Optimal Consumption with Intertemporal Substitution under Knightian Uncertainty*

Giorgio Ferrari[†] Hanwu Li[‡] Frank Riedel [§]

Abstract

In this paper, we study the intertemporal consumption and portfolio choice problem under Knightian uncertainty where preferences exhibit local intertemporal substitution in the sense of Hindy, Huang, and Kreps. The existence can be obtained based on a theorem of Komlós and its infinite-dimensional extension by Kabanov while the uniqueness follows by the strict concavity of the felicity function. We also establish the necessary and sufficient condition for optimality of the consumptions plan. In a Markovian setting, we solve this problem explicitly when the time horizon is infinite.

Key words: Hindy-Huang-Kreps preferences, Knightian uncertainty, Ambiguity aversion, Singular control
MSC-classification: 60H30

1 Introduction

Intertemporal utility functions which describe the preferences between the choice behavior of agents play a fundamental role in the study of dynamic economic models, such as the optimal consumption and portfolio choice problem. One of the widely used candidates is the time additive utility functional in the form of

$$\mathsf{E}[\widetilde{U}(C)] = \mathsf{E}\left[\int_0^T u(t, c_t)\mathrm{d}t\right],$$

where $c = \{c_t\}$ represents the consumption rate and u is the felicity function, and the expectation is taken under an appropriate probability measure which reflects the agent's probability assessments.

This model implies that the cumulative consumption plan $C_t = \int_0^t c_s \mathrm{d}s$ $(t \in [0, T])$ is absolutely continuous which excludes the case that the consumption plan occurs in gulps or in a singular way. Besides, there is no discrete time version of this utility functional. It is worth pointing out that the most important feature of this utility functional is the time additivity, which means that the overall utility associated with a consumption plan is the integral over all period utilities, and the period utility only depends on the consumption generated in this period. This structure leads to some limitations and restrictions. For example, consider the following two consumption plans: one is to have a good meal once everyday during the week and the other is to have such meals seven times in one day without any meals for the rest of the week. It is convincible that the agents in the real world would

*Financial support by the German Research Foundation (DFG) through the Collaborative Research Centre 1283 "Taming uncertainty and profiting from randomness and low regularity in analysis, stochastics and their applications" is gratefully acknowledged by the authors.
[†]Center for Mathematical Economics, Bielefeld University, giorgio.ferrari@uni-bielefeld.de.
[‡]Center for Mathematical Economics, Bielefeld University, hanwu.li@uni-bielefeld.de.
[§]Center for Mathematical Economics, Bielefeld University, frank.riedel@uni-bielefeld.de.

choose the first plan rather than the second one because of the substitution effects. However, due to the time additivity, the utility of both consumption plans are equal since each single meal contributes to the total utility separately.

In order to overcome the drawbacks of the time additive utility functional, Hindy, Huang and Kreps [9] proposed a new kind of utility functional whose period utilities rely on the so called current level of satisfaction Y^C, where Y^C is the weighted average of the past consumption

$$Y_t^C = \eta_t + \int_0^t \theta_{t,s} \mathrm{d}C_s, \quad t \in [0,T].$$

Here $\eta : [0,T] \to \mathbb{R}$ and $\theta : [0,T]^2 \to \mathbb{R}$ are nonnegative deterministic functions. Besides, η_t can be regarded as the level of satisfaction at time t given exogenously and $\theta_{t,s}$ represents the weight assigned at time t to the consumption plan generated before t. Then the intertemporal utility functional can be defined as the following

$$\mathsf{E}[U(C)] = \mathsf{E}\left[\int_0^T u(t, Y_t^C)\mathrm{d}t\right],$$

where u is a continuous felicity function which is increasing and concave in its second component. This new utility function is more suitable in the economical sense that it do not require the absolute continuity of consumption plans and it exhibits some desirable substitution properties.

Based on the Hindy-Huang-Kreps (HHK for short) utility, Bank and Riedel investigated the optimal consumption choice problem under both deterministic case [2] and under uncertainty [3]. Under both cases, they established the existence and uniqueness result for optimality. Besides, a version of Kuhn-Tucker theorem adapted to both situations, called the first-order condition, which is necessary and sufficient for optimality, was obtained. In the stochastic framework, applying the first-order condition, they constructed a process, called the minimal level of satisfaction, by a kind of backward equation which takes over the role of the Hamilton-Jacobi-Bellman equation. Finally, they proved that the consumption plan which tracks the minimal level of satisfaction is optimal for the original problem if the cost of this consumption equals to the initial wealth w.

It is important to note that [3] only deals with the optimal consumption problem under uncertainty other than under ambiguity, which means that the agent has full confidence in the probability distribution of the financial market. This assumption is strong which exclude the case that the agent may face Knightian uncertainty. Indeed, the ambiguity case may occur when the financial market is quite new such that there are not enough data available to conjecture the probability distribution with high confidence. Therefore, in this paper, we analyze the consumption behavior with the "HHK preferences" under Knightian uncertainty. The Knightian uncertainty is measured by a kind of nonlinear expectation, called g-expectation, denoted by $\mathcal{E}^g[\cdot]$, which was initiated by Peng [16]. It is natural to assume that the agent is ambiguity-averse such that he will use the variational preference to evaluate his profit. When g is a concave function, the g-expectation can be represented as a variational expectation. We also give some examples of financial market where the evaluation of the consumption should be modeled by g-expectation. Our problem is try to maximize the g-expected value of the utility function $U(C)$ induced by HHK preferences associated with consumption plan C subjects to some admissible constraints.

We first establish the existence and uniqueness result of optimal consumption plan under g-expectation framework. The method is similar with [3] by applying the Komlós theorem (see [12]) and its infinite-dimensional extension introduced by Kabanov [11]. The uniqueness can be derived easily from the fact that the felicity function u is strictly concave and the g-expectation satisfies the strict comparison property.

We then investigate the characterization of the optimal consumption plan, i.e., the first-order condition, which will be different since the one derived in [3] is based on the assumption that the financial market is complete. In our framework, the most important concept is the worst case scenario which is a probability measure P^{ξ} whose Girsanov kernel is ξ such that

$$\mathsf{E}^{\mathsf{P}^{\xi}}\left[U(C)+\int_0^T f(s,\xi_s)\mathrm{d}s\right]=\mathcal{E}^g[U(C)],$$

where f is the convex dual of g. Then we conclude that the first-order condition holds true under the worst case scenario, which degenerates to the results in [3] when $g=0$.

Although we get the existence of optimality by using Komlós' theorem, the construction is implicit which requires us to find a new approach for the optimal consumption plan. Motivated by [3], the optimal consumption plan can be characterized by a stochastic process, called the minimal level of satisfaction, which is a solution of a typical kind of backward equation derived from the first-order condition. This fundamental equation takes over the role of the Hamilton-Jacobi-Bellman equation in the dynamic programming approach. The optimal consumption means that the agent consumes just enough to make sure his level of satisfaction stays above the minimal level. Compared with [3], the main difference is that the state-price density varies when we choose different probability P_1, P_2 from the multiple priors $\mathcal{P}_1$, $\mathcal{P}_2$ for the utility and cost, respectively. In order to make sure optimality, P_1 should be the lowest utility probability and P_2 should be the highest cost probability.

We can also investigate the optimal consumption problem in a more general setting, i.e., the evaluation is not given by a g-expectation but a variational preference induced by an appropriate multiple priors and penalty function. In this framework, existence of optimal consumption plans and the sufficiency for optimality still hold, which makes the construction by the minimal level of satisfaction remains valid. We would like to give some concrete solutions for the infinite time case, where the interest rate and the dynamics of level of satisfaction are deterministic. Note that the multiple priors can be characterized by the Radon-Nikodym derivatives or the Girsanov kernels. Roughly speaking, the optimal minimal level of satisfaction can be interpreted as a function of the ratio between the Radon-Nikodym derivatives. We also show how the portfolio process that finances the optimal consumption changes with respect to the difference between multiple priors for utility and cost. An interesting phenomenon is that, if the multiple priors for the utility and cost has a common element P, the optimal minimal level of satisfaction is a deterministic function. In this case, the agent will not invest in the risk asset at all.

The paper is organized as follows. In Section 2, we formulate the utility maximization problem under Knightian uncertainty in details. Then we establish the existence and uniqueness result in Section 3. In Section 4, we investigate the first-order conditions for optimality. Section 5 is devoted to study the properties and the construction of the optimal consumption plan. In the last section, we study the utility maximization problem in a general framework and solve this problem explicitly when the time horizon is infinite.

2 Utility maximization problem under Knightian uncertainty

2.1 Problem formulation

Consider a filtered probability space $(\Omega,\mathcal{F}_T,(\mathcal{F}_t)_{t\in[0,T]},\mathsf{P}_0)$ satisfying the usual conditions of right continuity and completeness in which $B=\{B_t\}_{t\in[0,T]}$ is a d-dimensional Brownian motion. We aim to study the optimal consumption choice of an agent facing Knightian uncertainty whose preference is of the Hindy-Huang-Kreps type. We assume that the agent will evaluate his utility X and cost Y according to g-expectation $\mathcal{E}^g[\cdot]$ initiated by Peng [16], where $g:\Omega\times[0,T]\times\mathbb{R}^d\to\mathbb{R}$ is a function satisfying the following assumptions:

(A1) For any $z \in \mathbb{R}^d$, $g(\cdot, \cdot, z)$ is progressively measurable and

$$\mathsf{E}\left[\int_0^T |g(t,z)|^2 dt\right] < \infty,$$

where the expectation $\mathsf{E}[\cdot]$ is taken under probability P_0;

(A2) There exists some constant $\kappa > 0$, such that for any $(\omega, t) \in \Omega \times [0, T]$ and $z, z' \in \mathbb{R}^d$,

$$|g(\omega, t, z) - g(\omega, t, z')| \leq \kappa |z - z'|;$$

(A3) g is concave in z;

(A4) $g(\omega, t, 0) = 0$ for any $(\omega, t) \in \Omega \times [0, T]$.

Roughly speaking, a g-expectation is the first component of the solution to a backward stochastic differential equation with driver g. For more details, we may refer to the Appendix.

We now introduce the formulation of our utility maximization problem under Knightian uncertainty. Let the interest rate $r = \{r_t\}_{t \in [0,T]}$ be a bounded, progressively measurable process and g, h be two functions satisfying (A1)-(A4). An agent with initial wealth $w > 0$ will choose a consumption plan from the budget feasible set

$$\mathcal{A}_h(w) = \{C \in \mathcal{X} | \Psi(C) := \widetilde{\mathcal{E}}^h \left[\int_0^T \gamma_t \mathrm{d}C_t\right] \leq w\},$$

where $\gamma_t = \exp\left(-\int_0^t r_s ds\right)$, $\widetilde{\mathcal{E}}^h[\cdot] = -\mathcal{E}^h[-\cdot] = \mathcal{E}^{\widetilde{h}}[\cdot]$ (we define $\widetilde{h}(t,z) = -h(t,-z)$) and

$$\mathcal{X} = \{C | C \text{ is the distribution function of a nonnegative optional random measure}\}.$$

Remark 2.1 *Indeed, the function $\Psi(C)$ can be seen as the minimal initial capital needed to finance a consumption plan C. It is easy to check that $\mathcal{A}_h(w)$ is nonempty when h satisfies (A1)-(A4) since $\widetilde{\mathcal{E}}^h[0] = 0$.*

For a fixed consumption plan C, the agent's level of satisfaction is given by

$$Y_t^C = \eta_t + \int_0^t \theta_{t,s} \mathrm{d}C_s,$$

where $\eta : [0, T] \to \mathbb{R}_+$ and $\theta : [0, T]^2 \to \mathbb{R}_+$ are continuous. Indeed, the quantity $\theta_{t,s}$ can be seen as the weight assigned at time t to consumption made at time $s \leq t$ and η_t describes an exogenously level of satisfaction for time t. Then the agent's utility of the HHK type is of the form

$$U(C) = \int_0^T u(t, Y_t^C) \mathrm{d}t,$$

where $u : [0, T] \times \mathbb{R}_+ \to \mathbb{R}$ is a continuous felicity function which is increasing and concave in its second component. The goal of the agent is to maximize his expected utility over all budget feasible consumption plans, i.e.,

$$v_{g,h}(w) = \sup_{C \in \mathcal{A}_h(w)} V(C) = \sup_{C \in \mathcal{A}_h(w)} \mathcal{E}^g[U(C)]. \tag{2.1}$$

For simplicity, we always omit the subscripts g, h in $v_{g,h}$.

Remark 2.2 *(i) It is worth pointing out that the integration over time intervals is carried out including the finite boundaries. For the consumption plan C, we assume that it starts from initial value $C_{0-} := 0$. If $C_0 > 0$ which means that there is an initial consumption gulp, then the random measure $\mathrm{d}C$ is positive at time 0. For any other integrator I, we also assume that $I_{0-} := 0$ unless otherwise stated.*
(ii) A common type of η and θ is that $\eta_t := \eta e^{-\beta t}$ and $\theta_{t,s} = \beta e^{-\beta(t-s)}$ with constants $\eta \geq 0, \beta > 0$.

2.2 Financial market with g-expectation

This subsection explains why we use g-expectation to evaluate the utility and the cost. It is natural to assume that the agent is ambiguity averse and hence, the agent will evaluate his utility according to some variational preferences. By Proposition A.1, when g satisfies (A1)-(A3), the g-expectation coincides with a variational preference. It is reasonable to use g-expectation to evaluate the utility.

