

Gromov Hyperbolic Graphs Arising From Iterations

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Abstract

For a contractive iterated function system (IFS), it is known that there is a natural hyperbolic graph structure (augmented tree) on the symbolic space of the IFS that reflects the relationship among neighboring cells, and its hyperbolic boundary with the Gromov metric is Hölder equivalent to the attractor K [Ka, LW1, LW3]. This setup was taken up to study the probabilistic potential theory on K [KLW1, KL], and the bi-Lipschitz equivalence on K [LL]. In this paper, we formulate a broad class of hyperbolic graphs, called *expansive hyperbolic graphs*, to capture the most essential properties from the augmented trees and the hyperbolic boundaries (e.g., the special geodesics, bounded degree property, metric doubling property, and Hölder equivalence). We also study a new setup of “weighted” IFS and investigate its connection with the self-similar energy form in the analysis of fractals.

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1 Introduction

Let $\{S_i\}_{i=1}^N$ be a contractive iterated function system (IFS) on \mathbb{R}^d , and let K be the attractor. It is well-known that the IFS is associated with a finite word space Σ^* (symbolic space or coding space), and the limit set Σ^∞ is used to represent elements in K . With the intention to carry over the probabilistic potential theory to K , Denker and Sato [DS1, DS2] first constructed a special type of Markov chain $\{Z_n\}_{n=0}^\infty$ on Σ^* of the Sierpinski gasket (SG), and showed that the Martin boundary of $\{Z_n\}_{n=0}^\infty$ is homeomorphic to the SG. Motivated by this, Kaimanovich [Ka] introduced an “augmented tree” (Σ^*, E) by adding “horizontal” edges to the coding tree (Σ^*, E_v) according to

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the neighboring cells in each level of the SG. He showed that the graph is hyperbolic in the sense of Gromov [Gr, Wo], and the SG equipped with the Euclidean metric is Hölder equivalent to the hyperbolic boundary of (Σ^*, E) . He also suggested that this approach would also work for more general IFSs, and could provide another tool to study the geometry and analysis on fractals.

These initiatives were carried out by the authors in a series of papers [LW1, LW2, LW3, JLW, Wa, KLW1, KLW2, KL]. In [LW1, LW3], we investigated the systems of neighboring cells to general IFSs on which the hyperbolicity of the augmented trees is valid; it was shown that the hyperbolic boundaries and the attractors K are Hölder equivalent (or homeomorphic) under various circumstances. The Hölder equivalence was applied to the bi-Lipschitz classification of the totally disconnected self-similar sets [LL, DLL]. More importantly, this setup was taken up to study the probabilistic potential theory on the self-similar sets [KLW1]: for an augmented tree (X, E) defined by an IFS on \mathbb{R}^d satisfying the open set condition, we introduced a class of reversible transient random walks on (X, E) . By identifying the Martin boundary, the hyperbolic boundary and K , we obtain an induced Dirichlet form of Gagliardo-type on K . The relevant energy forms and function spaces were further studied in [KL].

The augmented tree has very rich structure inherited from the iterations and the attractors. In this paper, our first goal is to formulate a broad class of hyperbolic graphs to capture the most essential concepts, such as the special geodesics, the bounded degree property, and the doubling property of the Gromov metric on the hyperbolic boundaries, which were used extensively in the study of augmented trees. This new setup allows us to carry out the idea of augmentation much further, and beyond IFS. Besides extending the previous study on the identification of the hyperbolic boundaries with the attractors K , we are able to use the hyperbolic techniques developed to construct new metrics on K that are useful in the analysis of fractals.

Let (X, E) be a locally finite connected graph with a root $\vartheta \in X$. We have the decomposition $E = E_v \cup E_h$, the set of vertical edges (which does not necessarily form a tree) and the set of horizontal edges. Let $\mathcal{J}_m(x)$ denote the set of descendants in the m -th generation of $x \in X$. Denote by d and d_h the graph distances on (X, E) and (X, E_h) respectively, and let $|x| := d(\vartheta, x)$. We call a rooted graph *expansive* if for $x, y \in X$ with $|x| = |y|$,

$$d_h(x, y) > 1 \Rightarrow d_h(u, v) > 1, \quad \forall u \in \mathcal{J}_1(x), v \in \mathcal{J}_1(y),$$

and call it (m, k) -*departing* if

$$d_h(x, y) > k \Rightarrow d_h(u, v) > 2k, \quad \forall u \in \mathcal{J}_m(x), v \in \mathcal{J}_m(y).$$

Intuitively, the two definitions describe the distances of the descendants that are drifted apart compared to the non-neighboring predecessors. The expansive property provides a rather simple form of geodesics (*convex geodesics*, see Proposition 2.3, Figure 2), and the (m, k) -departing property gives more details of how the graph evolves. These two lead to the following criteria for the hyperbolicity (Theorem 2.11), which are crucial in the paper.

Theorem 1.1. *Let (X, E) be an expansive graph. The following assertions are equivalent.*

- (i) (X, E) is hyperbolic;
- (ii) $\exists L < \infty$ such that the lengths of all horizontal geodesics are bounded by L ;
- (iii) (X, E) is (m, k) -departing for some positive integers m and k .

(By a *horizontal geodesic*, we mean a geodesic in (X, E) consisting of edges in E_h only.) We will call such (X, E) an *expansive hyperbolic graph*: together with the expansiveness, (ii) gives a clear geometry of the geodesics in X ; the (m, k) -departing property in (iii) will serve as the workhorse in many of the proofs in this study.

We use ∂X to denote the hyperbolic boundary of (X, E) , which is a compact space with an associated metric θ_a , $a > 0$ small (*Gromov metric*, see Definition 2.4). By using the (m, k) -departing property, we obtain a sharp description of the equivalent rays in X that converge to the same boundary elements (Proposition 3.1) which will be used in a number of estimates. One of the main results is (Theorem 3.6)

Theorem 1.2. *Suppose (X, E) is an expansive hyperbolic graph, and has bounded degree (i.e., $\sup_{x \in X} \deg(x) < \infty$). Then $(\partial X, \theta_a)$ is a doubling metric space.*

Recall that an *augmented tree* of an IFS with attractor K is based on the tree of the symbolic space, together with the added horizontal edges that connect neighboring cells in the same level (see Appendix). This can easily be reformulated on any vertical rooted graph (X, E_v) : let (M, ρ) be a complete metric space, and let \mathcal{C}_M denote the family of nonempty compact subsets of M . We define an *index map* $\Phi : X \rightarrow \mathcal{C}_M$ that satisfies $\Phi(y) \subset \Phi(x)$ whenever $y \in \mathcal{J}_1(x)$, and $\bigcap_{i=0}^{\infty} \Phi(x_i)$ is a singleton for any geodesic ray $[x_i]_i$ from the root ϑ (see Definition 4.1); likewise, we also have an attractor K . This setup is very general, which includes all IFSs (where $\Phi(x)$ is defined as the cell K_x), as well as cases that are not from IFS, e.g., refinement systems of sets.

With the vertical graph (X, E_v) and the index map Φ , we define

$$E_h^{(\infty)} := \{(x, y) \in X \times X : |x| = |y|, x \neq y, \text{ and } \Phi(x) \cap \Phi(y) \neq \emptyset\}.$$

Let $E^{(\infty)} = E_v \cup E_h^{(\infty)}$, and call $(X, E^{(\infty)})$ an AI_∞ -graph (*augmented index graph of type- (∞)* , or *intersection type*). In the case that $\{\Phi(x)\}_{x \in X}$ is of exponential type- (b) (i.e., the diameter $|\Phi(x)|_\rho = O(e^{-b|x|})$ for some $b > 0$, as $|x| \rightarrow \infty$), for some fixed $\gamma > 0$, we define a horizontal edge set by

$$E_h^{(b)} (= E_h^{(b)}(\gamma)) := \{(x, y) \in X \times X : |x| = |y|, x \neq y, \text{ and } \text{dist}_\rho(\Phi(x), \Phi(y)) \leq \gamma e^{-b|x|}\},$$

Let $E^{(b)} = E_v \cup E_h^{(b)}$, and call $(X, E^{(b)})$ an AI_b -graph (*augmented index graph of type- (b)*).

Both AI_∞ - and AI_b -graphs are expansive. The condition that defines an AI_∞ -graph is more intuitive, and $E_h^{(\infty)}$ consists of fewer edges. However, concerning the hyperbolicity, the AI_b -graph has the advantage that we do not need to know the fine structure of K a priori. More precisely, by using Theorem 1.1, we show that (Theorem 4.5)

Theorem 1.3. *The AI_b -graph is $(m, 1)$ -departing for some $m \geq 1$, and hence hyperbolic. Moreover, the index map Φ induces a bijection $\kappa : \partial X \rightarrow K$ that is a Hölder equivalence, i.e., $\rho(\kappa(\xi), \kappa(\eta))^{a/b} \asymp \theta_a(\xi, \eta)$ for all $\xi, \eta \in \partial X$.*

(Here by $f \asymp g$ we mean that there exists $C \geq 1$ such that $C^{-1}f(x) \leq g(x) \leq Cf(x)$ for all variables x in a given domain.) The AI_∞ -graph is not always hyperbolic (Example 6.1); in order to have that, we need an additional separation condition (S_b) on $\{\Phi(x)\}_{x \in X}$ (Definition 5.1, which is satisfied by IFS of similitudes with the OSC).

Theorem 1.4. *Suppose $\{\Phi(x)\}_{x \in X}$ is of exponential type- (b) , and satisfies condition (S_b) for some $b > 0$. Then the AI_∞ -graph is hyperbolic, and the induced bijection $\kappa : \partial X \rightarrow K$ is Hölder continuous, i.e., $\rho(\kappa(\xi), \kappa(\eta))^{a/b} \leq C\theta_a(\xi, \eta)$ for all $\xi, \eta \in \partial X$.*

The proof of the theorem is in Theorem 5.4 and Corollary 4.7. We also provide an example (Example 6.2) to show that unlike the AI_b -graph, the Hölder continuity of κ for the AI_∞ -graph cannot be improved to Hölder equivalence.

For the augmented tree of an IFS, the *bounded degree* property is important because it allows us to consider certain random walks on it [KLW1, KL]. This property has been characterized in terms of the separation properties such as the open set condition and weak separation condition for IFSs [LW3, Wa]. In Section 5, we prove (Theorem 5.5)

Theorem 1.5. *The AI_b -graph has bounded degree if and only if condition (S_b) is satisfied. Also, the AI_∞ -graph has bounded degree provided that (S_b) is satisfied.*

In our previous consideration of IFS, the structure of augmented trees (or AI_∞ - and AI_b -graphs here) arose from the geometry of K_x under a given metric (usually Euclidean metric). In the analysis of fractals, there are situations that involve weighted IFS, which give rise to new metrics on K (e.g., the resistance metric in the study of Dirichlet form [Ki1, HW] or the metrics involve in the time change of Brownian motions [Ki2, Ki3, Ki4, GLQR]). This requires new graph structure to accommodate the new parameter of weights.

In this regard, we let $\{S_j\}_{j=1}^N$ be a contractive IFS on a complete metric space (M, ρ) , and let K be the attractor. Let $\mathbf{s} = (s_1, \dots, s_N)$, $s_j \in (0, 1)$, be a vector of weights of the maps S_j 's. We regroup the finite words in the symbolic space to form a new coding tree $(X(\mathbf{s}), E_v)$ (see (6.2)) such that in each level, the K_x 's have comparable weights. In this case the AI_∞ -graph is more natural for use (see Section 6). While we cannot check the hyperbolicity directly (as $\{K_x\}_{x \in X(\mathbf{s})}$ does not satisfy the separation condition (S_b) as in Theorem 1.4), we still obtain some rather satisfactory conclusions (Theorem 6.3) for the class of p.c.f. sets [Ki1].

Theorem 1.6. *Let $\{S_j\}_{j=1}^N$ be a contractive IFS that has the p.c.f. property. Then for any weight $\mathbf{s} \in (0, 1)^N$, the AI_∞ -graph $(X(\mathbf{s}), E^{(\infty)})$ is an $(m, 1)$ -departing expansive graph of bounded degree. Consequently the AI_∞ -graph is a hyperbolic augmented tree.*

The theorem is applied to study self-similar energy forms and resistance metrics in Section 6. By using the fact that the natural identification $\kappa : (\partial X, \theta_a) \rightarrow (K, \rho)$ is a homeomorphism, we can impose a new metric $\tilde{\theta}_a$ on K , as a consequence of Theorem 1.6. We show that *if K admits a regular harmonic structure, then $\tilde{\theta}_a$ is Hölder equivalent to the associated resistance metric on K* (Theorem 6.7).

We remark that in [Ki5], Kigami proposed another construction of metrics through the weighted trees associated with successive partitions on compact metrizable spaces, while he stated that such construction is possible if and only if the “resolution” graph (which is similar to our AI_∞ -graph here) is hyperbolic. He also studied different types of properties among metrics and measures (e.g., Lipschitz equivalence, Ahlfors-regularity, volume doubling, etc.) via the weight functions on trees.

In a forthcoming paper [KLWa], by showing that a hyperbolic graph is near-isometric to an expansive hyperbolic graph, we present in greater generality the framework of index maps and augmented index graphs. We can also extend the scope of underlying spaces to quasi-metric spaces

(particularly the spaces of homogeneous type [Ch,CW]), and study random walks on such hyperbolic graphs.

For the organization of the paper, we introduce the basics of expansive graphs and (m, k) -departing property in Section 2, and prove Theorem 1.1. We study the boundaries of hyperbolic expansive graphs in Section 3, and prove Theorem 1.2. In Section 4, we define the index maps, as well as the associated AI_b -graphs, AI_∞ -graphs, and prove the hyperbolicity of AI_b -graphs in Theorem 1.3. For the AI_∞ -graphs in Theorems 1.4 and 1.5, we need some separation properties of the index family, which are detailed in Section 5. In Section 6, we give the two examples as asserted above, and also apply the techniques developed to consider the weighted IFS. An appendix on the augmented tree defined by IFS of similitudes and some related results are included for the convenience of the reader.

2 Expansive graphs and hyperbolicity

A *graph* (X, E) is a countable set X of vertices together with a set E of edges which is a symmetric subset of $X \times X \setminus \Delta$ ($\Delta := \{(x, x) : x \in X\}$). It is called *locally finite* if for any vertex $x \in X$, $\deg(x) := \#\{y : (x, y) \in E\} < \infty$. For $x, y \in X$, we use $\pi(x, y)$ to denote the *geodesic* (path with smallest path length) from x to y , and define the *graph distance* $d(x, y)$ by the length of $\pi(x, y)$ if such path exists ($d(x, y) = \infty$ otherwise). If $d(x, y)$ is finite for all $x, y \in X$, we say that (X, E) is *connected*; in this case d is an integer-valued metric on X .

In this paper, we assume that (X, E) is a *rooted graph*, i.e., a locally finite connected graph in which a vertex $\vartheta \in X$ is fixed as a *root*. We write $|x| := d(\vartheta, x)$ for $x \in X$, and let $X_n := \{x \in X : |x| = n\}$. Then $X = \bigcup_{n=0}^{\infty} X_n$. We define a partial order \prec on X with $y \prec x$ if and only if x lies on some $\pi(\vartheta, y)$. For an integer $m \geq 0$ and $x \in X$, let

$$\mathcal{J}_m(x) := \{y \in X : y \prec x, |y| = |x| + m\}, \quad \mathcal{J}_{-m}(x) := \{z \in X : x \in \mathcal{J}_m(z)\}$$

be the m -th *descendant set* and the m -th *predecessor set* of x respectively; in general, $\mathcal{J}_m(x)$ is allowed to be empty. We also write $\mathcal{J}_*(x) := \{y \in X : y \prec x\}$ and $\mathcal{J}_{-*}(x) := \{z \in X : x \prec z\}$ for further use.

Let $E_v = \{(x, y) \in E : |x| - |y| = \pm 1\}$ and $E_h = \{(x, y) \in E : |x| = |y|\}$ denote the *vertical edge set* and the *horizontal edge set* respectively. Clearly $E = E_v \cup E_h$, and $E_v = \{(x, y) \in X \times X : x \in \mathcal{J}_1(y) \text{ or } y \in \mathcal{J}_1(x)\}$. We say that a rooted graph (X, E) is *vertical* if $E = E_v$. A (*rooted*) *tree* is a vertical rooted graph satisfying $\#\mathcal{J}_{-1}(x) = 1$ for all $x \in X \setminus \{\vartheta\}$.

