Hyperbolic graphs induced by iterations and applications in fractals

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Abstract

The paper is a survey of our recent and ongoing investigation on the class of Gromov hyperbolic graphs arising from iterated function systems (IFS) in the theory of fractals. The relations of the hyperbolic boundaries and the attractors of IFSs are discussed. The applications include Lipschitz equivalence of attractors as well as the discrete potential theory of random walks on such graphs.

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1 Introduction

With the intension to carry the probabilistic potential theory to the attractors of iterated function systems (IFS), Denker and Sato [10] first constructed a special type of (non-reversible) Markov chain $\{Z_n\}_{n=0}^\infty$ on the tree of symbolic space of the Sierpinski gasket

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(SG), and showed that the Martin boundary of $\{Z_n\}_{n=0}^\infty$ is homeomorphic to the SG. Motivated by this, Kaimanovich [30] introduced an"augmented tree" by adding "horizontal" edges to the coding tree according to the neighboring cells in each level of the SG, which was proved to be a hyperbolic graph in the sense of Gromov [23]. These ideas turned out to be very inspiring and far reaching in view of the very rich contents in the involved topics. In a series of papers, the initiatives were carried out and investigated in detail by the authors and their collaborators [9, 29, 36–39, 42, 45–52, 58]. Some related literatures include [2–6, 12, 15, 31, 33–35, 59].

In this paper, we attempt to give a brief account of our decade-long investigation as well as some ongoing works on the project of hyperbolic graphs on fractals. In Section 2, we recall some basic definitions for hyperbolic graphs, and characterize the hyperbolicity for the class of graphs in our consideration. In Section 3, we introduce the "augmented tree" for an iterated function system (IFS). We study its hyperbolicity using the criterion obtained in Section 2, and the Hölder equivalence of the hyperbolic boundary with the attractor of the IFS, in particular for the IFS of contractive similitudes. This is used to study the bi-Lipschitz equivalence of some totally disconnected self-similar sets in Section 4. In Section 5, we consider certain reversible random walk on the augment tree of the IFS of contractive similitudes. The Martin boundary, the hyperbolic boundary and the attractor are shown to be homeomorphic; the Martin kernel, the Naïm kernel and the induced energy form on the attractor are analyzed. In Section 6, we define a class of *expansive hyperbolic graphs* and a concept of index map. These unify various formulations of augmented trees, and include cases that are not governed by the IFS, such as refinement systems. Many of the properties studied in Section 3 are extended in this new setting. In Section 7, we provide some concluding remarks and future work of this study.

2 Hyperbolic graphs

We first define some basic notations for a *graph*. Let X be a countably infinite set. A (undirected) graph is a pair $(X, \mathcal{E})$, where $\mathcal{E}$ is a symmetric subset of $X \times X \setminus \{(x,x) : x \in X\}$. We call $x \in X$ a vertex and $(x,y) \in \mathcal{E}$ (also denoted by $x \sim y$) an edge. The *degree* of a vertex x is the total number of edges which connect to x and is denoted by $\deg(x)$. Throughout the paper, we assume that the graph is *locally finite*, i.e., $\deg(x) < \infty$ for all $x \in X$. For $x, y \in X$, a path from x to y is a finite sequence $\{x_0, x_1, \cdots, x_n\}$ such that $x = x_0,\ x_n = y$ and $(x_i, x_{i+1}) \in \mathcal{E}$, denoted by $p(x,y)$; we call $n = |p(x,y)|$ the *length* of the path ($\{x\}$ is a path with length 0 by convention). Moreover, if the above path has minimal length among all possible paths from x to y, then we say that the path is a *geodesic* and denote it by $\pi(x,y)$. We always assume that the graph is *connected*, that is, for any pair $x, y \in X$, there exists a path from x to y. Hence a graph induces an integer-valued metric $d(x,y)$ on X, which is the length of geodesic $\pi(x,y)$ from x to y.

Choose a reference vertex $\vartheta \in X$ and call it the root of the graph. We use $|x|$ to denote

$d(\vartheta, x)$. We can decompose $\mathcal{E}$ as: $\mathcal{E} = \mathcal{E}_h \cup \mathcal{E}_v$ where

$$\mathcal{E}_v = \{(x,y) \in \mathcal{E} : \ |x| - |y| = \pm 1\}, \quad \mathcal{E}_h = \{(x,y) \in \mathcal{E} : \ |x| = |y|\}.$$

An edge (x, y) in $\mathcal{E}_v$ is called a *vertical* edge ($x \sim_v y$), and an edge in $\mathcal{E}_h$ is called a *horizontal* edge; the notation $x \sim_h y$ means that $(x, y) \in \mathcal{E}_h$ or $x = y$; we also use $d_h(x, y)$ to denote the graph distance of the subgraph $(X, \mathcal{E}_h)$ (if there are no horizontal paths joining x, y, we let $d_h(x, y) = \infty$ by convention). We also decompose the vertex set X as $X = \bigcup_{n=0}^{\infty} X_n$, where the n-th level $X_n = \{x \in X : \ |x| = n\}$. By the local finiteness, it is easy to show (by induction) that X_n is a finite set for all $n \geq 0$. A (geodesic) *ray* $\pi(x_0, x_1, \cdots)$ is an infinite sequence with $x_0 = \vartheta$, $x_n \in X_n$ and $(x_n, x_{n+1}) \in \mathcal{E}_v$ for all $n \geq 0$. For $x, y \in X$ with $|y| - |x| =: m > 0$, we say y is an m-th *descendant* of x, or x is an m-th *predecessor* of y, if there exists a ray $\pi(\vartheta, \cdots, x, \cdots, y, \cdots)$ connecting them. Denote by $\mathcal{J}_m(x)$ and $\mathcal{J}_{-m}(x)$ the sets of m-th descendants and predecessors of x respectively, and let

$$\mathcal{J}_*(x) = \bigcup_{m \geq 1} \mathcal{J}_m(x), \quad \mathcal{J}_{-*}(x) = \bigcup_{m \geq 1} \mathcal{J}_{-m}(x).$$

Throughout, we assume that $\mathcal{J}_1(x) \neq \emptyset$ for all $x \in X$. We call $(X, \mathcal{E})$ a *tree* if $\mathcal{E} = \mathcal{E}_v$ and $\mathcal{J}_{-1}(x)$ is a singleton for all $x \in X \setminus \{\vartheta\}$.

Definition 2.1. [23] *Let $(X, \mathcal{E})$ be a graph with a root $\vartheta \in X$. Define the* Gromov product *of two vertices $x, y \in X$ by*

$$(x|y) = \frac{1}{2}(|x| + |y| - d(x, y)).$$

For $\delta \geq 0$, we say that $(X, \mathcal{E})$ is δ-hyperbolic *(with respect to ϑ) if*

$$(x|y) \geq \min\{(x|z), \, (z|y)\} - \delta, \qquad \forall \ x, y, z \in X. \tag{2.1}$$

$(X, \mathcal{E})$ is said to be hyperbolic *if it is δ-hyperbolic for some $\delta \geq 0$.*

The following justifies the definition of hyperbolicity [7, 60].

Proposition 2.2. *If X is δ-hyperbolic with respect to a particular $\vartheta \in X$, then it is 2δ-hyperbolic for any other fixed root $\vartheta' \in X$. Hence the hyperbolicity is independent of the choice of the root.*

It is easy to see that a tree is 0-hyperbolic and $(x|y)$ is the distance from ϑ to z, the confluence of x and y. For a δ-hyperbolic graph, the Gromov product $(x|y)$ is roughly the distance from ϑ to $\pi(x, y)$ in the following sense:

$$d(\vartheta, \pi(x, y)) - 2\delta - \frac{1}{2} \leq (x|y) \leq d(\vartheta, \pi(x, y)). \tag{2.2}$$

Indeed, the second inequality is trivial without requiring the hyperbolicity. For the first inequality, using (2.1) for $z \in \pi(x, y)$ we have

$$(x|y) \geq \min\{(x|z), (z|y)\} - \delta = \frac{1}{2}|z| + \frac{1}{2}\min\{|x| - d(x, z), |y| - d(y, z)\} - \delta.$$

Note that the two terms in $\min\{\cdots\}$ have the sum $|x| + |y| - d(x, y) = 2(x|y)$, therefore we can choose z such that $|x| - d(x, z) = |y| - d(y, z) = (x|y)$ if $(x|y)$ is an integer; and $|x| - d(x, z) = |y| - d(y, z) - 1 = (x|y) - \frac{1}{2}$ otherwise. It follows that

$$(x|y) \geq \frac{1}{2}|z| + \frac{1}{2}\big((x|y) - \frac{1}{2}\big) - \delta,$$

which implies $(x|y) \geq |z| - 2\delta - \frac{1}{2} \geq d(\vartheta, \pi(x, y)) - 2\delta - \frac{1}{2}$.

Note that in the inequality (2.2), we cannot omit $-\frac{1}{2}$: consider the simplest example, a triangle with $X = \{\vartheta, x, y\}$. Since $(\vartheta|x) = (\vartheta|y) = 0$ and $(x|y) = \frac{1}{2}$, (2.1) holds for $\delta = 0$, hence it is 0-hyperbolic. Note that $d(\vartheta, \pi(x, y)) = 1$, thus $d(\vartheta, \pi(x, y)) - 2\delta - \frac{1}{2} = (x|y)$. (In [60, Lemma 22.4], the $-\frac{1}{2}$ is missed in his estimate of (2.2).)

It is instructive to know that the notion of hyperbolicity is motivated by the "thin triangles" in the Poincaré disc model: a *geodesic triangle* in $(X, \mathcal{E})$ consists of three points $x, y, z \in X$ as vertices together with the three geodesic arcs $\pi(x, y), \pi(y, z), \pi(z, x)$ as sides; the triangle is called *δ-thin* if every point on any one of the sides is at distance at most δ to one of the other two sides. The following is the geometric characterization of hyperbolicity.

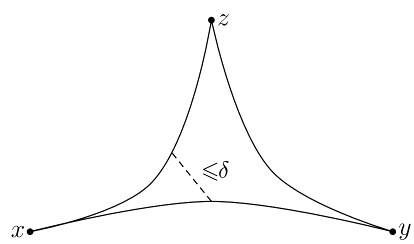

Figure 1: A δ-thin geodesic triangle.

Proposition 2.3. [7, 17, 60] *In a δ-hyperbolic graph $(X, \mathcal{E})$, every geodesic triangle is 8δ-thin. Conversely, if every geodesic triangle in $(X, \mathcal{E})$ is δ'-thin, then $(X, \mathcal{E})$ is $(3\delta' + \frac{1}{2})$-hyperbolic.*

For $a > 0$ small (say $e^{\delta a} < \sqrt{2}$), and $x, y \in X$, let

$$\theta_a(x, y) = \begin{cases} e^{-a(x|y)}, & x \neq y; \\ 0, & x = y. \end{cases} \tag{2.3}$$

Then by (2.1), $\theta_a(x,y) \le e^{\delta a}\max\{\theta_a(x,z),\theta_a(z,y)\}$ for all $x,y,z \in X$. It is known that $\theta_a(\cdot,\cdot)$ is not necessarily a metric (unless $\delta = 0$), but is always Lipschitz equivalent to a metric $\rho_a(\cdot,\cdot)$ on X (i.e., $C_1^{-1}\rho_a \le \theta_a \le C_1\rho_a$ holds for some constant $C_1 \ge 1$). Hence we can regard θ_a as a metric for convenience, and call it a *Gromov metric*. Note that if we choose another $b > 0$ for the Gromov metric, then $\theta_a = \theta_b^{a/b}$. By (2.3), it is clear that a sequence $\{x_n\}_{n=0}^\infty$ in X is a θ_a-Cauchy sequence if and only if $(x_m|x_n) \to \infty$ as $m,n \to \infty$.

Definition 2.4. *Denote by $\hat{X}$ the θ_a-completion of a hyperbolic graph X. We call $\partial X = \hat{X} \setminus X$ the* hyperbolic boundary *of X.*

The hyperbolic boundary ∂X is a compact set. It is useful to identify $\xi \in \partial X$ with the class of geodesic rays in X that converge to ξ. It is known that two rays $\pi(x_0,x_1,\cdots)$ and $\pi(y_0,y_1,\cdots)$ are equivalent as θ_a-Cauchy sequences if and only if

$$d(x_n,y_n) \le 4\delta \qquad \text{for all } n \ge 0.$$

(To prove the necessity, we let $n,m \ge 0$. Using (2.1), we have

$$\begin{aligned} n - \frac{1}{2}d(x_n,y_n) = (x_n|y_n) &\ge \min\{(x_n|x_{m+n}),(x_{n+m}|y_{n+m}),(y_{n+m}|y_n)\} - 2\delta \\ &= \min\{n,(x_{n+m}|y_{n+m})\} - 2\delta. \end{aligned}$$

Letting $m \to \infty$, we have $(x_{n+m}|y_{n+m}) \to \infty$ as the two rays are equivalent, hence $d(x_n,y_n) \le 4\delta$ follows.)

We can extend the Gromov product to $X \cup \partial X$ by letting

$$(x|\xi) = \inf\{\lim_{n\to\infty}(x|x_n)\}, \quad (\xi|\eta) = \inf\{\lim_{n\to\infty}(x_n|y_n)\},$$

where $x \in X$, $\xi,\eta \in \partial X$, and the infimum is taking over all geodesic rays $\pi(x_0,x_1,\cdots)$ and $\pi(y_0,y_1,\cdots)$ converging to ξ and η respectively; the Gromov metric on $X \cup \partial X$ is defined in the same way as in (2.3).

In the following we will consider a specific class of rooted graphs, and give a useful criterion for the hyperbolicity. For $x,y \in X$, we say that a geodesic $\pi(x,y) = [x_0,x_1,\cdots,x_n]$ is an *h-geodesic* if it consists of horizontal edges only, and a *v-geodesic* if $x_{i+1} \in \mathcal{J}_1(x_i)$ for all i, or $x_{i+1} \in \mathcal{J}_{-1}(x_i)$ for all i; it is called a *convex geodesic* if there exist $u,v \in \pi(x,y)$ such that

$$\pi(x,y) = \pi(x,u) \cup \pi(u,v) \cup \pi(v,y)$$

(also denoted by $\pi(x,u,v,y)$) in which $\pi(u,v)$ is an h-geodesic, and $\pi(x,u),\pi(v,y)$ are v-geodesics with $u \in \mathcal{J}_{-*}(x)$ and $v \in \mathcal{J}_{-*}(y)$ (one or two parts may vanish). Also between x,y, the convex geodesic may not be unique; by convention, we use the one such that $|u| = |v|$ is minimum (see Figure 2). Note that

$$(x|y) = \frac{1}{2}(|x| + |y| - d(x,y)) = |u| - \frac{1}{2}d_h(u,v) = (u|v). \tag{2.4}$$

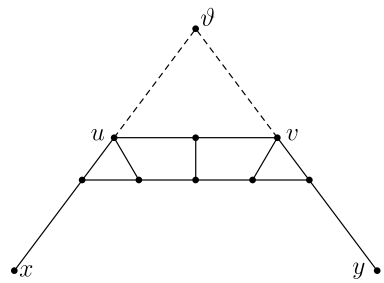

Figure 2: Two convex geodesics.

Lemma 2.5. *Let $(X,\mathcal{E})$ be a rooted graph such that $(X,\mathcal{E}_v)$ is a tree. Suppose the following property is satisfied: for any $x,y\in X$,*

(*) $\qquad x\sim_h y \quad \Rightarrow \quad x^{-1}\sim_h y^{-1}, \quad$ *(" $\sim_h$ " includes " $=$ "),*

where x^{-1} is the unique 1-st predecessor of x. Then any pair $x,y\in X$ can be joined by a convex geodesic.

Proof. The proof is quite simple: following [30], we can use the following moves repeatedly to change the geodesic without increasing the length: for $u,v\in\pi(x,y)$ with $(u,v)\in\mathcal{E}_h$,

$$[u,v,v^{-1}]\to[u,u^{-1},v^{-1}] \quad \text{and} \quad [u^{-1},u,v]\to[u^{-1},v^{-1},v].$$

Eventually we get a convex geodesic connecting x and y. □

The lemma allows us to have a good grasp of the geodesics in such graphs, and condition (*) is automatically satisfied for the augmented trees of the IFS in the next section. Our main theorem in this section is the following criterion for hyperbolicity, which relies on the convex geodesics.

Theorem 2.6. *Let $(X,\mathcal{E})$ be a rooted graph such that $(X,\mathcal{E}_v)$ is a tree and property (*) is satisfied. Then the following are equivalent.*

(i) $(X,\mathcal{E})$ is hyperbolic;

(ii) $\exists\ L<\infty$ such that the lengths of all h-geodesics are bounded by L.

