A RESTRICTED SUPERPOSITION PRINCIPLE FOR (NON-)LINEAR FOKKER–PLANCK–KOLMOGOROV EQUATIONS ON HILBERT SPACES

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ABSTRACT. We carefully combine three well-known results to ensure joint existence of solutions to Fokker–Planck–Kolmogorov equations and martingale problems allowing us to derive a version (covering a restricted subclass of solutions) of the Ambrosio–Figalli–Trevisan superposition principle that is valid on separable infinite-dimensional Hilbert spaces. Furthermore, we transfer this restricted superposition principle into a nonlinear setting.

1. INTRODUCTION

In [Tre16, Theorem 2.5, p. 7], D. Trevisan proved a superposition principle for linear Fokker–Planck–Kolmogorov equations (short: FPKEs) on $\mathbb{R}^n$, extending the prior seminal work of L. Ambrosio (see [Amb08]) and A. Figalli (see [Fig08]) substantially, assuming only fairly weak regularity and integrability conditions on the coefficients. An infinite-dimensional analogue on $\mathbb{R}^\infty$, equipped with the product topology (i.e. solutions of the martingale problem are probability measures on $C([0,T];\mathbb{R}^\infty)$), can be found in [Tre14, Section 7.1].

This article will focus on the Hilbert space case with respect to the norm topology, which is in itself a very different approach to the problem. In general, we have to impose stricter (but still commonly used) compactness assumptions to ensure that the constructed martingale solutions are supported on a path space with values in a separable Hilbert space and continuity with respect to the norm topology of e.g. another, larger separable Hilbert space instead of on $C([0,T];\mathbb{R}^\infty)$ with its componentwise continuity.

First of all, this direction for a generalization is interesting because many applications typically have their setting in Hilbert spaces for which the superposition principle on $\mathbb{R}^\infty$ is insufficient. Furthermore, the connection between probability solutions to FPKEs and martingale solutions in the original sense of Stroock–Varadhan (see [SV79]) to the corresponding martingale problem via the superposition principle is of most scientific value in a setting without uniqueness. Having uniqueness, the probability solution could directly be generated from the martingale solution through its time-marginal laws by simply setting $\mu_t := P \circ x(t)^{-1}$ and would, of course, coincide with any constructed solution to an FPKE. A non-unique Hilbert space setting is most prominently the case for d-dimensional stochastic Navier–Stokes equations (short: SNSEs) making it a prime candidate for an application of the methods studied.

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In the first part of this work (i.e. Sections 2–5) we will, however, take a step back from directly considering a superposition principle by instead combining and adapting well-known approaches from the literature ensuring "joint" existence (see Theorem 3.1 below) of our desired solutions. To be more precise, we use that, on the one hand, the authors of [BDRS15] (see also [BKRS15, Section 10.4]) construct a probability solution to an infinite-dimensional Cauchy problem as a weak limit of finite-dimensional solutions. On the other hand, in the article [GRZ09] (see also [RZZ15] for a more refined proof based on the same techniques in a setting with delay), the authors construct a martingale solution to a corresponding infinite-dimensional martingale problem on a separable Hilbert space in a very similar way using Galerkin approximations. This way, we will in particular answer the question if and how those constructions are linked.

In the second part, i.e. Section 6, we will use the scheme of proof presented in Section 5 to derive our version, i.e. a restricted version for a subclass of solutions, of a superposition principle as well as discuss implications for an application of Theorem 3.1 to SNSEs. Let us note that all four references [GRZ09], [RZZ15], [BDRS15], [BKRS15] feature SNSEs as an application making it an obvious choice for the latter in our case.

Before proceeding, we should point out that in general the superposition principle on a separable Hilbert space $\mathbb{H}$ does not always hold, allowing us to realize that a simple and direct adaption of Trevisan's result is not to be expected.

Example. Let $\mathbb{H} = \ell_2$ and let the Kolmogorov operator L be given by the generator of an Ornstein–Uhlenbeck process in ℓ_2 with a constant diffusion coefficient and a (unbounded) linear drift b, that satisfies Fernique's necessary and sufficient condition (guaranteeing the non-continuity of sample paths, see [Fer75, Theorem 1.3.2, p. 11] for the original theorem). Then the corresponding FPKE even has a stationary solution $\mu_t = \mu$ for every $t \geq 0$ with $\mu \in \mathcal{P}(\ell_2)$, but there exists no probability measure P on $C([0,\infty);\ell_2)$ with time-marginals equal to μ for every t.

Hence, the main result of this part (see Corollary 6.2 below) is only a restricted superposition principle on $\mathbb{H}$ to a subclass of solutions. In short, this means that for any given probability solution μ to an infinite-dimensional Cauchy problem, for which there already exists a subsequence of finite-dimensional solutions being created by Galerkin approximations and converging weakly to μ as well as the necessary integrability conditions and assumptions for the corresponding martingale problem, we immediately obtain a martingale solution P to the infinite-dimensional martingale problem satisfying $P \circ x(t)^{-1} = \mu_t$ for every $t \in [0,T]$. Obviously, we are interested in proving this restricted superposition principle for larger or at least easier to identify subclasses of solutions to an FPKE than a family of solutions that can be represented as limits of some certain subsequences. But, since this requires further research, we see our result as a first step and "proof of concept" in that direction.

In the third part (i.e. Sections 7 and 8) we adapt Corollary 6.2 to a nonlinear version of the restricted superposition principle on $\mathbb{H}$ (see Theorem 8.1 below). We make use of the idea in [BR20] and [BR18, Section 2] on "freezing" of a nonlinear solution. This means, that if we are given a probability solution μ to a nonlinear FPKE

$$\partial_t \mu = L_\mu^* \, \mu,$$

we fix this μ and consider the linear FPKE

$$\partial_t \varrho = L_\mu^* \, \varrho$$

for which μ is a particular solution. This allows us to apply results for linear Cauchy problems, but now with coefficients depending on some fixed measure μ_t. In our case, we will assume that the assumptions on our coefficients are uniform in the measure-component (see Subsection 7.3 below) and, hence, satisfy all assumptions necessary for Corollary 6.2. What's more, the martingale solution that we obtain for the martingale problem with coefficients $b(\cdot,\cdot,\mu_\cdot)$ and $\sigma(\cdot,\cdot,\mu_\cdot)$ is connected to McKean–Vlasov SDEs.

2. Framework

First, let us introduce the framework obtained by carefully combining both settings from [GRZ09, RZZ15] (in the simplified case where $\mathbb{Y}=\mathbb{H}$ and on a time interval $[0,T]$ instead of $[0,\infty)$) and [BDRS15].

Let $T>0$ and let $\mathbb{H}$ be a separable Hilbert space with inner product $\langle\cdot,\cdot\rangle_{\mathbb{H}}$ and norm $\|\cdot\|_{\mathbb{H}}$. Recall that all infinite-dimensional separable Hilbert spaces are isometrically isomorphic to ℓ^2 (see e.g. [Bre11, Remark 10, p. 144]) and that we can treat ℓ^2 as a subspace of $\mathbb{R}^\infty$, where $\mathbb{R}^\infty$, equipped with the product topology, is a Polish space. This means we consider the continuous and dense embedding

$$\ell^2 \subseteq \mathbb{R}^\infty$$

as it is done in [BDRS15]. Let $\{e_1,e_2,\dots\}$ be the standard orthonormal basis in ℓ^2. Then we define $\mathbb{H}_n := \operatorname{span}\{e_1,\dots,e_n\}$, for $n\in\mathbb{N}$.

To study martingale problems on $\mathbb{H}$, we introduce another separable Hilbert space $\mathbb{X}$ for which the embedding

$$\mathbb{X} \subseteq \mathbb{H} \simeq \mathbb{H}^* \subseteq \mathbb{X}^*$$

is continuous, dense and compact. Here, we write $\mathbb{X}^*$ for the dual space of $\mathbb{X}$ and $\|\cdot\|_{\mathbb{X}^*}$, $\|\cdot\|_{\mathbb{X}}$ denote their respective norms. In order to apply the methods of [GRZ09], we have to ensure that $\{e_1,e_2,\dots\}\subseteq\mathbb{X}$ holds and we have $\|\Pi_n z\|_{\mathbb{X}^*}\le\|z\|_{\mathbb{X}^*}$ for every $z\in\mathbb{X}^*$. Here, the projection $\Pi_n\colon\mathbb{X}^*\longrightarrow\mathbb{H}_n$ is defined by

$$\Pi_n z := \sum_{i=1}^{n} {}_{\mathbb{X}^*}\langle z,e_i\rangle_{\mathbb{X}}\, e_i, \quad z\in\mathbb{X}^*.$$

Therefore, we follow [AR89, Proposition 3.5, p. 424] and [Bre11, Remark 3, p. 136f] (in particular using that $\mathbb{X}\subseteq\mathbb{H}$ is compact) and identify $\mathbb{X}$ with the weighted ℓ^2-space $\ell^2(\lambda_i)$ for some sequence $(\lambda_i)_{i\in\mathbb{N}}$ with $\lim\limits_{i\to\infty}\lambda_i=\infty$ and $\lambda_i\ge 0$. By considering its dual $\ell^2\big(\frac{1}{\lambda_i}\big)$ we arrive at the embedding

$$\ell^2(\lambda_i)\subseteq\ell^2\subseteq\ell^2\big(\tfrac{1}{\lambda_i}\big)\subseteq\mathbb{R}^\infty,$$

where the dual pairing between $\ell^2(\lambda_i)$ and $\ell^2\big(\frac{1}{\lambda_i}\big)$ is given by

$${}_{\mathbb{X}^*}\langle z,v\rangle_{\mathbb{X}} = \sum_{i=1}^{\infty} z^i v^i,$$

for any $z\in\mathbb{X}^*$ and $v\in\mathbb{X}$.