We refer to the *horizontal distance* $d_h(\cdot, \cdot)$ as the graph distance on (X, E_h) . We write $x \sim_h y$ for each pair $(x, y) \in E_h \cup \Delta$. It is clear that $d_h(x, y) = \infty$ for $|x| \neq |y|$, and $d(x, y) \leq d_h(x, y)$. In the case that $d(x, y) = d_h(x, y)$, there is a geodesic $\pi(x, y)$ that lies in (X, E_h) , called a *horizontal geodesic* of (X, E) .

Definition 2.1. We call (X, E) an *expansive graph* if it is a rooted graph that satisfies for $x, y \in X$,

$$d_h(x, y) > 1 \Rightarrow d_h(u, v) > 1, \quad \forall u \in \mathcal{J}_1(x), v \in \mathcal{J}_1(y), \quad (2.1)$$

or equivalently if each $u \sim_h v$ implies $x \sim_h y$ whenever $x \in \mathcal{J}_{-1}(u)$ and $y \in \mathcal{J}_{-1}(v)$.

It follows that in such a graph (X, E) , if x, y are predecessors of u (i.e., $x, y \in \mathcal{J}_{-*}(u)$) with $|x| = |y|$, then $x \sim_h y$. It is easy to see that the above condition is also equivalent to

$$\max\{d_h(u, v), 1\} \geq d_h(x, y), \quad u \in \mathcal{J}_1(x), v \in \mathcal{J}_1(y). \quad (2.2)$$

Intuitively, in an expansive rooted graph the children are drifted farther apart than their non-neighboring parents (see Figure 1).

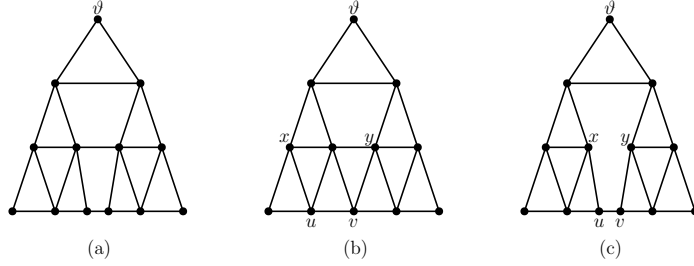


Figure 1: (a) is an expansive graph, while (b) and (c) are not.

Clearly every rooted tree is expansive. Note that an expansive rooted graph (X, E) is called a *pre-augmented (rooted) tree* in [LW3, KLW1] if the vertical part (X, E_v) is a tree. There are important examples in which the vertical parts of expansive graphs are not trees; in the following we display one of those, which arises from the well-known Bernoulli convolutions.

Example 2.2. Let $Y_0 = \{\vartheta\}$, $Y_n = \{x = \varepsilon_1 \cdots \varepsilon_n : \varepsilon_i = 0 \text{ or } 1\}$ for $n \geq 1$, and $Y = \bigcup_{n=0}^{\infty} Y_n$. Clearly there is a natural tree structure E_v^* on Y : $x = \varepsilon_1 \cdots \varepsilon_n \in \mathcal{J}_1(y)$ if and only if $y = \varepsilon_1 \cdots \varepsilon_{n-1}$. For $0 < \rho < 1$, let $I_n = \{\xi(x) = \sum_{i=1}^n \rho^i \varepsilon_i : x = \varepsilon_1 \cdots \varepsilon_n \in Y_n\}$. If $0 < \rho \leq 1/2$, then for each $n \geq 1$, $\xi : Y_n \rightarrow I_n$ is bijective. However for $1/2 < \rho < 1$, they may not be bijective, for example, if ρ equals the golden ratio $\frac{\sqrt{5}-1}{2}$ (the positive solution of $x^2 + x - 1 = 0$), then we have $\xi(011) = \xi(100)$.

We define an equivalence relation \simeq_n on Y_n by $x \simeq_n y$ if and only if $\xi(x) = \xi(y)$. Let X_n be the quotient of Y_n with respect to \simeq_n , and use E_v to denote the edge set on $X = \bigcup_{n=0}^{\infty} X_n$ induced by (Y, E_v^*) . Then (X, E_v) is a vertical graph, but not a tree unless all relations \simeq_n are trivial. We can augment (X, E_v) by adding a set of horizontal edges:

$$(x, y) \in E_h \Leftrightarrow x, y \in X_n, \text{ and } |\xi(x) - \xi(y)| \leq a\rho^n \text{ for some } n \geq 0,$$

where $a = \sum_{n=1}^{\infty} \rho^n = \frac{\rho}{1-\rho}$. Let $E = E_v \cup E_h$. Then (X, E) is an expansive graph, because for $x, y \in X_n$ with $|\xi(x) - \xi(y)| > a\rho^n$,

$$|\xi(u) - \xi(v)| > a\rho^n - \rho^{n+1} = a\rho^{n+1}, \quad u \in \mathcal{J}_1(x), v \in \mathcal{J}_1(y). \quad \square$$

For more discussion of this example in connection with the iterations and augmented trees, the reader can refer to Appendix and [Wa].

A geodesic path $[x_i]_i$ in a rooted graph (X, E) is said to be *convex* if $|x_i| \leq \frac{1}{2}(|x_{i-1}| + |x_{i+1}|)$ for all i . It is easy to see that for each convex geodesic $[x_i]_i$, there exist $u = x_k, v = x_\ell$ ($k \leq \ell$ and they can be equal) such that

$$\begin{cases} x_{i-1} \in \mathcal{J}_1(x_i), & i \leq k; \\ (x_{i-1}, x_i) \in E_h, & k < i \leq \ell; \\ x_{i-1} \in \mathcal{J}_{-1}(x_i), & i > \ell. \end{cases}$$

We denote a convex geodesic segment from x to y by $\pi(x, u, v, y)$; such geodesic may not be unique (see Figure 2).

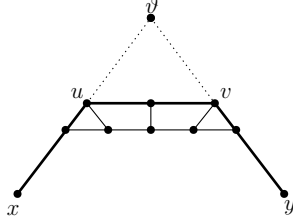


Figure 2: Two convex geodesics between x and y .

The following simple result is an important property of the expansive graphs.

Proposition 2.3. *Let (X, E) be an expansive graph. Then any two vertices $x, y \in X$ can be joined by a convex geodesic.*

Proof. Let π be a geodesic path from x to y . Then it is clear that π does not contain any segment $[u, v, w]$ such that $|u| = |w| = |v| - 1$; indeed in this case, we have $d_h(u, w) \leq 1$ by the expansive property, which leads to a contradiction. Next if π contains a segment $[u, v, w]$ such that $|v| = |w| = |u| + 1$ (or $|u| = |v| = |w| + 1$), then by the expansive property, we can replace this segment by $[u, w^-, w]$ (or $[u, u^-, w]$ respectively), where $w^- \in \mathcal{J}_{-1}(w)$ (see [Ka, Proposition 3.4]). Such a replacement does not change the end points or increase the length of the segment. Repeating this process, we get a convex geodesic from x to y eventually. \square

Definition 2.4. [Gr] *On a rooted graph (X, E) , we define the Gromov product of $x, y \in X$ to be*

$$(x|y) := \frac{1}{2}(|x| + |y| - d(x, y)).$$

(X, E) is said to be (Gromov) hyperbolic if there is $\delta \geq 0$ such that

$$(x|y) \geq \min\{(x|z), (z|y)\} - \delta, \quad \forall x, y, z \in X.$$

In this case for $a > 0$ with $e^{\delta a} < \sqrt{2}$, we can define a metric $\theta_a(\cdot, \cdot)$ (Gromov metric) on X by

$$\theta_a(x, y) = \inf\left\{\sum_{i=1}^n e^{-a(x_{i-1}|x_i)} : n \geq 1, x = x_0, x_1, \dots, x_n = y \in X\right\} \quad (2.3)$$

for distinct $x, y \in X$, and $\theta_a(x, x) = 0$ for $x \in X$. Let \widehat{X} denote the θ_a -completion of X , and call $\partial X := \widehat{X} \setminus X$ the hyperbolic boundary of (X, E) .

The reader can refer to [CDP, Gr, GH, Wo] for more details of the hyperbolic graphs and the hyperbolic boundaries. We observe that if (X, E) is expansive, then for $x, y \in X$ and a convex geodesic $\pi(x, u, v, y)$, we have

$$(x|y) = \frac{1}{2}(|x| + |y| - d(x, y)) = \frac{1}{2}(|u| + |v| - d_h(u, v)) = (u|v). \quad (2.4)$$

Next we introduce another important notion to describe the departing behavior of the descendant vertices in conjunction with the expansive property. Together they provide a useful criterion for the hyperbolicity of the expansive graphs.

Definition 2.5. Let m, k be two positive integers. A rooted graph (X, E) is said to be (m, k) -departing if for $x, y \in X$,

$$d_h(x, y) > k \Rightarrow d_h(u, v) > 2k, \quad \forall u \in \mathcal{J}_m(x), v \in \mathcal{J}_m(y). \quad (2.5)$$

It follows from the definitions that every $(1, 1)$ -departing graph is expansive; every rooted tree is (m, k) -departing for any m, k , and every finite graph is (m, k) -departing for sufficiently large m, k . However, an infinite expansive graph may not be (m, k) -departing for any m, k (see Figure 3 for a simple example; in that graph, if both x, y are on the left or right side, then $d_h(x, y) = d_h(u, v)$ for $u \in \mathcal{J}_m(x)$ and $v \in \mathcal{J}_m(y)$, $m \geq 1$).

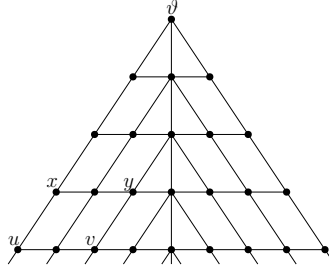


Figure 3: An expansive graph that is not (m, k) -departing for any m, k .

Lemma 2.6. Suppose (X, E) is (m, k) -departing. Then it is $(m, \ell k)$ -departing for any positive integer ℓ .

Proof. Let $x, y \in X$, $u \in \mathcal{J}_m(x)$ and $v \in \mathcal{J}_m(y)$ that satisfy $d_h(u, v) \leq 2\ell k$. Then there exists a set of vertices $\{z_0, z_1, \dots, z_s\} \subset X_{|x|+m}$ such that $z_0 = u$, $z_s = v$, $s \leq \ell$ and $d_h(z_{i-1}, z_i) \leq 2k$ for $i = 1, 2, \dots, s$. By the (m, k) -departing property, we have

$$d_h(x, y) \leq \sum_{i=1}^s d_h(z_{i-1}^{(-m)}, z_i^{(-m)}) \leq sk \leq \ell k,$$

where $z_i^{(-m)} \in \mathcal{J}_{-m}(z_i)$ and $z_0^{(-m)} = x$, $z_s^{(-m)} = y$. This completes the proof. \square

It follows that $(m, 1)$ -departing implies (m, k) -departing for any k . Also, it is straightforward to check inductively that $(1, k)$ -departing implies (m, k) -departing for any m . Hence we have

Corollary 2.7. Suppose (X, E) is $(1, 1)$ -departing. Then it is (m, k) -departing for any positive integers m, k .

We give two examples to illustrate the (m, k) -departing property.

Example 2.8. (SG graph) Let $X = \bigcup_{n=0}^{\infty} \Sigma^n$ with $\Sigma = \{0, 1, \dots, d\}$ ($d \geq 1$) be the symbolic space representing the d -dimensional Sierpinski gasket K , and let K_x be the cell associated to the word $x \in X$. Let E_v be the natural tree structure on X , and define $E_h = \{(x, y) \in X \times X : |x| = |y|, x \neq y, \text{ and } K_x \cap K_y \neq \emptyset\}$ [Ka]. Consider the rooted graph (X, E) with $E = E_v \cup E_h$. It is easy to show that (X, E) is expansive and $(1, 1)$ -departing.

Indeed, it is easy to check that $u \sim_h v$ if and only if either (i) $u = wi, v = wj$ for some $w \in X, i, j \in \Sigma$, or (ii) $u = wij^k, v = wji^k$ for some $w \in X, i, j \in \Sigma, i \neq j, k \geq 1$ [DS1, Ki1]. Note that (i) implies $u^- = v^-$, and (ii) implies $d_h(u^-, v^-) = 1$. Therefore $u \sim_h v$ implies $u^- \sim_h v^-$, i.e., (X, E) is expansive. Moreover, for any horizontal segment $[x, z, y]$, we can conclude that one of x^-, y^- must equal z^- , hence $d_h(x^-, y^-) \leq 1$. This shows that (X, E) is $(1, 1)$ -departing. \square

Example 2.9. (Hata tree) The Hata tree K is defined on \mathbb{C} by the iterated function system $S_1(z) = c\bar{z}$ and $S_2(z) = (1 - |c|^2)\bar{z} + |c|^2$ where $|c|, |1 - c| \in (0, 1)$ (see the left figure of Figure 4, the detailed description of K can be found in [Ki1, p.16]). The graph (X, E) is the symbolic space X with $E = E_v \cup E_h$, where $E_h = \{(x, y) \in X \times X : |x| = |y|, x \neq y, \text{ and } K_x \cap K_y \neq \emptyset\}$ (the right picture). It is clear that (X, E) is expansive.

From the graph, we see that $d_h(11, 22) = d_h(112, 221) = 2$ (the 212-branch sticks out and is thus not counted). Hence (X, E) is not $(1, 1)$ -departing. However, it is not hard to see that (X, E) is $(m, 1)$ -departing for $m \geq 2$ (check level 2, then use the similarity of the graph to conclude for all levels), since the further separation of descendants overcomes the missing count of those outlying branches. \square

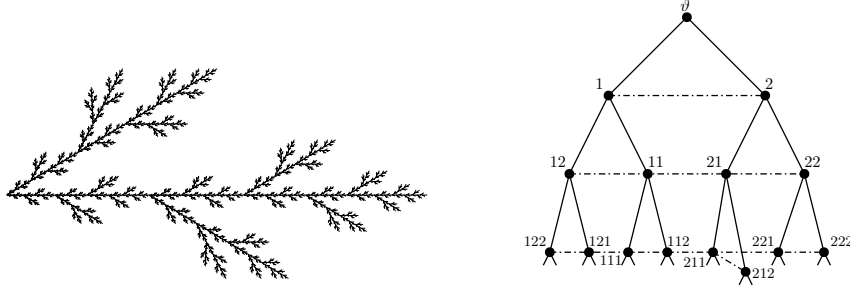


Figure 4: Hata tree ($c = 0.5 + 0.3i$) and the graph (X, E) .

Proposition 2.10. *Suppose (X, E) is (m, k) -departing, then the lengths of all horizontal geodesics in (X, E) are bounded by $L = L(m, k) := \lceil \frac{2m+1}{k} \rceil k + 2m$. In particular, for $(m, 1)$ -departing rooted graphs, $L = 4m + 1$.*

Proof. Suppose otherwise, then there exists a horizontal geodesic $\pi(x, y)$ with length $L + 1$. Note that $\pi(x, \vartheta) \cup \pi(\vartheta, y)$ is a path joining x and y . Comparing the lengths of two paths, we have $2|x| \geq L + 1 > 2m$, i.e., $|x| > m$. Let $x^{(-m)} \in \mathcal{J}_{-m}(x)$ and $y^{(-m)} \in \mathcal{J}_{-m}(y)$. Then there exists a horizontal path joining $x^{(-m)}$ and $y^{(-m)}$ (otherwise $d_h(x^{(-m)}, y^{(-m)}) = +\infty$, and Lemma 2.6 implies that $d_h(x, y) = +\infty$, a contradiction). Now consider a new path: $x \rightarrow x^{(-m)}$ (along horizontal edges) $\rightarrow y^{(-m)} \rightarrow y$, and by comparing the length of this new path with the one of geodesic $\pi(x, y)$ we have

$$2m + d_h(x^{(-m)}, y^{(-m)}) \geq L + 1.$$

It follows that $d_h(x^{(-m)}, y^{(-m)}) \geq \lceil \frac{2m+1}{k} \rceil k + 1 > \lceil \frac{2m+1}{k} \rceil k$. Making use of Lemma 2.6, we have $d_h(x, y) > 2\lceil \frac{2m+1}{k} \rceil k \geq L + 1$. This is a contradiction, and completes the proof. \square

The following theorem provides two useful criteria for the hyperbolicity of expansive graphs.

