Concentration on Poisson spaces via modified Φ-Sobolev inequalities

Anna Gusakova[*], Holger Sambale[†] and Christoph Thäle[‡]

Abstract

Concentration properties of functionals of general Poisson processes are studied. Using a modified Φ-Sobolev inequality a recursion scheme for moments is established, which is of independent interest. This is applied to derive moment and concentration inequalities for functionals on abstract Poisson spaces. Applications of the general results in stochastic geometry, namely Poisson cylinder models and Poisson random polytopes, are presented as well.

Keywords. Concentration inequalities, L^p-estimates, modified Φ-Sobolev inequalities, Poisson processes, stochastic geometry.
MSC. 60D05, 60G55, 60H05.

1 Introduction

The stochastic analysis of non-linear functionals of Poisson processes (so-called Poisson functionals) on general state spaces was subject of intensive investigation during the last decade. New tools have been developed and new applications have been found. Most notably, we mention here the very fruitful connection between Malliavin calculus on Poisson spaces with Stein's method for normal approximation and the various striking applications in stochastic geometry of the resulting abstract limit theorems. For further background material on this topic and for references we refer to the monograph of Last and Penrose [13] as well as to the collection edited by Peccati and Reitzner [14]. In contrast to the weak limit theorems just mentioned we are interested in concentration properties of Poisson functionals. In the past questions of this type for general Poisson functionals have been approached from three different angles. First, Wu [21] developed a modified log-Sobolev inequality to derive a collection of concentration bounds under rather restrictive conditions. This approach was largely extended by Bachmann and Peccati [2] who also studied various applications of such bounds to random geometric graphs. Using general covariance identities for exponential functions of Poisson processes Gieringer and Last [8] were able to derive a new set of concentration inequalities. They were particularly useful to study concentration properties of geometric functionals associated with the Poisson Boolean model [8] or with so-called Poisson cylinder processes [4] considered in stochastic geometry. Finally, based on a general transportation inequality concentration bounds for so-called convex functionals of Poisson processes were proved by Gozlan, Herry and Peccati [9]. They also gave applications to Poisson U-statistics.

It is the purpose of this paper to add another approach to concentration properties of Poisson functionals (see Section 2 for a formal definition of this notion). It is based on a modified Φ-Sobolev inequality for Poisson processes, which is due to Chafaï [7]. We formulate the result in Section 3 and include a short proof for completeness. Moreover, we shall demonstrate that these modified Φ-Sobolev inequalities interpolate between two well-known results, namely the Poincaré inequality and Wu's modified log-Sobolev inequality on Poisson spaces. From the modified Φ-Sobolev inequality we derive a recursive scheme of L^p-estimates for Poisson functionals by specializing the function Φ. They

[*]Ruhr University Bochum, Germany. Email: anna.gusakova@rub.de
[†]Bielefeld University, Germany. Email: hsambale@math.uni-bielefeld.de
[‡]Ruhr University Bochum, Germany. Email: christoph.thaele@rub.de

involve two different types of difference operators, which reflect the discrete nature of the underlying problem, and are of independent interest. In Section 5 we use the recursive scheme to derive moment and concentration bounds for Poisson functionals. In particular, we recover qualitatively the bounds of Bachmann and Peccati by our method. The approach just described is not new and has been used by Boucheron, Bousquet, Lugosi and Massart [6] for functionals of independent random variables. This note demonstrates that a similar methodology can successfully be implemented in the framework of Poisson functionals as well. In the final Section 6 we discuss two novel applications of our general results to models from stochastic geometry, namely Poisson polytopes and Poisson cylinder processes. While concentration bounds for geometric functionals of Poisson cylinder models are known from [4] and concentration inequalities for random polytopes can be found in [20], some of the estimates we prove are new. This is especially the case for the so-called intrinsic volumes of Poisson polytopes with vertices chosen from the boundary of a smooth convex body for which no concentration inequalities can be found in the existing literature.

Independently and, in a sense, in parallel with us Adamczak, Polaczyk and Strzelecki [1] have very recently obtained (among many other results) moment estimates and concentration bounds for Poisson functionals, which are similar to the ones we prove, see especially Section 4.7 in [1]. The main difference to their paper is that they use a so-called Beckner-type inequality as their starting point (a device we will be able to recover from our results as well, see Corollary 3.4), while we are building on a modified Φ-Sobolev inequality. Apart from this, both approaches eventually rely on the methods of Boucheron, Bousquet, Lugosi and Massart [6] and lead to very similar results. We will further comment on the differences within the text.

2 Poisson processes and Poisson functionals

Let $(\mathsf{X}, \mathcal{X})$ be a measurable space supplied with a σ-finite measure μ. By $\mathsf{N}(\mathsf{X})$ we denote the space of σ-finite counting measures on X. The σ-field $\mathcal{N}(\mathsf{X})$ is defined as the smallest σ-field on $\mathsf{N}(\mathsf{X})$ such that the evaluation mappings $\xi \mapsto \xi(B)$, $B \in \mathcal{X}$, $\xi \in \mathsf{N}(\mathsf{X})$ are measurable. A **point process** on X is a measurable mapping with values in $\mathsf{N}(\mathsf{X})$ defined over some fixed probability space $(\Omega, \mathcal{F}, \mathsf{P})$. By a **Poisson process** η on X with intensity measure μ we understand a point process with the following two properties:

(i) for any $B \in \mathcal{X}$ the random variable $\eta(B)$ is Poisson distributed with mean $\mu(B)$;

(ii) for any $n \in \mathsf{N}$ and pairwise disjoint sets $B_1, \dots, B_n \in \mathcal{X}$ the random variables $\eta(B_1), \dots, \eta(B_n)$ are independent.

For the existence and construction of Poisson processes we refer to the monograph [13]. A **Poisson functional** is a random variable F P-almost surely satisfying $F = f(\eta)$ for some measurable $f : \mathsf{N}(\mathsf{X}) \to \mathsf{R}$. In this case f is called a **representative** of F. If P_η denotes the distribution of the Poisson process η we will write $L^p(\mathsf{P}_\eta)$, $p \geq 0$, for the space of Poisson functionals F satisfying $\mathsf{E}|F|^p < \infty$. For a Poisson functional F with representative f and $x \in \mathsf{X}$ we define the **difference operator** $D_x F$ by putting

$$D_x F := f(\eta + \delta_x) - f(\eta),$$

where δ_x stands for the Dirac measure at x.

Next, we recall from [12, 22] the **Clark-Ocône representation** of Poisson functionals. For this we add a time component to our space X by putting $\mathsf{Y} := [0, 1] \times \mathsf{X}$. Further, we let $\lambda := \ell \otimes \mu$ be the product of the Lebesgue measure ℓ on $[0, 1]$ with the measure μ on X. By ζ we denote a Poisson process on Y with intensity measure λ and for $t \in [0, 1]$ we define the σ-field $\mathcal{F}_t = \sigma\{\zeta(A) : A \in \mathcal{B}([0, t]) \otimes \mathcal{X}\}$, where $\mathcal{B}([0, t])$ is the Borel σ-field on $[0, t]$. If $F \in L^2(\mathsf{P}_\zeta)$ then P-almost surely, one has that

$$F = \mathsf{E}[F] + \int_0^1 \int_{\mathsf{X}} \mathsf{E}[D_{(s,x)} F | \mathcal{F}_s] \, \widehat{\zeta}(\mathrm{d}(s, x)), \tag{2.1}$$

where $\widehat{\zeta} = \zeta - \lambda$ stands for the compensated Poisson process. Note that any Poisson functional $F = f(\eta)$ of the Poisson process η on the original space X can be identified with a Poisson functional $\widehat{F} = \widehat{f}(\zeta)$ on the time augmented space. In fact, if $f : \mathsf{N}(\mathsf{X}) \to \mathsf{R}$ is a representative of F then $\widehat{f} : \mathsf{N}([0,1] \times \mathsf{X}) \to \mathsf{R}$ given by $\widehat{f}(\xi) = f(\xi([0,1] \times \cdot\,))$ is a representative of $\widehat{F}$. Similarly, for the difference operator we have the relation $D_{(s,x)}\widehat{F} = D_x F$.

3 Modified Φ-Sobolev inequalities for Poisson functionals

We denote by $\mathcal{C}$ the space of functions $\Phi : \mathsf{R}_+ \to \mathsf{R}$ satisfying the following three properties:

(i) Φ is convex and continuous,

(ii) Φ is twice differentiable on $(0, \infty)$,

(iii) Φ is either affine, or Φ'' is strictly positive and $1/\Phi''$ is concave.

Typical examples of functions belonging to $\mathcal{C}$ are $\Phi_{\log}(x) = x \log x$ or $\Phi_r(x) = x^{2/r}$ with $r \in (1,2)$ and $x \in \mathsf{R}_+$. For $\Phi \in \mathcal{C}$ the Φ-*entropy* of a random variable F is defined as

$$\mathrm{Ent}_\Phi(F) := \mathsf{E}[\Phi(F)] - \Phi(\mathsf{E}[F]).$$

In particular, the classical entropy $\mathrm{Ent}(F) = \mathsf{E}[F \log F] - \mathsf{E}[F] \log(\mathsf{E}[F])$ of F is recovered by taking $\Phi = \Phi_{\log}$.
The following modified Φ-Sobolev inequality is due to Chafaï [7, Section 5.1] and generalizes the modified log-Sobolev inequality of Wu [21]. We include the argument for completeness.