Remark. It follows from Kuratowski's theorem (see e.g. [Kur66, p. 487f] or [Par67, Section I.3, p. 15ff]) that we have $\mathbb{X}\in\mathcal{B}(\mathbb{H})$, $\mathbb{H}\in\mathcal{B}(\mathbb{X}^*)$, $\mathbb{X}^*\in\mathcal{B}(\mathbb{R}^\infty)$ and $\mathcal{B}(\mathbb{X})=\mathcal{B}(\mathbb{H})\cap\mathbb{X}$, $\mathcal{B}(\mathbb{H})=\mathcal{B}(\mathbb{X}^*)\cap\mathbb{H}$, $\mathcal{B}(\mathbb{X}^*)=\mathcal{B}(\mathbb{R}^\infty)\cap\mathbb{X}^*$.

Remark. We see that the projection Π_n onto $\mathbb{H}_n$ in $\mathbb{X}^*$ in fact simplifies to

$$\Pi_n z = \sum_{i=1}^{n} {}_{\mathbb{X}^*}\langle z, e_i\rangle_{\mathbb{X}}\, e_i = \sum_{i=1}^{n} z^i e_i = (z^1, \dots, z^n, 0, \dots)$$

for any $z \in \mathbb{X}^*$.

In addition, let $\mathbb{U}$ be another separable Hilbert space with inner product $\langle\cdot,\cdot\rangle_{\mathbb{U}}$ and norm $\|\cdot\|_{\mathbb{U}}$.

Path-space: Let

$$\Omega := C\big([0,T]; \mathbb{X}^*\big)$$

be the space of all continuous functions from $[0,T]$ to $\mathbb{X}^*$. By $x\colon \Omega \longrightarrow \mathbb{X}^*$ we denote the canonical process on Ω given by $x(t,\omega) := \omega(t)$. We define the σ-algebra

$$\mathcal{F}_t := \sigma\big(x(s) \,\big|\, s \in [0,t]\big)$$

for every $t \in [0,T]$. For every $n \in \mathbb{N}$ we set

$$\Omega_n := C([0,T]; \mathbb{H}_n).$$

Furthermore, we denote by x_n the canonical process on Ω_n given by $x_n(t,\omega) := \omega(t)$ and define

$$\mathcal{F}_t^{(n)} := \mathcal{B}(C([0,t]; \mathbb{H}_n)).$$

Further spaces: Define

$$\mathbb{S} := C\big([0,T]; \mathbb{X}^*\big) \cap L^p([0,T]; \mathbb{H}),$$

where the $p \geq 2$ is later to be specified in our assumptions (see Subsection 2.3 below). Note that $\mathbb{S}$ is a Polish space.

We define the following classes of so-called finitely based functions given by

$$\begin{aligned} \mathcal{F}C^2(\{e_i\}) &:= \big\{f\colon \mathbb{R}^\infty \longrightarrow \mathbb{R} \,\big|\, f(y) = g\big(y^1, \dots, y^d\big), d \in \mathbb{N}, g \in C^2(\mathbb{R}^d)\big\}, \\ \mathcal{F}C_c^\infty(\{e_i\}) &:= \big\{f\colon \mathbb{R}^\infty \longrightarrow \mathbb{R} \,\big|\, f(y) = g\big(y^1, \dots, y^d\big), d \in \mathbb{N}, g \in C_c^\infty(\mathbb{R}^d)\big\} \end{aligned} \tag{1}$$

(see e.g. [MR92, p. 54] or [BKRS15, p. 404f]).

For two separable Hilbert spaces $\mathbb{H}_1$ and $\mathbb{H}_2$, let $L_2\big(\mathbb{H}_1; \mathbb{H}_2\big)$ be the space of all Hilbert–Schmidt operators from $\mathbb{H}_1$ to $\mathbb{H}_2$ with norm $\|\cdot\|_{L_2(\mathbb{H}_1;\mathbb{H}_2)}$.

By $\mathfrak{U}^\varrho$, for $\varrho \geq 1$, we denote the class of functions $\mathcal{N}\colon \mathbb{H} \longrightarrow [0,\infty]$ with the following properties:

(i) $\mathcal{N}(y) = 0$ implies $y = 0$,
(ii) $\mathcal{N}(cy) \leq c^\varrho \mathcal{N}(y)$ holds for every $c \geq 0$ and $y \in \mathbb{H}$,
(iii) the set $\{y \in \mathbb{H} \,|\, \mathcal{N}(y) \leq 1\}$ is compact in $\mathbb{H}$.

Remark. From properties (i)–(iii) we can conclude that any function in $\mathfrak{U}^\varrho$ is lower semi-continuous on $\mathbb{H}$. Furthermore, we can extend a function $\mathcal{N} \in \mathfrak{U}^\varrho$ to a $\mathcal{B}(\mathbb{X}^*)/\mathcal{B}([0,\infty])$-measurable one on $\mathbb{X}^*$ by setting $\mathcal{N}(y) = \infty$ for $y \in \mathbb{X}^* \setminus \mathbb{H}$. Note that $\mathcal{N}$, as a function on $\mathbb{X}^*$, is still lower semi-continuous since the embedding $\mathbb{H} \subseteq \mathbb{X}^*$ is continuous and compact.

Coefficients: Let the mappings

$$\begin{aligned} \sigma&\colon [0,T] \times \mathbb{H} \longrightarrow L_2(\mathbb{U}; \mathbb{H}), \\ b&\colon [0,T] \times \mathbb{H} \longrightarrow \mathbb{X}^* \end{aligned}$$

be Borel-measurable. In order to obtain components of those coefficients that are defined on $[0,T] \times \mathbb{R}^\infty$ for the Cauchy problem, we just extend b and σ by 0 on $\mathbb{R}^\infty \setminus \mathbb{H}$. This means, for every $i,j \in \mathbb{N}$, we consider the $\mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}^\infty)/\mathcal{B}(\mathbb{R})$-measurable mappings

$$a^{ij}\colon [0,T] \times \mathbb{R}^\infty \longrightarrow \mathbb{R},$$
$$b^{i}\colon [0,T] \times \mathbb{R}^\infty \longrightarrow \mathbb{R},$$

that are given by

$$a^{ij}(t,y) := \begin{cases} \frac{1}{2}\langle \sigma(t,y)\sigma(t,y)^* e_i, e_j\rangle_{\mathbb{H}}, & (t,y) \in [0,T] \times \mathbb{H}, \\ 0, & (t,y) \in [0,T] \times \mathbb{R}^\infty \setminus \mathbb{H} \end{cases}$$

and

$$b^{i}(t,y) := \begin{cases} {}_{\mathbb{X}^*}\langle b(t,y), e_i\rangle_{\mathbb{X}}, & (t,y) \in [0,T] \times \mathbb{H}, \\ 0, & (t,y) \in [0,T] \times \mathbb{R}^\infty \setminus \mathbb{H}. \end{cases}$$

In addition, we set

$$b_n := (b^1, \dots, b^n) \text{ and } A_n := (a^{ij})_{1 \le i,j \le n} \tag{2}$$

as well as

$$A := (a^{ij})_{1 \le i,j < \infty}.$$

Remark. We note that a^{ij} and b^i, regardless of our choice to simply extend them by 0 on $\mathbb{R}^\infty \setminus \mathbb{H}$, will still be admissible mappings to satisfy all necessary assumption from Subsection 2.3 below, because those assumptions are either imposed on $\mathbb{H}_n$ anyway or remain unchanged as e.g. symmetry or growth.

2.1. **Equation.** Let $x_0 \in \mathbb{H}$ and denote by δ_{x_0} the Dirac measure in this point. We will, for simplicity, reduce our calculations to this choice of an initial measure, but note that by integrating over all these measures, we can generalize our results.

Cauchy problem: Consider the following shorthand notation for a Cauchy problem for an infinite-dimensional linear Fokker–Planck–Kolmogorov equation given by

$$\begin{aligned} \partial_t \mu &= L^* \mu, \\ \mu_{\restriction t=0} &= \delta_{x_0}, \end{aligned} \tag{CP}$$

with respect to a nonnegative finite Borel measure μ of the form $\mu(\mathrm{d}t\,\mathrm{d}y) = \mu_t(\mathrm{d}y)\,\mathrm{d}t$ on $[0,T] \times \mathbb{R}^\infty$, where $(\mu_t)_{t \in [0,T]}$ is a family of Borel probability measures on $\mathbb{R}^\infty$. Furthermore, L^* is the formal adjoint of the associated Kolmogorov operator L to our FPKE, acting on finitely based functions $\varphi \in \mathcal{F}C^2(\{e_i\})$, which is given by

$$L\varphi(t,y) = \sum_{i,j=1}^{d} a^{ij}(t,y)\partial_{e_i}\partial_{e_j}\varphi(y) + \sum_{i=1}^{d} b^i(t,y)\partial_{e_i}\varphi(y),$$

for $(t,y) \in [0,T] \times \mathbb{R}^\infty$ and some $d \in \mathbb{N}$ depending on φ.