This was proved in [45, Theorem 2.3] based on the thin triangle characterization (Proposition 2.3). In the following, we use an algebraic argument by the Gromov products in Definition 2.1.

Proof. *(i)* $\Rightarrow$ *(ii)* : Let $\pi(x_0,x_1,\cdots,x_n)$ be an h-geodesic in $(X,\mathcal{E})$. Then for $i\in\{0,1,\cdots,n\}$, it follows from (2.1) and (2.4) that

$$|x_0|-\frac{n}{2}=(x_0|x_n)\ \geq\ \min\{(x_0|x_i),(x_i|x_n)\}-\delta\ =\ |x_0|-\max\Big\{\frac{i}{2},\frac{n-i}{2}\Big\}-\delta.$$

Therefore $n\leq\max\{i,n-i\}+2\delta$. By choosing $i=\lfloor n/2\rfloor$ we have $n\leq 4\delta+1$.

(ii) $\Rightarrow$ *(i)* : We prove that $(X,\mathcal{E})$ is $\frac{L}{2}$-hyperbolic. For $x,y,z\in X$, consider the convex geodesics $\pi(x,u,v,z),\pi(z,u',v',y)$ connecting x,z and z,y respectively (See Figure 3).

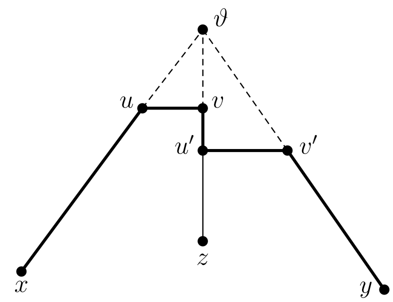

Figure 3: Illustration for the proof of (ii) ⇒ (i).

Without loss of generality, assume that $|u| \le |u'|$. Then $v = u'$ if $|u| = |u'|$, or $v \in \mathcal{J}_{-*}(u')$ if $|u| < |u'|$. We compute the length of the path from x to y passing through u, v, u', v' successively as

$$\begin{aligned} L' &:= d(x,u) + d(u,v) + d(v,u') + d(u',v') + d(v',y) \\ &\le (|x| - |u|) + d(u,v) + (|u'| - |v|) + L + (|y| - |u'|) \\ &= |x| + |y| - 2\big(|u| - \frac{1}{2}d(u,v)\big) + L \\ &= |x| + |y| - 2(x|z) + L \qquad \text{(by (2.4))}. \end{aligned}$$

On the other hand, using the definition of Gromov product, we have

$$L' \ge d(x,y) = |x| + |y| - 2(x|y).$$

Combining these two inequalities, we conclude that $(x|y) \ge (x|z) - \frac{L}{2}$. This implies that (2.1) is satisfied for $\delta = \frac{L}{2}$. □

3 IFS and augmented trees

We call a finite set of contractive maps $\{S_j\}_{j=1}^N$ $(N \ge 2)$ on $\mathbb{R}^d$ an *iterated function system* (IFS). It is well-known [13, 27] that there exists a unique nonempty compact set $K \subset \mathbb{R}^d$ such that $K = \bigcup_{j=1}^N S_j(K)$. We call the set K the *invariant set* (or *attractor*) of the IFS. In particular, for IFS of similitudes (also called self-similar IFS) (i.e., S_j are *similitudes*: $|S_j(x) - S_j(y)| = r_j|x - y|$), we call K a *self-similar set.*

Throughout the paper, we mainly consider the self-similar IFS $\{S_j\}_{j=1}^N$. Let $\Sigma = \{1, 2, \cdots, N\}$, and $\Sigma^* = \bigcup_{n=0}^\infty \Sigma^n$, where $\Sigma^0 = \{\vartheta\}$ contains the empty word ϑ only. Given $x = i_1 i_2 \cdots i_n \in \Sigma^n$, we denote $S_x = S_{i_1} \circ S_{i_2} \circ \cdots \circ S_{i_n}$, $K_x = S_x(K)$ (where S_ϑ is the identity map), and $r_x = r_{i_1} \cdots r_{i_n}$, the product of contractive ratios of the similitudes. Let $r_* = \min\{r_j : \ j \in \Sigma\}$.

The symbolic space Σ^* has a nature tree structure from each $x \in \Sigma^*$ to its descendants. But for $x \in \Sigma^n$, the diameter of K_x may vary greatly. On the other hand, *there are different ways to redefine the coding space for the attractor K of the IFS.* A commonly used tree structure which regroups the $x \in \Sigma^*$ in a more tractable manner is as follows: For $n \geq 1$, let

$$X_n := \{x = i_1 i_2 \cdots i_k \in \Sigma^* : \ r_x \leq r_*^n < r_{i_1 i_2 \cdots i_{k-1}}\}. \tag{3.1}$$

It can be proved that $X_n \cap X_{n+1} = \emptyset$ for each n. All maps in $\{S_x : \ x \in X_n\}$ have approximately equal contraction ratios ($\approx r_*^n$), and all cells in $\{K_x : \ x \in X_n\}$ have approximately equal diameters: $r_*^{n+1}|K| < |K_x| \leq r_*^n |K|$.

Let $X = \bigcup_{n=0}^{\infty} X_n$ (as usual $X_0 = \{\vartheta\}$). Then X has a natural tree structure, denoted by $\mathcal{E}_v$. It is direct to check that the total number of the 1-st level descendants of each x is less than $N^{1+\log r_*/\log r^*}$ (and $\geq N$) where $r^* = \max\{r_j : \ j \in \Sigma\}$. Note also that X can be a proper subset of Σ^*, but they define the same limit points on infinite paths. In order to describe the relation of the neighboring cells of $K_x, \ x \in X$, we augment the tree $(X, \mathcal{E}_v)$ by adding horizontal edges in the following two ways:

Definition 3.1. *On tree $(X, \mathcal{E}_v)$, we define a horizontal edge set by*

$$\mathcal{E}_h = \bigcup_{n=1}^{\infty} \{(x, y) \in X_n \times X_n : \quad x \neq y, \ K_x \cap K_y \neq \emptyset\};$$

for a fixed constant $\gamma > 0$, we define another horizontal edge set by

$$\widetilde{\mathcal{E}}_h = \bigcup_{n=1}^{\infty} \{(x, y) \in X_n \times X_n : \quad x \neq y, \ \mathrm{dist}(K_x, K_y) \leq \gamma r_*^n\}. \tag{3.2}$$

Denote $\mathcal{E} = \mathcal{E}_v \cup \mathcal{E}_h, \ \widetilde{\mathcal{E}} = \mathcal{E}_v \cup \widetilde{\mathcal{E}}_h$, and call $(X, \mathcal{E})$ and $(X, \widetilde{\mathcal{E}})$ augmented trees of $(X, \mathcal{E}_v)$.

Since the constant γ in $\widetilde{\mathcal{E}}_h$ has no real significance as long as it is positive, we omit it in the notation for brevity.

It is easy to check that both augmented trees have property (*) by observing that $\mathrm{dist}(K_x, K_y) \leq \mathrm{dist}(K_{xi}, K_{yj})$. The augmented tree $(X, \mathcal{E})$ is more naturally defined: it is our original consideration in [45], and has important applications (see Sections 4 and 5). But the hyperbolic property is not immediate: it needs additional conditions to control the fine structure of the attractor K, for which we will discuss in the sequel. The graph $(X, \widetilde{\mathcal{E}})$ is a relaxation of the intersection condition in $(X, \mathcal{E})$, enlarging the horizontal edge set $\mathcal{E}_h$. This allows us to have a more complete result on the hyperbolicity of the graph and the boundaries. The following is one of our main theorems.

Theorem 3.2. *For self-similar IFS, the augmented tree $(X, \widetilde{\mathcal{E}})$ is hyperbolic. Moreover, the hyperbolic boundary ∂X is Hölder equivalent to the self-similar set K, i.e., there exists a canonical bijection $\kappa : \ \partial X \to K$ such that*

$$C^{-1} \theta_a(\xi, \eta)^{\alpha} \leq |\kappa(\xi) - \kappa(\eta)| \leq C \theta_a(\xi, \eta)^{\alpha}, \quad \forall \ \xi, \eta \in \partial X, \tag{3.3}$$

where $C \geq 1$ is a constant and $\alpha = -\log r_*/a$.

The proof of the theorem is in [46, Theorems 1.2 and 1.3]. We will give another proof of this in Theorem 6.10 under a more general setup, which also includes the augmented tree $(\Sigma^*, \widetilde{\mathcal{E}})$ and some more general cases.

For the augmented tree $(X, \mathcal{E})$, we have

Proposition 3.3. *For self-similar IFS, if the augmented tree $(X, \mathcal{E})$ is hyperbolic, then there exists a bijective map $\kappa : \partial X \to K$ such that*

$$|\kappa(\xi) - \kappa(\eta)| \le C\theta_a(\xi, \eta)^\alpha, \quad \forall\ \xi, \eta \in \partial X,$$

where the constant $C > 0$ and $\alpha = -\log r_*/a$. In particular, the hyperbolic boundary ∂X is homeomorphic to the attractor K.

We do not know if $(X, \mathcal{E})$ is hyperbolic in general. However, we have simple conditions that guarantee sufficiently many interesting cases that $(X, \mathcal{E})$ is hyperbolic.

Proposition 3.4. *Suppose self-similar IFS has the following property:*
(H) $\exists\ C > 0\ \ni$ *for any $n > 0$ and $x, y \in X_n$, either*

$$K_x \cap K_y \ne \emptyset \quad or \quad \mathrm{dist}(K_x, K_y) \ge C r_*^n.$$

Then the augmented tree $(X, \mathcal{E})$ is hyperbolic. Moreover, the hyperbolic boundary ∂X is Hölder equivalent to the self-similar set K as in (3.3).

Proof. Let $0 < \gamma < C$, we define horizontal edge set $\tilde{\mathcal{E}}$ by (3.2). Then it is clear that the property (H) implies $\mathcal{E} = \tilde{\mathcal{E}}$, and the assertion follows from Theorem 3.2. □

Property (H) was introduced in [45], and also used in other applications [21, 26]. The property is satisfied for IFS of similitudes where the maps and the translations are defined by integral entries [45], so are the homogeneous p.c.f. IFS of similitudes [21]. There are also examples constructed so that (H) fails [45, 46, 58], including one that $(X, \mathcal{E})$ is hyperbolic (by Theorem 3.5 below), but ∂X is not Hölder equivalent to K [37].

One of the most fundamental conditions on the IFS of similitudes $\{S_i\}_{i=1}^N$ is the *open set condition* (OSC) [27], namely, there exists a non-empty bounded open set $O \subset \mathbb{R}^d$ such that $S_i(O) \subset O$ and the $S_j(O)$'s are disjoint. In such case, the Hausdorff dimension α of K is given by $\sum_{i=1}^N r_i^\alpha = 1$. It is also known that under the OSC, we can choose the open set O such that $O \cap K \ne \emptyset$ [56], which implies $0 < \mathcal{H}^\alpha(K)(< \infty)$. Moreover, the OSC is equivalent to [13, 46, 56]
(S) *for any $c > 0$, there exists $\ell = \ell(c) > 0$ such that any ball B of radius cr_*^n can intersect K_x with at most ℓ of $x \in X_n$.*

Theorem 3.5. *If the self-similar IFS satisfies either*

(i) the condition (S); or

(ii) the self-similar set K has a positive Lebesgue measure,

then the augmented tree $(X, \mathcal{E})$ is hyperbolic.

Proof. Assuming (i), to prove the hyperbolicity, it suffices to show that the lengths of the h-geodesics are bounded by some constant. Suppose otherwise, for any integer $m > 0$, there exists an h-geodesic $\pi(x_0, x_{3m}) = [x_0, x_1, \dots, x_{3m}]$ with $x_i \in X_n$. We consider the m-th predecessor $y_i = x_i^{[-m]}$. Let

$$p(y_0, y_{3m}) = [y_{i_0}, \dots, y_{i_k}] \tag{3.4}$$

with $y_{i_j} \in \{y_0, \dots, y_{3m}\}$ be the shortest horizontal path connecting y_0 and y_{3m}. By the geodesic property of $\pi(x_0, x_{3m})$, it is clear that

$$k = |p(y_0, y_{3m})| \geq |\pi(x_0, x_{3m})| - 2m = m. \tag{3.5}$$

Now choose $m \geq \ell$ such that $(3m+1)r_*^m \leq 1$, where $\ell = \ell(|K|)$ is as in condition (S). Let $D = \bigcup_{i=0}^{3m} K_{x_i}$. From $|K_{x_i}| \leq r_*^n |K|$ $(i = 0, 1, \cdots, 3m)$, it is direct to show that

$$|D| \leq (3m+1) r_*^n |K| \leq r_*^{n-m} |K|. \tag{3.6}$$

Note that $K_{x_i} \subset K_{y_i}$, we see that $K_{y_{i_j}} \cap D \neq \emptyset$ for each $j = 0, 1, \cdots, k$. It follows that

$$\#\{y \in X_{n-m} : \ K_y \cap D \neq \emptyset\} \geq k + 1 > m \geq \ell.$$

It contradicts condition (S) and the proof is completed.

For case (ii), we proceed as in (i). Let m be such that $|D| \leq r_*^{n-m}|K|$ (as in (3.6)), and let $D' = \bigcup_{i=0}^{k} K_{y_i}$. By using the horizontal path property of D, we see that D' is contained in the $(r_*^{n-m}|K|)$-neighborhood of D, hence,

$$|D'| \leq 2r_*^{n-m}|K| + |D| \leq C r_*^{n-m}.$$

Now consider (3.4). By the shortest path property, we see that for the even terms (or odd terms) of $\{y_{i_0}, \ \dots, \ y_{i_k}\}$, every pair are disjoint (otherwise, we can shorten the path). Observe that $\mathcal{L}(D') \leq C'|D'|^d$, we hence have

$$CC' r_*^{(n-m)d} \geq \mathcal{L}(D') \geq \sum_{i=0}^{\lfloor k/2 \rfloor} \mathcal{L}(K_{y_{2i}}) \geq (\lfloor k/2 \rfloor + 1)\mathcal{L}(K) r_*^{(n-m+1)d} > 0.$$

This is a contradiction, as $k \geq m$ (by (3.5)) and m can be arbitrary large. □

It follows that for the IFS of similitudes with OSC, the augmented graph $(X, \mathcal{E})$ is hyperbolic; the second condition applies to the well-known class of self-similar tilings (see e.g., [25, 40]).

We now consider the *bounded degree* (i.e., $\sup\{\deg(x) : \ x \in X\} < \infty$) property of the graphs. This property is important, especially when we consider random walks on graphs (see Section 5). In the following we will discuss this property in connection with the OSC. We need a simple lemma which can be proved by contrapositive argument.

Lemma 3.6. *Let $\{S_j\}_{j=1}^N$ be an IFS of contractive similitudes. Suppose that $(X, \widetilde{\mathcal{E}})$ is of bounded degree, then $S_x \neq S_y$ for any $x \neq y$ in X.*

Theorem 3.7. *Let $\{S_j\}_{j=1}^N$ be an IFS of contractive similitudes. Then $(X, \widetilde{\mathcal{E}})$ is of bounded degree if and only if the IFS satisfies the OSC.*

Proof. Assume that the IFS satisfies the OSC. Let $x \in X_n$ $(n \geq 1)$, then $|K_x| \leq r_*^n |K|$. Let B be a ball centered at some point in K_x with radius $(2\gamma + |K|)r_*^n$. Note that $K_y \cap B \neq \emptyset$ if $x \sim_h y$, or $x \sim_v y$ with $y \in X_{n+1}$. Also there is only one vertex $x^{[-1]} \in \mathcal{J}_{-1}(x)$. By making use of condition (S) from the OSC, we have the following estimate

$$\deg(x) \leq \#\{y : x \sim_h y\} + \#\{y : x \sim_v y\} \leq \ell(2\gamma + |K|) + \ell((2\gamma + |K|)r_*^{-1}) + 1.$$

Hence $(X, \widetilde{\mathcal{E}})$ is of bounded degree.