Proposition 3.1 (Modified Φ-Sobolev inequality). *Let η be a Poisson process on a measurable space X with σ-finite intensity measure μ. Let $F \in L^1(\mathsf{P}_\eta)$, $\Phi \in \mathcal{C}$ and assume that $F > 0$ P-almost surely. Then*

$$\mathrm{Ent}_\Phi(F) \le \mathsf{E}\Big[\int_{\mathsf{X}} \big(D_x \Phi(F) - \Phi'(F) D_x F \big)\, \mu(\mathrm{d}x) \Big].$$

Proof. We construct the time augmented space $\mathsf{Y} = [0,1] \times \mathsf{X}$ and let ζ be a Poisson process on Y with intensity measure $\lambda = \ell \otimes \mu$, where ℓ is the Lebesgue measure on $[0,1]$. By the discussion in the previous section it is sufficient to prove the inequality for Poisson functionals $F \in L^1(\mathsf{P}_\zeta)$ on the space Y. In addition, by a localization and approximation argument as in [21] we can assume that F satisfies the boundedness condition $1/N \le F \le N$ for some $N \in \mathsf{N}$, and eventually let $N \to \infty$. In particular, such F belong to $L^2(\mathsf{P}_\eta)$ so that the Clark-Ocône representation (2.1) is available for them (see [21] for a similar argument).
To F we associate the right-continuous martingale $M_t = M_t(F) := \mathsf{E}[F|\mathcal{F}_t]$, $t \in [0,1]$. Moreover, for $(s,x) \in \mathsf{Y}$ we put $d(s,x) := \mathsf{E}[D_{(s,x)}F|\mathcal{F}_s]$. Applying now Itô's formula for jump processes [5, Theorem 17.5], the Clark-Ocône representation (2.1) for Poisson functionals and using that $M_{t-} = M_t$ P-almost surely for ℓ-almost all t we conclude that

$$\begin{aligned}
\mathrm{Ent}_\Phi(F) &= \mathsf{E}[\Phi(M_1) - \Phi(M_0)] \\
&= \mathsf{E}\Big[\int_0^1 \int_{\mathsf{X}} \big(\Phi(M_{s-} + d(s,x)) - \Phi(M_{s-}) - \Phi'(M_{s-}) d(s,x) \big)\, \lambda(\mathrm{d}(s,x)) \Big] \\
&= \mathsf{E}\Big[\int_0^1 \int_{\mathsf{X}} \Psi(M_s, d(s,x))\, \lambda(\mathrm{d}(s,x)) \Big],
\end{aligned}$$

where

$$\Psi(x,y) := \Phi(x+y) - \Phi(x) - y\Phi'(x), \qquad x, y \in \mathsf{R}_+.$$

It is known from [7] that for $\Phi \in \mathsf{C}$ the function Ψ is convex on $\{(x,y) \in \mathsf{R}_+ \times \mathsf{R}_+ : x+y>0\}$. Thus, using Jensen's inequality we see that P-almost surely and for λ-almost all $(s,x) \in \mathsf{Y}$,

$$\Psi(M_s, d(s,x)) = \Psi(\mathsf{E}[F|\mathcal{F}_s], \mathsf{E}[D_{(s,x)}F|\mathcal{F}_s]) \le \mathsf{E}[\Psi(F, D_{(s,x)}F)|\mathcal{F}_s].$$

As a consequence, we conclude that

$$\begin{aligned}\mathrm{Ent}_\Phi(F) &\le \mathsf{E}\Big[\int_0^1 \int_{\mathsf{X}} \big(\Phi(F + D_{(s,x)}F) - \Phi(F) - \Phi'(F) D_{(s,x)}F\big)\,\lambda(\mathrm{d}(s,x))\Big]\\ &= \mathsf{E}\Big[\int_0^1 \int_{\mathsf{X}} \big(D_{(s,x)}\Phi(F) - \Phi'(F) D_{(s,x)}F\big)\,\lambda(\mathrm{d}(s,x))\Big].\end{aligned}$$

This proves the inequality for time augmented Poisson functionals and thus completes the proof. □

Remark 3.2. If the derivative $\Phi'(0)$ of Φ at 0 is well defined, we can relax the positivity assumption on F in the previous proposition by requiring only that $F \ge 0$ P-almost surely. We will take advantage of this fact from now on whenever we are dealing with Φ_r.

Corollary 3.3. *Fix $r \in (1,2)$ and let $F \in L^1(\mathsf{P}_\eta)$ be a Poisson functional satisfying $F \ge 0$ P-almost surely. Then*

$$\mathrm{Ent}_r(F) := \mathrm{Ent}_{\Phi_r}(F) \le \mathsf{E}\Big[\int_{\mathsf{X}} \big(D_x F^{\frac{2}{r}} - \frac{2}{r} F^{\frac{2}{r}-1}\,(D_x F)\big)\,\mu(\mathrm{d}x)\Big].$$

Proof. This is a direct consequence of Proposition 3.1 with $\Phi = \Phi_r$, since $\Phi'(x) = \frac{2}{r} x^{\frac{2}{r}-1}$. □

Note that if we let $r \to 1$, then $\mathrm{Ent}_r(F) \to \mathsf{V}(F)$, the variance of F. Using that for two Poisson functionals F, G and for all $x \in \mathsf{X}$ one has the product rule

$$D_x(FG) = (D_xF)G + F(D_xG) + (D_xF)(D_xG)$$

(compare with [13, Exercise 18.2]), it follows that, for $r = 1$,

$$D_x F^{\frac{2}{r}} - \frac{2}{r} F^{\frac{2}{r}-1}\,(D_xF) = D_xF^2 - 2F\,D_xF = 2F(D_xF) + (D_xF)^2 - 2F(D_xF) = (D_xF)^2.$$

Thus, as $r \to 1$, the modified Φ_r-Sobolev inequality turns into the L^1-version of the **Poincaré inequality** for Poisson functionals:

$$\mathsf{V}(F) \le \mathsf{E}\Big[\int_{\mathsf{X}} (D_xF)^2\,\mu(\mathrm{d}x)\Big], \tag{3.1}$$

see [11, Proposition 2.5] and [13, Theorem 18.7]. On the other hand, as $r \to 2$, we have that, for $x \in \mathsf{R}_+$,

$$\begin{aligned} x^{\frac{2}{r}} &= x - \Big(\frac{r}{2} - 1\Big) x \log x + O((r-2)^2),\\ -\frac{2}{r} x^{\frac{2}{r}-1} &= -1 + \Big(\frac{r}{2} - 1\Big)(\log x + 1) + O((r-2)^2), \end{aligned} \tag{3.2}$$

which implies that P-almost surely

$$\begin{aligned} &D_xF^{\frac{2}{r}} - \frac{2}{r}F^{\frac{2}{r}-1}\,(D_xF)\\ &= D_x\Big(F - \Big(\frac{r}{2} - 1\Big)F\log F\Big) - D_xF + \Big(\frac{r}{2} - 1\Big)(D_xF)(\log F + 1) + O((r-2)^2)\\ &= \Big(1 - \frac{r}{2}\Big)\Big(D_x(F\log F) - (\log F + 1)(D_xF) + O(r-2)\Big). \end{aligned}$$

As a consequence, we conclude that on the one hand side, as $r \to 2$,

$$\Big(1-\frac{r}{2}\Big)^{-1}\mathsf{E}\Big[\int_{\mathsf{X}}\big(D_xF^{\frac{2}{r}}-\frac{2}{r}F^{\frac{2}{r}-1}\,(D_xF)\big)\,\mu(\mathrm{d}x)\Big]$$
$$\to \mathsf{E}\Big[\int_{\mathsf{X}}\big(D_x(F\log F)-(\log F+1)(D_xF)\big)\,\mu(\mathrm{d}x)\Big].$$

On the other hand, using (3.2) again we see that, as $r \to 2$,

$$\Big(1-\frac{r}{2}\Big)^{-1}\mathrm{Ent}_r(F)=\Big(1-\frac{r}{2}\Big)^{-1}\Big(\mathsf{E}[F^{\frac{2}{r}}]-(\mathsf{E}[F])^{\frac{2}{r}}\Big)$$
$$=\mathsf{E}[F\log F]-\mathsf{E}[F]\log(\mathsf{E}[F])+O(r-2),$$

which implies

$$\Big(1-\frac{r}{2}\Big)^{-1}\mathrm{Ent}_r(F)\to\mathrm{Ent}(F),$$

as $r \to 2$, where $\mathrm{Ent}(F)=\mathrm{Ent}_{\Phi_{\log}}$ is the classical entropy of F. Summarizing, we conclude that, as $r \to 2$, the modified Φ_r-Sobolev inequality turns into Wu's **modified log-Sobolev inequality**

$$\begin{aligned}\mathrm{Ent}(F)&\le\mathsf{E}\Big[\int_{\mathsf{X}}\big(D_x(F\log F)-(\log F+1)(D_xF)\big)\,\mu(\mathrm{d}x)\Big]\\&=\mathsf{E}\Big[\int_{\mathsf{X}}\big(D_x\Phi_{\log}(F)-\Phi'_{\log}(F)(D_xF)\big)\,\mu(\mathrm{d}x)\Big],\end{aligned}\tag{3.3}$$

see [21, Theorem 1.1]. Against this light one can say that family of modified Φ_r-Sobolev inequalities for Poisson functionals from Corollary 3.3 interpolates between two classical inequalities for Poisson functionals, namely the Poincaré inequality ($r \to 1$, see (3.1)) and the modified log-Sobolev inequality ($r \to 2$, see (3.3)) for Poisson functionals.

Finally, let us note that the modified Φ_r-Sobolev inequality established in Corollary 3.3 implies the **Beckner-type inequality** Bec-$(2/r)$ used in [1, Section 4.7].

Corollary 3.4 (Beckner-type inequality). *Let $F \in L^1(\mathsf{P}_\eta)$ satisfy $F \ge 0$ P-almost surely and fix $r \in (1,2)$. Then F satisfies the Beckner-type inequality Bec-$(2/r)$, meaning that*

$$\mathrm{Ent}_r(F)\le\frac{6}{r}\mathsf{E}\Big[\int_{\mathsf{X}}(D_xF)\,(D_xF^{\frac{2}{r}-1})\,\mu(\mathrm{d}x)\Big].$$

Proof. This immediately follows from Corollary 3.3 once we have checked that

$$\mathsf{E}\Big[\int_{\mathsf{X}}\big(D_xF^{\frac{2}{r}}-\frac{2}{r}F^{\frac{2}{r}-1}\,(D_xF)\big)\,\mu(\mathrm{d}x)\Big]\le\frac{6}{r}\mathsf{E}\Big[\int_{\mathsf{X}}(D_xF)\,(D_xF^{\frac{2}{r}-1})\,\mu(\mathrm{d}x)\Big].$$

Setting $p := 2/r$, the latter follows from the pointwise inequality

$$a^p-b^p-pb^{p-1}(a-b)\le 3p(a^p+b^p-ab^{p-1}-ba^{p-1})$$

for any $a,b \ge 0$. Indeed, fixing $b \ge 0$ and rearranging, this inequality is equivalent to $\psi(a) \ge 0$ for $a \ge 0$, where

$$\psi(a):=(3p-1)a^p+(2p+1)b^p-2pab^{p-1}-3pba^{p-1}.$$

Now, for $a \ge 0$, one has that

$$\psi'(a)=(3p-1)pa^{p-1}-2pb^{p-1}-3p(p-1)ba^{p-2},$$

which has a unique zero at $a=b$ with $\psi(b)=0$. From this fact the claim follows. □

4 L^p-estimates for Poisson functionals

This section is devoted to L^p-estimates for functionals F of Poisson processes on a measurable space X. For this we use the modified Φ-Sobolev inequality for Poisson functionals developed in the previous section. As above we assume that η is a Poisson process on X and denote by μ its σ-finite intensity measure. For $x \in \mathsf{R}$ we shall write $x_+ := \max\{0, x\}$ and $x_- := \min\{0, x\}$ and for $p > 0$ and a random variable F we put $\|F\|_p := (\mathsf{E}[|F|^p])^{1/p}$.