Remark. As usual, L obviously also acts on finitely based functions that are in addition explicitly depending on time, because this time-dependence is "irrelevant" for the partial derivatives appearing in the operator. But we will not need the often used classes of time-dependent test functions and rather mostly apply L to functions $\varphi \in \mathcal{F}C_c^\infty(\{e_i\})$ in the following.

2.2. **Notion of solution.** Let us introduce the notion of a probability solution to a Cauchy problem (CP) and the notion of a martingale solution to the martingale problem associated to the same operator L in the sense of Stroock–Varadhan.

Definition 2.1 (probability solution). *A finite Borel measure μ on $[0,T]\times\mathbb{R}^\infty$ of the form $\mu(\mathrm{d}t\,\mathrm{d}y)=\mu_t(\mathrm{d}y)\,\mathrm{d}t$, where $(\mu_t)_{t\in[0,T]}$ is a family of Borel probability measures on $\mathbb{R}^\infty$, is called probability solution to Equation* (CP) *if the following conditions hold.*

(i) The functions a^{ij}, b^i are integrable with respect to the measure μ, i.e.

$$a^{ij}, b^i \in L^1([0,T]\times\mathbb{R}^\infty,\mu).$$

(ii) For every function $\varphi\in\mathcal{F}C_c^\infty(\{e_i\})$ we have

$$\int_{\mathbb{R}^\infty}\varphi(y)\,\mu_t(\mathrm{d}y)=\int_{\mathbb{R}^\infty}\varphi(y)\,\delta_{x_0}(\mathrm{d}y)+\int_0^t\int_{\mathbb{R}^\infty}L\varphi(s,y)\,\mu_s(\mathrm{d}y)\,\mathrm{d}s \tag{3}$$

for $\mathrm{d}t$*-a.e.* $t\in[0,T]$.

Definition 2.2 (martingale solution). *A probability measure $P\in\mathcal{P}(\Omega)$ is called martingale solution to the martingale problem with coefficients b and σ and initial value $x_0\in\mathbb{H}$ if the following conditions hold.*

(M1) $P\big[x(0)=x_0\big]=1$ *and*

$$P\Big[x\in\Omega\,\Big|\,\text{For } \mathrm{d}s\text{-a.e. } s\in[0,T]: x(s)\in\mathbb{H} \text{ and } \int_0^T\big\|b(s,x(s))\big\|_{\mathbb{X}^*}\,\mathrm{d}s+\int_0^T\big\|\sigma(s,x(s))\big\|^2_{L_2(\mathbb{U};\mathbb{H})}\,\mathrm{d}s<\infty\Big]=1.$$

(M2) *For every function $f\in\mathcal{F}C_c^\infty(\{e_i\})$ the process*

$$\mathbb{M}^f(t,x):=f(x(t))-f(x_0)-\int_0^t Lf(s,x(s))\,\mathrm{d}s,\quad t\in[0,T],$$

is an $(\mathcal{F}_t)$-martingale with respect to P.

It is important to note that in [GRZ09, Definition 3.1, p. 1730] Condition (M2) in the notion of a martingale solution is stated in a "weak formulation" involving the inner product. In fact, by using Itô's formula we can directly show that this implies our Condition **(M2)** used in Definition 2.2. Furthermore, we can completely drop Condition (M3) from [GRZ09] as a requirement for being a solution because we can simply transform it into an a priori energy estimate as already done in [RZZ15, Lemma 3.1, p. 368].

Let us now state what we exactly mean by a martingale problem in the original sense by Stroock–Varadhan arising from given coefficients b and σ and an initial value $x_0\in\mathbb{H}$. In concrete terms, we will consider the following problem:

Existence of a martingale solution $P\in\mathcal{P}(\mathbb{S})$ in the sense of Definition 2.2 for coefficients b and σ and with initial value $x_0\in\mathbb{H}$, (MP)

where $P\in\mathcal{P}(\mathbb{S})$ means that we are explicitly searching for solutions that also require paths from the path space $C\big([0,T];\mathbb{X}^*\big)$ to be of class $L^p([0,T];\mathbb{H})$.

Remark. We would like to stress that we do not focus on the, in the literature often naturally established, link to weak solutions of infinite-dimensional stochastic differential equations (see e.g. [RZZ15, Theorem 2.2, p. 364], which is substantially using [Ond05, Theorem 2, p. 1007]), and only consider martingale problems in the original sense of Stroock–Varadhan.

2.3. **Assumptions.** Before stating the necessary assumptions on the coefficients, let us quickly recall the meaning of a compact and a non-degenerate function.

Definition 2.3 (compact function, see e.g. [BKRS15, Definition 2.3.1, p. 62])**.** *A real-valued function f on a topological space is called compact if the sublevel sets $\{f \leq R\}$ are compact for any $R \in \mathbb{R}$.*

Definition 2.4 (non-degenerate function, see e.g. [BDR08, p. 410])**.** *A compact function $f \in C^2(\mathbb{R}^n)$ is called non-degenerate if there exists a sequence $(c_k)_{k\in\mathbb{N}}$ of numbers with $c_k \xrightarrow[k\to\infty]{} \infty$ such that the level sets $f^{-1}(c_k) = \{y \in \mathbb{R}^n \mid f(y) = c_k\}$ are C^1-surfaces.*

The following assumptions on our coefficients are, up to some minor modifications to ensure applicability of the used finite-dimensional results, directly taken from our main references.

(H1) For all $n \in \mathbb{N}$, the matrices $A_n = (a^{ij})_{1\leq i,j\leq n}$ are symmetric and nonnegative definite.

(H2) Let $\Theta\colon \mathbb{R}^\infty \longrightarrow [0,\infty]$ be a compact Borel function, bounded on bounded sets on each space $\mathbb{H}_n$, $n \in \mathbb{N}$, such that, for every $i \in \mathbb{N}$ and $j \leq i$,
- the functions $y \mapsto a^{ij}(t,y)$, $t \in [0,T]$, are equicontinuous on every set $\{\Theta \leq R\}$ with $R < \infty$ and also on every fixed ball in each $\mathbb{H}_n$,
- for every $t \in [0,T]$ the function $y \mapsto b^i(t,y)$ is continuous on every set $\{\Theta \leq R\}$ with $R < \infty$ and also on each $\mathbb{H}_n$.

(H3) There exist numbers $M_0, C_0 \geq 0$ and a compact Borel function $V\colon \mathbb{R}^\infty \longrightarrow [1,\infty]$ whose restrictions to $\mathbb{H}_n$ are of class $C^2(\mathbb{H}_n)$ and non-degenerate such that for all $y \in \mathbb{H}_n$, $n \in \mathbb{N}$, $t \in [0,T]$, we have

$$\sum_{i,j=1}^{n} a^{ij}(t,y)\partial_{e_i}V(y)\partial_{e_j}V(y) \leq M_0 V(y)^2,$$

$$LV(t,y) \leq C_0 V(y) - \Theta(y).$$

(H4) There exist constants $C_i \geq 0$ and $k_i \geq 0$ such that for all $i \in \mathbb{N}$ and $j \leq i$ we have

$$|a^{ij}(t,y)| + |b^i(t,y)| \leq C_i V(y)^{k_i}\big(1 + \kappa_i(\Theta(y))\Theta(y)\big),$$

for every $(t,y) \in [0,T] \times \mathbb{R}^\infty$, where κ_i is a bounded nonnegative Borel function on $[0,\infty)$ with $\lim_{s\to\infty} \kappa_i(s) = 0$.

(N) There exists a function $\mathcal{N} \in \mathfrak{U}^p$ for some $p \geq 2$ such that for every $n \in \mathbb{N}$ there exists a constant $C_n \geq 0$ with

$$\mathcal{N}(v) \leq C_n \|v\|_{\mathbb{H}_n}^p,$$

for any $v \in \mathbb{H}_n$.

(A1) (Demicontinuity) For any $v \in \mathbb{X}$, $t \in [0,T]$ and every sequence $(y_k)_{k\in\mathbb{N}}$ with $y_k \xrightarrow[k\to\infty]{} y$ in $\mathbb{H}$, we have

$$\lim_{k\to\infty} {}_{\mathbb{X}^*}\langle b(t,y_k), v\rangle_{\mathbb{X}} = {}_{\mathbb{X}^*}\langle b(t,y), v\rangle_{\mathbb{X}}$$

and

$$\lim_{k\to\infty} \big\|\sigma^*(t,y_k)(v) - \sigma^*(t,y)(v)\big\|_{\mathbb{U}} = 0.$$

(A2) (Coercivity) There exists a constant $\lambda_1 \geq 0$ such that for all $v \in \mathbb{X}$ and $t \in [0,T]$

$$_{\mathbb{X}^*}\langle b(t,v), v\rangle_{\mathbb{X}} \leq -\mathcal{N}(v) + \lambda_1(1 + \|v\|_{\mathbb{H}}^2)$$

holds.