Theorem 2.11. *Let (X, E) be an expansive graph. Then the following assertions are equivalent.*

- (i) (X, E) is hyperbolic;
- (ii) $\exists L < \infty$ such that the lengths of all horizontal geodesics are bounded by L ;
- (iii) (X, E) is (m, k) -departing for some positive integers m, k .

Proof. (i) \Leftrightarrow (ii) follows from a similar proof as in [LW1, Theorem 2.3].

(iii) \Rightarrow (ii) follows from Proposition 2.10.

(ii) \Rightarrow (iii): We claim that (X, E) is $(L+1, L+2)$ -departing. Indeed, let $x, y \in X$, $x' \in \mathcal{J}_{L+1}(x)$ and $y' \in \mathcal{J}_{L+1}(y)$ satisfying $L+2 < d_h(x', y') \leq 2(L+2)$ (see Figure 5). By Proposition 2.3, there exists a convex geodesic segment $\pi(x', u, v, y')$ between x' and y' , and $u \neq x'$ (by the first inequality and (ii)). Let $u, v \in X_j$. Then

$$2(L+2) \geq d_h(x', y') > d(x', y') = 2(|x'| - |j|) + d_h(u, v) \geq 2(|x'| - j).$$

As $\ell := |x'| - j \leq L+1 = |x'| - |x|$, we have $j \geq |x|$. Let $u' \in \mathcal{J}_*(x) \cap \mathcal{J}_{-\ell}(x') \cap X_j$ and $v' \in \mathcal{J}_*(y) \cap \mathcal{J}_{-\ell}(y') \cap X_j$. Since $x' \in \mathcal{J}_\ell(u) \cap \mathcal{J}_\ell(u')$, u and u' are predecessors of x' , and we have $u \sim_h u'$ by the expansive property. Similarly $v \sim_h v'$. Hence by (2.2) and (ii),

$$d_h(x, y) \leq \max\{d_h(u', v'), 1\} \leq d_h(u, v) + 2 \leq L+2.$$

This proves the claim. □

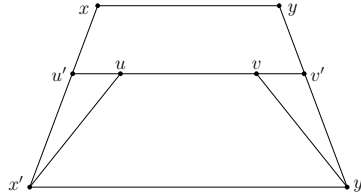


Figure 5: Illustration for the proof of Theorem 2.11.

We will call the graphs in the above theorem **expansive hyperbolic graphs**.

3 Hyperbolic boundaries

In an infinite graph (X, E) with root ϑ , we let

$$\mathcal{R}_v := \{\mathbf{x} = [x_i]_{i=0}^\infty : x_0 = \vartheta, \text{ and } x_{i+1} \in \mathcal{J}_1(x_i), \forall i \geq 0\} \quad (3.1)$$

denote the class of (geodesic) rays starting from the root ϑ . For brevity, we shall use the bold symbols such as $\mathbf{x}, \mathbf{y}, \mathbf{z}$ to denote the rays $[x_i]_i, [y_i]_i, [z_i]_i$ in \mathcal{R}_v respectively.

In this section, we assume that (X, E) is hyperbolic. It follows from (2.3) that

$$\theta_a(x, y) \asymp e^{-a(x|y)}, \quad \forall x, y \in X \quad (3.2)$$

(see [Wo, p.245]), and hence the topology on the θ_a -completion \widehat{X} is independent of the value of a (as $\theta_a(\cdot, \cdot) \asymp \theta_b(\cdot, \cdot)^{a/b}$ for $a, b > 0$); it is also known that both \widehat{X} and ∂X are compact.

Every ray in \mathcal{R}_v is θ_a -Cauchy. Note that for $\mathbf{x}, \mathbf{y} \in \mathcal{R}_v$, the Gromov product $(x_i|y_i)$ is increasing in i by the triangle inequality. By using this we define $(\mathbf{x}|\mathbf{y}) := \lim_{i \rightarrow \infty} (x_i|y_i)$. We say that $\mathbf{x}, \mathbf{y} \in \mathcal{R}_v$ are *equivalent* if they converge to the same point in ∂X ; this holds if and only if $(\mathbf{x}|\mathbf{y}) = \infty$. With such equivalence, the quotient of \mathcal{R}_v is identified with ∂X .

For an integer $k \geq 0$ and two rays $\mathbf{x}, \mathbf{y} \in \mathcal{R}_v$, we define

$$|\mathbf{x} \vee \mathbf{y}|_k := \sup\{i \geq 0 : d_h(x_i, y_i) \leq k\}. \quad (3.3)$$

Clearly if $E = E_v$, $|\mathbf{x} \vee \mathbf{y}|_k = \sup\{i \geq 0 : x_i = y_i\}$ for all $k \geq 0$, and equals $(\mathbf{x}|\mathbf{y})$ when (X, E) is a tree. Note that if $|\mathbf{x} \vee \mathbf{y}|_k = \infty$, i.e., $d_h(x_i, y_i) \leq k$ for all $i \geq 0$, then by

$$(x_i|y_i) = |x_i| - \frac{1}{2}d_h(x_i, y_i) \geq i - \frac{k}{2}, \quad (3.4)$$

we have $\lim_{i \rightarrow \infty} (x_i|y_i) = \infty$, i.e., \mathbf{x} and \mathbf{y} are equivalent.

Furthermore, if the hyperbolic graph (X, E) is expansive, then by Theorem 2.11, it is (m, k) -departing for some $m, k > 0$. This provides a more concrete characterization of the equivalence classes in \mathcal{R}_v as follows, which is used for some estimations in the sequel.

Proposition 3.1. *Suppose the rooted graph (X, E) is expansive and (m, k) -departing. Then there exists a constant $D_0 = D_0(m, k) > 0$ such that*

$$|(\mathbf{x}|\mathbf{y}) - |\mathbf{x} \vee \mathbf{y}|_k| \leq D_0, \quad \forall \mathbf{x}, \mathbf{y} \in \mathcal{R}_v. \quad (3.5)$$

Consequently, two rays \mathbf{x}, \mathbf{y} in \mathcal{R}_v are equivalent if and only if $d_h(x_i, y_i) \leq k$ for all $i \geq 0$.

Proof. Note that $|\mathbf{x} \vee \mathbf{y}|_k = \infty$ implies $(\mathbf{x}|\mathbf{y}) = \infty$. Hence we need only prove (3.5) for the case that $|\mathbf{x} \vee \mathbf{y}|_k < \infty$. Set $\ell := |\mathbf{x} \vee \mathbf{y}|_k + 1$. Then $d_h(x_{\ell-1}, y_{\ell-1}) \leq k$, and $d_h(x_\ell, y_\ell) > k$. Using (3.4) for $i = \ell - 1$, we have

$$(\mathbf{x}|\mathbf{y}) \geq (x_{\ell-1}|y_{\ell-1}) \geq \ell - 1 - k/2 = |\mathbf{x} \vee \mathbf{y}|_k - k/2.$$

On the other hand, by using the (m, k) -departing in Lemma 2.6 repeatedly, we have

$$d_h(x_{mj+\ell}, y_{mj+\ell}) > 2^j k, \quad \forall j \geq 1. \quad (3.6)$$

For $j \geq 0$, we choose $u, v \in X$ such that $\pi(x_{mj+\ell}, u, v, y_{mj+\ell})$ is a convex geodesic. Then $x_{|u|} \sim_h u$ and $v \sim_h y_{|v|}$ (by the remark after Definition 2.1). By Proposition 2.10, there exists $L = L(m, k) > 0$ such that

$$d_h(x_{|u|}, y_{|v|}) \leq d_h(x_{|u|}, u) + d_h(u, v) + d_h(v, y_{|v|}) \leq 1 + L + 1 = L + 2. \quad (3.7)$$

Let $j_0 = j_0(m, k)$ be the smallest integer such that $2^{j_0} k \geq L + 2$. It follows from (3.6) and (3.7) that $|u| = |v| \leq mj_0 + \ell - 1$. Hence by (2.4), we have

$$(x_{mj+\ell}|y_{mj+\ell}) = (u|v) = |u| - \frac{1}{2}d_h(u, v) \leq mj_0 + \ell - 1,$$

and this implies that $(\mathbf{x}|\mathbf{y}) \leq mj_0 + \ell - 1 = |\mathbf{x} \vee \mathbf{y}|_k + mj_0$. Setting $D_0 = \max\{k/2, mj_0\}$, inequality (3.5) follows. \square

By Proposition 3.1, we can extend the Gromov product to \widehat{X} by letting

$$(x|\xi) = \sup\{\lim_{i \rightarrow \infty} (x|x_i)\}, \quad (\xi|\eta) = \sup\{\lim_{i \rightarrow \infty} (x_i|y_i)\}, \quad (3.8)$$

where $x \in X$, $\xi, \eta \in \partial X$, and the supremum is taking over all rays $[x_i]_i$ and $[y_i]_i$ in \mathcal{R}_v that converge to ξ and η respectively; the differences of the limits for different rays are at most k if (X, E) is (m, k) -departing. Then the Gromov metric $\theta_a(\cdot, \cdot)$ on \widehat{X} satisfies the same estimate as in (3.2).

Throughout the rest of the paper, we will assume that $\mathcal{J}_1(x) \neq \emptyset$ for all $x \in X$. In this section we will investigate the metric doubling property of $(\partial X, \theta_a)$. For $x \in X$ and $\xi \in \partial X$, we extend the partial order \prec in Section 2 by writing $\xi \prec x$ if x lies on some ray $[x_i]_i \in \mathcal{R}_v$ that converges to ξ . Define

$$\mathcal{J}_\partial(x) := \{\xi \in \partial X : \xi \prec x\}$$

and call it the *cell* of x in ∂X . Then it is clear that each $\mathcal{J}_\partial(x)$ is a nonempty compact subset of ∂X . We remark that in the context of IFS with attractor K and an associated graph structure (X, E) of the symbolic space, $\mathcal{J}_\partial(x)$ plays the role of the x -cell K_x in K (see Appendix); this will be discussed in detail in Section 4.

Proposition 3.2. *Let (X, E) be an (m, k) -departing expansive graph. Then there exists a constant $\gamma > 0$ (depending on a) such that for $x, y \in X_n$, $n \geq 1$,*

$$d_h(x, y) > k \quad \Rightarrow \quad \text{dist}_{\theta_a}(\mathcal{J}_\partial(x), \mathcal{J}_\partial(y)) > \gamma e^{-an}.$$

Proof. For $\xi \in \mathcal{J}_\partial(x)$ and $\eta \in \mathcal{J}_\partial(y)$, we choose two rays $\mathbf{x}, \mathbf{y} \in \mathcal{R}_v$ that pass through x, y and converge to ξ, η respectively. Then $|\mathbf{x} \vee \mathbf{y}|_k < n$. It follows from (3.5) that

$$(\xi|\eta) \leq (\mathbf{x}|\mathbf{y}) + k \leq |\mathbf{x} \vee \mathbf{y}|_k + k + D_0 < n + k + D_0.$$

Hence $\theta_a(\xi, \eta) \geq ce^{-a(\xi|\eta)} > \gamma e^{-a|x|}$ with $\gamma = ce^{-a(k+D_0)}$. \square

For an integer $k \geq 0$, define the k -shadow of x in ∂X by

$$\mathcal{J}_\partial^k(x) := \bigcup\{\mathcal{J}_\partial(y) : d_h(x, y) \leq k\}. \quad (3.9)$$

(Hence $\mathcal{J}_\partial^0(x) = \mathcal{J}_\partial(x)$.)

Proposition 3.3. *Let (X, E) be an (m, k) -departing expansive graph. Then there exists a constant $C \geq 1$ (depending on a) such that*

$$B_{\theta_a}(\xi, C^{-1}e^{-a|x|}) \subset \mathcal{J}_\partial^k(x) \subset B_{\theta_a}(\xi, Ce^{-a|x|}), \quad \forall x \in X, \xi \in \mathcal{J}_\partial(x). \quad (3.10)$$

Proof. Suppose $x \in X_n$. For $\eta \in \partial X \setminus \mathcal{J}_\partial^k(x)$, we choose $y \in X_n$ with $\eta \in \mathcal{J}_\partial(y)$. Then $d_h(x, y) > k$, and by Proposition 3.2 we have

$$\theta_a(\xi, \eta) \geq \text{dist}_{\theta_a}(\mathcal{J}_\partial(x), \mathcal{J}_\partial(y)) > \gamma e^{-an}, \quad \forall \xi \in \mathcal{J}_\partial(x).$$

This shows that $B_{\theta_a}(\xi, \gamma e^{-a|x|}) \subset \mathcal{J}_\partial^k(x)$ for all $\xi \in \mathcal{J}_\partial(x)$.

For $\zeta \in \mathcal{J}_\partial^k(x)$, we choose $z \in X_n$ that satisfies $d_h(x, z) \leq k$ and $\zeta \in \mathcal{J}_\partial(z)$. Then it follows from (3.8) that

$$(\xi|\zeta) \geq (x|z) = n - \frac{1}{2}d(x, z) \geq n - \frac{k}{2}, \quad \forall \xi \in \mathcal{J}_\partial(x).$$

Hence $\theta_a(\xi, \zeta) \leq C_1 e^{-a(\xi|\zeta)} \leq C_2 e^{-a|x|}$ with $C_2 = C_1 e^{ak/2}$, and $\mathcal{J}_\partial^k(x) \subset B_{\theta_a}(\xi, C_2 e^{-a|x|})$ for all $\xi \in \mathcal{J}_\partial(x)$. \square

We remark that in (3.10), $\mathcal{J}_\partial^k(x)$ cannot be simply replaced by $\mathcal{J}_\partial(x)$, as is shown in the following simple example that the left inclusion does not hold.

Example 3.4. Let (X, E) be the rooted graph as in Figure 6. It is an expansive graph, in which any (nontrivial) horizontal geodesic has length 1. By Theorem 2.11, the graph is hyperbolic. Let $x = x_1$ be as in the figure. Then it is seen that $\mathcal{J}_\partial(x) = \{\xi\}$, where ξ is the limit of the ray $\mathbf{x} = [x_i]_{i=0}^\infty$ in ∂X . There is another ray $\mathbf{a} = [a_i]_{i=0}^\infty$ converging to ξ .

For $n \geq 1$, let η_n be the limit of the ray $[\vartheta, a_1, a_2, \dots, a_{n+1}, b_n, \dots]$ in ∂X . Then $\partial X = \{\xi, \eta_1, \eta_2, \dots\}$. By (3.8), we have $(\xi|\eta_n) = n + 1$ and $\theta_a(\xi, \eta_n) \asymp e^{-a(n+1)} \rightarrow 0$ as $n \rightarrow \infty$. We see that $\mathcal{J}_\partial(x)$ cannot contain any ball $B(\xi, r), r > 0$. \square

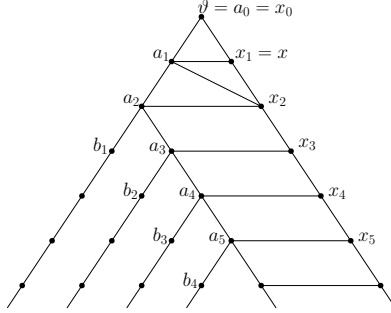


Figure 6: $\mathcal{J}_\partial(x)$ does not contain any ball.

Definition 3.5. [He] A metric space (M, ρ) is called (metric) doubling if there is an integer $\ell > 0$ such that for any $\xi \in M$ and $r > 0$, the ball $B_\rho(\xi, r)$ can be covered by a union of at most ℓ balls of radii $r/2$.

Theorem 3.6. Suppose (X, E) is an expansive hyperbolic graph, and has bounded degree (i.e., $\sup_{x \in X} \deg(x) < \infty$). Then the hyperbolic boundary $(\partial X, \theta_a)$ is doubling.