Proposition 4.1 (Recursive L^p-estimate). *Let $p \geq 2$ and $F \in L^1(\mathsf{P}_\eta)$ be a Poisson functional. Then*

$$\|(F - \mathsf{E}[F])_+\|_p^p \leq \|(F - \mathsf{E}[F])_+\|_{p-1}^p + (p-1)\|V^+\|_{p/2}\|(F - \mathsf{E}[F])_+\|_p^{p-2}, \tag{4.1}$$

where

$$V^+ := \int_{\mathsf{X}} (D_x F)_-^2 \, \mu(\mathrm{d}x) + \int_{\mathsf{X}} (F(\eta) - F(\eta - \delta_x))_+^2 \, \eta(\mathrm{d}x).$$

Proof. If $(F - \mathsf{E}[F])_+ \notin L^{p-1}(\mathsf{P}_\eta)$ there is nothing to prove. Hence, from now on we assume that $(F - \mathsf{E}[F])_+ \in L^{p-1}(\mathsf{P}_\eta)$. So, given the functional F with representative f consider another Poisson functional G with representative g defined as

$$g(\eta) := (f(\eta) - \mathsf{E}[F])_+^{p-1}.$$

It is clear that $G \geq 0$ P-almost surely. Thus, applying Corollary 3.3 to the Poisson functional G and with $r = 2 - \frac{2}{p}$ we obtain

$$\begin{aligned} \mathrm{Ent}_{2-2/p}(G) &= \mathsf{E}[(F - \mathsf{E}[F])_+^p] - (\mathsf{E}[(F - \mathsf{E}[F])_+^{p-1}])^{\frac{p}{p-1}} \\ &= \|(F - \mathsf{E}[F])_+\|_p^p - \|(F - \mathsf{E}[F])_+\|_{p-1}^p \\ &\leq \mathsf{E}\Big[\int_{\mathsf{X}} \big(D_x(F - \mathsf{E}[F])_+^p - \frac{p}{p-1}(F - \mathsf{E}[F])_+ D_x(F - \mathsf{E}[F])_+^{p-1}\big) \, \mu(\mathrm{d}x)\Big]. \end{aligned} \tag{4.2}$$

Using the definition of difference operator D_x we get

$$\begin{aligned} I(\eta, x) :&= D_x(F - \mathsf{E}[F])_+^p - \frac{p}{p-1}(F - \mathsf{E}[F])_+ D_x(F - \mathsf{E}[F])_+^{p-1} \\ &= (f(\eta + \delta_x) - \mathsf{E}[F])_+^p - (f(\eta) - \mathsf{E}[F])_+^p - \frac{p}{p-1}(f(\eta) - \mathsf{E}[F])_+(f(\eta + \delta_x) - \mathsf{E}[F])_+^{p-1} \\ &\quad + \frac{p}{p-1}(f(\eta) - \mathsf{E}[F])_+^p \\ &= \frac{1}{p-1}(f(\eta) - \mathsf{E}[F])_+\Big[(f(\eta) - \mathsf{E}[F])_+^{p-1} - (f(\eta + \delta_x) - \mathsf{E}[F])_+^{p-1}\Big] \\ &\quad + (f(\eta + \delta_x) - \mathsf{E}[F])_+^{p-1}\Big[(f(\eta + \delta_x) - \mathsf{E}[F])_+ - (f(\eta) - \mathsf{E}[F])_+\Big]. \end{aligned} \tag{4.3}$$

By the mean value theorem for the function x^{p-1} on the interval $J := \Big[\min((f(\eta) - \mathsf{E}[F])_+, (f(\eta + \delta_x) - \mathsf{E}[F])_+), \max((f(\eta) - \mathsf{E}[F])_+, (f(\eta + \delta_x) - \mathsf{E}[F])_+)\Big]$ we have

$$\begin{aligned} &\frac{1}{p-1}\Big[(f(\eta) - \mathsf{E}[F])_+^{p-1} - (f(\eta + \delta_x) - \mathsf{E}[F])_+^{p-1}\Big] \\ &\qquad = -\Big[(f(\eta + \delta_x) - \mathsf{E}[F])_+ - (f(\eta) - \mathsf{E}[F])_+\Big]c^{p-2} \end{aligned}$$

for some $c \in J$. Now consider two cases. If $(f(\eta + \delta_x) - \mathsf{E}[F])_+ - (f(\eta) - \mathsf{E}[F])_+ \geq 0$, then $(f(\eta + \delta_x) - \mathsf{E}[F])_+ \geq (f(\eta) - \mathsf{E}[F])_+$ and we obtain

$$\begin{aligned} &\frac{1}{p-1}\Big[(f(\eta) - \mathsf{E}[F])_+^{p-1} - (f(\eta + \delta_x) - \mathsf{E}[F])_+^{p-1}\Big] \\ &\qquad \leq -\Big[(f(\eta + \delta_x) - \mathsf{E}[F])_+ - (f(\eta) - \mathsf{E}[F])_+\Big](f(\eta) - \mathsf{E}[F])_+^{p-2}. \end{aligned}$$

On the other hand, if $(f(\eta+\delta_x)-\mathsf{E}[F])_+-(f(\eta)-\mathsf{E}[F])_+ \le 0$, then $(f(\eta+\delta_x)-\mathsf{E}[F])_+ \le (f(\eta)-\mathsf{E}[F])_+$ and we again obtain

$$\begin{aligned}\frac{1}{p-1}&\Big[(f(\eta)-\mathsf{E}[F])_+^{p-1}-(f(\eta+\delta_x)-\mathsf{E}[F])_+^{p-1}\Big]\\&\le -\Big[(f(\eta+\delta_x)-\mathsf{E}[F])_+-(f(\eta)-\mathsf{E}[F])_+\Big](f(\eta)-\mathsf{E}[F])_+^{p-2}.\end{aligned}$$

Substituting this into (4.3) and using the mean value theorem one more time we arrive at

$$\begin{aligned}I(\eta,x) &\le \Big((f(\eta+\delta_x)-\mathsf{E}[F])_+-(f(\eta)-\mathsf{E}[F])_+\Big)\Big((f(\eta+\delta_x)-\mathsf{E}[F])_+^{p-1}-(f(\eta)-\mathsf{E}[F])_+^{p-1}\Big)\\&\le (p-1)\Big[(f(\eta+\delta_x)-\mathsf{E}[F])_+-(f(\eta)-\mathsf{E}[F])_+\Big]^2 c^{p-2},\end{aligned}$$

where $c \in J$. Let us estimate the difference $\Big[(f(\eta+\delta_x)-\mathsf{E}[F])_+-(f(\eta)-\mathsf{E}[F])_+\Big]^2$. We again consider a number of cases.

1. If $f(\eta+\delta_x), f(\eta) \ge \mathsf{E}[F]$, then
$$\Big[(f(\eta+\delta_x)-\mathsf{E}[F])_+-(f(\eta)-\mathsf{E}[F])_+\Big]^2 = (D_xF)^2.$$
2. If $f(\eta+\delta_x) \ge \mathsf{E}[F] \ge f(\eta)$, then
$$\Big[(f(\eta+\delta_x)-\mathsf{E}[F])_+-(f(\eta)-\mathsf{E}[F])_+\Big]^2 = \Big[f(\eta+\delta_x)-\mathsf{E}[F]\Big]^2 \le (D_xF)^2.$$
3. If $f(\eta) \ge \mathsf{E}[F] \ge f(\eta+\delta_x)$, then
$$\Big[(f(\eta+\delta_x)-\mathsf{E}[F])_+-(f(\eta)-\mathsf{E}[F])_+\Big]^2 = \Big[f(\eta)-\mathsf{E}[F]\Big]^2 \le (D_xF)^2.$$
4. If $\mathsf{E}[F] \ge f(\eta+\delta_x), f(\eta)$, then
$$\Big[(f(\eta+\delta_x)-\mathsf{E}[F])_+-(f(\eta)-\mathsf{E}[F])_+\Big]^2 = 0 \le (D_xF)^2.$$

Thus

$$\begin{aligned}I(\eta,x) &\le (p-1)(D_xF)^2c^{p-2}\\&\le (p-1)(D_xF)_+^2(f(\eta+\delta_x)-\mathsf{E}[F])_+^{p-2}+(p-1)(D_xF)_-^2(f(\eta)-\mathsf{E}[F])_+^{p-2},\end{aligned}$$

and, by Fubini's theorem, we obtain

$$\begin{aligned}\mathrm{Ent}_{2-2/p}(G) &\le (p-1)\mathsf{E}\Big[\int_{\mathsf{X}}\Big((D_xF)_+^2(f(\eta+\delta_x)-\mathsf{E}[F])_+^{p-2}+(D_xF)_-^2(f(\eta)-\mathsf{E}[F])_+^{p-2}\Big)\,\mu(\mathrm{d}x)\Big]\\&= (p-1)\Big[\int_{\mathsf{X}}\mathsf{E}\Big((D_xF)_+^2(f(\eta+\delta_x)-\mathsf{E}[F])_+^{p-2}\Big)\,\mu(\mathrm{d}x)\\&\qquad+\mathsf{E}\Big((f(\eta)-\mathsf{E}[F])_+^{p-2}\int_{\mathsf{X}}(D_xF)_-^2\,\mu(\mathrm{d}x)\Big)\Big].\end{aligned}$$

In order to transform the first summand we use the Mecke formula for Poisson processes [13, Theorem 4.1] and by the definition of the difference operator we get

$$\begin{aligned}&\int_{\mathsf{X}}\mathsf{E}\Big((F(\eta+\delta_x)-F(\eta))_+^2(f(\eta+\delta_x)-\mathsf{E}[F])_+^{p-2}\Big)\,\mu(\mathrm{d}x)\\&\qquad= \mathsf{E}\int_{\mathsf{X}}\Big((F(\eta)-F(\eta-\delta_x))_+^2(f(\eta)-\mathsf{E}[F])_+^{p-2}\Big)\,\eta(\mathrm{d}x).\end{aligned}$$

This leads to the inequality

$$\begin{aligned}\mathrm{Ent}_{2-2/p}(G) &\le (p-1)\mathsf{E}\Big[(f(\eta)-\mathsf{E}[F])_+^{p-2}\Big(\int_{\mathsf{X}}\Big((F(\eta)-F(\eta-\delta_x))_+^2\,\eta(\mathrm{d}x)+\int_{\mathsf{X}}(D_xF)_-^2\,\mu(\mathrm{d}x)\Big)\Big]\\ &= (p-1)\mathsf{E}\Big[(F-\mathsf{E}[F])_+^{p-2}V^+\Big].\end{aligned}$$

In the last step we apply Hölder's inequality with $\frac{p}{p-2}$ and $\frac{p}{2}$ to obtain

$$\begin{aligned}\mathrm{Ent}_{2-2/p}(G) &= \|(F-\mathsf{E}[F])_+\|_p^p - \|(F-\mathsf{E}[F])_+\|_{p-1}^p\\ &\le (p-1)\|(F-\mathsf{E}[F])_+\|_p^{p-2}\|V^+\|_{p/2}.\end{aligned}$$

Together with (4.2) this completes the proof. □

5 Moment and concentration inequalities for Poisson functionals

The goal of this section is to derive moment and concentration inequalities for Poisson functionals $F = f(\eta)$, where η is a Poisson process with σ-finite intensity measure μ over some measurable space X. In our arguments we essentially follow [6]. In particular, let us introduce the constants

$$\kappa_p := \frac{1}{2}\Big(1-\Big(1-\frac{1}{p}\Big)^{p/2}\Big)^{-1}$$

for $p > 1$. It is not hard to check that κ_p is strictly increasing in p and that

$$\lim_{p\to 1}\kappa_p = \frac{1}{2}, \qquad \lim_{p\to\infty}\kappa_p = \frac{\sqrt{e}}{2(\sqrt{e}-1)} =: \kappa.$$

Furthermore, besides V^+ defined in Proposition 4.1 we shall need the quantities

$$\begin{aligned}V^- &:= \int_{\mathsf{X}}(D_xF)_+^2\,\mu(\mathrm{d}x) + \int_{\mathsf{X}}(F(\eta)-F(\eta-\delta_x))_-^2\,\eta(\mathrm{d}x),\\ V &:= \int_{\mathsf{X}}(D_xF)^2\,\mu(\mathrm{d}x) + \int_{\mathsf{X}}(F(\eta)-F(\eta-\delta_x))^2\,\eta(\mathrm{d}x).\end{aligned}$$

Let us also recall that $\|F\|_p = (\mathsf{E}[|F|^p])^{1/p}$ for $p \ge 1$.

