Product Innovation Investment under Technological Uncertainty and Financial Constraints*

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Abstract

We analyze the role of financial contraints for the speed of product innovation in a market characterized by technological and demand uncertainty. In a dynamic market setting we characterize the optimal R&D investment strategy of a monopolistic incumbent firm that can invest to develop a new product with uncertain demand. The size of the R&D investment flow determines the distribution of the stochastic innovation time and at the same time influences the dynamic evolution of firm's liquidity. If liquidity is negative the firm faces bankruptcy risk. We show that optimal investment is a U-shaped function of liquidity and characterize under which circumstances it is optimal for the firm to go into debt in order to speed up innovation. Furthermore, we show that, due to the existence of financial constraints, the relationship between the incumbent's profit on the existing market and the expected innovation time for the new product is non-monotone and follows a titled-z shape.

Keywords: Product Innovation, Financial Constraints, Optimal Investment, Uncertainty, Dynamic Optimization

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1 Introduction

Product Innovation is a crucial strategic activity for many firms. When firms are able to offer distinctive and innovative products, they not only extend their existing product lines, but also exert advantage against competitors. According to a widely circulated McKinsey survey, 84% of executives believe that innovation is critical for their business.¹ An innovation process takes time (Gee, 1978) and requires continuous financial investment from the firm. However, product innovation is associated with different types of uncertainties, in particular technological uncertainty and market uncertainty. Technological uncertainty implies it is difficult to predict the time and effort required for the successful innovation, and consequently, the firm has incomplete information about investment costs. Market uncertainty refers to the uncertain demand for the new product in case of a successful innovation, and leads to uncertain profitability. These two uncertainties affect the return to firms' product innovation investments.

Due to their risky nature, access to external financing for innovation projects is for many firms problematic (Brown et al., 2009) and therefore a large fraction of such projects have to be financed internally. This implies that innovation investment decisions of firms are often influenced by financial constraints. According to the data from CIS survey in Germany 2012-2014 (Behrens et al. (2017), 18.5% of innovative firms have sacrificed innovation projects due to lack of finance. 23% of all firms and 48% of firms in R&D-intensive industries would like to increase their innovation expenditures in case of an exogenous positive shock to their cash flow. Moreover, the combination of uncertainties, about the success of product innovation and the future profitability, might jeopardize firms' financial standing. Consider Kodak in 1996. Then CEO George Fisher knew that the company's core business might be invaded, or even replaced by digital photography. Kodak was so worried about the threat posed by the new technology that they invested more than \$2 billion in R&D for digital imaging. However, Kodak failed to anticipate how the market would develop and committed to product specifications which proved difficult to change. Hence they could not save their position in the traditional market but also failed to find a new market. In case a firm relies heavily on external financing for its R&D investments, when such disappointing market performance of the new product or cost overruns of the the innovation project accor, this can affect a firm's ability to roll over future debt and R&D investment might increase the bankruptcy risk for firms (Buddelmeyer et al. (2010)).

¹See https://hbr.org/2019/12/real-innovation-requires-more-than-an-rd-budget

The main agenda of this paper is to study how financial constraints and the induced risk of bankruptcy influence optimal investment strategies for product innovation by incumbent firms in a market. In particular, we analyze how optimal investment depends on the firms' financial standing, formally expressed as the firm's liquidity, as well as on its strength in the established market, formally expressed as the size of firm's profits on that market. Furthermore, we explore the role of frictions in access to external financing, expressed as the firm's bankruptcy risk when being unable to roll-over existing debt. Our focus on incumbent firms is motivated by empirical evidence that a large fraction of product innovations is developed by incumbents rather than by new market entrants (Chandy and Tellis (2000)).

Intuitively, when considering the effect of a firm's financial standing and strength on the established market on product innovation incentives, several effects come into play. First, with respect to the firm's available liquidity, a larger stock induces a lower demand for external financing, which reduces the bankruptcy risk and therefore the expected costs associated with investment. However, for firms with substantial negative liquidity in particular, a potential bankruptcy implies a reduction in expected costs of investment due to limited liability. Second, with respect to the firm's strength on the established market, at least in case the new product is a partial substitute for the firm's existing product, cannibalization arguments induce a negative relationship between the incumbent's strength on the established market and product innovation incentives, if there are be no financial frictions. Besides, higher profits of the incumbent on the established market reduces the need for external financing and therefore reduces the expected costs of innovation investment.

In this paper we disentangle these effects and shed light on their interplay in the framework of a simple dynamic market model, which incorporates technological and market uncertainty as well as bankruptcy risk. This risk is induced by the accumulation of debt due to the inability to internally finance innovation investments. By this approach, we bring together an industrial organization perspective that focuses on firms' innovation incentives, with a corporate finance perspective that emphasizes the impact of financial constraints. Building on our characterization of firms' optimal investment strategies we can also address how the interplay of profitability of established markets and access to external financing influence the speed of innovation and bankruptcy risk in an industry.

We consider a monopolistic firm offering an established product on a mature market with a constant demand function. The firm receives a continuous profit stream from its sales on this market and at the same time can invest to develop a new product, which is a partial substitute to the established product. The completion time of the new product development is stochastic and the innovation rate depends on the firm's R&D investment. Once the product development is completed the firm puts the new product on the market. Then the demand for the new product evolves stochastically, starting from a low level and approaching a long-term market size, which is higher than that of the established product. The dynamics of the firm's liquidity is driven by the difference between market profits and innovation investment plus dividends as well as by interest received or paid. It is assumed that the firm has access to external financing, such that liquidity can become negative. A firm with negative liquidity runs the risk of going bankrupt, and the risk increases with the size of negative liquidity. This formulation captures in reduced form that firms in debt might loose access to credit when trying to roll-over debt and that the probability for this to happen increases with the firm's leverage (see e.g. Sapienza (2002) for empirical evidence in this respect). It is also assumed that the firm follows a simple dividend policy, paying out a fixed fraction of its liquidity as dividend as long as liquidity is positive and not paying when liquidity is negative. The firm determines its R&D investment in order to maximize the expected discounted future dividend stream.

Our analysis shows that the optimal R&D strategy, as a function of the firm's liquidity, has a U-shape. Investments are highest when liquidity is either very high, which implies that the firm essentially faces no financial constraints, or when the firm is already heavily indebted, in which case a fast product innovation is the only chance to avoid future bankruptcy. For firms with a liquidity level between these two extremes two different scenarios might arise, depending on the efficiency of R&D investments, the profitability of the established market and the frictions in the firm's access to credit. In particular, a debt scenario or a no debt scenario might arise. In the former case it is optimal for a firm with positive initial liquidity to invest so heavily in R&D that it eventually accumulates debt and faces a positive bankruptcy probability, whereas in the latter case the firm never goes into debt if the initial liquidity is non-negative, and over time eliminates any potential initial debt. For large parts of the parameter range giving rise to a no debt scenario, the optimal R&D investment strategy is discontinuous at the liquidity level of zero and exhibits a downward jump there. The liquidity level of zero is then a stable fixed point of the liquidity dynamics under optimal investment, meaning that a firm with positive initial liquidity diminishes it to zero in order to speed up the innovation process, and then simply invests all incoming profits in R&D till the innovation is successful without relying on any external financing. Combining analytical results with an extensive numerical analysis we fully characterize how the occurrence of these scenarios depends on the interplay of the key model parameters, and how these scenarios influence the relationship between parameters and speed of innovation and the firm's bankruptcy probability. In particular, we show in such a scenario with financial constraints that, a highly non-monotone relationship between the profitability of the established market and the speed of innovation emerges. A higher profitability on the established market induces lower R&D investment and slower innovation if the firm is either in a debt scenario (arising for low levels of profitability) or in a no debt scenario with positive long-run liquidity (arising for high levels of profitability). Additionally, in a no debt scenario with long-run liquidity of zero there is a positive relationship between profitability of the established market and the speed of innovation.

Our results contribute to different streams of related literature. First, we extend analyses of product innovation incentives of incumbents in the absence of financial frictions. Dawid et al. (2015) and Dawid et al. (2020) show in a similar dynamic market setting and perfect financial markets, that the optimal product innovation investments of a monopolist respectively duopolist depend negatively on firm's production capacity on the established market. Therefore, there is a negative relationship between the cash-flow generated on the established market and innovation investment. Our analysis highlights, that this monotone relationship no longer holds if the assumption of perfect financial markets is dropped. Second, our findings are related to the long-lasting debate on the relationship between cash-flow and investment under financial constraints. Empirical results in that respect are mixed. Many paper starting with Fazzari et al. (1988, 2000) find that financially constrained firms have a stronger positive relationship between cash flow and investment compared to where financial constraints do not play a role. This view has been challenged on both theoretical and empirical grounds by Kaplan and Zingales (1997, 2000). Gomes (2001) and Alti (2003) have put forward models showing a positive relationship between cash-flow and investment in settings with perfect financial markets. In this realm of literature dealing with generic investments, typically in physical capital, several papers have also studied the relationship between R&D investment and cash flow, and in how far it is related to financial constraints. The general conclusion from this body of literature is that there is evidence that firms face constraints for financing R&D investments and this gives rise to high sensitivity of innovative firm's investment to cash flows (see Hall and Lerner (2010) for a survey of this literature).

Our paper complements this mainly empirical literature from a theoretical perspective and provides several innovative aspects. First, while these papers consider general R&D investment, we specifically focus on product innovation investments of incumbents by taking into account that, the firm's strength on the established market influences both the firm's revenues and its incentive to extend its product range. Second, our analysis characterizes the optimal investment strategy of the firm as a function of its current financial state, thereby capturing how the innovation investments of the firm evolves over time as its liquidity changes. This perspective allows us to show that, even within the same market environment the sign of the relationship between cash flow and R&D investment might change according to the firm's financial standing. Third, we characterize under which circumstances a rational incumbent should risk bankruptcy and jeopardize its position on the established market in order to pursue product innovation.

The remainder of the paper is organized as follows. In Section 2 we introduce our model. Analytical results characterizing the optimal investment strategy and the resulting liquidity dynamics are presented in Section 3. In Section 4 we extend these findings with an extensive numerical analysis illustrating the optimal investment strategies and the corresponding expected innovation times and bankruptcy probabilities for different parameter constellations. We provide concluding remarks in Section 5. All proofs are given in Appendix A, Appendix B contains additional numerical results and Appendix C a detailed description of our numerical method.

2 The Model

A monopoly firm is producing on an established market and developing at the same time a new product n, which is a partial substitute to the established product o. At the stochastic innovation time τ the firm can introduce the new product and afterwards is active in both markets. The inverse demand is assumed to be linear and of the form

$$p_o(t) = \alpha_o - q_o(t) - \eta q_n(t),$$

$$p_n(t) = \bar{\alpha}_n + \alpha_n - q_n(t) - \eta q_o(t).$$

Here $p_i(t)$, $i \in \{o, n\}$, denotes the price for product i and $q_i(t)$ the output of product $i \in \{o, n\}$ at time t. The parameter $\eta \in [0, 1)$ indicates the degree of horizontal differentiation between the two products. The consumers' maximal willingness to pay for the established product, α_o , is assumed to be constant. The maximal willingness to pay for the new product is denoted by $\bar{\alpha}_n + \alpha_n$, and α_n is assumed to evolve stochastically from the moment of successful innovation τ and to follow a mean-reverting stochastic process:

$$d\alpha_n = \delta \left(\tilde{\alpha}_n - \alpha_n \right) dt + \sigma \alpha_n dW(t), \quad \delta, \sigma > 0, \tag{1}$$

with $\alpha_n(\tau) = 0$. W(t) is a Wiener process, and $\tilde{\alpha}_n > \alpha_o - \bar{\alpha}_n$ captures the fact that the market potential of the new product is lager than that of the established product. The minimal market size $\bar{\alpha}_n > 0$ is sufficiently large to guarantee that the equilibrium output for the new product stays non-negative even for $\alpha_n = 0$.

If the firm cannot service its debt, it might have to exit the market, i.e., go bankrupt. The number of products, that firm is able to offer, is captured by $m(t) \in \{m_0, m_1, m_2\}$, which we denote as the *mode* of the problem. In mode m_0 the firm has exited the market and therefore $q_o = q_n = 0$, in mode m_1 the firm is active only on the established market, such that $q_o > 0, q_n = 0$ and in mode m_2 the firm has both products on the market, i.e. $q_o, q_n \ge 0$. At time t = 0, we have $m(0) = \{m_1\}$ indicating that the firm is active on the established market, and has not innovated yet.

At each time t the monopoly firm chooses the optimal output quantities taking into accout the current mode. For reasons of simplicity we normalize the unit costs of production to zero. For $m(t) = m_1$ standard calculations show that

$$q_o(m_1) = \frac{\alpha_o}{2}$$

Similarly, in mode m_2 the optimal output quantities are given by

$$q_o(\alpha_n, m_2) = \frac{\alpha_o - \eta(\bar{\alpha}_n + \alpha_n)}{2(1 - \eta^2)},$$
$$q_n(\alpha_n, m_2) = \frac{(\bar{\alpha}_n + \alpha_n) - \eta\alpha_o}{2(1 - \eta^2)},$$

whenever both expressions are non-negative. This is ensured by assuming that $\bar{\alpha}_n > \eta \alpha_o$ and that $\alpha_o - \eta(\bar{\alpha}_n + \tilde{\alpha}_n)$ is sufficiently large such that the probability that $\alpha_o < \eta(\bar{\alpha}_n + \alpha_n)$ is negligible. This generates the following market profits $\pi(\alpha_n, m)$ in the different modes:

$$\pi(\alpha_n, m_2) = \frac{(\bar{\alpha}_n + \alpha_n)^2 + \alpha_o^2 - 2\eta(\bar{\alpha}_n + \alpha_n)\alpha_o}{4 - 4\eta^2},$$

$$\pi(0, m_1) = \frac{\alpha_o^2}{4},$$

$$\pi(\alpha_n, m_0) = 0.$$

It is easy to verify that $\pi(\alpha_n, m_2) \ge \pi(0, m_1)$ with strict inequality whenever $q_n(\alpha_n, m_2) > 0$.

