theorem 3.1 *The function $\widehat{L}(\theta)$ satisfies the ultrametric inequality*

$$\widehat{L}(\theta_1+\theta_2)\leq\max\{\widehat{L}(\theta_1),\widehat{L}(\theta_2)\}. \tag{3.22}$$

Proof. In order to simplify notation we assume that $X=\mathbb{Q}_p$, the ring of p-adic numbers. Let $B\subset B'$ be two nearest neighboring balls centred at the neutral element. Notice that both B and B' are compact subgroups of the group $\mathbb{Q}_p$, say $B=p^{-k}\mathbb{Z}_p$ and $B'=p^{-k-1}\mathbb{Z}_p$.

Applying the Fourier transform to the both sides of equation (2.4) we get

$$\widehat{L}(\theta)\widehat{f_B}(\theta)=\lambda(B')\widehat{f_B}(\theta). \tag{3.23}$$

The measure $\omega_B=(\mathbf{1}_B m)/m(B)$ is the normalized Haar measure of the compact subgroup B, similarly $\omega_{B'}$. Since for any locally compact Abelian group, the Fourier transform of the normalized Haar measure of any compact subgroup A is the indicator of its annihilator group $A^{\perp}$, and in our particular case $B^{\perp}=p^k\mathbb{Z}_p$ and $(B')^{\perp}=p^{k+1}\mathbb{Z}_p$, we obtain

$$\widehat{f_B}(\theta)=\mathbf{1}_{B^{\perp}}(\theta)-\mathbf{1}_{(B')^{\perp}}(\theta)=\mathbf{1}_{\partial B^{\perp}}(\theta), \tag{3.24}$$

where $\partial B^{\perp}$ is the sphere $B^{\perp}\setminus(B')^{\perp}$.

Equations (3.24) and (2.4) imply that the function $\widehat{L}(\theta)$ takes constant value $\lambda(B')$ on the sphere $\partial B^{\perp}$, i.e. $\widehat{L}(\theta)=\psi(\|\theta\|_p)$ for some function $\psi(\tau)$ such that $\psi(0)=0$ and $\psi(+\infty)=+\infty$. Since $C\subset D$ implies $\lambda(C)>\lambda(D)$, the function $\psi(\tau)$ can be chosen to be continuous and increasing, so $\widehat{L}(\theta)=\psi(\|\theta\|_p)$ satisfies the ultrametric inequality (3.22) as claimed. ■

3.2 Locally bounded potentials

If we assume that the potential V is a locally bounded function then

$$(Hu)(x) := (Lu)(x) + V(x)u(x)$$

is a well defined symmetric operator $H : \mathcal{D} \to L^2(X, m)$. For the proof of the following theorem we refer to the paper [9, Theorem 3.1]

Theorem 3.2 *Assume that V is a locally bounded function, then*

1. The operator H is essentially self-adjoint.

2. If $V(x) \to +\infty$ as $x \to \varpi$, then the self-adjoint operator H has a compact resolvent. (Thus, its spectrum is discrete).

3. If $V(x) \to 0$ as $x \to \varpi$, then the essential spectrum of H coincides with the spectrum of L. (Thus, the spectrum of H is pure point and the negative part of the spectrum consists of isolated eigenvalues of finite multiplicity).

Remark 3.3 *For the classical Schrödinger operator $H = -\Delta + V$ defined on the set of test functions C_0^∞ the statement similar to the statement 1 of Theorem 3.2 is known as the Sears's theorem: H is essentially self-adjoint if the potential V admits a low bound*

$$V(x) \geq Q(|x|),$$

where $0 \leq Q(r) \in C(\mathbb{R}_+)$ a non-decreasing function such that

$$\int_0^\infty Q(r)^{-1/2} dr = \infty,$$

and it may fail to be essentially self-adjoint otherwise, see [10, Chapter II, Theorem 1.1 and Example 1.1].

3.3 Potentials with local singularities

If we are interested in potentials with local singularities, such as $V(x) = \|x\|_p^{-\beta}$, then certain local conditions on the potential V are necessary in order to prove that the quadratic form

$$Q(u, u) = Q_L(u, u) + Q_V(u, u) \tag{3.25}$$

defined on the set

$$dom(Q) := dom(Q_L) \cap dom(Q_V)$$

is a densly defined closed and bounded below quadratic form and whence it is associated to a bounded below self-adjoint operator H. That means precisely that there exists a constant $c > 0$ and a self-adjoint operator H such that the form $Q'(u, u) := Q(u, u) + c(u, u)$ (resp. the operator $H + c\mathrm{I}$) is non-negative definite and that

$$Q'(u, u) = ((H + c\mathrm{I})^{1/2} u, (H + c\mathrm{I})^{1/2} u) \tag{3.26}$$

for all $u \in dom(Q)$.

It is customary to write $H = L + V$, but it must be remembered that this is a quadratic form sum and not an operator sum as in the previous subsection.

Theorem 3.4 *If* $0 \le V \in L^1_{loc}(X,m)$*, then the quadratic form* (3.25) *is a regular Dirichlet form* [20]*. In particular, it is the form of a non-negative self-adjoint operator* H*,*

$$Q(u,u) = (H^{1/2}u, H^{1/2}u)$$

and the set $\mathcal{D}$ *is a core for* Q*.*

Proof. The set $\mathcal{D}$ beongs to both $dom(Q_L)$ and $dom(Q_V)$ hence Q is densly defined. Set $V_\tau = V \wedge \tau$ and define on the set $dom(Q_L)$ the form

$$Q^\tau(u,u) = Q_L(u,u) + Q_{V_\tau}(u,u).$$

Since V_τ is bounded the form Q^τ is closed. In particular, the function $u \to Q^\tau(u,u)$ is lower semicontinuous. Clearly $Q(u,u) = \sup\{Q^\tau(u,u) : \tau > 0\}$. It follows that the function $u \to Q(u,u)$ is lower semicontinuous. Hence by [14, Theorem 4.4.2] the form Q is closed, and thus it is the form of a non-negative definite self-adjoint operator H. Clearly the form Q is Markovian hence it is a Dirichlet form.

Let us show that $\mathcal{D}$ is a core for Q, i.e. Q is a regular Dirichlet form.

Step 1 For $u \in dom(Q)$ we set $u_n = ((-n) \vee u) \wedge n$, then $u_n \in dom(Q)$ and $Q(u-u_n, u-u_n) \to 0$, see [20, Theorem 1.4.2]. Therefore the set of bounded functions in $dom(Q)$ is a core for Q.

Step 2 Let B be a ball centred at the neutral element. Let $u \in dom(Q)$ be bounded and $u_B = 1_B \cdot u$. The function 1_B is in $dom(Q)$ (and even in $dom(L)$), whence applying [20, Theorem 1.4.2] we get: $u_B \in dom(Q)$ and

$$\sqrt{Q(u_B,u_B)} \le \sqrt{Q(u,u)} + \|u\|_\infty \cdot \sqrt{Q(1_B,1_B)}.$$

It is straightforward to show that

$$m(B)\lambda(B') \le Q(1_B,1_B) \le 2m(B)\lambda(B'),$$

where B' is the closest neighboring ball containing B, and $\lambda(B')$ is the eigenvalue of L corresponding to the ball B'. Thus, if we assume that

$$\lim_{B \nearrow X} m(B)\lambda(B') = 0 \tag{3.27}$$

(as it happens in the case of the operator $L = \mathfrak{D}^\alpha,\ \alpha > 1$) then the following contraction property holds

$$\limsup_{B \nearrow X} Q(u_B,u_B) \le Q(u,u). \tag{3.28}$$

Let $(R_\lambda)_{\lambda>0}$ be the Markov resolvent corresponding to Q. Let $Q_1(s,t) := Q(s,t)+(s,t)$. Then for any $v \in L^2(X,m)$,

$$Q_1(u_B, R_1 v) = (u_B, v) \to (u,v) = Q_1(u, R_1 v)$$

Since $R_1(L^2(X,m))$ is dense in $dom(Q)$ with respect to the metric Q_1, the sequence u_B weakly converges to u with respect to Q_1:

$$Q_1(u_B,w) \to Q_1(u,w), \quad \forall w \in dom(Q). \tag{3.29}$$

Using equations (3.28) and (3.29) we obtain:

$$\begin{aligned}\limsup_{B\nearrow X} Q_1(u-u_B,u-u_B) &= \limsup_{B\nearrow X}\left(Q_1(u,u)-2Q_1(u_B,u)+Q_1(u_B,u_B)\right)\\ &= Q_1(u,u)-2\lim_{B\nearrow X} Q_1(u_B,u)+\limsup_{B\nearrow X} Q_1(u_B,u_B)\\ &\le Q_1(u,u)-2Q_1(u,u)+Q_1(u,u)=0.\end{aligned}$$

Thus, if condition (3.27) holds, the set of bounded functions with compact support in $dom(Q)$ is a core for Q as desired.

Step 3 In general to prove contraction property (3.28) we proceed as follows. Any ball B centred at the neutral element is the compact subgroup of X. Since the Fourier transform of the normalized Haar measure of a compact subgroup is the indicator of its annihilator group, we obtain

$$\begin{aligned}Q_L(u_B,u_B) &= \int_{\widehat{X}} \widehat{L}(\theta)\,|\widehat{u_B}(\theta)|^2\,d\widehat{m}(\theta)\\ &= \int_{\widehat{X}} \widehat{L}(\theta)\,|\widehat{u}*\widehat{m}_{B^\perp}(\theta)|^2\,d\widehat{m}(\theta),\end{aligned}$$

where $\widehat{L}(\theta)$ is the sumbol of the multiplier L, $B^\perp$ is the annihilator group of the compact subgroup $B\subset X$ and $\widehat{m}_{B^\perp}$ is the normed Haar measure of $B^\perp$. The function $\widehat{L}(\theta)$ is an increasing function of the p-adic norm $\|\theta\|_p$ whence it satisfies the ultrametric inequality. Having this in mind and using the inequality

$$|\widehat{u}*\widehat{m}_{B^\perp}|^2 \le |\widehat{u}|^2 * \widehat{m}_{B^\perp}$$

we get

$$\begin{aligned}Q_L(u_B,u_B) &\le \int_{\widehat{X}} \widehat{L}(\theta)\left(|\widehat{u}|^2*\widehat{m}_{B^\perp}\right)(\theta)d\widehat{m}(\theta)\\ &= \int_{\widehat{X}} \widehat{L}(\theta)\left(\int_{B^\perp}|\widehat{u}(\theta+\zeta)|^2\,d\widehat{m}_{B^\perp}(\zeta)\right)d\widehat{m}(\theta)\\ &= \int_{B^\perp}\left(\int_{\widehat{X}} \widehat{L}(\theta+\zeta)\,|\widehat{u}(\theta)|^2\,d\widehat{m}(\theta)\right)d\widehat{m}_{B^\perp}(\zeta).\end{aligned}$$

Whence, using the ultrametric inequality (3.22), we obtain

$$\begin{aligned}Q_L(u_B,u_B) &\le \int_{B^\perp}\left(\int_{\widehat{X}} \max\left\{\widehat{L}(\theta),\widehat{L}(\zeta)\right\}|\widehat{u}(\theta)|^2\,d\widehat{m}(\theta)\right)d\widehat{m}_{B^\perp}(\zeta)\\ &\le \int_{B^\perp}\left(\int_{\widehat{X}}\left(\widehat{L}(\theta)+\widehat{L}(\zeta)\right)|\widehat{u}(\theta)|^2\,d\widehat{m}(\theta)\right)d\widehat{m}_{B^\perp}(\zeta)\\ &= Q_L(u,u)+\left(\int_{B^\perp}\widehat{L}(\zeta)d\widehat{m}_{B^\perp}(\zeta)\right)(u,u).\end{aligned}$$

When $B\nearrow X$ the measure $\widehat{m}_{B^\perp}$ converges weakly to the Dirac measure concentrated at the neutral element, whence we finally obtain the following inequality

$$\limsup_{B\nearrow X} Q_L(u_B,u_B)\le Q_L(u,u). \tag{3.30}$$

Evidently inequality (3.30) implies inequality (3.28) as desired.