Theorem 5.1 (Moment bounds). *Let $p \ge 2$ and $F \in L^1(\mathsf{P}_\eta)$. Then*

$$\|(F-\mathsf{E}[F])_+\|_p \le \sqrt{2\kappa p\|V^+\|_{p/2}} = \sqrt{2\kappa p}\,\|\sqrt{V^+}\|_p, \tag{5.1}$$

$$\|(F-\mathsf{E}[F])_-\|_p \le \sqrt{2\kappa p\|V^-\|_{p/2}} = \sqrt{2\kappa p}\,\|\sqrt{V^-}\|_p, \tag{5.2}$$

$$\|F-\mathsf{E}[F]\|_p \le \sqrt{8\kappa p\|V\|_{p/2}} = \sqrt{8\kappa p}\,\|\sqrt{V}\|_p. \tag{5.3}$$

Proof. To see (5.1), we prove the slightly sharper estimate

$$\|(F-\mathsf{E}[F])_+\|_p \le \sqrt{\Big(1-\frac{1}{p}\Big)2\kappa_p p\|V^+\|_{p/2}}$$

for any $p \ge 2$. To this end, setting $c_p := 2\|V^+\|_{p/2\vee 1}(1-1/(p\vee 2))$, we show by induction on k that for all $k \in \mathsf{N}$ and all $p \in (k, k+1]$,

$$\|(F-\mathsf{E}[F])_+\|_p \le \sqrt{p\kappa_{p\vee 2}c_p}. \tag{5.4}$$

To start, take $k = 1$, $F \in L^1(\mathsf{P}_\eta)$ and apply Hölder's inequality and the L^1-version of the Poincaré inequality for Poisson point processes (see (3.1) above). Together with the Mecke formula for Poisson processes [13, Theorem 4.1], this yields

$$\|(F - \mathsf{E}[F])_+\|_p \leq \|F - \mathsf{E}[F]\|_2 \leq \sqrt{\mathsf{E}\Big[\int_{\mathsf{X}} (D_x F)^2\,\mu(\mathrm{d}x)\Big]} = \sqrt{\|V^+\|_1} \leq \sqrt{p\kappa_{p\vee 2} c_p}.$$

In the induction step, we assume that $F \in L^{p-1}(\mathsf{P}_\eta)$ and that (5.4) holds for any integers smaller than some $k > 1$. Writing

$$x_p := \|(F - \mathsf{E}[F])_+\|_p^p (p\kappa_{p\vee 2} c_p)^{-p/2},$$

(5.4) is equivalent to $x_p \leq 1$ for any $p \in (k, k+1]$. Moreover, (4.1) can be rewritten as

$$x_p p^{p/2} c_p^{p/2} \kappa_p^{p/2} \leq x_{p-1}^{p/(p-1)} (p-1)^{p/2} c_{p-1}^{p/2} \kappa_{(p-1)\vee 2}^{p/2} + \frac{1}{2} x_p^{1-2/p} p^{p/2} c_p^{p/2} \kappa_p^{p/2-1}.$$

Using $c_{p-1} \leq c_p$ and $\kappa_{(p-1)\vee 2} \leq \kappa_p$, this reduces to

$$x_p \leq x_{p-1}^{p/(p-1)} \Big(1 - \frac{1}{p}\Big)^{p/2} + \frac{1}{2\kappa_p} x_p^{1-2/p}.$$

By induction, $x_{p-1} \leq 1$, and hence it follows that

$$x_p \leq \Big(1 - \frac{1}{p}\Big)^{p/2} + \frac{1}{2\kappa_p} x_p^{1-2/p}.$$

Noting that the function

$$f_p(x) := \Big(1 - \frac{1}{p}\Big)^{p/2} + \frac{1}{2\kappa_p} x^{1-2/p} - x$$

is strictly concave on R_+, positive at $x = 1$, $f_p(1) = 0$ and $f_p(x_p) \geq 0$, we obtain that $x_p \leq 1$, which completes the proof of (5.1).
The result in (5.2) simply follows by considering $-F$ instead of F in (5.1). Now (5.3) is easily seen by triangle inequality, combining (5.1) and (5.2) and using that $0 \leq V^+, V^- \leq V$:

$$\|F - \mathsf{E}[F]\|_p \leq \|(F - \mathsf{E}[F])_+\|_p + \|(F - \mathsf{E}[F])_-\|_p \leq 2\sqrt{2\kappa p \|V\|_{p/2}}.$$

This completes the proof. □

Remark 5.2. Theorem 5.1 yields the same L^p-bounds as [1, Proposition 4.17] up to constants, where $\sqrt{2\kappa}$ in (5.1) and $\sqrt{8\kappa}$ in (5.3) are both replaced by $D_{4.17} = \sqrt{6\kappa}$.

In particular, from the previous proposition we conclude the following concentration bounds for Poisson functionals, which recover Proposition 1.2, Corollary 3.3 (ii) and Corollary 3.4 (ii) from [2] (up to absolute constants).

Corollary 5.3 (Concentration bounds). *Let $F \in L^1(\mathsf{P}_\eta)$. If P-almost surely $V^+ \leq L$ or $V^- \leq L$ for some $L > 0$, we have that*

$$\mathsf{P}(F - \mathsf{E}[F] \geq t) \leq e^{-ct^2/L} \qquad \text{or} \qquad \mathsf{P}(F - \mathsf{E}[F] \leq -t) \leq e^{-ct^2/L} \tag{5.5}$$

for every $t \geq 4\sqrt{\kappa}$ and $c = \log(2)/(8\kappa)$, respectively. Moreover, if P-almost surely $V \leq L$ for some $L > 0$, we have that

$$\mathsf{P}(|F - \mathsf{E}[F]| \geq t) \leq 2e^{-ct^2/L} \tag{5.6}$$

for every $t \geq 0$ and $c = \log(2)/(16\kappa)$.

Proof. This follows from standard arguments. We include the proof to demonstrate how the constants we obtain come up. For simplicity and without loss of generality, let us assume that $L = 1$. First note that by Markov's inequality and Theorem 5.1,

$$\mathsf{P}(F - \mathsf{E}[F] \geq t) \leq \inf_{p \geq 2} \frac{\mathsf{E}[(F - \mathsf{E}[F])_+^p]}{t^p} \leq \inf_{p \geq 2} \Big(\frac{\sqrt{2\kappa p}}{t}\Big)^p.$$

Here, $\sqrt{2\kappa p}/t \leq 1/2$ iff $p \leq t^2/(8\kappa)$. Therefore, if $t^2/(8\kappa) \geq 2$, plugging in leads to

$$\mathsf{P}(F - \mathsf{E}[F] \geq t) \leq 2^{-t^2/(8\kappa)} = \exp\Big(-\frac{\log(2)}{8\kappa}t^2\Big).$$

By the same arguments we may prove the bound on $\mathsf{P}(F - \mathsf{E}[F] \leq -t)$ and consequently that

$$\mathsf{P}(|F - \mathsf{E}[F]| \geq t) \leq 4\exp\Big(-\frac{\log(2)}{8\kappa}t^2\Big)$$

for every $t \geq 0$, where we have chosen the factor 4 (instead of 2) to extend the bound trivially to the range $t^2/(8\kappa) < 2$. This may be adjusted to the bound stated above by recalling that for any two constants $\gamma_1 > \gamma_2 > 1$, we have that for all $r \geq 0$ and $\gamma > 0$

$$\gamma_1 \exp(-\gamma r) \leq \gamma_2 \exp\Big(-\frac{\log(\gamma_2)}{\log(\gamma_1)}\gamma r\Big)$$

whenever the left hand side is smaller or equal to 1. □

Remark 5.4. As demonstrated in the proof of (5.6) it is possible to extend the range of (5.5) to $t \geq 0$ by adding a prefactor 2 to the right hand sides of both inequalities and choosing $c = \log(2)/(16\kappa)$. We decided to keep the restricted range of possible values for t here and in what follows.

Remark 5.5 (Boundedness of F vs. boundedness of V^+)**.** Let us discuss the assumptions from Corollary 5.3 in more detail. Here we focus on the condition $V^+ \leq L$ and consider the simplest possible situation $\mathsf{X} = \{x\}$, i.e., η is a Poisson random variable with mean $\lambda > 0$. Clearly,

$$V^+ = \lambda(F(\eta + 1) - F(\eta))_-^2 + \eta(F(\eta) - F(\eta - 1))_+^2,$$

and if we moreover assume F to be non-decreasing, it therefore suffices to examine the quantities

$$V^+(k) = k(F(k) - F(k-1))^2, \qquad k \in \mathsf{N}.$$

First, the condition $V^+ \leq L$ fails for the function $F(k) = k$, which is a priori clear since it would yield sub-Gaussian tails for a Poisson variable. Formally, $V^+(k) = k$, i.e., V^+ is unbounded. In this situation, the condition $V^+ \leq L$ guarantees a suitable "deformation" of the Poisson random variable under consideration which in some sense "flattens" its tails. One may wonder whether boundedness of V^+ and boundedness of F coincide. It turns out that neither implication is true.
To see this, first consider the function $F(k) := \sum_{j=1}^k \frac{1}{j}$. Clearly, F is unbounded. However, it is easy to check that $V^+(k) = 1/k$ for any k, i.e., V^+ is bounded. On the other hand, consider the function

$$F(k) = \sum_{j=1}^{\lfloor k^{1/5} \rfloor} \frac{1}{\lfloor j^{1/5} \rfloor^2}.$$

By definition, $F(k)$ only changes its value if $k = m^5$ for some natural number m and in this case, the value $1/m^2$ is added. In particular, F is bounded. However, if $k = m^5$, we have $V^+(m^5) = m$, so that V^+ is unbounded.