Now, let us consider where g-expectation appears when evaluating the cost. Consider a financial market with $d+1$ assets. One of them is a riskless bond with price S^0 evolves according to the following equation

$$\mathrm{d}S_t^0 = rS_t^0 \mathrm{d}t, \tag{2.2}$$

where $r > 0$ is the interest rate. The price for the i-th stock is denoted by S^i and it satisfies the following stochastic differential equation

$$\mathrm{d}S_t^i = S_t^i \left[\mu^i \mathrm{d}t + \sum_{j=1}^{d} \sigma^{i,j} \mathrm{d}B_t^j \right], \tag{2.3}$$

where $\mu = (\mu^1, \cdots, \mu^d)$ represents the stock appreciation rates and $\sigma = (\sigma^{i,j})$ is the volatility matrix such that the inverse matrix σ^{-1} exists. Clearly, there exists a vector $\theta \in \mathbb{R}^d$ such that

$$\mu - r\mathbf{1} = \sigma\theta,$$

where $\mathbf{1}$ is the vector with each component equals to 1. The vector θ is called risk premium.

Consider an agent who decides how to invest in the financial market with portfolio $\pi_t = (\pi_t^1, \cdots, \pi_t^d)$ at time t, where π_t^i is the proportion of the wealth V_t to invest in the i-th stock, $i = 1, \cdots, d$ and $\pi_t^0 = 1 - \sum_{i=1}^d \pi_t^i$ is the proportion of the wealth to invest in the bond. The agent can also choose a consumption plan C, where C_t represents the total amount of consumption made before time t. Hence, it is a nondecreasing process. We assume that π and C are predictable since the agent can only make decisions with current information $\mathcal{F}_t$. Let $\bar{V}_t := e^{-rt}V_t$ be the present value. Suppose that the agent will consume all his wealth at the terminal time T.

Case 1. It is easy to check that

$$\mathrm{d}V_t = rV_t \mathrm{d}t - \mathrm{d}C_t + \pi_t \sigma V_t [\mathrm{d}B_t + \theta \mathrm{d}t]. \tag{2.4}$$

Hence, we have

$$\bar{V}_t = -\int_t^T \pi_t \sigma\theta \bar{V}_t \mathrm{d}t - \int_t^T \pi_s \sigma \bar{V}_s \mathrm{d}B_s + \int_t^T e^{-rs} \mathrm{d}C_s.$$

Let $g(z) = -\theta \cdot z$. By the definition of g-expectation, we have

$$\bar{V}_t = \mathcal{E}_t^g \left[\int_t^T e^{-rs} \mathrm{d}C_s \right].$$

Case 2. Let $d = 1$. Assume that the agent can borrow money at time t with interest rate $R > r$ (see Example 1.1 in [7]). Clearly, the amount borrowed at time t is $(1-\pi_t)^- V_t$ since the agent would not borrow money and invest money in the bond simultaneously. In this case, the triple of process $(\bar{V}, \pi, C)$ satisfies

$$\bar{V}_t = \int_t^T \left(\pi_s\sigma\theta - (R-r)(1-\pi_s)^-\right)\bar{V}_s \mathrm{d}s + \int_t^T \pi_s\sigma\bar{V}_s \mathrm{d}B_s + \int_t^T e^{-rs}\mathrm{d}C_s,$$

where $\bar{V}$ is the present value of wealth. Let $g(t, y, z) = \theta z - (R-r)(y - \frac{z}{\sigma})^-$. By the definition of g-expectation, we have

$$\bar{V}_t = \mathcal{E}^g_{t,T}\left[\int_t^T e^{-rs}\mathrm{d}C_s\right].$$

Case 3. Let $d = 1$. Assume that the risk premia for long and short positions are different and the difference between long and short positions is $\theta^1 - \theta^2$ (see Example 1.1 in [7]). Then the wealth V associated to the portfolio π and consumption plan C evolves according to the following equation

$$\mathrm{d}V_t = rV_t\mathrm{d}t + \sigma\theta^1\pi_t V_t\ dt + \sigma(\theta^1 - \theta^2)\pi_t^- V_t\mathrm{d}t + \sigma\pi_t V_t\mathrm{d}B_t - \mathrm{d}C_t,$$

which is equivalent to the following equation

$$\bar{V}_t = \int_t^T \sigma(\theta^2\pi_s^- - \theta^1\pi_s^+)\bar{V}_s\mathrm{d}s - \int_t^T \sigma\pi_s\bar{V}_s\mathrm{d}B_s - \int_t^T e^{-rs}\mathrm{d}C_s.$$

Let $g(z) = \theta^2 z^- - \theta^1 z^+$. Then we have

$$\bar{V}_t = \mathcal{E}^g_t\left[\int_t^T e^{-rs}\mathrm{d}C_s\right].$$

3 Existence and uniqueness

In this section, we investigate the existence and uniqueness result for the utility maximization problem (2.1). For this purpose, we need to propose the following assumption on the budget feasible set.

(H1) The family of budget feasible utilities $\{U(C), C \in \mathcal{A}_h(w)\}$ is uniformly P_0-square-integrable.

Remark 3.1 *According to Assumption (H1) and a priori estimates for BSDEs, we have*

$$v(w) = \sup_{C\in\mathcal{A}_h(w)} V(C) < \infty.$$

Before we study the existence and uniqueness of the optimal consumption plan, we recall the following technical lemma in [3].

Lemma 3.2 ([3]) **(i)** *There exists some constant $M \geq 0$, such that for any $C \in \mathcal{X}$ and $t \in [0, T]$,*

$$Y_t^C \leq M(1 + C_t);$$

(ii) *If $\{C^n\}_{n=1}^\infty \subset \mathcal{X}$ converges almost surely to $C \in \mathcal{X}$ in the weak topology of measures on $[0, T]$, then we have almost surely $Y_t^{C^n} \to Y_t^C$ for $t = T$ and for every point of continuity t of C.*

We first give a sufficient condition under which the Assumption (H1) holds. The proof of the sufficient condition needs the following lemma.

Lemma 3.3 *For any bounded progressively measurable processes ξ, σ and any constant p, the random variable* $\exp\big(p(\int_0^T \xi_s \mathrm{d}s + \int_0^T \sigma_s \mathrm{d}B_s)\big)$ *is integrable.*

Proof. It is easy to check that

$$\begin{aligned}
&\mathsf{E}\Big[\exp\Big(p(\int_0^T \sigma_t \mathrm{d}B_t + \int_0^T \xi_t \mathrm{d}t)\Big)\Big] \\
=&\mathsf{E}\Big[\exp\Big(\int_0^T p\sigma_t \mathrm{d}B_t - \int_0^T p^2\sigma_t^2 \mathrm{d}t\Big)\exp\Big(\int_0^T p^2\sigma_t^2 \mathrm{d}t + \int_0^T p\xi_t \mathrm{d}t\Big)\Big] \\
\le&\Big(\mathsf{E}\Big[\exp\big(\int_0^T 2p\sigma_t \mathrm{d}B_t - \frac{1}{2}\int_0^T (2p\sigma_t)^2 \mathrm{d}t\big)\Big]\Big)^{1/2}\Big(\mathsf{E}\Big[\exp\big(\int_0^T 2p^2\sigma_t^2 \mathrm{d}t + \int_0^T 2p\xi_t \mathrm{d}t\big)\Big]\Big)^{1/2} \\
=&\Big(\mathsf{E}\Big[\exp\big(\int_0^T 2p^2\sigma_t^2 \mathrm{d}t + \int_0^T 2p\xi_t \mathrm{d}t\big)\Big]\Big)^{1/2} \le C,
\end{aligned}$$

where we use the fact that $\{\exp\big(\int_0^t 2p\sigma_s \mathrm{d}B_s - \frac{1}{2}\int_0^t (2p\sigma_s)^2 \mathrm{d}s\big)\}_{t\in[0,T]}$ is a martingale by Novikov's condition. The proof is complete. ■

Lemma 3.4 *Suppose that the function h satisfies (A1)-(A4). The family of budget feasible utilities $\{U(C), C \in \mathcal{A}_h(w)\}$ is uniformly P_0-square-integrable if the following condition hold true: for some $\alpha \in (0, \frac{1}{2})$, the felicity function u satisfies the power growth condition*

$$|u(t, y)| \le M(1 + y^\alpha) \text{ for all } y \ge 0 \text{ uniformly in } t \in [0, T].$$

Proof. By the power growth condition and Lemma 3.2, it is easy to check that

$$U(0) \le U(C) \le M \int_0^T (1 + |Y_t^C|^\alpha) dt \le M(1 + C_T^\alpha).$$

Therefore, it suffices to show that the family $\{C_T^\alpha, C \in \mathcal{A}_h(w)\}$ is uniformly P_0-square-integrable. For any $p > 2$ with $\alpha p < 1$, by the Hölder inequality, we have

$$\mathsf{E}[C_T^{\alpha p}] \le \mathsf{E}\Big[C_T \frac{\mathrm{d}\mathsf{P}^\xi}{\mathrm{d}\mathsf{P}_0}\Big]^{\alpha p} \mathsf{E}\Big[\Big(\frac{\mathrm{d}\mathsf{P}^\xi}{\mathrm{d}\mathsf{P}_0}\Big)^{\frac{-\alpha p}{1-\alpha p}}\Big]^{1-\alpha p},$$

where $\xi \in D_h$ and $|\xi| \le \kappa$ (the definition of D_h can be found in Proposition A.1). By Lemma 3.3, $\mathsf{E}\Big[\Big(\frac{\mathrm{d}\mathsf{P}^\xi}{\mathrm{d}\mathsf{P}_0}\Big)^{\frac{-\alpha p}{1-\alpha p}}\Big] \le M$. Let l be the convex dual of h. Recalling Proposition A.1, we have

$$\begin{aligned}
\mathsf{E}\Big[C_T \frac{\mathrm{d}\mathsf{P}^\xi}{\mathrm{d}\mathsf{P}_0}\Big] &= \mathsf{E}^{\mathsf{P}^\xi}[C_T] \le M\mathsf{E}^{\mathsf{P}^\xi}\Big[\int_0^T \gamma_t \mathrm{d}C_t\Big] \\
&\le M\Big(\widetilde{\mathcal{E}}^h\Big[\int_0^T \gamma_t \mathrm{d}C_t\Big] + \mathsf{E}^{\mathsf{P}^\xi}\Big[\int_0^T l(s, \xi_s)\mathrm{d}s\Big]\Big).
\end{aligned}$$

Noting that $\xi \in D_h$, by Lemma 3.3, it follows that

$$\mathsf{E}^{\mathsf{P}^\xi}\Big[\int_0^T l(s, \xi_s)\mathrm{d}s\Big] = \mathsf{E}\Big[\frac{\mathrm{d}\mathsf{P}^\xi}{\mathrm{d}\mathsf{P}_0}\int_0^T l(s, \xi_s)\mathrm{d}s\Big] \le \Big(\mathsf{E}\Big[\Big(\frac{\mathrm{d}\mathsf{P}^\xi}{\mathrm{d}\mathsf{P}_0}\Big)^2\Big]\Big)^{1/2}\Big(T\mathsf{E}\Big[\int_0^T l^2(s, \xi_s)\mathrm{d}s\Big]\Big)^{1/2} < \infty. \tag{3.1}$$

All the above analysis implies that the family $\{C_T^\alpha, C \in \mathcal{A}_h(w)\}$ is p-integrable under P_0. Finally, we get the desired result. ■

Theorem 3.5 *Suppose that the functions g, h satisfy (A1)-(A4). Under Assumption (H1), the utility maximization problem* (2.1) *has a solution. Moreover, if $u(t,\cdot)$ is strictly concave for every $t \in [0,T]$ and $C \to Y^C$ is injective up to P_0-indistinguishability, the solution is unique.*

Proof. There exists a sequence $\{C^n\}_{n=1}^{\infty} \subset \mathcal{A}_h(w)$ such that

$$\sup_{C\in\mathcal{A}(w)} V(C) = \lim_{n\to\infty} V(C^n).$$

Since the interest rate is bounded, for any $\xi \in D_h$, we have

$$\begin{aligned}\sup_{C\in\mathcal{A}(w)} \mathsf{E}^{\mathsf{P}^\xi}[C_T] &\le M \sup_{C\in\mathcal{A}(w)} \mathsf{E}^{\mathsf{P}^\xi}\left[\int_0^T \gamma_t \mathrm{d}C_t\right] \\ &\le M\left\{\sup_{C\in\mathcal{A}(w)} \widetilde{\mathcal{E}}^h\left[\int_0^T \gamma_t d\ C_t\right] + \mathsf{E}^{\mathsf{P}^\xi}\left[\int_0^T l(s,\xi_s)\mathrm{d}s\right]\right\} < \infty,\end{aligned}$$

where we use Equation (3.1) in the last inequality. Then, by Kabanov's version of Komlós' theorem, there exists a subsequence, for simplicity, still denoted by $\{C^n\}$, such that P^ξ-a.s.

