PATHWISE UNIQUENESS FOR SDES WITH SINGULAR DRIFT AND NONCONSTANT DIFFUSION: A SIMPLE PROOF

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Abstract. A new proof of pathwise uniqueness for SDEs with Sobolev diffusion and integrable drift term is introduced by extending a method from E. Fedrizzi and F. Flandoli ([2]) to the case of nonconstant diffusion.

Keywords
Pathwise Uniqueness, Singular Drift, Sobolev Space, Krylov's Estimates

AMS Classification
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1. Introduction

Let us consider the following stochastic differential equation (SDE):

$$X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \quad t \in [0, T],$$

where $x \in \mathbb{R}^d$, b, σ are measurable functions from $[0, T] \times \mathbb{R}^d$ to $\mathbb{R}^d$, respectively $\mathbb{R}^{d\times m}$, and W is an m-dimensional standard Wiener process.
There are many papers which investigate the problem of existence or uniqueness of solutions for this kind of equation. In addition to the well known result for Lipschitz coefficients by K. Itô, [4], let us mention some of these results here. Strong existence and uniqueness have been obtained for example under local weak monotonicity and weak coercivity conditions on the coefficients. A proof can be found in Chapter 3 of the monograph by W. Liu and M. Röckner [6]. Furthermore, in their work [1], S. Fang and T. Zhang relaxed the Lipschitz conditions by a logarithmic factor. Moreover, A. Yu. Veretennikov proved strong existence and uniqueness for bounded measurable coefficients if the diffusion matrix is nondegenerated, continuous and Lipschitz continuous in the spacial variable, see [8]. In [3] I. Gyöngy and T. Martínez relaxed this to locally unbounded drifts, namely $b \in L^{2(d+1)}_{loc}(\mathbb{R}_+ \times \mathbb{R}^d)$ and b almost everywhere bounded by a constant plus some nonnegative function in $L^{d+1}(\mathbb{R}_+ \times \mathbb{R}^d)$.
In [5] N. Krylov and M. Röckner proved the existence of a unique strong solution up to some explosion time in the case where the diffusion coefficient σ is the unit matrix and the drift coefficient b is in $L^q_{\text{loc}}(\mathbb{R}_+; L^p_{\text{loc}}(\mathbb{R}^d))$ for some $p, q > 1$ fulfilling

$$\frac{d}{p} + \frac{2}{q} < 1. \tag{1}$$

If the diffusion is not constant and nondegenerate it is also possible to get strong existence and uniqueness results under similar conditions on the drift. The most general result can be found in the work of X. Zhang [11], respectively for the case $p = q$ see [10]. There, the drift is again in $L^q_{\text{loc}}(\mathbb{R}_+, L^p(\mathbb{R}^d))$ for $p, q > 1$ fulfilling (1). The diffusion coefficient is uniformly continuous in space, locally uniformly with respect to time, nondegenerated, bounded and the gradient is also in $L^q_{loc}(\mathbb{R}_+, L^p(\mathbb{R}^d))$. The idea of the proof is to remove the drift by the so-called Zvonkin transformation, see [12], and use known results for SDEs with zero drift. This transformation is based on the solution u to the equation

$$\partial_t u + \sum_{i=1}^{d} b^i \partial_{x_i} u + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} (\sigma\sigma^*)_{ij} \partial^2_{x_i x_j} u = 0, \qquad u(T, x) = x.$$

Then one gets a one-to-one correspondence between solutions X_t for the original SDE and solutions $u(t, X_t)$ for the transformed equation without drift term.
In the case of constant σ there is a much simpler proof for the pathwise uniqueness which is due to E. Fedrizzi and F. Flandoli (see [2]) under similar conditions as in [5]. They gave an elementary and short proof by developing another transformation of the SDE. The aim of this work is to extend their method to include the case of b and σ under the conditions as in [11]. The nonconstant diffusion leads to additional terms when performing the transformation of [2] which have to be controlled. One of the main tools to overcome these difficulties are Krylov Estimates but the price to pay is that we have to assume that (1) holds with 1/2 replacing 1 on its right hand side.
For simplicity we will state our result under global assumptions, but there are no difficulties to extend it by localization techniques, e.g. in the same way as in [11].

2. Preliminaries and main result

Definition 2.1. For $p, q \in (1, \infty)$ we define

$$\|f\|_{L^q_p(T)} := \left(\int_0^T \left(\int_{\mathbb{R}^d} |f(t,x)|^p \, dx \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}},$$

where $|\cdot|$ denotes the Hilbert–Schmidt norm.
We define $L^q_p(T)$ to be the space of measurable functions $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ (respectively $\mathbb{R}^{d\times m}$) such that $\|f\|_{L^q_p(T)} < \infty$.
Furthermore, we set

$$W^{1,2}_{q,p}(T) := \left\{ f : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d \;\middle|\; f, \partial_t f, \partial_x f, \partial^2_x f \in L^q_p(T) \right\},$$

where ∂_t, ∂_x, ∂^2_x denote the weak derivatives with respect to time, respectively space. The associated norm is given by

$$\|f\|_{W^{1,2}_{q,p}(T)} := \|f\|_{L^q_p(T)} + \|\partial_t f\|_{L^q_p(T)} + \|\partial_x f\|_{L^q_p(T)} + \|\partial^2_x f\|_{L^q_p(T)}.$$