Proof. Fix $a > 0$ with $e^{\delta a} < \sqrt{2}$, and suppose (X, E) is (m, k) -departing (by Theorem 2.11). Let $C \geq 1$ be the constant as in (3.10). For $r \in (0, 1]$, let $m^*(r)$ ($m_*(r)$) be the smallest (largest respectively) nonnegative integers such that

$$C^{-1}e^{-a(m^*(r)+1)} < r < 2Ce^{-a(m_*(r)-1)}.$$

It follows that $m_*(r) - m^*(r) < \lceil \frac{\log(2C^2)}{a} \rceil + 2 =: \ell_1$. For a ball $B_{\theta_a}(\xi, r) \subset \partial X$, let x be the vertex such that $|x| = m^*(r)$ and $\xi \in \mathcal{J}_\partial(x)$. We claim that $B_{\theta_a}(\xi, r) \subset \mathcal{J}_\partial^k(x)$. Indeed, if $m^*(r) = 0$, then $x = \vartheta$, and trivially $B_{\theta_a}(\xi, r) \subset \partial X = \mathcal{J}_\partial^k(\vartheta)$; if $m^*(r) \geq 1$, then by Proposition 3.3 we have $B_{\theta_a}(\xi, r) \subset B_{\theta_a}(\xi, C^{-1}e^{-am^*(r)}) \subset \mathcal{J}_\partial^k(x)$ as well. By the claim and Proposition 3.3, we have

$$\begin{aligned} B_{\theta_a}(\xi, r) &\subset \mathcal{J}_\partial^k(x) = \bigcup_{y: d_h(x, y) \leq k} \bigcup_{z \in \mathcal{J}_{\ell_1}(y)} \mathcal{J}_\partial(z) \\ &\subset \bigcup_{y: d_h(x, y) \leq k} \bigcup_{z \in \mathcal{J}_{\ell_1}(y)} B_{\theta_a}(\eta_z, Ce^{-am^*(r)}) \quad (\text{by (3.10)}) \\ &\subset \bigcup_{y: d_h(x, y) \leq k} \bigcup_{z \in \mathcal{J}_{\ell_1}(y)} B_{\theta_a}(\eta_z, r/2), \end{aligned}$$

where each η_z is chosen arbitrarily from $\mathcal{J}_\partial(z)$. Let $t = \sup_{x \in X} \deg(x)$, which is finite by assumption. Then

$$\#\{y : d_h(x, y) \leq k\} \leq t^k \quad \text{and} \quad \#\mathcal{J}_{\ell_1}(y) \leq t^{\ell_1}.$$

Hence the ball $B_{\theta_a}(\xi, r)$ is covered by a union of at most $t^{k+\ell_1}$ many balls of radii $r/2$, and $(\partial X, \theta_a)$ is doubling. \square

Remark 1. A similar result was proved in another setting by Bonk and Schramm [BS, Theorem 9.2]; here our proof on expansive hyperbolic graphs is more straightforward.

Remark 2. Note that the doubling property of $(\partial X, \theta_a)$ does not imply the bounded degree property of (X, E) . For example, if we take $E_h = \{(x, y) \in X \times X \setminus \Delta : |x| = |y|\}$, then it is trivially an expansive hyperbolic graph, and ∂X is a singleton (so it is doubling trivially); but $\deg(x) \geq \#X_n - 1$ for $x \in X_n$, which has no bound if $\#X_n$ tends to ∞ .

However, this converse can be verified with some further separation properties among the cells (Theorem 5.5).

4 Index maps and augmented graphs

In this section, we fix a complete metric space (M, ρ) , and let \mathcal{C}_M denote the family of all nonempty compact subsets of M . By our convention in last section, any graph (X, E) mentioned below is assumed to satisfy $\mathcal{J}_\partial(x) \neq \emptyset$ for all $x \in X$.

Definition 4.1. Let (X, E_v) be a vertical graph. A map $\Phi : X \rightarrow \mathcal{C}_M$ is called an index map (on (X, E_v) over (M, ρ)) if it satisfies

- (i) $\Phi(y) \subset \Phi(x)$ for all $x \in X$ and $y \in \mathcal{J}_1(x)$;
- (ii) $\bigcap_{i=0}^{\infty} \Phi(x_i)$ is a singleton for all $\mathbf{x} = [x_i]_i \in \mathcal{R}_v$.

In particular, such Φ is called saturated if (i) is strengthened to $\Phi(x) = \bigcup_{y \in \mathcal{J}_1(x)} \Phi(y)$.

We call $K := \bigcap_{n=0}^{\infty} \left(\bigcup_{x \in X_n} \Phi(x) \right)$ the attractor of Φ , and $K_x := \Phi(x) \cap K$ a cell of K .

Remark 1. An index map Φ induces another index map $\Phi' : X \rightarrow \mathcal{C}_K$ with $\Phi'(x) = K_x$; the image $\{K_x\}_{x \in X}$ is the family of cells of K indexed by X . Since Φ and Φ' behave the same at infinity (i.e., $\bigcap_{i=0}^{\infty} \Phi(x_i) = \bigcap_{i=0}^{\infty} \Phi'(x_i)$ for all $\mathbf{x} = [x_i]_i \in \mathcal{R}_v$), we use these two interchangeably.

Remark 2. For a saturated index map Φ , the attractor $K = \Phi(\vartheta)$, and the cell $K_x = \Phi(x)$ for all $x \in X$. In fact, every index map Φ induces a saturated index map $\tilde{\Phi} : X \rightarrow \mathcal{C}_K$ with

$$\tilde{\Phi}(x) := \bigcap_{n=0}^{\infty} \left(\bigcup_{y \in \mathcal{J}_n(x)} \Phi(y) \right), \quad x \in X, \quad (4.1)$$

which also behaves the same as Φ at infinity. It is clear that $\tilde{\Phi}(x) \subset K_x$ for all $x \in X$, but the reverse inclusion does not hold in general.

The index map Φ defines a mapping $\kappa_0 : \mathcal{R}_v \rightarrow K$ by

$$\{\kappa_0(\mathbf{x})\} = \bigcap_{i=0}^{\infty} \Phi(x_i), \quad \forall \mathbf{x} = [x_i]_i \in \mathcal{R}_v. \quad (4.2)$$

Using the local finiteness of (X, E_v) and a diagonal argument (cf. [Wo, LW3]), we can show that the image of κ_0 is equal to K . From Section 3, we see that for an expansive hyperbolic graph (X, E) , the hyperbolic boundary ∂X can be identified with a quotient set of \mathcal{R}_v . Hence the induced map

$\kappa : \partial X \rightarrow K$ from the quotient is well-defined if κ_0 satisfies: \mathbf{x}, \mathbf{y} are equivalent $\Rightarrow \kappa_0(\mathbf{x}) = \kappa_0(\mathbf{y})$; furthermore $\kappa : \partial X \rightarrow K$ is one-to-one if the converse is also satisfied. With these, we see that if κ_0 satisfies

$$\mathbf{x} \text{ and } \mathbf{y} \text{ are equivalent} \Leftrightarrow \kappa_0(\mathbf{x}) = \kappa_0(\mathbf{y}), \quad (4.3)$$

then it induces a bijection $\kappa : \partial X \rightarrow K$ via the quotient.

Definition 4.2. We call (X, E, Φ) an admissible index triple if it satisfies

- (i) (X, E) is an expansive hyperbolic graph;
- (ii) $\Phi : X \rightarrow \mathcal{C}_M$ is an index map on (X, E_v) over (M, ρ) ;
- (iii) $\kappa : \partial X \rightarrow K$ is a well-defined bijection, i.e., (4.3) holds.

In such case, (X, E) is said to be an admissible (augmented) graph (associated to Φ); if (X, E_v) is a tree, then we call (X, E) an admissible augmented tree.

By Remark 2,

$$\kappa(\mathcal{J}_\partial(x)) = \tilde{\Phi}(x) \subset \Phi(x), \quad \forall x \in X; \quad (4.4)$$

and the inclusion is an “=” if and only if the index map Φ is saturated. Via the bijection κ , the Gromov metric θ_a on ∂X defines naturally a metric $\tilde{\theta}_a$ on the attractor K by

$$\tilde{\theta}_a(\xi, \eta) = \theta_a(\kappa^{-1}(\xi), \kappa^{-1}(\eta)), \quad \xi, \eta \in K. \quad (4.5)$$

Proposition 4.3. For an admissible index triple, the bijection κ is a homeomorphism.

We will prove this in [KLWa], as the proof requires more preparatory work and we will not need the proposition here.

For a subset E in (M, ρ) , we denote the diameter of E by $|E|_\rho$ (or simply by $|E|$). In Definition 4.1, we see that the family $\{\Phi(x)\}_{x \in X}$ satisfies $\lim_{n \rightarrow \infty} \sup_{x \in \mathcal{J}_n} |\Phi(x)|_\rho = 0$. For $b \in (0, \infty)$, we say that $\{\Phi(x)\}_{x \in X}$ (or Φ) is of *exponential type-(b)* (under ρ) if the diameter $|\Phi(x)|_\rho$ is decreasing in a rate of $e^{-b|x|}$, i.e., $|\Phi(x)|_\rho = O(e^{-b|x|})$ as $|x| \rightarrow \infty$, and call Φ an *exponential type* if it is of type-(b) for some $b \in (0, \infty)$.

We mainly consider the following two classes of expansive graphs associated to index maps, which are motivated by the *augmented trees* of the IFS's [Ka, LW1, LW3, Wa] (see Appendix).

Definition 4.4. Let Φ be an index map on the vertical graph (X, E_v) . We define a horizontal edge set by

$$E_h^{(\infty)} := \bigcup_{n=1}^{\infty} \{(x, y) \in X_n \times X_n \setminus \Delta : \Phi(x) \cap \Phi(y) \neq \emptyset\}, \quad (4.6)$$

and let $E^\infty = E_v \cup E_h^{(\infty)}$. We call (X, E^∞) an AI_∞ -graph, augmented index graph of type-(∞) (or intersection type).

In addition, assume that Φ is of exponential type-(b). For a fixed $\gamma > 0$, we define

$$E_h^{(b)} := \bigcup_{n=1}^{\infty} \{(x, y) \in X_n \times X_n \setminus \Delta : \text{dist}_\rho(\Phi(x), \Phi(y)) \leq \gamma e^{-bn}\}. \quad (4.7)$$

Let $E^{(b)} = E_v \cup E_h^{(b)}$. We call $(X, E^{(b)})$ an AI_b -graph, augmented index graph of type-(b).

It is clear that both $(X, E^{(\infty)})$ and $(X, E^{(b)})$ are expansive. Comparing the two definitions, the AI_∞ -graph is more intuitive but needs more information on the neighborhood of $\Phi(x)$, $x \in X$ under the given metric ρ ; the AI_b -graph is more flexible on the neighboring cells, which actually makes it easier to handle. First we prove

Theorem 4.5. *For an index map Φ on (X, E_v) over (M, ρ) of exponential type- (b) , the associated AI_b -graph is $(m, 1)$ -departing, and is an admissible graph. Moreover, $\kappa : (\partial X, \theta_a) \rightarrow (K, \rho)$ is a Hölder equivalence with exponent b/a , i.e.,*

$$\rho(\kappa(\xi), \kappa(\eta))^{a/b} \asymp \theta_a(\xi, \eta), \quad \forall \xi, \eta \in \partial X. \quad (4.8)$$

Proof. It is easy to check from the definition of $E_h^{(b)}$ that $(X, E^{(b)})$ is expansive. To show that it is $(m, 1)$ -departing for some $m \geq 1$, let $\delta_0 := \sup_{z \in X} e^{b|z|} |\Phi(z)|$. Let $u \in \mathcal{J}_m(x)$ and $v \in \mathcal{J}_m(y)$ with $d_h(u, v) = 2$. Using the triangle inequality twice, we have

$$\text{dist}(\Phi(x), \Phi(y)) \leq \text{dist}(\Phi(u), \Phi(v)) \leq (2\gamma + \delta_0)e^{-b(|x|+m)} \leq \gamma e^{-b|x|},$$

where the positive integer m is chosen to give the last inequality, i.e., $(2\gamma + \delta_0)e^{-bm} \leq \gamma$. Therefore $x \sim_h y$, and this shows that (X, E) is $(m, 1)$ -departing. The hyperbolicity of AI_b -graph follows from Theorem 2.11.

By Proposition 3.1 (with $k = 1$) and (4.7), we see that two rays \mathbf{x}, \mathbf{y} are equivalent if and only if $\text{dist}(\Phi(x_i), \Phi(y_i)) \leq \gamma e^{-bi}$ for all i . This verifies (4.3) so that $\kappa : \partial X \rightarrow K$ is a well-defined bijection, and hence $(X, E^{(b)})$ is an admissible graph.

We now prove that κ is a Hölder equivalence. For distinct $\xi, \eta \in \partial X$, we take two rays $\mathbf{x}, \mathbf{y} \in \mathcal{R}_v$ that converge to ξ, η respectively with $(\xi|\eta) = (\mathbf{x}|\mathbf{y})$ (by (3.8) and the following remark). Let $n = |\mathbf{x} \vee \mathbf{y}|_1$ as in (3.3) with $k = 1$, i.e., $d_h(x_n, y_n) \leq 1$ and $d_h(x_{n+1}, y_{n+1}) \geq 2$. By Proposition 3.1, we have $|(\xi|\eta) - n| = |(\mathbf{x}|\mathbf{y}) - n| \leq D_0$ for some $D_0 := D_0(m, 1) > 0$. As $\kappa(\xi) \in \Phi(x_{n+1}) \subset \Phi(x_n)$ and $\kappa(\eta) \in \Phi(y_{n+1}) \subset \Phi(y_n)$, we get the lower bound of (4.8) by

$$\begin{aligned} \rho(\kappa(\xi), \kappa(\eta)) &\geq \text{dist}(\Phi(x_{n+1}), \Phi(y_{n+1})) \\ &\geq \gamma e^{-b(n+1)} \geq \gamma e^{-b(D_0+1)} e^{-b(\xi|\eta)} \geq C_1 \theta_a(\xi, \eta)^{b/a}, \end{aligned}$$

and the upper bound by

$$\begin{aligned} \rho(\kappa(\xi), \kappa(\eta)) &\leq |\Phi(x_n)| + \text{dist}(\Phi(x_n), \Phi(y_n)) + |\Phi(y_n)| \\ &\leq (2\delta_0 + \gamma)e^{-bn} \leq (2\delta_0 + \gamma)e^{bD_0} e^{-b(\xi|\eta)} \leq C_2 \theta_a(\xi, \eta)^{b/a}. \end{aligned}$$

This completes the proof. \square

Now we turn to study the AI_∞ -graphs. Unlike the AI_b -graph, the AI_∞ -graph is not always hyperbolic (see Example 6.1), and sufficient conditions for its hyperbolicity will be provided in Section 5. The following result shows that if it is hyperbolic, then the admissibility follows.

Proposition 4.6. *Suppose the AI_∞ -graph $(X, E^{(\infty)})$ is hyperbolic. Then for two rays $\mathbf{x}, \mathbf{y} \in \mathcal{R}_v$, the following assertions are equivalent.*

- (i) \mathbf{x} and \mathbf{y} are equivalent;
- (ii) $\kappa_0(\mathbf{x}) = \kappa_0(\mathbf{y})$;
- (iii) $d_h(x_i, y_i) \leq 1$ for all $i \geq 0$.

It follows from (4.3) that $(X, E^{(\infty)})$ is an admissible graph (Definition 4.2).

Proof. (iii) \Rightarrow (i) is clear. For (ii) \Rightarrow (iii), since $\kappa_0(\mathbf{x}) \in \Phi(x_i)$ and $\kappa_0(\mathbf{y}) \in \Phi(y_i)$, we have $\Phi(x_i) \cap \Phi(y_i) \neq \emptyset$ for all $i \geq 0$. This yields (iii) by the definition (4.6) of $E_h^{(\infty)}$.

For (i) \Rightarrow (ii), as $(X, E^{(\infty)})$ is expansive and hyperbolic, by Theorem 2.11(iii) and Proposition 3.1, there exists an integer $k > 0$ such that $d_h(x_i, y_i) \leq k$ for all $i \geq 0$. We show inductively that any such k will imply (ii). When $k = 1$, it follows that $\Phi(x_i) \cap \Phi(y_i) \neq \emptyset$ for all $i \geq 0$. By the compactness

of $\Phi(x_i) \cap \Phi(y_i)$, the intersection $(\bigcap_{i=0}^{\infty} \Phi(x_i)) \cap (\bigcap_{i=0}^{\infty} \Phi(y_i)) = \bigcap_{i=0}^{\infty} (\Phi(x_i) \cap \Phi(y_i)) \neq \emptyset$, hence $\kappa_0(\mathbf{x}) = \kappa_0(\mathbf{y})$.