$$\widetilde{C}_t^n := \frac{1}{n}\sum_{k=1}^n C_t^k \to C_t^*, \text{ as } n\to\infty$$

for $t = T$ and for every point of continuity t of C^*. Since P^ξ is equivalent to P_0, the above convergence holds P_0-a.s. We claim that $\{\widetilde{C}^n\}$ is also a maximizing sequence for problem (2.1). Indeed, the convexity of $\widetilde{\mathcal{E}}^h[\cdot]$ implies that $\widetilde{C}^n \in \mathcal{A}_h(w)$, for any $n \in \mathbb{N}$. Therefore, we have $V(\widetilde{C}^n) \le \sup_{C\in\mathcal{A}_h(w)} V(C)$. On the other hand, it is easy to check that

$$Y_t^{\widetilde{C}^n} = \frac{1}{n}\sum_{k=1}^n Y_t^{C^k}.$$

Since $u(t,\cdot)$ and $\mathcal{E}^g[\cdot]$ are both concave, we obtain that

$$V(\widetilde{C}^n) = \mathcal{E}^g[U(\widetilde{C}^n)] \ge \frac{1}{n}\sum_{k=1}^n \mathcal{E}^g[U(C^k)] = \frac{1}{n}\sum_{k=1}^n V(C^k).$$

Noting that $\{C^n\}$ is a maximizing sequence, it follows that

$$\liminf_{n\to\infty} V(\widetilde{C}^n) \ge \sup_{C\in\mathcal{A}_h(w)} V(C).$$

Therefore, the claim holds. We then show that C^* is optimal for problem (2.1). Since γ is continuous, we have

$$\lim_{n\to\infty}\int_0^T \gamma_t \mathrm{d}\widetilde{C}_t^n = \int_0^T \gamma_t \mathrm{d}C_t^*, \ \mathsf{P}_0\text{-a.s.}$$

It follows from Fatou's lemma and the convexity of $\widetilde{\mathcal{E}}^h[\cdot]$ that

$$\widetilde{\mathcal{E}}^h\left[\int_0^T \gamma_t \mathrm{d}C_t^*\right] \le \liminf_{n\to\infty} \widetilde{\mathcal{E}}^h\left[\int_0^T \gamma_t \mathrm{d}\widetilde{C}_t^n\right] \le \liminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \widetilde{\mathcal{E}}^h\left[\int_0^T \gamma_t \mathrm{d}C_t^n\right] \le w,$$

which implies that $C^* \in \mathcal{A}_h(w)$. Besides, by Lemma 3.2, we have $U(\widetilde{C}^n) \to U(C^*)$, P_0-a.s. Applying Assumption (H1) yields that $\mathsf{E}[|U(\widetilde{C}^n) - U(C^*)|^2] \to 0$. By the estimate for BSDE, we obtain that

$$V(\widetilde{C}^n) = \mathcal{E}^g[U(\widetilde{C}^n)] \to \mathcal{E}^g[U(C^*)] = V(C^*).$$

Recalling that $\{\widetilde{C}^n\}$ is a maximizing sequence of problem (2.1), it follows that C^* is optimal.

It remains to prove the uniqueness. If C_1 and C_2 are optimal and they are not indistinguishable, then the levels of satisfaction $Y^1 = Y^{C^1}$ and $Y^2 = Y^{C^2}$ are not indistinguishable neither. By a similar analysis as the proof of Theorem 2.3 in [3], on a set with positive probability, Y^1 and Y^2 differ on an open time interval. By the strict concavity of $u(t,\cdot)$ and strict comparison theorem, we have

$$\begin{aligned}
V(\frac{1}{2}(C^1+C^2)) &= \mathcal{E}^g\Big[\int_0^T u\big(t, \frac{1}{2}(Y_t^1+Y_t^2)\big)\mathrm{d}t\Big] \\
&> \mathcal{E}^g\Big[\int_0^T \frac{1}{2}\big(u(t,Y_t^1)+u(t,Y_t^2)\big)\mathrm{d}t\Big] \\
&\geq \frac{1}{2}\Big(\mathcal{E}^g\Big[\int_0^T u(t,Y_t^1)\mathrm{d}t\Big] + \mathcal{E}^g\Big[\int_0^T u(t,Y_t^2)\mathrm{d}t\Big]\Big) \\
&= \frac{1}{2}\big(V(C^1)+V(C^2)\big) = \sup_{C\in\mathcal{A}_h(w)} V(C),
\end{aligned}$$

which is a contradiction. ■

Remark 3.6 *We may check that if $\theta_{t,s} = \theta_t^1\theta_s^2$ for some strictly positive, continuous functions $\theta^1, \theta^2 : [0,T] \to \mathbb{R}^+$, then the mapping $C \mapsto Y^C$ is injective. Especially, if θ is of the form of Remark 2.2, the injectivity follows. Besides, Theorem 3.5 holds true even if g does not satisfy (A4).*

Remark 3.7 *We could also consider the utility maximization problem under general nonlinear expectations*

$$\begin{aligned}
\mathcal{E}[X] &= \inf_{\mathsf{P}\in\mathcal{P}_1}(\mathsf{E}^{\mathsf{P}}[X]+c_1(\mathsf{P})), \\
\widetilde{\mathcal{E}}[Y] &= \sup_{\mathsf{P}\in\mathcal{P}_2}(\mathsf{E}^{\mathsf{P}}[Y]-c_2(\mathsf{P})).
\end{aligned}$$

For this purpose, for any fixed constant $p > 1$, we assume that the multiple priors and penalty functions satisfy the following assumptions:

(i) $\sup_{P\in\mathcal{P}_1}\mathsf{E}\big[\big|\frac{\mathrm{d}\mathsf{P}}{\mathrm{d}\mathsf{P}_0}\big|^p\big] < \infty$, *where $p > 1$;*

(ii) $0 \leq \inf_{P\in\mathcal{P}_2} c_2(P) \leq \sup_{P\in\mathcal{P}_2} c_2(P) < \infty$.

Now, the budget feasible set can be defined as follows

$$\mathcal{A}(w) = \{C \in \mathcal{X} | \widetilde{\mathcal{E}}\Big[\int_0^T \gamma_t \mathrm{d}C_t\Big] \leq w\},$$

where γ is the discount factor associated with a bounded, progressively measurable interest rate r and $\mathcal{X}$ is the collection of all distribution functions of nonnegative optional random measures as in Section 2. By Assumption (ii), $\mathcal{A}(w)$ is non-empty for any given initial wealth $w > 0$. The level of satisfaction Y^C and the agent's utility $U(C)$ are the same with those in Section 2. The agent aims to maximize his expected utility over all budget feasible consumption plans and the value function is defined as the following:

$$v(w) = \sup_{C\in\mathcal{A}(w)} \mathcal{E}[U(C)]. \tag{3.2}$$

Suppose that the family of budget feasible utilities $\{U(C), C \in \mathcal{A}(w)\}$ is uniformly p^*-integrable under P_0, where $p^* = p/(p-1)$. Then, the utility maximization problem (3.2) has a solution. However, due to the lack of strict comparison property for $\mathcal{E}[\cdot]$, we do not have the uniqueness result as Theorem 3.5.

4 First-order condition for optimality

This section is devoted to the proof of first-order conditions for optimality. For any two functions g, h satisfying (A1)-(A3), let f, l be the convex dual respectively. Now for any budget feasible consumption plan C of problem (2.1). Set

$$\mathcal{P}_1(C) := \left\{ P^\xi | \xi \in D_g, \mathcal{E}^g[U(C)] = \mathsf{E}^{\mathsf{P}^\xi} \left[U(C) + \int_0^T f(s, \xi_s) \mathrm{d}s \right] \right\},$$
$$\mathcal{P}_2(C) := \left\{ P^\xi | \xi \in D_h, \widetilde{\mathcal{E}}^h \left[\int_0^T \gamma_t dC_t \right] = \mathsf{E}^{\mathsf{P}^\xi} \left[\int_0^T \gamma_t \mathrm{d}C_t - \int_0^T l(s, \xi_s) \mathrm{d}s \right] \right\}.$$

In fact, $\mathcal{P}_1(C)$ can be regarded as the collection of lowest-utility probabilities and $\mathcal{P}_2(C)$ the collection of highest-cost probabilities for the consumption plan C.

In order to obtain the first-order condition, we need to propose the following additional assumptions on the felicity function.

(H2) The felicity function u is strictly concave and differentiable in the second argument.

Theorem 4.1 *Suppose that the functions g, h satisfies Assumptions (A1)-(A4). Under Assumptions (H1)-(H2), a consumption plan C^* solves the utility maximization problem* (2.1) *if there exist some $\mathsf{P}_i := \mathsf{P}^{\xi^i} \in \mathcal{P}_i(C^*)$, $i = 1, 2$ such that*

(1) $\widetilde{\mathcal{E}}^h \big[\int_0^T \gamma_t \mathrm{d}C_t^* \big] = w$;

(2) $\mathsf{E}_t^{\mathsf{P}_2} \big[\frac{\mathrm{d}\mathsf{P}_1}{\mathrm{d}\mathsf{P}_2} \int_t^T \partial_y u(s, Y_s^*) \theta_{s,t} \mathrm{d}s \big] \le M \gamma_t$ *for any* $t \in [0, T]$ *a.s.;*

(3) $\mathsf{E}^{\mathsf{P}_1} \big[\int_0^T \int_t^T \partial_y u(s, Y_s^*) \theta_{s,t} \mathrm{d}s \mathrm{d}C_t^* \big] = M \mathsf{E}^{\mathsf{P}_2} \big[\int_0^T \gamma_t \mathrm{d}C_t^* \big]$,

where $M > 0$ is a finite Lagrange multiplier, $Y^* = Y^{C^*}$, $i = 1, 2$.

Proof. We first prove the sufficiency. Assume that C^* satisfies conditions (1)-(3) and consider another budget feasible consumption plan $C \in \mathcal{A}(w)$. For simplicity, set $Y = Y^C$ and

$$I = \mathsf{E}^{\mathsf{P}_1} \left[\int_0^T \int_0^s \partial_y u(s, Y_s^*) \theta_{s,t} \mathrm{d}C_t^* \mathrm{d}s \right] = \mathsf{E}^{\mathsf{P}_1} \left[\int_0^T \int_t^T \partial_y u(s, Y_s^*) \theta_{s,t} \mathrm{d}s \mathrm{d}C_t^* \right],$$
$$II = \mathsf{E}^{\mathsf{P}_1} \left[\int_0^T \int_0^s \partial_y u(s, Y_s^*) \theta_{s,t} \mathrm{d}C_t \mathrm{d}s \right] = \mathsf{E}^{\mathsf{P}_1} \left[\int_0^T \int_t^T \partial_y u(s, Y_s^*) \theta_{s,t} \mathrm{d}s \mathrm{d}C_t \right],$$

where we have used the Fubini theorem. Noting that $\mathsf{P}_2 \in \mathcal{P}_2(C^*)$, it is easy to check that

$$\begin{aligned} I = M \mathsf{E}^{\mathsf{P}_2} \left[\int_0^T \gamma_t \mathrm{d}C_t^* \right] &= M \mathsf{E}^{\mathsf{P}_2} \left[\int_0^T \gamma_t \mathrm{d}C_t^* + \int_0^T l(s, \xi_s^2) \mathrm{d}s - \int_0^T l(s, \xi_s^2) \mathrm{d}s \right] \\ &= M \widetilde{\mathcal{E}}^h \left[\int_0^T \gamma_t \mathrm{d}C_t^* \right] + M \mathsf{E}^{\mathsf{P}_2} \left[\int_0^T l(s, \xi_s^2) \mathrm{d}s \right] = M w + M \mathsf{E}^{\mathsf{P}_2} \left[\int_0^T l(s, \xi_s^2) \mathrm{d}s \right], \end{aligned}$$

and

$$
\begin{aligned}
II =&\mathsf{E}^{\mathsf{P}_2}\Big[\int_0^T \frac{\mathrm{d}\mathsf{P}_1}{\mathrm{d}\mathsf{P}_2}\int_t^T \partial_y u(s,Y_s^*)\theta_{s,t}\mathrm{d}s\mathrm{d}C_t\Big]\\
=&\mathsf{E}^{\mathsf{P}_2}\Big[\int_0^T \mathsf{E}_t^{\mathsf{P}_2}\Big[\frac{\mathrm{d}\mathsf{P}_1}{\mathrm{d}\mathsf{P}_2}\int_t^T \partial_y u(s,Y_s^*)\theta_{s,t}\mathrm{d}s\Big]\mathrm{d}C_t\Big]\\
\le& M\mathsf{E}^{\mathsf{P}_2}\Big[\int_0^T \gamma_t\mathrm{d}C_t+\int_0^T l(s,\xi_s^2)\mathrm{d}s-\int_0^T l(s,\xi_s^2)\mathrm{d}s\Big]\\
=&M\widetilde{\mathcal{E}}^h\Big[\int_0^T \gamma_t\mathrm{d}C_t\Big]+M\mathsf{E}^{\mathsf{P}_2}\Big[\int_0^T l(s,\xi_s^2)\mathrm{d}s\Big]\\
\le& Mw+M\mathsf{E}^{\mathsf{P}_2}\Big[\int_0^T l(s,\xi_s^2)\mathrm{d}s\Big],
\end{aligned}
$$

where we use Theorem (1.33) in Jacob [10] in the second equation. Noting that u is strictly concave in the second argument, we have

$$
\begin{aligned}
V(C^*)-V(C)\ge&\mathsf{E}^{\mathsf{P}_1}\Big[U(C^*)+\int_0^T f(s,\xi_s^1)\mathrm{d}s\Big]-\mathsf{E}^{\mathsf{P}_1}\Big[U(C)+\int_0^T f(s,\xi_s^1)\mathrm{d}s\Big]\\
=&\mathsf{E}^{\mathsf{P}_1}\Big[\int_0^T (u(s,Y_s^*)-u(s,Y_s))\mathrm{d}s\Big]\\
\ge&\mathsf{E}^{\mathsf{P}_1}\Big[\int_0^T \partial_y u(s,Y_s^*)(Y_s^*-Y_s)\mathrm{d}s\Big]\\
=&\mathsf{E}^{P_1}\Big[\int_0^T\int_0^s \partial_y u(s,Y_s^*)\theta_{s,t}(\mathrm{d}C_t^*-\mathrm{d}C_t)\mathrm{d}s\Big]\\
=&I-II\ge 0.
\end{aligned}
$$

