LARGE N LIMIT OF THE $O(N)$ LINEAR SIGMA MODEL VIA STOCHASTIC QUANTIZATION

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ABSTRACT. This article studies large N limits of a coupled system of N interacting Φ^4 equations posed over $\mathbb{T}^d$ for $d = 1, 2$, known as the $O(N)$ linear sigma model. Uniform in N bounds on the dynamics are established, allowing us to show convergence to a mean-field singular SPDE, also proved to be globally well-posed. Moreover, we show tightness of the invariant measures in the large N limit. For large enough mass, they converge to the (massive) Gaussian free field, the unique invariant measure of the mean-field dynamics, at a rate of order $1/\sqrt{N}$ with respect to the Wasserstein distance. We also consider fluctuations and obtain tightness results for certain $O(N)$ invariant observables, along with an exact description of the limiting correlations in $d = 1$.

CONTENTS

1. INTRODUCTION

In this paper, we consider the following system of equations on the d-dimensional torus $\mathbb{T}^d$ for $d = 1, 2$

$$\mathscr{L}\Phi_i = -\frac{1}{N}\sum_{j=1}^{N}\Phi_j^2\Phi_i + \xi_i, \quad \Phi_i(0) = \phi_i, \tag{1.1}$$

where $\mathscr{L} = \partial_t - \Delta + m$ with $m \geqslant 0$, $N \in \mathbb{N}$, and $i \in \{1, \cdots, N\}$. The collection $(\xi_i)_{i=1}^N$ consists of N independent space-time white noises on a stochastic basis $(\Omega, \mathcal{F}, \mathbf{P})$, and $(\phi_i)_{i=1}^N$ are random initial datum independent of $(\xi_i)_{i=1}^N$. In $d = 2$, the system (1.1) requires renormalization, and the formal product $\Phi_j^2\Phi_i$ will be interpreted as the Wick product $:\Phi_j^2\Phi_i:$ whose definition is postponed to Section 3.

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This system arises as the stochastic quantization of the following N-component generalization of the Φ^4_d model, given by the (formal) measure

$$\mathrm{d}\nu^N(\Phi) \stackrel{\text{def}}{=} \frac{1}{C_N}\exp\Big(-\int_{\mathbb{T}^d}\sum_{j=1}^N|\nabla\Phi_j|^2+m\sum_{j=1}^N\Phi_j^2+\frac{1}{2N}\Big(\sum_{j=1}^N\Phi_j^2\Big)^2\mathrm{d}x\Big)\mathcal{D}\Phi \tag{1.2}$$

over $\mathbb{R}^N$ valued fields $\Phi=(\Phi_1,\Phi_2,...,\Phi_N)$ and C_N is a normalization constant. In $d=2$, the interaction should be Wick renormalized $:\!\big(\sum_{j=1}^N\Phi_j^2\big)^2\!:$ for the measure to make sense. This is also referred to as the $O(N)$ linear sigma model, since this formal measure is invariant under a rotation of the N components of Φ. [1] This symmetry will play an important role throughout the paper.

Our focus in this article is on the asymptotic behavior as $N\to\infty$ of the system (1.1) and its invariant measures (1.2) as well as observables which preserve the $O(N)$ symmetry. Note that a factor $1/N$ has been introduced in front of the nonlinearity (resp. the quartic term in the measure), and heuristically, this compensates the sum of N terms so that one could hope to obtain an interesting limit as $N\to\infty$. The study of physically meaningful quantities associated with a quantum field theory model such as (1.2) as $N\to\infty$ is generally referred to as a *large N problem*; see Section 1.1 where we introduce more background, references in physics and mathematics, and different approaches to this problem. To the best of our knowledge, the present article provides the first rigorous results on large N problems in the formulation of stochastic quantization.

In Theorem 1.1 below, we study the $N\to\infty$ limit of each component in the Wick renormalized version of (1.1) in $d=2$, c.f. (3.1) below, and show that a suitable mean-field singular SPDE governs the limiting dynamics. Before giving the statement, let us first comment on the notion of solution used. Recall that the well-posedness of (1.1) in the case $N=1$ and $d=2$ (i.e. the dynamical Φ^4_2 model) is now well developed: two classical works being [AR91] where martingale solutions are constructed and [DPD03] where strong solutions are addressed, as well as the more recent approach to global well-posedness in [MW17b]. These results can be generalized to the vector case (with fixed $N>1$) without much extra effort. As in [DPD03] and [MW17b] the solutions are defined by the decomposition $\Phi_i=Z_i+Y_i$, where

$$\mathscr{L}Z_i=\xi_i, \tag{1.3}$$

$$\mathscr{L}Y_i=-\frac{1}{N}\sum_{j=1}^N(Y_j^2Y_i+Y_j^2Z_i+2Y_jY_iZ_j+2Y_j:\!Z_iZ_j\!:+:\!Z_j^2\!:Y_i+:\!Z_iZ_j^2\!:) \tag{1.4}$$

and $:\!Z_iZ_j\!:$, $:\!Z_iZ_j^2\!:$ are Wick renormalized products (see Section 3). For the uninitiated reader, note that (1.4) arises by inserting the decomposition of Φ_i into (1.1) and re-interpreting the ill-defined products Z_iZ_j and $Z_iZ_j^2$ that appear.