The transition from mode m_1 to m_2 corresponds to a successful innovation. The arrival process

of the innovation is assumed to be memoryless and the innovation rate, is given by $\lambda^{12} = \gamma_I I(t)$, where I(t) denotes the R&D investment by the monopoly firm and the *innovation parameter* $\gamma_I > 0$ captures the efficiency of the firm's R&D activities. R&D investment is associated with quadratic costs of the form $\frac{\xi}{2}I^2$, $\xi > 0$. The financial situation of the firm is expressed by its liquidity e(t), which evolves according to

$$\dot{e} = \pi(\alpha_n, m) - \frac{\xi}{2}I^2 - D + re, \qquad (2)$$

where r > 0 is the interest rate, and D(t) denotes the dividends paid out at time t to the shareholders. We assume the dividend policy $D(e, m) = \nu_m \max\{0, e\}$ with $\nu_m \in (0, 1)$. According to this policy the firm pays dividends as a fixed proportion of the positive liquidity reserve and does not pay any dividends if it has negative liquidity. It is worth mentioning that ν_m is assumed to be a constant, rather than a control variable for the firm in our model. Hence, in general this dividend policy is not intertemporally optimal. We make this simplification because our focus is on the innovation investment under bankruptcy threat rather than on the optimal dividend policy. Providing a characterization of the optimal dividend policy in this framework with demand uncertainty and endogenous investment is highly challenging and will be the topic of future work. Liquidity can become negative, in which case the firm is in debt. To capture the risk for a firm associated with external financing of its investment, it is assumed that there is a positive probability that an indebted firm is not able to extend its debt contracts, and has to exit the market. This probability is assumed to be increasing with the amount of debt.² More precisely, the transition rate (bankruptcy rate) from mode $m = \{m_1, m_2\}$ to $m = m_0$ is given by

 $\lambda^{k0} := \gamma_B \max[0, -e], \ k = 1, 2.$

This formulation can also be interpreted as a reduced form representation of a situation where the firm faces a fixed credit limit and is exposed to unanticipated financial shocks, e.g., with respect to its fixed costs. The *bankruptcy parameter* γ_B determines how tight the financial constraints are. The case $\gamma_B = 0$ indicates no financial constraint, and $\gamma_B \to \infty$ implies that the firm has no access to credit. Note that in our formulation the (expected) costs of external financing are completely captured by the dependence of the bankruptcy risk from liquidity, whereas the interest rate in the

²Empirical evidence that higher leverage and higher R&D investment increase the risk of loosing a banking relationship and having to exit the market are provided e.g. in Sapienza (2002) and Buddelmeyer et al. (2010)

liquidity dynamics (2) does not depend on the level of firm liquidity. Assuming that the interest the firm pays for loans also depends positively on its level of debt would certainly be a reasonable assumption, but we abstain from such a formulation here in order to keep the model as simple as possible. Our aim here is to study the optimal investment behavior of a firm who is aware that its expenditures for innovation activities might jeopardize its existence if these activities remain unsuccessful for too long. This effect is captured in the form of the bankruptcy rate.

Under these assumptions, the overall dynamics of the liquidity is given by the state dynamics

$$\dot{e} = \begin{cases} 0 & m = m_0, \\ \pi(0, m_1) - \frac{\xi}{2}I^2 - \nu_1 \max\{0, e\} + re & m = m_1, \\ \pi(\alpha_n, m_2) - \nu_2 \max\{0, e\} + re & m = m_2, \end{cases}$$
(3)

and the Markov process m in $\{m_0, m_1, m_2\}$ with transition states

$$\lambda^{ij} = \begin{cases} \gamma_I I & (i,j) = (m_1, m_2), \\ \gamma_B \max[0, -e] & (i,j) \in \{(m_2, m_0), (m_1, m_0)\}, \\ 0 & \text{else.} \end{cases}$$
(4)

Before the successful innovation in mode m_1 , the firm is constantly balancing the two effects from the innovation investment. One effect is that the investment decreases the firm's liquidity reserves and brings about a possible bankruptcy once the liquidity becomes negative. The other effect is the likelihood of transition into mode 2 that is boosted by investment. Note that a negative liquidity in mode 2 also implies a bankruptcy probability. Since the firm's profit flow in mode 2 is always non-negative, such negative liquidity has to be due to the result of the negative liquidity at the moment of transition from mode m_1 . In other words, there is a certain risk of bankruptcy even after successful innovation if the debt accumulated during the innovation phase is too high. We consider the investment problem of a decision maker with the objective of maximizing the expected dividend stream received by the firm's shareholders. Formally, this problem is given by

$$\max_{I(\cdot)} J = \mathbb{E}\left[\int_0^\infty e^{-rt} D(e, m) \mathrm{d}t\right].$$
(5)

subject to the state equation (3), the Markov process m(t) characterized by the transition rates (4) and the initial conditions $e(0) = e^{ini}$, $m(0) = m_1$. It should be noted that the firm's investment does not generate any expected revenue in mode m_2 , which trivially implies that I(t) = 0 for all t with $m(t) = m_2$. Hence, in what follows we will focus entirely on the characterization of optimal firm investment in mode m_1 . Due to the time autonomous structure of the problem, optimal investment depends only on the current state, but is independent of time. Hence, in what follows we will express the optimal investment in mode m_1 as a function of the state e and denote the optimal investment function by $\phi(e)$.

3 Analytical Results

In order to solve the firm's investment problem we use a Dynamic Programming approach and as a first step specify the Hamilton-Jacobi-Bellman (HJB) equations which characterize the value functions V_k in modes $m_k, k = 1, 2$. With respect to mode m_0 we note that no more dividends are paid once the firm is bankrupt, which means that the value function is given by $V_0(e, \alpha_n) = 0$ for all values of the state (e, α_n) . In mode m_2 , no more investments are made and hence there is no control for the decision maker to choose. Standard arguments (see e.g. Chapter 8 in Dockner et al. (2000)) show that under appropriate smoothness assumptions a function $V_2(e, \alpha_n)$, solving the HJB equation

$$rV_{2}(e,\alpha_{n}) = \nu_{2}\max\{0,e\} + \delta(\tilde{\alpha}_{n} - \alpha_{n})\frac{\partial V_{2}(e,\alpha_{n})}{\partial \alpha_{n}} + \frac{\sigma^{2}\alpha_{n}^{2}}{2}\frac{\partial^{2}V_{2}(e,\alpha_{n})}{\partial \alpha_{n}^{2}} + \frac{\partial V_{2}(e,\alpha_{n})}{\partial e}\dot{e} + \gamma_{B}\max\{0,-e\}(V_{0}(e,\alpha_{o}) - V_{2}(e,\alpha_{n}))$$

$$(6)$$

is the value function for the monopoly firm. The first term on the right hand side (RHS) is the dividend received by the share holders. The next three terms indicate the expected change in the value function due to the dynamics of market demand and the liquidity. The last term states the expected change in the value resulting from the possibility of bankruptcy in case of negative liquidity.

Considering now the investment problem in mode m_1 , we denote the value function in this mode as $V_1(e)$. Because the new market emerges only after successful innovation at time τ and by assumption $\alpha_n(\tau) = 0$, we write the value function in mode m_1 only as a function of the firm's liquidity e, but drop the second argument α_n . The corresponding HJB equation for a given dividend rate ν_1 in this mode can be written as

$$rV_1(e) = \max_I \left[\nu_1 \max\{e, 0\} + \frac{\mathrm{d}V_1(e)}{\mathrm{d}e} \dot{e} + \gamma_I I \left(V_2(e, 0) - V_1(e) \right) + \gamma_B \max\{0, -e\} \left(V_0(e) - V_1(e) \right) \right].$$
(7)

On the RHS, the expected change of the value comes from the changes of firm's liquidity, the possibility of transition into modes m_2 and m_0 . Using the Bellman equation for m_1 , i.e., equation (7), we obtain the following characterization of optimal investment.

Lemma 1. The optimal investment before innovation, i.e., in mode m_1 , is given by

$$\phi(e) = \frac{\gamma_I}{\xi} \frac{V_2(e,0) - V_1(e)}{dV_1(e)/de} > 0.$$
(8)

Lemma 1 can be easily derived by taking the first order derivative of the RHS of equation (7) with respect to I. It shows there are several factors that influence the firm's optimal innovation investment. R&D investment increases with respect to the innovation parameter γ_I , which determines the marginal effect of an increase of R&D investment on the innovation rate, and with respect to the jump in the value $(V_2(e, 0) - V_1(e))$ at the moment of successful innovation. The optimal innovation investment decreases with respect $\xi \frac{dV_1(e)}{de}$. To interpret this expression it should be noted that in light of (2) marginally increasing investment reduces the firm's liquidity by ξI and this decrease in liquidity is associated with a decrease in the firm owner's value of $\xi I \frac{dV_1(e)}{de}$. Hence, the coefficient $\xi \frac{dV_1(e)}{de}$ corresponds to the coefficient of the quadratic adjustment cost term in standard investment problems with convex investment costs. Thus, optimal investment in our setting can be interpreted in the usual way as the ratio of marginal (expected) returns to investment and the adjustment cost coefficient. In this respect it should be noted that the firm owner always profits from additional liquidity, which implies that we always have $dV_1(e)/de > 0$.

3.1 Scenario without effects of financial constraints

In order to better understand the effect of the bankruptcy threat for the optimal investment, we first study, as a benchmark, the special scenario without bankruptcy risk. In particular, we consider a situation where the initial liquidity is sufficiently large such that the firm never faces a positive bankruptcy probability even if it chooses its unconstrained optimal investment level. For such a scenario we can explicitly derive the value functions in both modes. Moreover, the firm's optimal innovation investment can also be calculated. These results are summarized in the following proposition.

Proposition 1. Assume that $e^{ini} > \tilde{e} = \max\left[\frac{\xi(I^{nc})^2 - \alpha_o^2/2}{2(r-\nu_1)}, 0\right]$ with

$$I^{nc} = \sqrt{\frac{r^2}{\gamma_I^2} + \frac{2rc}{\xi} - \frac{\alpha_o^2}{2\xi}} - \frac{r}{\gamma_I} > 0$$
(9)

and either $r > \nu_1$ or $\tilde{e} > 0$. Then the optimal investment in mode m_1 is constant over time with $I(t) = I^{nc}$ for all $t \in [0, \tau]$. In each mode liquidity changes monotonously over time and the value functions in the two modes are given by

$$V_1(e) = e + c + \frac{1}{\gamma_I^2} \left(r\xi - \sqrt{r^2 \xi^2 + 2cr\xi\gamma_I^2 - \frac{\xi\gamma_I^2 \alpha_o^2}{2}} \right),$$
(10)

$$V_2(e,\alpha_n) = \frac{\delta\tilde{\alpha}_n + (\bar{\alpha}_n - \alpha_o\eta)(r + 2\delta - \sigma^2)}{2(r+\delta)(1-\eta^2)(r+2\delta - \sigma^2)}\alpha_n + \frac{\alpha_n^2}{4(1-\eta^2)(r+2\delta - \sigma^2)} + e + c, \quad (11)$$

with

$$=\frac{\delta^2 \tilde{\alpha}_n^2 + \delta \tilde{\alpha}_n (\bar{\alpha}_n - \alpha_o \eta) (r + 2\delta - \sigma^2)}{2r(r+\delta)(1-\eta^2)(r+2\delta - \sigma^2)} + \frac{\bar{\alpha}_n^2 + \alpha_o^2 - 2\eta \alpha_o \bar{\alpha}_n}{4r(1-\eta^2)}$$
(12)

for all $e \geq e^{ini}$.

c

Proposition 1 covers two scenarios. First, if $r > \nu_1$ and $e^{ini} \ge \tilde{e}$ then liquidity grows throughout mode m_1 even if the firm chooses the unconstrained optimal investment level and therefore liquidity never becomes negative. In case $r < \nu_1$ and $\tilde{e} > 0$ we must have that $\alpha_o^2/4 > \xi(I^{nc})^2/2$, which means that the market profit in mode m_1 is sufficient to cover the investment costs under the unconstrained investment I^{nc} . At the liquidity level \tilde{e} the remaining part of the market profit plus earned interest is exactly equal to the firm's dividend payout. Hence, for any non-negative initial level of liquidity, in m_1 the liquidity of a firm investing I^{nc} converges to \tilde{e} and therefore never becomes negative. Since the firm no longer invests in mode m_2 liquidity stays non-negative throughout mode m_2 if it is non-negative at the time of the innovation.

Given that the firm's investment in mode m_1 is constant the expected innovation time can be easily calculated as

$$\mathbb{E}[\tau] = \int_0^\infty t \gamma_I I^{nc} \exp\left(-\gamma_I I^{nc} t\right) \mathrm{d}t = \frac{1}{\gamma_I I^{nc}}.$$
(13)

The value functions in both modes can be interpreted as a summation of the instantaneous liquidity reserve e and the discounted future profits. Since the interest rate is equal to the discount rate and the firm lives eternally, moving the payout of liquidity across time does not influence the value of

the discounted dividend stream of the firm owner as long as it is guaranteed that liquidity never becomes negative. Furthermore, as long as the firm does not face any (future) bankruptcy risk, the innovation investment is determined by the relationship between marginal costs and marginal future returns, but is independent of the liquidity and also of the dividend rate, see (9). From this equation also the following very intuitive effects of the key parameters on the unconstrained optimal investment level can be directly derived.

Corollary 1. Optimal R&D investment without bankruptcy risk, I^{nc} , increases with the efficiency of R&D (γ_I) but decreases with respect to investment costs (ξ) and the size of the established market (α_o). If $r + 2\delta > \sigma^2$, then optimal investment increases with the market potential of the new product ($\tilde{\alpha}$).

For the following analysis in particular the negative dependence of optimal R&D investment on the size of the established market is important. Intuitively, this dependence is due to a standard cannibalization effect. The introduction of the new product leads to a reduction in the price of the old product. Hence, the value of the new product introduction for the monopolist is smaller when the quantity of the established market product the firm sells is larger. If the monopolist has sufficiently large liquidity such that it can always internally finance its optimal R&D investments, this is the only effect induced by an increase of α_o . However, if the firm is financially constrained an increase of α_o also reduces the demand for external financing and hence the relationship between the optimal investment and the size of the established market is less clear cut. We now turn to analyzing this scenario where the firm has to take into account a potential bankruptcy risk.