The proof is complete. ■

Remark 4.2 *(i) If g does not satisfy (A4), Theorem 4.1 still holds.*
(ii) Let $\phi_t=\mathsf{E}_t^{\mathsf{P}_1}\big[\int_t^T \partial_y u(s,Y_s^*)\theta_{s,t}\mathrm{d}s\big]$. Applying the Bayes rule, the condition (2) is equivalent to the following

$$
\phi_t\le M\gamma_t\mathsf{E}_t^{\mathsf{P}_1}\Big[\frac{\mathrm{d}\mathsf{P}_2}{\mathrm{d}\mathsf{P}_1}\Big],\quad t\in[0,T].
$$

In fact, the quantity $\gamma_t\mathsf{E}_t^{\mathsf{P}_1}\big[\frac{\mathrm{d}\mathsf{P}_2}{\mathrm{d}\mathsf{P}_1}\big]$ is called the state-price density in [3].
(iii) In fact, if C^*, P_1 and P_2 satisfy conditions (2) and (3), then for any stopping time $S\le T$, we have

$$
\mathsf{E}_S^{\mathsf{P}_2}\Big[\int_S^T \widetilde{\phi}_t\mathrm{d}C_t^*\Big]=M\mathsf{E}_S^{\mathsf{P}_2}\Big[\int_S^T \gamma_t\mathrm{d}C_t^*\Big], \tag{4.1}
$$

where

$$
\widetilde{\phi}_t=\mathsf{E}_t^{\mathsf{P}_2}\Big[\frac{\mathrm{d}\mathsf{P}_1}{\mathrm{d}\mathsf{P}_2}\int_t^T \partial_y u(s,Y_s^*)\theta_{s,t}\mathrm{d}s\Big],\ t\in[0,T].
$$

We then give the proof of this statement. Noting that conditions (2) and (3) yield that for any stopping time $S\le T$,

$$
0=\mathsf{E}^{\mathsf{P}_2}\Big[\int_0^T(\widetilde{\phi}_t-M\gamma_t)\mathrm{d}C_t^*\Big]\le\mathsf{E}^{\mathsf{P}_2}\Big[\int_S^T(\widetilde{\phi}_t-M\gamma_t)\mathrm{d}C_t^*\Big]\le 0,
$$

which implies that $\mathsf{E}^{\mathsf{P}_2}\big[\int_S^T(\widetilde{\phi}_t - M\gamma_t)\mathrm{d}C_t^*\big]=0$. *It is easy to check that*

$$\mathsf{E}_S^{\mathsf{P}_2}\Big[\int_S^T \widetilde{\phi}_t \mathrm{d}C_t^*\Big] \leq M\mathsf{E}_S^{\mathsf{P}_2}\Big[\int_S^T \gamma_t \mathrm{d}C_t^*\Big].$$

If Equation (4.1) *does not hold, we assume that* $\mathsf{P}_2(A)>0$, *where*

$$A=\{\mathsf{E}_S^{\mathsf{P}_2}\Big[\int_S^T \widetilde{\phi}_t \mathrm{d}C_t^*\Big] < M\mathsf{E}_S^{\mathsf{P}_2}\Big[\int_S^T \gamma_t \mathrm{d}C_t^*\Big]\}.$$

By the strict comparison property, we have

$$\mathsf{E}^{\mathsf{P}_2}\Big[\int_S^T (\widetilde{\phi}_t - M\gamma_t)\mathrm{d}C_t^*\Big] < 0,$$

which leads to a contradiction.

Remark 4.3 *Consider the general utility maximization problem introduced in Remark 3.7. We then establish the characterization for optimality. Now, for any budget feasible consumption plan* $C\in\mathcal{A}_w$, *we define*

$$\mathcal{P}^1(C) := \{\mathsf{P}\in\mathcal{P}_1|\mathcal{E}[U(C)] = \mathsf{E}^{\mathsf{P}}[U(C)] + c_1(\mathsf{P})\},$$
$$\mathcal{P}^2(C) := \left\{\mathsf{P}\in\mathcal{P}_2|\widetilde{\mathcal{E}}\Big[\int_0^T \gamma_t \mathrm{d}C_t\Big] = \mathsf{E}^{\mathsf{P}}\Big[\int_0^T \gamma_t \mathrm{d}C_t\Big] - c_2(\mathsf{P})\right\}.$$

Let Assumption (H2) hold. Suppose that the family of budget feasible utilities $\{U(C), C\in\mathcal{A}(w)\}$ *is uniformly* p^**-integrable. A consumption plan* C^* *solves the utility maximization problem* (3.2) *if there exist some* $\mathsf{P}_i\in\mathcal{P}^i(C^*)$, $i=1,2$ *such that*

(1) $\widetilde{\mathcal{E}}\big[\int_0^T \gamma_t \mathrm{d}C_t^*\big] = w$;

(2) $\mathsf{E}_t^{\mathsf{P}_2}\big[\frac{\mathrm{d}\mathsf{P}_1}{\mathrm{d}\mathsf{P}_2}\int_t^T \partial_y u(s,Y_s^*)\theta_{s,t}\mathrm{d}s\big] \leq M\gamma_t$ *for any* $t\in[0,T]$ *a.s.;*

(3) $\mathsf{E}^{\mathsf{P}_1}\big[\int_0^T\int_t^T \partial_y u(s,Y_s^*)\theta_{s,t}\mathrm{d}s\mathrm{d}C_t^*\big] = M\mathsf{E}^{\mathsf{P}_2}\big[\int_0^T \gamma_t \mathrm{d}C_t^*\big]$,

where $M>0$ *is a finite Lagrange multiplier,* $Y^*=Y^{C^*}$, $i=1,2$.

5 The structure of optimal consumption plans

In this section, let us first study the optimal consumption problem dynamically and prove a version of the dynamic programming principle, which indicates that if a consumption plan is optimal at time zero, then it is also optimal at any latter time.

Proposition 5.1 *Suppose that the functions* g,h *satisfy (A1)-(A4). Let* $S\leq T$ *be a stopping time and* C^* *be the optimal consumption plan for the utility maximization problem* (2.1). *Set*

$$\mathcal{A}_S(C^*) = \{C\in\mathcal{X}|C|_{[0,S)}\equiv C^*|_{[0,S)}, \Psi_S(C)\leq\Psi_S(C^*)\},$$

where

$$\Psi_S(C) := \frac{1}{\gamma_S}\widetilde{\mathcal{E}}_S^g\Big[\int_S^T \gamma_t \mathrm{d}C_t\Big].$$

Consider the following optimal consumption problem

$$V_S := \operatorname*{ess\,sup}_{C\in\mathcal{A}_S(C^*)} \mathcal{E}^h_S[U(C)]. \tag{5.1}$$

Assume that the felicity function u satisfies $u(t,0)=0$ for any $t\in[0,T]$. Then the value function V is an $\mathcal{E}^g$-supermartingale in strong sense. Besides, $C^\star$ is optimal for (5.1).

Remark 5.2 *A process V is called an $\mathcal{E}^g$-supermartingale in strong sense if $V_\tau \in L^2(\mathcal{F}_\tau)$ for any stopping time τ and for any stopping times τ and σ taking values in $[0,T]$ with $\tau\le\sigma$, we have $\mathcal{E}^g_\tau[V_\sigma]\le V_\tau$.*

Proof. We first show that the family $\{\mathcal{E}^g_S[U(C)], C\in\mathcal{A}_S(C^*)\}$ is upward directed, where S is a stopping time. For any $C^i\in\mathcal{A}_S(C^*)$, $i=1,2$, set $C=C^1I_A+C^2I_{A^c}$, where $A=\{\mathcal{E}^g_S[U(C^1)]\ge \mathcal{E}^g_S[U(C^2)]\}$ is $\mathcal{F}_S$-measurable. Note that

$$\Psi_S(C)=\frac{1}{\gamma_t}\widetilde{\mathcal{E}}^h_S\Big[\int_S^T\gamma_r\mathrm{d}C^1_r\Big]I_A+\frac{1}{\gamma_t}\widetilde{\mathcal{E}}^h_S\Big[\int_S^T\gamma_r\mathrm{d}C^2_r\Big]I_{A^c}\le\Psi_S(C^*),$$

which implies that $C\in\mathcal{A}_S(C^*)$. It is easy to check that for any $s\in[0,T]$, $Y^C_s=Y^{C^1}_sI_A+Y^{C^2}_sI_{A^c}$. Since $u(s,0)=0$, it follows that $u(s,Y^c_s)=u(s,Y^{C^1}_s)I_A+u(s,Y^{C^2}_s)I_{A^c}$. Therefore, we have

$$\begin{aligned}\mathcal{E}^g_S[U(C)]=&\mathcal{E}^g_S[U(C^1)I_A+U(C^2)I_{A^c}]=\mathcal{E}^g_S[U(C^1)]I_A+\mathcal{E}^g_S[U(C^2)]I_{A^c}\\=&\mathcal{E}^g_S[U(C^1)]\vee\mathcal{E}^g_S[U(C^2)].\end{aligned}$$

The claim holds true. Thus there exists an increasing sequence $\{\mathcal{E}^g_S[U(C^n)]\}_{n=1}^\infty$ such that

$$V_S=\lim_{n\to\infty}\mathcal{E}^g_S[U(C^n)],$$

where $\{C^n\}_{n=1}^\infty\subset\mathcal{A}_S(C^*)$.

We then prove that for any stopping times τ,σ with $\tau\le\sigma$, $\mathcal{A}_\sigma(C^*)\subset\mathcal{A}_\tau(C^*)$. For any $C\in\mathcal{A}_\sigma(C^*)$, a simple calculation yields that

$$\begin{aligned}\Psi_\tau(C)=&\frac{1}{\gamma_\tau}\widetilde{\mathcal{E}}^h_\tau\Big[\int_\tau^T\gamma_r\mathrm{d}C_r\Big]=\frac{1}{\gamma_\tau}\widetilde{\mathcal{E}}^h_\tau\Big[\int_\tau^\sigma\gamma_r\mathrm{d}C^*_r+\gamma_\sigma\Psi_\sigma(C)\Big]\\\le&\frac{1}{\gamma_\tau}\widetilde{\mathcal{E}}^h_\tau\Big[\int_\tau^\sigma\gamma_r\mathrm{d}C^*_r+\gamma_\sigma\Psi_\sigma(C^*)\Big]=\Psi_\tau(C^*).\end{aligned}$$

Hence the conclusion follows. Now for any $\tau\le S$, it is easy to check that

$$\begin{aligned}\mathcal{E}^g_\tau[V_S]&=\mathcal{E}^g_\tau[\lim_{n\to\infty}\mathcal{E}^g_S[U(C^n)]]=\lim_{n\to\infty}\mathcal{E}^g_\tau[\mathcal{E}^g_S[U(C^n)]]\\&=\lim_{n\to\infty}\mathcal{E}^g_\tau[U(C^n)]\le\operatorname*{ess\,sup}_{C\in\mathcal{A}_\tau(C^*)}\mathcal{E}^g_\tau[U(C)]=V_\tau,\end{aligned}$$

where $\{C^n\}_{n=1}^\infty$ is taken as in the first step.

It remains to show that $C^\star$ is optimal for problem (5.1). By a similar analysis as the proof of Theorem 3.5, there exists a unique consumption plan $C^{S,\star}$ which is optimal for problem (5.1). Suppose that $C^{S,\star}$ and $C^\star$ are distinguishable on $[S,T]$. Consequently, we have

$$\mathcal{E}^g_S[U(C^{S,\star})]>\mathcal{E}^g_S[U(C^\star)].$$

Applying the strict comparison theorem for g-expectation yields that

$$\mathcal{E}^g[U(C^{S,\star})] = \mathcal{E}^g[\mathcal{E}^g_S[U(C^{S,\star})]] > \mathcal{E}^g[\mathcal{E}^g_S[U(C^\star)]] = \mathcal{E}^g[U(C^\star)],$$

which leads to a contradiction. ■

Theorem 3.5 indicates that the optimal consumption plan C^* exists while it does not give an explicit form of C^*. Motivated by [3] and the sufficiency of first-order conditions for optimality, we may construct C^* by a progressively measurable process L, called the minimal level of satisfaction, which is the solution of a backward equation (see Equation (5.2)). Obviously, since we consider the maximization problem (2.1) under multiple priors $\mathcal{P}$, the process L depends on the probability measure $\mathsf{P} \in \mathcal{P}$ as well as the Lagrange multiplier M as considered in [3]. We point out that the minimal level equation (5.2) in our method takes over the role of the Hamilton-Jacobi-Bellman equation in the dynamic programming approach.

In order to get the concrete form of the optimal consumption plan, we need the following dynamics for the level of satisfaction.