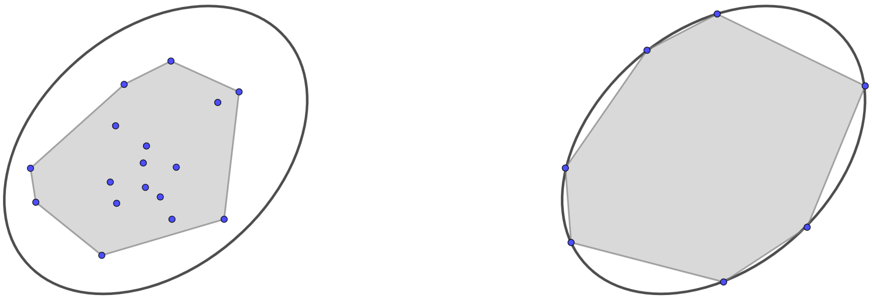

Figure 1: Illustration of the random polytope models (In) (left panel) and (Bd) (right panel) for $d = 2$ and with K being an ellipse.

Remark 5.6 (Self-bounding Poisson functionals). As demonstrated in [1, Proposition 4.18], Theorem 5.1 directly implies L^p-estimates and concentration inequalities for self-bounding Poisson functionals, where the latter also correspond to [2, Corollary 3.6]. In fact, if $F \in L^1(\mathsf{P}_\eta)$ and P-almost surely $V^+ \le cF^\alpha$ for some $\alpha \in [0, 2)$ and $c \in [0, \infty)$, then $\|(F - \mathsf{E}[F])_+\|_p \le 2\sqrt{2c\kappa p}\,(\mathsf{E}[F])^{\frac{\alpha}{2}} + (8c\kappa p)^{\frac{1}{2-\alpha}}$ for any $p \ge 2$ and

$$\mathsf{P}(F \ge \mathsf{E}[F] + t) \le 2\exp\Big\{ -C\min\Big(\frac{t^2}{c\cdot(\mathsf{E}[F])^\alpha}, \frac{t^{2-\alpha}}{c}\Big)\Big\}$$

for every $t \ge 0$, where $C \in (0, \infty)$ is an absolute numerical constant.

6 Applications to stochastic geometry models

Our goal in this section is to demonstrate how the general results obtained in the previous section can be applied to concrete problems arising in stochastic geometry. We will discuss two models, namely Poisson polytopes and Poisson cylinders and although concentration inequalities for a number of geometric functionals of these two models are already known in the literature, some of the concentration bounds we obtain are new or better than the already existing ones. For both applications we will discuss the relation between our results and those known from the literature.

6.1 Poisson polytopes

Our first application is concerned with convex hulls of Poisson processes in R^d, $d \ge 1$. To describe the set-up, in comparison to the existing literature we take a rather general point of view and let μ be a probability measure on R^d with the property that $\mu(H) = 0$ for each hyperplane $H \subset \mathsf{R}^d$. Now, consider a Poisson process in R^d with intensity measure $\gamma\mu$, where $\gamma \in (0, \infty)$ is some fixed intensity parameter. We are interested in concentration properties of the Poisson functional

$$F := \mu(\mathrm{conv}(\eta)), \tag{6.1}$$

where $\mathrm{conv}(\eta)$ stands for the convex hull of the support of η. Note that under our assumption on μ, $\mathrm{conv}(\eta)$ is P-almost surely a simplicial polytope in R^d, meaning that each of its faces is a simplex.

Proposition 6.1 (Concentration for the μ-content). *For F as defined by (6.1) one has that*

$$\mathsf{P}(F \ge \mathsf{E}[F] + t) \le e^{-ct^2/d} \qquad \text{and} \qquad \mathsf{P}(F \le \mathsf{E}[F] - t) \le e^{-ct^2/\gamma}$$

for every $t \ge 4\sqrt{\kappa}$, where $c = \log(2)/(8\kappa)$.

Proof. We start by investigating V^+, which in our case is given by

$$V^+ = \gamma \int_{\mathsf{R}^d} (D_xF)_-^2 \,\mu(\mathrm{d}x) + \int_{\mathsf{R}^d} (F(\eta) - F(\eta - \delta_x))_+^2 \,\eta(\mathrm{d}x) = \int_{\mathsf{R}^d} (F(\eta) - F(\eta - \delta_x))_+^2 \,\eta(\mathrm{d}x),$$

since P-almost surely $D_xF \geq 0$ for μ-almost all $x \in \mathsf{R}^d$. For $x \in \eta$ let $N(x)$ be the set of vertices of $\mathrm{conv}(\eta)$ which are connected to x by an edge. Clearly, $N(x) = ?$ iff x is not a vertex of $\mathrm{conv}(\eta)$. We now observe that P-almost surely $F(\eta) - F(\eta - \delta_x) \leq \mu(\mathrm{conv}(x \cup N(x)))$ for any $x \in \eta$, which implies that

$$V^+ \leq \int_{\mathsf{R}^d} \mu(\mathrm{conv}(x \cup N(x)))^2 \,\eta(\mathrm{d}x) \leq \int_{\mathsf{R}^d} \mu(\mathrm{conv}(x \cup N(x))) \,\eta(\mathrm{d}x),$$

because $\mu(\mathrm{conv}(x \cup N(x))) \leq 1$. Since $\mu(H) = 0$ for each hyperplane $H \subset \mathsf{R}^d$, $N(x)$ P-almost surely contains precisely d elements for any vertex of $\mathrm{conv}(\eta)$ (recall that indeed $\mathrm{conv}(\eta)$ is P-almost surely a simplicial polytope). This means that every point of $\mu(\mathrm{conv}(\eta))$ can be covered by at most d different simplices of the form $\mathrm{conv}(x \cup N(x))$ and, hence, the last integral is at most d times $\mu(\mathrm{conv}(\eta))$, i.e.,

$$V^+ \leq \int_{\mathsf{R}^d} \mu(\mathrm{conv}(x \cup N(x))) \,\eta(\mathrm{d}x) \leq d\mu(\mathrm{conv}(\eta)) \leq d.$$

Thus, the result follows from Corollary 5.3 by choosing $L = d$ there.
In the same manner we can consider V^- and since P-almost surely $D_xF \geq 0$ for μ-almost all $x \in \mathsf{R}^d$ we have

$$V^- = \gamma \int_{\mathsf{R}^d} (D_xF)_+^2 \,\mu(\mathrm{d}x).$$

Since μ is a probability measure on R^d we have

$$D_xF = \mu(\mathrm{conv}(\eta + \delta_x)) - \mu(\mathrm{conv}(\eta)) \leq 1$$

and, thus,

$$V^- \leq \gamma.$$

Applying Corollary 5.3 with $L = \gamma$ we obtain a lower tail. □

Let us consider two specific cases, well known from the existing literature, to which Proposition 6.1 can be applied, see Figure 1 for illustrations.

(In) Let $K \subset \mathsf{R}^d$ be a convex body with volume $\ell_d(K) \in (0, \infty)$ and choose for μ the restriction of the d-dimensional Lebesgue measure to K, normalized by $\ell_d(K)^{-1}$. In this case sub-Gaussian concentration bounds for F are known from [20], but they include a polynomial error term, see also [15]; for K being the d-dimensional unit ball, exponential concentration inequalities without polynomial error terms are the content of [10]. We remark that for the random polytope model (In) one has that

$$\ell_d(K) - \mathsf{E}[\ell_d(\mathrm{conv}(\eta))] = c_{d,K}\gamma^{-2/(d+1)}(1 + o(1)), \qquad \text{as} \qquad \gamma \to \infty,$$

if K has a C^2-smooth boundary with everywhere strictly positive Gaussian curvature and that

$$\ell_d(K) - \mathsf{E}[\ell_d(\mathrm{conv}(\eta))] = c'_{d,K}\gamma^{-1}(\log \gamma)^{d-1}(1 + o(1)), \qquad \text{as} \qquad \gamma \to \infty,$$

if K is a polytope. Here, $c_{d,K}$ and $c'_{d,K}$ are explicitly known constants depending on d and on K, see [15].

(Bd) Let $K \subset \mathsf{R}^d$ be a convex body with volume $\ell_d(K) \in (0,\infty)$ whose boundary is C^2-smooth with everywhere strictly positive Gaussian curvature and choose for μ the restriction of the $(d-1)$-dimensional Hausdorff measure $\mathcal{H}^{d-1}$ to the boundary ∂K of K, normalized by $\mathcal{H}^{d-1}(K)^{-1}$. Sub-Gaussian concentration bounds with again polynomial error terms for F in this situation were proved in [16], purely exponential inequalities were not known until now. Again, we remark that in this case

$$\ell_d(K) - \mathsf{E}[\ell_d(\mathrm{conv}(\eta))] = c''_{d,K}\gamma^{-2/(d-1)}(1+o(1)), \qquad \text{as} \qquad \gamma \to \infty,$$

where c''_K is another constant depending on d and K, see [15, 16].

Suppose now that we are in one of the two situations just described. In this case we can also consider the so-called intrinsic volumes of $\mathrm{conv}(\eta)$. In general, for a compact convex subset $K \subset \mathsf{R}^d$ the i-th intrinsic volume of K, $i \in \{0,1,\dots,d\}$, is defined as

$$V_i(K) := \binom{d}{i}\frac{\ell_d(B^d)}{\ell_i(B^i)\ell_{d-i}(B^{d-i})}\mathsf{E}[\ell_i(K|E)]$$

where B^k, $k \in \mathsf{N}$, stands for the k-dimensional unit ball, E is a uniformly distributed i-dimensional random linear subspace of R^d and $K|E$ denotes the orthogonal projection of K onto E. In particular, $V_d(K)$ is the volume of K, $\mathcal{H}^{d-1}(\partial K) = 2V_{d-1}(K)$ its surface content, $V_1(K)$ is a constant multiple of the mean width of K and $V_0(K) = 1$ as long as $K \neq ?$. The intrinsic volumes admit a unique additive extension to the class of finite unions of convex sets, which is denoted by the same symbol. For this and for further background material explaining the central role of intrinsic volumes in convex geometry we refer the reader to [17, Chapter 4] or [19, Chapter 14]. Our next result provides new concentration bounds for the surface area of $\mathrm{conv}(\eta)$, or equivalently, the intrinsic volume of order $d-1$, and in this case we can deal with both random polytope models (In) and (Bd).