We consider the SDE

$$X_t = x + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dW_s, \quad t \in [0, T], \tag{2}$$

where W is an m-dimensional standard Wiener process on a filtered probability space $(\Omega, (\mathcal{F}_t)_t, \mathbb{P})$, with $(\mathcal{F}_t)_t$ fulfilling the usual conditions, $x \in \mathbb{R}^d$ and $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ are measurable functions with the following properties:

Assumption 2.2. For some $p, q > 1$ with

$$\frac{d}{p} + \frac{2}{q} < \frac{1}{2},$$

we have

(c1) $b \in L^q_p(T)$,
(c2) σ is uniformly continuous in x, uniformly with respect to t, i.e. for all $\varepsilon > 0$ exists a $\delta > 0$ such that

$$\sup_{t\in[0,T]} |\sigma(t, x) - \sigma(t, y)| < \varepsilon \quad \text{for all } x, y \in \mathbb{R}^d \text{ with } |x - y| < \delta.$$

(c3) σ is nondegenerated, i.e. there exists a constant $c_\sigma > 0$ such that

$$\langle \sigma\sigma^*(t, x)\xi, \xi\rangle \geq c_\sigma \langle I\xi, \xi\rangle \qquad \forall\ \xi \in \mathbb{R}^d \quad \forall\ (t, x) \in [0, T] \times \mathbb{R}^d,$$

where σ^* denotes the transposed matrix of σ,
(c4) σ is bounded by a constant $\tilde{c}_\sigma$,
(c5) $\partial_x\sigma \in L^q_p(T)$.

Definition 2.3. (weak/strong solution) A weak solution for equation (2) is a pair (X, W) on a filtered probability space $(\Omega, (\mathcal{F}_t)_t, \mathbb{P})$ such that X is continuous, $(\mathcal{F}_t)_t$-adapted, fulfills

$$\mathbb{P}\left(\int_0^T |b(s, X_s)|\, ds < \infty\right) = 1, \tag{3}$$

$$\mathbb{P}\left(\int_0^T |\sigma(s, X_s)|^2\, ds < \infty\right) = 1, \tag{4}$$

W_t is an $\mathcal{F}_t$-Brownian motion and (X, W) satisfies equation (2) almost surely.
Given a Brownian motion W on a probability space, a strong solution for equation (2) is a continuous process X which is adapted to the filtration generated by W, fulfills (3), (4) and satisfies equation (2) almost surely.

Definition 2.4. (Pathwise Uniqueness) We say that pathwise uniqueness holds for equation (2) if for two weak solutions (X, W), $(\tilde{X}, \tilde{W})$, defined on the same probability space, we have that $X_0 = \tilde{X}_0$ and $W = \tilde{W}$ imply

$$\mathbb{P}\left(X_t = \tilde{X}_t \quad \forall\ t \in [0, T]\right) = 1.$$

Theorem 2.5. *Under Assumption 2.2, we have pathwise uniqueness for equation (2).*

Remark 1. One obtains the same result for $p, q > 2(d+1)$ if instead of Assumption 2.2 $(c2)$ σ is assumed to be continuous and such that for all $f \in L_p^q(T)$ there exists a solution to the equation

$$\partial_t u + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}(\sigma\sigma^*)_{ij}\partial^2_{x_i x_j} u = f \quad \text{on } [0,T], \quad u(T,x) = 0,$$

such that

$$\|u\|_{L_p^q(T)} \leq C\|f\|_{L_p^q(T)},$$

where C is independent of f and increasing in T. For details see [9].

In the following, whenever we speak of two solutions, we mean two weak solutions defined on the same probability space with the same Brownian motion.
Furthermore by $C > 0$ we always denote various finite constants, where we often indicate the dependence of parameters by writing them in brackets.

3. Transformation of the SDE

The following transformation works analogously to the transformation of [2] despite the appearance of additional terms in the partial differential equations and the stochastic integrals.
Assume that b and σ fulfill Assumption 2.2. Then by Theorem 10.3 and Remarks 10.4 and 10.5 in [5] for every $f \in L_p^q(T)$ there exists a solution $u \in W^{1,2}_{q,p}(T)$ to the equation

$$\partial_t u + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}(\sigma\sigma^*)_{ij}\partial^2_{x_i x_j} u = f \quad \text{on } [0,T], \quad u(T,x) = 0 \tag{5}$$

such that

$$\|u\|_{W^{1,2}_{q,p}(T)} \leq C\|f\|_{L_p^q(T)}, \tag{6}$$

where C does not depend on f and is increasing in T. Then by the Hölder continuity of $\partial_x u$, see [5] Lemma 10.2, we have

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |\partial_x u(t,x)| \leq C(p,q,\varepsilon,T)T^{\frac{\varepsilon}{2}}\|f\|_{L_p^q(T)} \tag{7}$$

for every $\varepsilon \in (0,1)$, which fulfills

$$\varepsilon + \frac{d}{p} + \frac{2}{q} < 1,$$

with $C(p,q,\varepsilon,T)$ increasing in T. We can therefore assume the constant in front of $\|f\|_{L_p^q(T)}$ to be as small as we want by choosing T appropriate which will be of importance in Lemma 3.1. Now, let U_b a solution to the equation

$$\partial_t u + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}(\sigma\sigma^*)_{ij}\partial^2_{x_i x_j} u = -b \quad \text{on } [0,T], \quad u(T,x) = 0. \tag{8}$$