(H3) The function $\eta : [0, T] \to \mathbb{R}$ and $\theta : [0, T]^2 \to \mathbb{R}$ are of following forms:

$$\eta_t = \eta \exp\Big(-\int_0^t \beta_s \mathrm{d}s\Big), \quad \theta_{t,s} = \beta_s \exp\Big(-\int_s^t \beta_r \mathrm{d}r\Big), \quad 0 \le s \le t \le T,$$

where $\beta = \{\beta_s\}_{s\in[0,T]}$ is a strictly positive, continuous function and the constant $\eta \ge 0$.

For each fixed $\xi^1 \in D_g$, $\xi^2 \in D_h$, $M > 0$ and stopping time $\tau < T$, consider the following equation

$$\mathsf{E}_\tau\Big[\int_\tau^T \frac{\mathrm{d}\mathsf{P}^1}{\mathrm{d}\mathsf{P}_0}\Big|_{\mathcal{F}_t} \partial_y u\Big(t, \sup_{\tau\le v\le t}\Big\{L_v \exp\Big(-\int_v^t \beta_s \mathrm{d}s\Big)\Big\}\Big)\theta_{t,\tau}\mathrm{d}t\Big] = M\gamma_\tau \frac{\mathrm{d}\mathsf{P}^2}{\mathrm{d}\mathsf{P}^0}\Big|_{\mathcal{F}_\tau}, \tag{5.2}$$

where P^i is the probability measure whose Girsanov kernel is ξ^i, $i = 1, 2$. By Theorem 3 in [1], the above equation admits a unique progressively measurable process $L = L^{M,\mathsf{P}^1,\mathsf{P}^2}$ with upper right-continuous paths with $L_T = 0$ for any stopping times $\tau < T$. We may construct two processes Y and C associated with this process L by setting

$$Y_t = Y_t^{L^{M,\mathsf{P}^1,\mathsf{P}^2}} = \exp\Big(-\int_0^t \beta_s \mathrm{d}s\Big)\Big(\eta \vee \sup_{0\le v\le t}\Big\{L_v^{M,\mathsf{P}^1,\mathsf{P}^2} \exp\Big(\int_0^v \beta_s \mathrm{d}s\Big)\Big\}\Big), t \in [0, T],$$

$$C_{0-}^{L^{M,\mathsf{P}^1,\mathsf{P}^2}} = 0, \quad C_t^{L^{M,\mathsf{P}^1,\mathsf{P}^2}} = \int_0^t Y_s^{L^{M,\mathsf{P}^1,\mathsf{P}^2}} \mathrm{d}s + \int_0^t \beta_s^{-1} \mathrm{d}Y_s^{L^{M,\mathsf{P}^1,\mathsf{P}^2}}, t \in [0, T].$$

According to Lemma 3.9 in [3], we derive that

(i) $Y^{L^{M,\mathsf{P}^1,\mathsf{P}^2}}$ is an adapted RCLL process of bounded variation with $Y^{L^{M,\mathsf{P}^1,\mathsf{P}^2}} \ge L^{M,\mathsf{P}^1,\mathsf{P}^2}$;

(ii) $C^{L^{M,\mathsf{P}^1,\mathsf{P}^2}}$ is right-continuous, nondecreasing and adapted. In other words, $C^{L^{M,\mathsf{P}^1,\mathsf{P}^2}} \in \mathcal{X}$;

(iii) The level of satisfaction induced by $C^{L^{M,\mathsf{P}^1,\mathsf{P}^2}}$, denoted by $Y^{C^{L^{M,\mathsf{P}^1,\mathsf{P}^2}}}$, coincides with $Y^{L^{M,\mathsf{P}^1,\mathsf{P}^2}}$ and is minimal in the following sense:

$$Y_t^{C^{L^{M,\mathsf{P}^1,\mathsf{P}^2}}} = Y_t^{L^{M,\mathsf{P}^1,\mathsf{P}^2}} = \inf_{C\in\mathcal{X}, Y^C \ge L} Y_t^C, \quad t \in [0, T].$$

Remark 5.3 *For any real valued, progressively measurable process L with upper-rightcontinuous paths, we may construct the process C^L as above. As in [3], we say that the consumption plan C^L tracks the level process L.*

Theorem 5.4 *Suppose that the functions g,h satisfy (A1)-(A4). Under Assumptions (H1)-(H3), if we can find some P^i whose Girsanov kernel is ξ^i, $i=1,2$ such that the following equations hold:*

$$\mathcal{E}^g\left[\int_0^T u\Big(t,Y_t^{L^{M,\mathsf{P}^1,\mathsf{P}^2}}\Big)\mathrm{d}t\right]=\mathsf{E}^{\mathsf{P}^1}\left[\int_0^T u\Big(t,Y_t^{L^{M,\mathsf{P}^1,\mathsf{P}^2}}\Big)\mathrm{d}t+\int_0^T f(r,\xi_r^1)\mathrm{d}r\right],$$
$$\widetilde{\mathcal{E}}^h\left[\int_0^T \gamma_t\mathrm{d}C_t^{L^{M,\mathsf{P}^1,\mathsf{P}^2}}\right]=\mathsf{E}^{\mathsf{P}^2}\left[\int_0^T \gamma_t\mathrm{d}C_t^{L^{M,\mathsf{P}^1,\mathsf{P}^2}}-\int_0^T l(r,\xi_r^2)\mathrm{d}r\right],$$

where $L^{M,\mathsf{P}^1,\mathsf{P}^2}$ is the solution to Equation (5.2). Then, the consumption plan $C^{L^{M,\mathsf{P}^1,\mathsf{P}^2}}$ is optimal for the utility maximization problem (2.1) with given initial capital $w=\Psi(C^{L^{M,\mathsf{P}^1,\mathsf{P}^2}})$.

Proof. The proof is similar with the one of Theorem 3.13 in [3], so we omit it. ■

Remark 5.5 *Consider the general utility maximization problem introduced in Remark 3.7. Since the sufficiency for optimality still holds as in Remark 4.3, we may provide the construction for the optimal consumption plan. For this purpose, we assume that the dynamics η and β for the level of satisfaction satisfy (H3). For any constant $M>0$ and any $\mathsf{P}^i\in\mathcal{P}_i$, $i=1,2$, let $L^{M,\mathsf{P}^1,\mathsf{P}^2}$ be the solution to Equation (5.2) and let $C^{M,\mathsf{P}^1,\mathsf{P}^2}$ be the consumption plan which tracks $L^{M,\mathsf{P}^1,\mathsf{P}^2}$. If $\mathsf{P}^i\in\mathcal{P}^i(C^{M,\mathsf{P}^1,\mathsf{P}^2})$, $i=1,2$, then $C^{M,\mathsf{P}^1,\mathsf{P}^2}$ is optimal for problem (3.2) with initial wealth given by $w=\widetilde{\mathcal{E}}\big[\int_0^T\gamma_t dC_t^{M,\mathsf{P}^1,\mathsf{P}^2}\mathrm{d}t\big]$.*

Remark 5.6 *Existence of the desired minimal level of satisfaction turns into some fixed point problem. Now we fix the Lagrange multiplier M in Equation (5.2). Choose $\xi^{1,1}\in D_g$, $\xi^{2,1}\in D_h$ and let $\mathsf{P}^{i,1}$ be the probability measure whose Girsanov kernel is given by $\xi^{i,1}$, $i=1,2$. Then, there exists a unique solution L^1 to Equation (5.2) with $\mathsf{P}^i=\mathsf{P}^{i,1}$, $i=1,2$. Let C^1 be the consumption plan which tracks L^1 and $\Gamma(C^1)=\int_0^T\gamma_t dC_t^1$. Consider the following two BSDEs*

$$Y_t^{1,1}=U(C^1)+\int_t^T g(s,Z_s^{1,1})ds-\int_t^T Z_s^{1,1}dB_s,$$
$$Y_t^{2,1}=\Gamma(C^1)-\int_t^T h(s,-Z_s^{2,1})ds-\int_t^T Z_s^{2,1}dB_s.$$

Let $\xi^{i,2}$ be the solution to the following equation, $i=1,2$

$$g(s,Z_s^{1,1})-Z_s^{1,1}\xi_s^{1,2}=f(s,\xi_s^{1,2}),$$
$$-h(s,-Z_s^{2,1})-Z_s^{2,1}\xi_s^{2,2}=-l(s,\xi_s^{2,2}).$$

Then, applying the Girsanov theorem, it is easy to check that

$$\mathcal{E}^g[U(C^1)]=\mathsf{E}^{\mathsf{P}^{1,2}}\left[U(C^1)+\int_0^T f(s,\xi_s^{1,2})\mathrm{d}s\right],$$
$$\widetilde{\mathcal{E}}^h[\Gamma(C^1)]=\mathsf{E}^{\mathsf{P}^{2,2}}\left[\Gamma(C^1)-\int_0^T l(s,\xi_s^{2,2})\mathrm{d}s\right],$$

where $\mathsf{P}^{i,2}$ is the probability measure with Girsanov kernel $\xi^{i,2}$, $i=1,2$. We define the mapping $I:D_g\times D_h\to D_g\times D_h$ as the following:

$$I(\xi^{1,1},\xi^{2,1})=(\xi^{1,2},\xi^{2,2}).$$

If I has a fixed point (ξ^1,ξ^2), then C^{ξ^1,ξ^2} which tracks the process L^{ξ^1,ξ^2} is optimal for the utility maximization problem (2.1), where L^{ξ^1,ξ^2} is the solution to Equation (5.2) with $\mathsf{P}^i=\mathsf{P}^{\xi^i}$, $i=1,2$.

In the following, we will investigate how the value function $v_{g,h}(w)$ depends on the functions g, h and the initial wealth w. It is easy to check that, for any fixed g, h, $v_{g,h}(w)$ in non-decreasing in w. Now, suppose that $g_1 \leq g_2$, $h_1 \leq h_2$ and g_i, h_i satisfy (A1)-(A4), $i = 1, 2$. It follows that $0 \leq f_1 \leq f_2$, $0 \leq l_1 \leq l_2$, where f_i, l_i are the convex dual of g_i, h_i, $i = 1, 2$, respectively. Recalling proposition A.1, we conclude that $D_{g_2} \subset D_{g_1}$ and $D_{h_2} \subset D_{h_1}$. Therefore, the functions g, h represents the ambiguity the agent faces. More precisely, the more ambiguity corresponds to smaller g, h. Hence, it is natural to conjecture that the value $v_{g,h}(w)$ is increasing with respect to g, h.

Proposition 5.7 *Suppose that $g_1 \leq g_2$ and g_i satisfies (A1)-(A4), $i = 1, 2$. Then, we have $v_{g_1}(w) \leq v_{g_2}(w)$.*

Proof. Since $g_1 \leq g_2$, $h_1 \leq h_2$, for any $X \in L^2(\mathcal{F}_T)$, we have $\mathcal{E}^{g_1}[X] \leq \mathcal{E}^{g_2}[X]$ and $\widetilde{\mathcal{E}}^{h_1}[X] \geq \widetilde{\mathcal{E}}^{h_2}[X]$. Hence, it is easy to check that $\mathcal{A}_{h_1}(w) \subset \mathcal{A}_{h_2}(w)$ and

$$v_{g_1,h_1}(w) = \sup_{C \in \mathcal{A}_{h_1}(w)} \mathcal{E}^{g_1}[U(C)] \leq \sup_{C \in \mathcal{A}_{h_2}(w)} \mathcal{E}^{g_2}[U(C)] = v_{g_2,h_2}(w).$$

■

6 Explicit Solution for the infinite time case

6.1 Setting and main result

In this section, we give some explicit solutions when the time horizon is infinite for some typical case. To this end, we assume that the felicity function is given by

$$u(t, y) = e^{-\delta t} \frac{1}{\alpha} y^{\alpha},$$

where $\delta > 0$ and $\alpha \in (0, 1)$. The interest rate is a constant $r > 0$ and the level of satisfaction evolves according to the following dynamics:

$$Y_t^C = \eta e^{-\beta t} + \int_0^t \beta e^{-\beta(t-s)} dC_s,$$

where $\eta, \beta > 0$. Now for any fixed $\kappa > 0$, let a, b, a', b' be four constants such that $-\kappa \leq a' < a < b < b' \leq \kappa$. The multiple priors are given by

$$\mathcal{P}^1 = \{P^{\xi} | \xi \text{ adpated with values in } [b, b'], \frac{\mathrm{d}\mathsf{P}^{\xi}}{\mathrm{d}\mathsf{P}_0}\bigg|_{\mathcal{F}_t} = \epsilon_t^{\xi}, 0 < t < \infty\},$$
$$\mathcal{P}^2 = \{P^{\xi} | \xi \text{ adpated with values in } [a', a], \frac{\mathrm{d}\mathsf{P}^{\xi}}{\mathrm{d}\mathsf{P}_0}\bigg|_{\mathcal{F}_t} = \epsilon_t^{\xi}, 0 < t < \infty\},$$

where $\epsilon_t^{\xi} = \epsilon_t^{1,\xi}$ and $\epsilon_t^{x_0,\xi} = x_0 \exp\left(\int_0^t \xi_s \mathrm{d}B_s - \frac{1}{2}\int_0^t \xi_s^2 \mathrm{d}s\right)$ for any $x_0 > 0$. Consider the following two nonlinear expectations

$$\mathcal{E}^1[X] := \inf_{P \in \mathcal{P}^1} \mathsf{E}^{\mathsf{P}}[X], \ \mathcal{E}^2[X] := \sup_{P \in \mathcal{P}^2} \mathsf{E}^{\mathsf{P}}[X].$$

The objective is to solve the following problem

$$\sup_{C \in \mathcal{A}(w)} \mathcal{E}^1\left[\int_0^{\infty} u(t, Y_t^C) \mathrm{d}t\right], \tag{6.1}$$

where $\mathcal{A}(w) = \{C \in \mathcal{X} | \mathcal{E}^2\big[\int_0^\infty \gamma_t \mathrm{d}C_t\big] \le w\}$. The sufficiency for optimality is a natural extension of Theorem 4.1 to the infinite time case. Following a similar construction as in the finite time case, we first solve the following backward equation for any stopping time τ

$$\mathsf{E}_\tau\left[\int_\tau^\infty \beta\exp(-\beta(t-\tau))\epsilon_t^{\xi^1}\partial_y u\Big(t, \sup_{\tau\le v\le t}\Big\{L_v\exp(-\beta(t-v))\Big\}\Big)\mathrm{d}t\right] = M\gamma_\tau\epsilon_\tau^{\xi^2}, \tag{6.2}$$

where $M > 0$ is a given constant and ξ^i are the Girsanov kernels such that $\mathsf{P}_i := \mathsf{P}^{\xi^i} \in \mathcal{P}^i$, $i = 1, 2$. The solution to Equation (6.2) is denoted by $L^{M,\mathsf{P}_1,\mathsf{P}_2}$. Let $C^{M,\mathsf{P}_1,\mathsf{P}_2}$ be the consumption plan which tracks the level process $L^{M,\mathsf{P}_1,\mathsf{P}_2}$ and $Y^{M,\mathsf{P}_1,\mathsf{P}_2}$ be the corresponding level of satisfaction. Similar as Theorem 5.4, we have the following result.