Inductively, suppose (ii) is verified when $k = n$ for some $n > 0$. Let $\mathbf{x}, \mathbf{y} \in \mathcal{R}_v$ satisfy $d_h(x_i, y_i) \leq n + 1$. Then for each $i \geq 0$, we choose $z_i \in X$ such that $d_h(x_i, z_i) \leq 1$ and $d_h(z_i, y_i) \leq n$. The sequence $\{z_i\}_{i=0}^{\infty}$ may not be a ray; however, using the local finiteness of (X, E_v) and a diagonal argument, we can choose a ray $\mathbf{w} \in \mathcal{R}_v$ such that each w_i contains an infinite subsequence of $\{z_i\}_{i=0}^{\infty}$, and the expansive property (2.2) implies that $d_h(x_i, w_i) \leq 1$ and $d_h(w_i, y_i) \leq n$ for all $i \geq 0$. Using the induction hypothesis, we have $\kappa_0(\mathbf{x}) = \kappa_0(\mathbf{w}) = \kappa_0(\mathbf{y})$, and the proof is completed by induction. \square

When Φ is of exponential type-(b), comparing the AI_{∞} -graph with the AI_b -graph, it is clear that $E^{(\infty)} \subset E^{(b)}$. The following is a consequence of Theorem 4.5.

Corollary 4.7. *Suppose the index map Φ is of exponential type-(b), and the associated AI_{∞} -graph $(X, E^{(\infty)})$ is hyperbolic. Then $\kappa : (\partial X, \theta_a) \rightarrow (K, \rho)$ is Hölder continuous with exponent b/a , i.e.,*

$$\rho(\kappa(\xi), \kappa(\eta))^{a/b} \leq C\theta_a(\xi, \eta), \quad \forall \xi, \eta \in \partial X. \quad (4.9)$$

Proof. We consider the associated AI_b -graph $(X, E^{(b)})$, and denote its graph distance and Gromov product by $d'(\cdot, \cdot)$ and $(\cdot|\cdot)'$ respectively. By Theorem 4.5, $(X, E^{(b)})$ is hyperbolic, and the bijection $\kappa' : \partial X' \rightarrow K$ satisfies $\rho(\kappa'(\xi), \kappa'(\eta)) \asymp e^{-b(\xi|\eta)'}$ for all $\xi, \eta \in \partial X'$. As $\kappa_0 = \kappa'_0$ on \mathcal{R}_v , it follows that $\partial X = \partial X'$ and $\kappa = \kappa'$.

Note that $E^{(\infty)} \subset E^{(b)}$. This implies $d(x, y) \geq d'(x, y)$, and

$$(x|y) = \frac{1}{2}(|x| + |y| - d(x, y)) \leq \frac{1}{2}(|x| + |y| - d'(x, y)) = (x|y)'$$

for all $x, y \in X$. Taking the limits in (3.8), we have $(\xi|\eta) \leq (\xi|\eta)'$, and

$$\theta_a(\xi, \eta) \geq c_1 e^{-a(\xi|\eta)} \geq c_1 e^{-a(\xi|\eta)'} \geq c_2 \rho(\kappa'(\xi), \kappa'(\eta))^{a/b} = c_2 \rho(\kappa(\xi), \kappa(\eta))^{a/b}.$$

This verifies (4.9), the Hölder continuity of κ . \square

In general, we cannot expect this κ to be a Hölder equivalence (see Example 6.2). In the following theorem, we give a characterization for the hyperbolicity, or equivalently the (m, k) -departing property, of the AI_{∞} -graph associated to a saturated index map (Remark 2 of Definition 4.1), together with the Hölder equivalence to hold.

Theorem 4.8. *Suppose Φ is a saturated index map on (X, E_v) over (M, ρ) . Then for $b \in (0, \infty)$ and an integer $k > 0$, the following assertions are equivalent.*

(i) *The AI_{∞} -graph $(X, E^{(\infty)})$ is (m, k) -departing for some $m > 0$, and $\kappa : (\partial X, \theta_a) \rightarrow (K, \rho)$ is a Hölder equivalence with exponent b/a , i.e.,*

$$\rho(\kappa(\xi), \kappa(\eta))^{a/b} \asymp \theta_a(\xi, \eta), \quad \forall \xi, \eta \in \partial X. \quad (4.10)$$

(ii) *Φ is of exponential type-(b) under ρ , and there exists $\gamma > 0$ such that $(X, E^{(\infty)})$ satisfies for $x, y \in X$,*

$$|x| = |y| \text{ and } d_h(x, y) > k \quad \Rightarrow \quad \text{dist}_{\rho}(\Phi(x), \Phi(y)) > \gamma e^{-b|x|}. \quad (4.11)$$

Proof. For (i) \Rightarrow (ii), since $(X, E^{(\infty)})$ is expansive and (m, k) -departing, it is hyperbolic (Theorem 2.11). By Proposition 4.6, it is an admissible graph, and the bijection $\kappa : \partial X \rightarrow K$ is well-defined. Using (4.4) and (4.10), we have

$$|\Phi(x)|_\rho = |\kappa(\mathcal{J}_\partial(x))|_\rho \leq C_1 |\mathcal{J}_\partial(x)|_{\theta_a}^{b/a} \leq C_1 e^{-b|x|}$$

for all $x \in X$. Therefore Φ is of exponential type- (b) under ρ . Moreover, for $x, y \in X_n$ with $d_h(x, y) > k$, it follows from (4.4), (4.10) and Proposition 3.2 that

$$\begin{aligned} \text{dist}_\rho(\Phi(x), \Phi(y)) &= \text{dist}_\rho(\kappa(\mathcal{J}_\partial(x)), \kappa(\mathcal{J}_\partial(y))) \\ &\geq C_2 \text{dist}_{\theta_a}(\mathcal{J}_\partial(x), \mathcal{J}_\partial(y))^{b/a} > C_2 \gamma_0 e^{-bn}. \end{aligned}$$

This proves (4.11) with $\gamma = C_2 \gamma_0$.

For (ii) \Rightarrow (i), the proof is similar to the one of Theorem 4.5 on the AI_b -graph. Let $\delta_0 = \sup_{z \in X} e^{b|z|} |\Phi(z)|_\rho$, which is finite as Φ is of exponential type- (b) . Let $u \in \mathcal{J}_m(x)$ and $v \in \mathcal{J}_m(y)$ with $d_h(u, v) = \ell < 2k$. Then $\Phi(u)$ and $\Phi(v)$ are joined by a chain $\{\Phi(u_j)\}_{j=0}^\ell$ with $u_0 = u$ and $u_\ell = v$, in which $|u_j| = |u| = |x| + m$ and $\Phi(u_{j-1}) \cap \Phi(u_j) \neq \emptyset$ for all $j \in \{1, 2, \dots, \ell\}$. Therefore,

$$\text{dist}_\rho(\Phi(x), \Phi(y)) \leq \text{dist}_\rho(\Phi(u), \Phi(v)) \leq \sum_{j=1}^{\ell-1} |\Phi(u_j)|_\rho \leq (2k-1) \delta_0 e^{-b(|x|+m)} \leq \gamma e^{-b|x|},$$

where the integer $m > 0$ is chosen to satisfy the last inequality, i.e., $(2k-1) \delta_0 e^{-bm} \leq \gamma$. By (4.11), we have $d_h(x, y) \leq k$, and this proves that $(X, E^{(\infty)})$ is (m, k) -departing.

To prove (4.10), for $\xi \neq \eta \in \partial X$, we take $\mathbf{x}, \mathbf{y} \in \mathcal{R}_v$ that converge to ξ, η respectively with $(\xi|\eta) = (\mathbf{x}|\mathbf{y})$, and let $n = |\mathbf{x} \vee \mathbf{y}|_k$ as in (3.3). It follows from Proposition 3.1 that $|(\xi|\eta) - n| = |(\mathbf{x}|\mathbf{y}) - n| \leq D_0$ for some $D_0 = D_0(m, k) > 0$. Using $\kappa(\xi) \in \Phi(x_{n+1})$, $\kappa(\eta) \in \Phi(y_{n+1})$, $d_h(x_{n+1}, y_{n+1}) > k$ and (4.11), we get the lower bound of (4.10) by

$$\begin{aligned} \rho(\kappa(\xi), \kappa(\eta)) &\geq \text{dist}_\rho(\Phi(x_{n+1}), \Phi(y_{n+1})) \\ &> \gamma e^{-b(n+1)} \geq \gamma e^{-b(D_0+1)} e^{-b(\xi|\eta)} \geq c \theta_a(\xi, \eta)^{b/a}; \end{aligned}$$

the upper bound is proved by Corollary 4.7. This completes the proof. \square

Remark. Letting $k = 1$ in (4.11), the condition becomes

$$|x| = |y| \quad \text{and} \quad \Phi(x) \cap \Phi(y) = \emptyset \quad \Rightarrow \quad \text{dist}(\Phi(x), \Phi(y)) > \gamma e^{-b|x|}. \quad (4.12)$$

This is the *condition (H)* in [LW1] in the setup where K is a self-similar set (see also Appendix, Theorem 7.1); the authors proved that this condition is sufficient for the Hölder equivalence between ∂X and K . Here a necessity part is also provided, together with an extra relation with the (m, k) -departing property (or hyperbolicity) of these graphs.

5 Separation conditions and doubling metrics

In this section, we aim to give some sufficient conditions for AI_∞ -graphs to be hyperbolic, and characterize the bounded degree property of AI_b - and AI_∞ -graphs; both involve some separation conditions on index maps and the doubling property of attractors.

Let Φ be an index map on a vertical graph (X, E_v) over a complete metric space (M, ρ) with attractor K . We call a map $\iota : X \rightarrow K$ a *projection* (with respect to Φ) if it satisfies $\iota(x) \in K_x$ ($:=$

$\Phi(x) \cap K$ for all $x \in X$. When there is no confusion, we shall denote the ball $B_\rho(\xi, r)$ by $B(\xi, r)$ for simplicity.

Definition 5.1. Let Φ be an index map with attractor K . For $b \in (0, \infty)$, we say that Φ (or $\{K_x\}_{x \in K}$) satisfies

(i) condition (S_b) if for any $c > 0$, there is a constant $\bar{\ell} = \bar{\ell}(c)$ such that

$$\#\{x \in X_n : K_x \cap F \neq \emptyset\} \leq \bar{\ell}, \quad \forall n \geq 0 \text{ and } F \subset M \text{ with } |F|_\rho < ce^{-bn}; \quad (5.1)$$

(ii) condition (B_b) if there exist a projection $\iota : X \rightarrow K$ and $c_0 \in (0, \infty)$ such that

$$B(\iota(x), c_0 e^{-b|x|}) \cap K \subset K_x, \quad \forall x \in X. \quad (5.2)$$

We remark that condition (S_b) is motivated from the open set condition on self-similar sets (see (7.3) in Appendix, and condition (S'_b) in Proposition 5.3). To study the above conditions, we need a preliminary result on doubling metric spaces (Lemma 5.2). For a subset $F \subset M$ and $r > 0$, define the r -covering number of F by

$$N_r^c(F) := \inf\{\#\Xi : \Xi \subset F \text{ and } F \subset \bigcup_{\xi \in \Xi} B(\xi, r)\}; \quad (5.3)$$

F is called *totally bounded* if the r -covering number is finite for all $r > 0$. We also define the r -packing number of F to be

$$N_r^p(F) := \sup\{\#\Xi : \Xi \subset F \text{ and } B(\xi, r) \cap B(\eta, r) = \emptyset, \forall \xi, \eta \in \Xi, \xi \neq \eta\},$$

and the r -separating number of F to be

$$N_r^s(F) := \sup\{\#\Xi : \Xi \subset F \text{ and } \rho(\xi, \eta) \geq r, \forall \xi, \eta \in \Xi, \xi \neq \eta\}.$$

Recall that the metric space (M, ρ) is doubling (Definition 3.5) if and only if

$$\sup\{N_r^c(B(\xi, 2r)) : \xi \in M, r > 0\} < \infty.$$

Lemma 5.2. Let (M, ρ) be a metric space. Then the inequalities

$$N_{2r}^s(F) \leq N_r^p(F) \leq N_r^c(F) \leq N_r^s(F) \quad (5.4)$$

hold for all totally bounded subset $F \subset M$ and $r > 0$. As a consequence, the following assertions are equivalent.

- (i) (M, ρ) is doubling (Definition 3.5).
- (ii) For some (\Leftrightarrow any) $t > 1$, $\widehat{N}^c(t) := \sup\{N_r^c(B(\xi, tr)) : \xi \in M, r > 0\} < \infty$.
- (iii) For some (\Leftrightarrow any) $t > 1$, $\widehat{N}^p(t) := \sup\{N_r^p(B(\xi, tr)) : \xi \in M, r > 0\} < \infty$.
- (iv) For some (\Leftrightarrow any) $t > 1$, $\widehat{N}^s(t) := \sup\{N_r^s(B(\xi, tr)) : \xi \in M, r > 0\} < \infty$.

The inequality (5.4) is straightforward by using the definitions, and the equivalence of (i)–(iv) follows directly from (5.4). We omit the detail.

Proposition 5.3. Let Φ be an index map with attractor K , and is of exponential type-(b). Suppose (K, ρ) is doubling, then the following conditions are equivalent:

- (i) condition (S_b) ;

(ii) condition (S'_b) : there exist $c_1 > 0$ and $\ell_1 > 0$ such that

$$\#\{x \in X_n : K_x \cap B(\xi, c_1 e^{-bn}) \neq \emptyset\} \leq \ell_1, \quad \forall n \geq 0, \xi \in K; \quad (5.5)$$

(iii) condition (S''_b) : there exist a projection $\iota : X \rightarrow K$, $c_2 > 0$ and $\ell_2 > 0$ such that

$$\#\{x \in X_n : \rho(\iota(x), \iota(y)) < c_2 e^{-bn}\} \leq \ell_2, \quad \forall n \geq 0, y \in X_n. \quad (5.6)$$

Proof. (i) \Rightarrow (ii) \Rightarrow (iii) is obvious. We need only show (ii) \Rightarrow (i) and (iii) \Rightarrow (ii).

(ii) \Rightarrow (i): For any $c > 0$ and $F \subset M$ with $|F|_\rho < c e^{-bn}$, we will show that (5.1) holds for some $\bar{\ell}$. Without loss of generality, we assume that $c > c_1$ and $K \cap F \neq \emptyset$. Fix a point $\xi \in K \cap F$, and denote the open ball $B(\xi, c e^{-bn}) \cap K$ in (K, ρ) by B_K . Since (K, ρ) has doubling property, by Lemma 5.2(ii), B_K can be covered by a union of $N_0 := \widehat{N}^c(\frac{c}{c_1})$ open balls $B(\xi_i, c_1 e^{-bn}) \cap K$ in (K, ρ) , where $\xi_i \in K$, $i = 1, \dots, N_0$, i.e.,

$$K \cap F \subset B_K \subset \bigcup_{i=1}^{N_0} (K \cap B(\xi_i, c_1 e^{-bn})).$$

It follows from (5.5) that

$$\#\{x \in X_n : K_x \cap F \neq \emptyset\} \leq \sum_{i=1}^{N_0} \#\{x \in X_n : K_x \cap B(\xi_i, c_1 e^{-bn}) \neq \emptyset\} \leq \ell_1 N_0.$$

We see that (5.1) holds for the constant $\bar{\ell} = \ell_1 N_0$.

(iii) \Rightarrow (ii): For $n \geq 0$ and $\xi \in K$, denote

$$X_n(\xi) := \{x \in X_n : K_x \cap B(\xi, c_1 e^{-bn}) \neq \emptyset\},$$

on which we define an edge set $\tilde{E} = \{(x, y) \in X_n(\xi) \times X_n(\xi) \setminus \Delta : \rho(\iota(x), \iota(y)) < c_2 e^{-bn}\}$. By (5.6), we see that the maximal degree in the graph $(X_n(\xi), \tilde{E})$ does not exceed $(\ell_2 - 1)$. Applying Brooks' Theorem on the chromatic number (cf. [LiW, Theorem 3.1]), there exists a coloring map $\mathcal{K} : X_n(\xi) \rightarrow \Sigma$ with $\#\Sigma \leq \ell_2$ such that $\mathcal{K}(x) \neq \mathcal{K}(y)$ if $(x, y) \in \tilde{E}$.