Using Itô's formula for functions in $W^{1,2}_{q,p}(T)$ (Proposition A.1) and that U_b is a solution to PDE (8), we get

$$U_b(t,X_t) = U_b(0,x) + \int_0^t \partial_x U_b(s,X_s) b(s,X_s)\,ds + \int_0^t \partial_x U_b(s,X_s)\sigma(s,X_s)\,dW_s - \int_0^t b(s,X_s)\,ds.$$

That implies

$$\int_0^t b(s,X_s)\,ds = U_b(0,x) - U_b(t,X_t) + \int_0^t \partial_x U_b(s,X_s) b(s,X_s)\,ds + \int_0^t \partial_x U_b(s,X_s)\sigma(s,X_s)\,dW_s.$$

Now, we define

$$\mathcal{T}(b) := \partial_x U_b \cdot b$$

and transform SDE (2) by replacing the drift term:

$$X_t = x + U_b(0,x) - U_b(t,X_t) + \int_0^t \mathcal{T}(b)(s,X_s)\,ds + \int_0^t \partial_x U_b(s,X_s)\sigma(s,X_s) + \sigma(s,X_s)\,dW_s. \tag{9}$$

Note, that $\mathcal{T}(b) \in L^q_p(T)$ since $\partial_x U_b$ is bounded and $b \in L^q_p(T)$. Next, let $U_{\mathcal{T}(b)}$ be a solution to the equation

$$\partial_t u + \frac{1}{2}\sum_{i=1}^d \sum_{j=1}^d (\sigma\sigma^*)_{ij} \partial^2_{x_i x_j} u = -\mathcal{T}(b) \quad \text{on } [0,T], \quad u(T,x) = 0.$$

Using again Itô's formula (Proposition A.1) and that $U_{\mathcal{T}(b)}$ solves the equation above, we get

$$U_{\mathcal{T}(b)}(t,X_t) = U_{\mathcal{T}(b)}(0,x) + \int_0^t \partial_x U_{\mathcal{T}(b)}(s,X_s) b(s,X_s)\,ds + \int_0^t \partial_x U_{\mathcal{T}(b)}(s,X_s)\sigma(s,X_s)\,dW_s - \int_0^t \mathcal{T}(b)(s,X_s)\,ds,$$

and therefore

$$\int_0^t \mathcal{T}(b)(s,X_s)\,ds = U_{\mathcal{T}(b)}(0,x) - U_{\mathcal{T}(b)}(t,X_t) + \int_0^t \partial_x U_{\mathcal{T}(b)}(s,X_s) b(s,X_s)\,ds + \int_0^t \partial_x U_{\mathcal{T}(b)}(s,X_s)\sigma(s,X_s)\,dW_s.$$

As before, we define

$$\mathcal{T}^2(b) := \partial_x U_{\mathcal{T}(b)} \cdot b$$

and replace the drift term in the transformed SDE (9):

$$X_t = x + U_b(0,x) + U_{\mathcal{T}(b)}(0,x) - U_b(t,X_t) - U_{\mathcal{T}(b)}(t,X_t) + \int_0^t \mathcal{T}^2(b)(s,X_s)\,ds + \int_0^t \partial_x U_b(s,X_s)\sigma(s,X_s) + \partial_x U_{\mathcal{T}(b)}(s,X_s)\sigma(s,X_s) + \sigma(s,X_s)\,dW_s.$$

Iteration yields after $n+1$ steps

$$X_t = x + \sum_{k=0}^n U_{\mathcal{T}^k(b)}(0,x) - \sum_{k=0}^n U_{\mathcal{T}^k(b)}(t,X_t) + \int_0^t \mathcal{T}^{n+1}(b)(s,X_s)\,ds + \int_0^t \sum_{k=0}^n \partial_x U_{\mathcal{T}^k(b)}(s,X_s)\sigma(s,X_s) + \sigma(s,X_s)\,dW_s \tag{10}$$

with the convention

$$\mathcal{T}^0(b) = b \quad \text{and} \quad \mathcal{T}^{k+1}(b) = \partial_x U_{\mathcal{T}^k(b)} \cdot b.$$

We define

$$U^{(n)}(t,x) := \sum_{k=0}^n U_{\mathcal{T}^k(b)}(t,x)$$

and therefore, SDE (10) becomes

$$X_t = x + U^{(n)}(0,x) - U^{(n)}(t,X_t) + \int_0^t \mathcal{T}^{n+1}(b)(s,X_s)\,ds + \int_0^t \left(\partial_x U^{(n)}(s,X_s) + I\right)\sigma(s,X_s)\,dW_s. \tag{11}$$

For two solutions $X_t^{(1)}$, $X_t^{(2)}$ we define

$$\begin{aligned} Y_t^{(i,n)} &:= X_t^{(i)} + U^{(n)}(t,X_t^{(i)}), \quad i=1,2, \text{ and} \\ b^{(n)}(t,x) &:= \mathcal{T}^{n+1}(b)(t,x), \\ \sigma^{(n)}(t,x) &:= \left(\partial_x U^{(n)}(t,x) + I\right)\sigma(t,x). \end{aligned}$$

Then equation (11) reads

$$Y_t^{(i,n)} = Y_0^{(i,n)} + \int_0^t b^{(n)}(s, X_s^{(i)})\, ds + \int_0^t \sigma^{(n)}(s, X_s^{(i)})\, dW_s. \tag{12}$$

The following Lemma summarizes some properties of the transformed equation which are necessary in the proof of pathwise uniqueness. It is similar to Lemma 7 in [2] and so is the proof.