Step 4 Now let $u \in dom(Q)$ be bounded and of compact support. Let B be a ball centred at the neutral element of X (a compact subgroup of X) and m_B be its normed Haar measure. We set $u^B = u * m_B$. The function u^B is locally constant and has a compact support, hence belongs to $dom(Q)$. We have $\widehat{u^B} = \widehat{u} \cdot 1_{B^\perp}$ whence

$$\left\|u - u^B\right\|_2^2 = \int\limits_{(B^\perp)^c} |\widehat{u}(\theta)|^2 \, d\widehat{m}(\theta)$$

which converges to zero as B converges to the trivial subgroup $\{e\}$. A similar argument establishes that

$$\lim_{B \to \{e\}} Q_L(u - u^B, u - u^B) = \lim_{B \to \{e\}} \int\limits_{(B^\perp)^c} \widehat{L}(\theta) \, |\widehat{u}(\theta)|^2 \, d\widehat{m}(\theta) = 0.$$

There exists a compact set K which contains the support of every $u - u^B$ when the diameter of B is less or equal one. Given $\varepsilon > 0$ there exists a decomposition $V|_K = V_1 + V_2$ such that $\|V_1\|_1 < \varepsilon$ and $V_2 \in L^\infty(X, m)$. We then have

$$\begin{aligned} Q_V(u - u^B, u - u^B) &= \int\limits_K V \left|u - u^B\right|^2 dm \\ &= \int\limits_K V_1 \left|u - u^B\right|^2 dm + \int\limits_K V_2 \left|u - u^B\right|^2 dm \\ &\leq 4\varepsilon \, \|u\|_\infty^2 + \|V_2\|_\infty \left\|u - u^B\right\|_2^2 . \end{aligned}$$

Therefore

$$\limsup_{B \to \{e\}} Q_V(u - u^B, u - u^B) \leq 4\varepsilon \, \|u\|_\infty^2$$

for all $\varepsilon > 0$. In other words

$$\lim_{B \to \{e\}} Q_1(u - u^B, u - u^B) = 0$$

and thus $\mathcal{D}$ is indeed a core for $Q = Q_L + Q_V$. ■

Remark 3.5 *It is clear that the above theorem can be extended to V which are bounded below and in $L^1_{loc}(X, m)$ by simply adding a large enough positive constant. If, however, we are interested in V with negative local singularities, then stronger local conditions on V are necessary in order to be able to prove that the form Q is closed.*

Definition 3.6 *Let $p \geq 1$ be fixed. We say that a potential V lies in $L^p + L^\infty$ if one can write $V = V' + V''$ where $V' \in L^p(X, m)$ and $V'' \in L^\infty(X, m)$. This decomposition is not unique, and, if it is possible at all, then one can arrange for $\|V'\|_p$ to be as small as one chooses.*

Theorem 3.7 *Let $L = \mathfrak{D}^\gamma$, and let $Q = Q_L + Q_V$ be quadratic form (3.25) where $V \in L^p + L^\infty$ for some $p > 1/\gamma$. Then:*

1. Q is a densly defined closed and bounded below form whence it is associated with a bounded below self-adjoint operator H.

2. If $2 \leq 1/\gamma < p$ then $dom(H) = dom(\mathfrak{D}^\gamma)$. The same is true if $1/\gamma < 2$ and $p = 2$.

Proof. The set $\mathcal{D}$ is in both $dom(Q_L)$ and $dom(Q_V)$ whence Q is densly defined. Given $\varepsilon > 0$ we may write $|V| = W + W'$ where $\|W\|_p < \varepsilon$ and $W' \in L^\infty(X,m)$. We claim that if $\varepsilon > 0$ is sufficiently small, then

$$\left\|W^{1/2}u\right\|_2^2 \leq \frac{1}{2}Q_L(u,u) + c_0\|u\|_2^2 \tag{3.31}$$

for some constant $c_0 > 0$ and all $u \in dom(Q_L)$.

Clearly inequality 3.31 yield that

$$\begin{aligned}\int |V|\,|u|^2\,dm &\leq \left\|W^{1/2}u\right\|_2^2 + \|W'\|_\infty \|u\|_2^2 \\ &\leq \frac{1}{2}Q_L(u,u) + c_1\|u\|_2^2\end{aligned}$$

for some constant $c_1 > 0$ and all $u \in dom(Q_L)$. Thus for $c_2 > 2c_1$ we get

$$\frac{1}{2}\left\{Q_L(u,u) + c_2\|u\|_2^2\right\} \leq Q(u,u) + c_2\|u\|_2^2 \leq \frac{3}{2}\left\{Q_L(u,u) + c_2\|u\|_2^2\right\}.$$

It follows that the quadratic form $u \to Q(u,u) + c_2\|u\|_2^2$ is non-negative and closed whence it is associated with a non-negative self-adjoint operator, which is clearly equal to $H + c_2 I$.

To prove the claim 3.31 we need some L^p-estimates. Recall that the number $D = 2/\gamma$ is called *the spectral dimension* related to the operator $\mathfrak{D}^\gamma$. The estimates $(E.1)$ and $(E.2)$ below are similar to the classical estimates for the Hamiltonian $-\Delta$ in the Eucledian space $\mathbb{R}^D$, see [14, Sec. 3.6].

E1. If $0 < \alpha \leq 1/(2\gamma)$ and $2 \leq p < 2/(1-2\alpha\gamma)$, then $(\mathfrak{D}^\gamma + \mathrm{I})^{-\alpha}$ is a bounded linear operator from $L^2(X,m)$ to $L^p(X,m)$. If $\alpha > 1/(2\gamma)$, then $(\mathfrak{D}^\gamma + \mathrm{I})^{-\alpha}$ is a bounded linear operator from $L^2(X,m)$ to $L^\infty(X,m)$.

E2. If $0 < \alpha \leq 1/(2\gamma)$ and $\mathcal{W} \in L^q(X,m)$ is a multiplication linear operator, then $\mathcal{A} := \mathcal{W}\cdot(\mathfrak{D}^\gamma + \lambda\mathrm{I})^{-\alpha}$ is a bounded linear operator on $L^2(X,m)$ provided $1/(\alpha\gamma) < q \leq \infty$. Moreover, there exists a constant $c > 0$ such that $\|\mathcal{A}\|_{L^2\to L^2} \leq c\|\mathcal{W}\|_q$ for all such $\mathcal{W}$. The same bound holds in the case $\alpha > 1/(2\gamma)$ and $q = 2$. In both cases the operator $\mathcal{A}$ is a compact operator on L^2. Moreover, $\lim_{\lambda\to\infty}\|\mathcal{W}\cdot(\mathfrak{D}^\gamma + \lambda\mathrm{I})^{-\alpha}\|_{L^2\to L^2} = 0$.

Proof of the statement E1. Assume first that $0 < \alpha \leq 1/(2\gamma)$. If we define the function $g(y) := (\|y\|_p^\gamma + 1)^{-\alpha}$ and assume that $1/(\alpha\gamma) < s \leq \infty$ then

$$\|g\|_s^s = \int_{\mathbb{Q}_p} \frac{dm(y)}{(\|y\|_p^\gamma + 1)^{\alpha s}} = \left(1 - \frac{1}{p}\right)\sum_{\tau=-\infty}^{\infty} \frac{p^\tau}{(p^{\tau\gamma}+1)^{\alpha s}} < \infty.$$

If $k = \widehat{(\mathfrak{D}^\gamma + I)^{-\alpha}f}$ and $f \in L^2$, then $k(y) = g(y)\widehat{f}(y)$. Putting $1/q = 1/s + 1/2$ we deduce that $1 < q \leq 2$ and

$$\|k\|_q \leq \|g\|_q \left\|\widehat{f}\right\|_2 = c_1\|f\|_2.$$

If $1/p + 1/q = 1$, then $2 \le p < \infty$ and it follows from the Hausdorff-Young theorem that
$$\left\|(\mathfrak{D}^\gamma + I)^{-\alpha} f\right\|_p = \left\|\widehat{k}\right\|_p \le \|k\|_q \le c_1 \|f\|_2 .$$
We have $1/p = 1 - 1/q = 1/2 - 1/s$ and $1/(\alpha\gamma) < s \le \infty$, whence p increases from 2 to $2/(1 - 2\alpha\gamma)$ as s decreases from ∞ to $1/(\alpha\gamma)$.

If $\alpha > 1/(2\gamma)$, then the function g defined above lies in L^2 and we deduce that
$$\|k\|_1 = \left\|g\widehat{f}\right\|_1 \le \|g\|_2 \left\|\widehat{f}\right\|_2 = c_2 \|f\|_2$$
whence as above
$$\left\|(\mathfrak{D}^\gamma + I)^{-\alpha} f\right\|_\infty = \left\|\widehat{k}\right\|_\infty \le \|k\|_1 \le c_2 \|f\|_2$$
as desired.