(A3) (Growth) There exist constants $\lambda_2, \lambda_3, \lambda_4 > 0$ and constants $\gamma' \geq \gamma > 1$ such that for all $y \in \mathbb{H}$ and $t \in [0,T]$ we have

$$\|b(t,y)\|_{\mathbb{X}^*}^{\gamma} \leq \lambda_2 \mathcal{N}(y) + \lambda_3(1 + \|y\|_{\mathbb{H}}^{\gamma'})$$

and

$$\|\sigma(t,y)\|_{L_2(\mathbb{U};\mathbb{H})}^2 \leq \lambda_4(1 + \|y\|_{\mathbb{H}}^2).$$

Furthermore, in order to guarantee that our initial measure δ_{x_0} satisfies all condition assumed in [BDRS15, Theorem 3.1, p. 1013], we assume that

$$W_k := \sup_{n\in\mathbb{N}} \|V(\cdot)^k \circ \Pi_n\|_{L^1(\delta_{x_0})} = \sup_{n\in\mathbb{N}} |V^k(\Pi_n x_0)| < \infty$$

holds for all $k \in \mathbb{N}$.

Remark. We note that e.g. in [BKRS15, Proposition 7.1.8, p. 293] we can find the idea for a transformation of a given Lyapunov function V to one that already satisfies integrability with respect to the initial measure in the finite-dimensional setting. Adapting this idea would be an option to actually drop the above assumption on W_k.

We also want to note that these assumptions are not supposed to be perfectly optimal and leave room for improvement and unification. In particular, Assumption **(A2)** would be a prime candidate to be transformed into a Lyapunov condition similar to Assumption **(H3)**. But for a start, we impose the combination of both sets of respective assumptions, because we are confident that coefficients in potential applications like SNSE will satisfy them anyway.

3. Existence result

Let us state the main result of this first part, which can be described as a "joint" existence theorem for probability and martingale solutions which are connected through their time-marginal laws.

Theorem 3.1. *Under the assumptions from Subsection 2.3 there exists a probability solution $\mu = \mu_t\,\mathrm{d}t$ on $[0,T] \times \mathbb{H}$ to the Cauchy problem* (CP) *in the sense of Definition 2.1 and a martingale solution $P \in \mathcal{P}(\mathbb{S})$ to the associated martingale problem* (MP) *in the sense of Definition 2.2, for which the time-marginal laws of P coincide with μ_t, i.e.*

$$P \circ x(t)^{-1} = \mu_t \tag{4}$$

holds for every $t \in [0,T]$.

In particular, the following estimates and equations hold. For every $q \geq 1$, we have

$$\mathbb{E}^P\left[\sup_{t\in[0,T]} \|x(t)\|_{\mathbb{H}}^{2q} + \int_0^T \|x(t)\|_{\mathbb{H}}^{2(q-1)} \mathcal{N}(x(t))\,\mathrm{d}t\right] < \infty. \tag{5}$$

Furthermore, for all $t \in [0,T]$ and $k \in \mathbb{N}$, we have

$$\int_{\mathbb{R}^\infty} V(y)^k\,\mu_t(\mathrm{d}y) + k\int_0^t \int_{\mathbb{R}^\infty} V(y)^{k-1}\Theta(y)\,\mu_s(\mathrm{d}y)\,\mathrm{d}s \leq N_k W_k, \tag{6}$$

where $N_k := M_k e^{M_k} + 1$ *and* $M_k := k(C_0 + (k-1)M_0)$*, as well as*

$$\mu_t(V < \infty) = 1 \tag{7}$$

for all $t \in [0,T]$ *and* $\mu_t(\Theta < \infty) = 1$ *for* $\mathrm{d}t$*-a.e.* $t \in [0,T]$*.*

Remark. Obviously, the respective assumptions from Subsection 2.3 directly ensure existence for both martingale and probability solutions individually, but without any additional information (e.g. on uniqueness) we a priori could not specify any such connection given in Equation (4) while also preserving knowledge about both solutions, i.e. Equation and Estimates (5)–(7), gained in their construction as limits of finite-dimensional approximations.

4. Auxiliary results

Before actually proving Theorem 3.1, let us discuss the three well-known results that we will combine for it in the following. We start with a short streamlined overview of the scheme of proof used in [GRZ09] and [BDRS15], on the one hand, to recall it for the reader and, on the other hand, to show its similarity and to give a clear idea on how to make use of it.

Existence of martingale solution: (Theorem 4.6 in [GRZ09, p. 1739])
First, the authors consider the finite-dimensional martingale problem on $\mathbb{H}_n$ with coefficients $\Pi_n b$ and $\Pi_n \sigma$ being created by the projections Π_n. By using well-known results in finite dimensions (see [SV79, Theorem 6.1.7, p. 144]), they deduce existence of martingale solutions, i.e. some $P_n \in \mathcal{P}(\Omega_n)$, for any $n \in \mathbb{N}$. From there they extend P_n to $\bar{P}_n \in \mathcal{P}(\Omega)$ and prove tightness of the family $(\bar{P}_n)_{n\in\mathbb{N}}$. Then they extract a subsequence of $(\bar{P}_n)_{n\in\mathbb{N}}$ converging weakly to a probability measure $P \in \mathcal{P}(\mathbb{S})$ that is a solution to the infinite-dimensional martingale problem with coefficients b and σ.

Existence of probability solution: (Theorem 3.1 in [BDRS15, p. 1013])
First, the authors consider the finite-dimensional Cauchy problem on $\mathbb{H}_n$ with coefficients A_n and b_n, which consist of the components a^{ij} and b^i up to n for any $n \in \mathbb{N}$. They prove existence of solutions $\mu_{t,n}$ by using finite-dimensional results (see [BDR08, Corollary 3.4, p. 415]). Then, after extending the family $(\mu_{t,n})_{n\in\mathbb{N}}$ to $(\bar{\mu}_{t,n})_{n\in\mathbb{N}}$ on $\mathbb{R}^\infty$ and proving tightness of $(\bar{\mu}_{t,n})_{n\in\mathbb{N}}$, they extract a subsequence that is weakly converging to a probability measure μ_t. Finally, they prove that $\mu = \mu_t\,\mathrm{d}t$ is a probability solution to the infinite-dimensional Cauchy problem with coefficients A and b.

Finite-dimensional superposition principle: Let us quickly state the finite-dimensional superposition principle proved in [BRS20, Theorem 1.1, p. 5], which is further weakening the integrability condition imposed in [Tre16]. This theorem is easier to use in our particular setting, making it our reference of choice for applying the superposition principle to probability solutions in finite dimensions later. First, let us recall the necessary assumptions stated in [BRS20]:

- **(S1)** The diffusion matrix $A_n = (a^{ij})_{1\le i,j\le n}$ is symmetric and nonnegative definite.
- **(S2)** For every ball $U \subseteq \mathbb{R}^n$ we have
$$a^{ij}, b^i \in L^1([0,T] \times U, \mu_{t,n}\,\mathrm{d}t).$$
- **(S3)** The integrability condition
$$\int_0^T \int_{\mathbb{R}^n} \frac{\|A_n(t,y)\| + |\langle b_n(t,y), y\rangle_{\mathbb{R}^n}|}{(1+\|y\|_{\mathbb{R}^n})^2}\,\mu_{t,n}(\mathrm{d}y)\,\mathrm{d}t < \infty$$
holds.

Their finite-dimensional superposition principle can then be written as follows:

Theorem 4.1 (see [BRS20, Theorem 1.1, p. 5]). *Suppose that $(\mu_{t,n})_{t\in[0,T]}$ is a "narrowly continuous" solution to the finite-dimensional Cauchy problem on $[0,T]$ with initial measure ν and Assumptions **(S1)** – **(S3)** are fulfilled. Then there exists a Borel probability measure P_n^ν on $C([0,T];\mathbb{R}^n)$ such that the following properties hold:*

(m1) *For all Borel sets $B \subseteq \mathbb{R}^n$ we have $P_n^\nu\big[x_n \in C([0,T];\mathbb{R}^n) \mid x_n(0) \in B\big] = \nu(B)$.*

(m2) *For every function $f \in C_c^\infty(\mathbb{R}^n)$, the function*

$$(x_n, t) \longmapsto f(x_n(t)) - f(x_n(0)) - \int_0^t Lf(s, x_n(s))\,\mathrm{d}s$$

is a martingale with respect to the measure P_n^ν and the natural filtration $\mathcal{F}_t^{(n)} = \sigma(x_n(s) \mid s \in [0,t])$.

(m3) *For every function $f \in C_c^\infty(\mathbb{R}^n)$, the equality*

$$\int_{\mathbb{R}^n} f(y)\,\mu_{t,n}(\mathrm{d}y) = \int_{C([0,T];\mathbb{R}^n)} f(x_n(t))\,P_n^\nu(\mathrm{d}x_n)$$

holds for all $t \in [0,T]$.

Here, the term "narrowly continuous" means that we have continuity with respect to the weak topology.

5. Proof of Theorem 3.1

A more elaborate version with all details can be found in [Die20, Section 6.4, p. 81ff].

Proof. Let us divide the proof into nine steps.