3.2 Scenario with effects of financial constraints

If the unconstrained level of investment cannot be internally financed through profits on the established market, the monopolist, even if it initially does not have any debt, faces financial constraints. Building on Proposition 1 the following corollary shows that the efficiency of R&D activities as well as the size of the established market play a key role in determining whether financial constraints are relevant for the firm.

Corollary 2. If $2rc \leq \alpha_o^2$ or $2rc > \alpha_o^2$ and $\gamma_I \leq \underline{\gamma}_I$ with

$$\underline{\gamma}_I = \frac{r\alpha_o \sqrt{2\xi}}{2rc - \alpha_o^2} \tag{14}$$

and c given by (12), then $\tilde{e} = 0$. The optimal investment reads $\phi(e) = I^{nc}$ for all $e \ge 0$ and for all $e^{ini} \ge 0$ liquidity e(t) stays non-negative for all $t \ge 0$ and either converges towards the steady state $\tilde{e} \ge 0$ (for $r < \nu_1$) or diverges towards infinity while the firm is in mode m_1 (for $r \ge \nu_1$). If $\gamma_I > \gamma_I$ then either $\phi(e) < I^{nc}$ for some liquidity $e \ge 0$ or for some $e^{ini} \ge 0$ there is a positive probability for bankruptcy under the investment strategy $\phi(.)$, or both.

In what follows we will focus on the case with $2rc > \alpha_o^2$ and $\gamma_I > \underline{\gamma}_I$, where the inability to internally finance the unconstrained optimal investment also affects the investment of a firm without initial debt. Intuitively for $2rc \leq \alpha_o^2$ the attractiveness of the new market is so low relative to the established one that, for any effectiveness of R&D the unconstrained R&D investment can be financed internally by market profits and therefore financial constraints have little relevance. In order to restrict attention to scenarios where financial constraints have potential impact we make the following formal assumption.

Assumption 1. Throughout the following analysis it is assumed that $2rc > \alpha_o^2$.

The existence of bankruptcy risk makes the characterization of the optimal investment strategy much more challenging compared to the case without such risk. Formally, this is due to the fact that the last terms on the right hand side of the HJB equations (6) and (7), which disappear if only positive values of e are considered, prevent us from obtaining closed form solutions for the value functions in modes m_1 and m_2 on the entire state space.

3.2.1 Post-innovation

Since market profit in mode m_2 is non-negative and the firm makes no investments, the value function $V_2(e, \alpha_n)$ can be explicitly calculated for a positive initial liquidity e > 0. Given $\nu_2 > r$ liquidity in the long run oscilates around the positive steady state $e_2^* = \pi(\tilde{\alpha}, m_2)/(\nu_2 - r) > 0$. For $r \ge \nu_2$, liquidity would diverge to positive infinity, but it is clear that such a dividend policy would be sub-optimal. Because liquidity never decreases in mode m_2 , the bankruptcy rate is zero for all $t \ge \tau$ if $e(\tau) > 0$, and the value function $V_2(e, \alpha_n)$ has the same expression as equation (11) for $e \ge 0$. Since for negative liquidity no dividends are paid, these considerations and the fact that (11) does not depend on ν_2 show that also in the presence of bankruptcy risk, the value functions V_1 and V_2 and also the optimal investment strategy do not depend on the value of ν_2 . With respect to $V_2(e, \alpha_n)$ for e < 0, the non-linear form of the HJB equation does not allow us to obtain a closed form solution. Therefore, we have to resort to numerical calculations to determine the value function.

3.2.2 Pre-innovation R&D Investment

Taking into account Corollary 2, it is in general it is not clear whether for $\gamma_I > \underline{\gamma}_I$ the liquidity stays non-negative under the optimal investment strategy even if it starts evolving from a non-negative initial level. Since the problem in mode m_1 is an optimal control problem with one-dimensional state-space the liquidity trajectory under the optimal control has to be monotonous (see Hartl (1987)). Therefore, the analysis of the locations of the steady states of the problem provide clear insights on whether liquidity might become negative if the monopolist invests optimally. The following lemma provides a characterization of steady states candidates e^* under the assumption of differentiability of the value function at e^* .

Lemma 2. Assume that e^* is a steady state of the liquidity dynamics under the optimal investment strategy $\phi(e)$ in mode m_1 and that the associated value function $V_1(e)$ is differentiable at e^* . Then the following conditions have to be satisfied:

$$\frac{\alpha_o}{2} - \frac{\xi \Phi^2(e^*)}{2} + re^* - \nu_1 \max\{0, e^*\} = 0, \tag{15}$$

$$\xi\phi(e^*)\frac{\mathrm{d}V_1(e^*)}{\mathrm{d}e} - \gamma_I (V_2(e^*, 0) - V_1(e^*)) = 0, \tag{16}$$

$$rV_1(e^*) = \max\{0, \nu_1 e^*\} + \gamma_I \phi(e^*) (V_2(e^*, 0) - V_1(e^*)) - \gamma_B \max\{0, -e^*\} V_1(e^*),$$
(17)

$$\gamma_{I}\phi(e^{*})\left(\frac{\partial V_{2}(e^{*},0)}{\partial e} - \frac{\mathrm{d}V_{1}(e^{*})}{\mathrm{d}e}\right) + \nu_{1} \mathcal{I}_{[e^{*}\geq 0]} - r\frac{\mathrm{d}V_{1}(e^{*})}{\mathrm{d}e} + \gamma_{B}\left(\mathcal{I}_{[e^{*}\leq 0]}V_{1}(e^{*}) - \max\{0,-e^{*}\}\frac{\mathrm{d}V_{1}(e^{*})}{\mathrm{d}e}\right) = 0.$$
(18)

This system of necessary conditions is derived by taking into account the steady state condition $\dot{e} = 0$ (15), the first order condition for investment (16), the HJB equation at the steady state (17) and the state derivative of the HJB equation at the steady state (18). Once the problem in mode m_2 is solved and $V_2(e, \alpha_n)$ is known, then there are four unknowns in the above equations, e^* , $\Phi(e^*)$, $V_1(e^*)$, and $dV_1(e^*)/de$. Though closed form solutions to this system of equations in general cannot be obtained, Lemma 2 provides the basis for identifying via numerical analysis all candidates for steady states with local differentiability of the value function.

Before applying this lemma in the numerical analysis we first derive conditions under which zero liquidity is a steady state. In light of Corollary 2 we already know that this can only happen for $\gamma_I \geq \underline{\gamma}_I$. Due to the kink in the dividend policy and the bankruptcy rate at e = 0 we must expect that in general the value function V_1 is not differentiable at e = 0. Hence, Lemma 2 cannot be directly applied and we must resort to a viscosity solution of the HJB equation when determining the value function of the problem (see e.g. Bardi and Capuzzo-Dolcetta (2008)). Based on this we can characterize the conditions under which $e^* = 0$ is a steady state under optimal investment.

Proposition 2. The liquidity $e^* = 0$ is a stable steady state under the optimal investment strategy $\phi(e)$ in mode m_1 if

$$\gamma_I \in \left[\underline{\gamma}_I, \bar{\gamma}_I\right] \tag{19}$$

with $\underline{\gamma}_{I}$ given by (14) and

$$\bar{\gamma}_I = \frac{(r + \gamma_B c) \,\alpha_o \sqrt{2\xi}}{2rc - \alpha_o^2},\tag{20}$$

where c is given by (12). Optimal $R \mathcal{E}D$ investment in the steady state is then given by

$$\phi(0) = \sqrt{\frac{2\pi(0,m_1)}{\xi}}$$
(21)

and for $\gamma_I \in (\underline{\gamma}_I, \overline{\gamma}_I)$ optimal investment is discontinuous at e = 0 such that

$$\lim_{\epsilon \to 0+} \phi(-\epsilon) < \phi(0) < \lim_{\epsilon \to 0+} \phi(\epsilon).$$

Proposition 2 gives the upper and lower bounds for γ_I such that at e = 0 choosing investment, that is exactly covered by market profits, can be optimal. Whereas the lower bound $\underline{\gamma}_I$ does not depend on the bankruptcy parameter, the upper bound in (19) is influenced by γ_B . This is quite intuitive since the bankruptcy parameter only becomes relevant if the firm's liquidity becomes negative. Proposition 2 also implies that if e = 0 is a steady state then the optimal investment strategy $\phi(e)$ exhibits a jump at this value of the liquidity. Clearly, this jump is due to fact that, as soon as liquidity becomes negative, an increase of investment increases the bankruptcy risk and therefore the incentive to invest is lower compared to a situation where no such effect on bankruptcy risk exists. For γ_I in the interval (19) the optimal investment without consideration of the effect on bankruptcy risk is larger than the profit on the established market, whereas optimal investment taking into account the effect on bankruptcy risk is below market profit. In such a scenario, for positive initial liquidity the firm invests above the profit on the established market until liquidity has been depleted to zero and then reduces investment such that it equals the current profit. If γ_I is sufficiently large, then for small negative liquidity optimal investment, even under the consideration of its effect on bankruptcy risk, is larger than what can be internally financed by the profits on the established market. In this scenario the optimal investment strategy induces that the firm goes into debt even if it starts with a non-negative liquidity. This intuition is formalized in the following corollary, which follows directly from Proposition 2 together with Corollary 2.

Corollary 3. For $\gamma_I < \underline{\gamma}_I$, then under the optimal investment $\dot{e}(0) > 0$. For $\gamma_I > \overline{\gamma}_I$, we have $\dot{e}(0) < 0$.

It follows directly from Proposition 2 that, for a given value of $\gamma_I > \underline{\gamma}_I$, there is a positive threshold for the bankruptcy risk parameter γ_B , such that e = 0 is a steady state of the liquidity dynamics under optimal investment only if γ_B is above that threshold.

Corollary 4. For $\gamma_I > \underline{\gamma}_I$ there exists a unique threshold $\bar{\gamma}_B$ such that $e^* = 0$ is a stable steady state if and only if $\gamma_B \geq \bar{\gamma}_B$.

In the following section we will use this corollary to distinguish between scenarios where a firm with positive initial liquidity eventually accumulates debt or keeps a non-negative liquidity.

Before numerically exploring in the next section additional properties of the optimal investment policy and the resulting innovation rate and liquidity dynamics, we conclude this analytical section by briefly discussing the implications of the a variation of the dividend rate ν_1 in mode m_1 . In particular, we show in the following proposition that if the monopolist never enters the negative liquidity domain in mode m_1 , then it is optimal to delay all dividend payments till mode m_2 .

Proposition 3. Denote by $\tilde{\phi}(e)$ the optimal solution to the problem (5) under $\nu_1 = 0$. If $\dot{e} \ge 0$ at e = 0, i.e. $\alpha_o^2 \ge 2\xi \tilde{\phi}(0)^2$, then for any value of $e^{ini} \ge 0$ the maximal value for the firm owner under $\nu_1 = 0$ is larger or equal than the maximal value that can be obtained for any $\nu_1 \ge 0$.

The intuition for this result is that while investing the firm should keep as high a liquidity as possible in order to avoid going into debt. Paying out dividends during mode m_1 could either make the firm go into debt or restrict its possible innovation choice. Both of these effects are associated with costs for the firm and hence reduce the expected dividend stream. In case the firm optimally avoids to go into debt under $\nu_1 = 0$, there is no bankruptcy risk and therefore no costs associated with delaying the payout of dividends to mode m_2 . As discussed above, this is due to the fact that interest and discount rate coincide and that the firm has a positive income stream in mode m_2 . The condition that liquidity stays non-negative under the optimal investment is crucial for the claim of Proposition 3. If initial liquidity is positive but at some point becomes negative under the optimal investment strategy, it can no longer be claimed that in general $\nu_1 = 0$ is optimal. In such a scenario it might be profitable for the owner to receive dividends before a potential bankruptcy, which would stop all dividend flows. As mentioned above, studying the optimal (liquidity dependent) dividend policy is not the focus of our analysis. The main purpose of Proposition 3 is to provide some foundation for the fact that throughout the numerical analysis we will assume that $\nu_1 = 0$.

4 Numerical Analysis

It is challenging to get closed form solutions for the value function and investment when $\gamma_B > 0$. This is because of the non-linear form of the HJB equations, especially for e < 0. In order to analyze the effect of the bankruptcy threat on optimal investment, we need to numerically determine the value function of $V_1(e)$, which requires to approximate $V_2(e, \alpha_n)$ first. To achieve this goal, we resort to numerical methods. More specifically, we rely on a collocation method to calculate the approximate solution for $V_2(e, \alpha_n)$ for e < 0 and $V_1(e)$. Details of our approach, building on Vedenov and Miranda (2001) and Dawid et al. (2015), are provided in Appendix C. When applying the numerical method, we encounter two technical challenges. The first is that the collocation method operates on a finite state space, but in our model the state space for liquidity is infinite. The second challenge is that, the denominator term $dV_1(e)/de$ in the optimal control (8) could be close to 0, especially when the initial liquidity is very negative and the bankruptcy probability is very large. This would make the optimal control I explode and the numerical calculations difficult. In order to solve these two technical problems, we propose a transformation from the state space of liquidity e to a state space of z according to $z(e) = (1 + \exp(-\lambda e))^{-1} \in (0, 1)$ with $0 < \lambda < 1$. The new state space of the problem is the interval (0, 1), and therefore a bounded interval, which makes the application of the collocation method easier. Then $e(z) = \frac{-1}{\lambda} \ln\left(\frac{1}{z} - 1\right)$ and the calculations are carried out in the state space $(\alpha_n, z) \in [0, \alpha^u] \times [z_l, z_u]$ after innovation in mode m_2 , and in the state space $z \in [z_l, z_u]$ before innovation in mode m_1 , where $\alpha^u > \tilde{\alpha}$ is chosen sufficiently large, z_l is close to zero and z_u close to one. Note that after this transition, the dynamics read

$$\dot{z}(e) = \lambda z(e)(1 - z(e))\dot{e},$$

while \dot{e} is given by (2). The denominator of the optimal investment is

$$\frac{\mathrm{d}V_1\left(e(z)\right)}{\mathrm{d}z} = \frac{\mathrm{d}V_1\left(e(z)\right)/\mathrm{d}e}{\lambda z(1-z)}.$$

For strongly negative liquidity e, which corresponds to z(e) close to zero, both the nominator $dV_1(e)/de$ and denominator $\lambda z(1-z)$ after the transition are close to 0. The technical problem associated with a small value of the derivative of the value function can thereby be alleviated. In this section, we focus on the influence of the bankruptcy parameter γ_B and the firm's strength on the established market α_o on the firm's investment, the expected innovation time and the bankruptcy probabilities.