The mean-field SPDE formally associated to (1.1) takes the form

$$\mathscr{L}\Psi_i=-\mathbf{E}[\Psi_i^2]\Psi_i+\xi_i,\quad \Psi_i(0)=\psi_i. \tag{1.5}$$

On the formal level this equation arises naturally: assuming the initial conditions $\{\phi_i\}_{i=1}^N$ are exchangeable, the components $\{\Phi_i\}_{i=1}^N$ will have identical laws, so that replacing the empirical average $\frac{1}{N}\sum_{j=1}^N\Phi_j^2$ in (1.1) by its mean and re-labelling Φ as Ψ leads us to (1.5). In two space dimensions, (1.5) is a singular SPDE where the ill-defined non-linearity depends on the law of the solution and similar to (1.1) it also requires a renormalization. Postponing for the moment a more complete discussion of this point, we now state our first main result.

Theorem 1.1 (Large N limit of the dynamics for $d=2$)**.** *Let $(\phi_i^N,\psi_i)_{i=1}^N$ be random initial datum with components in $\mathbf{C}^{-\kappa}$ for some small $\kappa>0$ and all moments uniformly finite. Assume that for*

[1]The word “linear” here only means that the target space $\mathbb{R}^N$ is a linear space. “Nonlinear” sigma models on the other hand refers to similar models where the target space is subject to certain nonlinear constraints, e.g. Φ takes value in a sphere in $\mathbb{R}^N$ or more generally in a manifold.

each $i \in \mathbb{N}$, ϕ_i^N converges to ψ_i in $L^p(\Omega; \mathbf{C}^{-\kappa})$ for all $p > 1$, $\frac{1}{N}\sum_{i=1}^N \|\phi_i^N - \psi_i\|_{\mathbf{C}^{-\kappa}}^p \to^{\mathbf{P}} 0$ and $(\psi_i)_i$ are iid. Here $\to^{\mathbf{P}}$ means the convergence in probability.

Then for each component i and all $T > 0$, the solution Φ_i^N defined by (1.3)-(1.4) with initial datum ϕ_i^N converges in probability to Ψ_i in $C([0,T], \mathbf{C}^{-1}(\mathbb{T}^2))$ as $N \to \infty$, where Ψ_i is the unique solution to the mean-field SPDE formally described by

$$\mathscr{L}\Psi_i = -\mathbf{E}[\Psi_i^2 - Z_i^2]\Psi_i + \xi_i, \quad \Psi_i(0) = \psi_i, \tag{1.6}$$

and Z_i is the stationary solution to (1.3). Furthermore, under the additional hypothesis that $(\phi_i^N, \psi_i)_{i=1}^N$ are exchangeable, for each $t > 0$ it holds that

$$\lim_{N\to\infty} \mathbf{E}\|\Phi_i^N(t) - \Psi_i(t)\|_{L^2(\mathbb{T}^2)}^2 = 0. \tag{1.7}$$

In Section 5 we actually prove this convergence result under more general conditions for initial data (see Assumption 5.1). Along the way to Theorem 1.1, we prove new uniform in N bounds through suitable energy estimates on the remainder equation (1.4). We are inspired in part by the approach in [MW17b], but subtleties arise as we track carefully the dependence of the bounds on N. Indeed, the natural approach (e.g. [MW17b] for dynamical Φ_2^4 model) to obtain global in time bounds for fixed N is to exploit the damping effect from $Y_j^2 Y_i$. However, the extra factor $1/N$ before the nonlinear terms makes this effect weaker as N becomes large. In fact, the moral is that we cannot exploit the strong damping effect at the level of a fixed component Y_i, rather we're forced to consider aggregate quantities, and ultimately we focus on the empirical average of the L^2-norm (squared) instead of the L^p-norm, $p > 2$, c.f. Lemma 3.3 and Remark 3.6. This is natural on one hand due to the coupling of the components, but also for the slightly more subtle point that we ought to respect the structure of the mean-field SPDE (1.6), for which the damping effect seems to hold only in the mean square sense, not at the path-by-path level.

In this direction, we now discuss a bit more the solution theory for the mean-field SPDE (1.6). While the notion of solution we use is again via the Da-Prato/Debussche trick, the well-posedness theory for (1.5) requires more care than for Φ_2^4 since we cannot proceed by pathwise arguments alone. In fact, similar to (1.3)-(1.4), we understand (1.6) via the decomposition $\Psi_i = Z_i + X_i$ with X_i satisfying

$$\mathscr{L}X_i = -(\mathbf{E}[X_j^2]X_i + \mathbf{E}[X_j^2]Z_i + 2\mathbf{E}[X_jZ_j]X_i + 2\mathbf{E}[X_jZ_j]Z_i). \tag{1.8}$$

Here we actually introduce an independent copy (X_j, Z_j) of (X_i, Z_i), which turns out to be useful for both the local and global well-posedness of (1.6). Indeed, one point is that the term $\mathbf{E}[X_jZ_j]Z_i$ in (1.8) cannot be understood in a classical sense; however we can view it as a conditional expectation $\mathbf{E}[X_jZ_jZ_i|Z_i]$ and use properties of the Wick product Z_iZ_j to give a meaning to this, c.f. Lemma 4.1. Furthermore, to obtain global bounds, using this independent copy allows us to approach the a priori estimates for (1.8) much like the uniform in N bounds for (1.4). Indeed, after taking expectation, $\mathbf{E}[X_j^2]X_i$ in (1.8) also plays the role of the damping mechanism, which helps us to obtain uniform bounds on the mean-squared L^2-norm of X_i c.f. Lemma 4.3.

Theorem 1.1 can be viewed as a *mean field* limit result in the context of singular SPDE systems. Our proof is indeed inspired by certain mean field limit techniques, and we combine them with a priori estimates that are specific to our model - see the discussion above Theorem 5.1 for a more detailed discussion on this strategy. We will provide more background discussion on mean field limits below in Section 1.2. By a classical coupling argument, this result also yields a propagation of chaos type statement: if the initial condition is asymptotically chaotic (i.e. independent components as $N \to \infty$) then, although the Φ-system is interacting, as $N \to \infty$ the limiting system becomes decoupled ([Jab14, Def.3,Def.5]).