Proof of the statement E2. If $0 < \alpha \le 1/(2\gamma)$, then
$$\left\|\mathcal{W}\cdot(\mathfrak{D}^\gamma + I)^{-\alpha} f\right\|_2 \le \|\mathcal{W}\|_q \left\|(\mathfrak{D}^\gamma + I)^{-\alpha} f\right\|_p$$
provided $1/2 = 1/p + 1/q$. The condition $2 \le p < 2/(1 - 2\alpha\gamma)$ is equivalent to $1/(\alpha\gamma) < q \le \infty$. We apply the statement E1 to get the desired conclusion. The case $\alpha > 1/(2\gamma)$ is similar,
$$\left\|\mathcal{W}\cdot(\mathfrak{D}^\gamma + I)^{-\alpha} f\right\|_2 \le \|\mathcal{W}\|_2 \left\|(\mathfrak{D}^\gamma + I)^{-\alpha} f\right\|_\infty .$$
To prove compactness of the operator $\mathcal{A} = \mathcal{W}\cdot(\mathfrak{D}^\gamma + I)^{-\alpha}$ we choose a sequence $\mathcal{W}_n \in \mathcal{D}$ such that $\mathcal{W}_n \to \mathcal{W}$ in L^q. Let Φ_n be a strictly increasing function such that $\Phi_n(\tau) = \tau^\gamma$ for $0 \le \tau \le n$ and $\Phi_n(\tau) \asymp e^\tau$ as $\tau \to \infty$. If we set $\mathcal{A}_n = \mathcal{W}_n\cdot(\Phi_n(\mathfrak{D}) + I)^{-\alpha}$ then $\mathcal{A}_n \to \mathcal{A}$ in the operator norm. Since the set of compact operators is closed under norm limits, it is sufficient to prove that each $\mathcal{A}_n$ is a Hilbert-Schmidt operator. Each operator $\mathcal{A}_n$ is unitary equivalent to the integral operator $\widehat{\mathcal{A}_n} : \widehat{u} \to \widehat{\mathcal{A}_n u}$ which has the kernel
$$\widehat{\mathcal{A}_n}(\theta, \zeta) = \widehat{\mathcal{W}_n}(\theta - \zeta)(\Phi_n(\|\zeta\|) + 1)^{-\alpha} := \widehat{\mathcal{W}_n}(\theta - \zeta)\mathcal{G}(\zeta)$$
so that the Hilbert-Schmidt norm $\left\|\widehat{\mathcal{A}_n}\right\|$of the operator $\widehat{\mathcal{A}_n}$ is
$$\left\|\widehat{\mathcal{A}_n}\right\| = \|\mathcal{W}_n\|_2 \|\mathcal{G}\|_2 < \infty.$$
Thus the operator $\mathcal{A} = \mathcal{W}\cdot(\mathfrak{D}^\gamma + I)^{-\alpha}$ is endeed a compact operator.

Let us turn to the proof of the claim 3.31. To prove the claim in the case $0 < \gamma \le 1$ and $p > 1/\gamma$ we write
$$\begin{aligned}
\left\|W^{1/2}u\right\|_2^2 &= \left\|W^{1/2} \cdot (\mathfrak{D}^\gamma + I)^{-1/2} \cdot (\mathfrak{D}^\gamma + I)^{1/2} u\right\|_2^2 \\
&\le \left\|W^{1/2} \cdot (\mathfrak{D}^\gamma + I)^{-1/2}\right\|_{L^2\to L^2}^2 \left\|(\mathfrak{D}^\gamma + I)^{1/2} u\right\|_2^2 \\
&= \left\|W^{1/2} \cdot (\mathfrak{D}^\gamma + I)^{-1/2}\right\|_{L^2\to L^2}^2 \left(Q_L(u,u) + \|u\|_2^2\right) \\
&\le c \left\|W^{1/2}\right\|_q^2 \left(Q_L(u,u) + \|u\|_2^2\right) \le \frac{1}{2} Q_L(u,u) + c_1 \|u\|_2^2
\end{aligned}$$

provided $\varepsilon > 0$ is chosen small enough and $q = 2p > 2/\gamma$ as in the statement E2 with $\alpha = 1/2$.

The case $\gamma > 1$ is similar: The restriction $p > 1/\gamma$ becomes $p \geq 1$. We set $Y = \{|V| > \tau\}$ and $W = |V| 1_Y$. By Markov inequality $m(Y) \leq \tau^{-p} \|V\|_p^p < \infty$ whence $\|W\|_1 = o(1)$ as $\tau \to \infty$. In particular, $W^{1/2} \in L^2$ and $\left\|W^{1/2}\right\|_2 = o(1)$ as $\tau \to \infty$. Applying the second part of the statement E2 with $\alpha = 1/2$ and $q = 2$ we come to the conclusion

$$\begin{aligned}\left\|W^{1/2}u\right\|_2^2 &\leq c\left\|W^{1/2}\right\|_2^2 \left(Q_L(u,u) + \|u\|_2^2\right) \\ &\leq \frac{1}{2}Q_L(u,u) + c_1\|u\|_2^2,\end{aligned}$$

as desired.

To prove that $dom(H) = dom(\mathfrak{D}^\gamma)$ we first write $V = V' + V''$, where $V' \in L^p(X,m)$ and $V'' \in L^\infty(X,m)$. The statement E2 yields that

$$\lim_{t\to\infty} \left\|V' \cdot (\mathfrak{D}^\gamma + t\mathrm{I})^{-1}\right\|_{L^2\to L^2} = 0.$$

We also have

$$\left\|V'' \cdot (\mathfrak{D}^\gamma + t\mathrm{I})^{-1}\right\|_{L^2\to L^2} \leq \|V''\|_\infty \left\|(\mathfrak{D}^\gamma + t\mathrm{I})^{-1}\right\|_{L^2\to L^2} = t^{-1}\|V''\|_\infty$$

for all $t > 0$, so

$$\lim_{t\to\infty} \left\|V \cdot (\mathfrak{D}^\gamma + t\mathrm{I})^{-1}\right\|_{L^2\to L^2} = 0.$$

For any $1 > \delta > 0$ small enough we conclude that if $t > 0$ is large enough then

$$\|Vf\|_2 \leq \delta\|\mathfrak{D}^\gamma f\|_2 + t\delta\|f\|_2$$

for all $f \in dom(\mathfrak{D}^\gamma)$. Thus V is a relatively bounded perturbation of $\mathfrak{D}^\gamma$with a relative bound $\delta < 1$ whence $dom(\mathfrak{D}^\gamma + V) = dom(\mathfrak{D}^\gamma)$ by an application of [14, Theorem 1.4.2]. The proof is now completed ∎

Next we discuss several results giving information about the negative part of the spectrum of H.

Theorem 3.8 *Let $L = \mathfrak{D}^\gamma$ and let $V \in L^p(X,m)$ for some $p > 1/\gamma$. Then:*

1. The operator $H = L + V$ has essential spectrum equals to the spectrum of L. In particular, if H has any negative spectrum, then it consists of a sequence of negative eigenvalues of finite multiplicity. If this sequence is infinite then it converges to zero.

2. Suppose that there exists an open set $U \subset X$ on which V is negative. If E_λ is the bottom of the spectrum of the operator $H_\lambda = L + \lambda V$, then $E_\lambda \leq 0$ for all $\lambda \geq 0$ and $\lim_{\lambda\to\infty} E_\lambda = -\infty$.

Proof. 1. By Theorem 3.7, if $c > 0$ is large enough then the operator $H + cI$ is non-negative and

$$\frac{1}{2}\left\|(L+cI)^{1/2}u\right\|_2 \leq \left\|(H+cI)^{1/2}u\right\|_2 \leq \frac{3}{2}\left\|(L+cI)^{1/2}u\right\|_2 \tag{3.32}$$

for all $u \in dom(Q_L)$. Let us define $\Delta := (L+cI)^{-1} - (H+cI)^{-1}$, then

$$\Delta = (L+cI)^{-1}V(H+cI)^{-1} = ABCDE$$

where $A = (L+cI)^{-1/2}$, $B = (L+cI)^{-1/2}|V|^{1/2}$, $C = sign(V)B^*$, $D = (L+cI)^{1/2}(H+cI)^{-1/2}$ and $E = (H+cI)^{-1/2}$. It is clear that A and E are bounded operators on L^2, B^* and C are compact operators on $L^2((X,m)$, see the statement E2 in the proof of Theorem 3.7, and D is a bounded operator on $L^2(X,m)$ by equation (3.32). Thus, as a product of compact and bounded operators, the difference of two resolvents Δ is a compact operator on L^2. By perturbation theory of linear operators, H and L have the same essential spectrum, see e.g. [25]. Since $Spec_{ess}(L) = Spec(L) \subset [0,\infty[$, any negative point in the spectrum of H must be an isolated eigenvalue of finite multiplicity. Any limit of negative eigenvalues lies in the essential spectrum whence the only possible limit is zero.

2. We use the first statement to prove that $E_\lambda \leq 0$ for all $\lambda \geq 0$. Observe that

$$E_\lambda = \inf\{Q_L(u,u) + \lambda Q_V(u,u) : u \in \mathcal{D} \text{ and } \|u\|_2 = 1\} \tag{3.33}$$

because $\mathcal{D}$ is a core for $Q_L + \lambda Q_V$. Let us choose $u \in \mathcal{D}$ having support in the set U, then as $\lambda \to \infty$ we get

$$\begin{aligned} E_\lambda &\leq Q_L(u,u) + \lambda Q_V(u,u) \\ &= Q_L(u,u) - \lambda \int_U |V|\,|u|^2\, dm \to -\infty \end{aligned}$$

as was claimed. ■

3.4 Positivity of the spectrum

We prove here criteria for positivity of the spectum of the operator $H = L + V$. We use the notion of *the square of gradient* $\Gamma(u,v)$ defined as follows: for all $u, v \in \mathcal{D}$ we set

$$\Gamma(u,v) := \frac{1}{2}\{uLv + vLu - L(uv)\}. \tag{3.34}$$

It is straightforward to show that the following identities hold true:

$$Q_L(u,v) = \int_X \Gamma(u,v)dm, \tag{3.35}$$

$$Q_L(uv,w) = \int_X v\Gamma(u,w)dm + \int_X u\Gamma(v,w)dm, \tag{3.36}$$

$$\begin{aligned} &\int_X v\Gamma(u^2,w)dm - 2\int_X vu\Gamma(u,w)dm \\ &= \frac{1}{2}\int_{X\times X} (u(y)-u(x))^2 (w(y)-w(x))(v(y)-v(x))\, J(x-y)dm(x)dm(y). \end{aligned} \tag{3.37}$$

Here $J(x-y)$ is the jump kernel associated with the (non-local) operator L, see equations (3.18) and (3.20). In particular, we have

$$\begin{aligned} &\int_X w\Gamma(u^2,w)dm - 2\int_X wu\Gamma(u,w)dm \\ &= \frac{1}{2}\int_{X\times X} (u(y)-u(x))^2 (w(y)-w(x))^2\, J(x-y)dm(x)dm(y) \geq 0. \end{aligned} \tag{3.38}$$

The identities listed above can be extended to the set of all bounded functions u, v and w from $dom(Q_L)$. We refer to [20, Sec. 5].