To prove the converse, it suffices to show that the condition (S) holds. Suppose otherwise, there exists a constant $c > 0$ such that for any $\ell > 0$, there exist n and a ball $B \subset \mathbb{R}^d$ with radius cr_*^n satisfying

$$\#\{x \in X_n : \ K_x \cap B \neq \emptyset\} > \ell.$$

Let $X_{n,B}$ denote the set in the above inequality, and let $D = \bigcup\{K_x : \ x \in X_{n,B}\}$. Then

$$|D| \leq 2|K|r_*^n + cr_*^n = (2|K| + c)r_*^n.$$

We can choose k_0 independent of n such that $\{B_1, B_2, \cdots, B_{k_0}\}$ is a family of open balls with radius $\gamma r_*^n/2$ and covers D (where γ is in the definition (3.2) of $\widetilde{\mathcal{E}}_h$). There exists a B_i that intersects at least $\ell' = \lfloor \ell/k_0 \rfloor$ of K_x $(x \in X_{n,B})$, say, $K_{x_1}, K_{x_2}, \cdots, K_{x_{\ell'}}$. Then $\mathrm{dist}(K_{x_i}, K_{x_j}) \leq \gamma r_*^n$ for $1 \leq i, j \leq \ell'$. Hence $x_i \sim_h x_j$ if $i \neq j$. It follows that

$$\deg(x_i) \geq \ell' - 1, \quad i = 1, 2, \cdots, \ell'.$$

Since ℓ can be arbitrarily large and k_0 is a fixed constant, we see that ℓ' can be arbitrarily large. This contradicts that the graph is of bounded degree, and the condition (S) follows. Hence the IFS satisfies the OSC. □

Note that $\mathcal{E}_h \subset \widetilde{\mathcal{E}}_h$, as a direct consequence of the above theorem, we have

Corollary 3.8. *Let $\{S_j\}_{j=1}^N$ be an IFS of contractive similitudes satisfying the OSC, then the graph $(X, \mathcal{E})$ is of bounded degree.*

There are variations of the IFS, that also fall into this framework of augmented trees. An easy example is the IFS $\{S_i\}_{i=1}^2$, with $S_1(x) = \frac{1}{2}x, S_2(x) = \frac{1}{2}(x+1)$, then the augmented tree has boundary $[0, 1]$. If we identify the two end vertices on each level of the tree, then we get a new hyperbolic graph with the boundary homeomorphic to a circle. A less trivial one is $S_1(x) = rx, S_2(x) = rx + (1 - r)$ where $r = (\sqrt{5} - 1)/2$ is the golden ratio (corresponding to the Bernoulli convolution). Note that for $x, y \in X_n$, $x \neq y$, S_x

can equal S_y (e.g., $S_{122} = S_{211}$). We can identify these indices and form a new hyperbolic graph. In this case, the vertical part of the graph is not a tree. This has been discussed in detail in [58] as quotient graphs for the IFSs with a *weak separation condition (WSC)* defined in [41]. It is known that the OSC is equivalent to the WSC together with $S_x \neq S_y$ for all $x \neq y$ [62]. Also in [48], the Moran sets and the hyperbolicity were studied. In the following we will introduce yet another type of IFS, the *weighted IFS*, which is connected to the study of energy forms on fractals. All these considerations can be embraced in a general setup of expansive hyperbolic graphs in Section 6.

To consider the weighted IFS, we start by defining another regrouping of indices in the symbolic space Σ^* as in the following. Let $\mathbf{s} = (s_1, s_2, \cdots, s_N)$ be a vector with $0 < s_i < 1$ for all $i \in \Sigma$. Denote $s_* = \min\{s_i : \ i \in \Sigma\}$ and $s^* = \max\{s_i : \ i \in \Sigma\}$. For $x = i_1 i_2 \cdots i_n \in \Sigma^*$, write $s_x = s_{i_1} s_{i_2} \cdots s_{i_n}$. We regroup Σ^* by setting $X_0(\mathbf{s}) = \{\vartheta\}$, and for $n \geq 1$, let

$$X_n(\mathbf{s}) := \{x = i_1 i_2 \cdots i_k \in \Sigma^* : \ s_x \leq s_*^n < s_{i_1} s_{i_2} \cdots s_{i_{k-1}}\}.$$

Then $X(\mathbf{s}) = \bigcup_{n=0}^{\infty} X_n(\mathbf{s})$ has a natural tree structure denoted by $\mathcal{E}_v$. We also define horizontal edge sets as in Definition 3.1 to get the augmented trees $(X(\mathbf{s}), \mathcal{E})$ and $(X(\mathbf{s}), \widetilde{\mathcal{E}})$.

For $(X(\mathbf{s}), \tilde{\mathcal{E}})$, Theorem 6.10 in Section 6 will yield the hyperbolicity as well as the Hölder equivalence of $\partial X(\mathbf{s})$ and K. The more interesting and useful case is on $(X(\mathbf{s}), \mathcal{E})$. For this, we can only prove the hyperbolicity for a restrictive class of self-similar sets, called *post critically finite* (p.c.f.) sets, defined by Kigami [32]. The crucial property of p.c.f. set is that the intersection of two cells K_i, K_j, $i \neq j$ has at most finitely many points. We have the following theorem (see [37]).

Theorem 3.9. *Let $\{S_j\}_{j=1}^N$ be a contractive IFS that satisfies the p.c.f. property. Then the augmented tree $(X(\mathbf{s}), \mathcal{E})$ is hyperbolic, and is of bounded degree. Moreover, the embedding $\kappa : (\partial X(\mathbf{s}), \theta_a) \to (K, |\cdot|)$ is a Hölder continuous homeomorphism.*

Let θ_a be the Gromov metric on $\partial X(\mathbf{s})$, then it induces a new metric $\tilde{\theta}_a$ on K via the homeomorphism κ. We consider the metric space $(K, \tilde{\theta}_a)$. Let α be the positive number satisfying $\sum_{j=1}^N s_j^\alpha = 1$. We define the self-similar measure $\mu_{\mathbf{s}}$ to be the unique Borel probability measure which satisfies the following identity

$$\mu_{\mathbf{s}}(\cdot) = \sum_{j=1}^{N} s_j^\alpha \mu_{\mathbf{s}}(S_j^{-1}(\cdot)) \ .$$

The following proposition provides an interesting relation of above self-similar measure and the induced Gromov metric on K.

Proposition 3.10. *Let $\{S_j\}_{j=1}^N$ be a contractive IFS that satisfies the p.c.f. property. For $\mathbf{s} \in (0,1)^N$, the self-similar measure $\mu_{\mathbf{s}}$ is Ahlfors-regular with exponent $(-\alpha \log s_*/a)$ with respect to the new metric space $(K, \tilde{\theta}_a)$, i.e.,*

$$\mu_{\mathbf{s}}(B_{\tilde{\theta}_a}(\xi, r)) \asymp r^{-\alpha \log s_*/a}, \qquad \forall\ \xi \in K,\ r \in (0,1).$$

On a p.c.f. set, we consider the *energy form* $(\mathfrak{E}, \mathcal{D})$ defined through a *harmonic structure* with weight $\mathbf{s} \in (0,1)^N$ on K [32, 55], which satisfies the following self-similarity:

$$\mathfrak{E}[u] = \sum\nolimits_{j\in\Sigma} s_j^{-1} \mathfrak{E}[u \circ S_j], \qquad u \in \mathcal{D},$$

where $\mathcal{D} = \{u \in C(K) : \mathfrak{E}[u] < \infty\}$ is the domain of $\mathfrak{E}$, and $C(K)$ is the set of continuous functions on K. Also define the *effective resistance* between two nonempty disjoint compact subsets $F, G \subset K$ by

$$R(F, G) = \big(\inf\{\mathfrak{E}[u] : u = 1 \text{ on } F,\ u = 0 \text{ on } G\}\big)^{-1}.$$

Then $R(\xi, \eta)(:= R(\{\xi\}, \{\eta\})$ for $\xi, \eta \in K)$ is a metric on K (*resistance metric* [32]).

Theorem 3.11. *Let K be a connected p.c.f. set that admits a harmonic structure with weight $\mathbf{s} \in (0,1)^N$, and let R be the resistance metric of the associated self-similar energy form. Then the metric $\tilde{\theta}_a$ on K induced by $(X(\mathbf{s}), \mathcal{E})$ satisfies*

$$\tilde{\theta}_a(\xi, \eta) \asymp R(\xi, \eta)^{-a/\log s_*}, \qquad \forall\ \xi, \eta \in K.$$

The interested reader can refer to [37] for the details.

4 Lipschitz equivalence of self-similar sets

Recall that two compact metric spaces (X, d_1) and (Y, d_2) are said to be *Lipschitz equivalent*, denoted by $X \simeq Y$, if there exists a bi-Lipschitz map $\sigma : X \to Y$, i.e., σ is a bijection and there exists a constant $C > 0$ such that

$$C^{-1} d_1(x, y) \le d_2(\sigma(x), \sigma(y)) \le C d_1(x, y), \quad \text{for all } x, y \in X.$$

Lipschitz classification of fractals was first started by Falconer and Marsh [14] on Cantor-type sets under the strong separation condition. The recent interest was due to Rao, Ruan and Xi [43] on their solution to an open question of David and Semmes, so called the $\{1,3,5\} - \{1,4,5\}$ problem, i.e., subdivide $[0,1]$ into five equal size subintervals, and pick the $1,3,5$-th subintervals and the $1,4,5$-th subintervals (the 4-th and 5-th subintervals have nonvoid intersection) to form the respective IFSs and self-similar sets K_1, K_2. They used a technique of graph directed system to show that K_1 and K_2 are Lipschitz equivalent. The result stimulates a lot of interest and generalizations. In this section, we discuss a different approach to this type of Lipschitz equivalence problem through the augmented trees, hyperbolic graphs and hyperbolic boundaries [9, 51]. More developments can be found in [48–50, 52].

We will consider the self-similar IFS $\{S_i\}_{i=1}^N$ with equal contractive ratio. Let $\Sigma = \{1, 2, \cdots, N\}$, $X_n = \Sigma^n$ and $X = \bigcup_{n=0}^\infty \Sigma^n$. Let $(X, \mathcal{E})$ be the augmented tree as in Definition 3.1. For convenience, we call $(X, \mathcal{E})$ an N-ary augmented tree. We say that T

is an X_n-*horizontal component* if $T \subset X_n$ is a maximal connected subset with respect to $\mathcal{E}_h$. In this case, we denote by $T_{\mathcal{D}}$ the set of all descendants of T (including T itself), i.e.,

$$T_{\mathcal{D}} = \{x \in X : \ x|_n \in T\}$$

where $x|_n$ is the initial segment of x with length n. Obviously, $T_{\mathcal{D}}$ should be a connected subgraph of X, and if $(X, \mathcal{E})$ is hyperbolic then $T_{\mathcal{D}}$ is also hyperbolic. Indeed suppose x, y are children of T and let γ be a horizontal geodesic in $(X, \mathcal{E})$ with end points at x, y respectively. Then the parents of γ in T form a connected horizontal path (by property (*) in Lemma 2.5), hence is in T (because T is a connected component). This implies that γ is also a horizontal geodesic in $T_{\mathcal{D}}$. Applying this argument to each level in $T_{\mathcal{D}}$, we see that the lengths of horizontal geodesics in $T_{\mathcal{D}}$ are uniformly bounded, inherited from $(X, \mathcal{E})$.

We let $\mathcal{F}_n$ denote the family of all X_n-horizontal components, and let $\mathcal{F} = \bigcup_{n \geq 0} \mathcal{F}_n$. Note that for distinct $T, T' \in \mathcal{F}_n$, the subgraphs $T_{\mathcal{D}}, T'_{\mathcal{D}}$ are disjoint. Denote T by $\lfloor x \rfloor$ for $x \in T$, we can define a graph structure on $\mathcal{F}$ as: $\lfloor x \rfloor$ and $\lfloor y \rfloor$ are connected by an edge if and only if $(u, v) \in \mathcal{E}_v$ for some $u \in \lfloor x \rfloor$ and $v \in \lfloor y \rfloor$; we denote this graph by X_Q (see the left two graphs of Figure 4). It is clear that X_Q defined above is a tree, and we call it the *quotient tree* of X.

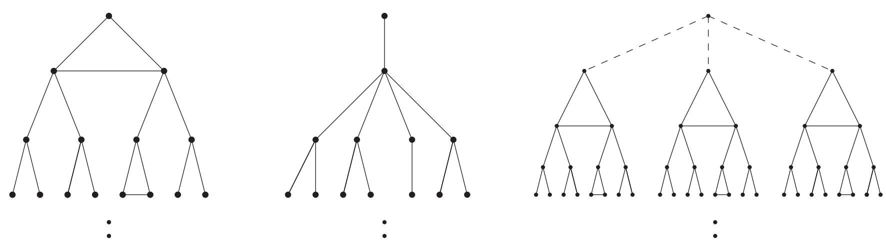

Figure 4: A rooted graph X, the quotient tree X_Q and the union of three copies of X.

For $T, T' \in \mathcal{F}$, we say that T and T' are equivalent, denoted by $T \sim T'$, if there exists a graph isomorphism $g : T_{\mathcal{D}} \to T'_{\mathcal{D}}$, i.e., the map g and its inverse map both preserve the vertical and horizontal edges of $T_{\mathcal{D}}$ and $T'_{\mathcal{D}}$. We denote the equivalence class by $[T]$.

Definition 4.1. *We call an augmented tree $(X, \mathcal{E})$* simple *if the equivalence classes in $\mathcal{F}$ is finite. Let $[T_1], \dots, [T_m]$ be the equivalence classes in $\mathcal{F} \setminus \{\vartheta\}$, and let a_{ij}, where $1 \leq i, j \leq m$, denote the cardinality of the horizontal components of the 1-st level descendants of $T \in [T_i]$ that belong to $[T_j]$. We call $A = [a_{ij}]$ the* incidence matrix *of $(X, \mathcal{E})$.*

Proposition 4.2. *A simple augmented tree $(X, \mathcal{E})$ is always hyperbolic.*

Proof. Note that for each horizontal geodesic $\pi(x, y)$ in X, the horizontal part must be contained in a horizontal component of the augmented tree. Since there are finitely many equivalence classes $[T]$ of horizontal components, and each T contains finitely many vertices, it follows that $\pi(x, y)$ is uniformly bounded, and hence $(X, \mathcal{E})$ is hyperbolic. □

Definition 4.3. *Let X, Y be two hyperbolic graphs. We say that σ is a near-isometry from X to Y if there exist finite subsets $E \subset X$, $F \subset Y$, and $c > 0$ such that $\sigma : X \setminus E \to Y \setminus F$ is a bijection and satisfies*

$$\left| d(\sigma(x), \sigma(y)) - d(x, y) \right| < c.$$

The following two propositions are easy consequences from the definitions [9, 51].

Proposition 4.4. *Let X, Y be two hyperbolic augmented trees. Suppose there exists a near-isometry from X to Y, then $\partial X \simeq \partial Y$.*

It is clear that $\partial(X, \mathcal{E}_v)$ is an N-ary Cantor set. Our aim is to show that a simple augmented tree $(X, \mathcal{E})$ is near-isometric to $(X, \mathcal{E}_v)$; by the above proposition, $\partial(X, \mathcal{E}_v) \simeq \partial(X, \mathcal{E})$. In the following we develop some tools to construct such near-isometry.

Proposition 4.5. *Let $(X, \mathcal{E})$ be a simple N-ary augmented tree, let $[T_1], \ldots, [T_m]$ be the equivalence classes with incidence matrix A, and let $\mathbf{u} = [u_1, \ldots, u_m]^t$ where $u_i = \# T_i$. Then $A\mathbf{u} = N\mathbf{u}$.*

Suppose $(X_i, \mathcal{E}_i), 1 \le i \le \ell$ are augmented trees with roots ϑ_i. Let $\widehat{X} = (\bigcup_{i=1}^{\ell} X_i) \cup \{\vartheta\}$ where ϑ is an additional vertex. We equip $\widehat{X}$ with an edge set $\widehat{\mathcal{E}}$ that consists of all $\mathcal{E}_i$ and the new edges joining ϑ and ϑ_i. Then $(\widehat{X}, \widehat{\mathcal{E}})$ forms a new connected graph and each $(X_i, \mathcal{E}_i)$ becomes its subgraph (see Figure 4). We call $(\widehat{X}, \widehat{\mathcal{E}})$ *the union of* $\{X_i\}_{i=1}^{\ell}$. It follows that [9, Proposition 2.8]

Proposition 4.6. *Let $(X, \mathcal{E})$ be an N-ary augmented tree such that $\partial(X, \mathcal{E}) \simeq \partial(X, \mathcal{E}_v)$. Suppose $(X_i, \mathcal{E}_i), 1 \le i \le \ell$, are copies of $(X, \mathcal{E})$, and $(\widehat{X}, \widehat{\mathcal{E}})$ is the union of $\{(X_i, \mathcal{E}_i)\}_{i=1}^{\ell}$. Then $\partial(\widehat{X}, \widehat{\mathcal{E}}) \simeq \partial(X, \mathcal{E})$.*

The following notions of rearrangeable matrix [8, 51] and quasi-rearrangeable matrix [9] are the most important technical devices in constructing the near-isometry between $(X, \mathcal{E}_v)$ and $(X, \mathcal{E})$.