The second part of this paper (Section 6) is concerned with equilibrium theories, namely stationary solutions, invariant measures, and large N convergence. For $N = 1$, the long-time behavior of the solutions was investigated in [RZZ17] and [TW18]. In the vector valued setting, by lattice approximation (see [HM18a, ZZ18, GH18]), strong Feller property in [HM18b] and irreducibility in [HS19] it can be shown that ν^N is the unique invariant measure to (1.1) and the law of $\Phi_i(t)$ converges to ν^N as $t \to \infty$. Our goal then is to study the large N limit of the $O(N)$ linear sigma model ν^N. Our second main result yields the convergence of the unique invariant measure ν^N of (1.1) to the invariant measure of (1.6), provided the mass is sufficiently large.

To state the result, consider the projection onto the i^{th} component,

$$\Pi_i : \mathcal{S}'(\mathbb{T}^d)^N \to \mathcal{S}'(\mathbb{T}^d), \qquad \Pi_i(\Phi) \stackrel{\text{def}}{=} \Phi_i. \tag{1.9}$$

Noting that ν^N is a measure on $\mathcal{S}'(\mathbb{T}^d)^N$, we define the marginal law $\nu^{N,i} \stackrel{\text{def}}{=} \nu^N \circ \Pi_i^{-1}$. Furthermore, consider

$$\Pi^{(k)} : \mathcal{S}'(\mathbb{T}^d)^N \to \mathcal{S}'(\mathbb{T}^d)^k, \qquad \Pi^{(k)}(\Phi) = (\Phi_i)_{1\leqslant i\leqslant k} \tag{1.10}$$

and define the marginal law of the first k components by $\nu_k^N \stackrel{\text{def}}{=} \nu^N \circ (\Pi^{(k)})^{-1}$.

Theorem 1.2 (Large N limit of the invariant measures)**.** *There exists $m_0 > 0$ such that the following results hold:*

- *The Gaussian free field $\nu \stackrel{def}{=} \mathcal{N}(0, (m-\Delta)^{-1})$ is an invariant measure for (1.6).*
- *The sequence of probability measures $(\nu^{N,i})_{N\geqslant 1}$ are tight on $\mathbf{C}^{-\kappa}$ for $\kappa > 0$.*
- *For $m \geqslant m_0$, the Gaussian free field $\mathcal{N}(0, (m-\Delta)^{-1})$ is the unique invariant measure to equation (1.6).*
- *For $m \geqslant m_0$, $\nu^{N,i}$ converges to ν and ν_k^N converges to $\nu \times \cdots \times \nu$, as $N \to \infty$. Furthermore, $\mathbb{W}_2(\nu^{N,i}, \nu) \lesssim N^{-\frac{1}{2}}$.*

These statements will follow from Theorem 6.9, Theorem 6.4 and Theorem 6.11. Here $\mathbb{W}_2(\nu_1, \nu_2)$ is the $\mathbf{C}^{-\kappa}$-Wasserstein distance defined before Theorem 6.11. The Gaussian free field limit is expected (at a heuristic level) by physicists e.g. [Wil73] and also in mathematical physics [Kup80b]. [2] Our result Theorem 1.2 provides a precise justification provided $m \geqslant m_0$, with the convergence rate $N^{-\frac{1}{2}}$ (which is expected to be optimal, see for instance [JW18, Remark 4]) in terms of Wasserstein distance. The large m assumption could also be formulated as a small nonlinearity assumption - see Remark 6.12.

Note that the study of ergodicity properties of the dynamic (1.6) is nontrivial. In fact, the dynamic for Ψ depends on the law of Ψ itself, so the associated semigroup is generally nonlinear (see Section 6.1). As a result, the general ergodic theory for Markov process (see e.g. [DPZ96], [HMS11], [HM18b]) could not be directly applied here. Instead, we prove the solutions to (1.6) converge to the limit directly as time goes to infinity, which requires $m \geqslant m_0$.

We now comment on our approach to the fourth part of Theorem 1.2. It would be natural to try and use Theorem 1.1 together with the tightness result from the second part of Theorem 1.2 to derive the convergence of $\nu^{N,i}$ to ν directly (see e.g. [HM18a]). However, it is not clear to the authors how to implement this strategy in the present setting. Indeed, to apply Theorem 1.1, it is important that each component ψ_i of the initial data is independent of each other. However, we are not able to deduce that an arbitrary limit point ν^* has this property. If we use $P_t^*\nu^*$ to denote the marginal distribution of the solution to (1.6) starting from the initial distribution ν^*, we cannot write $P_t^*\nu^*$ as $\int (P_t^*\delta_\psi)\nu^*(\mathrm{d}\psi)$ due to the lack of linearity, which makes it difficult to overcome the assumption of independence. Alternatively, we follow the idea in [GH18] and construct a jointly stationary process (Φ, Ψ) whose components satisfy (1.1) and (1.6), respectively. In this case $\Psi = Z$, since the Gaussian free field gives the unique invariant measure to (1.6). We then establish the convergence of $\nu^{N,i}$ to ν by deriving suitable uniform estimates on the stationary process.