The operator $(L, \mathcal{D})$ can be extended to each of the Banach spaces $C_\infty(X)$ and $L^q(X, m)$, $1 \leq q < \infty$, as minus Markov generator. The extended operators we denote L_∞ and L_q respectively.

Theorem 3.9 *Assume that the quadratic form $Q = Q_L + Q_V$ defines a bounded below self-adjoint operator H (see e.g. Theorem 3.7). If there exists a function $0 < f \in dom(L_\infty)$ such that the inequality*

$$V(x) \geq -\frac{Lf(x)}{f(x)}$$

holds almost everywhere, then $Spec(H) \subseteq [0, \infty)$.

Proof. *Let us assume first that f is a locally constant function. Let us put $W_f := (-Lf)/f$ and let $\varphi \in \mathcal{D}$. If we put $\psi := \varphi/f \in \mathcal{D}$, then using equations (3.35)-(3.38) we get*

$$\begin{aligned} Q(\varphi, \varphi) &= \int_X (\varphi L\varphi + V\varphi^2) dm \geq \int_X (\varphi L\varphi + W_f \varphi^2) dm \\ &= \int_X (L\varphi + W_f\varphi)\varphi dm = \int_X (\psi Lf - 2\Gamma(f, \psi) + fL\psi + W_f f\psi) f\psi dm. \end{aligned}$$

Since $Lf + W_f f = 0$ the right-hand side RHS of the inequality from above can be written as

$$\begin{aligned} RHS &= \int_X (-2\psi f\Gamma(f, \psi) + f^2\psi L\psi) dm \\ &= \int_X -2\psi f\Gamma(f, \psi) dm + Q_L(f^2\psi, \psi). \end{aligned}$$

It follows that

$$\begin{aligned} Q(\varphi, \varphi) &\geq \int_X -2\psi f\Gamma(f, \psi) dm + Q_L(f^2\psi, \psi) \\ &= \int_X \{-2\psi f\Gamma(f, \psi) + f^2\Gamma(\psi, \psi) + \psi\Gamma(f^2, \psi)\} dm \\ &= \int_X f^2\Gamma(\psi, \psi) dm + \int_X \{-2\psi f\Gamma(f, \psi) + \psi\Gamma(f^2, \psi)\} dm \\ &\geq \int_X f^2\Gamma(\psi, \psi) dm \geq 0. \end{aligned}$$

Thus $Q(\varphi, \varphi) \geq 0$ for all $\varphi \in \mathcal{D}$. Since such functions φ form a core for Q, the result follows by an application of the variational formula (3.33).

In general one can choose a sequence of locally constant functions f_n such that $W_{f_n} \to W_f$ locally uniformly in X, for instance one can choose a δ-sequence $\phi_n \in \mathcal{D}_+$ and set $f_n := f * \phi_n$. Then setting $\psi_n := \varphi/f_n$ we get

$$\begin{aligned} Q(\varphi, \varphi) &= \int_X (\varphi L\varphi + V\varphi^2) dm \geq \int_X (\varphi L\varphi + W_f\varphi^2) dm \\ &= \lim_{n\to\infty} \int_X (\varphi L\varphi + W_{f_n}\varphi^2) dm \geq \limsup_{n\to\infty} \int_X f_n^2\Gamma(\psi_n, \psi_n) dm \geq 0. \end{aligned}$$

The proof of the theorem is finished. ■

Corollary 3.10 *Assume that* $0 < \alpha < 1$ *and that the following inequality*

$$V_-(x) \leq \left(\Gamma_p\left(\frac{1+\alpha}{2}\right)\right)^2 \|x\|_p^{-\alpha}$$

holds almost everywhere, then

$$Spec(\mathfrak{D}^\alpha + V) \subseteq [0, \infty).$$

Proof. Let us set $\mathfrak{u}_\beta(x) := \|x\|_p^\beta$. By [43, Sec. 8.1, Eq. (1.6)], the function $\mathfrak{u}_\beta$ defines a distribution (a generalized function) which is holomorphic on β everywhere on the real line. The operator $\mathfrak{D}^\alpha : \psi \to \mathfrak{D}^\alpha\psi$ can be defined as convolution of distributions $\mathfrak{u}_{-\alpha-1}/\Gamma_p(-\alpha)$ and ψ, see [43, Sec. 9].

We claim that for all $\beta \neq \alpha$,

$$\frac{\mathfrak{D}^\alpha \mathfrak{u}_\beta}{\mathfrak{u}_\beta} = \frac{\Gamma_p(\beta+1)}{\Gamma_p(\beta+1-\alpha)}\mathfrak{u}_{-\alpha}. \tag{3.39}$$

The case $\beta = 0$ is trivial. For $\beta \neq 0$ we apply the Fourier transform argument. Remind that the Fourier transform $f \to \widehat{f}$ is a linear isomorphism of $\mathcal{D}' \to \mathcal{D}'$. By virtue of the results of [43, Sec. 7.5], the equation

$$\widehat{\mathfrak{u}_{\gamma-1}}(\xi) = \Gamma_p(\gamma)\mathfrak{u}_{-\gamma}(\xi) \tag{3.40}$$

holds true for all $\gamma \neq 1$. Applying equation (3.40) we obtain

$$\begin{aligned}
\widehat{\mathfrak{D}^\alpha \mathfrak{u}_\beta}(\xi) &= \mathfrak{u}_\alpha(\xi)\widehat{\mathfrak{u}_\beta}(\xi) = \mathfrak{u}_\alpha(\xi)\widehat{\mathfrak{u}_{\beta+1-1}}(\xi) \\
&= \mathfrak{u}_\alpha(\xi)\Gamma_p(\beta+1)\mathfrak{u}_{-\beta-1}(\xi) = \Gamma_p(\beta+1)\mathfrak{u}_{-(1+\beta-\alpha)}(\xi) \\
&= \frac{\Gamma_p(\beta+1)}{\Gamma_p(\beta+1-\alpha)}\Gamma_p(\beta+1-\alpha)\mathfrak{u}_{-(1+\beta-\alpha)}(\xi) \\
&= \frac{\Gamma_p(\beta+1)}{\Gamma_p(\beta+1-\alpha)}\widehat{\mathfrak{u}_{(1+\beta-\alpha)-1}}(\xi) = \frac{\Gamma_p(\beta+1)}{\Gamma_p(\beta+1-\alpha)}\widehat{\mathfrak{u}_{\beta-\alpha}}(\xi),
\end{aligned}$$

so by the unicity theorem the desired result follows.

For $\phi \in \mathcal{D}_+$ and $\beta := (\alpha-1)/2$ we define the following function

$$W_\phi := \frac{\Gamma_p(\beta+1)}{\Gamma_p(\beta+1-\alpha)}\frac{\mathfrak{u}_{\beta-\alpha} * \phi}{\mathfrak{u}_\beta * \phi} = \left(\Gamma_p\left(\frac{1+\alpha}{2}\right)\right)^2 \frac{\mathfrak{u}_{-\frac{1+\alpha}{2}} * \phi}{\mathfrak{u}_{-\frac{1-\alpha}{2}} * \phi}.$$

Equation (3.39) shows that W_ϕ belongs to $C_\infty(X)$ and $W_\phi = Lf/f$ for some $0 < f \in dom(L_\infty)$, so applying Theorem 3.9 we get

$$Q_{W_\phi}(\varphi, \varphi) \leq Q_{\mathfrak{D}^\alpha}(\varphi, \varphi),$$

for all $\varphi \in \mathcal{D}$. Let us choose a sequence $\{B_n : n = 1, 2, ...\}$ of balls centred at the neutral element 0 such that $\cap_{n=1}^\infty B_n = \{0\}$ and set $\phi_n = 1_{B_n}/m(B_n)$. Clearly $\phi_n * f$ converges to f for any continuous function f, whence

$$W_{\phi_n}(x) \to W(x) = \left(\Gamma_p\left(\frac{1+\alpha}{2}\right)\right)^2 \frac{\mathfrak{u}_{-\frac{1+\alpha}{2}}(x)}{\mathfrak{u}_{-\frac{1-\alpha}{2}}(x)} = \left(\Gamma_p\left(\frac{1+\alpha}{2}\right)\right)^2 \|x\|_p^{-\alpha}.$$

Applying now Fatou lemma we conclude that for all $\varphi \in \mathcal{D}$,

$$Q_W(\varphi,\varphi) \le Q_{\mathfrak{D}^\alpha}(\varphi,\varphi).$$

It follows that for all $\varphi \in \mathcal{D}$,

$$-Q_V(\varphi,\varphi) \le Q_{V_-}(\varphi,\varphi) \le Q_W(\varphi,\varphi) \le Q_{\mathfrak{D}^\alpha}(\varphi,\varphi),$$

or equivalently,

$$Q(\varphi,\varphi) := Q_{\mathfrak{D}^\alpha}(\varphi,\varphi) + Q_V(\varphi,\varphi) \ge 0.$$

The set $\mathcal{D}$ forms a core for $Q(\varphi,\varphi)$, for reasongs which depend upon which assumption we make on V, and the proof is completed by an application of the variational formula (3.33). ■

The following results show that the crucial issue for the existence of negative eigenvalues in Theorem 3.8 for all $\lambda > 0$ is the rate at which the potential $V(x)$ converges to 0 as $\|x\|_p \to \infty$.