Step 1: Starting point

We are given an initial value $x_0 \in \mathbb{H}$ and coefficients b and σ on $[0,T]\times\mathbb{H}$ which directly allow us to study the martingale problem (MP) on $\mathbb{H}$. As described in Section 2, we then also consider the components b^i and a^{ij} that are extended to $[0,T]\times\mathbb{R}^\infty$ by 0 for Equation (CP) on $[0,T]\times\mathbb{R}^\infty$ with initial measure δ_{x_0} at the same time. Note that these extensions still satisfy all assumptions that we have imposed in Subsection 2.3.

Now, for every $n \in \mathbb{N}$, we project the coefficients and initial value/measure down onto $\mathbb{H}_n$ via the projections Π_n to obtain coefficients $\Pi_n b$ and $\Pi_n\sigma$ for the finite-dimensional martingale problem, which we can state as

$$\begin{array}{c}\text{Existence of a martingale solution } P_n \in \mathcal{P}(\Omega_n) \text{ in the sense of Definition 2.2}\\ \text{for coefficients } \Pi_n b \text{ and } \Pi_n\sigma \text{ and with initial value } \Pi_n x_0 \in \mathbb{H}_n,\end{array} \tag{MP$_n$}$$

as well as $\Pi_n b$ and $\Pi_n A\,\Pi_n^*$ for the finite-dimensional Cauchy problem, that can be written in short-hand notation as

$$\begin{aligned}\partial_t \mu_n &= L^*\mu_n,\\ \mu_{n\,\restriction t=0} &= \delta_{x_0}\circ\Pi_n^{-1}.\end{aligned} \tag{CP$_n$}$$

Here, the operator L, acting on functions $\varphi \in C^2(\mathbb{H}_n)$, is given by

$$L\varphi(t,y) = \sum_{i,j=1}^n a^{ij}(t,y)\partial_{e_i}\partial_{e_j}\varphi(y) + \sum_{i=1}^n b^i(t,y)\partial_{e_i}\varphi(y),$$

for $(t,y) \in [0,T]\times\mathbb{H}_n$.

Since we have assumed **(H1)**–**(H4)**, we can conclude, as in [BDRS15, p. 1014], existence of solutions $\mu_{t,n}$ to Equation (CP_n) for any $n \in \mathbb{N}$ with the property that the function

$$t \mapsto \int_{\mathbb{H}_n} \zeta(y)\, \mu_{t,n}(\mathrm{d}y) \tag{8}$$

is continuous on $t \in [0,T]$ for every $\zeta \in C_c^\infty(\mathbb{H}_n)$ by using [BDR08, Corollary 3.4, p. 415]. In fact, the coefficients A_n and b_n are exactly $\Pi_n b$ and $\Pi_n A\, \Pi_n^*$ in our setting. To those probability measures $\mu_{t,n}$ on $\mathbb{H}_n$ we will now apply the superposition principle.

Step 2: Application of Theorem 4.1

Let us fix some $n \in \mathbb{N}$ for the moment and specify how to exactly apply the finite-dimensional superposition principle on $\mathbb{H}_n$. We start with a solution $\mu_{t,n}$ to Equation (CP_n) with initial distribution $\delta_{x_0} \circ \Pi_n^{-1}$. Now, let us check the necessary assumptions for using Theorem 4.1. Assumption **(S1)** follows from Assumption **(H1)**, Assumption **(S2)** is fulfilled, since we have $a^{ij}, b^i \in L^1_{\mathrm{loc}}(\mu_{t,n}\,\mathrm{d}t)$ for our solution $\mu_{t,n}$ by definition, and Assumption **(S3)** for the projected coefficients $\Pi_n b$ and $\Pi_n A\, \Pi_n^*$ can be derived by in particular using Assumption **(H4)**.

Note that there are two small but apparent differences in the notions of a probability solution in finite-dimensions used in [BRS20] and [BDRS15] (which is in fact constructed in the sense of [BDR08, p. 397f]). First, the finite-dimensional analogue of Equation (3) has to hold for every $t \in [0,T]$ in [BRS20]. This can be concluded by using Lemma 2.1 in [BDR08, p. 399] (including the explanation about the limit on p. 400). Second, continuity with respect to the weak topology of a probability solution is part of its definition in [BRS20]. For a solution constructed in [BDRS15], this follows from Equation (8) by a standard approximation argument enlarging the space of test functions.

Therefore, we can apply Theorem 4.1 and conclude that there exists a probability measure P_n on Ω_n such that Conditions **(m1)** and **(m2)** hold. In addition, P_n also satisfies Condition **(m3)**, i.e. for the time-marginal laws we have

$$P_n \circ x_n(t)^{-1} = \mu_{t,n}$$

for every $t \in [0,T]$. This implies in particular, that P_n is a martingale solution to the martingale problem (MP_n) satisfying Conditions **(M1)** and **(M2)** of Definition 2.2.

Step 3: Tightness of $(\bar{P}_n)_{n\in\mathbb{N}}$

Collect the family $(P_n)_{n\in\mathbb{N}}$ of all probability measures obtained by the application of the superposition principle for each $n \in \mathbb{N}$. Since they are solutions to (MP_n) satisfying Conditions **(M1)** and **(M2)** (and by using e.g. an adaption of the proof in [RY99, Chapter VII, §2, p. 293ff] to obtain the "weak formulation" of condition (M2) in [GRZ09]), we are actually in the same situation as in the proof of Theorem 4.6 in [GRZ09, p. 1739ff] (see also the proof of Theorem 2.1 in [RZZ15, p. 364ff]).

There (see p. 1739), the authors conclude existence of such probability measures (with no additional property except for being a solution to (MP_n)) from a finite-dimensional result in the appendix, which is based on [SV79, Theorem 6.1.7, p. 144]. In our case, these solutions are just directly "created" by the superposition principle. Hence, since we have assumed **(A1)**–**(A3)**, we can repeat all calculations including the extension of P_n to $\bar{P}_n$ (see p. 1741) and the proof of tightness of the family $(\bar{P}_n)_{n\in\mathbb{N}}$ (see p. 1742).

Step 4: Tightness of $(\bar{\mu}_{t,n})_{n\in\mathbb{N}}$
After extending the measures $\mu_{t,n}$ on $\mathbb{H}_n$ to $\bar{\mu}_{t,n}$ on $\mathbb{R}^\infty$, we can simply follow [BDRS15, p. 1014f] and prove tightness of the family of probability measures $(\bar{\mu}_{t,n})_{n\in\mathbb{N}}$ for every fixed $t \in [0,T]$ the same way as it is done there.

Step 5: Weak convergence on a joint subsequence
By using a "diagonal argument", we can modify the index set of the two tight families, which is necessary for selecting a joint index set on which both subsequences converge to a limit helping us to prove that Equation (4) holds. Recall that a subset of a tight set of measures is by definition still a tight set and we lose no additional properties by dropping some indices. More importantly, we can verify that both proofs remain unchanged after that point by considering a smaller index set.

Hence, we can choose a set of indices for which both tight families have a convergent subsequence with the same indices. Note that we ensure in the process, that the subsequence of probability solutions converges weakly for any $t \in [0,T]$ (see [Die20, p. 54ff] for more details). Let us, for simplicity, denote these joint subsequences by $(\bar{P}_{n_k})_{k\in\mathbb{N}}$ and $(\bar{\mu}_{t,n_k})_{k\in\mathbb{N}}$ and their limits by P and μ_t, respectively.

Step 6: μ is a solution
For proving that $\mu = \mu_t\,\mathrm{d}t$ is a probability solution to the Cauchy problem (CP) on $[0,T]\times\mathbb{R}^\infty$ in the sense of Definition 2.1 we can again follow the proof in [BDRS15] (starting on p. 1015). In particular, Estimate (6) and Equation (7) hold.

Step 7: P is a solution
We can carry over the calculations from [GRZ09] (see p. 1736ff) in order to prove that $P \in \mathcal{P}(\mathbb{S})$ is a solution to the martingale problem (MP) with initial measure δ_{x_0}. In particular, Estimate (5) holds.

Step 8: Time-marginal laws for the limit
Finally, we have to prove that

$$P \circ x(t)^{-1} = \mu_t$$

holds for every $t \in [0,T]$. In fact, let $f \in \mathcal{F}$, where $\mathcal{F} \subseteq \mathcal{F}C_c^\infty(\{e_i\})$ is a measure-separating family on $\mathbb{R}^\infty$ (always exists, see Lemma A.1 in the appendix). Then f is of the form

$$f(y) = g\big(y^1,\dots,y^d\big), \quad y \in \mathbb{R}^\infty,$$

for some $d \in \mathbb{N}$ and $g \in C_c^\infty(\mathbb{R}^d)$.

Note that Condition **(m3)** not only holds for functions in $C_c^\infty(\mathbb{R}^n)$, but also by approximation for functions in $C_b^\infty(\mathbb{R}^n)$. Furthermore, for $n \geq d$, a function in $C_c^\infty(\mathbb{R}^d)$ treated as a function on $\mathbb{R}^n$ is of class $C_b^\infty(\mathbb{R}^n)$.