Proposition 6.2 (Concentration for the surface area). *Suppose that we are in one of the situations (In) or (Bd) and consider the Poisson functional $F := \mathcal{H}^{d-1}(\partial\,\mathrm{conv}(\eta))$. Then*

$$\mathsf{P}(F \geq \mathsf{E}[F]+t) \leq \exp\Big(-\frac{c\,t^2}{d\mathcal{H}^{d-1}(\partial K)^2}\Big), \qquad \mathsf{P}(F \leq \mathsf{E}[F]-t) \leq \exp\Big(-\frac{c\,t^2}{\gamma\mathcal{H}^{d-1}(\partial K)^2}\Big)$$

for every $t \geq 4\sqrt{\kappa}$, where $c = \log(2)/(8\kappa)$.

Proof. Consider the functional

$$V^+ = \gamma\int_{\mathsf{R}^d}(D_xF)_-^2\,\mu(\mathrm{d}x) + \int_{\mathsf{R}^d}(F(\eta)-F(\eta-\delta_x))_+^2\,\eta(\mathrm{d}x),$$

where μ is the normalized Lebesgue measure on K for model (In) and the normalized $(d-1)$-dimensional Hausdorff measure on ∂K for model (Bd). Given a point $x \in \eta$ denote by $M(x,\eta)$ the union of all facets of $\mathrm{conv}(\eta)$ which contain x, by $N(x,\eta)$ the set of vertices of $\mathrm{conv}(\eta)$ which are connected to x by an edge and by $T(x,\eta) := \mathrm{conv}(N(x,\eta)\cup x)\cap(\eta-\delta_x)$ the set of points of η different from x which are lying inside the simplex $\mathrm{conv}(N(x)\cup x)$. By additivity property of the surface area we have

$$\begin{aligned} D_xF &= \mathcal{H}^{d-1}(\partial\,\mathrm{conv}(\eta+\delta_x)) - \mathcal{H}^{d-1}(\partial\,\mathrm{conv}(\eta)) \\ &= \mathcal{H}^{d-1}(M(x,\eta+\delta_x)) - \mathcal{H}^{d-1}(\mathrm{conv}(T(x,\eta+\delta_x))\cap\mathrm{conv}(\eta)) \\ &\geq 0, \end{aligned}$$

which holds P-almost surely for ℓ_d-almost all $x \in K$ if we consider model (In) or for $\mathcal{H}^{d-1}$-almost all $x \in \partial K$ in case of model (Bd). Hence,

$$V^+ = \int_{\mathsf{R}^d}(F(\eta)-F(\eta-\delta_x))_+^2\,\eta(\mathrm{d}x).$$

Further we have

$$\begin{aligned}F(\eta) - F(\eta - \delta_x) &= \mathcal{H}^{d-1}(M(x,\eta)) - \mathcal{H}^{d-1}(\operatorname{conv}(T(x,\eta)) \cap \operatorname{conv}(\eta - \delta_x)) \\ &\le \mathcal{H}^{d-1}(M(x,\eta)) \\ &\le 2V_{d-1}(\operatorname{conv}(N(x,\eta) \cup x)) \\ &\le \mathcal{H}^{d-1}(\partial K),\end{aligned}$$

where in the third step we used the equality

$$2V_{d-1}(\operatorname{conv}(N(x,\eta) \cup x)) = \mathcal{H}^{d-1}(M(x,\eta)) + \mathcal{H}^{d-1}(\partial \operatorname{conv}(N(x,\eta)))$$

and in the last step we used the monotonicity of the intrinsic volume V_{d-1} under set inclusion on the space of convex sets.
Finally, we conclude that

$$\begin{aligned}V^+ &\le \int_{\mathbb{R}^d} \mathcal{H}^{d-1}(M(x,\eta))^2\, \eta(\mathrm{d}x) \\ &\le \mathcal{H}^{d-1}(\partial K) \int_{\mathbb{R}^d} \mathcal{H}^{d-1}(M(x,\eta))\, \eta(\mathrm{d}x) \\ &\le d\mathcal{H}^{d-1}(\partial K)\mathcal{H}^{d-1}(\partial \operatorname{conv}(\eta)) \\ &\le d\mathcal{H}^{d-1}(\partial K)^2,\end{aligned}$$

where in the third step we used the fact that every facet of $\operatorname{conv}(\eta)$ is $\mathbb{P}$-almost surely a $(d-1)$-dimensional simplex and, hence, can be covered by at most d different sets of the type $M(x,\eta)$. Applying Corollary 5.3 with $L = d\mathcal{H}^{d-1}(\partial K)^2$ we finish the proof of the upper tail.
By analogy we consider

$$V^- = \gamma \int_{\mathbb{R}^d} (D_x F)_+^2\, \mu(\mathrm{d}x) \le \gamma \mathcal{H}^{d-1}(\partial K)^2,$$

and the lower tail follows from Corollary 5.3 with $L = 2\gamma\mathcal{H}^{d-1}(\partial K)^2$ as well. □

It should be noted that the d-th intrinsic volume (i.e., the usual volume) and the $(d-1)$-st intrinsic volume (i.e., the surface area) are very special functionals in the list of the intrinsic volumes. Nevertheless, one could expect that a similar concentration inequality will hold for $F := V_i(\operatorname{conv}(\eta))$ and all $i \in \{1, \dots, d\}$. However, the main obstacle in proving such an inequality for $i < d-1$ is that the intrinsic volumes V_i are not necessarily positive and not monotone under set inclusion for the class of finite unions of convex sets. That means that the difference operator $D_x F$ can be unbounded or even negative. However, we can avoid this situation if we consider the model (Bd) for which we have the following result, whose proof requires additional geometric efforts in comparison to those of Proposition 6.1 and Proposition 6.2. We emphasize that the concentration bounds we obtain are again new.

Proposition 6.3 (Concentration for the intrinsic volumes). *For the model (Bd) consider the Poisson functional $F := V_i(\operatorname{conv}(\eta))$ for some $i \in \{1, \dots, d\}$. Then*

$$\mathbb{P}(F \ge \mathbb{E}[F] + t) \le \exp\Big\{ -\frac{ct^2}{(i+1)V_i(K)^2}\Big\}, \qquad \mathbb{P}(F \le \mathbb{E}[F] - t) \le \exp\Big\{ -\frac{ct^2}{\gamma V_i(K)^2}\Big\}$$

for every $t \ge 4\sqrt{\kappa}$, where $c = \log(2)/(8\kappa)$.

Proof. We start by noting that since the intensity measure of the Poisson process η is concentrated on the boundary ∂K of K and by using the additivity property of intrinsic volumes we have that, $\mathbb{P}$-almost surely and for $\mathcal{H}^{d-1}$-almost all $x \in \partial K$,

$$\begin{aligned}D_x F &= V_i(\operatorname{conv}(\eta + \delta_x)) - V_i(\operatorname{conv}(\eta)) \\ &= V_i(\operatorname{conv}(N(x) \cup x)) - V_i(\operatorname{conv}(N(x))),\end{aligned}$$

where, as before, $N(x)$ denotes the set of vertices of $\mathrm{conv}(\eta)$ which are connected to x by an edge. Next, we recall from [19, Equation (14.14)] that for a polytope $P \subset \mathsf{R}^d$ the i-th intrinsic volume can be represented as

$$V_i(P) = \sum_{G \in \mathcal{F}_i(P)} \ell_i(G)\, \gamma(G, P),$$

where $\mathcal{F}_i(P)$ is the set of all i-dimensional faces of P and $\gamma(G,P)$ stands for the external angle at the face G with respect to P. The latter is just the solid angle of the convex cone of normal vectors of P at an arbitrary point from the relative interior of F, see [19, Equation (14.10)]. Applying this formula to the d-dimensional polytope $P_x := \mathrm{conv}(N(x) \cup x)$ and to the $(d-1)$-dimensional polytope $\hat{P}_x := \mathrm{conv}(N(x))$, we deduce that

$$\begin{aligned} D_x F &= \sum_{G \in \mathcal{F}_i(P_x)} \ell_i(G)\, \gamma(G, P_x) - \sum_{G \in \mathcal{F}_i(\hat{P}_x)} \ell_i(G)\, \gamma(G, \hat{P}_x) \\ &= \sum_{\substack{G \in \mathcal{F}_i(P_x) \\ x \in G}} \ell_i(G)\, \gamma(G, P_x) + \sum_{\substack{G \in \mathcal{F}_i(P_x) \\ x \notin G}} \ell_i(G)\, [\gamma(G, P_x) - \gamma(G, \hat{P}_x)]. \end{aligned}$$

We shall argue now that the second sum in the last expression is non-positive. To do this, we denote by $n_1(G), \ldots, n_{d-i}(G) \in \mathsf{R}^d$ the outer unit normal vectors of the $d-i$ facets $H_1, \ldots, H_{d-i} := \hat{P}_x$ of P_x containing a face $G \in \mathcal{F}_i(P_x)$, and by $\hat{n}_1(G), \ldots, \hat{n}_{d-i-1}(G) \in \mathrm{aff}(G) \subset \mathsf{R}^d$ (where $\mathrm{aff}(G)$ denotes the affine hull of G) the outer unit normal vectors of the $d-i-1$ facets $\hat{H}_1 = H_1 \cap \hat{P}_x, \ldots, \hat{H}_{d-i-1} = H_{d-i-1} \cap \hat{P}_x$ of $\hat{P}_x$ containing G. Then

$$\gamma(G, P_x) = \alpha_{d-i}(\mathrm{pos}(n_1(G), \ldots, n_{d-i}(G)))$$

and

$$\gamma(G, \hat{P}_x) = \alpha_{d-i-1}(\mathrm{pos}(\hat{n}_1(G), \ldots, \hat{n}_{d-i-1}(G))) = \alpha_{d-i}(\mathrm{pos}(\hat{n}_1(G), \ldots, \hat{n}_{d-i-1}(G), \pm n_{d-i}(G))),$$

where $\mathrm{pos}(\,\cdot\,)$ stands for the positive hull and $\alpha_k(\,\cdot\,) := \mathcal{H}^{k-1}(\,\cdot\, \cap \mathsf{S}^{d-1})/\omega_k$ with $\omega_k := H^{\,k}(\mathsf{S}^k)$ denoting the k-dimensional Hausdorff measure of k-dimensional unit sphere S^k. Moreover, we have that $n_j(G) = \hat{n}_j(G) - n(G)$ for each $j \in \{1, \ldots, d-i-1\}$. Finally, by definition of the positive hull we have

$$\mathrm{pos}(\hat{n}_1(G), \ldots, \hat{n}_{d-i-1}(G), \pm n_{d-i}(G)) = \Big\{ \sum_{k=1}^{d-i-1} \alpha_k \hat{n}_k(G) + \beta n_{d-i}(G) \colon \alpha_1, \ldots, \alpha_{n-i-1} \geq 0, \beta \in \mathsf{R} \Big\}$$

and, similarly,

$$\begin{aligned} &\mathrm{pos}(n_1(G), \ldots, n_{d-i}(G)) \\ &\quad = \Big\{ \sum_{k=1}^{d-i} \alpha_k n_k(G) \colon \alpha_1, \ldots, \alpha_{n-i} \geq 0 \Big\} \\ &\quad = \Big\{ \sum_{k=1}^{d-i-1} \alpha_k \hat{n}_k(G) + \Big(\alpha_{n-i} - \sum_{k=1}^{d-i-1} \alpha_k \Big) n_{d-i}(G) \colon \alpha_1, \ldots, \alpha_{n-i} \geq 0 \Big\}. \end{aligned}$$