Example 3.11 *Let $0 < \alpha < 1$ and let $H_\lambda = \mathfrak{D}^\alpha - \lambda V$ where $V(x) = (\|x\|_p + 1)^{-\beta}$ for some $0 < \beta < 1$ and $\lambda > 0$. If $\beta \ge \alpha$ then Theorem 3.8 and Corollary 3.10 are applicable and there exists a positive threshold for the existence of negative eigenvalues of H_λ. If $0 < \beta < \alpha$ the result is totally different.*

Theorem 3.12 *In the notation of Example 3.11 assume that $0 < \beta < \alpha$, then H_λ has non-empty negative spectrum for all $\lambda > 0$.*

Proof. Let $f := \mathfrak{D}^{-\alpha} 1_B$ where B is a ball centred at the neutral element which we will specify later. The function f belongs to $\mathrm{dom}(\mathfrak{D}^\alpha)$ and calculations based on the spectral resolution formula and equation (2.12) show that

$$\begin{aligned}
\mathfrak{D}^{-\alpha} 1_B / m(B) &= \mathfrak{D}^{-\alpha} \sum_{T:\, B \subseteq T} f_T = \sum_{T:\, B \subseteq T} \mathfrak{D}^{-\alpha} f_T \\
&= \sum_{T:\, B \subseteq T} \left(\frac{m(T')}{p} \right)^\alpha f_T = \sum_{T:\, B \subseteq T} m(T)^\alpha \left(\frac{1_T}{m(T)} - \frac{1_{T'}}{m(T')} \right) \\
&= m(B)^{\alpha - 1} \sum_{T:\, B \subseteq T} \left(\frac{m(T)}{mB)} \right)^{\alpha - 1} \left(1_T - \frac{1}{p} 1_{T'} \right).
\end{aligned}$$

In particular, $W := (\mathfrak{D}^\alpha f)/f$ is given by

$$W = \frac{1_B}{\mathfrak{D}^{-\alpha} 1_B} = \frac{p - p^\alpha}{p - 1} \frac{1_B}{m(B)^\alpha} = \frac{p - p^\alpha}{p - 1} \frac{1_B}{\mathrm{diam}(B)^\alpha}.$$

If $\lambda > 0$ and $0 < \beta < \alpha$, there exists a ball B such that $\mathrm{diam}(B)$ is large enough so that

$$W(x) < \frac{\lambda}{(\|x\|_p + 1)^\beta} = \lambda V(x)$$

for all $x \in \mathbb{Q}_p$. Hence, as f belongs to $\mathrm{dom}(\mathfrak{D}^\alpha)$, we obtain

$$\begin{aligned}
Q_\lambda(f,f) &= Q_{\mathfrak{D}^\alpha}(f,f) - Q_{\lambda V}(f,f) < Q_{\mathfrak{D}^\alpha}(f,f) - Q_W(f,f) \\
&= (\mathfrak{D}^\alpha f, f) - (W \cdot f, f) = 0
\end{aligned}$$

and an application of the Rayleigh-Ritz formulae yields the desired result. ■

4 An example

In this section we study the quadratic form $Q_H(u,u)$ defined by the Hamiltonian $H = L + V$. We provide our calculations assuming that $L = \mathfrak{D}^\alpha$ and $V(x) = b\,\|x\|_p^{-\alpha}$ for $0 < \alpha < 1$ and $b \geq b_*$, a critical value which will be specified later.

4.1 The Dirichlet form

We regard the function $h(x) = \|x\|_p^\beta$ as a distribution, see [43]. For $\beta \neq \alpha$ equation (3.39) shows that (in the sence of distributions)

$$Lh(x) = \frac{\Gamma_p(\beta+1)}{\Gamma_p(\beta+1-\alpha)}\,\|x\|_p^{\beta-\alpha}.$$

In particular, for $\beta > \alpha - 1$ the distributions $h(x)$ and $Lh(x)$ are regular (generated by locally integrable functions) and the function

$$V(x) := -\frac{Lh(x)}{h(x)} = -\frac{\Gamma_p(\beta+1)}{\Gamma_p(\beta+1-\alpha)}\,\|x\|_p^{-\alpha} \tag{4.41}$$

belongs to $L^1_{loc}(X,m)$, so it defines a regular distribution as well.

Theorem 4.1 *For $\alpha - 1 < \beta < \alpha$ the following statements hold true:*

1. For $0 < \beta < \alpha$ the function $V(x)$ is strictly positive and belongs to $L^1_{loc}(X,m)$, so H is a minus Markovian generator by Theorem 3.4. Moreover, for any $b > 0$ there exists $0 < \beta < \alpha$, a solution of the equation

$$-\frac{\Gamma_p(\beta+1)}{\Gamma_p(\beta+1-\alpha)} = b, \tag{4.42}$$

such that $V(x) = b\,\|x\|_p^{-\alpha}$ for this value of β.

2. For $\alpha - 1 < \beta < 0$ the function $V(x)$ is strictly negative, so H is not a minus Markovian generator. However, for these values of β

$$V_-(x) = -V(x) \leq \left(\Gamma_p\left(\frac{1+\alpha}{2}\right)\right)^2 \|x\|_p^{-\alpha},$$

so H is a non-negative definite operator by Corollary 3.10. Moreover, for any $0 > b \geq b_* := -\{\Gamma_p((1+\alpha)/2)\}^2$ there exist $\alpha - 1 < \beta_1 \leq (\alpha-1)/2$ and $(\alpha-1)/2 \leq \beta_2 < 0$, solutions of equation (4.42), such that $V(x) = b\,\|x\|_p^{-\alpha}$ for these two values of β.

Proof. To prove the theorem we set $\vartheta = \beta + (1-\alpha)/2$ and write

$$-\frac{\Gamma_p(\beta+1)}{\Gamma_p(\beta+1-\alpha)} = -\Gamma_p\left(\frac{1+\alpha}{2}+\vartheta\right)\Gamma_p\left(\frac{1+\alpha}{2}-\vartheta\right) := C_\alpha(\vartheta).$$

The function $C_\alpha(\vartheta)$ is even, continuous and increasing on each interval $[0,(1+\alpha)/2[$ and $](1+\alpha)/2,+\infty[$. Using the very definition of the function $\Gamma_p(\xi)$ it is straightforward to show that the following properties hold true:

1. $C_\alpha(0) = -\{\Gamma_p((1+\alpha)/2)\}^2, C_\alpha((1-\alpha)/2) = 0,$

2. $C_\alpha((1+\alpha)/2 - 0) = +\infty, C_\alpha((1+\alpha)/2 + 0) = -\infty,$

3. $C_\alpha(+\infty) = -p^\alpha < C_\alpha(0).$

Clearly (1)-(3) imply the result. The proof of the theorem is finished. ■

Let us choose $h(x) = \|x\|_p^\beta$ with $(\alpha-1)/2 < \beta < \alpha$. Then Theorem 4.1 applies, so $H = L + V$ is a non-negative definite self-adjoint operator acting in $L^2(X, m)$.

According to our choice $h^2 \in L^1_{loc}(X, m)$, so $h^2 m$ is a Radon measure. In particular, this allows us to define an isometry $U : L^2(X, h^2 m) \to L^2(X, m)$ by setting $U : g \to hg$. Consider a non-negative self-adjoint operator

$$\mathcal{H} := U^{-1} \circ H \circ U$$

and let $Q_{\mathcal{H}}(u,u) = (\mathcal{H}^{1/2}u, \mathcal{H}^{1/2}u)$ be the associated quadratic form. We have $Q_H = Q_L + Q_V$ whence

$$\begin{aligned} Q_{\mathcal{H}}(u,u) &= Q_H(hu,hu) = Q_L(hu,hu) + Q_V(hu,hu) \\ &= \frac{1}{2}\int_X\int_X (h(x)u(x) - h(y)u(y))^2 J(x,y)dm(y)dm(x) \\ &+ \int_X V(x)u^2(x)h^2(x)dm(x) \end{aligned}$$

where

$$J(x,y) = -\frac{1}{\Gamma_p(-\alpha)}\frac{1}{\|x-y\|_p^{1+\alpha}}\ . \tag{4.43}$$

Theorem 4.2 *Assume that $(\alpha-1)/2 < \beta < \alpha$. Then $Q_{\mathcal{H}}(u,u)$ is a Dirichlet form in $L^2(X, h^2 m)$. Moreover, $\mathcal{D} \subset dom(Q_{\mathcal{H}})$ and for $u \in \mathcal{D}$,*

$$Q_{\mathcal{H}}(u,u) = \frac{1}{2}\int_X\int_X (u(x) - u(y))^2 J(x,y)h(y)dm(y)h(x)dm(x). \tag{4.44}$$

Proof. Let us prove that $\mathcal{D} \subset dom(Q_{\mathcal{H}})$. It is enough to show that $Q_{\mathcal{H}}(u,u)$ is finite for $u = \mathbf{1}_B$, $B \in \mathcal{B}$. We have $Q_{\mathcal{H}}(u,u) = Q_L(hu,hu) + Q_V(hu,hu)$. Since $V(x) = b\,\|x\|_p^{-\alpha}$ we get for $\beta > (\alpha-1)/2$:

$$|Q_V(hu,hu)| = |b|\int_B \|x\|_p^{-\alpha+2\beta}\,dm(x) < \infty.$$

Let us assume first that $0 \notin B$, then clearly $hu \in \mathcal{D}$. Since $\mathcal{D} \subset dom(L)$,

$$Q_L(hu,hu) = (Lhu, hu) < \infty.$$

Assume now that $0 \in B$ and set $h_B := h\mathbf{1}_B$, then

$$\begin{aligned} Q_L(hu,hu) &= \frac{1}{2}\iint (h_B(x) - h_B(y))^2 J(x,y)dm(x)dm(y) \\ &= \iint\limits_{(x,y)\in B\times B:\ \|x\|_p<\|y\|_p} (h(x) - h(y))^2 J(x,y)dm(x)dm(y) \\ &+ \int_B h^2(x)dm(x)\int_{B^c} J(x,y)dm(y). \end{aligned}$$

The second term, call it II, is finite. Indeed, we have

$$II = \int_B h^2(x)dm(x) \int_{B^c} J(0,z)dm(z) < \infty.$$

Without loss of generality we may assume that $diam(B) = 1$. By the ultrametric inequality, $\|x\|_p < \|y\|_p$ implies that $\|x-y\|_p = \|y\|_p$, so the first term, call it I, can be estimated as follows:

$$\begin{aligned}
I &= -\frac{1}{\Gamma_p(-\alpha)} \sum_{k=1}^{\infty}\sum_{l=1}^{k} \int_{\|x\|_p = p^{-k}} dm(x) \int_{\|y\|_p = p^{-k+l}} dm(y) \left(\|x\|_p^{\beta} - \|y\|_p^{\beta}\right)^2 \|y\|_p^{-(1+\alpha)} \\
&= -\frac{1}{\Gamma_p(-\alpha)} \left(1 - \frac{1}{p}\right)^2 \sum_{k=1}^{\infty}\sum_{l=1}^{k} p^{-k} p^{-k+l} p^{-(1+\alpha)(-k+l)} \left(p^{-k\beta} - p^{(-k+l)\beta}\right)^2 \\
&= -\frac{1}{\Gamma_p(-\alpha)} \left(1 - \frac{1}{p}\right)^2 \sum_{k=1}^{\infty} p^{-k(1-\alpha+2\beta)} \sum_{l=1}^{k} p^{-l\alpha} \left(1 - p^{l\beta}\right)^2 .
\end{aligned}$$

That I is finite for $(\alpha-1)/2 < \beta < \alpha$ follows by inspection.