Definition 4.7. *Let $\mathbf{a} = [a_1, \ldots, a_m]$ and $\mathbf{u} = [u_1, \ldots, u_m]^t$ be in $\mathbb{N}^m$. For $N > 0$, we say that $\mathbf{a}$ is $(N, \mathbf{u})$-rearrangeable if there exist $p > 0$ and a non-negative integral $p \times m$ matrix C (rearranging matrix) such that*

$$\mathbf{a} = \underbrace{[1, \ldots, 1]}_{p} C \quad \text{and} \quad C\mathbf{u} = \underbrace{[N, \ldots, N]}_{p}^t. \tag{4.1}$$

(In this case $\mathbf{au} = pN$, and $u_i \le N$.) We say that $\mathbf{a}$ is $(N, \mathbf{u})$-quasi-rearrangeable if the second identity is replaced by $C\mathbf{u} \le [N, \ldots, N]^t$.

A matrix A is said to be $(N, \mathbf{u})$-rearrangeable (quasi-rearrangeable) if each row vector in A is $(N, \mathbf{u})$-rearrangeable (quasi-rearrangeable). (Note that the p and C in each row may be different.)

To realize the above definition, let us assume that there are m different kinds of objects, each kind has cardinality a_i and each one of the same kind has weight u_i (assume that the gcd of the u_i's is 1), hence the total weight is $\sum_i a_i u_i = pN$. The rearranging matrix C is a way to divide these objects into p groups (first identity in (4.1)) such that every entry of a row represents the number of each kind in the group, and the total weight of the objects in the group is N (the second identity in (4.1)).

Remark. The main purpose of this rearrangeable matrix A is to modify the horizontal edges of the offsprings of a component T so that each component T_i in the offsprings has the same parent $x \in T$ (when $\gcd(\mathbf{u}) = 1$) . This rearrangement gives a near-isometry of $(X, \mathcal{E})$ to $(X, \mathcal{E}_v)$, and hence $\partial(X, \mathcal{E}) \simeq \partial(X, \mathcal{E}_v)$ (by Proposition 4.4). In the following we also consider A^k and the same idea holds with the k-th generation.

Recall that a non-negative matrix A is called *primitive* if $A^n > 0$ for some n, and is called *irreducible* if for any (i, j), there exists $k > 0$ such that the (i, j)-entry of A^k is positive. In [9], we proved

Proposition 4.8. *Let A be an $m \times m$ primitive matrix and $\mathbf{u} \in \mathbb{N}^m$. Let $u = \gcd(\mathbf{u})$,*
(i) if $A\mathbf{u} = N\mathbf{u}$, then there exists $k > 0$ such that A^k is $(uN^k, \mathbf{u})$-rearrangeable;
(ii) if $A\mathbf{u} \leq N\mathbf{u}$, then there exists $k > 0$ such that A^k is $(uN^k, \mathbf{u})$-quasi-rearrangeable.
In both cases, the corresponding matrix C_i for each row of A^k is of size $(u_i/u) \times m$.

It is well-known that for any non-negative matrix A, it can be brought into the form of the upper triangular block by a permutation matrix P,

$$P^t A P = \begin{bmatrix} A_1 & & * \\ & \ddots & \\ \mathbf{0} & & A_r \end{bmatrix}$$

where each A_i is a square matrix that is either irreducible or zero, $i = 1, \ldots, r$. We give a stronger result that for certain power A^ℓ, the block matrices are primitive, if not zero. The lemma has independent interest and might be useful elsewhere.

Lemma 4.9. *Let A be a non-negative matrix, then we have*
(i) if A^n is irreducible for any $n \geq 1$, then A is primitive;
(ii) there is $\ell \geq 1$ such that the block matrices lying in the diagonal of the canonical form of A^ℓ are either primitive or 0.

Proposition 4.10. *Let $(X, \mathcal{E})$ be a simple N-ary augmented tree, and assume that the incidence matrix A is primitive. Then $\partial(T_{\mathcal{D}}, \mathcal{E}) \simeq \partial(X, \mathcal{E}_v)$ for any horizontal component $T \in \mathcal{F}$.*

Proof. Here we only sketch the main idea. Since the incidence matrix A of $(X, \mathcal{E})$ is primitive, A is also an incidence matrix of the subgraph $T_{\mathcal{D}}$. Let $\{[T_1], \cdots, [T_m]\}$ be the equivalence classes that are in $T_{\mathcal{D}}$, let $u_i = \#T_i$ be the number of vertices in T_i, and let $u = \gcd(\mathbf{u})$.

By Propositions 4.5 and 4.8, there exists k such that A^k is $(uN^k, \mathbf{u})$-rearrangeable. Without loss of generality we can assume that $k = 1$. Hence for each T_i, there is a C_i rearranging its descendants $T_i\Sigma$ into $p_i = u_i/u$ groups consisting of components T_j's, which are denoted by $\mathcal{V}_{i,k}, 1 \le k \le p_i$; the number of vertices in $\mathcal{V}_{i,k}$ is uN. We denote this process by Step I.

Let $\ell = \#T$, and let Y be the union of ℓ copies of $(X, \mathcal{E}_v)$. Let $\mathcal{E}'$ be an augmented structure on Y by adding horizontal edges that joining u consecutive vertices in each level (see the right figure in Figure 5). We call this Step II. (Note that number of vertices in the n-th level is ℓN^{n-1} and u divides ℓ.) Then

$$\partial(Y, \mathcal{E}') \simeq \partial(Y, \mathcal{E}_v) \simeq \partial(X, \mathcal{E}_v)$$

as the first $\simeq$ follows from a direct check that the identity map is a near-isometry, and the second $\simeq$ follows from Proposition 4.6.

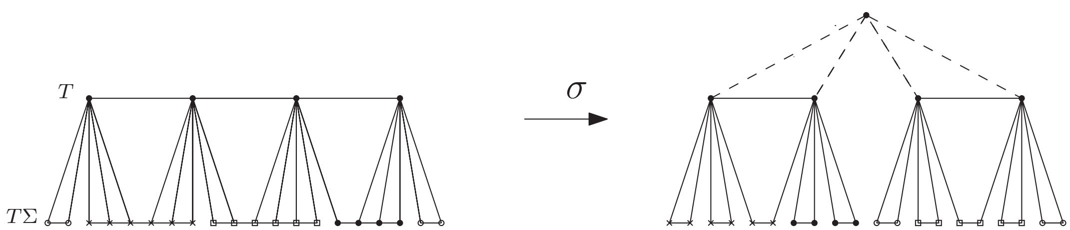

Figure 5: An illustration of $\sigma : (T_{\mathcal{D}}, \mathcal{E}) \to (Y, \mathcal{E}')$ with $u = 2, \ell = 4$, the $\bullet, \times, \circ, \square$ denote four kinds of components.

With this setup, we can define a map $\sigma : (T_{\mathcal{D}}, \mathcal{E}) \to (Y, \mathcal{E}')$ as follows. On the first level, let σ be any bijection from T to Y_1. Suppose we have defined σ on T_i of $T_{\mathcal{D}}$ in the n-th level, we can define σ on $T_i\Sigma$ by first applying Step I of rearrangement to obtain $\{\mathcal{V}_{i,k}\}_{k=1}^{p_i}$, then assigning the vertices of $\mathcal{V}_{i,k}$ to the descendants of $\sigma(T_i)$ and applying Step II (see Figure 5). It follows from the rearrangement property that each $\sigma(\mathcal{V}_{i,k})$ are descendants of u consecutive vertices in $\sigma(T_i)(\subset Y_n)$ (see Theorem 3.7 in [51] for detail). By the same proof as Theorem 3.7 in [51], that σ is a near-isometry, and hence $\partial(T_{\mathcal{D}}, \mathcal{E}) \simeq \partial(Y, \mathcal{E}') \simeq \partial(X, \mathcal{E}_v)$. $\square$

We continue the construction of the near-isometry $\sigma : (X, \mathcal{E}) \to (X, \mathcal{E}_v)$ with the following incidence matrix A.

Lemma 4.11. *Let $(X, \mathcal{E})$ be a simple N-ary augmented tree with equivalence classes $\{[T_1], \dots, [T_m]\}$, and the incidence matrix is of the form*

$$A = \begin{bmatrix} A_1 & A_3 \\ 0 & A_2 \end{bmatrix}$$

where A_1, A_2 are non-zero matrices with orders r and $m-r$ respectively. Let $u_i = \#T_i$, $\mathbf{u}_1 = [u_1, \dots, u_r]^t$ and $u = gcd(\mathbf{u})$. Suppose

(i) A_1 is $(uN, \mathbf{u}_1)$-quasi-rearrangeable;

(ii) for $i = r+1, \dots, m$, there exist near-isometries $\sigma_i : ((T_i)_{\mathcal{D}}, \mathcal{E}) \to (Y_i, \mathcal{E}_v)$.

Then there exists a near-isometry $\sigma : (X, \mathcal{E}) \to (X, \mathcal{E}_v)$, hence $\partial(X, \mathcal{E}) \simeq \partial(X, \mathcal{E}_v)$.

Proof. For convenience, we assume that A_1 is $(N, \mathbf{u}_1)$-quasi-rearrangeable, i.e., $\gcd(\mathbf{u}_1) = 1$; the general case follows from the same argument of Step II in last proposition. We will use (i) and (ii) to construct a near-isometry $\sigma : (X, \mathcal{E}) \to (X, \mathcal{E}_v)$. We write $X_1 = (X, \mathcal{E})$ and $X_2 = (X, \mathcal{E}_v)$. Let $\sigma(\vartheta) = \vartheta$ and $\sigma(i) = i,\ i \in \Sigma$. Suppose σ has been defined on Σ^n such that

(1) for component $T \in [T_i],\ i \le r$, $\sigma(T)$ has the same parent, i.e., $\sigma(x)^{-1} = \sigma(y)^{-1}$ for all $x, y \in T \subset \Sigma^n$.

(2) for component $T \in [T_i],\ i \ge r+1$, $\sigma(x) = \sigma_i(x)$ for $x \in T_{\mathcal{D}}$.

To define the map σ on Σ^{n+1}, we note that if $T \subset \Sigma^n$ in (2), then σ is well-defined by σ_i. If $T \subset \Sigma^n$ in (1), without loss of generality, we let $T \in [T_1]$. Then T gives rise to horizontal components in Σ^{n+1}, we group them into $\mathcal{Z}_{1,j}, j = 1, \cdots m$ according to the components belonging to $[T_j]$.

By the quasi-rearrangeable property of A_1 (assumption (i)), for the row vector $\mathbf{a}_1 = [a_{11}, \dots, a_{1r}]$, there exists a nonnegative integral matrix $C = [c_{sj}]_{u_1 \times r}$ such that

$$\mathbf{a}_1 = \mathbf{1}C \quad \text{and} \quad C\mathbf{u}_1 \le [N, \dots, N]^t.$$

By using this, we can decompose $\mathbf{a}_1$ into u_1 groups as follows. Note that a_{1j} denotes the number of horizontal components that belong to $[T_j]$. For each $1 \le s \le u_1$, we choose $c_{sj}, 1 \le j \le r$, of those components that are of size u_j respectively, and denote this collection by Λ_s. Then $\bigcup_{j=1}^r \mathcal{Z}_{1,j}$ can be rearranged into u_1 groups

$$\bigcup_{j=1}^r \mathcal{Z}_{1,j} = \Lambda_1 \ \cup \ \cdots \cup \Lambda_{u_1}, \tag{4.2}$$

and the total number of vertices in each group is $\le N$.

For the component $T = \{\mathbf{i}_1, \dots, \mathbf{i}_{u_1}\} \subset \Sigma^n$ in $(X, \mathcal{E})$, we have defined $\sigma(T) = \{\mathbf{j}_1 = \sigma(\mathbf{i}_1),\ \dots,\ \mathbf{j}_{u_1} = \sigma(\mathbf{i}_{u_1})\}$ in $(X, \mathcal{E}_v)$ by induction. In view of (4.2), we define σ on $\bigcup_{j=1}^r \mathcal{Z}_{1,j}$ by assigning vertices in Λ_s (cardinality $\le N$) to the descendants of $\mathbf{j}_s$ (cardinality N) in a one-to-one manner; for the remaining $T' \in \bigcup_{j=r+1}^m \mathcal{Z}_{1,j}$ (maybe empty), say $T' \in [T_j]$ and $j \ge r+1$, we define for $x \in T'$, $\sigma(x)$ to be any point in $\sigma(T)\Sigma \setminus \bigcup_{j=1}^r \sigma(\mathcal{Z}_{1,j})$ to fill up the $\sigma(T)\Sigma$. We also use σ_i to induce a near-isometry $\sigma : T_{\mathcal{D}} \to (\sigma(T))_{\mathcal{D}}$. We apply the same construction of σ on the offsprings of every component in Σ^{n+1}. Inductively, σ can be defined from X_1 to X_2.

We omit the proof that σ is a near-isometry, the reader can refer to [9, Lemma 4.4] for detail. □

Theorem 4.12. *Let K be a self-similar set generated by an equicontractive IFS $\{S_i\}_{i=1}^N$. If the augmented tree $(X, \mathcal{E})$ is simple, then $\partial(X, \mathcal{E}) \simeq \partial(X, \mathcal{E}_v)$.*

Proof. Let $\{[T_1], \ldots, [T_m]\}$ be the equivalence classes of horizontal components, $u_i = \#T_i$, and A the associated incidence matrix. By Lemma 4.9, there exist $\ell \geq 1$ and a permutation matrix P such that

$$P^t A^\ell P = \begin{bmatrix} A_1 & & * \\ & \ddots & \\ \mathbf{0} & & A_k \end{bmatrix}$$

where A_i are either 0 or primitive. From the definition of incidence matrix, we see that $A_k \neq 0$, hence is primitive. Without loss of generality, we let $\ell = 1$.

If $k = 1$, then $A = A_1$ is primitive. For any horizontal component $T \subset \Sigma$, $T_{\mathcal{D}}$ has incidence matrix A also. Hence by Proposition 4.10 that $\partial(T_{\mathcal{D}}, \mathcal{E}) \simeq \partial(X, \mathcal{E}_v)$. As Σ is the disjoint union of such T, it follows that $\partial(X, \mathcal{E}) = \partial(\cup(T_{\mathcal{D}}, \mathcal{E})) \simeq \partial(X, \mathcal{E}_v)$.

If $k = 2$, let A_1, A_2 correspond to $\{[T_1], \ldots, [T_r]\}$, and $\{[T_{r+1}], \ldots, [T_r]\}$ respectively. If $A_1 = \mathbf{0}$, we can take A_2 as the incidence matrix of $(X, \mathcal{E})$ by removing finitely many vertices that belong to $[T_i], 1 \leq i \leq r$. By Proposition 4.10, the result follows. If $A_1 \neq \mathbf{0}$, then Proposition 4.10 implies that assumption (ii) in Lemma 4.11 is satisfied; the other assumptions also follow readily, and the theorem follows. The general case that $k \geq 2$ follows by applying the above argument inductively. □

By applying Proposition 3.4 and Theorem 4.12, we obtain

Theorem 4.13. *Let K and K' be self-similar sets that are generated by two IFSs that have the same number of similitudes, the same contraction ratio, and satisfy condition (H) (in Proposition 3.4). Suppose further the two augmented trees are simple. Then $K \simeq K'$.*

As an illustration, we consider the following example with IFS $\{S_i\}_{i=1}^4$ defined by $\{J_i\}_{i=1}^4$ as in Figure 6. Let r be the contraction ratio of the IFS. Then the self-similar set K is Lipschitz equivalent to the canonical 4-ary cantor set of contraction ratio r. (For detail, please see [51, Example 5.4])

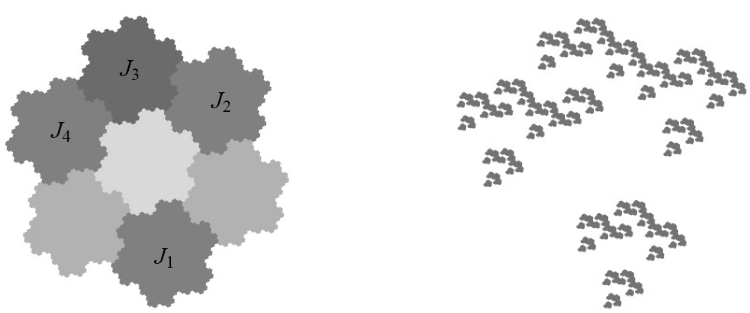

Figure 6: The IFS is defined by $\{J_i\}_{i=1}^4$; the attractor K is Lipschitz equivalent to the canonical 4-ary Cantor set of the same contraction ratio.