Theorem 6.1 *Suppose that $\mathsf{P}_i \in \mathcal{P}^i(C^{M,\mathsf{P}_1,\mathsf{P}_2})$, $i = 1, 2$, where for any admissible consumption plan C*

$$\mathcal{P}^1(C) := \left\{\mathsf{P}\in\mathcal{P}^1 \Big| \mathcal{E}^1\left[\int_0^\infty u(t, Y_t^C)\mathrm{d}t\right] = \mathsf{E}^\mathsf{P}\left[\int_0^\infty u(t, Y_t^C)\mathrm{d}t\right]\right\},$$
$$\mathcal{P}^2(C) := \left\{\mathsf{P}\in\mathcal{P}^2 \Big| \mathcal{E}^2\left[\int_0^\infty \gamma_t dC_t\right] = \mathsf{E}^\mathsf{P}\left[\int_0^\infty \gamma_t \mathrm{d}C_t\right]\right\}.$$

Then, $C^{M,\mathsf{P}_1,\mathsf{P}_2}$ is an optimal consumption plan to (6.1) with $w = \mathcal{E}^2\big[\int_0^\infty \gamma_t \mathrm{d}C_t^{M,\mathsf{P}_1,\mathsf{P}_2}\big]$. Consequently, $L^{M,\mathsf{P}_1,\mathsf{P}_2}$ is called the optimal minimal level of satisfaction.

Remark 6.2 *Let $g(z) = bz^+ - b'z^-$ and $h(z) = a'z^+ - az^-$. Then, by the representation of g-expectation, we have*

$$\mathcal{E}^1[X] = \mathcal{E}^g[X],\ \mathcal{E}^2[X] = \widetilde{\mathcal{E}}^h[X].$$

Hence, the financial market is the one introduced as Case 3 in Subsection 2.2.

We then give the main result in this section.

Theorem 6.3 *Suppose that $\delta > \alpha r + \frac{\alpha(a-b)^2}{2(1-\alpha)}$. The optimal minimal level of satisfaction L^K is given by*

$$L_t^K = \left(Ke^{(\delta-r)t}\frac{\epsilon_t^a}{\epsilon_t^b}\right)^{\frac{1}{\alpha-1}} = K^{\frac{1}{\alpha-1}}\exp\left\{\frac{a-b}{\alpha-1}B_t - \frac{1}{2(\alpha-1)}[(a^2-b^2)-2(\delta-r)]t\right\},$$

where K is a constant to be determined by the initial wealth w.

Remark 6.4 *Theorem 4.7 in [3] requires that the investor's rate of time preference δ satisfy*

$$\delta > \alpha r + (1-\alpha)\pi\left(\frac{\alpha\theta'}{1-\alpha}\right) + \alpha\pi(-\theta'), \tag{6.3}$$

where $\pi(\cdot)$ is the Laplace exponent of a Lévy process X and θ' can be interpreted as the market price of risk. Compared with this result, the Lévy process X in our context is in fact a Brownian motion B and $\theta' = b - a$. Then, the right-hand side of Equation (6.3) becomes $\alpha r + \frac{\alpha(a-b)^2}{2(1-\alpha)}$.

6.2 Proof of the main result

The proof of Theorem 6.3 will be divided into the following parts. First, for any fixed $\mathsf{P}^{\xi^i} \in \mathcal{P}^i$ with ξ^i being a constant, $i = 1, 2$, we solve the backward Equation (6.2) explicitly. Then, we need to verify that P^a is the highest-cost probability and P^b is the lowest-utility probability for the consumption C^K that tracks L^K defined in Theorem 6.3.

6.2.1 Solving Equation (6.2)

Lemma 6.5 *For any fixed constants M, ξ^i and probability measures $\mathsf{P}^i := \mathsf{P}^{\xi^i}$, $i = 1, 2$, the solution to Equation* (6.2) *is*

$$L_t := L_t^{M,\xi^1,\xi^2} = \left(K e^{(\delta - r)t} \frac{\epsilon_t^{\xi^2}}{\epsilon_t^{\xi^1}} \right)^{\frac{1}{\alpha-1}},$$

where K is a constant satisfying

$$K\beta \mathsf{E}\bigg[\int_0^\infty e^{-(\delta+\alpha\beta)t} \inf_{0\le v\le t} \Big\{ e^{(\delta+\beta(\alpha-1)-r)v} \epsilon_v^{\xi^2} \frac{\epsilon_t^{\xi^1}}{\epsilon_v^{\xi^1}} \Big\} \mathrm{d}t \bigg] = M.$$

Proof. By the strong Markov property and change of variable, we have

$$\begin{aligned}
&\mathsf{E}_\tau\bigg[\int_\tau^\infty \epsilon_t^{\xi^1} \partial_y u\Big(t, \sup_{\tau\le v\le t} \Big\{ L_v \exp\big(-\int_v^t \beta_s ds \big) \Big\} \Big) \theta_{t,\tau} \mathrm{d}t \bigg] \\
=&\mathsf{E}_\tau\bigg[\int_\tau^\infty \epsilon_t^{\xi^1} e^{-\delta t} \beta e^{-\beta(t-\tau)} \inf_{\tau\le v\le t} \Big\{ K e^{(\delta-r)v} e^{-\beta(\alpha-1)(t-v)} \frac{\epsilon_v^{\xi^2}}{\epsilon_v^{\xi^1}} \Big\} \mathrm{d}t \bigg] \\
=&K\beta e^{-r\tau} \mathsf{E}_\tau\bigg[\int_0^\infty e^{-(\delta+\alpha\beta)t} \inf_{0\le v\le t} \Big\{ e^{(\delta+\beta(\alpha-1)-r)v} \epsilon_{v+\tau}^{\xi^2} \frac{\epsilon_{t+\tau}^{\xi^1}}{\epsilon_{v+\tau}^{\xi^1}} \Big\} \mathrm{d}t \bigg] \\
=&K\beta e^{-r\tau} \mathsf{E}\bigg[\int_0^\infty e^{-(\delta+\alpha\beta)t} \inf_{0\le v\le t} \Big\{ e^{(\delta+\beta(\alpha-1)-r)v} \epsilon_v^{x_2,\xi^2} \frac{\epsilon_t^{x_1,\xi^1}}{\epsilon_v^{x_1,\xi^1}} \Big\} \mathrm{d}t \bigg] \bigg|_{x_1=\epsilon_\tau^{\xi^1}, x_2=\epsilon_\tau^{\xi^2}} \\
=&K\beta e^{-r\tau} \mathsf{E}\bigg[\int_0^\infty e^{-(\delta+\alpha\beta)t} \inf_{0\le v\le t} \Big\{ e^{(\delta+\beta(\alpha-1)-r)v} \epsilon_v^{\xi^2} \frac{\epsilon_t^{\xi^1}}{\epsilon_v^{\xi^1}} \Big\} \mathrm{d}t \bigg] \epsilon_\tau^{\xi^2}.
\end{aligned}$$

Hence, the result follows. ■

Remark 6.6 *It is easy to verify that*

$$\mathsf{E}\bigg[\int_0^\infty e^{-(\delta+\alpha\beta)t} \inf_{0\le v\le t} \Big\{ e^{(\delta+\beta(\alpha-1)-r)v} \epsilon_v^{\xi^2} \frac{\epsilon_t^{\xi^1}}{\epsilon_v^{\xi^1}} \Big\} \mathrm{d}t \bigg] \le \frac{1}{\delta+\alpha\beta}.$$

Consequently, we have $K \ge \frac{M(\delta+\alpha\beta)}{\beta}$.

6.2.2 Verifying the lowest-utility and highest-cost probability

For simplicity, set

$$\theta = \frac{a-b}{\alpha-1} (>0), \quad \lambda = \frac{1}{2(\alpha-1)}[(a^2-b^2) - 2(\delta-r)].$$

For any constant ξ, let $x_+^\alpha(\xi)$, $x_+(\xi)$ be the larger solution to the following two equations $h^{\alpha,\xi}(x) = 0$, $h^\xi(x) = 0$, respectively, where

$$h^{\alpha,\xi}(x) = \frac{1}{2}\alpha^2\theta^2x^2 - \alpha(\lambda-\beta-\theta\xi)x - (\delta+\alpha\beta), \tag{6.4}$$

and

$$h^\xi(x) = \frac{1}{2}\theta^2x^2 - (\lambda-\beta-\theta\xi)x - (r+\beta). \tag{6.5}$$

Let C^K be the consumption plan which tracks L^K defined in Theorem 6.3 and Y^K be the corresponding level of satisfaction. By the results in [3], for some suitable constant K, C^K is optimal for the following linear problem

$$v^{a,b}(w) = \sup_{C\in\mathcal{A}^a(w)} \mathsf{E}^b\left[\int_0^\infty u(t, Y_t^C)\mathrm{d}t\right],$$

where $\mathcal{A}^a(w) = \{C \in \mathcal{X} | \mathsf{E}^a[\int_0^\infty \gamma_s \mathrm{d}C_s] \le w\}$, E^a, E^b are the expectations taken under P^a, P^b, respectively. Besides, the expected utility and expected cost associated with consumption plan C^K satisfy

$$\phi^b(\eta) := \mathsf{E}^b\left[\int_0^\infty u(t, Y_t^K)\mathrm{d}t\right] = \frac{1}{\alpha(\delta+\alpha\beta)}\begin{cases} \eta^\alpha + \frac{1}{x_+^\alpha(b)-1}K^{\frac{\alpha x_+^\alpha(b)}{\alpha-1}}\eta^{\alpha(1-x_+^\alpha(b))}, & \eta > K^{\frac{1}{\alpha-1}}; \\ \frac{x_+^\alpha(b)}{x_+^\alpha(b)-1}K^{\frac{\alpha}{\alpha-1}}, & \eta \le K^{\frac{1}{\alpha-1}}, \end{cases} \tag{6.6}$$

and

$$\psi^a(\eta) := \mathsf{E}^a\left[\int_0^\infty e^{-rt}\mathrm{d}C_t^K\right] = \frac{1}{\beta}\begin{cases} \frac{1}{x_+(a)-1}K^{\frac{x_+(a)}{\alpha-1}}\eta^{1-x_+(a)}, & \eta > K^{\frac{1}{\alpha-1}}; \\ \frac{x_+(a)}{x_+(a)-1}K^{\frac{1}{\alpha-1}} - \eta, & \eta \le K^{\frac{1}{\alpha-1}}. \end{cases} \tag{6.7}$$

For any constant $\eta > 0$, set

$$\phi^{\xi^1}(\eta) = \mathsf{E}\bigg[\int_0^\infty \epsilon_t^{\xi^1} u(t, Y_t^K)\mathrm{d}t\bigg],$$
$$\psi^{\xi^2}(\eta) = \mathsf{E}\bigg[\int_0^\infty e^{-rt}\epsilon_t^{\xi^2}\mathrm{d}C_t^K\bigg].$$

Lemma 6.7 *Under the same assumption as Theorem 6.3, we have*

$$\phi^b(\eta) = \inf_{\xi^1\in[b',b],adapted}\phi^{\xi^1}(\eta),\ \ \psi^a(\eta) = \sup_{\xi^2\in[a,a'],adapted}\psi^{\xi^2}(\eta).$$

Besides, the values are finite.