Our next result is concerned with observables, in the stationary setting. In QFT models with continuous symmetries, physically interesting quantities involve more than just a component of the field itself but also quantities composed by the fields which preserve the symmetries, called invariant

[2] In [Wil73] it was written that "If one now looks at vacuum expectation values of individual Φ fields, all diagrams vanish like $1/N$ (at least), except for the free-field terms." In the Introduction of [Kup80b] it was mentioned that "the $1/N$ expansion predicts that the theory is close to Gaussian as N becomes large enough", but this reference did not intend to prove this statement (see Section 1.1 below).

observables. These acquire the same interest in SPDE (a natural example being the gauge invariant observables, e.g. [She18, Section 2.4]). In the present setting of (1.1), a natural quantity that is invariant under $O(N)$-rotation is the "length" of Φ; another being the quartic interaction in (1.2). We thus consider the following two $O(N)$ invariant observables: for $\Phi \backsimeq \nu^N$

$$\frac{1}{\sqrt{N}}\sum_{i=1}^N :\Phi_i^2: , \qquad \frac{1}{N} :\Big(\sum_{i=1}^N \Phi_i^2\Big)^2: . \tag{1.11}$$

Here the precise definition is given in Section 6.3. One could consider more general renormalized polynomials of $\sum_i \Phi_i^2$ but we choose to focus on the above two in this article. We establish the large N tightness of these observables as random fields in suitable Besov spaces by using iteration to derive improved uniform estimates in the stationary case.

Theorem 1.3. *Suppose that $\Phi \backsimeq \nu^N$. For m large enough, the following result holds for any $\kappa > 0$:*

- $\frac{1}{\sqrt{N}}\sum_{i=1}^N :\Phi_i^2:$ *is tight in $B^{-2\kappa}_{2,2}$.*
- $\frac{1}{N} :(\sum_{i=1}^N \Phi_i^2)^2:$ *is tight in $B^{-3\kappa}_{1,1}$.*

Note that the physics literature usually considers integrated quantities, i.e. partition function of correlations of these observables. Our SPDE approach allows us to study these observables as random fields with precise regularity as $N \to \infty$ which is new. The above result can also be shown in dimension one, and in fact we can obtain more refined information about the limit in this case. This is the focus of the final part of our paper. In $d = 1$ the equations are less singular and uniform estimates are simpler; we provide in Section 7 and Appendix B the arguments in $d = 1$ that simplify in comparison to $d = 2$ (while skipping details that follow essentially the same way as $d = 2$). A reader might also use Section 7 with the proof in Appendix B to grasp the main ideas of our analysis before embarking on the more challenging case of $d = 2$.

Moreover, in $d = 1$, we investigate the *nontrivial statistics* of the large N limit of the $O(N)$ invariant observables. To this end, we suitably recenter the system (1.1), that is, the Wick ordered system defined by subtracting the finite variance of the stationary solution Z_i (see (7.9) for the precise definition), and also Wick order the observables (see (7.10)). We show that although for large enough m the invariant measure of Φ_i converges as $N \to \infty$ to the invariant measure of Z_i i.e. Gaussian free field, the limits of the observables (1.11) have different laws than those if Φ_i in (1.11) were replaced by Z_i:

$$\frac{1}{\sqrt{N}}\sum_{i=1}^N :Z_i^2: , \qquad \frac{1}{N} :\Big(\sum_{i=1}^N Z_i^2\Big)^2: . \tag{1.12}$$

Theorem 1.4 (Large N limit in $d = 1$)**.** *Under the same distributional assumption on the initial conditions as in Theorem 1.1 together with the additional hypothesis that for some $q > 1$, $\mathbf{E}[\|\phi_i^N\|_{L^2}^{2q} + \|\psi_i\|_{L^2}^4 + \|\psi_i\|_{L^\infty}^2] \lesssim 1$, and $\mathbf{E}\|\phi_i^N - \psi_i\|_{L^2}^2 \to 0$ as $N \to \infty$, for every $i \in \mathbb{N}$, $t > 0$, the solution $\Phi_i^N(t)$ to (1.1) starting from ϕ_i^N converge in $L^2(\Omega; C([0,T]; L^2(\mathbb{T})))$ to the solution of*

$$\mathscr{L}\Psi_i = -\mathbf{E}[\Psi_i^2]\Psi_i + \xi_i, \tag{1.13}$$

with $\Psi_i(0) = \psi_i$, where ξ_i are the space-time white noises as in (1.1).

Moreover, there is a unique constant $\mu > 0$ such that the Gaussian free field $\nu \stackrel{def}{=} \mathcal{N}(0, (m-\Delta+\mu)^{-1})$ is an invariant measure to (1.13). The sequence of measures $(\nu^{N,i})_{N\geqslant 1}$ are tight, and for large enough m, the above Gaussian free field is the unique invariant measure to equation (1.13), $\nu^{N,i}$ converges to ν and ν_k^N converges to $\nu \times \cdots \times \nu$, as $N \to \infty$.

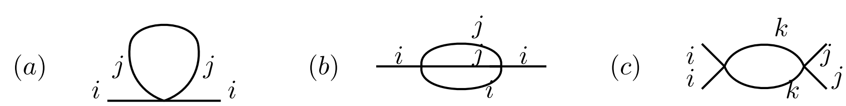
Finally, when the system (1.1) is Wick ordered (see (7.9)), for m large enough, the stationary solution Φ_i converges to the stationary solution Z_i as $N \to \infty$; the observables (1.11) are tight in L^2 and L^1 respectively. However, the limiting laws of the observables (1.11) are different from those of (1.12); in fact, we have an explicit formula for the Fourier transform of the two point correlation function of $\frac{1}{\sqrt{N}}\sum_{i=1}^N :\Phi_i^2:$ as $N \to \infty$ given by $2\widehat{C^2}/(1+\widehat{C^2})$, where $C = (m-\Delta)^{-1}$ and $\widehat{C}$ is the

Fourier transform, and the formula for $\mathbf{E}\frac{1}{N}:\!\big(\sum_{i=1}^N \Phi_i^2\big)^2\!:$ *as* $N\to\infty$ *given by* $-2\sum_{k\in\mathbb{Z}}\widehat{C^2}(k)^2/(1+\widehat{C^2}(k))$.