5 Random walks on augmented trees

In this section we discuss a class of random walks on the augmented trees such that the Martin boundaries are identified with the hyperbolic boundaries, and the attractor K. This allows us to carry the discrete potential theory as well as the induced energy forms (Dirichlet forms) on K [36,38,39]. We use the notation $f \asymp g$ to mean that there exists a constant $C > 0$ such that $C^{-1}g(x) \leq f(x) \leq Cg(x)$ for any variable x in a given domain.

We recall some basic notions of discrete potential theory (see [11,60]). Let $(X, \mathcal{G})$ be a locally finite connected graph with the root $\vartheta \in X$. A *(reversible) random walk* on $(X, \mathcal{G})$ is a Markov chain $\{Z_n\}_{n=0}^{\infty}$ with the state space X and the transition probability given by $P(x, y) = \frac{c(x,y)}{m(x)}$, $x, y \in X$, where the *conductance* $c(\cdot, \cdot)$ is a nonnegative symmetric function on $X \times X$ that satisfies $c(x, y) > 0$ if and only if $(x, y) \in \mathcal{G}$, and $m(x) := \sum_{y \in X} c(x, y)$ is the *total conductance* at x. A function $f : X \to \mathbb{R}$ is called *P-harmonic* if $\sum_{y \in X} P(x, y)f(y) = f(x)$ for all $x \in X$; the *graph energy* of f is defined by

$$\mathfrak{E}_X[f] = \frac{1}{2} \sum_{x,y \in X} c(x, y)|f(x) - f(y)|^2.$$

We assume that $\{Z_n\}$ is *transient*, i.e., the *Green function* $G(x, y) = \sum_{n=0}^{\infty} P^n(x, y)$ is finite for all $x, y \in X$, where P^n is the n-step transition probability which can be defined inductively by $P^{n+1}(x, y) = \sum_{z \in X} P(x, z)P^n(z, y)$ with P^0 being the identity matrix on X. Write $\mathbb{P}(\cdot \mid Z_0 = x)$ as $\mathbb{P}_x(\cdot)$ for short. We denote by $F(x, y) = \mathbb{P}_x(\exists\ n \geq 0$ such that $Z_n = y)$ the *ever-visiting probability* from x to y; it is known that $G(x, y) = F(x, y)G(y, y)$, and $F(x, y) \geq F(x, z)F(z, y)$ for all $x, y, z \in X$. The *Martin kernel* is defined as

$$K(x, y) = \frac{G(x, y)}{G(\vartheta, y)} = \frac{F(x, y)}{F(\vartheta, y)}, \qquad x, y \in X.$$

Following [11,60], we define the *Martin metric* $\rho_M(\cdot, \cdot)$ on X by

$$\rho_M(x, y) = \sum_{u \in X} a(u)\big(|K(u, x) - K(u, y)| + |\chi_u(x) - \chi_u(y)|\big),$$

where $a : X \to (0, \infty)$ satisfies $\sum_{u \in X} \frac{a(u)}{F(\vartheta,u)} < \infty$, and χ_u is the indicator function at u. In view of

$$K(u, x) = \frac{F(u, x)}{F(\vartheta, x)} \leq \frac{1}{F(\vartheta, u)},$$

we see that $\rho_M(\cdot, \cdot)$ is well defined and is a metric on X.

Definition 5.1. *Let $\widehat{X}_M$ be the completion of (X, ρ_M). We call $\mathcal{M} = \widehat{X}_M \setminus X$ the* Martin boundary *of $\{Z_n\}$.*

Note that the completion coincides with the minimal compactification of X such that for every $x \in X$, $K(x, \cdot)$ extends continuously to $\widehat{X}_M$ [60]. Under this topology (or the Martin metric on $\widehat{X}_M$), the trajectory $\{Z_n\}$ converges to an $\mathcal{M}$-valued random variable Z_∞ almost surely. Let ν denote the *hitting distribution* of Z_∞ on $\mathcal{M}$ when $Z_0 = \vartheta$. For

$\xi \in \mathcal{M}$, the *ξ-process* is defined as the random walk $\{Z_n^\xi\}_{n=0}^\infty$ on $(X, \mathcal{G})$ with the transition probability $P^\xi(x,y) = P(x,y)\frac{K(y,\xi)}{K(x,\xi)},\ x, y \in X$ (P^ξ is a transition probability since $K(\cdot,\xi)$ is P-harmonic), and the corresponding hitting distribution is denoted by ν^ξ. The *minimal Martin boundary* $\mathcal{M}_{\min}$ of $\{Z_n\}$ is the collection of all $\xi \in \mathcal{M}$ such that ν^ξ is the point mass at ξ; it is known that $\nu(\mathcal{M} \setminus \mathcal{M}_{\min}) = 0$.

To obtain the energy form on $\mathcal{M}$, we define the *Naïm kernel* by

$$\Theta(x,y) = \frac{K(x,y)}{G(x,\vartheta)} = \frac{F(x,y)}{F(x,\vartheta)G(\vartheta,\vartheta)F(\vartheta,y)}, \qquad x, y \in X.$$

Clearly $\Theta(\cdot,\cdot)$ is symmetric on $X \times X$, and can be extended continuously to $X \times \mathcal{M}$ as the Martin kernel $K(\cdot,\cdot)$ does. The extension on $\mathcal{M} \times \mathcal{M} \setminus \Delta$ (here $\Delta := \{(\xi,\xi) : \xi \in \mathcal{M}\}$) is formulated by

$$\Theta(\xi,\eta) = \lim_{m\to\infty} \sum\nolimits_{z\in X_m} \ell_m^\xi(z)\Theta(z,\eta), \qquad \xi \neq \eta \in \mathcal{M}, \tag{5.1}$$

where $\ell_m^\xi(z) = \mathbb{P}_\vartheta^\xi\big(\exists\ n \geq 0 \ni Z_n^\xi = z, Z_k^\xi \notin X_m\ \forall\ k > n\big)$ is the *last-visiting probability* on X_m of the ξ-process [57]; the limit exists as the sum is increasing in m.

For a ν-integrable function u on $\mathcal{M}$, its *Poisson integral* Hu is given by

$$(Hu)(x) = \int_{\mathcal{M}} K(x,\xi)u(\xi)d\nu(\xi), \qquad x \in X.$$

Note that $K(\cdot,\xi)$ is P-harmonic for all $\xi \in \mathcal{M}$, so is Hu. For $u \in L^2(\mathcal{M},\nu)$, we define the *induced energy* of u by

$$\mathfrak{E}_{\mathcal{M}}[u] = \mathfrak{E}_X[Hu] = \frac{1}{2}\sum\nolimits_{x,y\in X} c(x,y)|Hu(x) - Hu(y)|^2.$$

The domain of the quadratic form $\mathfrak{E}_{\mathcal{M}}$ is $\mathcal{D}_{\mathcal{M}} = \{u \in L^2(\mathcal{M},\nu) : \mathfrak{E}_{\mathcal{M}}[u] < \infty\}$.

Theorem 5.2. (Silverstein [57]) *The induced energy has the expression*

$$\mathfrak{E}_{\mathcal{M}}[u] = \frac{m(\vartheta)}{2}\iint_{\mathcal{M}\times\mathcal{M}\setminus\Delta} |u(\xi) - u(\eta)|^2\Theta(\xi,\eta)d\nu(\xi)d\nu(\eta), \qquad u \in \mathcal{D}_{\mathcal{M}}.$$

For a hyperbolic graph $(X, \mathcal{G})$, we need some hypotheses on $\{Z_n\}$ to identify $\mathcal{M}$ with the hyperbolic boundary ∂X:

(p_0) $\quad p_* := \inf_{(x,y)\in\mathcal{E}} P(x,y) > 0$;

(SI) *(strong isoperimetry)* $\sup\{\frac{m(F)}{c(\partial F)}$ *: F is a finite subset of $X\} < \infty$, where $m(F) = \sum_{x\in F} m(x)$ and $c(\partial F) = \sum_{x\in F, y\notin F} c(x,y)$.*

The (p_0) implies that $\deg(x) \leq (\min_{y\in X: x\sim y} P(x,y))^{-1} \leq p_*^{-1}$ for all $x \in X$, hence $(X, \mathcal{G})$ has bounded degree. Also, it is known that (SI) yields the transience of $\{Z_n\}$ [60].

Theorem 5.3. (Ancona [1]) *Let $(X, \mathcal{G})$ be a hyperbolic graph. Suppose $\{Z_n\}$ is a random walk on $(X, \mathcal{G})$ satisfying (p_0) and (SI). Then there exists a constant $C_0 \geq 1$ such that*

$$F(x, y) \leq C_0 F(x, z) F(z, y) \tag{5.2}$$

whenever $x, y, z \in X$ and z lies on some $\pi(x, y)$. Moreover, the Martin boundary $\mathcal{M}$ equals $\mathcal{M}_{\min}$, and is homeomorphic to the hyperbolic boundary ∂X.

To apply the above theorem, we provide a sufficient condition for $\{Z_n\}$ satisfying (SI). For $x \in X \setminus \{\vartheta\}$, define the *return ratio* at $x \in X$ by

$$\lambda(x) := \frac{\mathbb{P}_x(|Z_1| = |x| - 1)}{\mathbb{P}_x(|Z_1| = |x| + 1)} = \frac{\sum_{z \in \mathcal{J}_{-1}(x)} c(x, z)}{\sum_{y \in \mathcal{J}_1(x)} c(x, y)}.$$

Following a similar proof as [38, Theorem 5.1], we have

Proposition 5.4. *Suppose a random walk $\{Z_n\}$ on a rooted graph $(X, \mathcal{G})$ satisfies (p_0) and $\sup_{x \in X \setminus \{\vartheta\}} \lambda(x) < 1$. Then $\{Z_n\}$ has strong isoperimetry (SI).*

To obtain explicit estimates of Martin kernels and Naïm kernels, we will consider a class of random walks satisfying

(R_λ) *(constant return ratio) $\lambda(x) \equiv \lambda \in (0, 1)$ for all $x \in X \setminus \{\vartheta\}$.*

With condition (R_λ), by counting the time instants $n_0 = 0$ and $n_k = \inf\{\ell > n_{k-1} : |Z_\ell| \neq |Z_{\ell-1}|\}$ for $k \geq 1$ inductively, the sequence $\{|Z_{n_k}|\}_{k=0}^\infty$ is a birth and death chain on the nonnegative integers with the transition probability $\widetilde{P}(0, 1) = 1$, $\widetilde{P}(m, m-1) = \frac{\lambda}{1+\lambda}$, and $\widetilde{P}(m, m+1) = \frac{1}{1+\lambda}$ for $m \geq 1$; it follows that

$$F(x, \vartheta) = \widetilde{F}(|x|, 0) = \lambda^{-|x|}, \qquad \forall\, x \in X. \tag{5.3}$$

Definition 5.5. *Let $\lambda \in (0, 1)$. A random walk $\{Z_n\}$ on $(X, \mathcal{G})$ is said to be* λ-natural (λ-NRW) *if it satisfies (p_0) and (R_λ).*

Remark. The above definition generalizes the NRW (and quasi-NRW) in [38], which was defined by self-similar measures of natural weight (with doubling property respectively). We will see in the following Theorem 5.6 that the new NRW is characterized by the doubling regular Borel measures. Note also that this definition is equivalent to the NRW in [39] by Theorem 5.6.

From Proposition 5.4 we know that every λ-NRW satisfies (SI), and if $(X, \mathcal{G})$ is hyperbolic, then Theorem 5.3 applies.

In the rest of this section, we will consider the λ-NRW $\{Z_n\}$ on the augmented tree $(X, \widetilde{\mathcal{E}})$ associated to a self-similar set K (Definition 3.1). Recall that a regular Borel measure μ on K is called *(volume) doubling* (VD) if there exists $C \geq 1$ such that

$$0 < \mu(B(\xi, 2r)) \leq C\mu(B(\xi, r)) < \infty, \qquad \forall\, \xi \in K,\ r > 0.$$

The following theorem improves and strengthens [38, Theorem 4.8].

Theorem 5.6. *Let $\{S_j\}_{j=1}^N$ be an IFS of contractive similitudes with attractor K, and let $(X, \widetilde{\mathcal{E}})$ be an associated augmented tree in Definition 3.1. Then $(X, \widetilde{\mathcal{E}})$ admits a λ-NRW if and only if $\{S_j\}_{j=1}^N$ satisfies the OSC. Moreover, the conductance of the λ-NRW satisfies*

$$c(x, x^-) = \lambda^{-|x|}\mu(K_x), \qquad \forall\, x \in X \setminus \{\vartheta\}, \tag{5.4}$$

for some doubling measure μ on K with

$$\mu(K_x \cap K_y) = 0, \qquad \forall\, x \neq y \in X \text{ with } |x| = |y|. \tag{5.5}$$

Proof. For the first statement, as the (p_0) implies the bounded degree property of $(X, \widetilde{\mathcal{E}})$, by Theorem 3.7, the OSC on $\{S_j\}_{j=1}^N$ is necessary for possessing a λ-NRW.

To prove the sufficiency, we assume the OSC, and use the α-Hausdorff measure $\mathcal{H}^\alpha$ to construct a λ-NRW, where α is the Hausdorff dimension of K. It is known that under the OSC, $0 < \mathcal{H}^\alpha(K_x) = r_x^\alpha \mathcal{H}^\alpha(K) < \infty$ holds for all $x \in X$, and $\mathcal{H}^\alpha(K_x \cap K_y) = 0$ for all distinct $x, y \in X$ with $|x| = |y|$ [13, 56, 62]. For $x \in X \setminus \{\vartheta\}$ and $(x, y) \in \widetilde{\mathcal{E}}_h$, we let

$$c(x, x^-) = \lambda^{-|x|}\mathcal{H}^\alpha(K_x), \qquad c(x, y) = \lambda^{-|x|}\sqrt{\mathcal{H}^\alpha(K_x)\mathcal{H}^\alpha(K_y)}.$$

Then for $x \in X \setminus \{\vartheta\}$,

$$\lambda(x) = \frac{c(x, x^-)}{\sum_{y \in \mathcal{J}_1(x)} c(x, y)} = \frac{\lambda^{-|x|}\mathcal{H}^\alpha(K_x)}{\lambda^{-(|x|+1)}\sum_{y \in \mathcal{J}_1(x)} \mathcal{H}^\alpha(K_y)} = \lambda.$$

To verify condition (p_0), as the OSC is equivalent to the bounded degree property of $(X, \widetilde{\mathcal{E}})$ (Theorem 3.7), let $\ell = \sup_{x \in X} \#\{z : (x, z) \in \widetilde{\mathcal{E}}_h\} < \infty$. Note that for $x \in X_n$, we have $r_*^{n+1} < r_x \le r_*^n$ by (3.1). Therefore

$$\begin{aligned} m(x) &= c(x, x^-) + \sum\nolimits_{y \in \mathcal{J}_1(x)} c(x, y) + \sum\nolimits_{z \in X : (x,z) \in \widetilde{\mathcal{E}}_h} c(x, z) \\ &= \lambda^{-n}\mathcal{H}^\alpha(K) \cdot \big(r_x^\alpha + \lambda^{-1} r_x^\alpha + \sum\nolimits_{z \in X : (x,z) \in \widetilde{\mathcal{E}}_h} r_x^{\alpha/2} r_z^{\alpha/2}\big) \\ &\le \lambda^{-n} r_*^{\alpha n}\mathcal{H}^\alpha(K) \cdot (1 + \lambda^{-1} + \ell), \end{aligned}$$

and $c(x, y) \ge \lambda^{-n} r_*^{\alpha(n+1)}\mathcal{H}^\alpha(K) \min\{1, \lambda^{-1} r_*^\alpha\}$ for $y \in X$ with $(x, y) \in \widetilde{\mathcal{E}}$.

This proves the (p_0) by $P(x, y) = \frac{c(x,y)}{m(x)} \ge r_*^\alpha (1 + \lambda^{-1} + \ell)^{-1} \min\{1, \lambda^{-1} r_*^\alpha\}$ for all $(x, y) \in \widetilde{\mathcal{E}}$. Hence such conductance c defines a λ-NRW.