Proof. Notice that $Y_t^K = e^{-\beta t}\{\eta\vee\sup_{0\le v\le t}(L_v^K e^{\beta v})\} = e^{-\beta t}\{\eta\vee\sup_{0\le v\le t}K^{\frac{1}{\alpha-1}}\exp(\theta B_v - (\lambda-\beta)v)\}$. Applying the Girsanov Theorem and Tonelli Theorem, it is easy to check that

$$\begin{aligned}\phi^{\xi^1}(\eta) &= \int_0^\infty \frac{1}{\alpha}e^{-(\delta+\alpha\beta)t}\mathsf{E}\bigg[\epsilon_t^{\xi^1}\{\eta^\alpha \vee \sup_{0\le v\le t}K^{\frac{\alpha}{\alpha-1}}\exp(\alpha\theta B_v - \alpha(\lambda-\beta)v)\}\bigg]\mathrm{d}t \\ &= \int_0^\infty \frac{1}{\alpha}e^{-(\delta+\alpha\beta)t}\mathsf{E}\bigg[\{\eta^\alpha \vee \sup_{0\le v\le t}K^{\frac{\alpha}{\alpha-1}}\exp(\alpha\theta B_v - \alpha(\lambda-\beta)v + \alpha\theta\int_0^v \xi_s^1\mathrm{d}s)\}\bigg]\mathrm{d}t.\end{aligned}$$

Clearly, for any adapted ξ^1 taking values between b' and b, we have $\phi^b(\eta) \le \phi^{\xi^1}(\eta) \le \phi^{b'}(\eta)$. The assumption $\delta > \alpha r + \frac{\alpha(a-b)^2}{2(1-\alpha)}$ implies that $h^{\alpha,b}(1) < 0$. Hence, $x_+^\alpha(b) > 1$. Recalling Equation (6.6), $\phi^b(\eta)$ is finite.

Then, we calculate $\psi^{\xi^2}(\eta)$. Noting that $\mathrm{d}C_t^K = Y_t^K\mathrm{d}t + \frac{1}{\beta}\mathrm{d}Y_t^K = \frac{1}{\beta}e^{-\beta t}\mathrm{d}(e^{\beta t}Y_t^K)$. Simple calculation yields that

$$\begin{aligned}&\int_0^T e^{-rt}\epsilon_t^{\xi^2}\mathrm{d}C_t^K = \int_0^T \frac{1}{\beta}e^{-(r+\beta)t}\epsilon_t^{\xi^2}\mathrm{d}(e^{\beta t}Y_t^K) \\ =&\frac{1}{\beta}e^{-rT}\epsilon_T^{\xi^2}Y_T^K - \frac{\eta}{\beta} - \frac{1}{\beta}\int_0^T e^{\beta t}Y_t^K\mathrm{d}(e^{-(r+\beta)t}\epsilon_t^{\xi^2}) \\ =&\frac{1}{\beta}e^{-rT}\epsilon_T^{\xi^2}Y_T^K - \frac{\eta}{\beta} + (1+\frac{r}{\beta})\int_0^T e^{-rt}\epsilon_t^{\xi^2}Y_t^K\mathrm{d}t - \frac{1}{\beta}\int_0^T e^{-rt}\xi_t^2\epsilon_t^{\xi^2}Y_t^K\mathrm{d}B_t.\end{aligned} \tag{6.8}$$

Set $\widetilde{\psi}^{\xi^2}(\eta) = \mathsf{E}\big[\int_0^\infty e^{-rt}\epsilon_t^{\xi^2} Y_t^K \mathrm{d}t\big]$. Similar with the analysis for $f^{\xi^1}(\eta)$, for any adapted process ξ^2 taking values in $[a, a']$, we have

$$\widetilde{\psi}^{\xi^2}(\eta) = \int_0^\infty e^{-(r+\beta)t}\mathsf{E}\Big[\eta \vee \sup_{0\le v\le t} K^{\frac{1}{\alpha-1}} \exp(\theta B_v - (\lambda-\beta)v + \theta\int_0^v \xi_s^2 \mathrm{d}s)\Big]\mathrm{d}t.$$

Hence, for any adapted process ξ^2 with values in $[a, a']$, we have $\widetilde{\psi}^{a'}(\eta) \le \widetilde{\psi}^{\xi_2}(\eta) \le \widetilde{\psi}^{a}(\eta)$. Similar calculation as the proof of Proposition 5.8 in [8] implies that, for $\xi^2 \in \{a, a'\}$,

$$\widetilde{\psi}^{\xi^2}(\eta) = \frac{1}{\beta+r}\begin{cases} \frac{1}{x_+(\xi^2)-1}K^{\frac{x_+(\xi^2)}{\alpha-1}}\eta^{1-x_+(\xi^2)} + \eta, & \eta > K^{\frac{1}{\alpha-1}}; \\ \frac{x_+(\xi^2)}{x_+(\xi^2)-1}K^{\frac{1}{\alpha-1}}, & \eta \le K^{\frac{1}{\alpha-1}}. \end{cases}$$

The assumption $\delta > \alpha r + \frac{\alpha(a-b)^2}{2(1-\alpha)}$ yields that $h^a(1) < 0$ and $h^{a'}(1) < 0$. Therefore, we have $x_+(a) > 1$ and $x_+(a') > 1$. Hence, the quantity $\widetilde{\psi}^{\xi^2}(\eta)$ is positive and finite for any adapted $\xi^2 \in [a, a']$. By a similar analysis as the proof of Lemma 4.9 in [3], we have

$$\lim_{T\to\infty} e^{-rT}\epsilon_T^{\xi^2} Y_T^K = 0.$$

Taking expectation on both sides of (6.8) and letting T go to infinity, we obtain that

$$\psi^{\xi^2}(\eta) = (1 + \frac{r}{\beta})\widetilde{\psi}^{\xi^2}(\eta) - \frac{\eta}{\beta}.$$

It is obvious that $\psi^{a'}(\eta) \le \psi^{\xi^2}(\eta) \le \psi^a(\eta)$. The proof is complete. ■

Proof of Theorem 6.3. By Lemma 6.5 and 6.7, we know that L^K is the solution to (6.2) with $\mathsf{P}_1 = \mathsf{P}^b$, $\mathsf{P}_2 = \mathsf{P}^a$ and $\mathsf{P}^b \in \mathcal{P}^1(C^K)$, $\mathsf{P}^a \in \mathcal{P}^2(C^K)$. By Theorem 6.1, it remains to find an appropriate constant K, such that $w = \psi^a(\eta)$. By simple calculation, we have

$$K = \begin{cases} \Big(\frac{x_+(a)-1}{x_+(a)}(\beta w + \eta)\Big)^{\alpha-1}, & w \ge \frac{\eta}{\beta(x_+(a)-1)}; \\ (\beta(x_+(a)-1)\eta^{x_+(a)-1}w)^{\frac{\alpha-1}{x_+(a)}}, & \text{otherwise}. \end{cases}$$

■

6.3 Examples with different parameters

In this subsection, we first show that the constraint for the parameters δ, a, b, α, r in Theorem 6.3 are necessary to make Problem (6.1) well-posed. Then we study how the optimal consumption plan behaves when the multiple priors $\mathcal{P}^1$, $\mathcal{P}^2$ overlap.

Proposition 6.8 *Suppose that $\delta \le \alpha r + \frac{\alpha(a-b)^2}{2(1-\alpha)}$. Then, the utility maximization problem (6.1) is ill-posed, which means that we may find a consumption plan with finite cost but the induced utility is infinite.*

Proof. The proof is similar with Proposition 4.13 in [3]. For readers' convenience, we give a short proof here for the case that $\delta < \alpha r + \frac{\alpha(a-b)^2}{2(1-\alpha)}$.

In fact, since $\delta < \alpha r + \frac{\alpha(a-b)^2}{2(1-\alpha)}$, it is easy to check that $\underline{\delta} < \bar{\delta}$, where

$$\underline{\delta} = \frac{(a-b)^2}{2(1-\alpha)} - \frac{1-\alpha}{\alpha}\delta,\ \bar{\delta} = \frac{\alpha(a-b)^2}{2(1-\alpha)} + (\alpha-1)r.$$

Now, we may find some $\hat{\delta} \in (\underline{\delta}, \bar{\delta})$. Set

$$\hat{L}_t^K = \left(Ke^{\hat{\delta}t}\frac{\epsilon_t^a}{\epsilon_t^b}\right)^{\frac{1}{\alpha-1}}.$$

Let $\hat{Y}^K$, $\hat{C}^K$ be the level of satisfaction and consumption plan induced by the process $\hat{L}^K$, respectively. By a similar analysis as the proof of Lemma 6.7, we have

$$\mathcal{E}^2\left[\int_0^\infty e^{-rt}d\hat{C}_t^K\right] = \sup_{\mathsf{P}\in\mathcal{P}^2}\mathsf{E}^{\mathsf{P}}\left[\int_0^\infty e^{-rt}d\hat{C}_t^K\right] = \mathsf{E}\left[\int_0^\infty \epsilon_t^a e^{-rt}d\hat{C}_t^K\right].$$

Besides,

$$\mathsf{E}\left[\int_0^\infty \epsilon_t^a e^{-rt}\mathrm{d}\hat{C}_t^K\right] = \frac{1}{\beta}\begin{cases}\frac{1}{\hat{x}_+(a)-1}K^{\frac{\hat{x}_+(a)}{\alpha-1}}\eta^{1-\hat{x}_+(a)}, & \eta > K^{\frac{1}{\alpha-1}};\\ \frac{\hat{x}_+(a)}{\hat{x}_+(a)-1}K^{\frac{1}{\alpha-1}}-\eta, & \eta \le K^{\frac{1}{\alpha-1}},\end{cases}$$

where $\hat{x}_+(a)$ is the larger solution to the following equation

$$\hat{h}^a(x) = \frac{1}{2}\theta^2x^2 - (\hat{\lambda}-\beta-a\theta)x - (r+\beta)$$

and

$$\hat{\lambda} = \frac{1}{2(\alpha-1)}[(a^2-b^2)-2\hat{\delta}].$$

Since $\hat{\delta} < \bar{\delta}$, it is easy to check that $\hat{h}^a(1) < 0$, which implies that $\hat{x}_+(a) > 1$. Therefore, we may choose some appropriate K, such that

$$\mathcal{E}^2\left[\int_0^\infty e^{-rt}d\hat{C}_t^K\right] = w.$$

Now, let us calculate the corresponding expected utility. First, applying a similar analysis yields that

$$\mathcal{E}^1\left[\int_0^\infty u(t,\hat{Y}_t^K)dt\right] = \inf_{\mathsf{P}\in\mathcal{P}^1}\mathsf{E}^{\mathsf{P}}\left[\int_0^\infty u(t,\hat{Y}_t^K)dt\right] = \mathsf{E}\left[\int_0^\infty \epsilon_t^b u(t,\hat{Y}_t^K)dt\right].$$

However, due to the fact that $\underline{\delta} < \hat{\delta}$, we have $\hat{h}^{\alpha,b}(1) > 0$, where

$$h^{\alpha,b}(x) = \frac{1}{2}\alpha^2\theta^2x^2 - \alpha(\hat{\lambda}-\beta-b\theta)x - (\delta+\alpha\beta).$$

Consequently,

$$\mathsf{E}\left[\int_0^\infty \epsilon_t^b u(t,\hat{Y}_t^K)dt\right] = +\infty.$$

Hence, this problem is ill-posed. ■

Proposition 6.9 *Suppose that there exists some probability P such that $\mathsf{P}\in\mathcal{P}^1\cap\mathcal{P}^2$ and $\alpha r < \delta$. Then, the optimal consumption plan for the utility maximization problem (6.1) is a deterministic process.*

Proof. It is easy to check that $L_v^K = (Ke^{(\delta-r)v})^{\frac{1}{\alpha-1}}$ is the solution to (6.2) with $\mathsf{P}^{\xi^1} = \mathsf{P}^{\xi^2} = \mathsf{P}$ and Lagrange multiplier $M = \frac{K\beta}{r+\beta}$. Clearly, the consumption plan tracks this L^K is a deterministic function and hence, the utility and the cost induced by this consumption plan are deterministic. Therefore, we have

$$\inf_{\mathsf{P}\in\mathcal{P}^1} \mathsf{E}^{\mathsf{P}}\left[\int_0^\infty u(t,Y_t^K)\mathrm{d}t\right] = \int_0^\infty u(t,Y_t^K)\mathrm{d}t = \mathsf{E}^{\mathsf{P}}\left[\int_0^\infty u(t,Y_t^K)\mathrm{d}t\right],$$
$$\sup_{P\in\mathcal{P}^2} \mathsf{E}^{\mathsf{P}}\left[\int_0^\infty \gamma_t \mathrm{d}C_t^K\right] = \int_0^\infty \gamma_t \mathrm{d}C_t^K = \mathsf{E}^{\mathsf{P}}\left[\int_0^\infty \gamma_t \mathrm{d}C_t^K\right],$$

which implies that $\mathsf{P}\in\mathcal{P}^1(C^K)\cap\mathcal{P}^2(C^K)$.