Since we have $\bar{\mu}_{t,n_k} \xrightarrow[k\longrightarrow\infty]{w} \mu_t$ on $\mathbb{R}^\infty$, we know that

$$\int_{\mathbb{R}^\infty} h(y)\,\mu_t(\mathrm{d}y) = \lim_{k\to\infty}\int_{\mathbb{R}^\infty} h(y)\,\bar{\mu}_{t,n_k}(\mathrm{d}y)$$

is fulfilled for every $h \in C_b(\mathbb{R}^\infty)$, i.e. in particular for f. In addition, we have that $\bar{P}_{n_k} \xrightarrow[k\longrightarrow\infty]{w} P$ on Ω, which means that

$$\int_\Omega h(\omega)\,P(\mathrm{d}\omega) = \lim_{k\to\infty}\int_\Omega h(\omega)\,\bar{P}_{n_k}(\mathrm{d}\omega)$$

holds for every $h \in C_b(\Omega)$. Consequently, it is also true for the mapping given by $\omega \longmapsto g\big(\omega(t)^1, \dots, \omega(t)^d\big)$ for every $t \in [0,T]$.

Then we obtain

$$\begin{aligned}
\int_{\mathbb{R}^\infty} f(y)\,\mu_t(\mathrm{d}y) &= \lim_{k\to\infty} \int_{\mathbb{R}^\infty} g\big(y^1,\dots,y^d\big)\,\bar{\mu}_{t,n_k}(\mathrm{d}y)\\
&= \lim_{k\to\infty} \int_{\mathbb{H}_{n_k}} g\big(y^1,\dots,y^d\big)\,\mu_{t,n_k}(\mathrm{d}y) = \lim_{k\to\infty} \int_{\Omega_{n_k}} g\big(\omega(t)^1,\dots,\omega(t)^d\big)\,P_{n_k}(\mathrm{d}\omega)\\
&= \lim_{k\to\infty} \int_{\Omega} g\big(\omega(t)^1,\dots,\omega(t)^d\big)\,\bar{P}_{n_k}(\mathrm{d}\omega) = \int_{\Omega} g\big(\omega(t)^1,\dots,\omega(t)^d\big)\,P(\mathrm{d}\omega)\\
&= \int_{\Omega} f(\omega(t))\,P(\mathrm{d}\omega) = \int_{\mathbb{S}} f(x(t))\,P(\mathrm{d}x)
\end{aligned}$$

for any $t \in [0,T]$. Since $f \in \mathcal{F}$ separates measures on $\mathbb{R}^\infty$ (and all of its subsets), the assertion follows.

Step 9: μ_t are probability measures on $\mathbb{H}$
From Estimate (5) we can follow that

$$\mathbb{E}^P\Big[\sup_{t\in[0,T]} \|x(t)\|_{\mathbb{H}}^{2q}\Big] < \infty$$

holds for every $q \geq 1$, where we made use of the lower semi-continuity of the norm $\|\cdot\|_{\mathbb{H}}$ as an extended function on $\mathbb{X}^*$ and, therefore, of the supremum. Consequently, $P \circ x(t)^{-1}$ is a probability measure on $\mathbb{H}$ for every $t \in [0,T]$, hence by Step 8 so is μ_t. □

6. Conclusions and consequences

The first corollary will highlight the fact that we can directly conclude continuity for the mapping $t \mapsto \mu_t$ with respect to the topology generated by finitely based functions from Equation 4.

Corollary 6.1. *For solutions P and μ constructed in Theorem 3.1, Equation (4) implies that the mapping $t \mapsto \mu_t$ from $[0,T]$ to $\mathcal{P}(\mathbb{H})$ is continuous with respect to the topology generated by the class $\mathcal{F}C_c^\infty(\{e_i\})$ of finitely based functions, i.e. that the mapping*

$$t \mapsto \int_{\mathbb{H}} f(y)\,\mu_t(\mathrm{d}y)$$

is continuous for every $f \in \mathcal{F}C_c^\infty(\{e_i\})$.

Proof. From Theorem 3.1 we are given measures $\mu_t \in \mathcal{P}(\mathbb{H})$, $t \in [0,T]$, and $P \in \mathcal{P}(\mathbb{S})$ that satisfy Equation (4).

We know that for $P \in \mathcal{P}(\mathbb{S})$ the canonical process x on $\mathbb{S}$ is in particular a mapping in the path space $C\big([0,T];\mathbb{X}^*\big)$. This means that for any $f \in \mathcal{F}C_c^\infty(\{e_i\})$, and with it some $d \in \mathbb{N}$ and $g \in C_c^\infty(\mathbb{R}^d)$, the mapping

$$t \longmapsto g\big({}_{\mathbb{X}^*}\langle x(t), e_1\rangle_{\mathbb{X}}, \dots, {}_{\mathbb{X}^*}\langle x(t), e_d\rangle_{\mathbb{X}}\big)$$

is continuous. Hence, the mapping $t \mapsto \int_{\mathbb{S}} f(x(t))\,P(\mathrm{d}x)$ is continuous for any $f \in \mathcal{F}C_c^\infty(\{e_i\})$, which yields the assertion. □

The next corollary is an obvious consequence of the proof of Theorem 3.1. By upfront assuming existence of a probability solution to (CP) as well as all desired properties for it, we can leave out Assumptions **(H2)**–**(H4)**. Instead, we only ensure the application of the superposition principle directly by assumption, where in particular Assumption **(H1)** is a part of.

Corollary 6.2. *Let Assumptions **(N)**, **(A1)**, **(A2)**, **(A3)** and **(H1)** be fulfilled. Assume there exists a probability solution $\mu = \mu_t\,\mathrm{d}t$ on $[0,T]\times\mathbb{H}$ to the Cauchy problem* (CP) *in the sense of Definition 2.1 and a subsequence $(\mu_{t,n_k})_{k\in\mathbb{N}}$ on $\mathbb{H}_{n_k}$ of a family of Borel probability measures on $\mathbb{H}_n$ with the following properties:*

- *The measures $(\mu_{t,n_k})_{k\in\mathbb{N}}$ are solutions to the finite-dimensional Cauchy problems* (CP_n) *on $\mathbb{H}_{n_k}$ with the property that the mapping*
$$t \mapsto \int_{\mathbb{H}_{n_k}} \zeta(y)\,\mu_{t,n_k}(\mathrm{d}y)$$
is continuous on $[0,T]$ for every $\zeta \in C_c^\infty(\mathbb{H}_{n_k})$.
- *For the family $(\bar{\mu}_{t,n_k})_{k\in\mathbb{N}}$ of extended measures to $\mathbb{H}$, we have $\bar{\mu}_{t,n_k} \xrightarrow[k\to\infty]{w} \mu_t$ for every $t\in[0,T]$.*
- *The integrability condition*
$$\int_0^T \int_{\mathbb{H}_{n_k}} \frac{\|\Pi_{n_k} A(t,y)\Pi_{n_k}^*\| + |\langle \Pi_{n_k} b(t,y), y\rangle_{\mathbb{H}_{n_k}}|}{(1+\|y\|_{\mathbb{H}_{n_k}})^2}\,\mu_{t,n_k}(\mathrm{d}y)\,\mathrm{d}t < \infty$$
holds for every $k\in\mathbb{N}$.

Then there exists a martingale solution $P\in\mathcal{P}(\mathbb{S})$ to the martingale problem (MP) *in the sense of Definition 2.2, for which Equation* (4) *holds for every $t\in[0,T]$.*

Remark. The statements of Corollary 6.1 remains valid in the setting of this corollary.

Remark. Let us note, that it is not sufficient to just restrict the measures μ_t to the finite-dimensional spaces $\mathbb{H}_n$, by e.g. considering the push-forward measures $\mu_t\circ\Pi_n^{-1}$, in order to get a weakly convergent subsequence, because these measures not necessarily form a solution to the Cauchy problems (CP_n) with coefficients $\Pi_n b$ and $\Pi_n A\,\Pi_n^*$. We refer to [BKRS15, Section 10.2, p. 413ff] for more details on this kind of equation.

Proof of Corollary 6.2. We can just repeat the proof of Theorem 3.1, because this time we are simply given an explicit family $(\mu_{t,n_k})_{k\in\mathbb{N}}$ of solutions to the finite-dimensional Cauchy problem (CP_n) on $\mathbb{H}_{n_k}$ for which we already know that $(\bar{\mu}_{t,n_k})_{k\in\mathbb{N}}$ converges weakly to the given solution μ of the Cauchy problem (CP).

In particular, all steps necessary to derive finite-dimensional probability solutions are redundant, because we have directly assumed all desired properties for μ and $\bar{\mu}_{n_k}$. Furthermore, we can apply the finite-dimensional superposition principle, because we have ensured Condition **(S3)** directly by assumption. □

Application to d-dimensional stochastic Navier–Stokes equations:
The articles [BDRS15, Example 3.5, p. 17f] (which can partly also be found in [BKRS15, Example 10.1.6, p. 411f and Example 10.4.3, p. 425f]), [GRZ09, Chapter 6, p. 1749ff] and [RZZ15, Section 5.1, p. 377f] contain extensive calculations on d-dimensional SNSEs that remain valid in our case.

We realize that we can mimic the proof of Theorem 3.1 by constructing a probability solution via Galerkin approximations as in [BDRS15] while simultaneously obtaining finite-dimensional martingale solutions and studying their limit.