For the second part, let c' be the conductance of a λ-NRW. Write $q_x = c'(x, x^-)\lambda^{|x|}$ for $x \in X \setminus \{\vartheta\}$, and $q_\vartheta = m'(\vartheta)\lambda$. By $\lambda(x) \equiv \lambda$, we have $q_x = \sum_{y \in \mathcal{J}_1(x)} q_y$ for all $x \in X$. For $n \ge 0$ and a Borel set $E \subset K$, define

$$\mu_n(E) = \inf\{\sum\nolimits_{i=1}^{\ell} q_{x_i} : E \subset \bigcup\nolimits_{i=1}^{\ell} K_{x_i},\ x_i \in X_n\}.$$

Clearly $\mu_n(E)$ is decreasing in n. Let $\mu(E) = \lim_{n\to\infty} \mu_n(E)$. Then it is standard to check that μ is a regular Borel measure on K, and from the OSC, it follows that

$$\mu_n(K_x) = \sum\nolimits_{y \in \mathcal{J}_{n-|x|}(x)} q_y = q_x, \qquad \forall\, x \in X,\ n \ge |x|,$$

therefore $\mu(K_x) = q_x$ for all $x \in X$. Denote $U_x = K \setminus \big(\bigcup_{y\in X_{|x|}\setminus\{x\}} K_y\big)$. Then $U_x \subset K_x$, we have

$$q_x = \mu(K) - \sum\nolimits_{y\in X_{|x|}\setminus\{x\}} \mu(K_y) \le \mu(U_x) \le \mu(K_x) = q_x.$$

Hence $\mu(U_x) = \mu(K_x) = q_x$, and this proves (5.5).

Finally we show that μ is doubling. Let $p_* = \inf_{(x,y)\in\widetilde{\mathcal{E}}} P(x,y)$ (> 0 by the (p_0)). Note that for $x \in X \setminus \{\vartheta\}$,

$$q_{x^-} < m'(x^-)\lambda^{|x|-1} = \frac{c'(x^-,x)\lambda^{|x|-1}}{P(x^-,x)} \le \frac{q_x}{\lambda p_*}, \tag{5.6}$$

and for $(x,y) \in \widetilde{\mathcal{E}}_h$,

$$q_y < m'(y)\lambda^{|y|} = \frac{c'(y,x)\lambda^{|y|}}{P(y,x)} < \frac{m'(x)\lambda^{|x|}}{P(y,x)} = \frac{c'(x,x^-)\lambda^{|x|}}{P(y,x)P(x,x^-)} \le \frac{q_x}{p_*^2}. \tag{5.7}$$

Suppose $\xi \in K$ and $0 < r \le |K|$. Let n_1 be the integer such that $r_*^{n_1}|K| < r \le r_*^{n_1-1}|K|$. Choose $x \in X_{n_1}$ such that $\xi \in K_x$. As $|K_x| \le r_*^{n_1}|K| < r$, we have $K_x \subset B(\xi, r)$, and

$$\mu(B(\xi,r)) \ge \mu(K_x) = q_x.$$

Let $\gamma > 0$ be as in (3.2). If $\gamma \le 2r$, then from the choice of n_1 we see that $\gamma \le 2|K|r_*^{n_1-1}$, therefore $n_1 \le \frac{\log(\gamma/2|K|)}{\log r_*} + 1 := m_0$. Using (5.6) repeatedly, we have

$$\mu(B(\xi,2r)) \le \mu(K) = q_\vartheta < (\lambda p_*)^{-n_1} q_x \le (\lambda p_*)^{-m_0}\mu(B(\xi,r)).$$

If $\gamma > 2r$, we let n_2 be the maximal integer such that $\gamma \cdot r_*^{n_2} \ge 2r$ and $n_2 \le n_1$. Then either $\gamma \cdot r_*^{n_2+1} < 2r \le 2|K|r_*^{n_1-1}$ or $n_2 = n_1$ holds true, which implies $0 \le n_1 - n_2 \le \max\{0, \frac{\log(\gamma/2|K|)}{\log r_*} + 2\} =: m_1$. Let u be the unique $(n_1 - n_2)$-th predecessor of x. Denote $T_u = \bigcup_{v\in X: u\sim_h v} K_v$. For $\eta \notin T_u$, as $|\eta - \xi| \ge \mathrm{dist}(\eta, K_u) > \gamma \cdot r_*^{n_2} \ge 2r$, we see that $B(\xi, 2r) \subset T_u$. It follows from (5.7) and (5.6) that

$$\begin{aligned}\mu(B(\xi,2r)) &\le \mu(T_u) = q_u + \sum\nolimits_{v\in X:(u,v)\in\widetilde{\mathcal{E}}_h} q_v \\ &< (1+\ell p_*^{-2})q_u < (1+\ell p_*^{-2})(\lambda p_*)^{-(n_1-n_2)} q_x \\ &\le (1+\ell p_*^{-2})(\lambda p_*)^{-m_1}\mu(B(\xi,r)).\end{aligned}$$

Hence μ is doubling, and completes the proof. □

Without loss of generality, we will assume that the doubling measure μ in (5.4) satisfies $\mu(K) = 1$. For a λ-NRW $\{Z_n\}$ on $(X, \widetilde{\mathcal{E}})$, by applying Theorems 3.2 and 5.3, the Martin boundary $\mathcal{M} = \mathcal{M}_{\min}$, and is homeomorphic to the hyperbolic boundary ∂X as well as the attractor K. From now on we will identify K with $\mathcal{M}$, and regard the hitting distribution ν as a probability measure on K. Using (5.3) and a time reversal argument on $\{Z_n\}$, we have

$$F(\vartheta, x) \asymp \mathbb{P}_\vartheta(Z_{\tau_m} = x) = \mu(K_x), \qquad \forall\ x \in X_m,\ m \ge 0, \tag{5.8}$$

where $\tau_m = \inf\{n \ge 0 : Z_n \in X_m\}$ is the *first hitting time* for X_m (see [38] for details).

Theorem 5.7. *Let $\{Z_n\}$ be a λ-NRW on an augmented tree $(X, \widetilde{\mathcal{E}})$ associated to a self-similar set K. Then $\mathcal{M}$, ∂X and K are homeomorphic. Moreover, the hitting distribution ν equals the probability doubling measure μ given in* (5.4).

Proof. We only need to prove $\nu = \mu$. Let U_x and T_x be as in the proof of the above theorem. We fix a projection $\iota : X \to K$ that satisfies $\iota(x) \in U_x$ for each $x \in X$, and let

$$T_x^{(m)} = \bigcup\nolimits_{y \in \mathcal{J}_m(x)} T_y, \qquad x \in X,\ m \geq 0.$$

Then it is clear that $\bigcap_{m=0}^{\infty} T_x^{(m)} = K_x$. For any m fixed, as $\mathrm{dist}(K \setminus T_x^{(m)}, K_x) > 0$, the event $Z_\infty \in K_x$ implies that $\iota(Z_n)$ lies eventually in $T_x^{(m)}$. Using Fatou's lemma together with (5.5) and (5.8), we have

$$\begin{aligned}
\nu(K_x) = \mathbb{P}_\vartheta(Z_\infty \in K_x) &\leq \mathbb{E}_\vartheta\big(\liminf_{n\to\infty} \chi_{T_x^{(m)}}(\iota(Z_n))\big) \\
&\leq \liminf_{n\to\infty} \mathbb{P}_\vartheta(\iota(Z_n) \in T_x^{(m)}) \\
&\leq \liminf_{\ell\to\infty} \mathbb{P}_\vartheta(\iota(Z_{\tau_\ell}) \in T_x^{(m)}) = \mu(T_x^{(m)}).
\end{aligned}$$

Letting $m \to \infty$, it follows $\nu(K_x) \leq \mu(K_x)$ for all $x \in X$, which implies $\nu(F) \leq \mu(F)$ for any Borel set $F \subset K$; the same "$\leq$" holds for $K \setminus F$. Hence $\nu(F) = \mu(F)$, and completes the proof. $\square$

For distinct $\xi, \eta \in X \cup K (\approx \widehat{X})$, we define

$$p_\mu(\xi, \eta) = \sup\{\mu(K_z) : z \in X \text{ and lies on some geodesic } \pi(\xi, \eta)\}.$$

It is easy to see that the supremum can be reached at a vertex on the horizontal segment of some convex geodesic between ξ and η. Moreover, this $p_\mu(\cdot, \cdot)$ satisfies the estimate

$$p_\mu(\xi, \eta) \asymp V(\xi, \eta) := \mu(B(\xi, |\xi - \eta|)), \qquad \forall\, \xi, \eta \in K,\ \xi \neq \eta. \tag{5.9}$$

Theorem 5.8. *Let $\{Z_n\}_{n=0}^{\infty}$ be a λ-NRW on an augmented tree $(X, \widetilde{\mathcal{E}})$. Then*

$$F(x, y) \asymp \lambda^{|x| - (x|y)} \mu(K_y) p_\mu(x, y)^{-1}, \qquad \forall\, x, y \in X.$$

Consequently, the Martin kernel satisfies the estimate

$$K(x, \eta) \asymp \lambda^{|x| - (x|\eta)} p_\mu(x, \eta)^{-1}, \qquad \forall\, x \in X,\ \eta \in X \cup K.$$

Proof. For $x, y \in X$, let $\pi(x, u, v, y)$ be a convex geodesic on which a vertex w lies on the horizontal segment $\pi(u, v)$ with $\mu(K_w) = p_\mu(x, y)$. By observing that $F(x, z) \geq F(x, z)F(z, y)$ for $z \in X$ together with (5.2) in Theorem 5.3, we have

$$F(x, y) \asymp F(x, u)F(u, v)F(v, y) \asymp \frac{F(x, \vartheta)}{F(u, \vartheta)} \cdot F(u, v) \cdot \frac{F(\vartheta, y)}{F(\vartheta, v)}.$$

As the length of $\pi(u,v)$ does not exceed L (Theorem 2.6), we see that $p_*^L \le F(u,v) \le 1$ (here $p_* := \inf_{(x,y)\in\widetilde{\mathcal{E}}} P(x,y) > 0$), and the volume doubling property of μ implies that $\mu(K_v) \asymp \mu(K_w) = p_\mu(x,y)$. From the estimates (5.3) and (5.8), it follows that

$$F(x,y) \asymp \lambda^{|x|-|u|}\mu(K_y)\mu(K_v)^{-1} \asymp \lambda^{|x|-(x|y)}\mu(K_y)p_\mu(x,y)^{-1}.$$

This leads to the estimate $K(x,y) = \frac{F(x,y)}{F(\vartheta,y)} \asymp \lambda^{|x|-(x|y)}p_\mu(x,y)^{-1}$ for $x,y \in X$, and passing the limit along some ray $[y_i]_i$ that converges to η, it can be extended to $X \times K$. □

From the above estimates, it is easy to see that $\Theta(x,\eta) = \frac{K(x,\eta)}{F(x,\vartheta)G(\vartheta,\vartheta)} \asymp \lambda^{-(x|\eta)}p_\mu(x,\eta)^{-1}$ for all $x \in X$ and $\eta \in K$. By using a similar technique as in [38, Theorem 6.3], we can analyze the limit in (5.1), and extend such Naïm kernel estimate to $K \times K$.

Theorem 5.9. *Let $\{Z_n\}_{n=0}^\infty$ be a λ-NRW on an augmented tree $(X,\widetilde{\mathcal{E}})$. Then the Naïm kernel satisfies the estimate*

$$\Theta(\xi,\eta) \asymp \lambda^{-(\xi|\eta)}p_\mu(\xi,\eta)^{-1}, \qquad \forall\ \xi,\eta \in K,\ \xi \ne \eta.$$

Consequently, by (3.3) *and* (5.9)*, we have*

$$\Theta(\xi,\eta) \asymp \frac{1}{V(\xi,\eta)|\xi-\eta|^\beta}, \qquad \forall\ \xi,\eta \in K,\ \xi \ne \eta,$$

where μ is the doubling measure associated with the λ-NRW as in (5.4)*, $V(\xi,\eta) := \mu(B(\xi,|\xi-\eta|))$, and $\beta = \frac{\log\lambda}{\log r_*}$.*

In particular, if μ is chosen to be the normalized α-Hausdorff measure on K (where α is the Hausdorff dimension of K), then $\mu(B(\xi,r)) \asymp r^\alpha$ for all ball $B(\xi,r) \subset K$, and the above estimate becomes $\Theta(\xi,\eta) \asymp |\xi-\eta|^{-(\alpha+\beta)}$ [38]. Applying Silverstein's Theorem (Theorem 5.2) together with Theorems 5.7 and 5.9, we get

Theorem 5.10. *Let $\{Z_n\}_{n=0}^\infty$ be a λ-NRW on an augmented tree $(X,\widetilde{\mathcal{E}})$. Then the induced energy form $(\mathfrak{E}_K,\mathcal{D}_K)$ satisfies*

$$\mathfrak{E}_K[u] := \mathfrak{E}_X[Hu] \asymp \iint_{K\times K\setminus\Delta} \frac{|u(\xi)-u(\eta)|^2}{V(\xi,\eta)|\xi-\eta|^\beta}d\mu(\xi)d\mu(\eta), \qquad \forall\ u \in \mathcal{D}_K,$$

where $\mathcal{D}_K := \{u \in L^2(K,\mu) : \mathfrak{E}_K[u] < \infty\}$, and $\beta = \frac{\log\lambda}{\log r_*}$.

Remark. The above theorems also hold for $(X,\mathcal{E})$ with property (H), as in this case $\mathcal{E} = \tilde{\mathcal{E}}$ as in Proposition 3.4.

The domain $\mathcal{D}_K$ is equal to a *Besov space* $\Lambda_{2,2}^{\beta/2}$ (see [38,39]); it is decreasing in β, and can be trivial (i.e., consists of only constant functions) when β is large. As $\mathfrak{E}_K$ defines a symmetric bilinear form $\mathfrak{E}_K(\cdot,\cdot)$ via the standard polarization, we are interested in the conditions for $(\mathfrak{E}_K, \mathcal{D}_K)$ to be nontrivial, or becomes a *regular (non-local) Dirichlet form* on $L^2(K,\mu)$ [16]. The key for this is to determine the value of the *critical exponents*

$$\beta^\sharp := \sup\{\beta > 0 : \dim(\Lambda_{2,2}^{\beta/2} \cap C(K)) > 1\}, \qquad \text{and}$$
$$\beta^* := \sup\{\beta > 0 : \Lambda_{2,2}^{\beta/2} \cap C(K) \text{ is dense in } C(K) \text{ with the supremum norm}\},$$

where $C(K)$ is the family of all continuous functions on K. It is known that $2 \le \beta^* \le \beta^\sharp$ in general; for classical domains in $\mathbb{R}^d$ with Lebegue measure, $\beta^* = \beta^\sharp = 2$; for Cantor-type sets, $\beta^* = \beta^\sharp = \infty$; for the d-dimensional Sierpiński gasket with α-Hausdorff measure (here $\alpha = \frac{\log(d+1)}{\log 2}$ is the Hausdorff dimension), $\beta^* = \beta^\sharp = \frac{\log(d+3)}{\log 2}$ [28]; some examples with $\beta^* < \beta^\sharp$ are provided in [20, 36]. Moreover, if K satisfies a *chain condition* in [18], then $\beta^* \le \beta^\sharp \le \bar{d}_\mu + 1$, where $\bar{d}_\mu$ is the *upper dimension* given by

$$\bar{d}_\mu = \inf\{\alpha > 0 : \exists\, c > 0 \text{ such that } \mu(B(\xi, r)) \ge c r^\alpha \ \forall\, \xi \in K \text{ and } r \in (0,1)\}.$$

It is also known that $\mathcal{D}_K \subset C(K)$ when $\beta > \bar{d}_\mu$ (i.e., $\lambda < r_*^{\bar{d}_\mu}$). As a consequence, in the case that $\beta^* > \bar{d}_\mu$, $(\mathfrak{E}_K, \mathcal{D}_K)$ is a regular Dirichlet form for any $\beta \in (\bar{d}_\mu, \beta^*)$.