Thus,

$$\mathrm{pos}(n_1(G), \ldots, n_{d-i}(G)) \subseteq \mathrm{pos}(\hat{n}_1(G), \ldots, \hat{n}_{d-i-1}(G), \pm n_{d-i}(G))$$

and $\gamma(G, P_x) \leq \gamma(G, \hat{P}_x)$ for all $G \in \mathcal{F}_i(P_x)$ with $x \notin G$. This shows that, indeed, the second sum in the above representation of $D_x F$ is non-positive, and hence that

$$D_x F \leq \sum_{\substack{G \in \mathcal{F}_i(P_x) \\ x \in G}} \ell_i(G)\, \gamma(G, P_x)$$

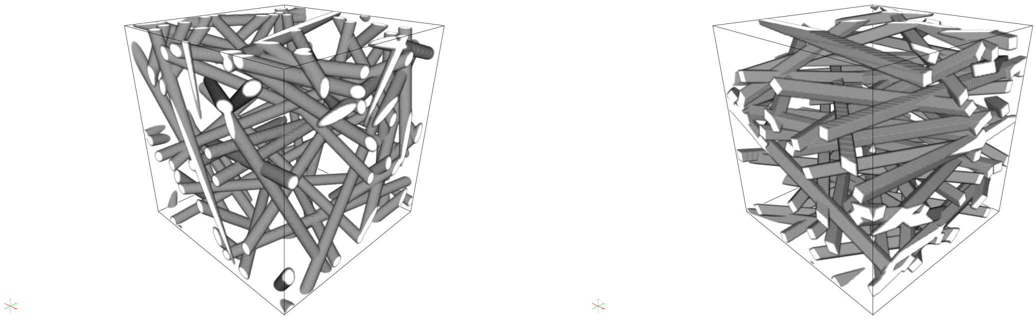

Figure 2: Simulations of two Poisson cylinder models with $d = 3$, $k = 1$ and with different base distributions (circles in the left panel and squares in the right panel). They were kindly provided by Claudia Redenbach (Kaiserslautern).

P-almost surely for $\mathcal{H}^{d-1}$-almost all $x \in \partial K$. Then by monotonicity of the intrinsic volumes under set inclusion on the space of convex sets we conclude that P-almost surely,

$$\begin{aligned} V^+ &= \int_{\partial K} (F(\eta) - F(\eta - \delta_x))_+^2 \, \eta(\mathrm{d}x) \\ &= \int_{\partial K} \big(V_i(\mathrm{conv}(N(x) \cup x)) - V_i(\mathrm{conv}(N(x)))\big)_+^2 \, \eta(\mathrm{d}x) \\ &\le V_i(K) \int_{\partial K} \sum_{\substack{G \in \mathcal{F}_i(P_x) \\ x \in G}} \ell_i(G) \, \gamma(G, P_x) \, \eta(\mathrm{d}x) \\ &= V_i(K) \int_{\partial K} \sum_{\substack{G \in \mathcal{F}_i(\mathrm{conv}(\eta)) \\ x \in G}} \ell_i(G) \, \gamma(G, \mathrm{conv}(\eta)) \, \eta(\mathrm{d}x) \end{aligned}$$

where in the second step we used the fact that intrinsic volumes of a convex set are non-negative. By construction of the model (Bd), each face $G \in \mathcal{F}_i(\mathrm{conv}(\eta))$ is P-almost surely an i-dimensional simplex with precisely $i + 1$ vertices. Thus,

$$V^+ \le (i+1)V_i(K) \sum_{G \in \mathcal{F}_i(\mathrm{conv}(\eta))} \ell_i(G) \, \gamma(G, \mathrm{conv}(\eta)) = (i+1)V_i(K)V_i(\mathrm{conv}(\eta)) \le (i+1)V_i(K)^2.$$

The bound for the upper tail now follows from Corollary 5.3 by choosing $L = (i+1)V_i(K)^2$ there. The proof of the lower tail follows from the fact that

$$V^- = \gamma \mathcal{H}^{d-1}(\partial K)^{-1} \int_{\partial K} (D_x F)^2 \, \mathcal{H}^{d-1}(\partial x) \le \gamma V_i(K)^2$$

and once again from Corollary 5.3, this time applied with $L = \gamma V_i(K)^2$. □

6.2 Poisson cylinder models

Our second application deals with the Poisson cylinder model for which a number of concentration properties for various geometric functionals have recently been studied in [4]. We refer to that paper and the literature cited therein for further references and background material. To define the model, fix some space dimension $d \ge 1$ and another dimension parameter $k \in \{0, 1, \ldots, d-1\}$. By $e_1, \ldots, e_d$ we denote the standard orthonormal basis in R^d and let $G(d,k)$ be the Grassmannian of all k-dimensional linear subspaces of R^d. We identify each element $L \in G(d,k)$ with a representative of the equivalence class $\mathbf{M}_L$ of orthogonal matrices $M_L \in \mathrm{SO}(d)$ satisfying $L = M_L E_k$, where $E_k := \mathrm{span}(e_{d-k+1}, \ldots, e_d)$; for concreteness we choose the lexicographically smallest element $\mathrm{lex\,min}\,\mathbf{M}_L$ from $\mathbf{M}_L$. Then we put $\mathrm{SO}_{d,k} := \{\mathrm{lex\,min}\,\mathbf{M}_L : L \in G(d,k)\}$ and define the product space $\mathsf{M}_{d,k} := \mathrm{SO}_{d,k} \times \mathcal{C}'_{d-k}$, where $\mathcal{C}'_{d-k}$

denotes the space of non-empty compact subsets of $\mathsf{R}^{d-k} = \mathrm{span}(e_1, \dots, e_{d-k}) \subset \mathsf{R}^d$. The elements of that space describe the direction ($\mathrm{SO}_{d,k}$-component) and the basis ($\mathcal{C}'_{d-k}$-component) of a cylinder. Next, we let Q be a probability measure on $\mathsf{M}_{d,k}$ and consider a Poisson process η in the product space $\mathsf{R}^{d-k} \times \mathsf{M}_{d,k}$ with intensity measure $\gamma\ell_{d-k} \otimes \mathsf{Q}$, where $\gamma \in (0, \infty)$ is an intensity parameter and ℓ_{d-k} denotes the Lebesgue measure on R^{d-k}. The associated random union set

$$Z := \bigcup_{(x,\theta,K)\in\eta} Z(x, \theta, K) \qquad \text{with} \qquad Z(x, \theta, K) := \theta((x + K) \times E_k),$$

is called a **Poisson cylinder model**, see Figure 2 for two simulations. The measure Q describes the joint distribution of the direction and the basis of a typical cylinder of the model. We emphasize that in the special case where $k = 0$ the Poisson cylinder model reduces to the classical **Boolean model**, which is included in our discussion as well, see [19] for background material on Boolean models. For a test set $W \in \mathcal{C}'_d$ we are interested in the volume of Z that can be observed in W, that is,

$$F := \ell_d(Z \cap W). \tag{6.2}$$

From [4, Equation (3.1)] we have that $\mathsf{E}[F] = \ell_d(W)(1 - e^{-\gamma \mathsf{E}[\ell_{d-k}(\Xi)]})$. Concentration properties for F around its mean have recently been studied in [4, Section 3] (and [8] for $k = 0$) based on concentration inequalities deduced from general covariance identities for Poisson functionals. The purpose of this section is to demonstrate that Corollary 5.3 can also be used to derive concentration bounds for F, where in addition to [4] we assume that the typical cylinder base has uniformly bounded volume. To formulate our result, let Q_* the marginal distribution of Q on the $\mathcal{C}'_{d-k}$-coordinate and let Ξ be a random element with distribution Q_*, the so-called typical cylinder base.

Proposition 6.4 (Concentration for the volume). *Consider a Poisson cylinder model as described above and suppose that the typical cylinder base Ξ satisfies $\ell_{d-k}(\Xi) \le A$ P-almost surely for some $A < \infty$. Then for F as defined by* (6.2) *one has that*

$$\mathsf{P}(F \ge \mathsf{E}[F] + t) \le \exp\Big\{ - \frac{c\,t^2}{A\,\mathrm{diam}(W)^k\,\ell_d(W)} \Big\},$$
$$\mathsf{P}(F \le \mathsf{E}[F] - t) \le \exp\Big\{ - \frac{c\,t^2}{\gamma\,A^2\,\mathrm{diam}(W)^k\,\ell_d(W)} \Big\}$$

for every $t \ge 4\sqrt{\kappa}$, where $c = \log(2)/(8\kappa)$.

Proof. In view of Corollary 5.3 the bound for the upper tail follows once we prove that $V^+ \le A\,\mathrm{diam}(W)^k\,\ell_d(W)$. To establish this inequality we recall the definition of V^+:

$$V^+ = \gamma \int_{\mathsf{M}_{d,k}} \int_{\mathsf{R}^{d-k}} (D_{(x,\theta,K)}F)_-^2\,\ell_{d-k}(\mathrm{d}x)\mathsf{Q}(\mathrm{d}(\theta, K))$$
$$+ \int_{\mathsf{M}_{d,k}} \int_{\mathsf{R}^{d-k}} (F(\eta) - F(\eta - \delta_{(x,\theta,K)}))_+^2\,\eta(\mathrm{d}(x, \theta, K)).$$