Lemma 3.1. *Let* $(c1) - (c4)$ *of Assumption 2.2 be fulfilled and* $X_t^{(1)}$, $X_t^{(2)}$ *be two solutions to* (2)*. Then there exists* $0 < T_0 \le T$ *such that for all* $T' \in (0, T_0]$ *we have*

(i) $\|b^{(n)}\|_{L_p^q(T')} \le \frac{1}{2^{n+1}} \|b\|_{L_p^q(T')}$,

(ii) $\sum_{k=0}^{n} \sup_{(t,x)\in[0,T']\times\mathbb{R}^d} |\partial_x U_{\mathcal{T}^k(b)}(t,x)| \le \frac{1}{2}$,

(iii) $\|\partial_x^2 U^{(n)}\|_{L_p^q(T')} \le C$ *for some constant* $C > 0$*, independent of* n*, and*

(iv) $\left|Y_t^{(1,n)} - Y_t^{(2,n)}\right| \le \frac{3}{2}\left|X_t^{(1)} - X_t^{(2)}\right|$,

$$\left|X_t^{(1)} - X_t^{(2)}\right| \le 2\left|Y_t^{(1,n)} - Y_t^{(2,n)}\right| \quad \textit{for all } t \in (0, T'].$$

4. Pathwise uniqueness

We now prove Theorem 2.5. It works analogously to [2], based on Lemma 3.1 and three results, namely Lemmas 4.1, 4.2 and Proposition 4.3, which are similar to [2] but with different proofs. This is due to the fact that in our framework the solution is in general not a Brownian motion. For reasons of readability we defer the proofs to the next section.

Proof. (Proof of Theorem 2.5 for small T) In the following, we denote by x^i the i-th entry of a vector $x \in \mathbb{R}^d$. Let Assumption 2.2 be fulfilled and $X_t^{(1)}$, $X_t^{(2)}$ be two solutions to (2). Furthermore, let $T := T_0$ from Lemma 3.1 and $Y_t^{(i,n)}$ given by (12). By Itô's formula and an application of the inequality of Cauchy and Schwarz we then have

$$\begin{aligned} d\left|Y_t^{(1,n)} - Y_t^{(2,n)}\right|^2 \le{}& 2\left|Y_t^{(1,n)} - Y_t^{(2,n)}\right| \left|b^{(n)}(t, X_t^{(1)}) - b^{(n)}(t, X_t^{(2)})\right| dt \\ &+ 2\left\langle Y_t^{(1,n)} - Y_t^{(2,n)}, \left(\sigma^{(n)}(t, X_t^{(1)}) - \sigma^{(n)}(t, X_t^{(2)})\right) dW_t \right\rangle \\ &+ \left|\sigma^{(n)}(t, X_t^{(1)}) - \sigma^{(n)}(t, X_t^{(2)})\right|^2 dt. \end{aligned} \tag{13}$$

Moreover, with

$$A_t^{(n)} := \int_0^t \frac{\left|\sigma^{(n)}(s, X_s^{(1)}) - \sigma^{(n)}(s, X_s^{(2)})\right|^2}{\left|Y_s^{(1,n)} - Y_s^{(2,n)}\right|^2} \mathbf{1}_{\{Y_s^{(1,n)} \ne Y_s^{(2,n)}\}} ds,$$

we have

$$d\left(e^{-A_t^{(n)}}\left|Y_t^{(1,n)}-Y_t^{(2,n)}\right|^2\right)=e^{-A_t^{(n)}}d\left|Y_t^{(1,n)}-Y_t^{(2,n)}\right|^2 - \left|Y_t^{(1,n)}-Y_t^{(2,n)}\right|^2 e^{-A_t^{(n)}}dA_t^{(n)},$$

since the quadratic covariation is zero due to the monotonicity of $e^{-A_t^{(n)}}$. Now, we use inequality (13) to conclude that

$$\begin{aligned}
&d\left(e^{-A_t^{(n)}}\left|Y_t^{(1,n)}-Y_t^{(2,n)}\right|^2\right)\\
&\leq 2e^{-A_t^{(n)}}\left|Y_t^{(1,n)}-Y_t^{(2,n)}\right|\left|b^{(n)}(t,X_t^{(1)})-b^{(n)}(t,X_t^{(2)})\right|\,dt\\
&\quad+2e^{-A_t^{(n)}}\left\langle Y_t^{(1,n)}-Y_t^{(2,n)},\left(\sigma^{(n)}(t,X_t^{(1)})-\sigma^{(n)}(t,X_t^{(2)})\right)\,dW_t\right\rangle
\end{aligned}$$