Since the function $u \to Q_{\mathcal{H}}(u,u)$ is lower semi-continuous, equation (4.44) is enough to prove for $u = \mathbf{1}_B$ where B is a ball such that $0 \notin B$. In this case the function $h_B = h\mathbf{1}_B$ belongs to $\mathcal{D}$. Let us consider the distribution $f_\gamma(x) = \|x\|_p^{\gamma-1}/\Gamma_p(\gamma)$. According to [43, Section IX], $h(x) = \Gamma_p(\beta+1) f_{\beta+1}$ and $-Lh = f_{-\alpha} * \Gamma_p(\beta+1) f_{\beta+1}$ whence, setting $C := \Gamma_p(\beta+1)$, we get

$$\begin{aligned}
Q_V(hu,hu) &= \int (-Lh) h_B dm = ((-Lh) * h_B)(0) \\
&= C((f_{-\alpha} * f_{\beta+1}) * h_B)(0) = C((f_{\beta+1} * (f_{-\alpha} * h_B))(0) \\
&= \int h(-Lh_B)dm = -\iint (h_B(x) - h_B(y)) h(x) J(x,y) dm(x)dm(y)
\end{aligned}$$

and by symmetry

$$Q_V(hu,hu) = -\iint (h_B(y) - h_B(x)) h(y) J(x,y) dm(x)dm(y).$$

Thus finally we get

$$Q_V(hu,hu) = -\frac{1}{2}\iint (h_B(x) - h_B(y))(h(x) - h(y)) J(x,y) dm(x)dm(y). \tag{4.45}$$

On the other hand, for u as above,

$$Q_L(hu,hu) = \frac{1}{2}\iint (h_B(x) - h_B(y))^2 J(x,y) dm(x)dm(y). \tag{4.46}$$

Clearly equations (4.45) and (4.46) yield equation (4.44).

Thus, the quadratic form $Q_{\mathcal{H}}(u,u)$ is densely defined, closed, non-negative definite, and Markovian. That means that $Q_{\mathcal{H}}(u,u)$ is a Dirichlet form in $L^2(X, h^2 m)$ as claimed.
■

Definition 4.3 *A Dirichlet form $Q(u,u)$ relative to $L^2(X,\mu)$ (respectively, a symmetric Markovian semigroup $(P_t)_{t>0}$ in $L^2(X,\mu)$) is called transient if the associated resolvent $(G_\lambda)_{\lambda>0}$ can be extended for the value $\lambda = 0$ as a self-adjoint (possibly unbounded) operator $G_0 = \int_0^\infty P_t dt$ such that $\mathbf{1}_K \in dom(G_0)$ for every compact set $K \subset X$.*

One can show that the form $Q(u,u)$ is transient if and only if the following condition holds: for every compact set $K \subset X$ there exists a constant $C_K > 0$ such that

$$\int_X |u|\, d\mu \leq C_K \sqrt{Q(u,u)},\ \forall u \in dom(Q)\ ^3.$$

Theorem 4.4 *In the setting of Theorem 4.2:*

1. *There exists a hierarhical Laplacian $\mathcal{L}$, related to the (non-homogeneous) ultrametric measure space (X, hm), such that*

$$Q_{\mathcal{H}}(u,u) = Q_{\mathcal{L}}(u,u),\ \forall u \in L^2(X,hm) \cap L^2(X,h^2 m).$$

2. *$\mathcal{D} \subset dom(Q_{\mathcal{L}})$ is a core of $Q_{\mathcal{L}}$ (i.e. $Q_{\mathcal{L}}(u,u)$ is a regular Dirichlet form in $L^2(X,hm)$).*

3. *The Dirichlet form $Q_{\mathcal{L}}$ relative to $L^2(X,hm)$ is transient. In particular, the Dirichlet form $Q_{\mathcal{H}}$ relative to $L^2(X,h^2m)$ is transient as well.*

Proof. Consider the function

$$J(B) := -\frac{1}{\Gamma_p(-\alpha)} \frac{1}{m(B)^{1+\alpha}},\ B \in \mathcal{B},$$

defined on the set $\mathcal{B}$ of all open balls. Since in the p-adic metric $m(B) = diam(B)$ for any ball B, we get

$$J(x,y) = J(x \curlywedge y)$$

where $x \curlywedge y$ is the minimal ball which contains x and y. Consider also the Radon measure $\widetilde{m} = hm$. We claim that the following properties hold true:

(i) $S \subset T \Longrightarrow J(S) > J(T)$ and $J(T) \to 0$ as $T \to X$.

(ii) $\widetilde{\lambda}(B) := \sum_{S\colon B \subseteq S} \widetilde{m}(S)\,(J(S) - J(S')) < \infty$ for any $B \in \mathcal{B}$.

(iii) $\widetilde{\lambda}(B) \to +\infty$ as $B \to \{x\}$ for any $x \in X$.

The property (i) is evident. To prove (ii) we write

$$\begin{aligned}\widetilde{\lambda}(B) &= -\frac{1}{\Gamma_p(-\alpha)} \left(1 - \frac{1}{p^{1+\alpha}}\right) \sum_{S\colon B \subseteq S} \frac{\widetilde{m}(S)}{m(S)^{1+\alpha}} \\ &= (p^\alpha - 1) \sum_{S\colon B \subseteq S} \frac{\widetilde{m}(S)}{m(S)^{1+\alpha}}.\end{aligned}$$

[3] This condition of transience was first introduced by A. Beurling and J. Deny in the unreplacable paper [12]. It is slightly more restrictive than the definition of transience given in [20, Section 1.5].

Next, using the identity

$$\int f(\|x\|_p)dm(x) = \left(1 - \frac{1}{p}\right) \sum_{\gamma=-\infty}^{\infty} f(p^\gamma)p^\gamma,$$

we obtain that if $0 \in S$ then

$$\widetilde{m}(S) = \frac{p-1}{p-p^{-\beta}} m(S)^{1+\beta}, \tag{4.47}$$

so

$$\frac{\widetilde{m}(S)}{m(S)^{1+\alpha}} = \frac{p-1}{p-p^{-\beta}} \frac{1}{m(S)^{\alpha-\beta}}. \tag{4.48}$$

Clearly equality (4.48) implies (ii). On the other hand, for $B \in \mathcal{B}(x)$ small enough we have

$$\widetilde{\lambda}(B) \geq (p^\alpha - 1)\frac{\widetilde{m}(B)}{m(B)^{1+\alpha}} > (p^\alpha - 1)m(B)^{-\alpha} \min_{y \in B} \|y\|_p^\beta \tag{4.49}$$

and

$$\min_{y \in B} \|y\|_p^\beta = \begin{cases} \|x\|_p^\beta & \text{if} \quad x \neq 0 \\ (m(B))^\beta & \text{if} \quad x = 0 \end{cases}, \tag{4.50}$$

so (4.49) and (4.50) imply (iii).

According to [3, Section 2], properties $(i) - (iii)$ imply that the operator

$$\mathcal{L}u(x) = \int (u(x) - u(y))\, J(x,y)d\widetilde{m}(y) \tag{4.51}$$

is a hierarhical Laplacian in $L^2(X, \widetilde{m})$. In particular, $\mathcal{D} \subset dom(\mathcal{L})$ and for $u \in \mathcal{D}$ we have

$$Q_{\mathcal{L}}(u,u) = \frac{1}{2}\int\int (u(x) - u(y))^2 J(x,y)d\widetilde{m}(y)d\widetilde{m}(x) = Q_{\mathcal{H}}(u,u).$$

That $\mathcal{D}$ is a core of $Q_{\mathcal{L}}$ follows from the fact that $\mathcal{L}$, as a hierarchical Laplacian, is essentially self-adjoint. Indeed, in this case $(Q_{\mathcal{L}}, dom(Q_{\mathcal{L}}))$ coinsides with the minimal extension of $(Q_{\mathcal{L}}, \mathcal{D})$ which has $\mathcal{D}$ as a core.

The proof of the fact that the Markovian semigroup $(e^{-t\mathcal{L}})_{t>0}$ is transient, i.e. that $\mathbf{1}_K$ belongs to $dom(G_0)$ for any compact set K, we postpone to the next section (Theorem 4.5). Let us show how to derive the Beurling-Deny condition of transience from the transience of the semigroup $(e^{-t\mathcal{L}})_{t>0}$. For any $u \in dom(Q_{\mathcal{L}})$ we have $|u| \in dom(Q_{\mathcal{L}})$ and $Q_{\mathcal{L}}(|u|,|u|) \leq Q_{\mathcal{L}}(u,u)$. Also $v := G_0\mathbf{1}_K$ is in $dom(\mathcal{L})$ and $\mathcal{L}v = \mathbf{1}_K$ whence

$$\begin{aligned}\int_K |u|\, d\widetilde{m} &= Q_{\mathcal{L}}(|u|, v) \\ &\leq \sqrt{Q_{\mathcal{L}}(v,v)}\sqrt{Q_{\mathcal{L}}(u,u)}.\end{aligned}$$

Setting $C_K := \sqrt{Q_{\mathcal{L}}(v,v)}$ we get the desired result. The proof is finished. ■

4.2 The Green function $g_{\mathcal{L}}(x,y)$

In what follows we assume that $(\alpha-1)/2<\beta<\alpha$. The Markovian resolvent $G_\lambda = (\mathcal{L}+\lambda \mathrm{I})^{-1}$, $\lambda>0$, acts in Banach spaces $C_\infty(X)$ and $L^p(X,\widetilde{m})$, where $\widetilde{m}=hm$, as a bounded operator and admits the following representation

$$G_\lambda u(x)=\int g_{\mathcal{L}}(\lambda,x,y)u(y)d\widetilde{m}(y).$$

Here $g_{\mathcal{L}}(\lambda,x,y)$, the so called λ-Green function, is a continuous function taking finite values outside the diagonal set. As a function of λ it decreases, so the limit (finite or infinite)

$$g_{\mathcal{L}}(x,y):=\lim_{\lambda\to 0} g_{\mathcal{L}}(\lambda,x,y)$$

exists. The function $g_{\mathcal{L}}(x,y)$ is called the Green function of the operator $\mathcal{L}$.