Now, since for $\ell_{d-k} \otimes \mathsf{Q}$-almost all $(x, \theta, K) \in \mathsf{R}^{d-k} \times \mathsf{M}_{d,k}$,

$$\begin{aligned} D_{(x,\theta,K)}F &= \ell_d((Z \cup Z(x, \theta, W)) \cap W) - \ell_d(Z \cap W) \\ &= \ell_d(Z(x, \theta, K) \cap W) - \ell_d(Z \cap Z(x, \theta, K) \cap W) \ge 0 \end{aligned}$$

holds P-almost surely, we have that $(D_{(x,\theta,K)}F)_-^2 = 0$, implying that the first term in the representation of V^+ vanishes. On the other hand, for $(x, \theta, K) \in \eta$ we have that P-almost surely

$$\begin{aligned} F(\eta) - F(\eta - \delta_{(x,\theta,K)}) &= \ell_d((Z_{-(x,\theta,K)} \cup Z(x, \theta, K)) \cap W) - \ell_d(Z_{-(x,\theta,K)} \cap W) \\ &= \ell_d(Z(x, \theta, K) \cap W) - \ell_d(Z_{-(x,\theta,K)} \cap Z(x, \theta, K) \cap W) \\ &= \ell_d((Z(x, \theta, K) \cap W) \setminus Z_{-(x,\theta,K)}) \ge 0, \end{aligned}$$

where $Z_{-(x,\theta,K)} := \bigcup_{(y,\phi,L)\in\eta-\delta_{(x,\theta,K)}} Z(y,\phi,L)$ stands for the cylinder model based on η but with the cylinder corresponding to (x,θ,K) removed. Thus, using the assumption on the boundedness of the volume of the typical cylinder base, we find that P-almost surely

$$\begin{aligned}
&\int_{\mathsf{M}_{d,k}}\int_{\mathsf{R}^{d-k}} (F(\eta)-F(\eta-\delta_{(x,\theta,K)}))_+^2\,\eta(\mathrm{d}(x,\theta,K))\\
&\quad\le \int_{\mathsf{M}_{d,k}}\int_{\mathsf{R}^{d-k}} \ell_d((Z(x,\theta,K)\cap W)\setminus Z_{-(x,\theta,K)})^2\,\eta(\mathrm{d}(x,\theta,K))\\
&\quad\le A\,\mathrm{diam}(W)^k\int_{\mathsf{M}_{d,k}}\int_{\mathsf{R}^{d-k}} \ell_d((Z(x,\theta,K)\cap W)\setminus Z_{-(x,\theta,K)})\,\eta(\mathrm{d}(x,\theta,K))\\
&\quad= A\,\mathrm{diam}(W)^k\,F\\
&\quad\le A\,\mathrm{diam}(W)^k\,\ell_d(W).
\end{aligned}$$

Now, Corollary 5.3 can be applied with $L = A\,\mathrm{diam}(W)^k\,\ell_d(W)$. This concludes the proof for the upper tail.
To obtain the bound for the lower tail we notice that, since $F(\eta)-F(\eta-\delta_{(x,\theta,K)})\ge 0$ P-almost surely for $(x,\theta,K)\in\eta$,

$$\begin{aligned}
V^- &= \gamma\int_{\mathsf{M}_{d,k}}\int_{\mathsf{R}^{d-k}} \big(\ell_d(Z(x,\theta,K)\cap W)-\ell_d(Z\cap Z(x,\theta,K)\cap W)\big)^2\,\ell_{d-k}(\mathrm{d}x)\mathsf{Q}(\mathrm{d}(\theta,K))\\
&\le \gamma\int_{\mathsf{M}_{d,k}}\int_{\mathsf{R}^{d-k}} \ell_d(Z(x,\theta,K)\cap W)^2\,\ell_{d-k}(\mathrm{d}x)\mathsf{Q}(\mathrm{d}(\theta,K))\\
&\le \gamma\,A\,\mathrm{diam}(W)^k\int_{\mathsf{M}_{d,k}}\int_{\mathsf{R}^{d-k}} \ell_d(Z(x,\theta,K)\cap W)\,\ell_{d-k}(\mathrm{d}x)\mathsf{Q}(\mathrm{d}(\theta,K))
\end{aligned}$$

holds P-almost surely. For fixed $(\theta,K)\in\mathsf{M}_{d,k}$ the inner integral can be evaluated by Fubini's theorem, which yields

$$\int_{\mathsf{R}^{d-k}} \ell_d(Z(x,\theta,K)\cap W)\,\ell_{d-k}(\mathrm{d}x) = \ell_{d-k}(K)\,\ell_d(W),$$

(this formula is also a special case of [18, Theorem 2], which is stated there under the (in this situation unneessary) assumption that both W and K are convex). Thus,

$$V^- \le \gamma\,A\,\mathrm{diam}(W)^k\,\ell_d(W)\,\mathsf{E}[\ell_{d-k}(\Xi)] \le \gamma\,A^2\,\mathrm{diam}(W)^k\,\ell_d(W),$$

where we used our assumption on the volume of the typical cylinder base. The bound for the lower tail now follows from Corollary 5.3 with $L=\gamma\,A^2\,\mathrm{diam}(W)^k\,\ell_d(W)$. This completes the proof. □

To compare the result of Proposition 6.4 with [4, Corollary 4.5] we focus on the upper tail and choose a $(d-k)$-dimensional ball B_ϱ^{d-k} of radius $\varrho>0$ as our typical cylinder base and for the window W a d-dimensional ball of radius $R>0$. In this case [4, Corollary 4.5] yields that

$$\mathsf{P}(F\ge \mathsf{E}[F]+t)\le \exp\Big\{\frac{t}{a}-\Big(b+\frac{t}{a}\Big)\log\Big(1+\frac{t}{ab}\Big)\Big\}$$

for all $t\ge 0$, where $a=A\,\mathrm{diam}(W)^k$ with $A=\ell_{d-k}(B_\varrho^{d-k})$ and $b>0$ is another constant depending on ϱ, R and the intensity γ whose value is not relevant for our purpose. At first sight, this bound seems worse than the one in Proposition 6.4. However, since $ab\ge\mathsf{E}[F]$ as shown in the proof of [4, Corollary 4.3], we have that

$$\mathsf{P}(F\ge \mathsf{E}[F]+t)\le \exp\Big\{\frac{t}{a}-\frac{\mathsf{E}[F]}{a}\Big(1+\frac{t}{\mathsf{E}[F]}\Big)\log\Big(1+\frac{t}{\mathsf{E}[F]}\Big)\Big\},\qquad t\ge 0.$$

Using now the elementary inequality $(1+x)\log(1+x) \geq x+\frac{1}{2}x^2/(1+x/3)$, valid for $x \geq 0$, we conclude that

$$\mathsf{P}(F \geq \mathsf{E}[F]+t) \leq \exp\Big\{-\frac{t^2}{2a(\mathsf{E}[F]+\frac{t}{3})}\Big\}, \qquad t \geq 0. \tag{6.3}$$

To compare (6.3) with the bound from Proposition 6.4 we put $p := \mathsf{E}[F]/\ell_d(W)$ and observe that in the relevant regime where $t \leq \ell_d(W) - \mathsf{E}[F]$ (for larger t the probability $\mathsf{P}(F \geq \mathsf{E}[F]+t)$ is zero by construction), one has that $2a(\mathsf{E}[F]+\frac{t}{3}) \leq 2a(p\ell_d(W)+\frac{1}{3}(1-p)\ell_d(W)) = \frac{2}{3}a\ell_d(W)(2p+1)$. Moreover, the inequality $\frac{2}{3}a\ell_d(W)(2p+1) \leq \frac{a}{c}\ell_d(W)$ is always satisfied, since $\frac{2}{3}(2p+1) \leq 2 < \frac{1}{c} \approx 14.66$ and $0 \leq p \leq 1$. As a consequence,

$$\exp\Big\{-\frac{t^2}{2a(\mathsf{E}[F]+\frac{t}{3})}\Big\} \leq \exp\Big\{-\frac{c\,t^2}{a\,\ell_d(W)}\Big\}$$

for all relevant values of t. In other words, the concentration bound from [4] is in this case always better than the one implied by Proposition 6.4 by at least a constant factor.

Acknowledgement

A.G. was partially supported by the Deutsche Forschungsgemeinschaft (DFG) via RTG 2131 *High-Dimensional Phenomena in Probability – Fluctuations and Discontinuity*. H.S. was supported by the Deutsche Forschungsgemeinschaft (DFG) via CRC 1283 *Taming Uncertainty and Profiting from Randomness and Low Regularity in Analysis, Stochastics and their Applications*.

References

[1] R. Adamczak, B. Polaczyk and M. Strzelecki: Modified log-Sobolev inequalities, Beckner inequalities and moment estimates. arXiv: 2007.10209.

[2] S. Bachmann and G. Peccati: Concentration bounds for geometric Poisson functionals: Logarithmic Sobolev inequalities revisited. Electron. J. Probab. **21**, no. 6, 44 pp. (2016).

[3] S. Bachmann and M. Reitzner: Concentration for Poisson U-statistics: Subgraph counts in random geometric graphs. Stochastic Processes Appl. **128**, 3327–3352 (2018).

[4] A. Baci, C. Betken, A. Gusakova and C. Thäle: Concentration inequalities for functionals of Poisson cylinder processes. arXiv:1908.02112.

[5] R.F. Bass: *Stochastic Processes*. Cambridge University Press (2011).

[6] S. Boucheron, O. Bousquet, G. Lugosi and P. Massart: Moment inequalities for functions of independent random variables. Ann. Probab. **31**, 514–560 (2005).

[7] D. Chafaï: Entropies, convexity, and functional inequalities. On Φ-entropies and Φ-Sobolev inequalities. J. Math. Kyoto Univ. **44**, 325–363 (2004).

[8] F. Gieringer and G. Last: Concentration inequalities for measures of a Boolean model. ALEA Lat. Am. J. Probab. Math. Stat. **15**, 151–166 (2018)

[9] N. Gozlan, R. Herry and G. Peccati: Transport inequalities for random point measures. arXiv: 2002.04923.

[10] J. Grote and C. Thäle: Concentration and moderate deviations for Poisson polytopes and polyhedra. Bernoulli **24**, 2811–2841 (2018).

[11] G. Last, G. Peccati and M. Schulte: Normal approximation on Poisson spaces: Mehler's formula, second order Poincaré inequalities and stabilization. Probab. Theory Related Fields **165**, 667–723 (2016).

[12] G. Last and M.D. Penrose: Martingale representation for Poisson processes with applications to minimal variance hedging. Stochastic Processes Appl. **121**, 1588–1606 (2011).

[13] G. Last and M.D. Penrose: *Lectures on the Poisson Process*. Cambridge University Press (2017).

[14] G. Peccati and M. Reitzner (editors): *Stochastic Analysis for Poisson Point Processes*. Bocconi & Springer (2016).

[15] M. Reitzner: Random polytopes. In W. Kendall and I. Molchanov (editors) *New Persiectives in Stochastic Geometry*, Oxford University Press (2010).

[16] R.M. Richardson, V.H. Vu and L. Wu: An inscribing model for random polytopes. Discrete Comput. Geom. **39**, 469–499 (2008).

[17] R. Schneider: *Convex Bodies: The Brunn–Minkowski Theory*. 2nd edition, Cambridge University Press (2013).

[18] R. Schneider and W. Weil: Translative and kinematic integral formulae for curvature measures. Math. Nachr. **129**, 67–80 (1986).

[19] R. Schneider and W. Weil: *Stochastic and Integral Geometry*. Springer (2008).

[20] V.H. Vu: Sharp concentration of random polytopes. Geom. Funct. Anal. **15**, 1284 – 1318 (2005).

[21] L. Wu: A new modified logarithmic Sobolev inequality for Poisson point processes and several applications. Probab. Theory Related Fields **118**, 427–438 (2000).

[22] X. Zhang: Clark-Ocone formula and variational representation for Poisson functionals. Ann. Probab. **37**, 506–529 (2009).