and thus,

$$\begin{aligned}
&\mathbb{E}\left[e^{-A_t^{(n)}}\left|Y_t^{(1,n)}-Y_t^{(2,n)}\right|^2\right]\\
&\leq \mathbb{E}\left[\left|Y_0^{(1,n)}-Y_0^{(2,n)}\right|^2\right]\\
&\quad+2\mathbb{E}\left[\int_0^t e^{-A_s^{(n)}}\left|Y_s^{(1,n)}-Y_s^{(2,n)}\right|\left|b^{(n)}(s,X_s^{(1)})-b^{(n)}(s,X_s^{(2)})\right|\,ds\right]\\
&\quad+2\mathbb{E}\left[\int_0^t e^{-A_s^{(n)}}\left\langle Y_s^{(1,n)}-Y_s^{(2,n)},\left(\sigma^{(n)}(s,X_s^{(1)})-\sigma^{(n)}(s,X_s^{(2)})\right)\,dW_s\right\rangle\right].
\end{aligned}$$

With the help of Lemma 3.1, we get

$$\begin{aligned}
&\mathbb{E}\left[e^{-A_t^{(n)}}\left|Y_t^{(1,n)}-Y_t^{(2,n)}\right|^2\right]\\
&\leq \frac{9}{4}\left|x^{(1)}-x^{(2)}\right|^2\\
&\quad+3\mathbb{E}\left[\int_0^t \left|X_s^{(1)}-X_s^{(2)}\right|\left|b^{(n)}(s,X_s^{(1)})-b^{(n)}(s,X_s^{(2)})\right|\,ds\right]\\
&\quad+2\mathbb{E}\left[\int_0^t e^{-A_s^{(n)}}\left\langle Y_s^{(1,n)}-Y_s^{(2,n)},\left(\sigma^{(n)}(s,X_s^{(1)})-\sigma^{(n)}(s,X_s^{(2)})\right)\,dW_s\right\rangle\right].
\end{aligned}\tag{14}$$

Summarizing, for two solutions with the same initial values, we have for all $t \leq T$

$$\begin{aligned}
\mathbb{E}\left[\left|X_t^{(1)}-X_t^{(2)}\right|\right] &\leq \mathbb{E}\left[2\left|Y_t^{(1,n)}-Y_t^{(2,n)}\right|\right]\\
&= 2\mathbb{E}\left[e^{\frac{1}{2}A_t^{(n)}}e^{-\frac{1}{2}A_t^{(n)}}\left|Y_t^{(1,n)}-Y_t^{(2,n)}\right|\right]\\
&\leq 2\mathbb{E}\left[e^{A_t^{(n)}}\right]^{\frac{1}{2}}\mathbb{E}\left[e^{-A_t^{(n)}}\left|Y_t^{(1,n)}-Y_t^{(2,n)}\right|^2\right]^{\frac{1}{2}}.
\end{aligned}$$

With inequality (14) we obtain

$$\mathbb{E}\left[\left|X_t^{(1)} - X_t^{(2)}\right|\right]$$

(15)

$$\le 2\mathbb{E}\left[e^{A_t^{(n)}}\right]^{\frac{1}{2}} \left(3\mathbb{E}\left[\int_0^T \left|X_s^{(1)} - X_s^{(2)}\right| \left|b^{(n)}(s, X_s^{(1)}) - b^{(n)}(s, X_s^{(2)})\right| \, ds\right]\right.$$
$$\left. + 2\mathbb{E}\left[\int_0^t e^{-A_s^{(n)}} \left\langle Y_s^{(1,n)} - Y_s^{(2,n)}, \left(\sigma^{(n)}(s, X_s^{(1)}) - \sigma^{(n)}(s, X_s^{(2)})\right) dW_s\right\rangle\right]\right)^{\frac{1}{2}}.$$

Note that the second expectation term vanishes due to the martingale property of the stochastic integral which is well defined as $\sigma^{(n)}$ is bounded and $|Y_t^{(1,n)} - Y_t^{(2,n)}|^2$ is integrable by the following Lemma.

Lemma 4.1. *Let* $(c1) - (c4)$ *of Assumption 2.2 be fulfilled. If* X_t *is a solution to SDE* (2)*, we have*

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X_t|\right] < \infty \quad \text{and} \quad \sup_{t\in[0,T]} \mathbb{E}\left[|X_t|^2\right] < \infty.$$

Therefore, by (15) we have

$$\mathbb{E}\left[\left|X_t^{(1)} - X_t^{(2)}\right|\right]$$
$$\le C\mathbb{E}\left[e^{A_t^{(n)}}\right]^{\frac{1}{2}} \mathbb{E}\left[\int_0^T \left|X_s^{(1)} - X_s^{(2)}\right| \left|b^{(n)}(s, X_s^{(1)}) - b^{(n)}(s, X_s^{(2)})\right| \, ds\right]^{\frac{1}{2}}$$
$$\le C\mathbb{E}\left[e^{A_t^{(n)}}\right]^{\frac{1}{2}} \mathbb{E}\left[\int_0^T \left|X_s^{(1)} - X_s^{(2)}\right|^2 ds\right]^{\frac{1}{4}} \mathbb{E}\left[\int_0^T \left|b^{(n)}(s, X_s^{(1)}) - b^{(n)}(s, X_s^{(2)})\right|^2 ds\right]^{\frac{1}{4}}$$
$$\le C\mathbb{E}\left[e^{A_t^{(n)}}\right]^{\frac{1}{2}} \mathbb{E}\left[\int_0^T \left|b^{(n)}(s, X_s^{(1)}) - b^{(n)}(s, X_s^{(2)})\right|^2 ds\right]^{\frac{1}{4}}$$

for all $n \in \mathbb{N}$, where the last inequality follows from Lemma 4.1. The proof of pathwise uniqueness is complete if we show that the first term is uniformly bounded in n and that the second term converges to zero. These assertions are given by the next two statements which we also prove in the next section.