The application of our scheme of proof, therefore, extends the individual results of [BDRS15] and [GRZ09] on existence of a probability measure μ solving the Cauchy problem (CP) and a martingale solution P solving the martingale problem (MP) by their connection through Equation (4). This means that the solution constructed in [BDRS15, Example 3.5, p. 17f] is in fact identical with the time-marginals of a solution to the corresponding martingale problem. In particular, this implies that the mapping $t \longmapsto \mu_t$ is continuous with respect to the topology generated by finitely based functions.

7. Nonlinear framework

As a generalization we will now consider nonlinear Fokker–Planck–Kolmogorov equations in infinite dimensions.

Our setting remains, for the most part, identical to Section 2. Let us focus on what exactly changes in the nonlinear case. For some fixed $T > 0$, let the mappings

$$b\colon [0,T] \times \mathbb{H} \times \mathcal{P}(\mathbb{H}) \longrightarrow \mathbb{X}^*,$$
$$\sigma\colon [0,T] \times \mathbb{H} \times \mathcal{P}(\mathbb{H}) \longrightarrow L_2(\mathbb{U},\mathbb{H})$$

be Borel-measurable, where $\mathcal{P}(\mathbb{H})$ denotes the space of all Borel probability measures on $\mathbb{H}$. By extending those mappings by 0 as before, we again obtain mappings

$$a^{ij}\colon [0,T] \times \mathbb{R}^\infty \times \mathcal{P}(\mathbb{R}^\infty) \longrightarrow \mathbb{R},$$
$$b^{i}\colon [0,T] \times \mathbb{R}^\infty \times \mathcal{P}(\mathbb{R}^\infty) \longrightarrow \mathbb{R}$$

as coefficients. To be more precise, we now set

$$b^i(t,y,\varrho) := \begin{cases} {}_{\mathbb{X}^*}\langle b(t,y,\varrho), e_i\rangle_{\mathbb{X}}, & (t,y,\varrho) \in [0,T]\times\mathbb{H}\times\mathcal{P}(\mathbb{H}),\\ 0, & \text{else}\end{cases}$$

and

$$a^{ij}(t,y,\varrho) := \begin{cases} \frac{1}{2}\langle \sigma(t,y,\varrho)\sigma(t,y,\varrho)^* e_i, e_j\rangle_{\mathbb{H}}, & (t,y,\varrho) \in [0,T]\times\mathbb{H}\times\mathcal{P}(\mathbb{H}),\\ 0, & \text{else.}\end{cases}$$

Again, let $A(t,y,\varrho) := \big(a^{ij}(t,y,\varrho)\big)_{1\le i,j<\infty}$ be our diffusion matrix.

7.1. **Equation.** Consider the following shorthand notation for a Cauchy problem for a nonlinear Fokker–Planck–Kolmogorov equation given by

$$\begin{aligned} \partial_t \mu &= L^*_\mu\, \mu,\\ \mu_{\restriction t=0} &= \delta_{x_0}, \end{aligned} \tag{NCP}$$

with respect to a nonnegative finite Borel measure μ of the form $\mu(\mathrm{d}t\,\mathrm{d}y) = \mu_t(\mathrm{d}y)\,\mathrm{d}t$ on $[0,T]\times\mathbb{R}^\infty$, where $(\mu_t)_{t\in[0,T]}$ is a family of Borel probability measures on $\mathbb{R}^\infty$. Here, L^*_μ is the formal adjoint of the operator L_μ, acting on functions $\varphi \in \mathcal{F}C^2(\{e_i\})$, which is given by

$$L_\mu\varphi(t,y) = \sum_{i,j=1}^d a^{ij}(t,y,\mu_t)\partial_{e_i}\partial_{e_j}\varphi(y) + \sum_{i=1}^d b^i(t,y,\mu_t)\partial_{e_i}\varphi(y)$$

for $(t,y)\in[0,T]\times\mathbb{R}^\infty$ and for some $d\in\mathbb{N}$ depending on φ.

7.2. **Notion of solution.** The notion of a probability solution in the nonlinear case is analogue to Definition 2.1. We only have to consider coefficients explicitly depending on a measure.

Definition 7.1. *(probability solution, nonlinear) A finite Borel measure μ on $[0,T]\times\mathbb{R}^\infty$ of the form $\mu(\mathrm{d}t\,\mathrm{d}y)=\mu_t(\mathrm{d}y)\,\mathrm{d}t$, where $(\mu_t)_{t\in[0,T]}$ is a family of Borel probability measures on $\mathbb{R}^\infty$, is called probability solution to the Cauchy problem* (NCP) *if*

(i) The functions a^{ij}, b^i are integrable with respect to the measure μ, i.e.

$$a^{ij}(\cdot,\cdot,\mu_\cdot),b^i(\cdot,\cdot,\mu_\cdot)\in L^1([0,T]\times\mathbb{R}^\infty,\mu).$$

(ii) For every function $\varphi\in\mathcal{F}C_c^\infty(\{e_i\})$ we have

$$\int_{\mathbb{R}^\infty}\varphi(y)\,\mu_t(\mathrm{d}y)=\int_{\mathbb{R}^\infty}\varphi(y)\,\delta_{x_0}(\mathrm{d}y)+\int_0^t\int_{\mathbb{R}^\infty}L_\mu\varphi(s,y)\,\mu_s(\mathrm{d}y)\,\mathrm{d}s$$

for every $t\in[0,T]$.

The notion of a martingale solution remains completely unchanged. We will simply consider martingale problems with "special" functions in the place of b and σ that explicitly depend on a fixed measure, i.e. $b(\cdot,\cdot,\mu_\cdot)$ and $\sigma(\cdot,\cdot,\mu_\cdot)$, in the following. Hence, the martingale problem we study can be stated as:

Existence of a martingale solution $P\in\mathcal{P}(\mathbb{S})$ in the sense of Definition 2.2 for the coefficients $b(\cdot,\cdot,\mu_\cdot)$ and $\sigma(\cdot,\cdot,\mu_\cdot)$ and with initial value $x_0\in\mathbb{H}$. (NMP)

Remark. As before, by following [RZZ15, Theorem 2.2, p. 364] and using [Ond05, Theorem 2, p. 1007], we can establish the connection to a weak solution of the corresponding McKean–Vlasov SDE.

7.3. **Nonlinear assumptions.** The following assumptions on the coefficients b and σ are directly adapted from those in Subsection 2.3 by making the estimates uniform in the newly added dependence on measures.

(NN) Assume there exists a function $\mathcal{N}\in\mathfrak{U}^p$ for some $p\geq 2$ such that for every $n\in\mathbb{N}$ there exists a constant $C_n\geq 0$ with

$$\mathcal{N}(y)\leq C_n\|y\|_{\mathbb{H}_n}^p,$$

for any $y\in\mathbb{H}_n$.

(NA1) (Demicontinuity) For any $v\in\mathbb{X}$, $t\in[0,T]$, $\varrho\in\mathcal{P}(\mathbb{H})$ and every sequence $(y_k)_{k\in\mathbb{N}}$ with $y_k\xrightarrow[k\to\infty]{}y$ in $\mathbb{H}$, we have

$$\lim_{k\to\infty}{}_{\mathbb{X}^*}\langle b(t,y_k,\varrho),v\rangle_{\mathbb{X}}={}_{\mathbb{X}^*}\langle b(t,y,\varrho),v\rangle_{\mathbb{X}}$$

and

$$\lim_{k\to\infty}\left\|\sigma^*(t,y_k,\varrho)(v)-\sigma^*(t,y,\varrho)(v)\right\|_{\mathbb{U}}=0.$$

(NA2) (Coercivity) There exists a constant $\lambda_1\geq 0$ such that for all $y\in\mathbb{X}$, $t\in[0,T]$ and $\varrho\in\mathcal{P}(\mathbb{H})$

$${}_{\mathbb{X}^*}\langle b(t,y,\varrho),y\rangle_{\mathbb{X}}\leq-\mathcal{N}(y)+\lambda_1(1+\|y\|_{\mathbb{H}}^2)$$

holds.

(NA3) (Growth) There exist constants $\lambda_2, \lambda_3, \lambda_4 > 0$ and constants $\gamma' \geq \gamma > 1$ such that for all $y \in \mathbb{H}$, $t \in [0,T]$ and $\varrho \in \mathcal{P}(\mathbb{H})$ we have

$$\|b(t,y,\varrho)\|_{\mathbb{X}^*}^{\gamma} \leq \lambda_2 \mathcal{N}(y) + \lambda_3(1 + \|y\|_{\mathbb{H}}^{\gamma'})$$

and

$$\|\sigma(t,y,\varrho)\|_{L_2(\mathbb{U};\mathbb{H})}^2 \leq \lambda_4(1 + \|y\|_{\mathbb{H}}^2).$$

(NH1) For all $n \in \mathbb{N}$, $\varrho \in \mathcal{P}(\mathbb{H})$, the matrices $(a^{ij}(\cdot,\cdot,\varrho))_{1\leq i,j\leq n}$ are symmetric and nonnegative definite.

8. Nonlinear result

Let us state the main result of this third part, which is a direct adaption of Corollary 6.2. To be more precise, it is a superposition principle for a given probability solution μ on $[0,T] \times \mathbb{H}$ to a nonlinear Cauchy problem yielding existence of a martingale solution whose time-marginals are equal to μ_t.