4.1 Parameter calibration

Our numerical analysis is based on a standard parameter setting shown in the following table. Based on this standard parameter setting we will analyze the effects of variations of several of these parameters, in particular γ_B , γ_I and α_o .

$\nu_1 = 0$	pre-innovation dividend rate	$\delta = 1.55$	adjustment speed for α_n to reach $\tilde{\alpha}_n$
$\nu_2 = 0.2$	post-innovation dividend rate	$\sigma = 0.1$	uncertainty in new market dynamics
$\alpha_o = 0.8$	size of the old market	r = 0.02	interest rate
$\bar{\alpha}_n = 0.6$	base size of the new market	$\gamma_B = 0.05$	bankruptcy parameter
$\tilde{\alpha}_n = 0.8$	expansion of new market	$\gamma_I = 0.1$	efficiency of innovation
$\eta = 0.5$	horizontal differentiation	$\xi = 0.025$	invesment costs
$\lambda = 0.5$	parameter for state-space		
	transformation		

 Table 1: Parameter values

Although this parameter setting is not based on a systematic empirical calibration for a specific setting, they have been chosen with clear theoretical and empirical foundations in mind. As mentioned above, our choice of $\nu_1 = 0$ is based on Proposition 3, whereas as discussed in Section 3.2.1 the choice of ν_2 does not affect any of our results.

The reason behind the choice of parameter values for α_o , $\bar{\alpha}_n$, $\tilde{\alpha}_n$ and η is the resulting demand elasticity. Empirical evidence indicates that the unitary elasticity is reasonable for many established consumption goods. For the established market without the influence by the new product, the chosen parameter values would yield the price elasticity before innovation as

$$-\left.\left(\frac{\mathrm{d}p_o}{\mathrm{d}q_o}\right)^{-1}\left(\frac{p_o}{q_o}\right)\right|_{q_o(m_1)} = 1,$$

and the price elasticity for the new product, in the long-run when $\alpha_n = \tilde{\alpha}_n$, to be equal to

$$-\left.\left(\frac{\partial p_n}{\partial q_n}\right)^{-1}\left(\frac{p_n}{q_n}\right)\right|_{q_o(\tilde{\alpha}_n,m_2);\ q_n(q_o(\tilde{\alpha}_n,m_2))} = 1.05.$$

The parameter values for σ and δ are chosen in a way that the expected duration in mode m_2 until the new product price reaches its peak $\bar{\alpha}_n + \tilde{\alpha}_n$ is approximately 2.5 years, which is consistent with empirical observations about the time till full development of the demand for a new product in industries like the car industry (Volpato and Stocchetti, 2008).

Parameters γ_B , γ_I and ξ are calibrated such that for the default set of parameter values, the average innovation time is 2 to 2.5 years, which is consistent with empirical data about the average length of innovation projects (Behrens et al., 2017).

In the following analysis, we first calculate the value function $V_2(e, \alpha_n)$ for mode m_2 . Using the estimated values $V_2(e, 0)$, we then numerically determine the (approximate) value function $V_1(e)$ in mode m_1 . This allows us then to analyze the influence of γ_B and α_o on the optimal investment, the liquidity dynamics, expected innovation time and bankruptcy probability.

4.2 Post-innovation

There is no more investment after the successful innovation, i.e., no control to be chosen by the firm. The value function $V_2(e, \alpha_n)$ in mode m_2 is shown in Figure 1. $V_2(e, \alpha_n)$ increases with both the market demand for new product α_n , and the liquidity e. When e < 0, Figure 1 also indicates the influence of the bankruptcy risk on the value in mode m_2 . Note that without bankruptcy risk (i.e. $\gamma_B = 0$) the value function $V_2(e, \alpha_n)$ is linear in e with a slope of 1, and has the same functional form for both positive and negative liquidity reserves. However, for $\gamma_B = 0.05$, which is the case depicted in Figure 1, $V_2(e, \alpha_n)$ is in the negative domain convex-convace with respect to e and clearly below the value that would emerge for $\gamma_B = 0$. This highlights that the bankruptcy risk decreases the value, when the liquidity reserves are negative, and that the size of the negative effect of the bankruptcy risk depends in a non-linear way on the liquidity.



Figure 1: Value function $V_2(e, \alpha_n)$.

4.3 Pre-innovation

We now turn to the analysis of optimal investment during the innovation phase in mode m_1 . First, we note for further reference that in our default parameter setting the optimal R&D investment level in the absence of financial constraints is $I^{nc} = 4.93$, and the minimal amount of liquidity at which this investment can be financed internally is given by $\tilde{e} = 7.2$. It's worth mentioning that Assumption 1 is satisfied for this parameter setting. Based on Proposition 1 and Corollary 3 it is clear that the optimal investment strategy and the induced dynamics depend crucially on the fact whether e = 0 is a steady state of the liquidity dynamics under optimal investment or not. Hence, we first characterize regions in the parameter space where e = 0 is a steady state. Since in these scenarios a firm with non-negative initial liquidity never goes into debt under optimal investment we refer to these cases as no debt scenarios. On the contrary we label situations where optimal investment implies that the firm should enter the negative domain of the liquidity as debt scenarios.

4.3.1 Debt vs. No Debt Scenarios

Corollary 4 implies that if all other parameters are given according to their default values we have a debt scenario for $\gamma_B < \bar{\gamma}_B = 0.0069$, whereas a no debt scenario arises for $\gamma_B \ge 0.0069$. With respect to our second key parameter, α_o , the effect of a parameter variation on the occurrence of



Figure 2: Influence of the bankruptcy risk parameter γ_B and the firm's strength in the old market α_o on liquidity 0 as a steady state, i.e., $e^* = 0$.

the no debt scenario is less clear cut, since both boundaries $\underline{\gamma}_I$ and $\bar{\gamma}_I$ in Proposition 2 depend in a highly non-linear way on α_o . In order to gain insights how increasing the size of the established market affects the occurrence of the no debt scenario and how this effect depends on the value of the bankruptcy parameter, we show in Figure 2 the influence of γ_B and α_o on the occurrence of the no debt scenario. Specifically, the shaded area shows the combination of γ_B and α_o such that $e^* = 0$ is a steady state for the standard parameter setting. In our analysis, we assume that $0.7 \leq \alpha_o \leq 2.8$ to make sure that the output quantities for both the old and the new products to be non-negative after innovation. The shaded area is bounded from above by $\alpha_o = 0.992$ and below by $\alpha_o = 0.7$. It can be clearly seen that for sufficiently large values of γ_B it is never optimal for the firm to go into debt, however for values of the bankruptcy parameter below approximately $\gamma_B = 0.012$ the firm avoids to go into debt only if the size of the established market is sufficiently large. Two effects, both pointing in the same direction, drive this result. First, due to the cannibalization effect the optimal innovation investment becomes lower if α_o grows, and, second, the profit on the established market grows with α_o and therefore the firm is able to internally finance larger investments.

4.3.2 Effect of the Bankruptcy Risk

We are now in a position to characterize the shape of the optimal investment strategy $\phi(e)$ and to explore how this optimal strategy changes if the bankruptcy risk parameter grows. The optimal strategies depicted in this and the following sections have all been calculated based on the numerically determined value functions of the problem, as described in Appendix C.

In Figure 3 we show the optimal investment strategies for our standard parameter setting and different values of γ_B . As noted above, for our standard parameter setting $e^* = 0$ is a steady state



Figure 3: Effect of the bankruptcy risk parameter γ_B on the optimal investment strategy $\phi(e)$ for debt scenarios (a) and no debt scenarios (b).

whenever $\gamma_B \geq \bar{\gamma}_B = 0.0069$. In panel (a) we show the optimal investment strategy for values of γ_B below this threshold, i.e. debt scenarios, whereas in panel (b) the optimal investment strategy in no debt scenarios are depicted. The value functions $V_1(e)$ corresponding to these optimal investment strategies can be found in Appendix B.

Figure 3 illustrates our theoretical result that the optimal investment strategy exhibits a downward jump at zero liquidity in the no debt scenario. Furthermore, it shows that investment is continuous at e = 0 in the debt scenario. Intuitively, one might expect that even in the debt scenario investment changes discontinuously when the bankruptcy risk kicks in at e = 0, but under the optimal strategy the firm at a time t, when e(t) is still positive, already foresees that liquidity will turn negative in the future and therefore already takes into account that current investment will influence future bankruptcy risk.

A main insight from Figure 3 is that the optimal investment strategy $\phi(e)$ is U-shaped when $\gamma_B > 0$ for both the debt and the no debt scenarios. When the initial liquidity is positive and large, the optimal investment is not influenced by the bankruptcy threat and the optimal investment is equal to that with no bankruptcy risks, i.e., I^{nc} , as given in equation (9). When the initial liquidity is positive but close to 0, we can observe that the firm's optimal investment decreases, but for different reasons in the two scenarios. For the debt scenario the firm has an incentive to delay the point in time when liquidity becomes negative and thus the bankruptcy threat arises, and the firm does this by reducing its investment. Moreover, Figure 3a shows that the larger γ_B is, the steeper is the decrease of $\phi(e)$ as liquidity approaches zero. For the no debt scenario $\phi(e)$ decreases as e approaches zero because the firm anticipates the downward jump of investment once e = 0



Figure 4: Effect of bankruptcy risk parameter γ_B on the liquidity dynamics $\dot{e}(e)$ for debt scenarios (a) and no debt scenarios (b).

is reached, and in light of the convex investment costs smoothes this investment path by reducing investment already before the zero liquidity steady state is reached. Since in the no debt scenario liquidity never becomes negative for $e^{ini} \ge 0$ it is evident that the branch of $\phi(e)$ for $e \ge 0$ does not change if γ_B is varied. Furthermore, considering the significantly negative liquidity levels, the firm invests more the larger the negative liquidity is. The intuition for this behavior is that if the firm is deeply in debt, then there is a large probability that the firm will go bankrupt if it does not innovate quickly, thereby generating higher profits. The amount of debt the firm holds at the time of bankruptcy does not influence owners' value (in any case it is zero due to limited liability of owners) and therefore it is optimal to invest heavily in order to try to speed up innovation.

Figure 4 shows the values of \dot{e} under optimal investment for the debt and no debt scenarios. There is always a positive steady state, e^* , which is unstable. In the debt scenario this is the only steady state and the liquidity decreases for any $e < e^*$ (see Figure 4a). Hence, the liquidity diverges to $-\infty$ in the long run as long as the firm is in mode m_1 , i.e. neither has innovated nor gone bankrupt. For larger values of the bankruptcy parameter γ_B , i.e. for the no debt scenario, two additional steady states emerge (see 4b). The locally stable steady state at e = 0 and an unstable negative steady state constituting the lower boundary of the basin of attraction of e = 0. Hence, if the initial liquidity of the firm is negative, but the amount of debt is small, then it is optimal for the firm to choose a sufficiently small R&D investment such that its debt is reduced to zero over time.

Figure 5 illustrates these findings by showing the dynamics of liquidity and optimal investment for an initial liquidity of e(0) = 1. The figure highlights that even in the debt scenario (i.e., for



Figure 5: Effect of the bankruptcy risk parameter γ_B on the dynamics of liquidity and optimal investment for an initial liquidity of e(0) = 1.

 $\gamma_B = 0.001, 0.005$) the firm accumulates debt rather slowly, and once entering the negative liquidity domain the firm chooses an investment level that is almost constant over time and substantially below the unconstrained optimal level $I^{nc} = 4.93$. For the case where the bankruptcy risk parameter is sufficiently large to induce the no debt scenario (i.e., $\gamma_B = 0.01$), the downward jump in investment, once liquidity hits zero, can be clearly seen in Figure 5b. As is illustrated in panel (a) of the figure, this downward jump indeed implies that liquidity stays constant at the steady state level of $e^* = 0$. Overall, figure 5b also illustrates that an increase of γ_B has a negative impact on the firm's level of investment throughout time, where this effect becomes more pronounced as liquidity gets close to zero.

To conclude our analysis for the effect of an increase in the bankruptcy risk parameter we now consider the impact of γ_B on the expected innovation time and the actual ex-ante expected probability for the firm to go bankrupt. Restricting attention to scenarios with a non-negative initial firm liquidity, it follows directly from our previous analysis that, if $\gamma_B \geq \bar{\gamma}_B$, then we are in a no debt scenario, where the bankruptcy probability is zero and the exepcted innovation time does not depend on the actual value of γ_B . The latter observation is due to the fact that in the no debt scenario the level of investment for non-negative liquidity is not influenced by γ_B . This in confirmed in Figure 6a, which also shows that as long as we remain in the debt scenario the firm's expected innovation time increases with γ_B , due to the negative effect of this parameter on investment.

With respect to the bankruptcy probability an inverse U-shaped relationship with γ_B emerges (see Figure 6b). As long as γ_B is small, the direct effect of an increase of this parameter dominates,



Figure 6: Effect of the bankruptcy risk parameter γ_B on the expected innovation time $E(\tau)$ and the bankruptcy probability for a given the initial liquidity e(0) = 0.1.

thereby leading to a higher bankruptcy probability. However, as discussed above, such an increase induces a reduction of firm investment and therefore a slower build-up of debt, which reduces the bankruptcy probability. As γ_B grows this effect starts to dominate and the bankruptcy probability decreases with γ_B . As γ_B crosses the threshold $\bar{\gamma}_B$, and we enter the no debt scenario, the negative effect on investment is so strong that the firm never accumulates any debt and hence the bankruptcy probability is zero.