Lemma 4.2. *Let* $(c1) - (c4)$ *of Assumption 2.2 be fulfilled and* $X_t^{(1)}$, $X_t^{(2)}$ *be two solutions of* (2)*. Then we have*

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^T \left|b^{(n)}(t, X_t^{(1)}) - b^{(n)}(t, X_t^{(2)})\right|^2 dt\right] = 0.$$

Proposition 4.3. *Let Assumption 2.2 be fulfilled and* $X_t^{(1)}$, $X_t^{(2)}$ *be two solutions to* (2)*. Then there exists a constant* $C > 0$ *such that*

$$\mathbb{E}\left[e^{A_T^{(n)}}\right] \le C \qquad \text{uniformly for all } n \in \mathbb{N}.$$

Hence, we proved

$$X_t^{(1)} = X_t^{(2)} \quad \mathbb{P}\text{-a.s.} \quad \forall\, t \in [0,T].$$

Thus,

$$\mathbb{P}\left(X_t^{(1)} = X_t^{(2)} \ \forall\, t \in \mathbb{Q} \cap [0,T] \right) = 1$$

and by continuity of the solutions we obtain

$$\mathbb{P}\left(X_t^{(1)} = X_t^{(2)} \ \forall\, t \in [0,T] \right) = 1.$$

□

Remark 2. The interval of pathwise uniqueness can easily be extended to arbitrarily large T by means of a time-shift argument.

5. Proofs of auxiliaries

Proof. (Proof of Lemma 4.1) We have

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X_t| \right] \le |x| + \mathbb{E}\left[\int_0^T |b(s,X_s)|\, ds \right] + \mathbb{E}\left[\sup_{t\in[0,T]} \left| \int_0^t \sigma(s,X_s)\, dW_s \right| \right].$$

Then applications of a Krylov estimate, namely Theorem 2.2 in [11] to the first expectation term and of the inequality of Burkholder, Davis and Gundy (see e.g. [7] Corollary IV.4.2) to the second yield

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X_t| \right] \le |x| + C\|b\|_{L_p^q(T)} + C\mathbb{E}\left[\left(\int_0^T \sigma(s,X_s)^2\, ds \right)^{\frac{1}{2}} \right].$$

Since σ is bounded and $b \in L_p^q(T)$, this is finite. Furthermore,

$$\sup_{t\in[0,T]} \mathbb{E}\left[|X_t|^2 \right] \le 2|x|^2 + 4 \sup_{t\in[0,T]} \mathbb{E}\left[\left| \int_0^t b(s,X_s)\, ds \right|^2 \right] + 4 \sup_{t\in[0,T]} \mathbb{E}\left[\left| \int_0^t \sigma(s,X_s)\, dW_s \right|^2 \right].$$

We apply Hölder's inequality to the first expectation and the multidimensional Itô Isometry to the second one to receive

$$\sup_{t\in[0,T]} \mathbb{E}\left[|X_t|^2 \right] \le 2|x|^2 + 4T\mathbb{E}\left[\int_0^T |b(s,X_s)|^2\, ds \right] + 4 \sup_{t\in[0,T]} \mathbb{E}\left[\int_0^t |\sigma(s,X_s)|^2\, ds \right].$$

Again, we use Theorem 2.2 of [11] and Assumption 2.2 (c1), (c4) to obtain that this is finite. □

The following proof of the convergence of the drift term becomes simple with the help of the Krylov estimate Theorem 2.2 of [11]. The price to pay is the factor two in the assumptions on p and q.

Proof. (Proof of Lemma 4.2) Theorem 2.2 of [11] for $\frac{p}{2}$, $\frac{q}{2}$ and an application of Lemma 3.1 yields

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \left|b^{(n)}(t,X_t^{(1)}) - b^{(n)}(t,X_t^{(2)})\right|^2 \, dt\right] \\
&\leq 2\mathbb{E}\left[\int_0^T \left|b^{(n)}(t,X_t^{(1)})\right|^2 \, dt\right] + 2\mathbb{E}\left[\int_0^T \left|b^{(n)}(t,X_t^{(2)})\right|^2 \, dt\right] \\
&\leq C(d,p,q,T,c_\sigma,\tilde{c}_\sigma,\|b\|_{L_p^q(T)})\|b^{(n)}\|_{L_p^q(T)}^2 \\
&\leq C(d,p,q,T,c_\sigma,\tilde{c}_\sigma,\|b\|_{L_p^q(T)})\frac{1}{2^{2(n+1)}}\|b\|_{L_p^q(T)}^2 \\
&\xrightarrow{n\to\infty} 0.
\end{aligned}$$