Since Φ is of exponential type-(b), there exists a constant $\delta_0 > 0$ such that $|\Phi(x)|_\rho \leq \delta_0 e^{-b|x|}$ for all $x \in X$. Consider the discrete sets

$$\mathcal{M}^{(i)} := \{\iota(x) : x \in X_n(\xi), \mathcal{K}(x) = i\}, \quad i \in \Sigma.$$

Then $\rho(\iota(x), \iota(y)) \geq c_2 e^{-bn}$ for $x \neq y \in \mathcal{M}^{(i)}$. By the triangle inequality, it is easy to see that $\mathcal{M}^{(i)} \subset B(\xi, (c_1 + \delta_0) e^{-bn})$ holds for all $i \in \Sigma$. Using Lemma 5.2(iii), we have

$$\#X_n(\xi) = \sum_{i \in \Sigma} \#\mathcal{M}^{(i)} \leq \ell_2 N_{c_2 e^{-bn}}^s(B(\xi, (c_1 + \delta_0) e^{-bn}) \cap K) \leq \ell_2 \widehat{N}^s((c_1 + \delta_0)/c_2) < \infty.$$

This completes the proof by letting $\ell_1 = \ell_2 \widehat{N}^s((c_1 + \delta_0)/c_2)$. \square

The following theorem provides sufficient conditions for AI_∞ -graphs to be hyperbolic, and completes the study of AI_∞ -graphs in the last section.

Theorem 5.4. *Let Φ be an index map with attractor K , and is of exponential type-(b). If either*

(i) *condition (S_b) is satisfied; or*

(ii) *the attractor (K, ρ) is doubling, and condition (B_b) is satisfied,*

then the AI_∞ -graph is hyperbolic, and hence an admissible graph.

Proof. We prove the hyperbolicity of (X, E^∞) by using a similar argument as in [LW3, Theorem 1.2]. Suppose it is not hyperbolic. Then by Theorem 2.11, for any integer $m > 0$, there exists a horizontal geodesic $[x_0, x_1, \dots, x_{3m}]$ with length $3m$. Clearly $|x_0| =: n > m$, and by the expansive property (2.1), there is a horizontal path $[y_0, y_1, \dots, y_\ell]$ in X_{n-m} (see Figure 7) such that

- (a) $y_0 \in \mathcal{J}_{-m}(x_0)$, $y_\ell \in \mathcal{J}_{-m}(x_{3m})$, and $y_i \in \bigcup_{j=0}^{3m} \mathcal{J}_{-m}(x_j)$ for all $0 < i < \ell$;
- (b) $(y_i, y_j) \in E_h$ if and only if $|i - j| = 1$.

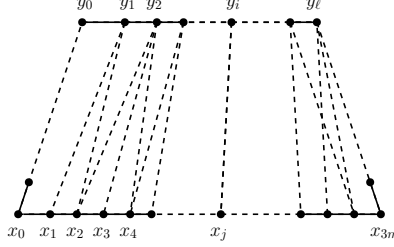


Figure 7: Two horizontal paths $[x_j]_j$ and $[y_i]_i$.

As $3m = d(x_0, x_{3m}) \leq d(x_0, y_0) + d(y_0, y_\ell) + d(y_\ell, x_{3m}) \leq m + \ell + m$, we have $\ell \geq m$. Let $F := \bigcup_{i=0}^{\ell} \Phi(y_i)$. Note that by (a), for any $\eta \in F$, there exists $k \in \{0, 1, \dots, 3m\}$ such that both $\Phi(x_k)$ and η are contained in some $\Phi(y_i)$. Thus for any projection $\iota : X \rightarrow K$,

$$\begin{aligned} \rho(\iota(x_0), \eta) &\leq \sum_{j=0}^{k-1} \rho(\iota(x_j), \iota(x_{j+1})) + |\Phi(y_i)| \\ &\leq 2k\delta_0 e^{-bn} + \delta_0 e^{b(m-n)} \leq \delta_0(6me^{-bm} + 1)e^{b(m-n)}, \end{aligned}$$

where $\delta_0 := \sup_{z \in X} e^{b|z|} |\Phi(z)| < \infty$. Take m large enough such that $6me^{-bm} < 1$. Then

$$F = \bigcup_{i=0}^{\ell} \Phi(y_i) \subset B(\iota(x_0), 2\delta_0 e^{b(m-n)}) =: B.$$

- (i) If condition (S_b) is satisfied, by noting that $|F| \leq 4\delta_0 e^{b(m-n)} =: ce^{b(m-n)}$, we have

$$\ell + 1 \leq \#\{y \in X_{n-m} : \Phi(y) \subset F\} \leq \#\{y \in X_{n-m} : K_y \cap F \neq \emptyset\} \leq \bar{\ell}(c).$$

This is impossible since $\ell \geq m$ can be arbitrarily large.

- (ii) If (K, ρ) is doubling and condition (B_b) holds, then we choose the above ι to satisfy (5.2). By (b), the ball B contains at least $\lfloor \ell/2 \rfloor + 1 > m/2$ mutually disjoint balls $B(\iota(y_{2i}), c_0 e^{b(m-n)})$, $0 \leq i \leq \lfloor \ell/2 \rfloor$. On the other hand by considering the packing number and the constant of doubling (see Lemma 5.2(iii)), we have

$$N_{e^{b(m-n)}}^p(B \cap K) \leq \widehat{N}^p(2\delta_0/c_0) < \infty.$$

As m can be arbitrarily large, this is impossible. Hence (X, E^∞) is hyperbolic in either case. By Proposition 4.6, the AI_∞ -graph is an admissible graph. \square

Recall that for an expansive hyperbolic graph with bounded degree, the hyperbolic boundary possesses the doubling property (Theorem 3.6). With the separation property, we have a stronger result for AI_b -graphs as well as a sufficient condition for AI_∞ -graphs.

Theorem 5.5. *Let Φ be an index map of exponential type-(b). Then the AI_b -graph has bounded degree if and only if condition (S_b) is satisfied. As a consequence, (S_b) condition is sufficient for the AI_∞ -graph to have bounded degree.*

Proof. It is clear that the last statement follows from the first one, since $E^\infty \subset E^{(b)}$.

To show the necessity of the first part, by Theorem 4.5, the AI_b -graph is hyperbolic, and the hyperbolic boundary $(\partial X, \theta_a)$ is Hölder equivalent to the attractor (K, ρ) . From Theorem 3.6 we know that $(\partial X, \theta_a)$ has the doubling property, which is preserved by the Hölder equivalence κ , and thus (K, ρ) is also doubling.

Next we will show that the index map Φ satisfies condition (S_b'') . We fix an arbitrary projection ι from X to K . For $x, y \in X_n$ with $\rho(\iota(x), \iota(y)) < \gamma e^{-bn}$ (where $\gamma > 0$ is the constant in (4.7), the definition of $E_h^{(b)}$), we have $x \sim_h y$ since $\text{dist}_\rho(\Phi(x), \Phi(y)) \leq \rho(\iota(x), \iota(y)) < \gamma e^{-bn}$. Hence for all $y \in X_n$,

$$\#\{x \in X_n : \rho(\iota(x), \iota(y)) < \gamma e^{-bn}\} \leq \#\{x \in X_n : x \sim_h y\} \leq \text{deg}(y).$$

As $\ell_2 := \sup_{y \in X} \text{deg}(y) < \infty$, (5.6) holds for $c_2 = \gamma$. Making use of Proposition 5.3, we see that condition (S_b) is satisfied.

For the sufficiency, we calculate the degree of a fixed vertex $y \in X_n$. Set $\delta_0 := \sup_{z \in X} e^{b|z|} |\Phi(z)|$. For $x \in \mathcal{J}_1(y)$, we see that $K_x \subset \Phi(x) \subset \Phi(y)$. As $|\Phi(y)| \leq \delta_0 e^{-bn} =: c e^{-b(n+1)}$, condition (S_b) implies that

$$\#\mathcal{J}_1(y) \leq \#\{x \in X_{n+1} : K_x \cap \Phi(y) \neq \emptyset\} \leq \bar{\ell}(c).$$

For $x \in \mathcal{J}_{-1}(y)$, it follows that

$$\Phi(y) \subset \Phi(x) \subset F := \{\xi \in M : \rho(\xi, \Phi(y)) \leq \delta_0 e^{b(1-n)}\}.$$

Using the triangle inequality, we get $|F| < 3\delta_0 e^{b(1-n)} =: c' e^{b(1-n)}$, and thus by condition (S_b) ,

$$\#\mathcal{J}_{-1}(y) \leq \#\{x \in X_{n-1} : K_x \cap F \neq \emptyset\} \leq \bar{\ell}(c').$$

For $x \in X_n$ with $(x, y) \in E_h$, using the triangle inequality we have

$$K_x \subset \Phi(x) \subset G := \{\xi \in M : \rho(\xi, \Phi(y)) \leq (\gamma + \delta_0) e^{-bn}\}.$$

It follows in a similar way that $|G| < 3(\gamma + \delta_0) e^{-bn} =: c'' e^{-bn}$, and hence

$$\#\{x \in X : (x, y) \in E_h\} \leq \#\{x \in X_n : K_x \cap G \neq \emptyset\} \leq \bar{\ell}(c'').$$

As $\text{deg}(y) = \#\mathcal{J}_{-1}(y) + \#\mathcal{J}_1(y) + \#\{x \in X : (x, y) \in E_h\}$, we conclude from the above estimates that $\text{deg}(y)$ is uniformly bounded by $\bar{\ell}(c) + \bar{\ell}(c') + \bar{\ell}(c'')$ for all $y \in X$, so that the AI_b -graph is of bounded degree. We complete the proof. \square

As a consequence of Theorems 3.6, 5.5 and Proposition 5.3, we have

Corollary 5.6. *Let Φ be an index map with attractor K , and is of exponential type-(b). Then the AI_b -graph has bounded degree if and only if the attractor (K, ρ) is doubling, and condition (S_b'') (or (S_b')) in Proposition 5.3 is satisfied.*

6 Examples and more on IFSs

We first give an example that the AI_∞ -graph is not hyperbolic.

Example 6.1. (An anisotropic binary partition of unit square) We define a binary subdivision scheme to partition $[0, 1]^2$ iteratively into subrectangles. By a Type-I subdivision, we mean dividing a rectangle J horizontally into two equal subrectangles; Type-II means dividing J vertically instead. Let $\mathcal{L} = \{\frac{1}{2}\ell(\ell + 1) : \text{integer } \ell > 0\} = \{1, 3, 6, \dots\}$. Now let $J_\vartheta = [0, 1]^2$. Suppose we have defined J_x with $x = x_1 \cdots x_n$, $x_i = 0, 1$. If $n + 1 \in \mathcal{L}$, we use Type-I subdivision on J_x to obtain J_{x0} and J_{x1} ; otherwise, we use Type-II for the subdivision (see Figure 8).

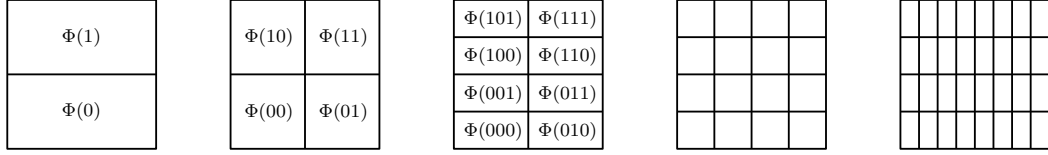


Figure 8: The binary partition $\{\Phi(x)\}_{x \in X_n}$, $n = 1, 2, 3, 4, 5$; $1, 3 \in \mathcal{L}$.

Let (X, E_v) be the corresponding *binary tree*, on which we define the index map $\Phi(x) = J_x$ and consider the AI_∞ -graph (X, E) . Fix an integer $\ell > 0$. Let $n := \frac{\ell(\ell+1)}{2}$, $x = 0^n$ and $y = 1010^2 1 \cdots 0^{\ell-1} 1$. Then $\Phi(x) = [0, 2^{-n+\ell}] \times [0, 2^{-\ell}]$ and $\Phi(y) = [0, 2^{-n+\ell}] \times [1 - 2^{-\ell}, 1]$, which are the rectangles in the lower-left and the upper-left corners of $[0, 1]^2$ respectively. It is clear that $d_h(x, y) = 2^\ell - 1$. Taking $u = 0^{n+\ell} \in \mathcal{J}_\ell(x)$ and $v = 1010^2 1 \cdots 0^{\ell-1} 10^\ell \in \mathcal{J}_\ell(y)$, we can also check that $d_h(u, v) = d_h(x, y) = 2^\ell - 1$. This shows that (X, E) is not (m, k) -departing whenever $m \leq \ell$ and $k \leq 2^\ell - 2$. As ℓ can be arbitrary, (X, E) is not hyperbolic by Theorem 2.11. \square

Our next example gives an IFS that is homogeneous (the contraction ratios are equal) and satisfies the OSC. The associated AI_∞ -graph is hyperbolic (by Theorem 5.4(i)), but the hyperbolic boundary is not Hölder equivalent to the attractor. This shows that (unlike the AI_b -graphs) the one-sided Hölder inequality in Corollary 4.7 cannot be improved.

Example 6.2. In $M = \mathbb{R}^2$, let $p_1 = (0, 0)$, $p_2 = (1, 0)$, $p_3 = (3, 0)$, $p_4 = (\eta, 2)$, $p_5 = (0, 3)$ and $p_6 = (3, 3)$, where

$$\eta := \sum_{\ell=0}^{\infty} (4^{-n_\ell} + 4^{-n_\ell-1}) \quad \text{with} \quad n_\ell = 1 + \frac{\ell(\ell+7)}{2}, \quad \ell = 0, 1, \dots$$

Let $S_j(x) = \frac{1}{4}(x + p_j)$, $j \in \Sigma = \{1, 2, \dots, 6\}$, and let K be the self-similar set of the IFS $\{S_j\}_{j=1}^6$. We set $X = \Sigma^*$ (the symbolic space) and $\Phi(x) = S_x([0, 1]^2)$, $x \in X$. Clearly $\bigcup_{j=1}^6 \Phi(j) \subset [0, 1]^2$, and every $\Phi(x)$ is a square with side length $4^{-|x|}$ (see Figure 9). Hence Φ is an index map of exponential type- (b) with $b = \ln 4$, and it is easy to see that the associated saturated map $\tilde{\Phi}(x) = S_x(K)$ for any $x \in X$ (see (4.1)).

Fix an integer $\ell > 0$. Let $x = 5221^2 221^3 \cdots 221^{\ell+1}$ and $y = 45^{n_\ell-1}$. Clearly $|x| = |y| = n_\ell$, and $\Phi(y)$ has the same upper-left corner $\zeta = (\eta/4, 3/4)$ as $\Phi(4)$. Moreover, we can calculate that the lower-left corner of $\Phi(x)$ is $(\xi/4, 3/4)$, where

$$\xi := \sum_{k=0}^{\ell-1} (4^{-n_k} + 4^{-n_k-1}) = \eta - \sum_{k=\ell}^{\infty} (4^{-n_k} + 4^{-n_k-1}).$$

Let $u = x23$ and $v = y56$. Then $|u| = |v| = n_\ell + 2$, the lower-right corner of $\Phi(u)$ is $(\xi/4 + 2 \cdot 4^{-n_\ell-1}, 3/4)$, and the upper-left corner of $\Phi(v)$ is $(\eta/4 + 4^{-n_\ell-1} - 4^{-n_\ell-2}, 3/4)$. It follows that

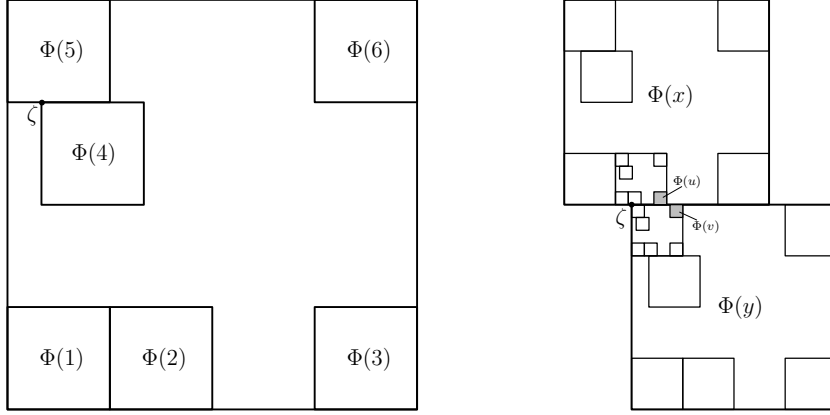


Figure 9: The squares in Example 6.2.