4.4 Effect of α_o

The size of the established market α_o determines the quantity sold by the firm on the established market and also the associated profit. In particular, this parameter therefore influences the firm's ability to finance innovation expenditures internally. Understanding how optimal innovation investments depend on α_o allows us to gain insight on the question under which circumstances larger sales and higher profits on the established market lead to higher R&D investments and faster innovation.

Figure 7, shows the optimal investment strategy as a function of liquidity for $\gamma_B = 0.005$ and different values of α_o . Whereas in panel (a) the entire relevant part of the state space is shown, and in panel (b) we zoom in to liquidity values close to zero. First, it should be noted that for the default value $\alpha_o = 0.8$ we are in a debt scenario because $\gamma_B = 0.005 < 0.0069 = \bar{\gamma}_B$. However, increasing the market size of the established market to $\alpha_o = 0.85$ lowers the threshold to $\bar{\gamma}_B = 0.0048$ such that a no debt scenario arises for $\gamma_B = 0.005$. Hence, the investment strategy is continuous at zero liquidity for $\alpha_o = 0.7, 0.8$, but exhibits a jump for $\alpha_o = 0.85, 0.9$.

Concerning the effect of α_o on the level of investment, it becomes clear that if liquidity is



Figure 7: Effect of the established market size α_o on the optimal investment I(e) for the bankruptcy risk parameter $\gamma_B = 0.005$. Panel (a) shows the entire relevant part of the state space whereas panel (b) zooms into the neighborhood of e = 0.

strongly positive or strongly negative the optimal R&D investment is smaller the larger the established market is. For large liquidity, where the consideration of financial constraints hardly influence investment, this is due to a standard cannibalization effect. The larger quantity of the established product that the firm sells, the stronger negative implication the drop in the price of the established product has, which is triggered by product innovation. Hence, large sales on the established market reduce the incentive to invest in the development of the new product. This result is consistent with Dawid et al. (2015), which shows that, in the absence of financial constraints, a larger production capacity on the established market induces lower investment in new product development. The cannibalization effect is also present in the case of negative liquidity, however here it is complemented by a second effect. If the firm has negative liquidity then an increase in investment instantaneously increases the firm's bankruptcy rate. The larger the established market is the larger is the loss in expected future dividends induced by bankruptcy. Hence, an increase in α_o has a negative effect on R&D expenditures. We refer to this effect as the *bankruptcy loss effect*.

A third effect of an increase of α_o is that it pushes up the limit of the firm's expenditure that can be financed internally and therefore reduces the amount of debt needed for a certain investment size. This effect, which we label as the *financing effect*, increases the optimal size of R&D investment. A close look at the optimal investment I(e) around e = 0 reveals that, this effect may dominate canibalization in this part of the state space. In particular, in the no debt scenario, where $e^* = 0$ is a stable steady state ($\alpha_o = 0.85, 0.9$ in Figure 7b), a larger value of α_o induces higher R&D investments. This is quite obvious in the steady state $e^* = 0$, where investment is given by $\alpha_o^2/4$, and it also holds in an interval around zero liquidity. However, in a debt scenario ($\alpha_o = 0.7, 0.8$



Figure 8: Effect of the established market size α_o on the expected innovation time $E[\tau]$ under bankruptcy risk parameters $\gamma_B \in \{0.05, 0.005\}$ and initial liquidity $e(0) \in \{0, 1\}$.

in Figure 7b), the cannibalization and bankruptcy loss effects dominate, and a larger size of the established market induces lower product innovation investments also around e = 0. Intuitively, the main difference to the no debt scenario is that the bankruptcy loss effect is present here, whereas in the no debt scenario this effect is absent for any non-negative liquidity and negligible for small negative liquidity, because under the optimal investment the negative liquidity quickly disappears.

The interplay of these three effects determines how an increase in the size of the established market α_o influences the firm's expected innovation time. Figure 8 depicts the expected innovation time as a function of α_o under four cases with different values of the bankruptcy risk parameter γ_B and different initial liquidity e(0). Focusing first on the case where the firm's initial liquidity is zero, it follows directly from our analysis above that for $\gamma_B = 0.05$ the state $e^* = 0$ is a stable steady state for all values of $\alpha_o \in [0.7, 0.992]$, where for $\alpha_o > 0.992$ we have $\bar{\gamma}_I > \gamma_I = 0.1$. Hence, for $\alpha_o \in [0.7, 0.992]$ R&D investment is constant over time and equal to $\phi(e(t)) = \phi(0) = \alpha_o/\sqrt{2\xi} \ \forall t \ge$ 0, see (21). Since this expression increases with α_o , the expected innovation time decreases with α_o , as can be seen in the dashed grey line in Figure 8. For $\alpha_o > 0.992$ the state e = 0 is no longer a steady state, but starting from e = 0 liquidity grows over time in mode m_1 , where investment is constant at $\phi(e(t)) = I^{nc} \ \forall t \ge 0$. Due to the cannibalization effect I^{nc} decreases with α_o , so on this interval the expected innovation time grows when the size of the established market becomes larger. If we assume a lower value of the bankruptcy risk parameter ($\gamma_B = 0.005$), then we get qualitatively the same picture as above as long as $\alpha_o \ge 0.845$, which is the threshold where $\bar{\gamma}_I = \gamma_I$ for this value of γ_B . For $\alpha_o \in [0.845, 0.992]$ zero liquidity is a stable steady state, whereas for $\alpha_o \geq 0.992$ liquidity grows and investment is constant at the unconstrained optimum. However, for $\alpha_o < 0.845$ we are in the debt scenario and the firm accumulates debt over time. Consistent with the intuition developed above, in this interval the expected innovation time increases with α_o since the combination of the cannibalization and bankruptcy loss effects reduces overall R&D investment. Hence, for low values of the bankruptcy risk parameter the relationship between the size of the established market and expected innovation time is characterized by a highly non-monotone tilted z-shaped pattern (the solid grey line in Figure 8). If firm's initial liquidity is sufficiently large (e(0) = 1), then the probability that the firm innovates before liquidity gets close to zero is so large that for most parts of the considered range of α_o values it does not matter how large the bankruptcy risk parameter is (compare the dashed and solid red lines in Figure 8). Therefore, the cannibalization effect dominates and the expected innovation time grows with α_{o} . Only for very low values of α_o around 0.7 the financing effect starts to have a sizeable impact. In this region the expected innovation time is clearly larger under a higher bankruptcy risk parameter. The intuition for this observation is that in light of such a small size of the established market, a large fraction of the firm's investment has to be financed from the existing stock of liquidity rather than from instantaneous profit and therefore liquidity decreases fast. Hence, the effects driving incentives around e = 0 become relevant with a higher probability and also with a lower associated discount factor.

5 Conclusions

This paper is one of the first to explicitly incorporate the bankruptcy risk associated with prolonged investments in uncertain innovation projects in a dynamic market model. We analyze the optimal product innovation investment strategy of a monopolistic firm facing technological and demand uncertainty as well as financial constraints. The firm can finance investments externally, however faces a bankruptcy risk that grows with the size of the firm's debt. We analytically characterize scenarios in which it is optimal for the monopolist to refrain from the accumulation of any debt, thereby treating non-negativity of liquidity as a binding constraint, and scenarios where accumulating a positive amount of debt is optimal. Combining these insights with an extensive numerical analysis we show that the optimal investment strategy is U-shaped as a function of the firm's liquidity, such that investments are lowest around zero liquidity. We argue that this shape is driven by the interplay of three effects, the well-known cannibalization effect, the bankruptcy loss effect and the financing effect. Due to the induced adjustment of firm's investment strategy, an increase of the bankruptcy risk parameter has a non-monotone inverse U-shape effect on the actual bankruptcy probability of the firm. Finally, we show that there is a highly non-monotone relationship between the profitability of the established market for the firm and the expected innovation time under optimal investment. In that respect our paper contributes from a dynamic theoretical perspective to the long lasting debate in the economics of innovation about the relationship between the profitability of a firm and its innovation incentives. In particular, we show that (assuming low initial liquidity and a low bankruptcy risk parameter) for very low and very high levels of profitability of the established market, increasing the size of the established market delays innovation, whereas for intermediate levels innovation is sped up if profits on the established market go up. Hence, our analysis delivers several testable empirical implications, which we plan to address in future work using firm-level data.

The framework developed in this paper can be extended in several directions, thereby allowing to address a number of important issues that were put aside in our analysis. First and foremost, we have considered a monopolistic firm and therefore have abstracted from the effect of strategic competition. On the one hand, competition should generate incentives to preempt the competitor and therefore increases the willingness of firms to take on debt. On the other hand, particularly in markets without strong patent protection, there exists risk that even after winning the innovation race the competitor might catch-up, and thereby eliminate pioneering profits. Such risk could make the accumulation of a large debt prior to innovation substantially more risky compared to the monopoly case. Addressing these issues in an oligopolistic framework of a multi-mode differential game is a natural extension to our work here. A second restriction of our analysis is that we have not fully characterized the combination of optimal investment and optimal dividend policy. Although we are confident that our qualitative insights about optimal R&D investment fully carry over to a setting where the dividend strategy is fully state-dependent, potentially singular and intertemporally optimal, determining such an optimal policy gives rise to a highly challenging singular control problem, whose solution would be an important technical contribution. Finally, in this paper we have assumed that the firm has access to credit at a given interest rate even if it already has accumulated substantial debt. Alternatively, one could assume that the interest rate grows with the level of debt, thus there is a maximal level of debt under which the firm still can get additional credit, or both. Whereas the addition of an upper bound for debt should hardly influence our results, endogenizing the interest rate, either assuming a competitive credit market or

a potential debtor with some market power, would enrich the analysis and allow additional insights on the robustness of the U-shaped investment pattern identified here.

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A Proofs

Proof of Lemma 1. Taking the derivative of both sides of HJB (7) with respect to I yields the following equation

$$\gamma_I(V_2(e,0) - V_1(e)) - \xi I \frac{\mathrm{d}V_1(e)}{\mathrm{d}e} = 0$$

which leads to

$$I = \frac{\gamma_I}{\xi} \frac{V_2(e,0) - V_1(e)}{\mathrm{d}V_1(e)/\mathrm{d}e}$$

Taking into account that $\frac{dV_1(e)}{de} > 0$ for all e shows that also the second order condition is satisfied. Furthermore, it follows from $\pi(\alpha_n, m_2) \ge \pi(0, m_1) \forall \alpha_n \ge 0$ with strict inequality for some $\alpha_n > 0$ that $V_2(e, 0) > V_1(e)$ for all e and therefore $\phi(e) > 0$.

Assuming that $e^{ini} \ge 0$ is sufficiently large such that $e(t) \ge 0$ for all t and the firm never faces a positive bankruptcy probability. In such a case the HJB (6) in mode 2 can be rewritten as

$$rV_{2}(e,\alpha_{n}) = \nu_{2}e + \delta(\tilde{\alpha}_{n} - \alpha_{n})\frac{\partial V_{2}(e,\alpha_{n})}{\partial \alpha_{n}} + \frac{\sigma^{2}\alpha_{n}^{2}}{2}\frac{\partial^{2}V_{2}(e,\alpha_{n})}{\partial \alpha_{n}^{2}} + \frac{\partial V_{2}(e,\alpha_{n})}{\partial e}\left(\frac{(\bar{\alpha}_{n} + \alpha_{n})^{2} + \alpha_{o}^{2} - 2\eta(\bar{\alpha}_{n} + \alpha_{n})\alpha_{o}}{4(1 - \eta^{2})} - \nu_{2}e + re\right).$$
(22)

Assume $V_2(e, \alpha_n)$ takes the form of

$$V_2(e,\alpha_n) = a_2\alpha_n^2 + a_1\alpha_n + be + c,$$
(23)

with the unknown coefficients a_1 , a_2 , b and c that need to be determined. Substituting (23) into (22) and comparing the coefficients of 1, α_n , α_n^2 and e on both sides of the equation yields the values of a_1 , a_2 , b and c and thus leads to the expression (11). Among the two solutions of this system of equations only the one with $V_1(0) < V_2(0,0)$ is relevant. A similar method can also be applied in mode m_1 to solve the HJB equation of (7), i.e.,

$$rV_1(e) = \nu_1 e + \frac{\mathrm{d}V_1(e)}{\mathrm{d}e} \left(\frac{\alpha_o^2}{4} + \frac{\gamma_I^2}{2\xi} \left(\frac{V_2(e,0) - V_1(e)}{\mathrm{d}V_1(e)/\mathrm{d}e}\right)^2 - \nu_1 e + re\right).$$

Assuming a value function of the form $V_1(e) = e + \tilde{c}$ yields after the comparison of coefficients the expression (10). Substituting equations (10) and (11) into (8) yields that the optimal investment without bankruptcy risk is equal to (9). Note that the constant term \tilde{c} in the value function $V_1(e)$

is the smaller root of a quadratic equation. It follows from $V_1(e) < V_2(e, 0)$ that the smaller root has to be considered.