□

Proof. (Proof of Proposition 4.3) Considering $\sigma^{(n)}$ we find that:

$$\partial_{x_i}\sigma^{(n)} = \left(\partial_{x_i}\partial_x U^{(n)}\right)\sigma + \partial_x U^{(n)}\partial_{x_i}\sigma + \partial_{x_i}\sigma.$$

We use that σ is bounded and $\partial_x\sigma \in L_p^q(T)$, that $\partial_x U^{(n)}$ is uniformly bounded by $\frac{1}{2}$ and $\partial_x^2 U^{(n)}$ is equibounded in $L_p^q(T)$ (see Lemma 3.1) to deduce that

$$\|\partial_x\sigma^{(n)}\|_{L_p^q(T)} \leq C \text{ uniformly in } n.$$

Additionally, $\sigma^{(n)}$ is continuous, since $\partial_x U^{(n)}$ is Hölder continuous. Then there exists a sequence of continuous functions $(u_m)_m$, which are differentiable with respect to x in the ordinary sense, such that $u_m \to \sigma^{(n)}$ uniformly on $[0,T]\times\mathbb{R}^d$ and

$$\|\partial_x u_m\|_{L_p^q(T)} \leq \|\partial_x\sigma^{(n)}\|_{L_p^q(T)} \quad \forall\, m \in \mathbb{N}.$$

The existence of such a function can be obtained by mollification. Define $X_t^\lambda := \lambda X_t^{(1)} + (1-\lambda)X_t^{(2)}$. Then we have with Lemma 3.1 (iv) and uniform convergence

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\int_0^T \frac{\left|\sigma^{(n)}(t,X_t^{(1)}) - \sigma^{(n)}(t,X_t^{(2)})\right|^2}{\left|Y_t^{(1,n)} - Y_t^{(2,n)}\right|^2}\mathbf{1}_{\{Y_t^{(1,n)}\neq Y_t^{(2,n)}\}}\, dt\right)\right] \\
&\leq \mathbb{E}\left[\exp\left(4\int_0^T \frac{\left|\sigma^{(n)}(t,X_t^{(1)}) - \sigma^{(n)}(t,X_t^{(2)})\right|^2}{\left|X_t^{(1)} - X_t^{(2)}\right|^2}\mathbf{1}_{\{X_t^{(1)}\neq X_t^{(2)}\}}\, dt\right)\right]. \\
&= \lim_{m\to\infty}\mathbb{E}\left[\exp\left(4\int_0^T \frac{\left|u_m(t,X_t^{(1)}) - u_m(t,X_t^{(2)})\right|^2}{\left|X_t^{(1)} - X_t^{(2)}\right|^2}\mathbf{1}_{\{X_t^{(1)}\neq X_t^{(2)}\}}\, dt\right)\right]
\end{aligned}$$

$$\leq \lim_{m\to\infty} \mathbb{E}\left[\exp\left(4\int_0^T\int_0^1 |\partial_x u_m(t,X_t^\lambda)|^2\, d\lambda\, dt\right)\right]$$
$$\leq \lim_{m\to\infty} \int_0^1 \mathbb{E}\left[\exp\left(4\int_0^T |\partial_x u_m(t,X_t^\lambda)|^2\, dt\right)\right] d\lambda.$$

Now, choose $\mu > 0$ so small that $(d/p + 2/q)(1+\mu) < 1/2$ holds. Then we have for $\beta > 0$ with Young's and Hölder's inequality

$$\mathbb{E}\left[\exp\left(\int_0^T \frac{\left|\sigma^{(n)}(t,X_t^{(1)}) - \sigma^{(n)}(t,X_t^{(2)})\right|^2}{\left|Y_t^{(1,n)} - Y_t^{(2,n)}\right|^2}\mathbf{1}_{\{Y_t^{(1,n)}\neq Y_t^{(2,n)}\}}\, dt\right)\right]$$

$$\leq \lim_{m\to\infty}\int_0^1 \mathbb{E}\left[\exp\left(\frac{1}{\mu+1}\left(\beta\int_0^T |\partial_x u_m(t,X_t^\lambda)|^2\, dt\right)^{1+\mu} + \frac{\mu}{1+\mu}\left(\frac{4}{\beta}\right)^{\frac{1+\mu}{\mu}}\right)\right] d\lambda \tag{16}$$

$$\leq \exp\left(\frac{\mu}{1+\mu}\left(\frac{4}{\beta}\right)^{\frac{1+\mu}{\mu}}\right) \cdot \lim_{m\to\infty}\int_0^1 \mathbb{E}\left[\exp\left(\int_0^T \frac{\beta^{1+\mu}}{1+\mu}T^{\frac{\mu}{1+\mu}}|\partial_x u_m(t,X_t^\lambda)|^{2(1+\mu)}\, dt\right)\right] d\lambda. \tag{17}$$