$S_u(K) \cap S_v(K) = \Phi(u) \cap \Phi(v) = \emptyset$. By checking the two nearest corners of $\Phi(u)$ and $\Phi(v)$, we see that $\text{dist}(\tilde{\Phi}(u), \tilde{\Phi}(v)) = \text{dist}(\Phi(u), \Phi(v))$, hence

$$\begin{aligned} \text{dist}(S_u(K), S_v(K)) &= \text{dist}(\Phi(u), \Phi(v)) \\ &= (\eta/4 + 4^{-n_\ell-1} - 4^{-n_\ell-2}) - (\xi/4 + 2 \cdot 4^{-n_\ell-1}) \\ &= \sum_{k=\ell+1}^{\infty} (4^{-n_k-1} + 4^{-n_k-2}) < 4^{-n_{\ell+1}} = 4^{-\ell-2} e^{-b|u|}. \end{aligned} \quad (6.1)$$

Consider the AI_∞ -graph (X, E) associated to $\tilde{\Phi}$, i.e.,

$$E_h = \{(x, y) \in X \times X \setminus \Delta : |x| = |y|, S_x(K) \cap S_y(K) \neq \emptyset\}.$$

As the IFS $\{S_j\}_{j=1}^6$ satisfies the OSC, so that condition (S_b) in Definition 5.1 is satisfied, by Theorem 5.4, (X, E) is hyperbolic. Since neither $\Phi(u)$ nor $\Phi(v)$ intersects other cells in $\{\Phi(z)\}_{z \in X_{n_\ell+2}}$, we have $d_h(u, v) = \infty$. As ℓ can be arbitrarily large, (6.1) implies that the condition (4.11) fails for any $k \geq 1$. By Theorem 4.8, the hyperbolic boundary $(\partial X, \theta_a)$ of (X, E) is not Hölder equivalent to K . \square

We now return to the setup in which $\{S_j\}_{j=1}^N$ is a contractive IFS on a complete metric space (M, ρ) (see Appendix). In the previous studies, the augmented tree was established according to the geometric sizes of K_x [LW1, LW3]. Within the framework of augmented index graphs (Section 4), we are able to extend the consideration to some weighted IFSs. In the rest of this section, we are given a vector of weights $\mathbf{s} \in (0, 1)^N$ on $\{S_j\}_{j=1}^N$ (instead of contraction ratio $\mathbf{r} = (r_j)_{j=1}^N$), and expect some new metric induced on K .

Let Σ^* be the symbolic space of a contractive IFS $\{S_j\}_{j=1}^N$. Let $\mathbf{s} = (s_1, s_2, \dots, s_N)$ be a vector of weights on Σ with $s_j \in (0, 1)$ (not necessarily probability weight). Write $s_* := \min_{j \in \Sigma} \{s_j\}$, and $s_x := s_{i_1} s_{i_2} \cdots s_{i_m}$ for $x = i_1 i_2 \cdots i_m \in \Sigma^*$ ($s_\emptyset = 1$ by convention). We consider a regrouping on Σ^* by setting $X_0 := \{\emptyset\}$, and for $n \geq 1$,

$$X_n = X_n(\mathbf{s}) := \{x = i_1 i_2 \cdots i_m \in \Sigma^* : s_x \leq s_*^n < s_{i_1} s_{i_2} \cdots s_{i_{m-1}}\}. \quad (6.2)$$

Let $X = X(\mathbf{s}) := \bigcup_{n=0}^{\infty} X_n$ denote the *modified coding space* with respect to \mathbf{s} . This X has a natural tree structure E_v that consists of edges between each $x = i_1 i_2 \cdots i_m \in X_n$ and $y =$

$i_1 i_2 \cdots i_m i_{m+1} \cdots i_k \in X_{n+1}$. Let $\Phi(x) = S_x(K)$, $x \in X$. Then Φ is a saturated index map on (X, E_v) over (K, ρ) .

Let $r^* = \max_{j \in \Sigma} \{r_j\}$ be the maximal contraction ratio of $\{S_j\}_{j \in \Sigma}$. Then it is clear that $\{\Phi(x)\}_{x \in X(\mathbf{s})}$ is always of exponential type- (b) with $b = |\log r^*|$. From (4.5) and Theorem 4.5, the associated AI_b -graph $(X(\mathbf{s}), E^{(b)})$ brings a metric $\tilde{\theta}_a$ on K that is Hölder equivalent to the original ρ , which is similar to the previous investigation; also, note that the index map Φ on $(X(\mathbf{s}), E^{(b)})$ rarely satisfies the separation conditions in Section 5, and hence we cannot expect that $(X(\mathbf{s}), E^{(b)})$ has bounded degree. The more interesting question is to investigate the hyperbolicity of AI_∞ -graph $(X(\mathbf{s}), E^{(\infty)})$ without assuming any separation property. The problem is difficult in general. However, for post critically finite (p.c.f.) sets, we have some rather complete conclusions, as well as a new connection to the harmonic structure and resistance networks in analysis on fractals.

We recall the notion of p.c.f. sets (without assuming self-similarity) [Kil]. Let $\Sigma^\infty := \{i_1 i_2 \cdots : i_k \in \Sigma, k \geq 1\}$ be the set of infinite words, and let $\varpi : \Sigma^\infty \rightarrow K$ be the natural surjection defined by

$$\{\varpi(i_1 i_2 \cdots)\} = \bigcap_{k=1}^{\infty} S_{i_1 i_2 \cdots i_k}(K).$$

The *shift map* $\sigma : \Sigma^\infty \rightarrow \Sigma^\infty$ is given by $\sigma(i_1 i_2 i_3 \cdots) = i_2 i_3 i_4 \cdots$. Define the *critical set* and the *post critical set* by

$$\mathcal{C} := \varpi^{-1}\left(\bigcup_{i,j \in \Sigma, i \neq j} (S_i(K) \cap S_j(K))\right) \quad \text{and} \quad \mathcal{P} := \bigcup_{n=1}^{\infty} \sigma^n(\mathcal{C}) \quad (6.3)$$

respectively. We call the IFS $\{S_j\}_{j=1}^N$ (or K) *post critically finite* (p.c.f.) if \mathcal{P} is a finite set. Also we define $V_0 = \varpi(\mathcal{P})$ as the *boundary* of K . A p.c.f. set K has the property that every two cells $S_i(K), S_j(K)$, $i, j \in \Sigma, i \neq j$ are disjoint or intersect at finitely many points. An important consequence is that [Kil, Proposition 1.3.5] for any distinct x, y in the same $X_n(\mathbf{s})$,

$$S_x(K) \cap S_y(K) = S_x(V_0) \cap S_y(V_0).$$

Theorem 6.3. *Let $\{S_j\}_{j=1}^N$ be a contractive IFS that satisfies the p.c.f. property. Then there exists an integer $m > 0$ such that for any $\mathbf{s} \in (0, 1)^N$, the AI_∞ -graph $(X(\mathbf{s}), E^{(\infty)})$ is an $(m, 1)$ -departing expansive graph, and hence $(X(\mathbf{s}), E^{(\infty)})$ is admissible. Moreover, it has bounded degree.*

Proof. It is clear that the AI_∞ -graph $(X(\mathbf{s}), E^{(\infty)})$ is expansive. We show that $(X(\mathbf{s}), E^{(\infty)})$ is $(m, 1)$ -departing for some positive integer m , i.e.,

$$d_h(x, y) = 2 \text{ with } |x| = |y| \geq m \quad \Rightarrow \quad d_h(x^{(-m)}, y^{(-m)}) \leq 1,$$

where $\mathcal{J}_{-m}(x) = \{x^{(-m)}\}$.

Set $\ell_* := \min_{\xi \neq \eta \in V_0} \{\rho(\xi, \eta)\}$, $\ell^* := \max_{\xi, \eta \in V_0} \{\rho(\xi, \eta)\}$ and $m := \lfloor \frac{\log(\ell_*/\ell^*)}{\log r^*} \rfloor + 1$, where $r^* = \max_i \{r_i\}$. Let $[x, z, y]$ be a horizontal geodesic. Suppose the statement is not true. Then $d_h(x^{(-m)}, y^{(-m)}) = 2$, so that $[x^{(-m)}, z^{(-m)}, y^{(-m)}]$ is also a horizontal geodesic. As $d_h(x, y) = 2$, we can choose distinct ξ_1, ξ_2 such that $\xi_1 \in S_x(K) \cap S_z(K)$ and $\xi_2 \in S_y(K) \cap S_z(K)$. Observe that

$$S_x(V_0) \cap S_z(V_0) = S_x(K) \cap S_z(K) \subset S_{x^{(-m)}}(K) \cap S_{z^{(-m)}}(K) = S_{x^{(-m)}}(V_0) \cap S_{z^{(-m)}}(V_0).$$

Thus $\xi_1 \in S_z(V_0) \cap S_{z^{(-m)}}(V_0)$, and so does ξ_2 . Let $w \in \Sigma^*$ satisfy $z = z^{(-m)}w$. It follows that

$$\xi_i \in S_z(V_0) \cap S_{z^{(-m)}}(V_0) = S_{z^{(-m)}}(V_0 \cap S_w(V_0)), \quad i = 1, 2.$$

Therefore there exists $\eta_i \in V_0$ such that $S_w(\eta_i) \in V_0$ (and $S_z(\eta_i) = \xi_i$) for $i = 1, 2$. The fact that $w \in \bigcup_{k \geq m} \Sigma^k$ implies

$$\ell_* \leq \rho(S_w(\eta_1), S_w(\eta_2)) \leq (r^*)^m \rho(\eta_1, \eta_2) \leq (r^*)^m \ell^*.$$

Thus $m \leq \frac{\log(\ell_*/\ell^*)}{\log r^*}$, contradicting the choice of m above. Hence $(X(\mathbf{s}), E^{(\infty)})$ is $(m, 1)$ -departing, and is an admissible augmented tree (Theorem 2.11 and Proposition 4.6).

Finally, we prove the bounded degree property. Set $s^* := \max_{j \in \Sigma} \{s_j\}$. For $x \in X_n(\mathbf{s})$ and $xv \in X_{n+1}(\mathbf{s})$, using (6.2) we have

$$s_v = s_{xv} \cdot s_x^{-1} \geq s_*^{n+2} \cdot s_*^{-n} = s_*^2.$$

Therefore $v \in \bigcup_{k=1}^{m'} \Sigma^k$ where $m' := \lfloor \frac{2 \log s_*}{\log s^*} \rfloor$, and hence $\#\mathcal{J}_1(x) \leq \#\left(\bigcup_{k=1}^{m'} \Sigma^k\right) < N^{m'+1}$. Using (6.3), it follows that $\sup_{\xi \in K} \{\#\varpi^{-1}(\xi)\} \leq \#\mathcal{C} < \infty$. For $x, y \in X(\mathbf{s})$, note that $S_x(K) \cap S_y(K) \subset S_x(V_0)$. Therefore

$$\#\{y \in X : (x, y) \in E_h\} \leq \#\varpi^{-1}(S_x(V_0)) \leq \#V_0 \cdot \sup_{\xi \in K} \{\#\varpi^{-1}(\xi)\} \leq \#V_0 \cdot \#\mathcal{C}.$$

As $\deg(x) \leq 1 + \#\mathcal{J}_1(x) + \#\{y \in X : (x, y) \in E_h\}$ for all $x \in X$, the graph $(X(\mathbf{s}), E^{(\infty)})$ has bounded degree. \square

Consequently, the map $\kappa : \partial X(\mathbf{s}) \rightarrow K$ is a bijection as in Definition 4.2 (actually κ is a homeomorphism by Proposition 4.3). Hence the Gromov metric θ_a on ∂X induces a new metric $\tilde{\theta}_a$ on K by (4.5). Next we show that the index map Φ over $(K, \tilde{\theta}_a)$ satisfies the condition (B_a) in Definition 5.1.

Proposition 6.4. *Let $\{S_j\}_{j=1}^N$ be a contractive p.c.f. IFS. For $\mathbf{s} \in (0, 1)^N$, let $\tilde{\theta}_a$ be the metric on K induced by $(X(\mathbf{s}), E^{(\infty)})$. Then there exists $c_0 > 0$ such that for any $x \in X(\mathbf{s})$, $S_x(K)$ contains a ball of radius $c_0 e^{-a|x|}$ in $(K, \tilde{\theta}_a)$ (i.e., condition (B_a)).*

Proof. Note that $(X(\mathbf{s}), E^{(\infty)})$ is $(m, 1)$ -departing (Theorem 6.3). By Theorem 4.8, there exists $\gamma > 0$ such that for $x, y \in X(\mathbf{s})$,

$$|x| = |y| \text{ and } S_x(K) \cap S_y(K) = \emptyset \Rightarrow \text{dist}_{\tilde{\theta}_a}(S_x(K), S_y(K)) > \gamma e^{-a|x|}. \quad (6.4)$$

Let $\ell := \lfloor \frac{\log(\#\mathcal{C})}{\log N} \rfloor + 1$, where \mathcal{C} is the critical set. Then for $x \in X$, there is $y \in \mathcal{J}_\ell(x)$ such that $S_y(K) \subset S_x(K \setminus V_0)$. Choose $\iota(x) \in S_y(K)$ arbitrarily. It follows from (6.4) that

$$\inf_{\eta \in K \setminus S_x(K)} \{\tilde{\theta}_a(\iota(x), \eta)\} \geq \text{dist}_{\tilde{\theta}_a}(S_y(K), \bigcup_{z \in X_{|x|} \setminus \{x\}} S_z(K)) > \gamma \cdot e^{-a(|x|+\ell)}.$$

Hence $B_{\tilde{\theta}_a}(\iota(x), c_0 e^{-a|x|}) \subset S_x(K)$ with $c_0 = \gamma e^{-a\ell}$. This completes the proof. \square

Let $\alpha = \alpha(\mathbf{s})$ be the positive number such that $\sum_{j \in \Sigma} s_j^\alpha = 1$, and let $\mu_{\mathbf{s}}$ be the *self-similar measure* with respect to the vector of probability weights $(s_1^\alpha, s_2^\alpha, \dots, s_N^\alpha)$, i.e., the unique regular Borel probability measure on K that satisfies

$$\mu_{\mathbf{s}}(\cdot) = \sum_{j \in \Sigma} s_j^\alpha \cdot \mu_{\mathbf{s}}(S_j^{-1}(\cdot)). \quad (6.5)$$

In particular if the IFS is p.c.f., then $\mu_{\mathbf{s}}(S_x(K)) = s_x^\alpha$ for all $x \in X$.

Proposition 6.5. *Let $\{S_j\}_{j=1}^N$ be a contractive IFS that satisfies the p.c.f. property. For $\mathbf{s} \in (0, 1)^N$, the self-similar measure $\mu_{\mathbf{s}}$ is Ahlfors-regular with exponent $(-\alpha \log s_*/a)$ on $(K, \tilde{\theta}_a)$, i.e.,*

$$\mu_{\mathbf{s}}(B_{\tilde{\theta}_a}(\xi, r)) \asymp r^{-\alpha \log s_*/a}, \quad \forall \xi \in K, r \in (0, 1). \quad (6.6)$$

Proof. Consider the AI_{∞} -graph $(X(\mathbf{s}), E^{\infty})$. For $x \in X(\mathbf{s})$, set

$$\Phi^1(x) := \bigcup \{\Phi(y) : d_h(x, y) \leq 1\} = \kappa(\mathcal{J}_{\tilde{\theta}}^1(x)),$$

where the index map $\Phi(x) = S_x(K)$. By Proposition 3.3 and Theorem 6.3, there is $C_0 \geq 1$ such that

$$B_{\tilde{\theta}_a}(\xi, C_0^{-1}e^{-a|x|}) \subset \Phi^1(x) \subset B_{\tilde{\theta}_a}(\xi, C_0e^{-a|x|}), \quad \forall x \in X(\mathbf{s}), \xi \in \Phi(x).$$

Using $s_*^{\alpha(|x|+1)} < s_x^{\alpha} = \mu_{\mathbf{s}}(\Phi(x)) \leq \mu_{\mathbf{s}}(\Phi^1(x)) \leq ts_*^{\alpha|x|}$ where $t := \sup_{x \in X(\mathbf{s})} \deg(x) (< \infty$ by Theorem 6.3), we have

$$\begin{cases} \mu_{\mathbf{s}}(B_{\tilde{\theta}_a}(\xi, C_0^{-1}e^{-an})) \leq ts_*^{\alpha n}, \\ \mu_{\mathbf{s}}(B_{\tilde{\theta}_a}(\xi, C_0e^{-an})) \geq s_*^{\alpha(n+1)}, \end{cases} \quad \forall \xi \in K, n \geq 0.$$

This proves (6.6). \square

In the following, we show that the metric measure space $(K, \tilde{\theta}_a, \mu_{\mathbf{s}})$ plays a special role in connection with the study of local regular Dirichlet forms (which give a Laplacian) on K with a regular harmonic structure. This will also extend a consideration by Hu and Wang [HW], in which they studied the relation between the resistance metric R and the Euclidean metric for IFS on \mathbb{R}^d .