Theorem 8.1. *Let the assumptions from Subsection 7.3 be fulfilled. Assume there exists a probability solution $\mu = \mu_t \, \mathrm{d}t$ on $[0,T] \times \mathbb{H}$ to the nonlinear Cauchy problem* (NCP) *in the sense of Definition 7.1 and subsequence $(\mu_{t,n_k})_{k\in\mathbb{N}}$ on $\mathbb{H}_{n_k}$ of a family of Borel probability measures on $\mathbb{H}_n$ with the following properties:*

- *The measures $(\mu_{t,n_k})_{k\in\mathbb{N}}$ are solutions to the finite-dimensional nonlinear Cauchy problems with coefficients $\Pi_{n_k} b(\cdot,\cdot,\mu_{\cdot,n_k})$ and $\Pi_{n_k} A(\cdot,\cdot,\mu_{\cdot,n_k})\Pi_{n_k}^*$ on $\mathbb{H}_{n_k}$ with the property that the mapping*

$$t \mapsto \int_{\mathbb{H}_{n_k}} \zeta(y)\, \mu_{t,n_k}(\mathrm{d}y)$$

  is continuous on $[0,T]$ for every $\zeta \in C_c^\infty(\mathbb{H}_{n_k})$.
- *For the family $(\bar{\mu}_{t,n_k})_{k\in\mathbb{N}}$ of extended measures to $\mathbb{H}$, we have $\bar{\mu}_{t,n_k} \xrightarrow[k\to\infty]{w} \mu_t$ for every $t \in [0,T]$.*
- *The integrability condition*

$$\int_0^T \int_{\mathbb{H}_{n_k}} \frac{\|\Pi_{n_k} A(t,y,\mu_{t,n_k})\Pi_{n_k}^*\| + |\langle \Pi_{n_k} b(t,y,\mu_{t,n_k}), y\rangle_{\mathbb{H}_{n_k}}|}{(1 + \|y\|_{\mathbb{H}_{n_k}})^2}\, \mu_{t,n_k}(\mathrm{d}y)\, \mathrm{d}t < \infty$$

  holds for every $k \in \mathbb{N}$.

Then there exists a martingale solution $P \in \mathcal{P}(\mathbb{S})$ to the martingale problem (NMP) *in the sense of Definition 2.2, for which*

$$P \circ x(t)^{-1} = \mu_t$$

holds for every $t \in [0,T]$.

As mentioned before, we follow the ideas presented in [BR20] and [BR18, Section 2] on dealing with nonlinearity for the proof.

Proof of Theorem 8.1. Given solutions μ to the nonlinear Cauchy problem (NCP) and μ_{n_k}, $k \in \mathbb{N}$, to the finite-dimensional nonlinear Cauchy problems, we "freeze" all of these measures and consider linear FPKEs of the form

$$\partial_t \varrho = L_\mu^* \, \varrho$$

and, for $k \in \mathbb{N}$,

$$\partial_t \varrho = L^*_{\mu_{n_k}} \varrho.$$

But then, μ and μ_{n_k}, for $k \in \mathbb{N}$, are again particular solutions of these linear FPKEs. Since our assumptions from Section 7.3 are uniform in the dependence on the measure, all assumptions from Corollary 6.2 hold for our new coefficients $b(\cdot,\cdot,\mu_\cdot)$ and $\sigma(\cdot,\cdot,\mu_\cdot)$ in the linear case. Consequently, we can just apply Corollary 6.2 and obtain the desired martingale solution P in the sense of Definition 2.2. □

Appendix A. Measure-separating families of finitely based functions

Let us follow up on the lemma on countable measure-separating families of finitely based functions, which can be proved by using similar techniques as in [MR92, p. 119].

Lemma A.1. *There exits a countable family $\mathcal{F}$ of functions in $\mathcal{F}C_c^\infty(\{e_i\})$, which*

i) separates points in $\mathbb{R}^\infty$,
ii) separates measures on $\mathcal{B}(\mathbb{R}^\infty)$ (i.e. for any two measures $\mu_1, \mu_2 \in \mathcal{B}(\mathbb{R}^\infty)$ with $\mu_1 \neq \mu_2$ there exists $f \in \mathcal{F}$ such that $\int_{\mathbb{R}^\infty} f \,\mathrm{d}\mu_1 \neq \int_{\mathbb{R}^\infty} f \,\mathrm{d}\mu_2$).

Proof. Let us begin by first leaving out the countability and showing the following simplified claim instead.

Claim: There exists a family $\tilde{\mathcal{F}} \subseteq \mathcal{F}C_c^\infty(\{e_i\})$ that separates points in $\mathbb{R}^\infty$.

Proof of Claim: Let $y_1, y_2 \in \mathbb{R}^\infty$ with $y_1 \neq y_2$. Consequently, there exists some $d \in \mathbb{N}$ such that their d-th component differs, i.e. $y_1^d \neq y_2^d$. We consider

$$\Pi_d^\infty y_1 = (y_1^1, \dots, y_1^d) \in \mathbb{R}^d \quad \text{and} \quad \Pi_d^\infty y_2 = (y_2^1, \dots, y_2^d) \in \mathbb{R}^d,$$

where Π_d^∞ is the projection onto $\mathbb{R}^d$, and obtain $\Pi_d^\infty y_1 \neq \Pi_d^\infty y_2$. By using the fact that points in $\mathbb{R}^d$ can be separated by functions of class $C_c^\infty(\mathbb{R}^d)$, there exists some $f \in C_c^\infty(\mathbb{R}^d)$ such that $f(\Pi_d^\infty y_1) \neq f(\Pi_d^\infty y_2)$. Hence, we can just consider the finitely based function $\xi\colon \mathbb{R}^\infty \longrightarrow \mathbb{R}$ given by $\xi(y) := f(\Pi_d^\infty(y))$, $y \in \mathbb{R}^\infty$. The family $\tilde{\mathcal{F}}$ is then chosen to be the collection of all such functions ξ. □

Now let us start with i): By using the claim from above, it remains to show that there exists a subset $\mathcal{F}$ of the family $\tilde{\mathcal{F}}$ that is in fact countable.

For any function $f \in \mathcal{F}C_c^\infty(\{e_i\})$, let $(f,f)\colon \mathbb{R}^\infty \times \mathbb{R}^\infty \longrightarrow \mathbb{R} \times \mathbb{R}$ be the function given by $(f,f)(y_1,y_2) := (f(y_1), f(y_2))$. Set $D_{\mathbb{R}^\infty} := \{(y_1,y_2) \in \mathbb{R}^\infty \times \mathbb{R}^\infty \mid y_1 = y_2\}$ to be the diagonal of $\mathbb{R}^\infty \times \mathbb{R}^\infty$ and analogously let $D_{\mathbb{R}}$ be the diagonal of $\mathbb{R} \times \mathbb{R}$. Then, since the functions $f \in \tilde{\mathcal{F}}$ separate points in $\mathbb{R}^\infty$, the equation

$$\big(\mathbb{R}^\infty \times \mathbb{R}^\infty\big) \setminus D_{\mathbb{R}^\infty} = \bigcup_{f \in \tilde{\mathcal{F}}} (f,f)^{-1}(\mathbb{R} \times \mathbb{R} \setminus D_{\mathbb{R}}) \tag{9}$$

holds.

Furthermore, note that $\mathbb{R}^\infty \times \mathbb{R}^\infty$ is a Polish space and, hence, a strongly Lindelöf space. Since $\big(\mathbb{R}^\infty \times \mathbb{R}^\infty\big) \setminus D_{\mathbb{R}^\infty}$ is an open subset of $\mathbb{R}^\infty \times \mathbb{R}^\infty$, the open cover on the right hand side of Equation (9) has a countable subcover, i.e. there exists a countable family $\mathcal{F} \subseteq \mathcal{F}C_c^\infty(\{e_i\})$ such that

$$\big(\mathbb{R}^\infty \times \mathbb{R}^\infty\big) \setminus D_{\mathbb{R}^\infty} = \bigcup_{f \in \mathcal{F}} (f,f)^{-1}(\mathbb{R} \times \mathbb{R} \setminus D_{\mathbb{R}}).$$

The assertion follows.

Finally let us prove ii): Without loss of generality we can assume that $\mathcal{F}$ is a multiplicative system. By a monotone class argument it then remains to prove that $\mathcal{F}$ generates $\mathcal{B}(\mathbb{R}^\infty)$. Since the functions in $\mathcal{F}$ are continuous, we immediately have $\sigma(\mathcal{F}) \subseteq \mathcal{B}(\mathbb{R}^\infty)$.

In addition, we consider the measurable function id mapping from the Polish space $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty))$ to the space $(\mathbb{R}^\infty, \sigma(\mathcal{F}))$ equipped with the countably generated σ-algebra $\sigma(\mathcal{F})$. By Kuratowski's theorem (see e.g. [Kur66, p. 487f] or [Par67, Section I.3, p. 15ff]) it follows that id^{-1} is $\sigma(\mathcal{F})/\mathcal{B}(\mathbb{R}^\infty)$-measurable, which implies that $\mathcal{B}(\mathbb{R}^\infty) \subseteq \sigma(\mathcal{F})$. $\square$

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