As a last step we verify that for any $e > \tilde{e}$ indeed e(t) > 0 holds under the optimal investment strategy. Taking into account that liquidity is positive we have in mode m_2

$$\dot{e} = (r - \nu_2)e + \pi(\alpha_n, m_2).$$

Since $\pi(\alpha_n, m_2) > 0$ it follows that $\dot{e} > 0$ for sufficiently small positive values of e and therefore liquidity stays positive if it is positive at the time $t = \tau$ of the innovation. Considering m_1 we have under the optimal investment

$$\dot{e} = (r - \nu_1)e + \frac{\alpha_o^2}{4} - \frac{\xi}{2}(I^{nc})^2, \qquad (24)$$

which is non-negative due to our assumptions that $e \geq \tilde{e}$. Hence, e(t) > 0 holds also in mode m_1 . \Box

Proof of Corollary 1. First it should be noted that $4rc - \alpha_o^2 > 0$, which can be verified by inserting (12) for c. Taking this into account, we have

$$\frac{\partial I^{nc}}{\partial \gamma_I} = -\frac{r}{\gamma_I^2} \underbrace{\left(\frac{2r/\gamma_I}{2\sqrt{\left(\frac{r}{\gamma_I}\right)^2 + \frac{4rc - \alpha_o^2}{2\xi}}} - 1\right)}_{<0} > 0.$$

The monotonicity of I^{nc} with respect to ξ follows directly from $4rc - \alpha_o^2 > 0$ and considering the effect of an increase of $\tilde{\alpha}$ it follows directly from (12) that under the assumption $r + 2\delta > \sigma^2$ we have $\frac{\partial c}{\partial \tilde{\alpha}_n} > 0$, which implies that I^{nc} increases with $\tilde{\alpha}_n$. Finally, considering the effect of a change of α_o we obtain

$$\frac{\partial I^{nc}}{\partial \alpha_o} = \frac{1}{2\sqrt{\left(\frac{r}{\gamma_I}\right)^2 + \frac{4rc - \alpha_o^2}{2\xi}}} \frac{1}{2\xi} \left(4r\frac{\partial c}{\partial \alpha_o} - 2\alpha_o\right) < 0.$$

The sign of the term in the bracket is negative because

$$4r\frac{\partial c}{\partial \alpha_o} - 2\alpha_o = -\frac{4\eta\delta\tilde{\alpha}_n}{2(r+\delta)(1-\eta^2)} + \frac{2\alpha_o}{1-\eta^2} - \frac{2\eta\bar{\alpha}_n}{1-\eta^2} - 2\alpha_o$$
$$= -\frac{4\eta\delta\tilde{\alpha}_n}{2(r+\delta)(1-\eta^2)} + \frac{2\eta}{1-\eta^2}\left(\eta\alpha_o - \bar{\alpha}_n\right) < 0,$$

due to our assumption that $\bar{\alpha}_n > \eta \alpha_o$.

Proof of Corollary 2. For $2rc = \alpha_o$ we obtain

$$\frac{\xi(I^{nc})^2}{2} = \frac{\alpha_o^2}{4} + \frac{\xi r}{\gamma_I} \left(\frac{r}{\gamma_I} - \sqrt{\frac{r^2}{\gamma_I^2} + \frac{\alpha_o^2}{2\xi}}\right) < \frac{\alpha_o^2}{4}$$

and therefore $\frac{\alpha_o^2}{4} > \frac{\xi(I^{nc})^2}{2}$ holds for all $\gamma_I \ge 0$. The unconstrained investment I^{nc} is an increasing function of c and therefore $\frac{\alpha_o^2}{4} > \frac{\xi(I^{nc})^2}{2}$ holds for all $\gamma_I \ge 0$ whenever $2rc \le \alpha_o$. If $2rc > \alpha_o$ the value of of γ_I follows directly from inserting (9) into the inequality $\frac{\alpha_o^2}{4} \ge \frac{\xi(I^{nc})^2}{2}$ and solving for γ_I .

For $\gamma_I \leq \underline{\gamma}_I$ optimality of I^{nc} for all $e \geq 0$ follows directly from Proposition 1. The resulting liquidity given by (24) is positive for all $e \geq 0$ if $r - \nu_1 > 0$. For $r < \nu_1$ the equation $\dot{e} = 0$ has a unique positive solution $e^* = \frac{\alpha_o^2 - 2\xi(I^{nc})^2}{4(\nu_1 - r)}$. If $\gamma_I > \underline{\gamma}_I$ then for e = 0 we have $\dot{e} = \frac{\alpha_o^2}{4} - \frac{\xi(I^{nc})^2}{2} < 0$. Hence, either the firm chooses $\phi(e) < I^{nc}$ on some interval $(-\epsilon, 0]$ or for $e^{ini} = 0$ we have e(t) < 0for all $t \geq 0$, which implies that there is a positive probability that the firm goes bankrupt before it moves to mode m_2 .

Proof of Proposition 2. First we note that it follows from Corollary 2 that e = 0 is not a steady state for $\gamma_I < \underline{\gamma}_I$ and that for $\gamma_I \geq \underline{\gamma}_I$ we have $\dot{e}(0) \leq 0$, where $\dot{e}(e)$ denotes the value of \dot{e} at liquidity level e. Furthermore, we have $\lim_{\gamma_I \to 0} \phi(e) = 0$ for all e. Denoting the limit from below of optimal investment at e = 0 by $\phi(0)^- = \lim_{\epsilon \to 0} \phi(-\epsilon)$, this implies that $\phi(0)^- < \sqrt{\frac{2\pi(0,m_1)}{\xi}}$ and therefore $\lim_{\epsilon \to 0} \dot{e}(-\epsilon) > 0$ for sufficiently small values of γ_I .

Next we show that there is a single value of γ_I for which under the optimal investment strategy we can have $\phi(0)^- = \phi(0) = \sqrt{\frac{2\pi(0,m_1)}{\xi}}$ and that this value is given by $\gamma_I = \bar{\gamma}_I$. Using (21) and the fact the value functions in both modes have to be continuous we conclude that in such a scenario we must have

$$\frac{\gamma_I(V_2(0,0) - V_1(0))}{\xi dV_1(0)/de^-} = \sqrt{\frac{2\pi(0,m_1)}{\xi}}.$$
(25)

In order to determine $dV_1(0)/de^-$ we take the left derivative of the HJB equation at e = 0 to obtain

$$\gamma_I \sqrt{\frac{2\pi(0,m_1)}{\xi}} \left(\frac{\partial V_2(0,0)}{\partial e^-} - \frac{\mathrm{d}V_1(0)}{\mathrm{d}e^-}\right) + \gamma_B V_1(0) + \underbrace{\left(\pi(0,m_1) - \frac{\xi(\phi(0)^-)^2}{2}\right)}_{=0} \frac{\mathrm{d}^2 V_1(0)}{\mathrm{d}e^2} = 0,$$

Since $V_2(e,0)$ is smooth at e=0, it holds that $\partial V_2(0,0)/\partial e=1$ and we can get

$$\frac{\mathrm{d}V_1(0)}{\mathrm{d}e^-} = \frac{\gamma_I \sqrt{\frac{2\pi(0,m_1)}{\xi}} + \gamma_B V_1(0)}{\gamma_I \sqrt{2\pi(0,m_1)/\xi}}.$$
(26)

Furthermore, (16) yields that the value function in mode m_1 at the steady state $e^* = 0$ equals to

$$V_1(0) = \frac{\gamma_I V_2(0,0) \sqrt{2\pi(0,m_1)/\xi}}{r + \gamma_I \sqrt{2\pi(0,m_1)/\xi}}.$$
(27)

Inserting this into (25) yields

$$\frac{r\gamma_I V_2(0,0)}{\xi \left(r + \gamma_I \sqrt{2\pi (0,m_1)/\xi}\right) \frac{\mathrm{d}V_1(0)}{\mathrm{d}e^-}} = \sqrt{\frac{2\pi (0,m_1)}{\xi}}$$

and using (26) we obtain

$$\frac{r\gamma_I V_2(0,0)}{\xi \left(r + \gamma_I \sqrt{2\pi(0,m_1)/\xi} + \gamma_B V_2(0,0)\right)} = \sqrt{\frac{2\pi(0,m_1)}{\xi}}.$$

Solving for γ_I yields a unique solution, which is given by (20). Hence, $\lim_{e\to 0^-} \phi(e) = \phi(0) = \sqrt{\frac{2\pi(0,m_1)}{\xi}}$ can only hold if $\gamma_I = \bar{\gamma}_I$.

Using this inight we can now show that we have $\phi(0)^- > \sqrt{\frac{2\pi(0,m_1)}{\xi}}$ if and only if $\gamma_I > \bar{\gamma}_I$. We first show that this inequality holds for a sufficiently large value of γ_I . Consider an arbitrary fixed value of $\gamma_B = \tilde{\gamma}_B$. In light of Assumption 1 we have $\lim_{\gamma_I \to \infty} I^{NC} > \sqrt{\frac{2\pi(0,m_1)}{\xi}}$ and therefore for a sufficiently large $\gamma_I > \bar{\gamma}_I$ the inequality $I^{NC} > \sqrt{\frac{2\pi(0,m_1)}{\xi}}$ holds. Note that both sides in this inequality are independent from γ_B . We now show by contradiction that for this value of γ_I we must have $\phi(0)^- > \sqrt{\frac{2\pi(0,m_1)}{\xi}}$. To do this, we assume that $\phi(0)^- \le \sqrt{\frac{2\pi(0,m_1)}{\xi}}$. Since $\gamma_I \neq \bar{\gamma}_I$ we know that this weak inequality cannot hold as equality and therefore we must have that $\phi(0)^- < \sqrt{\frac{2\pi(0,m_1)}{\xi}}$. Keeping γ_I fixed, it is straight forward to see that for any e < 0 we have $\lim_{\gamma_B \to 0} \phi(e) = I^{NC} > \sqrt{\frac{2\pi(0,m_1)}{\xi}}$. Therefore, for sufficiently small $\gamma_B > 0$ we must have $\phi(0)^- > \sqrt{\frac{2\pi(0,m_1)}{\xi}}$, whereas by assumption we have $\phi(0)^- < \sqrt{\frac{2\pi(0,m_1)}{\xi}}$ for $\gamma_B = \tilde{\gamma}_B$. Since $\phi(e)$ changes continuously with γ_B , this implies that there must exist a value $\hat{\gamma}_B \in (0, \tilde{\gamma}_B)$ such that $\phi(0)^- = \sqrt{\frac{2\pi(0,m_1)}{\xi}}$ for $\gamma_B = \hat{\gamma}_B$. According to our arguments above, this can only hold if $\gamma_I = \bar{\gamma}(\hat{\gamma}_B)$, where for expositional reasons we now write $\bar{\gamma}_I$ explicitly as a function of γ_B . The threshold $\bar{\gamma}_I(\gamma_B)$ is an increasing function of γ_B , and therefore we have

$$\gamma_I > \bar{\gamma}_I(\tilde{\gamma}_B) > \bar{\gamma}_I(\hat{\gamma}_B),$$

which contradicts $\gamma_I = \bar{\gamma}(\hat{\gamma}_B)$. Hence, we have shown that $\phi(0)^- > \sqrt{\frac{2\pi(0,m_1)}{\xi}}$ has to hold for a sufficiently large γ_I . Summarizing, we have shown that $\phi(0)^- < (>)\sqrt{\frac{2\pi(0,m_1)}{\xi}}$ has to hold for sufficiently small (large) γ_I and that $\phi(0)^- = \sqrt{\frac{2\pi(0,m_1)}{\xi}}$ can only hold if $\gamma_I = \bar{\gamma}_I$. Taking into account the continuity of $\phi(e)$ with respect to γ_I this implies that

$$< \sqrt{\frac{2\pi(0,m_1)}{\xi}} \qquad \gamma_I < \bar{\gamma}_I$$

$$\phi(0)^- = \sqrt{\frac{2\pi(0,m_1)}{\xi}} \qquad \gamma_I = \bar{\gamma}_I$$

$$> \sqrt{\frac{2\pi(0,m_1)}{\xi}} \qquad \gamma_I > \bar{\gamma}_I.$$
(28)

Using the notation $\phi(0)^+ = \lim_{\epsilon \to 0} \phi(\epsilon)$ analogous arguments show that

We are now in a position to show that for $\gamma_I \in [\underline{\gamma}_I, \overline{\gamma}_I]$ the optimal investment at e = 0 is given by $\phi(0) = \sqrt{\frac{2\pi(0,m_1)}{\xi}}$. Since the value function $V_1(e)$ in general can have a kink at e = 0, we are searching for a viscosity solution to the HJB equation, see Bardi and Capuzzo-Dolcetta (2008). Therefore, we have to show that for a continuous value function satisfying the HJB equation (7) on $e \neq 0$ and (27) for e = 0 the first order condition (16)

$$\phi(0) = \sqrt{\frac{2\pi(0, m_1)}{\xi}} = \frac{\gamma_I(V_2(0, 0) - V_1(0))}{\xi\kappa}$$
(30)

holds for some $\kappa \in [dV_1(0)/de^+, dV_1(0)/de^-,]$ where $dV_1(0)/de^- = \lim_{\epsilon \to 0} dV_1(-\epsilon)/de$ is the one sided derivative from below and $dV_1(0)/de^+$ that from above. We know that $dV_1(0)/de^+ \leq dV_1(0)/de^-$, which follows from $\phi(0)^- \leq \phi(0)^+$ and

$$\phi(0)^{-} = \frac{\gamma_{I}(V_{2}(0,0) - V_{1}(0))}{\xi dV_{1}(0)/de^{-}}, \quad \phi(0)^{+} = \frac{\gamma_{I}(V_{2}(0,0) - V_{1}(0))}{\xi dV_{1}(0)/de^{+}}.$$

Such a value of κ exists if and only if $\sqrt{\frac{2\pi(0,m_1)}{\xi}} \in [\phi(0)^-, \phi(0)^+]$. Taking into account (28) and (29) this is true if and only if $\gamma_I \in [\underline{\gamma}_I, \overline{\gamma}_I]$.

Proof of Proposition 3. For $\gamma_I \leq \underline{\gamma_I}$ liquidity stays positive for any $e^{ini} \geq 0$ under the unconstrained optimal investment level I^{nc} and $\nu_1 = 0$. Hence in this case $\tilde{\phi}(e) = I^{nc}$ and the firm does not have any bankruptcy risk. Accordingly the value function is equal to that of the problem without financial constraints (10) and therefore $V_{1,0}(e) \geq V_{1,\nu_1}(e)$ for all $e \geq 0$, where $V_{1,\nu_1}(e)$ denotes the value function of problem (5) under the dividend rate ν_1 . For $\gamma_I > \underline{\gamma_I}$ there is no positive steady state for any value of ν_1 and therefore we always have $\dot{e} \leq 0$ at e = 0. Assume that for the optimal investment strategy $\phi_0(e)$ under $\nu_1 = 0$ we have $\alpha_o^2/4 = \xi\phi_0(0)^2/2$ and therefore $\dot{e} = 0$ at e = 0. We first show that this implies $\dot{e} = 0$ at e = 0 for an optimal strategy ϕ_{ν_1} under any dividend rate $\nu_1 \geq 0$. Assume that there would be a $\tilde{\nu}_1 > 0$ such that $\dot{e} < 0$ at e = 0 under strategy $\phi_{\tilde{\nu}_1}$. Since the value of ν_1 has no impact on the payoff stream for $e \in (-\infty, 0]$ and under $\phi_{\tilde{\nu}_1}$ and $e^{ini} = 0$ liquidity would be negative for all t, it follows from the optimality of $\phi_0(e)$ that $V_{1,0}(0) \geq V_{1,\tilde{\nu}_1}(0)$. Hence choosing investment of $\phi_0(0)$ at e = 0 is also an optimal strategy ϕ_0 , then for any $\nu_1 \geq 0$ there exists an optimal strategy ϕ_{ν_1} such that $e(t) \geq 0$, $\forall t$ holds for any $e^{ini} \geq 0$.