Theorem 4.5 *The Green function $g_{\mathcal{L}}(x,y)$ is a continuous function taking finite values off the diagonal set. Moreover, the following relationship holds:*

$$g_{\mathcal{L}}(x,y)\asymp \frac{\|x-y\|_p^{\alpha-1}}{\left(\|x\|_p\vee\|y\|_p\right)^{2\beta}}, \tag{4.52}$$

or equivalently

$$\frac{g_{\mathcal{L}}(x,y)}{g_L(x,y)}\asymp\left(\frac{1}{\|x\|_p}\wedge\frac{1}{\|y\|_p}\right)^{2\beta}. \tag{4.53}$$

Proof. Let us assume that X is equipped with the ultrametric $d(x,y)=p^{-\alpha}\|x-y\|_p^\alpha$, intrinsic for the hierarchical Laplacian L, and define the following variables

$$F(x,R)=\left(\int_R^\infty\left(\frac{1}{m(B_r(x))}\int_{B_r(x)}hdm\right)\frac{dr}{r^2}\right)^{-1}$$

and

$$\widetilde{d}(x,y)=F(x,d(x,y)). \tag{4.54}$$

Since for each fixed x the function $R\to F(x,R)$ is continuous, strictly increasing, 0 at 0 and ∞ at ∞, $\widetilde{d}(x,y)$ is an ultrametric on X. Let $\widetilde{B}_{\widetilde{R}}(x)$ be a $\widetilde{d}$-ball of radius $\widetilde{R}$ centred at x. Then $\widetilde{B}_{\widetilde{R}}(x)=B_R(x)$ whenever

$$\widetilde{R}=F(x,R).$$

Since L is a hierarchical Laplacian acting in $L^2(X,m)$ and $d(x,y)$ is its intrinsic ultrametric, we have (see [5, equation (3.11)])

$$\begin{aligned}J(x,y)&=\int_{d(x,y)}^\infty\frac{1}{m(B_R(x))}\frac{dR}{R^2} \\ &=\int_{\widetilde{d}(x,y)}^\infty\frac{1}{\widetilde{m}(\widetilde{B}_{\widetilde{R}}(x))}\frac{d\widetilde{R}}{\widetilde{R}^2}.\end{aligned} \tag{4.55}$$

It follows that $\widetilde{d}(x,y)$ is intrinsic ultrametric corresponding to the hierarchical Laplacian $\mathcal{L}$ and

$$\begin{aligned}\widetilde{V}(x,\widetilde{R}) &:= \widetilde{m}(\widetilde{B}_{\widetilde{R}}(x)) \\ &= \widetilde{m}(B_R(x)) = \int_{B_R(x)} h dm\end{aligned}$$

is its volume-function. We claim that

$$\frac{\widetilde{m}(B_R(x))}{m(B_R(x))} \asymp \begin{cases} m(B_R(x))^{\beta} & \text{if} \quad d(0,x) \leq R \\ h(x) & \text{if} \quad d(0,x) > R \end{cases}. \tag{4.56}$$

Indeed, if $d(0,x) \leq R$ then $B_R(x) = B_R(0)$, so applying (4.47), we get

$$\begin{aligned}\frac{\widetilde{m}(B_R(x))}{m(B_R(x))} &= \frac{1}{m(B_R(x))}\int_{B_R(x)} h dm \\ &= \frac{1}{m(B_R(0))}\int_{B_R(0)} h dm \\ &= \frac{p-1}{p-p^{-\beta}} m(B_R(0))^{\beta} = \frac{p-1}{p-p^{-\beta}} m(B_R(x))^{\beta}.\end{aligned}$$

On the other hand, if $d(0,x) > R$ then from $y \in B_R(x)$ we get that $d(y,0) = d(x,0)$, so

$$\begin{aligned}\frac{\widetilde{m}(B_R(x))}{m(B_R(x))} &= \frac{1}{m(B_R(x))}\int_{B_R(x)} h(y) dm(y) \\ &= \frac{1}{m(B_R(x))}\int_{B_R(x)} h(x) dm(y) = h(x).\end{aligned}$$

Notice that asymptotic relationship (4.56) holds uniformly in x and R in the sence that the corresponding two sided inequality contains constants which do not depend on x and R. In turn, (4.56) implies the following (uniform) asymptotic relationship:

$$\widetilde{R} = F(x,R) \asymp \begin{cases} R/h(x) & \text{if} \quad R < d(0,x) \\ R^{\frac{\alpha-\beta}{\alpha}} & \text{if} \quad R \geq d(0,x) \end{cases} \tag{4.57}$$

Indeed, if $d(0,x) \leq R$ then

$$\begin{aligned}\int_R^{\infty} \frac{\widetilde{m}(B_r(x))}{m(B_r(x))}\frac{dr}{r^2} &\asymp \int_R^{\infty} m(B_r(x))^{\beta}\frac{dr}{r^2} \\ &\asymp \int_R^{\infty} r^{-\left(2-\frac{\beta}{\alpha}\right)} dr \asymp R^{-\left(1-\frac{\beta}{\alpha}\right)},\end{aligned}$$

so

$$\widetilde{R} := F(x,R) \asymp R^{1-\frac{\beta}{\alpha}}.$$

If $d(0,x) \geq R$ then for some constants $C_1, C_2 > 0$,

$$\begin{aligned}\int_R^{\infty} \frac{\widetilde{m}(B_r(x))}{m(B_r(x))}\frac{dr}{r^2} &= \int_R^{d(0,x)} \frac{\widetilde{m}(B_r(x))}{m(B_r(x))}\frac{dr}{r^2} + \int_{d(0,x)}^{\infty} \frac{\widetilde{m}(B_r(x))}{m(B_r(x))}\frac{dr}{r^2} \\ &= C_1 d(0,x)^{\frac{\beta}{\alpha}}\left(\frac{1}{R} - \frac{1}{d(0,x)}\right) + \frac{C_2}{d(0,x)^{1-\frac{\beta}{\alpha}}} \\ &\asymp \frac{d(0,x)^{\frac{\beta}{\alpha}}}{R} \asymp \frac{h(x)}{R},\end{aligned}$$

so

$$\widetilde{R} := F(x,R) \asymp \frac{R}{h(x)}.$$

Furthermore, asymptotic relationships (4.56) and (4.57) yield the following (uniform) asymptotic relationship

$$\begin{aligned}\widetilde{V}(x,\widetilde{R}) &= \widetilde{m}(B_R(x)) \qquad (4.58)\\ &\asymp \begin{cases} h(x)R^{\frac{1}{\alpha}} & \text{if} \quad R < d(0,x) \\ R^{\frac{1+\beta}{\alpha}} & \text{if} \quad R \geq d(0,x) \end{cases},\end{aligned}$$

or equivalently, we get

$$\widetilde{V}(x,\widetilde{R}) \asymp \begin{cases} h(x)^{1+\frac{1}{\alpha}}\widetilde{R}^{\frac{1}{\alpha}} & \text{if} \quad \widetilde{R} < \widetilde{d}(0,x) \\ \widetilde{R}^{\frac{1+\beta}{\alpha-\beta}} & \text{if} \quad \widetilde{R} \geq \widetilde{d}(0,x) \end{cases}. \qquad (4.59)$$

1. Let us consider the case $\|x-y\|_p = \|x\|_p \vee \|y\|_p$. Then clearly $d(x,y) = d(0,x) \vee d(0,y)$, and similar equation holds in $\widetilde{d}$ metric. If $R \geq d(0,x)$ then

$$\widetilde{R} := F(x,R) \asymp R^{1-\frac{\beta}{\alpha}} \qquad (4.60)$$

and

$$\widetilde{V}(x,\widetilde{R}) \asymp R^{\frac{1+\beta}{\alpha}} \asymp \widetilde{R}^{\frac{1+\beta}{\alpha-\beta}}, \qquad (4.61)$$

Equation (4.61) implies the following two results:

1. Since $\delta := \frac{1+\beta}{\alpha-\beta} > 1$, the function $\widetilde{R} \to 1/\widetilde{V}(x,\widetilde{R})$ is integrable at ∞ for any fixed x, so the Markovian semigroup $(e^{-t\mathcal{L}})_{t>0}$ (equivalently, the Dirichlet form $Q_{\mathcal{L}}$) is transient (see [5, Theorem 2.28]) as it has been stated in Theorem 4.4.

2. The fact that $\widetilde{V}(x,\widetilde{R}) \asymp \widetilde{R}^{\delta}$, $\delta > 1$, for $\widetilde{R} \geq \widetilde{d}(0,x)$, yield the following asymptotic relationship

$$g_{\mathcal{L}}(x,y) = \int_{\widetilde{d}(x,y)}^{\infty} \frac{d\widetilde{R}}{\widetilde{V}(x,\widetilde{R})} \asymp \frac{\widetilde{d}(x,y)}{\widetilde{V}(x,\widetilde{d}(x,y))}, \qquad (4.62)$$

or equivalently, see equations (4.60) and (4.61),

$$g_{\mathcal{L}}(x,y) \asymp \|x-y\|_p^{\alpha-1-2\beta} = \frac{\|x-y\|_p^{\alpha-1}}{\left(\|x\|_p \vee \|y\|_p\right)^{2\beta}} \qquad (4.63)$$

provided $\|x\|_p \leq \|x-y\|_p$. Similarly, by symmetry, relationship (4.63) holds provided $\|y\|_p \leq \|x-y\|_p$. Thus finally, the assumption $\|x-y\|_p = \|x\|_p \vee \|y\|_p$ implies (4.63), as it was claimed.

2. Let us consider the case $\|x-y\|_p < \|x\|_p \vee \|y\|_p$. In this case we have: $\|x\|_p = \|y\|_p$ and $\|x-y\|_p < \|x\|_p$, similar relations hold in d and $\widetilde{d}$ metrics. Having this in mind we write

$$g_{\mathcal{L}}(x,y) = \int_{\widetilde{d}(x,y)}^{\infty} \frac{d\widetilde{R}}{\widetilde{V}(x,\widetilde{R})} = \left(\int_{\widetilde{d}(x,y)}^{\widetilde{d}(0,x)} + \int_{\widetilde{d}(0,x)}^{\infty}\right) \frac{d\widetilde{R}}{\widetilde{V}(x,\widetilde{R})} = I + II.$$

Since $\widetilde{d}(0,x) \leq \widetilde{R}$ implies $\widetilde{V}(x,\widetilde{R}) \asymp \widetilde{R}^{\frac{1+\beta}{\alpha-\beta}}$, we get

$$II \asymp \frac{\widetilde{d}(0,x)}{\widetilde{V}(x,\widetilde{d}(0,x))} \asymp \frac{1}{\widetilde{d}(0,x)^{\frac{1-\alpha+2\beta}{\alpha-\beta}}}.$$

To estimate the first term we write

$$I = \int_{\widetilde{d}(x,y)}^{\widetilde{d}(0,x)} \frac{d\widetilde{R}}{\widetilde{V}(x,\widetilde{R})} \asymp \frac{1}{h(x)^{1+\frac{1}{\alpha}}} \int_{\widetilde{d}(x,y)}^{\widetilde{d}(0,x)} \frac{d\widetilde{R}}{\widetilde{R}^{\frac{1}{\alpha}}}$$

and

$$\begin{aligned}\frac{1}{h(x)^{1+\frac{1}{\alpha}}} \int_{\widetilde{d}(x,y)}^{\widetilde{d}(0,x)} \frac{d\widetilde{R}}{\widetilde{R}^{\frac{1}{\alpha}}} &= \frac{1}{h(x)^{1+\frac{1}{\alpha}}} \left(\frac{1}{\widetilde{d}(x,y)^{\frac{1}{\alpha}-1}} - \frac{1}{\widetilde{d}(0,x)^{\frac{1}{\alpha}-1}} \right) \\ &= \frac{\widetilde{d}(x,y)^{1-\frac{1}{\alpha}}}{h(x)^{1+\frac{1}{\alpha}}} \left(1 - \left(\frac{\widetilde{d}(x,y)}{\widetilde{d}(0,x)} \right)^{\frac{1}{\alpha}-1} \right).\end{aligned}$$