Case 1. Suppose that $\delta < r+(1-\alpha)\beta$. In this case, $Y_t^K = (e^{-\beta t}\eta)\vee(K^{\frac{1}{\alpha-1}}e^{\frac{\delta-r}{\alpha-1}t}) = (e^{-\beta t}\eta)\vee L_t^K$. It remains to find K such that the cost $\int_0^\infty \gamma_t dC_t^K$ equals to w, where

$$\int_0^\infty e^{-rt}\mathrm{d}C_t^K = \begin{cases} K^{\frac{1}{\alpha-1}}\frac{(1-\alpha)(\beta+r)}{\beta(\delta-\alpha r)} - \frac{\eta}{\beta}, & \eta \le K^{\frac{1}{\alpha-1}}; \\ \frac{(1-\alpha)\beta+r-\delta}{\beta(\delta-\alpha r)}\eta^{\frac{\alpha r-\delta}{(1-\alpha)\beta+r-\delta}}K^{-\frac{r+\beta}{(1-\alpha)\beta+r-\delta}}, & \eta > K^{\frac{1}{\alpha-1}}. \end{cases}$$

Simple calculation implies that

$$K = \begin{cases} \left[\frac{(\eta+\beta w)}{(1-\alpha)(\beta+r)}\right]^{\alpha-1}, & w \ge \frac{\beta\eta(1-\alpha)+\eta(r-\delta)}{\beta(\delta-\alpha r)}; \\ \left[\frac{\beta w(\delta-\alpha r)}{(1-\alpha)\beta+r-\delta}\right]^{-\frac{(1-\alpha)\beta+r-\delta}{r+\beta}}\eta^{\frac{\delta-\alpha r}{\delta+r}}, & \text{otherwise}. \end{cases}$$

Case 2. Suppose that $\delta \ge r+(1-\alpha)\beta$. In this case, it is easy to check that the optimal consumption C^* should be such that $C_t^* = C_0^* = w$. That is, the agent will consume all his initial wealth at the original time. ∎

6.4 Portfolio financing the optimal consumption

In this subsection, we will find the portfolio process π in order to finance the optimal consumption plan. For this purpose, let V_t^a, V_t be the present values of the future consumption C^K taken under probability P^a and multiple priors $\mathcal{P}^2$, respectively, i.e.,

$$V_t^a = \mathsf{E}_t^a\left[\int_t^\infty e^{-r(s-t)}\mathrm{d}C_s^K\right] = \mathsf{E}_t\left[\int_t^\infty \frac{\epsilon_s^a}{\epsilon_t^a}e^{-r(s-t)}\mathrm{d}C_s^K\right],$$
$$V_t = \operatorname*{ess\,sup}_{\mathsf{P}^\xi\in\mathcal{P}^2}\mathsf{E}_t^\xi\left[\int_t^\infty e^{-r(s-t)}\mathrm{d}C_s^K\right] = \operatorname*{ess\,sup}_{\mathsf{P}^\xi\in\mathcal{P}^2}\mathsf{E}_t\left[\int_t^\infty \frac{\epsilon_s^\xi}{\epsilon_t^\xi}e^{-r(s-t)}\mathrm{d}C_s^K\right].$$

In fact, by the definition of g-expectation, we have

$$V_t = \mathcal{E}_t^{\widetilde{h}}\left[\int_t^\infty e^{-r(s-t)}\mathrm{d}C_s^K\right],$$

where $\widetilde{h}(z) = a'z^+ - az^-$.

Lemma 6.10 *In fact, we have*

$$V_t = V_t^a = e^{\theta B_t-\lambda t}\psi^a(e^{-\theta B_t+\lambda t}Y_t^K).$$

Proof. By Equation (6.8), we have

$$V_t^a = \frac{(\beta+r)e^{rt}}{\beta\varepsilon_t^a}\mathsf{E}_t\left[\int_t^\infty e^{-rs}\varepsilon_s^a Y_s^K\mathrm{d}s\right] - \frac{Y_t^K}{\beta} := \widetilde{V}_t^a - \frac{Y_t^K}{\beta}.$$

Applying Markov property implies that

$$\begin{aligned}
\widetilde{V}_t^a =&\frac{(\beta+r)e^{rt}}{\beta\varepsilon_t^a}\mathsf{E}_t\left[\int_t^\infty e^{-(r+\beta)s}\varepsilon_s^a\left\{\eta\vee\sup_{0\leq v\leq s}K^{\frac{1}{\alpha-1}}\exp(\theta B_v-(\lambda-\beta)v)\right\}\mathrm{d}s\right]\\
=&\frac{(\beta+r)}{\beta}\mathsf{E}_t\left[\int_0^\infty e^{-(r+\beta)s}\frac{\varepsilon_{s+t}^a}{\varepsilon_t^a}\left\{Y_t^K\vee e^{-\beta t}\sup_{t\leq v\leq s+t}K^{\frac{1}{\alpha-1}}\exp(\theta B_v-(\lambda-\beta)v)\right\}\mathrm{d}s\right]\\
=&\frac{(\beta+r)}{\beta}\mathsf{E}_t\left[\int_0^\infty e^{-(r+\beta)s}\frac{\varepsilon_{s+t}^a}{\varepsilon_t^a}\left\{Y_t^K\vee e^{-\lambda t}\sup_{0\leq v\leq s}K^{\frac{1}{\alpha-1}}\exp(\theta B_{v+t}-(\lambda-\beta)v)\right\}\mathrm{d}s\right]\\
=&\frac{(\beta+r)}{\beta}e^{\theta B_t-\lambda t}\mathsf{E}_t\left[\int_0^\infty e^{-(r+\beta)s}\frac{\varepsilon_{s+t}^a}{\varepsilon_t^a}\left\{\eta\vee\sup_{0\leq v\leq s}K^{\frac{1}{\alpha-1}}\exp(\theta(B_{v+t}-B_t)-(\lambda-\beta)v)\right\}\mathrm{d}s\right]\Bigg|_{\eta=\widetilde{Y}_t^K}\\
=&\frac{(\beta+r)}{\beta}e^{\theta B_t-\lambda t}\mathsf{E}\left[\int_0^\infty e^{-(r+\beta)s}\varepsilon_s^a\left\{\eta\vee\sup_{0\leq v\leq s}K^{\frac{1}{\alpha-1}}\exp(\theta B_v-(\lambda-\beta)v)\right\}\mathrm{d}s\right]\Bigg|_{\eta=\widetilde{Y}_t^K}\\
=&e^{\theta B_t-\lambda t}\left(\psi^a(e^{-\theta B_t+\lambda t}Y_t^K)+\frac{1}{\beta}e^{-\theta B_t+\lambda t}Y_t^K\right),
\end{aligned}$$

where $\widetilde{Y}_t^K=e^{-\theta B_t+\lambda t}Y_t^K$. All the above analysis yields that $V_t^a=e^{\theta B_t-\lambda t}\psi^a(e^{-\theta B_t+\lambda t}Y_t^K)$. Then, we follow a similar analysis as the proof of Theorem 5.11 in [8] and conclude that $V_t=V_t^a$. ■

Recall that the financial market is given as Case 3 in Subsection 2.2. The portfolio financing C^K should be chosen as follows.

Proposition 6.11 *Under the same assumption as Theorem 6.3, we have*

$$\pi=\frac{\theta x_+(a)}{\sigma}.$$

Proof. It is easy to check that for any $t\geq 0$, $e^{-\theta B_t+\lambda t}Y_t^K\geq K^{\frac{1}{\alpha-1}}$. By Lemma 6.10 and Equation (6.7), applying Itô's formula, we have

$$dV_t=\theta x_+(a)V_t\mathrm{d}B_t+\text{terms of bounded variation}.$$

Compared with Equation (2.4), the portfolio $\pi=\theta x_+(a)/\sigma$. ■

Remark 6.12 *Suppose that $\hat{\delta}=\beta+\frac{\delta-r}{\alpha-1}>0$. We define*

$$F(\theta):=\theta x_+(a)=\frac{1}{2(1-\alpha)}\theta-\frac{\hat{\delta}}{\theta}-\sqrt{(\frac{1}{2(1-\alpha)}\theta-\frac{\hat{\delta}}{\theta})^2+2(r+\beta)}.$$

It is easy to check that $F'(\theta)\geq 0$ and $\lim_{\theta\to 0}F(\theta)=0$. From the economic point of view, the more the difference between multiple priors for utility and cost, the more the investment for the risk assets.

Appendix A

Consider a filtered probability space $(\Omega,\mathcal{F}_T,(\mathcal{F}_t)_{t\in[0,T]},\mathsf{P}_0)$ satisfying the usual conditions of right continuity and completeness in which $B=\{B_t\}_{t\in[0,T]}$ is a d-dimensional Brownian motion. For any terminal value $X\in L^2(\mathcal{F}_T)$, the collection of all $\mathcal{F}_T$-measurable and square integrable random variables, consider the following BSDE

$$Y_t^X=X+\int_t^T g(s,Z_s^X)\mathrm{d}s-\int_t^T Z_s^X\mathrm{d}B_s.$$

By the results in [15], there exists a unique pair of solution (Y^X, Z^X) under Assumptions (A1)-(A2). Then we define the g-conditional expectation for X as the following

$$\mathcal{E}^g_{t,T}[X] = Y^X_t.$$

For simplicity, we denote $\mathcal{E}^g_{0,T}[X]$ by $\mathcal{E}^g[X]$. The g-expectation coincides with a variational preference in the following sense, which can be found in [7].

Proposition A.1 *Suppose that g satisfies (A1)-(A3). For each $\omega \in \Omega$, $t \in [0, T]$ and $\theta \in \mathbb{R}^d$, let*

$$f(t, \omega, \theta) = \sup_{z \in \mathbb{R}^d} (g(t, \omega, z) - z \cdot \theta)$$

be the convex dual of g. Denote by D_g be the collection of all progressively measurable processes $\{\xi_t\}$ such that

$$\mathsf{E}\left[\int_0^T |f(s, \xi_s)|^2 ds\right] < \infty.$$

Let τ be a stopping time satisfying $0 \le t \le \tau \le T$. For each $\mathcal{F}_\tau$-measurable and square integrable random variable X, we have the following representation

$$\mathcal{E}^g_{t,\tau}[X] = \operatorname*{ess\,inf}_{\xi \in D_g} \{\mathsf{E}^{\mathsf{P}^\xi}_t[X] + \alpha_{t,\tau}(\xi)\},$$

where $\mathsf{E}^{\mathsf{P}^\xi}_t[\cdot]$ is the conditional expectation, the penalty function is defined as follows

$$\alpha_{t,\tau}(\xi) = \mathsf{E}^{\mathsf{P}^\xi}_t\left[\int_t^\tau f(s, \xi_s) \mathrm{d}s\right],$$

and the probability measure P^ξ satisfies

$$\frac{\mathrm{d}\mathsf{P}^\xi}{\mathrm{d}\mathsf{P}_0} = \exp\left(\int_0^T \xi_s dB_s - \frac{1}{2}\int_0^T \xi_s^2 ds\right).$$

Recall that one of the most important properties of the classical conditional expectation is time consistency (or tower property). In fact, this property still hold for the g-conditional expectations. More precisely, we have the following proposition. For more details, please refer to the papers [16] and [5].

Proposition A.2 *Suppose that g satisfies (A1)-(A3). The conditional g-expectation satisfies the following properties:*

(1) *Strict comparison: if $X \le Y$, then $\mathcal{E}^g_{t,T}[X] \le \mathcal{E}^g_{t,T}[Y]$. Furthermore, if $\mathsf{P}_0(X < Y) > 0$, then $\mathcal{E}^g_{t,T}[X] < \mathcal{E}^g_{t,T}[Y]$;*

(2) *Time consistency: for any $0 \le s \le t \le T$, $\mathcal{E}^g_{s,T}[\mathcal{E}^g_{t,T}[X]] = \mathcal{E}^g_{s,T}[X]$;*

(3) *Concavity: $\mathcal{E}^g_{t,T}[\cdot]$ is concave, i.e., for any $X, Y \in L^2(\mathcal{F}_T)$ and $\lambda \in [0, 1]$, we have $\mathcal{E}^g_{t,T}[\lambda X + (1 - \lambda)Y] \ge \lambda \mathcal{E}^g_{t,T}[X] + (1 - \lambda)\mathcal{E}^g_{t,T}[Y]$;*

(4) *Fatou's lemma: Suppose that for any $n \in \mathbb{N}$, $\mathcal{E}^g[X_n]$ exists and $X_n \ge X$ (respectively, $X_n \le X$), where $X \in L^2(\mathcal{F}_T)$. Then, we have*

$$\liminf_{n\to\infty} \mathcal{E}^g[X_n] \ge \mathcal{E}^g[\liminf_{n\to\infty} X_n] \quad (\textit{respectively, } \limsup_{n\to\infty} \mathcal{E}^g[X_n] \le \mathcal{E}^g[\limsup_{n\to\infty} X_n]).$$

If we assume additionally that the function g satisfies (A4), it is easy to check that for any $X \in L^2(\mathcal{F}_{T_1}) \subset L^2(\mathcal{F}_{T_2})$, where $T_1 \le T_2$, we have

$$\mathcal{E}^g_{t,T_1}[X] = \mathcal{E}^g_{t,T_2}[X].$$

In this case, we denote $\mathcal{E}^g_{t,T}[X]$ by $\mathcal{E}^g_t[X]$. The advantage of using g which satisfies condition (A4) additionally lies in the fact that it preserves almost all properties as the classical expectation except the linearity.

Proposition A.3 *Suppose that g satisfies (A1), (A2) and (A4). The conditional g-expectation satisfies the following:*

(1) *Translation invariance: if $Z \in L^2(\mathcal{F}_t)$, then for all $X \in L^2(\mathcal{F}_T)$, $\mathcal{E}^g_t[X+Z] = \mathcal{E}^g_t[X] + Z$;*

(2) *Local property: for an event $A \in \mathcal{F}_t$, we have $\mathcal{E}^g_t[XI_A + YI_{A^c}] = \mathcal{E}^g_t[X]I_A + \mathcal{E}^g_t[Y]I_{A^c}$;*

(3) *Constant preserving: if $X \in L^2(\mathcal{F}_t)$, we have $\mathcal{E}^g_t[X] = X$.*

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