Now consider such an optimal strategy ϕ_{ν_1} for some ν_1 . Consider an arbitrary $e^{ini} \geq 0$ and denote by $\tilde{e}(t)$ the liquidity trajectory under the optimal policy and by $\tilde{D}(t) = \nu_1 \tilde{e}(t) \geq 0$ the dividend stream and by $\tilde{\Phi}(t) = \phi_{\nu_1}(e(t))$ the investment stream. Based on our considerations above, we have $\tilde{e}(t) \geq 0$ for all t. Now consider an alternative dividend and investment trajectory of the form $\hat{D}(t) = 0$ and $\hat{\Phi}(t) = \tilde{\Phi}(t)$ for all t. The corresponding expected values of the total dividend (in both modes) are denoted by \tilde{J} and \hat{J} . Then showing that $\hat{J} \geq \tilde{J}$ proves the claim of Proposition 3.

To show that $\hat{J} \geq \tilde{J}$ holds we first observe that both trajectories give rise to exactly the same trajectory of innovation rates, which implies that under both trajectories the distribution of the innovation time τ is identical. Hence, for establishing $\hat{J} \geq \tilde{J}$ it is sufficient to show that $\hat{J}(\tilde{\tau}) \geq \tilde{J}(\tilde{\tau})$ holds for any realisation $\tilde{\tau}$ of the stochastic innovation time. Here $\hat{J}(\tilde{\tau})$ and $\tilde{J}(\tilde{\tau})$ denote the values conditional on the innovation time, given by

$$\begin{split} \tilde{J}(\tilde{\tau}) &= \int_0^{\tilde{\tau}} e^{-rt} \tilde{D}_t dt + e^{-r\tilde{\tau}} V_2(\tilde{e}(\tilde{\tau}), 0), \\ \hat{J}(\tilde{\tau}) &= e^{-r\tilde{\tau}} V_2(\hat{e}(\tilde{\tau}), 0). \end{split}$$

Furthermore, liquidity dynamics under the two trajectories read

$$\dot{\tilde{e}} = \pi(0, m_1) - \frac{\xi}{2}\tilde{\Phi}_t^2 - \tilde{D}_t + r\tilde{e}, \text{ and } \tilde{e}(0) = e^{ini},$$

and

$$\dot{\hat{e}} = \pi(0, m_1) - \frac{\xi}{2}\tilde{\Phi}_t^2 + r\hat{e}$$
, and $\hat{e}(0) = e^{ini}$

The difference between the two liquidity streams can be written as $\Delta e_t = \hat{e}_t - \tilde{e}_t$ and $\Delta e(0) = 0$. Then,

$$\dot{\Delta e} = \tilde{D} + r\Delta e.$$

and it follows that

$$\Delta e(t) = \int_0^t \exp{(rt - r\rho)\tilde{D}(\rho)} d\rho.$$

Using this and noting that $V_2(e, 0)$ is linear with slope 1 in e for all $e \ge 0$ (see (11)), we obtain

$$\begin{aligned} \hat{J}(\tilde{\tau}) - \tilde{J}(\tilde{\tau}) &= e^{-r\tilde{\tau}} (V_2(\hat{e}(\tilde{\tau}), 0) - V_2(\tilde{e}(\tilde{\tau}), 0)) - \int_0^{\tilde{\tau}} e^{-rt} \tilde{D}(t) dt \\ &= e^{-r\tilde{\tau}} \Delta e(\tilde{\tau}) - \int_0^{\tilde{\tau}} e^{-rt} \tilde{D}(t) dt \\ &= 0 \end{aligned}$$

Hence, $\hat{J} \ge \tilde{J}$ and this completes the proof.

B Value Function in Mode m_1

Figure 9 shows the value function $V_1(e)$, calculated through our numerical procedure, for our default parameter setting and different values of γ_B .

The figure shows that $V_1(e)$ increases with liquidity e, which is quite easy to understand because more liquidity induces higher dividends and a higher fraction of internally financed investments. Note that when $\gamma_B = 0$, the value function is linear with respect to e and has the functional form of (10). For a positive initial liquidity, the value function is not influenced by γ_B because there is no bankruptcy threat. Whereas for the negative liquidity, there is a positive possibility for the firm to go bankrupt, and a higher γ_B decreases the value. For $e \to -\infty$ the value function $V_1(e)$ goes to zero for any $\gamma_B > 0$ since expected time till bankruptcy goes to zero.



Figure 9: Effect of the bankruptcy risk parameter γ_B on the mode 1 value function $V_1(e)$

C Details of the Numerical Procedure

In order to numerically determine a Markov Perfect Equilibrium strategy profile for the entire game, we first need to calculate a solution for value function $V_2(\alpha_n, e)$ in mode m = 2 that solves the HJB equation (6). Based on the derived solution $\hat{V}_2(\alpha_n, e)$, we then numerically calculate the (approximate) value function $V_1(e)$ as the solution to the HJB equation (7) in mode m_1 .

C.1 Post-innovation mode m_2

Note that for the state space of $e \ge 0$, the analytical solution of $V_2^+(\alpha_n, e)$ is (11), hence we only need to calculate $V_2^-(\alpha_n, e)$ for $e \le 0$. Note that $V_2(\alpha_n, e)$ in our model is a continuous function in e, implying that $V_2^+(\alpha_n, 0) = V_2^-(\alpha_n, 0)$. In order to better estimate the value function in mode m = 1, we have proposed a transformation from the state space of liquidity e to a state space of z according to the transformation rule $z(e) = (1 + \exp(-\lambda e))^{-1}$ with $\lambda \in (0, 1)$. The state space of $e \in (-\infty, +\infty)$ corresponds to $z \in (0, 1)$, and the negative liquidity corresponds to $z \in (0, 0.5]$. Thus, our numerical calculation is carried out in the state space $(\alpha_n, z) \in [0, \bar{\alpha}_n] \times (0, 0.5]$. From $e(z) = (\ln z - \ln (1 - z))/\lambda$, after the transition, it holds that

$$\frac{\partial V_2^-(\alpha_n, z)}{\partial z} = \frac{\partial V_2(\alpha_n, e)}{\partial e} \frac{\mathrm{d}e(z)}{\mathrm{d}z} = \frac{1}{\lambda z(1-z)} \frac{\partial V_2(\alpha_n, e)}{\partial e},$$

and the value function $V_2^-(\alpha_n, z)$ satisfies the revised HJB as

$$rV_2^-(\alpha_n, z) = \delta\left(\tilde{\alpha}_n - \alpha_n\right) \frac{\partial V_2^-(\alpha_n, z)}{\partial \alpha_n} + \frac{\sigma^2 \alpha_n^2}{2} \frac{\partial^2 V_2^-(\alpha_n, z)}{\partial \alpha_n^2} + \frac{\gamma_B}{\lambda} V_2^-(\alpha_n, z) \ln\left(\frac{z}{1-z}\right)$$
(31)

$$+\lambda z(1-z)\frac{\partial V_2^-(\alpha_n,z)}{\partial z}\left(\frac{(\bar{\alpha}_n+\alpha_n)^2+\alpha_o^2-2\eta\alpha_o(\bar{\alpha}_n+\alpha_n)}{4(1-\eta^2)}+\frac{r}{\lambda}\ln\left(\frac{z}{1-z}\right)\right)\ .$$

In order to solve this nonlinear partial different equation, we resort to the numerical collocation method to calculate an approximate solution $\hat{V}_2^-(\alpha_n, z)$.

In a given state space $[\alpha_n^l, \alpha_n^u] \times [z^l, z^u]$ with l and u denoting the lower and the upper boundary for the corresponding interval, we first construct a sparse grid of collocation nodes $\mathcal{N} = \mathcal{N}_{\alpha} \times \mathcal{N}_z$, where $\mathcal{N}_{\alpha} = {\{\alpha_n^i\}_{i=1,...,n_{\alpha}}}$ and $\mathcal{N}_z = {\{z^j\}_{j=1,...,n_z}}$, and α_n^i and z^j are defined as

$$\alpha_n^i = \frac{\alpha_n^u + \alpha_n^l}{2} + \frac{\alpha_n^u - \alpha_n^l}{2} \cos\left(\frac{(n_\alpha - i + 0.5)\pi}{n_\alpha}\right),\tag{32}$$

$$z^{j} = \frac{z^{u} + z^{l}}{2} + \frac{z^{u} - z^{l}}{2} \cos\left(\frac{(n_{z} - j + 0.5)\pi}{n_{z}}\right).$$
(33)

Then we construct a set of basis functions $\{b_{k_{\alpha},k_{z}}(\alpha_{n},z)\}_{\{k_{\alpha}=1,\dots,n_{\alpha}\}\times\{k_{z}=1,\dots,n_{z}\}}$ corresponding to our Chebyshev sparse grid such that

$$b_{k_{\alpha},k_{z}}(\alpha_{n},z) = T_{k_{\alpha}-1}\left(-1+\frac{2\left(\alpha_{n}-\alpha_{n}^{l}\right)}{\alpha_{n}^{u}-\alpha_{n}^{l}}\right) \times T_{k_{z}-1}\left(-1+\frac{2\left(z-z^{l}\right)}{z^{u}-z^{l}}\right),$$

and function $T_k(x)$ is the Chebyshev polynomial of of degree k defined on the interval [0, 1]. For the given state space of $[0, \bar{\alpha}_n] \times (0, 0.5]$, our calculation is carried out in the space of $[0, \alpha_n^u] \times [z^l, 0.5]$: $\alpha_n^l = 0$ represents that $\alpha_n = 0$ at the moment the mode jumps from m = 1 to m = 2, and $z^u = 0.5$ corresponds to an upper boundary of e = 0. In order to make sure the calculated value function is continuous at e = 0, we specify further that

$$z^{j} = \begin{cases} \frac{0.5+z^{l}}{2} + \frac{0.5-z^{l}}{2} \cos\left(\frac{(n_{z}-j+0.5)\pi}{n_{z}}\right) & 1 \le j \le n_{z} - 1, \\ 0.5 & j = n_{z}. \end{cases}$$
(34)

The value function is assumed to take the form of

$$\hat{V}_2^-(\alpha_n, z) = \sum_{k_\alpha=1}^{n_\alpha} \sum_{k_z=1}^{n_z} c_{k_\alpha, k_z} \times b_{k_\alpha, k_z}(\alpha_n, z) = \vec{c}^\top \cdot \vec{b}(\alpha_n, z),$$

where \vec{c} and \vec{b} are column vectors with a length of $n_{\alpha}n_z$ such that $\vec{c}_k = c_{k_{\alpha},k_z}$ and $\vec{b}_k(\alpha_n,z) = b_{k_{\alpha},k_z}(\alpha_n,z)$ with $k = (k_z - 1)n_z + k_{\alpha}$ for $k_{\alpha} \in \{1,...,n_{\alpha}\}$ and $k_z \in \{1,...,n_z\}$. \vec{c} and $\vec{b}(\alpha,z)$ together can capture all the polynomial elements in the value function given a pair of $\{\alpha,z\}$. We

aim to determine the weight vector of \vec{c} such that the (approximate) value function $\hat{V}_2^-(\alpha_n, z)$ satisfies the HJB equation (31) on the collocation nodes $\{\alpha_n^i, z^j\}$ with $i \in \{1, ..., n_\alpha\}$ and $j \in \{1, ..., n_z - 1\}$ in \mathcal{N} . For the other n_α nodes with $i \in \{1, ..., n_\alpha\}$ and $j = n_z$ in \mathcal{N} , we have $\hat{V}_2^-(\alpha_n^i, z^j = 0.5) = V_2^+(\alpha_n^i, e = 0)$ to make $V_2(\alpha_n, e)$ continuous. In total there are $n_\alpha n_z$ number of nodes, implying $n_\alpha n_z$ number of equations.