We provide an approach to these critical exponents by using the networks of NRWs (see details in [36]). For each $\lambda \in (0,1)$, in view of Theorem 5.6, we fix a conductance $c^{(\lambda)}(\cdot,\cdot)$ on $(X, \widetilde{\mathcal{E}})$ that defines a λ-NRW with a given doubling measure μ:

$$c^{(\lambda)}(x, x^-) = \lambda^{-|x|}\mu(K_x),\ x \in X \setminus \{\vartheta\}, \quad c^{(\lambda)}(x,y) = \lambda^{-|x|}\sqrt{\mu(K_x)\mu(K_y)},\ (x,y) \in \widetilde{\mathcal{E}}_h. \tag{5.10}$$

For $m \ge 1$, by restricting the graph energy to $\bigcup_{i=0}^m X_i$, we let

$$\mathfrak{E}_{X,m}^{(\lambda)}[f] = \frac{1}{2}\sum\nolimits_{x,y\in X: |x|, |y| \le m} c^{(\lambda)}(x,y)|f(x) - f(y)|^2$$

for a real function f on X, and define the *level-m resistance* by

$$R_m^{(\lambda)}(x,y) = \big(\inf\{\mathfrak{E}_{X,m}^{(\lambda)}[f] : f(x) = 1,\ f(y) = 0\}\big)^{-1}, \qquad x, y \in X_m.$$

To represent the resistance on K by a limit, we choose a sequence $\{\kappa_m\}_{m=0}^\infty$, in which κ_m is a map from K to X_m, satisfying that for any $\xi \in K (\approx \partial X)$, $\{\kappa_m(\xi)\}_{m=0}^\infty$ is a geodesic ray converging to ξ. Define

$$R^{(\lambda)}(\xi, \eta) := \liminf_{m\to\infty} R_m^{(\lambda)}(\kappa_m(\xi), \kappa_m(\eta)), \qquad \xi, \eta \in K.$$

With the assumption $\lambda < r_*^{\bar{d}_\mu}$, it can be proved that the above limit always exists (hence the "lim inf" can be replaced by "lim"), and is independent of the choice of $\{\kappa_m\}$; in this case, $R^{(\lambda)}(\xi,\eta) > 0$ if and only if there exists $u \in \mathcal{D}_K$ such that $u(\xi) \ne u(\eta)$.

We will further assume that the measure μ is *self-similar*, i.e., $\mu(\cdot) = \sum_{j\in\Sigma} p_j \mu(S_j^{-1}(\cdot))$ for some set $\{p_j\}_{j\in\Sigma}$ of positive probability weights. For $j \in \Sigma$, denote by j^∞ the unique fixed point of the contractive similitude S_j, i.e., $\{j^\infty\} = \bigcap_{n=0}^\infty K_{j^n}$.

Theorem 5.11. *Suppose $\{S_j\}_{j\in\Sigma}$ is an IFS of contractive similitudes satisfying the OSC, and μ is a doubling self-similar measure on the attractor K. For $\lambda \in (0,1)$, let $\{Z_n\}$ be the λ-NRW on $(X, \widetilde{\mathcal{E}})$ defined by the conductance $c^{(\lambda)}(\cdot,\cdot)$ as in (5.10). Then $\mathcal{D}_K \cap C(K)$ consists of only constant functions if $R^{(\lambda)}(i^\infty, j^\infty) = 0$ for all $i,j \in \Sigma$, and the converse is also true for $\lambda \in (0, r_*^{\bar{d}_\mu})$.*

Consequently, $\beta^\sharp = \frac{\log \lambda^\sharp}{\log r_*}$ if

$$\lambda^\sharp := \sup\{\lambda > 0 : R^{(\lambda)}(i^\infty, j^\infty) = 0,\ \forall\, i,j \in \Sigma\} \in (0, r_*^{\bar{d}_\mu}),$$

and $\beta^\sharp = \infty$ if the above set of λ is empty.

We also have a result for β^* when K is a p.c.f. set that satisfies

($\star$) *there exist constants $r_0, C > 0$ such that for any $i,j \in \Sigma$ and $\zeta \in K_i \cap K_j$,*

$$|\xi - \zeta| + |\zeta - \eta| \le C|\xi - \eta| \quad \text{whenever} \quad \xi \in K_i \cap B(\zeta, r_0) \text{ and } \eta \in K_j \cap B(\zeta, r_0).$$

This condition ($\star$) is fulfilled for most of familiar p.c.f. sets including all nested fractals. Let V_0 denote the projection of the post critical set, known as the *boundary* of K.

Theorem 5.12. *With the same assumption as in Theorem 5.11, assume further that K is p.c.f. and satisfies ($\star$). If $\lambda \in (0, r_*^{\bar{d}_\mu})$, and for some $\varepsilon \in (0,\lambda)$,*

$$R^{(\lambda-\varepsilon)}(\xi,\eta) > 0, \qquad \forall\, \xi,\eta \in V_0,\ \xi \ne \eta,$$

then $\mathcal{D}_K$ is dense in $C(K)$, and hence $(\mathfrak{E}_K, \mathcal{D}_K)$ is a regular non-local Dirichlet form.

Consequently, $\beta^* = \frac{\log \lambda^*}{\log r_*}$ if

$$\lambda^* := \inf\{\lambda > 0 : R^{(\lambda)}(\xi,\eta) > 0,\ \forall\, \xi,\eta \in V_0,\ \xi \ne \eta\} \in [0, r_*^{\bar{d}_\mu}),$$

and $\beta^* \le \frac{\log \lambda^*}{\log r_*}$ otherwise.

6 Expansive hyperbolic graphs

We establish a class of graphs called *expansive hyperbolic graphs*, which covers the various augmented trees considered, and includes cases not governed by the IFS, like refinement systems. It has the potential to have broader applications.

Definition 6.1. *We call a rooted graph $(X, \mathcal{E})$ an* expansive graph *if it satisfies for $x, y \in X$ with $|x| = |y|$,*

$$d_h(x,y) > 1 \ \Rightarrow\ d_h(u,v) > 1, \quad \forall\, u \in \mathcal{J}_1(x),\ v \in \mathcal{J}_1(y),$$

or equivalently if each $u \sim_h v$ with $u \in \mathcal{J}_1(x)$ and $v \in \mathcal{J}_1(y)$ implies that $x \sim_h y$.

It is easy to see that the above condition is also equivalent to

$$\max\{d_h(u,v),1\} \geq d_h(x,y), \quad u \in \mathcal{J}_1(x),\ v \in \mathcal{J}_1(y). \tag{6.1}$$

Intuitively, in an expansive rooted graph the children are drifted farther apart than their non-neighboring parents. Note that the expansive property is also equivalent to property (*) in Lemma 2.5 if the vertical subgraph $(X, \mathcal{E}_v)$ is a tree.

There are important cases that the vertical parts of expansive graphs are not trees: for example, the treatment of the IFS with a weak separation condition by taking quotients of vertices on the augmented tree $(X, \mathcal{E})$ [58]. By the same argument as in Lemma 2.5, we see that any two vertices $x, y \in X$ can be connected by a convex geodesic. To study the hyperbolicity of $(X, \mathcal{E})$, we introduce one more definition.

Definition 6.2. *Let m, k be two positive integers. A rooted graph $(X, \mathcal{E})$ is said to be* (m,k)-departing *if for $x, y \in X$,*

$$d_h(x,y) > k \ \Rightarrow \ d_h(u,v) > 2k, \quad \forall\ u \in \mathcal{J}_m(x),\ v \in \mathcal{J}_m(y).$$

It follows from the definitions that every $(1,1)$-departing graph is expansive; every rooted tree is (m,k)-departing for any m, k. However, an infinite expansive graph may not be (m,k)-departing for any m, k. It is direct to check that

$$(m,k)\text{-departing} \ \Rightarrow \ (m,\ell k)\text{-departing, and } (\ell' m, k)\text{-departing} \quad \forall\ \ell, \ell' > 1. \tag{6.2}$$

In particular, $(1,1)$-departing implies (m,k)-departing for any $m, k \geq 1$. As an example, we can show that the augmented tree $(X, \mathcal{E})$ of the Sierpinski gasket (see [30]) is $(1,1)$-departing. With a little more work, we can show that the augmented tree $(X, \mathcal{E})$ of the Hata tree (see [32]) is $(2,1)$-departing, but not $(1,1)$-departing.

The (m,k)-departing property provides very useful criteria to check the hyperbolicity.

Theorem 6.3. *Let $(X, \mathcal{E})$ be an expansive graph. Then the following are equivalent.*

(i) $(X, \mathcal{E})$ is hyperbolic;

(ii) $\exists\ L < \infty$ such that the lengths of all h-geodesics are bounded by L;

(iii) $(X, \mathcal{E})$ is (m,k)-departing for some positive integers m, k.

Proof. $(i) \Leftrightarrow (ii)$ follows from a similar proof as in Theorem 2.6.

$(ii) \Rightarrow (iii)$: we claim that $(X, \mathcal{E})$ is $(L+1, L+2)$-departing. Indeed, let $x, y \in X$, $x' \in \mathcal{J}_{L+1}(x)$ and $y' \in \mathcal{J}_{L+1}(y)$ satisfying $L + 2 < d_h(x', y') \leq 2(L+2)$ (see Figure 7). By the expansive property (as in Proposition 2.5), there exists a convex geodesic $\pi(x', u, v, y')$ between x' and y', and $u \neq x'$ (by the first inequality and (ii)). Let $u, v \in X_j$, then

$$2(L+2) \geq d_h(x', y') > d(x', y') = 2(|x'| - |j|) + d_h(u,v) \geq 2(|x'| - j).$$

As $\ell := |x'| - j \le L + 1 = |x'| - |x|$, we have $j \ge |x|$. Let $u' \in \mathcal{J}_*(x) \cap \mathcal{J}_{-*}(x') \cap X_j$ and $v' \in \mathcal{J}_*(y) \cap \mathcal{J}_{-*}(y') \cap X_j$. Since $x' \in \mathcal{J}_\ell(u) \cap \mathcal{J}_\ell(u')$, u and u' are predecessors of x', and we have $u \sim_h u'$ by the expansive property. Similarly $v \sim_h v'$. Hence by (6.1) and (ii),

$$d_h(x,y) \le \max\{d_h(u',v'),1\} \le d_h(u,v) + 2 \le L+2.$$

This proves the claim.

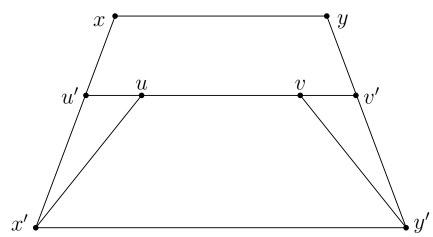

Figure 7: Illustration for the proof of $(ii) \Rightarrow (iii)$.

$(iii) \Rightarrow (ii)$: suppose $(X,\mathcal{E})$ is (m,k)-departing. If (ii) does not hold, then there exists $x, y \in X$, $|x| = |y|$, and horizontal geodesic $\pi(x,y)$ such that $|\pi(x,y)| = 2m + \ell k + 1$, where $\ell > (2m+1)/k$ is an integer. It is clear that $2|x| = d(x,\vartheta) + d(\vartheta,y) \ge d(x,y) > 2m$, which implies $|x| > m$. Let $x^{[-m]} \in \mathcal{J}_{-m}(x)$ and $y^{[-m]} \in \mathcal{J}_{-m}(y)$. Computing the length of the path joining x and y with x to $x^{[-m]}$, horizontally to $y^{[-m]}$, then to y, we have

$$m + d_h(x^{[-m]}, y^{[-m]}) + m \ge d(x,y) = 2m + \ell k + 1.$$

It follows that $d_h(x^{[-m]}, y^{[-m]}) > \ell k$. Hence by (6.2), we have $d_h(x,y) = d(x,y) > 2\ell k$, i.e., $2m + \ell k + 1 > 2\ell k$, which leads to $\ell < (2m+1)/k$, a contradiction. □

We will call the graph in the above theorem an **expansive hyperbolic graph**.

The (m,k)-departing property also provides a useful estimate of the Gromov product. Let $\mathcal{R}_v = \{\mathbf{x} = [x_i]_{i=0}^\infty : x_0 = \vartheta, \text{ and } x_{i+1} \in \mathcal{J}_1(x_i), \ \forall\ i \ge 0\}$ be the set of all rays in $(X,\mathcal{E})$. For any two rays $\mathbf{x}, \mathbf{y} \in \mathcal{R}_v$, we define $|\mathbf{x} \vee \mathbf{y}|_j = \sup\{i \ge 0 : d_h(x_i, y_i) \le j\}$.

Lemma 6.4. *Suppose $(X,\mathcal{E})$ is expansive and (m,k)-departing. Then there exists $D_0 > 0$ (depends on m,k) such that*

$$\big|(\mathbf{x}|\mathbf{y}) - |\mathbf{x} \vee \mathbf{y}|_k\big| \le D_0, \qquad \forall\ \mathbf{x}, \mathbf{y} \in \mathcal{R}_v.$$

Moreover, two rays are equivalent if and only if $d_h(x_i, y_i) \le k$ for all i.

On the hyperbolic boundary $(\partial X, \theta_a)$, we define

$$\mathcal{J}_\partial(x) = \{\xi \in \partial X : \exists \text{ ray } \pi(\vartheta, \cdots, x, \cdots) \text{ that converges to } \xi\}, \qquad x \in X,$$

to be the set of descendants of x in ∂X. Under the Gromov metric θ_a, $|\mathcal{J}_\partial(x)| \le Ce^{-a|x|}$ and is compact. This $\mathcal{J}_\partial(x)$ acts as the K_x in the augmented tree of the IFS. By using Lemma 6.4, we have an analog of the condition (H) in Section 3.

Proposition 6.5. *Let $(X, \mathcal{E})$ be an (m, k)-departing expansive graph. Then there exists a constant $\gamma > 0$ (depends on a) such that for $x, y \in X_n, n \geq 1$,*

$$d_h(x, y) > k \quad \Rightarrow \quad \mathrm{dist}_{\theta_a}(\mathcal{J}_\partial(x), \mathcal{J}_\partial(y)) > \gamma e^{-an}.$$

A metric space (M, ρ) is called a *doubling metric space* [24] if there exists an integer $\ell > 0$ such that for any $\xi \in M$ and $r > 0$, the ball $B_\rho(\xi, r)$ can be covered by a union of not more than ℓ balls of radius $r/2$. We prove the following interesting theorem using Theorem 6.3, Lemma 6.4 and Proposition 6.5.

Theorem 6.6. *Suppose $(X, \mathcal{E})$ is a hyperbolic expansive graph and has bounded degree. Then the hyperbolic boundary $(\partial X, \theta_a)$ is doubling.*

In the rest of this section, we will consider a generalization of the augmented trees. Fix a complete metric space (M, ρ), and let $\mathcal{C}_M$ denote the family of all nonempty compact subsets of M. By our convention in Section 2, $\mathcal{J}_\partial(x) \neq \emptyset$ for all $x \in X$.

Definition 6.7. *Let $(X, \mathcal{E}_v)$ be a vertical rooted graph. A map $\Phi : X \to \mathcal{C}_M$ is called an* index map *(on $(X, \mathcal{E}_v)$ over (M, ρ)) if it satisfies*

(i) $\Phi(y) \subset \Phi(x)$ for all $x \in X$ and $y \in \mathcal{J}_1(x)$;

(ii) $\bigcap_{i=0}^\infty \Phi(x_i)$ is a singleton for all $\mathbf{x} = [x_i]_i \in \mathcal{R}_v$.

We call $K := \bigcap_{n=0}^\infty \left(\bigcup_{x \in X_n} \Phi(x)\right)$ the attractor *of Φ, and $K_x := \Phi(x) \cap K$ a* cell *of K. We also call the index map* saturated *if $\Phi(x) = \bigcup_{y \in \mathcal{J}_1(x)} \Phi(y)$.*

Remark 1. For an index map Φ, let $\Phi_\partial(x) := \bigcap_{n=0}^\infty \left(\bigcup_{y \in \mathcal{J}_n(x)} \Phi(y)\right), x \in X$. Then Φ_∂ is a saturated index map, and $\Phi_\partial(x) \subset \Phi(x)$. If Φ is saturated, then $\Phi(x) = K_x = \Phi_\partial(x)$.