These results are proved in Theorems 7.1– 7.5. The last statement on correlation formulas of the observables are known – first heuristically by physicists who expressed these formulas in terms of the sum of "bubble" diagrams, and then derived in [Kup80b, Eq. (15)] using constructive field theory techniques such as "chessboard estimates". Our new proofs of these correlation formulas using PDE methods are quite simple and straightforward once all the a priori estimates are available. We expect that these methods can be applied to study more $O(N)$ invariant observables and higher order correlations; we will pursue these in future work.

Let's also mention the three dimensional construction of local solutions [Hai14, GIP15, CC18], global solutions [MW17a, GH19, AK17, MW18], as well as a priori bounds in fractional dimension $d<4$ by [CMW19], though we only focus on $d=1,2$ in this paper. It would also be interesting to see if our methodology could be used to study limits of other singular SPDE systems as dimensionality of the target space tends to infinity, such as coupled dynamical Φ^4_3, [3] coupled KPZ systems [FH17], random loops in N dimensional manifolds [BGHZ19, Hai16, RWZZ20, CWZZ18] and Yang-Mills model [CCHS20] with N dimensional Lie groups (or abelian case [She18] with Higgs field generalized to value in $\mathbb{C}^N$). These are of course left to further work.

1.1. **Large N problem in QFT: background and motivation.** Large N methods (or "$1/N$ expansions") in theoretical physics are ubiquitous and are generally applied to models where dimensionality of the target space is large. It was first used in [Sta68] for spin models, and then developed in quantum field theories (QFT) which was pioneered by [Wil73] (Φ^4 type and Fermionic models), [GN74] (Fermionic models), [t'H74] (Yang-Mills model), and the idea was soon popularized and extended to many other systems - see [BW93] for an edited comprehensive collection of articles on large N as applied to a wide spectrum of problems in quantum field theory and statistical mechanics; see also the review articles [Wit80], [Col88, Chapter 8] and [MZJ03] for summaries of the progress. Loosely speaking, in terms of our model (1.2), the ordinary QFT perturbative calculation of for instance a two-point correlation of Φ_i is given by sum of Feynman graphs with two external legs and degree-4 internal vertices, each vertex carrying two distinct summation variables and a factor $1/N$ that represents the interaction $\frac{1}{N}\sum_{i,j}\Phi_i^2\Phi_j^2$, such as (a) (b) below



Heuristically, graph (a) is of order $\frac{1}{N}\sum_j \approx O(1)$ and graph (b) is of order $\frac{1}{N^2}\sum_j \approx O(\frac{1}{N})$. The philosophy of [Wil73] is that graphs with "self-loops" such as (a) get *cancelled* by Wick renormalization, and all other graphs with internal vertices including (b) are at least of order $O(1/N)$ and thus vanish, so the theory would be asymptotically Gaussian free field - which is what we prove in Theorem 1.2. Two-point correlation of observables such as $\frac{1}{\sqrt{N}}\sum_{i=1}^N :\!\Phi_i^2\!:$ on the other hand may have $O(1)$ contributions as shown in graph (c), [4] which is the heuristic behind the existence of a nontrivial correlation structure for such observables as in Theorem 1.4. The "$1/N$ expansion" is a re-organization of the series in the parameter $1/N$, with each term typically being a (formal) sum of infinitely many orders of the ordinary perturbation theory. Besides directly examining the perturbation theory, alternative (and more systematic) methodologies of analyzing such expansion were discovered in physics, for instance a method via "dual" field [CJP74], [MZJ03, Section 2], via Schwinger-Dyson equations [Sym77], or via stochastic quantization (with references below).

[3] In fact, we have obtained some partial results for coupled dynamical Φ^4_3, such as convergence of invariant measures to the Gaussian free field.

[4] but there are infinitely many $O(1)$ graphs.

Rigorous study of large N in mathematical physics was initiated by Kupiainen [Kup80c, Kup80b, Kup80a]. The literature most related to the present article is [Kup80b], which studied the QFT in continuum in $d=2$ given by (1.2), and proved that the $1/N$ expansion of the pressure (i.e. vacuum energy, or log of partition per area) is asymptotic, and each order in this expansion can be described by sums of infinitely many Feynman diagrams of certain types. Borel summability of $1/N$ expansion of Schwinger functions for this model was discussed in [BR82].

In [Kup80c] Kupiainen also proved that on the lattice with fixed lattice spacing, the large N expansion of correlation functions of the N-component nonlinear sigma model (which simplifies to "spherical model" as $N\to\infty$) is asymptotic above the spherical model criticality; asymptoticity was later extended to Borel summability by [FMR82]. Large N limit and expansion for Yang-Mills model has also been rigorously studied: see [Lév11] (also [AS12]) for convergence of Wilson loop observables to master field in the continuum plane, and [Cha19] (resp. [CJ16]) for computation of correlations of the Wilson loops in the large N limit (resp. $1/N$ expansion) which relates to string theory.