Finally, since $\|x\|_p = \|y\|_p$ and $\|x-y\|_p < \|x\|_p$, we have

$$\begin{aligned}g_{\mathcal{L}}(x,y) &= I + II \\ &\asymp \frac{\widetilde{d}(x,y)^{1-\frac{1}{\alpha}}}{h(x)^{1+\frac{1}{\alpha}}} \left(1 - \left(\frac{\widetilde{d}(x,y)}{\widetilde{d}(0,x)} \right)^{\frac{1}{\alpha}-1} \right) + \frac{1}{\widetilde{d}(0,x)^{\frac{1-\alpha+2\beta}{\alpha-\beta}}} \\ &= \frac{\widetilde{d}(x,y)^{1-\frac{1}{\alpha}}}{h(x)^{1+\frac{1}{\alpha}}} \left(\left(1 - \left(\frac{\widetilde{d}(x,y)}{\widetilde{d}(0,x)} \right)^{\frac{1}{\alpha}-1} \right) + \frac{\widetilde{d}(x,y)^{\frac{1}{\alpha}-1} h(x)^{1+\frac{1}{\alpha}}}{\widetilde{d}(0,x)^{\frac{1-\alpha+2\beta}{\alpha-\beta}}} \right).\end{aligned}$$

According to (4.60) $h(x) \asymp \widetilde{d}(0,x)^{\frac{\beta}{\alpha-\beta}}$ whence

$$\frac{h(x)^{1+\frac{1}{\alpha}}}{\widetilde{d}(0,x)^{\frac{1-\alpha+2\beta}{\alpha-\beta}}} \asymp \frac{\widetilde{d}(0,x)^{\frac{\beta}{\alpha-\beta}(1+\frac{1}{\alpha})}}{\widetilde{d}(0,x)^{\frac{1-\alpha+2\beta}{\alpha-\beta}}} \asymp \frac{1}{\widetilde{d}(0,x)^{\frac{1}{\alpha}-1}}$$

and thus, using (4.57), we get

$$\begin{aligned}g_{\mathcal{L}}(x,y) &\asymp \frac{\widetilde{d}(x,y)^{1-\frac{1}{\alpha}}}{h(x)^{1+\frac{1}{\alpha}}} \asymp \left(\frac{d(x,y)}{h(x)} \right)^{1-\frac{1}{\alpha}} \frac{1}{h(x)^{1+\frac{1}{\alpha}}} \\ &= \frac{d(x,y)^{1-\frac{1}{\alpha}}}{h(x)^2} \asymp \frac{\|x-y\|_p^{\alpha-1}}{\|x\|_p^{2\beta}} = \frac{\|x-y\|_p^{\alpha-1}}{\left(\|x\|_p \vee \|y\|_p \right)^{2\beta}}.\end{aligned}$$

The proof of the theorem is finished. ■

4.3 The Green function $g_H(x, y)$

Throughout this section we assume that $(\alpha - 1)/2 \leq \beta < \alpha$ and that b is a solution of equation (4.42). Then, by Theorem 4.1), the operator

$$H = \mathfrak{D}^{\alpha} + b \left\| x \right\|_p^{-\alpha}$$

is a self-adjoint and non-negative definite operator acting in $L^2(X, m)$. Notice that b is an increasing continuous function of β which fulfill the whole range $[b_*, +\infty)$, where $b_* = -\{\Gamma_p\left((1+\alpha)/2\right)\}^2$. In particular, $b < 0$ for $(\alpha - 1)/2 \leq \beta < 0$ and $b \geq 0$ otherwise.

Theorem 4.6 *The equation $Hu = v$ has a unique solution*

$$u(x) = \int_X g_H(x, y) v(y) dm(y),$$

where

$$g_H(x, y) = h(x) g_{\mathcal{L}}(x, y) h(y).$$

We call $g_H(x, y)$ the Green function of the operator H, or the fundamenthal solution of the equation $Hu = v$.

Proof. We know that $\mathcal{L} : \mathcal{D} \to L^2(X, hm) \cap C_{\infty}(X)$. Let us show that $\mathcal{L} : \mathcal{D} \to L^q(X, m)$, $\forall 1 \leq q \leq \infty$. It is enough to check this property for $\psi = \mathbf{1}_B$, the indicator of an open ball B. In this case there exists a constant $C > 0$ such that as $x \to \infty$ the following asymptotic relationship holds:

$$\begin{aligned} \mathcal{L}\psi(x) &= -\int_B J(x, y) h(y) dm(y) \\ &= -\frac{1}{\Gamma_p(-\alpha)} \frac{1}{\left\| x \right\|_p^{1+\alpha}} \int_B h dm \asymp \frac{C}{\left\| x \right\|_p^{1+\alpha}} \end{aligned}$$

Clearly this relationship and the fact that $\mathcal{L}\psi(x)$ is bounded proofs the claim.

In particular, $\mathcal{L}\psi \in L^2(X, m)$ and therefore $\frac{1}{h}\mathcal{L}\psi \in L^2(X, h^2 m)$ for any $\psi \in \mathcal{D}$. Having this in mind we do our computations for $\varphi, \psi \in \mathcal{D}$:

$$\begin{aligned} |Q_{\mathcal{H}}(\varphi, \psi)| &= |Q_{\mathcal{L}}(\varphi, \psi)| = |(\mathcal{L}\psi, \varphi)_{L^2(hm)}| \\ &= \left| \left(\frac{1}{h}\mathcal{L}\psi, \varphi \right)_{L^2(h^2 m)} \right| \leq \left\| \frac{1}{h}\mathcal{L}\psi \right\|_{L^2(h^2 m)} \|\varphi\|_{L^2(h^2 m)}. \end{aligned}$$

That means that $\varphi \to Q_{\mathcal{H}}(\varphi, \psi)$ is a bounded linear functional in $L^2(X, h^2 m)$ for any $\psi \in \mathcal{D}$. This fact, in turn, implies that $\mathcal{D} \subset dom(\mathcal{H})$ and

$$\mathcal{H}\psi = \frac{1}{h}\mathcal{L}\psi, \ \forall \psi \in \mathcal{D}. \tag{4.64}$$

Let us consider the equation $\mathcal{H}u = v$ for $v \in \mathcal{D}$. Since $\mathcal{D} \subset dom(\mathcal{H})$ we have

$$(\mathcal{H}u, \psi)_{L^2(X, h^2 m)} = (u, \mathcal{H}\psi)_{L^2(X, h^2 m)}, \ \forall \psi \in \mathcal{D}.$$

Applying equation (4.64) we get

$$(\mathcal{H}u,\psi)_{L^2(X,h^2m)} = \left(u,\frac{1}{h}\mathcal{L}\psi\right)_{L^2(X,h^2m)} = (u,\mathcal{L}\psi)_{L^2(X,hm)}.$$

On the other hand, we have

$$(\mathcal{H}u,\psi)_{L^2(X,h^2m)} = (v,\psi)_{L^2(X,h^2m)} = (hv,\psi)_{L^2(X,hm)}.$$

Our calculations from above show that for Hölder conjugated (p,q) we have

$$\left|(u,\mathcal{L}\psi)_{L^2(X,hm)}\right| = |(hv,\psi)_{L^2(X,hm)}| \le \|hv\|_{L^p(X,hm)} \|\psi\|_{L^q(X,hm)}.$$

It follows that if we choose $1 < p < \frac{1+\alpha}{1-\alpha}$, then $\psi \to (u,\mathcal{L}\psi)_{L^2(X,hm)}$ is a bounded linear functional in $L^q(X,hm)$ provided $q = \frac{p}{p-1}$, i.e. $\frac{1}{2}\left(1+\frac{1}{\alpha}\right) < q < \infty$.

As $(e^{-t\mathcal{L}})_{t>0}$ is a continuous symmetric Markovian semigroup an application of the Riesz-Thorin interpolation theorem shows that it can be extended to all $L^q(X,hm)$ as a continuous contraction semigroup. Let $\mathcal{L}_q$ be its minus infinitesimal generator, then $\mathcal{L}_q$ extends $\mathcal{L}$, and $\mathcal{L}_q^* = \mathcal{L}_p$.

All the above shows that u must belong to the set $dom(\mathcal{L}_p)$ and $\mathcal{L}_p u = hv$. The equation $\mathcal{L}_p u = hv$ has unique solution

$$\begin{aligned} u(x) &= \int_X g_{\mathcal{L}}(x,y)(hv)(y)h(y)dm(y) \\ &= \int_X g_{\mathcal{L}}(x,y)v(y)h^2(y)dm(y). \end{aligned}$$

It follows that the operator $\mathcal{H}$ acting in $L^2(X,h^2m)$ admits a Green function $g_{\mathcal{H}}(x,y)$ and that $g_{\mathcal{H}}(x,y)$ coincides with the function $g_{\mathcal{L}}(x,y)$, the Green function of the operator $\mathcal{L}$ acting in $L^2(X,hm)$:

$$g_{\mathcal{H}}(x,y) = g_{\mathcal{L}}(x,y). \tag{4.65}$$

Finally, let us consider the equation $Hu = v$. Since $H = U\circ\mathcal{H}\circ U^{-1}$, we get $\mathcal{H}(U^{-1}u) = U^{-1}v$. It follows that

$$(U^{-1}u)(x) = \int_X g_{\mathcal{H}}(x,y)(U^{-1}v)(y)h(y)^2dm(y),$$

or equivalently

$$u(x) = \int_X h(x)g_{\mathcal{H}}(x,y)h(y)v(y)dm(y).$$

That means that equation $Hu = v$ admits a fundamenthal solution

$$\begin{aligned} g_H(x,y) &:= h(x)g_{\mathcal{H}}(x,y)h(y) \\ &= h(x)g_{\mathcal{L}}(x,y)h(y), \end{aligned}$$

thanks to (4.65). The proof of the theorem is finished. ■

Corollary 4.7 *The Green function $g_H(x,y)$ is a continuous function taking finite values off the diagonal set. Moreover, the following relationship holds:*

$$g_H(x,y) \asymp \frac{\|x\|_p^{\beta} \|x-y\|_p^{\alpha-1} \|y\|_p^{\beta}}{\left(\|x\|_p \vee \|y\|_p\right)^{2\beta}}, \tag{4.66}$$

or equivalently,

$$\frac{g_H(x,y)}{g_L(x,y)} \asymp \left(\frac{\|x\|_p}{\|y\|_p} \wedge \frac{\|y\|_p}{\|x\|_p}\right)^{\beta}.[4]$$

Proof. Follows directly from Theorem 4.5 and Theorem 4.6. ■

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