The index map Φ defines a mapping $\kappa_0 : \mathcal{R}_v \to K$ by

$$\{\kappa_0(\mathbf{x})\} = \bigcap_{i=0}^\infty \Phi(x_i), \qquad \forall\, \mathbf{x} \in \mathcal{R}_v.$$

Using the local finiteness of $(X, \mathcal{E}_v)$ and a diagonal argument (see [46, 60]), we can show that the image of κ_0 is equal to K. Note that for a hyperbolic expansive graph $(X, \mathcal{E})$, the hyperbolic boundary ∂X can be identified with a quotient set of $\mathcal{R}_v$. Hence the induced $\kappa : \partial X \to K$ is well-defined if κ_0 satisfies: $\kappa_0(\mathbf{x}) = \kappa_0(\mathbf{y})$ provided that $\mathbf{x}$ and $\mathbf{y}$ are equivalent; furthermore $\kappa : \partial X \to K$ is one-to-one if the converse is also satisfied. With these, we see that $\kappa : \partial X \to K$ is a well-defined bijection if

$$\mathbf{x}, \mathbf{y} \text{ are equivalent} \quad \Leftrightarrow \quad \kappa_0(\mathbf{x}) = \kappa_0(\mathbf{y}).$$

Definition 6.8. *We call $(X, \mathcal{E}, \Phi)$ an* admissible index triple *if $(X, \mathcal{E})$ is an expansive hyperbolic graph, $\Phi : X \to \mathcal{C}_M$ is an index map on $(X, \mathcal{E}_v)$ over (M, ρ), and $\kappa : \partial X \to K$ is well-defined and is a bijection. In such case, $(X, \mathcal{E})$ is said to be an* admissible graph *(with respect to Φ).*

Remark 2. For an admissible index triple $(X, \mathcal{E}, \Phi)$, if Φ is saturated, then by Remark 1, we have $\Phi(x) = \Phi_\partial(x) = \kappa(\mathcal{J}_\partial(x))$, and the Gromov metric θ_a implies

$$|\Phi(x)|_{\tilde{\theta}_a} = |\mathcal{J}_\partial(x)|_{\theta_a} \leq Ce^{-a|x|}$$

where $\tilde{\theta}_a$ is the metric on K induced by θ_a via the bijection $\kappa : \partial X \to K$.

For a subset A in (M, ρ), we denote the diameter of A by $|A|_\rho$ (or simply by $|A|$). In Definition 6.7, we see that the family $\{\Phi(x)\}_{x\in X}$ satisfies $\lim_{n\to\infty}\sup_{x\in X_n}|\Phi(x)|_\rho = 0$. For $b \in (0, \infty)$, we say that $\{\Phi(x)\}_{x\in X}$ (or Φ) is of *exponential type*-(b) (under ρ) if the diameter $|\Phi(x)|_\rho$ is decreasing in a rate of $e^{-b|x|}$, i.e., $|\Phi(x)|_\rho = O(e^{-b|x|})$ as $|x| \to \infty$, and call Φ an *exponential type* if it is of type-(b) for some $b \in (0, \infty)$.

The following two classes of rooted graphs are our main consideration of the index triples, which generalize the two augmented trees in Definition 3.1: the exponential type-(b) corresponds to the r_* in (3.2) with $r_* = e^{-b}$.

Definition 6.9. *Let Φ be an index map on the vertical rooted graph $(X, \mathcal{E}_v)$. We define a horizontal edge set by*

$$\mathcal{E}_h^{(\infty)} := \bigcup\nolimits_{n=1}^{\infty} \{(x, y) \in X_n \times X_n \setminus \Delta : \Phi(x) \cap \Phi(y) \neq \emptyset\},$$

and let $\mathcal{E}^{(\infty)} = \mathcal{E}_v \cup \mathcal{E}_h^{(\infty)}$. We call $(X, \mathcal{E}^{(\infty)})$ an AI_∞-graph*,* **augmented index graph of type-**(∞) *(or* intersection type*).*

Suppose in addition Φ is of exponential type-(b). Then for a fixed $\gamma > 0$, we define

$$\mathcal{E}_h^{(b)} := \bigcup\nolimits_{n=1}^{\infty} \{(x, y) \in X_n \times X_n \setminus \Delta : \mathrm{dist}_\rho(\Phi(x), \Phi(y)) \leq \gamma e^{-bn}\}, \tag{6.3}$$

and let $\mathcal{E}^{(b)} = \mathcal{E}_v \cup \mathcal{E}_h^{(b)}$. We call $(X, \mathcal{E}^{(b)})$ an AI_b-graph, **augmented index graph of type-**(b)*.*

It is clear that both $(X, \mathcal{E}^{(b)})$ and $(X, \mathcal{E}^{(\infty)})$ are expansive. First we consider the AI_b-graphs.

Theorem 6.10. *For an index map Φ on $(X, \mathcal{E}_v)$ over (M, ρ) of exponential type-(b), the associated AI_b-graph is $(m, 1)$-departing for some positive integer m, and is an admissible graph. Moreover, $\kappa : (\partial X, \theta_a) \to (K, \rho)$ is a Hölder equivalence, i.e.,*

$$\rho(\kappa(\xi), \kappa(\eta))^{a/b} \asymp \theta_a(\xi, \eta), \qquad \forall\ \xi, \eta \in \partial X. \tag{6.4}$$

Proof. To show that it is $(m, 1)$-departing for some $m \geq 1$, let $\delta_0 := \sup_{z\in X} e^{b|z|}|\Phi(z)|$. Let $u \in \mathcal{J}_m(x)$ and $v \in \mathcal{J}_m(y)$ with $d_h(u, v) = 2$. Using the triangle inequality twice, we have

$$\mathrm{dist}(\Phi(x), \Phi(y)) \leq \mathrm{dist}(\Phi(u), \Phi(v)) \leq (2\gamma + \delta_0)e^{-b(|x|+m)} \leq \gamma e^{-b|x|},$$

where the positive integer m is chosen to give the last inequality, i.e., $(2\gamma+\delta_0)e^{-bm} \le \gamma$. Therefore $x \sim_h y$, and this shows that $(X, \mathcal{E})$ is $(m,1)$-departing. The hyperbolicity of AI_b-graph follows from Theorem 6.3.

By Lemma 6.4 (with $k=1$) and (6.3), we see that two rays $\mathbf{x}, \mathbf{y}$ are equivalent if and only if $\mathrm{dist}(\Phi(x_i), \Phi(y_i)) \le \gamma e^{-bi}$ for all i (i.e., $\kappa_0(\mathbf{x}) = \kappa_0(\mathbf{y})$). This implies $\kappa : \partial X \to K$ is a well-defined bijection, and $(X, \mathcal{E}^{(b)})$ is an admissible graph.

We now prove that κ is a Hölder equivalence. For distinct $\xi, \eta \in \partial X$, we take two rays $\mathbf{x}, \mathbf{y} \in \mathcal{R}_v$ that converge to ξ, η respectively with $(\xi|\eta) = (\mathbf{x}|\mathbf{y})$. Let $n = |\mathbf{x} \vee \mathbf{y}|_1$ as in Lemma 6.4 with $k=1$, i.e., $d_h(x_n, y_n) \le 1$ and $d_h(x_{n+1}, y_{n+1}) \ge 2$. By Lemma 6.4, we have $|(\xi|\eta) - n| = |(\mathbf{x}|\mathbf{y}) - n| \le D_0$ for some $D_0 > 0$. As $\kappa(\xi) \in \Phi(x_{n+1}) \subset \Phi(x_n)$ and $\kappa(\eta) \in \Phi(y_{n+1}) \subset \Phi(y_n)$, we get the lower bound of (6.4) by

$$\begin{aligned}\rho(\kappa(\xi), \kappa(\eta)) &\ge \mathrm{dist}_\rho(\Phi(x_{n+1}), \Phi(y_{n+1})) \\ &\ge \gamma e^{-b(n+1)} \ge \gamma e^{-b(D_0+1)} e^{-b(\xi|\eta)} \ge C_1 \theta_a(\xi,\eta)^{b/a},\end{aligned}$$

and the upper bound by

$$\begin{aligned}\rho(\kappa(\xi), \kappa(\eta)) &\le |\Phi(x_n)| + \mathrm{dist}(\Phi(x_n), \Phi(y_n)) + |\Phi(y_n)| \\ &\le (2\delta_0+\gamma)e^{-bn} \le (2\delta_0+\gamma)e^{bD_0} e^{-b(\xi|\eta)} \le C_2 \theta_a(\xi,\eta)^{b/a}.\end{aligned}$$

This completes the proof. □

Now we turn to the study of the AI_∞-graphs. Unlike the AI_b-graph, the AI_∞-graph is not always hyperbolic.

Proposition 6.11. *Suppose the index map Φ is of exponential type-(b), and the associated AI_∞-graph $(X, \mathcal{E}^{(\infty)})$ is hyperbolic. Then $(X, \mathcal{E}^{(\infty)})$ is an admissible graph, and $\kappa : (\partial X, \theta_a) \to (K, \rho)$ is Hölder continuous, i.e.,*

$$\rho(\kappa(\xi), \kappa(\eta))^{a/b} \le C \theta_a(\xi, \eta), \qquad \forall\ \xi, \eta \in \partial X. \tag{6.5}$$

Proof. Note that $(X, \mathcal{E}^{(\infty)})$ is a subgraph of $(X, \mathcal{E}^{(b)})$. On the AI_b-graph $(X, \mathcal{E}^{(b)})$, let us denote its graph distance and Gromov product by $d'(\cdot,\cdot)$ and $(\cdot|\cdot)'$ respectively. By Theorem 6.10, $\kappa' : (\partial X', \theta'_a) \to K$ is a bijection, and satisfies $\rho(\kappa'(\xi), \kappa'(\eta)) \asymp e^{-b(\xi|\eta)'}$ for all $\xi, \eta \in \partial X'$. As $\kappa = \kappa'$ on $\mathcal{R}_v$, it follows that $\kappa = \kappa'$ on $\partial X = \partial X'$ is a well-defined bijection, and $(X, \mathcal{E}^{(\infty)})$ is admissible.

From $\mathcal{E}^{(\infty)} \subset \mathcal{E}^{(b)}$, it follows that $d(x,y) \ge d'(x,y)$, and

$$(x|y) = \frac{1}{2}(|x| + |y| - d(x,y)) \le \frac{1}{2}(|x| + |y| - d'(x,y)) = (x|y)' \quad \forall\ x, y \in X.$$

Taking limits, we have $(\xi|\eta) \le (\xi|\eta)'$, and

$$\theta_a(\xi,\eta) \ge c_1 e^{-a(\xi|\eta)} \ge c_1 e^{-a(\xi|\eta)'} \ge c_2 \rho(\kappa'(\xi), \kappa'(\eta))^{a/b} = c_2 \rho(\kappa(\xi), \kappa(\eta))^{a/b}.$$

This verifies (6.5), the Hölder continuity of κ. □

The following is a characterization of the Hölder equivalence of ∂X to K for the AI_∞-graph associated to some saturated Φ.

Theorem 6.12. *Suppose Φ is a saturated index map on $(X, \mathcal{E}_v)$ over (M, ρ). Then for $b \in (0, \infty)$ and an integer $k > 0$, the following assertions are equivalent.*

(i) The AI_∞-graph $(X, \mathcal{E}^{(\infty)})$ is (m, k)-departing for some $m > 0$, and $\kappa : (\partial X, \theta_a) \to (K, \rho)$ is a Hölder equivalence with exponent b/a, i.e.,

$$\rho(\kappa(\xi), \kappa(\eta))^{a/b} \asymp \theta_a(\xi, \eta), \qquad \forall\ \xi, \eta \in \partial X.$$

(ii) Φ is of exponential type-(b) under ρ, and there exists $\gamma > 0$ such that $(X, \mathcal{E}^{(\infty)})$ satisfies for $x, y \in X$,

$$|x| = |y| \ \text{ and } \ d_h(x, y) > k \quad \Rightarrow \quad \mathrm{dist}_\rho(\Phi(x), \Phi(y)) > \gamma e^{-b|x|}.$$

Remark 3. For $k = 1$, the above condition (ii) is just the condition (H) on self-similar sets in Section 3. In comparison with Theorem 3.2, the above theorem gives a more complete criterion for the AI_∞-graph on Hölder equivalence of ∂X to K.

In the following, we give two sufficient conditions for hyperbolicity of the AI_∞-graph. We first define two separation conditions. Let Φ be an index map on a vertical graph $(X, \mathcal{E}_v)$ over a complete metric space (M, ρ) with attractor K. We call a map $\iota : X \to K$ a *projection* (with respect to Φ) if it satisfies $\iota(x) \in K_x (:= \Phi(x) \cap K)$ for all $x \in X$.

Definition 6.13. *For $b \in (0, \infty)$, we say that Φ (or $\{K_x\}_{x \in X}$) satisfies*

(i) condition (S_b)*: if for any $c > 0$, there is a constant $\bar{\ell} = \bar{\ell}(c)$ such that*

$$\#\{x \in X_n : K_x \cap F \neq \emptyset\} \leq \bar{\ell}, \quad \forall\ n \geq 0 \text{ and } F \subset M \text{ with } |F|_\rho < ce^{-bn};$$

(ii) condition (B_b)*: if there exist a projection $\iota : X \to K$ and $c_0 \in (0, \infty)$ such that*

$$B_\rho(\iota(x), c_0 e^{-b|x|}) \cap K \subset K_x, \qquad \forall\ x \in X.$$

Note that (S_b) is an analog of (S) in Section 3, and they are equivalent when M is $\mathbb{R}^d$ (see [44, Theorem 2.1(iii),(iv)]) or any other doubling metric space. It is well-known that for the OSC on self-similar sets in $\mathbb{R}^d$, the desired open set O can be chosen to satisfy $O \cap K \neq \emptyset$ [56]. Then by taking $\xi \in O \cap K$, a ball $B_\rho(\xi, r) \subset O$ and $\iota(x) = S_x(\xi)$ for $x \in X$, we see that the OSC implies condition (B_b).

Similar to Theorem 3.5, we have

Theorem 6.14. *Let Φ be an index map with attractor K, and is of exponential type-(b). If either*

(i) condition (S_b) is satisfied; or

(ii) the attractor (K, ρ) is doubling, and condition (B_b) is satisfied,

then the AI_∞-graph is hyperbolic and hence an admissible graph.

By using (S_b), we can also obtain an analog of Theorem 3.7.

Theorem 6.15. *Let Φ be an index map with attractor K, and is of exponential type-(b). Then the AI_b-graph has bounded degree if and only if condition (S_b) is satisfied. Also the (S_b) is sufficient for the AI_∞-graph to have bounded degree.*

7 Remarks and future work

In Section 3, we provided some sufficient conditions for the hyperbolicity of augmented tree $(X, \mathcal{E})$. It is interesting to find an example of a self-similar IFS that gives a non-hyperbolic $(X, \mathcal{E})$. (Note that in [37], we constructed an AI_∞-graph (Definition 6.7) that is not hyperbolic; however, the example is not self-similar.)

In the application of Gromov hyperbolic graphs to the Lipschitz equivalence problem in Section 4, we only showed the case that the self-similar sets have equal contraction ratios. Actually, by some minor modification of the matrix rearrangeable technique, we can also use it to deal with more general self-similar sets, say, IFS with multiple contraction ratios, and even with substantial overlaps [49]. Furthermore, this technique can be used for classification of certain fractal squares with nice overlapping structures [52].

In Section 5, we discussed the λ-NRW on the hyperbolic graph $(X, \widetilde{\mathcal{E}})$ and obtained a non-local regular Dirichlet form $\mathfrak{E}_K$ (i.e., the induced energy form) on the attractor K with domain $\Lambda_{2,2}^{\beta/2}$ where $\beta = \frac{\log \lambda}{\log r_*}$ (Theorem 5.10). There is further functional relationship of the graph energy $\mathfrak{E}_X$ and the induced energy $\mathfrak{E}_K$ studied in [36, Section 3]. By varying the return ratio λ, we obtain a critical exponent λ^*, which is of crucial importance: for the classical examples in analysis on fractals, this value gives another Besov space $\Lambda_{2,\infty}^{\beta^*/2}$ as the domain of the local regular Dirichlet form (LRDF) (or equivalently, Laplacian) [28, 53, 54].

Recently, Gregor'yan and Yang [19] gave an analytic proof of the existence of the LRDF on the Sierpinski carpet using the Γ-convergence of the $\Lambda_{2,2}^{\beta/2}$-norm to $\Lambda_{2,\infty}^{\beta/2}$-norm as $\beta \nearrow \beta^*$. This approach was used in [19, 20, 22, 61] to p.c.f. sets, and some non-p.c.f. sets. It will be interesting to find out any limiting r.w. of λ-NRW as $\lambda \searrow \lambda^*$ that yields $\beta \nearrow \beta^*$. Also it is important to investigate the relationship of the Dirichlet form obtained from this approach of random walks on hyperbolic graphs and the one from the classical discrete approximation in analysis of fractals.

In Section 6, we used index maps to establish the AI_b-graphs as the generalization of the augmented tree $(X, \widetilde{\mathcal{E}})$ in Section 3. It should be possible to consider the λ-NRW on the AI_b-graph, and expect the same estimate for the induced energy forms. Also in here we have not yet discussed the near-isometry between hyperbolic graphs; in fact, the transform allows us to extend the scope to large classes of hyperbolic graphs, and enables us to consider the λ-NRW in these more general hyperbolic graphs. We expect the volume doubling property of the hitting distribution, and a similar estimate for the Naïm kernel in term of the Gromov metric. We will discuss this in detail in a forthcoming paper.

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