Large N problems in the stochastic quantization formalism have also been discussed in the physics literature, for instance [Alf83, AS83], [DH87, Section 8]. [MZJ03, Section 5.1] is close to our setting; it makes an "ansatz" that $\frac{1}{N}\sum_{j=1}^N \Phi_j^2$ in (1.1) would self-average in the large N limit to a constant; our present paper justifies this ansatz and in the non-equilibrium setting generalizes it.

In summary, the study of large N problems in QFT is motivated by the following properties (among others). The first property is simplification or solvability as $N\to\infty$. This is the motivation ever since the earliest literature [Sta68] as aforementioned: the model studied therein becomes a simplified, solvable model as $N\to\infty$ known as the Berlin-Kac spherical model. In our setting this simplification or solvability heuristics are reflected by the Gaussian free field asymptotic as well as the rigorous derivation of exact formula (which would not be possible for finite N) for certain correlation of observables in Theorem 1.2 and Theorem 1.4. Another property is that when N is large, $1/N$ serves as a natural perturbation parameter in QFT models, as already discussed above. Of course this went much farther than just simplifying things later when applied to more sophisticated models like gauge theory, for which $1/N$ expansions led to the discovery of so called gauge-string duality as mentioned above.

1.2. **Mean field limits.** As mentioned above the proof of our main theorems borrows some ingredients from mean field limit theory (MFT). To the best of our knowledge, the study of mean field problems originated from McKean [McK67]. Typically, a mean field problem is concerned with a system of N particles interacting with each other, which is often modeled by a system of stochastic *ordinary* differential equations, for instance, driven by independent Brownian motions. A prototype of such systems has the form $\mathrm{d}X_i=\frac{1}{N}\sum_j f(X_i,X_j)\mathrm{d}t+\mathrm{d}B_i$, see for instance the classical reference by Sznitman [Szn91, Sec I(1)], and in the $N\to\infty$ limit one could obtain decoupled SDEs each interacting with the law of itself: $\mathrm{d}Y_i=\int f(Y_i,y)\mu(\mathrm{d}y)\mathrm{d}t+\mathrm{d}B_i$ where $\mu(\mathrm{d}y)$ is the law of Y_i. So just as in QFT the motivation of MFT is also a simplification of an N-body system to a one-body equation which interacts with itself, i.e. the system is factorized.

In simple situations the interaction f is assumed to be "nice", for instance globally Lipschitz ([McK67]); much of the literature aims to prove such limits under more general assumptions on the interaction, see [Szn91] for a survey. [5] Our Theorem 1.1 can be viewed as a result of this flavor, in an SPDE setting, and in fact the starting point of our proof is indeed close in spirit to [Szn91, Sec I(1)] where one subtracts X_i from Y_i to cancel the noise and then bound a suitable norm of the difference.

We note that mean field limits are studied under much broader frameworks or scopes of applications, such as mean field limit in the context of rough paths (e.g. [CL15, BCD18, CDFM18]), mean field games (e.g. survey [LL07]), quantum dynamics (e.g. [ESY10] and references therein). We do not

[5] In the context of SDE systems, one also considers the empirical measures of the particle configurations, and aims to show their convergence as $N\to\infty$ to the McKean-Vlasov PDEs, which are typically deterministic. Note that in this paper we do not consider the "analogue" of McKean-Vlasov PDE (which would be infinite dimensional) in the context of our model.

intend to have a comprehensive list, but rather refer to survey articles [Gol16, Jab14] and the book [Spo91, Chapter 8] besides [Szn91].

The study of mean field limit for SPDE systems also has precursors, see for instance the book [KX95, Chapter 9] or [ES13]. However these results make strong assumptions on the interactions of the SPDE systems such as linear growth and globally Lipschitz drift, and certainly do not cover the singular regime where renormalization is required as in our case.

1.3. **Structure of the paper.** This paper is organized as follows. In Section 2, we collect the notations and useful lemmas used throughout the paper. Sections 3-5 are devoted to proof of Theorem 1.1. First in Section 3.1 we recall the definition of the renormalization for Z_i, which satisfies the linear equation (1.3). Then a uniform in N estimate for the average of the L^2-norm of Y_i, the solutions to equation (1.4), is derived in Section 3.2. Local well-posedness to equation (1.6) is proved in Section 4.1. Global well-posedness to equation (1.6) is proved in Section 4.2 by combining a uniform L^p-estimate with Schauder theory. The difference estimate for $\Phi_i - \Psi_i$ is given in Section 5, which gives the proof of Theorem 1.1.

Section 6 is concerned with the proof of Theorem 1.2 and Theorem 1.3. In Section 6.1, uniqueness of invariant measures to (1.6) for large m is proved. The convergence of invariant measures from $\nu^{N,i}$ to the Gaussian free field ν is shown in Section 6.2 by comparing the stationary solutions (Φ_i, Z_i). Section 6.3 is devoted to the study of the observables and the proof of Theorem 1.3.

Finally in Section 7 all the results have been stated in the one dimensional case. In Section 7.1 the convergence of the dynamics is stated. Section 7.2 is the corresponding part of the uniqueness of the invariant measure and the convergence of the invariant measures. Section 7.3 mainly concentrates on the proof of the nontriviality of the statistics of the observables. In Appendix A we give the proof of global well-posedness of equation (1.4). In Appendix B we give the proof of the result in Sections 7.1-7.2. In Appendix C the application of Dyson-Schwinger equations has been derived, which is useful in studying the limiting law of the observables.