Furthermore, we have with Theorem 2.2 from [11] for all $0 \leq t_0 \leq T$

$$\mathbb{E}\left[\int_{t_0}^T \frac{\beta^{1+\mu}}{1+\mu}T^{\frac{\mu}{1+\mu}}\left|\partial_x u_m(t,X_t^\lambda)\right|^{2(1+\mu)}\, dt \,\middle|\, \mathcal{F}_{t_0}\right]$$
$$\leq C(d,p,q,\mu,T,c_\sigma,\tilde{c}_\sigma,\|b\|_{L_p^q(T)})\frac{\beta^{1+\mu}}{1+\mu}T^{\frac{\mu}{1+\mu}}\||\partial_x u_m|^{2(1+\mu)}\|_{L^{\frac{q}{2(1+\mu)}}_{\frac{p}{2(1+\mu)}}(T)}$$
$$\leq C(d,p,q,\mu,T,c_\sigma,\tilde{c}_\sigma,\|b\|_{L_p^q(T)})\beta^{1+\mu}\|\partial_x\sigma^{(n)}\|^{2(1+\mu)}_{L_p^q(T)}.$$

Since $\|\partial_x\sigma^{(n)}\|_{L_p^q(T)}$ is equibounded, we can choose β so small that this is less than some $0 < \alpha < 1$ for all $n \in \mathbb{N}$. Then we have by Lemma A.2 and inequality (17) that

$$\mathbb{E}\left[e^{A_T^{(n)}}\right] \leq \exp\left(\frac{\mu}{1+\mu}\left(\frac{4}{\beta}\right)^{\frac{1+\mu}{\mu}}\right)\frac{1}{1-\alpha}$$
$$\leq C,$$

where C does not depend on n. □

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Appendix A.

Proposition A.1. *(Itô's formula) Let* (c1) − (c4) *of Assumption 2.2 be fulfilled,* X_t *a solution to* (2) *and* $u \in W^{1,2}_{q,p}(T)$*. Then for* $0 \le s \le t \le T$ *we have*

$$\begin{aligned} u(t,X_t) = u(s,X_s) &+ \int_s^t \partial_t u(r,X_r)\,dr + \int_s^t \partial_x u(r,X_r) b(r,X_r)\,dr \\ &+ \int_s^t \partial_x u(r,X_r)\sigma(r,X_r)\,dW_r \\ &+ \frac{1}{2}\int_s^t \sum_{i=1}^d \sum_{j=1}^d (\sigma\sigma^*(r,X_r))_{ij}\partial^2_{x_i x_j} u(r,X_r)\,dr \qquad \mathbb{P}\text{-almost surely.} \end{aligned}$$

This result can be obtained by approximation with smooth functions as in [5, Theorem 3.7] with the help of [11, Theorem 2.2].

Lemma A.2. *Let* $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ *be a nonnegative measurable function and* γ *an arbitrary stopping time. Assume that* X_t *is an adapted process such there exists a constant* $\alpha < 1$ *with*

$$\mathbf{1}_{\{t_0 \le \gamma\}} \mathbb{E}\left[\int_{t_0}^{T\wedge\gamma} f(t,X_t)\,dt \,\middle|\, \mathcal{F}_{t_0} \right] \le \alpha \qquad \mathbb{P}\text{-a. s.} \;\; \forall\, 0 \le t_0 \le T.$$

Then we have

$$\mathbb{E}\left[\exp\left(\int_0^{T\wedge\gamma} f(t,X_t)\,dt \right) \right] \le \frac{1}{1-\alpha}.$$

This is a slightly more general version of Khasminski's Lemma which can be obtained by rewriting the exponential series and using properties of the conditional expectation.

References

[1] S. Fang and T. Zhang, *A study of a class of stochastic differential equations with non-Lipschitzian coefficients*, Probab. Theory Related Fields 132 (2005), pp. 356–390

[2] E. Fedrizzi and F. Flandoli, *Pathwise uniqueness and continuous dependence of SDEs with non-regular drift*, Stochastics 83 (2011), pp. 241–257

[3] I. Gyöngy and T. Martínez, *On stochastic differential equations with locally unbounded drift*, Czechoslovak Math. J. 51(126) (2001), pp. 763–783

[4] K. Itô, *On a stochastic integral equation*, Proc. Japan Acad. 22 (1946), pp. 32–35

[5] N.V. Krylov and M. Röckner, *Strong solutions of stochastic equations with singular time dependent drift*, Probab. Theory Related Fields 131 (2005), pp. 154–196

[6] W. Liu and M. Röckner, *Stochastic partial differential equations: an introduction*, Springer, Cham 2015

[7] D. Revuz and M. Yor, *Continuous martingales and Brownian motion*, Springer 2005

[8] A.Yu. Veretennikov, *Strong solutions of some stochastic equations*, Uspekhi Mat. Nauk 33 (1978), pp. 173–174

[9] K. von der Lühe, *Pathwise uniqueness for stochastic differential equations with singular drift and nonconstant diffusion*, Ph.D. thesis, Universität Bielefeld, 2017

[10] X. Zhang, *Strong solutions of SDES with singular drift and Sobolev diffusion coefficients*, Stochastic Process. Appl. 115 (2005), pp. 1805–1818

[11] X. Zhang, *Stochastic homeomorphism flows of SDEs with singular drifts and Sobolev diffusion coefficients*, Electron. J. Probab. 16 (2011), pp. no. 38, 1096–1116

[12] A.K. Zvonkin, *A transformation of the phase space of a diffusion process that will remove the drift*, Mat. Sb. (N.S.) 93(135) (1974), pp. 129–149, 152

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