We first recall some notations. A (*discrete*) *Laplacian* $H = [H_{pq}]_{p, q \in V_0}$ on V_0 is a non-positive definite matrix on V_0 that satisfies $\sum_{q \in V_0} H_{pq} = 0$ for all $p \in V_0$, and $H_{pq} \geq 0$ for all distinct $p, q \in V_0$. For a weight vector $\mathbf{s} \in (0, 1)^N$, let $V_n = V_n(\mathbf{s}) := \bigcup_{x \in X_n(\mathbf{s})} S_x(V_0)$ for $n \geq 1$, and $V_* = V_*(\mathbf{s}) := \bigcup_{n=0}^{\infty} V_n(\mathbf{s})$. Denote the collection of real-valued functions on V_n (or V_*) by $\ell(V_n)$ (or $\ell(V_*)$ respectively). For a Laplacian H on V_0 , we have

$$(u, Hv) = \sum_{p \in V_0} u(p) \left(\sum_{q \in V_0} H_{pq} v(q) \right), \quad u, v \in \ell(V_0).$$

We define the energy form \mathcal{E}_n on V_n by

$$\mathcal{E}_0[u] = -(u, Hu), \quad \mathcal{E}_n[u] = \sum_{x \in X_n(\mathbf{s})} s_x^{-1} \mathcal{E}_0[u \circ S_x], \quad \text{for } u \in \ell(V_n), n \geq 1$$

(i.e., $\mathcal{E}_n[u] := \frac{1}{2} \sum_{x \in X_n(\mathbf{s})} s_x^{-1} \sum_{p, q \in V_0} H_{pq} |u(S_x(p)) - u(S_x(q))|^2$). We say that the pair (H, \mathbf{s}) is a *regular harmonic structure* [Kil] of K if

$$\min\{\mathcal{E}_{n+1}[v] : v \in \ell(V_{n+1}), v = u \text{ on } V_n\} = \mathcal{E}_n[u], \quad \forall n \geq 0, u \in \ell(V_n). \quad (6.7)$$

This implies that for $u \in \ell(V_*)$, $\{\mathcal{E}_n[u]\}_{n=0}^{\infty}$ is an increasing sequence (here in each $\mathcal{E}_n[u]$, u is restricted on V_n), and

$$\mathcal{E}[u] := \lim_{n \rightarrow \infty} \mathcal{E}_n[u] = \sup_{n \geq 1} \mathcal{E}_n[u], \quad u \in \ell(V_*). \quad (6.8)$$

The $u \in \ell(V_*)$ can be extended continuously to a function on K if $\mathcal{E}[u] < \infty$. This defines the local regular Dirichlet form $(\mathcal{E}, \mathcal{D})$ with $\mathcal{D} := \{u \in C(K) : \mathcal{E}[u] < \infty\}$, where $C(K)$ denotes the space of continuous functions on K . This energy form is *self-similar* in the sense that for any $n \geq 1$,

$$\mathcal{E}[u] = \sum_{x \in X_n(\mathbf{s})} s_x^{-1} \mathcal{E}[u \circ S_x], \quad \forall u \in \mathcal{D}. \quad (6.9)$$

Define the *effective resistance* between two nonempty compact subsets $F, G \subset K$ by

$$R(F, G) = (\inf\{\mathcal{E}[u] : u = 1 \text{ on } F, \text{ and } u = 0 \text{ on } G\})^{-1}$$

if F and G are disjoint, and $= 0$ otherwise.

Lemma 6.6. *Let K be a connected p.c.f. set that possesses a regular harmonic structure (H, \mathbf{s}) . Then there exists $\gamma' > 0$ such that for any $x, y \in X(\mathbf{s})$ with $|x| = |y|$, the inequality*

$$R(S_x(K), S_y(K)) \geq \gamma' \cdot s_*^{|x|}$$

holds whenever $S_x(K) \cap S_y(K) = \emptyset$.

Proof. Let $x, y \in X_n(\mathbf{s})$ with $S_x(K) \cap S_y(K) = \emptyset$. The p.c.f. property implies

$$R(S_x(K), S_y(K)) = R(S_x(V_0), S_y(V_0)) \geq R(S_x(V_0), V_n \setminus S_x(V_0)). \quad (6.10)$$

By (6.7) and (6.8), there exists a function u on K such that $u = 1$ on $S_x(V_0)$, $u = 0$ on $V_n \setminus S_x(V_0)$, and

$$\mathcal{E}_n[u] = \mathcal{E}[u] = R(S_x(V_0), V_n \setminus S_x(V_0))^{-1}. \quad (6.11)$$

Let $\mathfrak{H}(x) := \{z \in X_n(\mathbf{s}) \setminus \{x\} : S_z(K) \cap S_x(K) \neq \emptyset\}$. Using the bounded degree property of the AI_∞ -graph (Theorem 6.3), we have

$$\#\mathfrak{H}(x) = \#\{z \in X(\mathbf{s}) : (z, x) \in E_h\} < \sup_{x \in X(\mathbf{s})} \{\deg(x)\} =: t < \infty.$$

It follows that

$$\mathcal{E}_n[u] = \frac{1}{2} \sum_{z \in \mathfrak{H}(x)} s_z^{-1} \sum_{p, q \in V_0} H_{pq} |u(S_z(p)) - u(S_z(q))|^2 \leq \gamma'^{-1} s_*^{-n},$$

where $\gamma'^{-1} := t s_*^{-1} (\frac{\#V_0}{2})^2 \max\{H_{pq} : p \neq q \in V_0\}$. This together with (6.10) and (6.11) proves the lemma. \square

For $\xi, \eta \in K$, we write $R(\xi, \eta)$ instead of $R(\{\xi\}, \{\eta\})$ for short. It is well-known that $R(\cdot, \cdot)$ is a metric on K , called the *resistance metric*. By definition, for compact subsets $F, G \subset K$ we have

$$\text{dist}_R(F, G) := \inf\{R(\xi, \eta) : \xi \in F, \eta \in G\} \geq R(F, G). \quad (6.12)$$

As a consequence of Lemma 6.6, we see that the metric space (K, R) with the index map Φ has the property in (6.4).

Theorem 6.7. *Let K be a connected p.c.f. set that possesses a regular harmonic structure (H, \mathbf{s}) . Then the metric $\tilde{\theta}_a$ induced by $(X(\mathbf{s}), E^{(\infty)})$ satisfies*

$$\tilde{\theta}_a(\xi, \eta) \asymp R(\xi, \eta)^{-a/\log s_*}, \quad \forall \xi, \eta \in K. \quad (6.13)$$

Proof. Firstly we prove that the index map Φ is of exponential type-(b) under R , where $b := |\log s_*|$. For this, let $x \in X_n(\mathbf{s})$ and $\xi, \eta \in S_x(K)$. Then for $u \in \mathcal{D}$,

$$\begin{aligned} |u(\xi) - u(\eta)|^2 &= |u(S_x(\xi')) - u(S_x(\eta'))|^2 \quad (\text{here } S_x(\xi') = \xi \text{ and } S_x(\eta') = \eta) \\ &\leq R(\xi', \eta') \mathcal{E}[u \circ S_x] \leq |K|_R \mathcal{E}[u \circ S_x] \\ &\leq |K|_R \cdot s_x \mathcal{E}[u] \leq |K|_R \cdot s_*^n \mathcal{E}[u], \end{aligned}$$

where the diameter $|K|_R < \infty$ (cf. [Kil, Theorem 3.3.4]), and the third inequality follows from the the energy self-similar identity (6.9). Therefore, by using an equivalent expression of the effective resistance [Kil],

$$R(\xi, \eta) = \sup\left\{\frac{|u(\xi) - u(\eta)|^2}{\mathcal{E}[u]} : u \in \mathcal{D}, \mathcal{E}[u] \neq 0\right\} \leq |K|_R \cdot s_*^n = |K|_R \cdot e^{-bn}.$$

This proves that $|\Phi(x)|_R \leq |K|_R \cdot e^{-b|x|}$ for all $x \in X$.

By (6.12) and Lemma 6.6, the index map Φ satisfies (4.11) with $k = 1$ under R . It follows from Theorem 4.8 that the bijection $\kappa : (\partial X, \theta_a) \rightarrow (K, R)$ satisfies (4.10) with $\rho = R$. From the definition (4.5) of $\tilde{\theta}_a$, we see that $\tilde{\theta}_a(\cdot, \cdot) \asymp R(\cdot, \cdot)^{a/b}$ on K . \square

7 Appendix: IFSs and augmented trees

In this Appendix, for the convenience of the reader, we summarize some notations and known facts on iterated function systems, as well as some background of this paper.

Let (M, ρ) be a complete metric space, and let $\{S_j\}_{j=1}^N$ ($N \geq 2$) be a *contractive iterated function system* (IFS) on (M, ρ) [Kil], i.e., each $S_j : M \rightarrow M$ satisfies

$$r_j := \sup\left\{\frac{\rho(S_j(\xi), S_j(\eta))}{\rho(\xi, \eta)} : \xi, \eta \in M, \xi \neq \eta\right\} < 1. \quad (7.1)$$

Then there exists a unique nonempty compact set $K \subset M$ satisfying $K = \bigcup_{j=1}^N S_j(K)$, called the *attractor* of $\{S_j\}_{j=1}^N$; K is called a *self-similar set* if $M = \mathbb{R}^n$ and the S_j 's are similitudes, i.e., $|S_j(\xi) - S_j(\eta)| = r_j |\xi - \eta|$.

Let the *alphabet set* $\Sigma := \{1, 2, \dots, N\}$. Write $\Sigma^0 := \{\vartheta\}$ (ϑ is the *empty word*), and for $n \geq 1$, $\Sigma^n := \{x = i_1 \cdots i_k \cdots i_n : i_k \in \Sigma, \forall k\}$. Let $\Sigma^* := \bigcup_{n=0}^{\infty} \Sigma^n$ denote the *symbolic space* of finite words. This Σ^* has a natural N -ary tree structure with the root ϑ . For $x = i_1 \cdots i_n \in X$, write $r_x := r_{i_1} r_{i_2} \cdots r_{i_n}$, $S_x := S_{i_1} \circ S_{i_2} \circ \cdots \circ S_{i_n}$ and $K_x := S_x(K)$ for short.

For a self-similar set K of a *homogeneous* IFS $\{S_j\}_{j=1}^N$ (i.e., $r_j = r$ for all j), coding the iterations by the tree of symbolic space Σ^* is natural, as each K_x with $x \in \Sigma^n$ has a constant diameter $r^n |K|$. But in a non-homogeneous case, the diameters of the cells on each level are not comparable. A common way is to regroup the indices as follows: let $r_* = \min_{j \in \Sigma} \{r_j\}$, $X_0 = \{\vartheta\}$,

$$X_n = \{x = i_1 i_2 \cdots i_k \in \Sigma^* : r_x \leq r_*^n < r_{i_1} r_{i_2} \cdots r_{i_{k-1}}\}, \quad n \geq 1, \quad (7.2)$$

and $X = \bigcup_{n=0}^{\infty} X_n$. This X has a natural tree structure E_v , and the diameters of the cells in $\{K_x\}_{x \in X_n}$ are comparable with r_*^n .

A contractive IFS of similitudes is said to satisfy the *open set condition* (OSC) if there is a bounded nonempty open set U such that $S_j(U) \subset U$ for all $j \in \Sigma$, and $S_i(U) \cap S_j(U) = \emptyset$ for all $i \neq j$. The OSC is one of the most fundamental conditions in fractal geometry. For a self-similar set K , the OSC yields an explicit expression of the Hausdorff dimension s of K by $\sum_{j=1}^N r_j^s = 1$, and K supports the s -Hausdorff measure. Furthermore, the OSC is equivalent to the following property (cf. [Sc, Theorem 2.2]): for any $c > 0$, there is a constant $\ell = \ell(c)$ such that

$$\forall \eta \in K \text{ and integer } n > 0, B(\eta, cr_*^n) \cap K_x \neq \emptyset \text{ for at most } \ell \text{ of } x \in X_n. \quad (7.3)$$

We augment the tree (X, E_v) by a set of horizontal edges

$$E_h = \bigcup_{n=1}^{\infty} \{(x, y) \in X_n \times X_n \setminus \Delta : K_x \cap K_y \neq \emptyset\}, \quad (7.4)$$

and let $E = E_v \cup E_h$ (cf. [Ka, LW1]). It has been proved that

Theorem 7.1. [LW1] *Let $\{S_j\}_{j=1}^N$ be an IFS of contractive similitudes that satisfies the OSC. Then (X, E) is a hyperbolic graph, and for the hyperbolic boundary ∂X , the canonical identification $\kappa : \partial X \rightarrow K$ is a homeomorphism. Furthermore, κ is a Hölder equivalence if the IFS satisfies the condition (H), i.e., there exists a constant $c > 0$ such that*

$$\forall n > 0, x, y \in X_n, K_x \cap K_y = \emptyset \Rightarrow \text{dist}(K_x, K_y) \geq cr_*^n. \quad (7.5)$$

The identification of K and ∂X has been applied to study the Lipschitz equivalence of self-similar sets [LL, DLL]. If we enlarge the horizontal edge set to be

$$E'_h = \bigcup_{n=1}^{\infty} \{(x, y) \in X_n \times X_n \setminus \Delta : \text{dist}(K_x, K_y) \leq \gamma r_*^n\} \quad (7.6)$$

for some $\gamma > 0$ and let $E' = E_v \cup E'_h$ (cf. [LW3]), then we can improve Theorem 7.1 as

Theorem 7.2. [LW3] *For any IFS $\{S_j\}_{j=1}^N$ of similitudes, (X, E') is a hyperbolic graph, and the canonical identification $\iota : \partial X \rightarrow K$ is a Hölder equivalence. Moreover, (X, E') has bounded degree if and only if $\{S_j\}_{j=1}^N$ satisfies the OSC.*

In [KLW1], thanks to the bounded degree property, we can introduce a class of transient reversible random walks on (X, E') such that the Martin boundary, ∂X and K are homeomorphic, by which we obtain a jump kernel (i.e., Naïm kernel) to study the induced energy form on K .

For IFS $\{S_j\}_{j=1}^N$ with overlaps, it is possible that $S_x = S_y$ for some different $x, y \in X$, where X is defined in (7.2). For example, let $M = \mathbb{R}$, $S_1(x) = rx$ and $S_2(x) = rx + (1 - r)$, where $r = \frac{\sqrt{5}-1}{2}$ is the golden ratio. Then $S_{122} = S_{211}$ (see also Example 2.2). In this case, we can define an equivalence relation \simeq on X by $x \simeq y$ if and only if $S_x = S_y$. Then there is a natural vertical graph (X^\sim, E_v^\sim) as the quotient of (X, E_v) with respect to \simeq , which is not a tree unless the relation \simeq is trivial [LW3]. It was proved in [Wa] that the associated augmented tree of (7.4) in (X^\sim, E_v^\sim) is hyperbolic if the self-similar set K has positive Lebesgue measure, or the IFS $\{S_j\}_{j=1}^N$ satisfies the *weak separation condition* (WSC) (cf. [LN], [LW, Theorem 2.1]), i.e., the condition (7.3) with X_n replaced by the quotient X_n / \simeq . Some more variants were discussed in [Wa].

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