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2. Preliminary

Throughout the paper, we use the notation $a \lesssim b$ if there exists a constant $c > 0$ such that $a \leqslant cb$, and we write $a \simeq b$ if $a \lesssim b$ and $b \lesssim a$. Given a Banach space E with a norm $\|\cdot\|_E$ and $T > 0$, we write $C_T E = C([0, T]; E)$ for the space of continuous functions from $[0, T]$ to E, equipped with the supremum norm $\|f\|_{C_T E} = \sup_{t\in[0,T]} \|f(t)\|_E$. For $p \in [1, \infty]$ we write $L^p_T E = L^p([0, T]; E)$ for the space of L^p-integrable functions from $[0, T]$ to E, equipped with the usual L^p-norm. Let $\mathcal{S}'$ be the space of distributions on $\mathbb{T}^d$. We use $(\Delta_i)_{i\geqslant -1}$ to denote the Littlewood–Paley blocks for a dyadic partition of unity. Besov spaces on the torus with general indices $\alpha \in \mathbb{R}$, $p, q \in [1, \infty]$ are defined as the completion of C^∞ with respect to the norm

$$\|u\|_{B^\alpha_{p,q}} := (\sum_{j\geqslant -1} (2^{j\alpha} \|\Delta_j u\|^q_{L^p})^{1/q},$$

and the Hölder-Besov space $\mathbf{C}^\alpha$ is given by $\mathbf{C}^\alpha = B^\alpha_{\infty,\infty}$. We will often write $\|\cdot\|_{\mathbf{C}^\alpha}$ instead of $\|\cdot\|_{B^\alpha_{\infty,\infty}}$.

Set $\Lambda=(1-\Delta)^{\frac12}$. For $s\geqslant 0$, $p\in[1,+\infty]$ we use H^s_p to denote the subspace of L^p, consisting of all f which can be written in the form $f=\Lambda^{-s}g, g\in L^p$ and the H^s_p norm of f is defined to be the L^p norm of g, i.e. $\|f\|_{H^s_p}:=\|\Lambda^s f\|_{L^p}$. For $s<0$, $p\in(1,\infty)$, H^s_p is the dual space of H^{-s}_q with $\frac1p+\frac1q=1$. Set $H^s:=H^s_2$.

The following embedding results will be frequently used (e.g. [Tri78]).

Lemma 2.1. *(i) Let $1\leqslant p_1\leqslant p_2\leqslant\infty$ and $1\leqslant q_1\leqslant q_2\leqslant\infty$, and let $\alpha\in\mathbb{R}$. Then $B^\alpha_{p_1,q_1}\subset B^{\alpha-d(1/p_1-1/p_2)}_{p_2,q_2}$. (cf. [GIP15, Lemma A.2])*

(ii) Let $s\in\mathbb{R}$, $1<p<\infty$, $\epsilon>0$. Then $H^s_2=B^s_{2,2}$, and $B^s_{p,1}\subset H^s_p\subset B^s_{p,\infty}\subset B^{s-\epsilon}_{p,1}$. (cf. [Tri78, Theorem 4.6.1])

(iii) Let $1\leqslant p_1\leqslant p_2<\infty$ and let $\alpha\in\mathbb{R}$. Then $H^\alpha_{p_1}\subset H^{\alpha-d(1/p_1-1/p_2)}_{p_2}$.

Here $\subset$ means continuous and dense embedding.

We recall the following interpolation inequality and multiplicative inequality for the elements in H^s_p:

Lemma 2.2. *(i) Suppose that $s\in(0,1)$ and $p\in(1,\infty)$. Then for $f\in H^1_p$*

$$\|f\|_{H^s_p}\lesssim\|f\|_{L^p}^{1-s}\|f\|^s_{H^1_p}.$$

(cf. [Tri78, Theorem 4.3.1])

(ii) Suppose that $s>0$ and $p\in[1,\infty)$. It holds that

$$\|\Lambda^s(fg)\|_{L^p}\lesssim\|f\|_{L^{p_1}}\|\Lambda^s g\|_{L^{p_2}}+\|g\|_{L^{p_3}}\|\Lambda^s f\|_{L^{p_4}},\tag{2.1}$$

with $p_i\in(1,\infty], i=1,...,4$ such that

$$\frac1p=\frac1{p_1}+\frac1{p_2}=\frac1{p_3}+\frac1{p_4}.$$

(cf. see [GO14, Theorem 1])

(iii) (Gagliardo-Nirenberg inequality) For $s\in[0,1)$, $\alpha\in(0,1)$, $r\geqslant1$,

$$\|u\|_{H^s_q}\lesssim\|u\|^\alpha_{H^1}\|u\|^{1-\alpha}_{L^r}\tag{2.2}$$

with $\frac1q=\frac sd+\alpha(\frac12-\frac1d)+\frac{1-\alpha}{r}$.

Lemma 2.3. *(i) Let $\alpha,\beta\in\mathbb{R}$ and $p,p_1,p_2,q\in[1,\infty]$ be such that $\frac1p=\frac1{p_1}+\frac1{p_2}$. The bilinear map $(u,v)\mapsto uv$ extends to a continuous map from $B^\alpha_{p_1,q}\times B^\beta_{p_2,q}$ to $B^{\alpha\wedge\beta}_{p,q}$ if $\alpha+\beta>0$. (cf. [MW17b, Corollary 2])*

(ii) (Duality.) Let $\alpha\in(0,1)$, $p,q\in[1,\infty]$, p' and q' be their conjugate exponents, respectively. Then the mapping $(u,v)\mapsto\langle u,v\rangle=\int uv dx$ extends to a continuous bilinear form on $B^\alpha_{p,q}\times B^{-\alpha}_{p',q'}$, and one has $|\langle u,v\rangle|\lesssim\|u\|_{B^\alpha_{p,q}}\|v\|_{B^{-\alpha}_{p',q'}}$ (cf. [MW17b, Proposition 7]).