Furthermore, for $i \in \{1, ..., n_{\alpha}\}$ and $j \in \{1, ..., n_z - 1\}$ we introduce four $n_{\alpha}(n_z - 1) \times n_{\alpha}n_z$ matrices **B**, **B**^{α}, **B**^{α}, and **B**^z with entries

$$\mathbf{B}_{s,k} = b_k(\alpha_n^i, z^j), \quad \mathbf{B}_{s,k}^{\alpha} = \frac{\partial b_k(\alpha_n^i, z^j)}{\partial \alpha_n}, \quad \mathbb{B}_{s,k}^{\alpha} = \frac{\partial^2 b_k(\alpha_n^i, z^j)}{\partial \alpha_n^2}, \quad \mathbf{B}_{s,k}^z = \frac{\partial b_k(\alpha_n^i, z^j)}{\partial z},$$

where $s = (j-1)n_z + i$ denotes node s. These four matrices capture the values of all base functions and their partial derivatives at the nodes in \mathcal{N} that are not on the boundary of $z^{n_z} = 0.5$. For each node $\{\alpha_n^i, z^j\}$ with $i \in \{1, ..., n_\alpha\}$ and $j \in \{1, ..., n_z - 1\}$, we define the following four column vectors in such a way that $\vec{g}^{\dot{z}}$ captures the dynamics of the liquidity, \vec{g}^{α^2} captures the quadratic of α_n , \vec{g}^{α} captures α_n , and \vec{g}^z captures z. Specifically these four vectors read

$$\vec{\mathbf{g}}_{s}^{\alpha^{2}} = \left(\alpha_{n}^{i}\right)^{2}, \quad \vec{\mathbf{g}}_{s}^{\alpha} = \alpha_{n}^{i}, \quad \vec{\mathbf{g}}_{s}^{e} = \frac{1}{\lambda} \ln\left(\frac{z^{j}}{1-z^{j}}\right),$$
$$\vec{\mathbf{g}}_{s}^{\dot{z}} = \lambda z^{j} \left(1-z^{j}\right) \left(\frac{\left(\bar{\alpha}_{n}+\alpha_{n}^{i}\right)^{2}+\alpha_{o}^{2}-2\eta(\bar{\alpha}_{n}+\alpha_{n}^{i})\alpha_{o}}{4(1-\eta^{2})}+\frac{r}{\lambda} \ln\left(\frac{z^{j}}{1-z^{j}}\right)\right),$$

and $s \in \{1, ..., n_{\alpha}(n_z - 1)\}$. Thus \vec{c} has to be chosen to solve

$$r\mathbf{B}\cdot\vec{c} = \delta\tilde{\alpha}_n\mathbf{B}^{\alpha}\cdot\vec{c} - \delta\vec{g}^{\alpha}\cdot\mathbf{B}^{\alpha}\cdot\vec{c} + \frac{\sigma^2}{2}\vec{g}^{\alpha^2}\cdot\mathbb{B}^{\alpha}\cdot\vec{c} + \vec{g}^{\dot{z}}\cdot\mathbf{B}^{z}\cdot\vec{c} + \gamma_B\vec{g}^{e}\cdot\mathbf{B}\cdot\vec{c} .$$
(35)

and in addition

$$\vec{c}^{\top}\vec{b}(\alpha^{i},0.5) = V_{2}^{+}(\alpha^{i},0), \quad i \in \{1,...,n_{\alpha}\}.$$
(36)

There are in total $n_{\alpha}n_z$ linear equations when combining (35) and (36), which can be solved using standard solvers. Note that there is no control in mode m_2 , implying that solving these $n_{\alpha}n_z$ equations yields the solution \vec{c} . Thus, we can write the calculated value function in mode 2 as

$$\hat{V}_{2}(\alpha_{n}, e) = \begin{cases} \hat{V}_{2}^{-}(\alpha_{n}, z(e)) & e < 0 \\ \\ V_{2}^{+}(\alpha_{n}, e) & e \ge 0 \end{cases}.$$

C.2 Pre-innovation mode m_1

There are several differences in mode m_1 compared with that in m_2 : $V_1(e)$ is only defined on the domain of e, and the control is captured by

$$\phi(e) = \frac{\gamma_I}{\xi} \frac{V_2(0, e) - V_1(e)}{\mathrm{d}V_1(e)/\mathrm{d}e}.$$

In order to numerically calculate for $V_1(e)$, we carry out the same transformation as for mode m_2 from the state space of e to the state space of $z \in (0, 1)$ according to e(z). After the transition, the optimal control can be rewritten as

$$\phi(e(z)) = \frac{\gamma_I}{\xi} \frac{\hat{V}_2(0, e(z)) - V_1(e(z))}{\lambda z (1 - z) dV_1(e(z))/dz}$$

We proceed with the same collocation method as in mode m_2 , but on just one dimensional state space of z. Because the HJB in mode m_1 has different expressions for the positive and negative e, we need to calculate the value function separately for $V_1^+(z)$ with $z \in [0.5, 1)$, corresponding to $e \ge 0$, and for $V_1^-(z)$ with $z \in (0, 0.5]$, corresponding to $e \le 0$. The value function $V_1(e(z))$ has to be continuous on the entire state space, however, there might exist a kink for $V_1(e(z))$ at z = 0.5and a jump in the control function $\phi(e(z))$ because of different HJB expressions and the difference in $dV_1(e(z))/de$ for positive and negative e. As has been shown in our analysis in Section3 such discontinuity can arises only if $e^* = 0$ is a stable steady state, which according to Proposition 2, happens if and only if $\gamma_I \in [\underline{\gamma}_I, \overline{\gamma}_I]$. If $\gamma_I \ge \underline{\gamma}_I$ we are in the no debt scenario and the interval $z \in [0.5, 1]$ is invariant under the state dynamics under optimal investment. For $\gamma_I \ge \overline{\gamma}_I$ the value function V_1^+ of the problem is given in closed form by (10). For $\gamma_I \in [\underline{\gamma}_I, \overline{\gamma}_I)$. Finally, for $\gamma_I > \overline{\gamma}_I$ only the interval $z \in [0, 0.5]$ is invariant. In any case the HJB equations on the positive domain, given by

$$rV_{1}^{+}(z) = \nu_{1}e(z) + \lambda z(1-z)\frac{\mathrm{d}V_{1}^{+}(z)}{\mathrm{d}z} \left(\frac{\alpha_{o}^{2}}{4} - \frac{\xi}{2}\phi^{2}(e(z)) + (r-\nu_{1})e(z)\right) + \gamma_{I}\phi(e(z))\left(V_{2}(0,e(z)) - V_{1}^{+}(z)\right), \quad z \in [0.5,1].$$
(37)

and on the negative domain, given by

$$rV_{1}^{-}(z) = \lambda z(1-z) \frac{\mathrm{d}V_{1}^{-}(z)}{\mathrm{d}z} \left(\frac{\alpha_{o}^{2}}{4} - \frac{\xi}{2} \phi^{2}(e(z)) + re(z) \right) + \gamma_{I} \phi(e(z)) \left(V_{2}(0, e(z)) - V_{1}^{-}(z) \right) + \gamma_{B} e(z) V_{1}^{-}(z), \quad z \in [0, 0.5]$$

$$(38)$$

are solved separately in our numerical procedure.

With respect to the boundary conditions and the sequence of the numerical calculation of $V_1^$ and V_1^+ three cases have to be distinguished:

1. For $\gamma_I \in [\underline{\gamma}_I, \overline{\gamma}_I]$ both intervals $z \in [0, 0.5]$ and $z \in [0.5, 1]$ are invariant under the state dynamics under optimal investment and $e^* = 0$ is a stable steady state. Hence,

$$V_1^{-}(0.5) = V_1^{+}(0.5) = \int_0^\infty e^{-(r+\gamma_I\phi(e(0.5)))t} \gamma_I\phi(e(0.5))V_2^{+}(0,0)dt = \frac{\gamma_I\alpha_o V_2^{+}(0,0)}{r\sqrt{2\xi} + \gamma_I\alpha_o} .$$
 (39)

with $\phi(e(0.5)) = \phi(0) = \alpha_o/\sqrt{2\xi}$ has to hold both for V_1^- and V_1^+ .

- 2. For $\gamma_I > \bar{\gamma}_I$ we have $\dot{e} < 0$ under optimal investment at e = 0. Hence, the interval $z \in [0, 0.5]$ is invariant under the state dynamics under optimal investment. Therefore, we first numerically find a function \hat{V}_1^- (approximately) solving (38), where no explicit boundary conditions are imposed.³ Then, as a second step, we numerically determine a solution of (37) with the boundary condition $V_1^+(0.5) = \hat{V}_1^-(0.5)$.
- 3. For $\gamma_I < \underline{\gamma}_I$ we have $\dot{e} > 0$ under optimal investment at e = 0. Hence, the interval $z \in [0.5, 1]$ is invariant under the state dynamics under optimal investment and the value function V_1^+ of the problem is given in closed form by (10). The value function on the negative domain is determined as the solution of (38) with boundary condition $V_1^-(0.5) = V_1^+(0.5)$.

As has been explained in the main text, in this paper we only consider the first two of these three

³Formally, we have the boundary condition $\lim_{z\to 0} V_1^-(z) = 0$ and we check in our numerical solution that $V_1^-(z^l)$ becomes small for a sufficiently small lower bound of the state interval considered in the numerical approximation of \hat{V}_1^- (see below).

cases, since in case 3. financial constraints are irrelevant for $e(0) \ge 0$. In what follows we just describe our algorithm for the first case, the procedure in the second case is analogous.

In order to calculate an (approximate) value function $\hat{V}_1^+(z)$ that makes (37) hold on the interval $z \in [0.5, z^u)$, we first construct a set of collocation nodes $\mathcal{N}_z = \{z^j\}_{j=1,\dots,n_z}$. The idea is similar as to construct the grid in mode m = 2 except in mode m = 1 that $n_\alpha = 1$ and $\alpha^u = \alpha^l = 0$. Thus, the corresponding set of base functions is denoted by $\{b_{1,k_z}(0,z)\}_{k_z=1,\dots,n_z}$. In order to be able to incorporate the boundary condition (39) at z = 0.5, we further specify that

$$z^{j} = \begin{cases} 0.5 & j = 1, \\ \frac{z^{u} + 0.5}{2} + \frac{z^{u} - 0.5}{2} \cos\left(\frac{(n_{z} - j + 0.5)\pi}{n_{z}}\right) & 1 < j \le n_{z} \end{cases}$$

Similarly to mode m_2 we consider an (approximate) value function of the form

$$\hat{V}_{1}^{+}(z) = \sum_{k_{z}=1}^{n_{z}} c_{k_{z}}^{pos} \times b_{1,k_{z}}(0,z) = c^{\vec{pos}^{\top}} \cdot b^{\vec{pos}}(z)$$

where $c^{\vec{pos}} = (c_k^{pos})_{k=1}^{n_z}$ and $b^{\vec{pos}}(z) = (b_k^{pos}(z))_{k=1}^{n_z}$ are column vectors with a length of n_z and $b_k^{pos}(z) = b_{1,k}(0, z)$. Finding the solution is equivalent to determine the vector $c^{\vec{pos}}$ such that $\hat{V}_1^+(z)$ satisfies the HJB (37) on the collocation nodes $z^j \in \mathcal{N}_z$ and $j \in \{2, ..., n_z\}$. Furthermore, \hat{V}_1^+ has to satisfy (39) at node z^1 . Thus, there are in total n_z equations and to be solved with n_z unknowns in vector $c^{\vec{pos}}$. It should be noted that, contrary to mode m_2 , the right hand side of the HJB equations in this mode contain terms with the optimal control $\phi(z^j)$ at the considered node, where the optimal control function ϕ depends on the value function V_1 and its state derivative.

We use an iterative algorithm to solve this system of equations. In particular, we consider a sequence of vectors $c^{\vec{pos}}(it)$, with $it \in \{0, 1, ...\}$ is the indicator for the iterations. In iteration $it \geq 1$, we calculate for all nodes z^j in \mathcal{N}_z , i.e., $j \in \{1, ..., n_z\}$, the optimal control as

$$\vec{\phi}(it) = \frac{\gamma_I}{\xi} \operatorname{Diag}\left(\left(V_2(0, \vec{\mathbf{g}}^z) - \mathbf{B} \cdot c^{\vec{pos}}(it-1)\right) \cdot \left(\vec{\mathbf{g}}^z \cdot \mathbf{I} \cdot \mathbf{B}^z \cdot c^{\vec{pos}}(it-1)\right)^{-1}\right),\tag{40}$$

where $\text{Diag}(\mathbf{X})$ generates a column vector with elements on the diagonal line of \mathbf{X} , \mathbf{I} is a $n_z \times n_z$ identity matrix, $\vec{\mathbf{g}}^z$ is of length n_z with $\vec{\mathbf{g}}_j^z = \lambda z^j (1 - z^j)$ and \mathbf{B} and \mathbf{B}^z are such that

$$\mathbf{B}_{j,k} = b_k^{pos}(z^j), \quad \mathbf{B}_{j,k}^z = \frac{\mathrm{d}b_k^{pos}(z^j)}{\mathrm{d}z}, \quad j,k \in \{1,..,n_z\}.$$

Substitution of $\vec{\phi}(it)_{j \in \{2,...,n_z\}}$ at node z^j into HJB (37) together with the boundary condition at node z_1 generates a system of n_z linear equations in $c^{\vec{pos}}(it)$, which can be solved by standard methods. This gives the value of $c^{\vec{pos}}(it)$ and updated optimal controls at each node under this new coefficient vector. The iteration is stoped once after inserting these updated controls into HJB equations the maximal absolute difference between the right and left hand side of (37) across nodes is sufficiently small. Overall, the numerical details can be summarized as follows:

- (1) Choose n_z and calculate the nodes in \mathcal{N}_z . Choose the stopping criterion ϵ .
- (2) Calculate **B**, \mathbf{B}^z and $\vec{\mathbf{g}}^z$.
- (3) Choose $c^{\vec{pos}}(0)$.
- (4) Calculate the optimal control $\vec{\phi}(0)$.
- (5) While the stopping criteria is not satisfied, iterate the following steps for it = 1, 2, ...
 - (a) Calculate $c^{\vec{pos}}(it)$ by solving the combined n_z equations: (39) for node $z^1 = 0.5$ and (37) for node z^j using $\vec{\phi_j}(it-1), j \in \{2, ..., n_z\}$.
 - (b) Calculate the optimal control $\vec{\phi}(it)$.
 - (c) Calculate the difference $\Delta_j(it)$ between left and right hand side of (39) for node z^1 and (37) for node z^j using $\vec{\phi}(it)$ and $c^{\vec{pos}}(it)$.
 - (d) Checking the stopping criteria of $\max_{j \in \{1,...,n_z\}} \left[\left| \Delta_j(it) \right| / (\mathbf{B} \cdot c^{\vec{pos}}(it))_j \right] < \epsilon.$
- (6) Set the value function $\hat{V}_1^+(z) = c^{\vec{pos}^{\top}}(it) \cdot b^{\vec{pos}}(z)$ and calculate the optimal control $\phi(e(z))$ by $\hat{V}_1^+(z)$.

The numerical calculation of $\hat{V}_1^-(z)$ with $z \in (z^l, 0.5]$ is analogous to the numerical calculation for $\hat{V}_1^+(z)$ and we do not repeat the details here. The numerical approximation for value function $V_1(e)$ then reads

$$\hat{V}_1(e) = \begin{cases} \hat{V}_1^+(z(e)) & e \ge 0, \\ \\ \hat{V}_1^-(z(e)) & e < 0. \end{cases}$$