We recall the following smoothing effect of the heat flow $S_t=e^{t(\Delta-m)}$, $m\geqslant0$ (e.g. [GIP15, Lemma A.7], [MW17b, Proposition 5]).

Lemma 2.4. *Let $u\in B^\alpha_{p,q}$ for some $\alpha\in\mathbb{R},p,q\in[1,\infty]$. Then for every $\delta\geqslant0$*

$$\|S_tu\|_{B^{\alpha+\delta}_{p,q}}\lesssim t^{-\delta/2}\|u\|_{B^\alpha_{p,q}},$$

where the proportionality constant is independent of t.

Lemma 2.5. *For $s\in(0,1)$*

$$\langle g,f\rangle\lesssim\big(\|\nabla g\|^s_{L^1}\|g\|^{1-s}_{L^1}+\|g\|_{L^1}\big)\|f\|_{\mathbf{C}^{-s}}.$$

Proof. This follows from Lemma 2.3 which states that $\langle g,f\rangle$ is a continuous bilinear form on $B^s_{1,1}\times\mathbf{C}^{-s}$, together with [MW17b, Proposition 8] which states that $\|g\|_{B^s_{1,1}}\lesssim\|\nabla g\|^s_{L^1}\|g\|^{1-s}_{L^1}+\|g\|_{L^1}$. □

We also recall the following comparison test result, which has been proved in [TW18, Lemma 3.8].

Lemma 2.6. *Let $f : [0,T] \to [0,\infty)$ differentiable such that for every $t \in [0,T]$*

$$\frac{\mathrm{d}f}{\mathrm{d}t} + c_1 f^2 \leqslant c_2.$$

Then for $t > 0$

$$f(t) \leqslant \Big(t^{-1}\frac{2}{c_1}\Big) \vee \Big(\frac{2c_2}{c_1}\Big)^{\frac{1}{2}}.$$

Finally, recall that for mean-zero independent random variables $U_1, \dots, U_N$ taking values in a Hilbert space H, we have

$$\mathbf{E}\Big\|\sum_{i=1}^N U_i\Big\|_H^2 = \mathbf{E}\sum_{i=1}^N \|U_i\|_H^2. \tag{2.3}$$

This simple fact is important for us since the square of the sum on the l.h.s. of (2.3) appears to have "N^2 terms" but under expectation it's only a sum of N terms, in certain sense giving us a "factor of $1/N$".

3. Uniform in N bounds on the dynamical linear sigma model

In this section, we obtain new estimates on the Wick renormalized version of (1.1), given by

$$\mathscr{L}\Phi_i = -\frac{1}{N}\sum_{j=1}^N :\Phi_j^2\Phi_i: + \xi_i, \quad \Phi_i(0) = \phi_i. \tag{3.1}$$

The notion of solution to (3.1) is the same as in [DPD03] and [MW17b], where the case $N = 1$ is treated. For a fixed N, these well-posedness results are easy to generalize to the present setting, so we only give the statement here and refer the reader to Appendix A for the proof. Our primary goal in this section is rather to obtain bounds which are stable with respect to the number of components N, which we will send to infinity in Section 5.

As is well known, it's natural to consider initial datum to (3.1) belonging to a negative Hölder space with exponent just below zero. We will be slightly more general and consider random initial datum of the form $\phi_i = z_i + y_i$ satisfying $\mathbf{E}\|z_i\|_{\mathbf{C}^{-\kappa}}^p \lesssim 1$ for $\kappa > 0$ small enough and every $p > 1$, and $\mathbf{E}\|y_i\|_{L^2}^2 \lesssim 1$, where the implicit constants are independent of i, N.

The notion of solution to (3.1) is based on the now classical trick of Da-Prato and Debussche, c.f. [DPD03]. Namely, we say that Φ_i is a solution to (3.1) provided the decomposition $\Phi_i = Z_i + Y_i$ holds, where Z_i is a solution to the linear SPDE

$$\mathscr{L}Z_i = \xi_i, \quad Z_i(0) = z_i, \tag{3.2}$$

and Y_i is a weak solution to the remainder equation

$$\begin{aligned} \mathscr{L}Y_i &= -\frac{1}{N}\sum_{j=1}^N (Y_j^2 Y_i + Y_j^2 Z_i + 2Y_jY_iZ_j + 2Y_j :Z_iZ_j: + :Z_j^2: Y_i + :Z_iZ_j^2:), \\ Y_i(0) &= y_i. \end{aligned} \tag{3.3}$$

The notation $:Z_iZ_j:$, $:Z_j^2:$ and $:Z_iZ_j^2:$ denotes a renormalized product of Wick type which will be defined in Section 3.1 below.

3.1. **Renormalization.** To define the renormalized products appearing in (3.3), it is convenient to make a further splitting of Z_i relative to the corresponding stationary solution to (3.2), which we will denote by $\tilde{Z}_i$. For $\tilde{Z}_i$, these products have a canonical definition that we now recall. Namely, let $\xi_{i,\varepsilon}$ be a space-time mollification of ξ_i defined on $\mathbb{R}\times\mathbb{T}^2$ and let $\tilde{Z}_{i,\varepsilon}$ be the stationary solution to $\mathscr{L}\tilde{Z}_{i,\varepsilon}=\xi_{i,\varepsilon}$. For convenience, we assume that all the noises are mollified with a common bump function. In particular, $\tilde{Z}_{i,\varepsilon}$ are i.i.d. mean zero Gaussian. For $k\geqslant 1$ and $i_1,\dots,i_k\in\{1,\cdots,N\}$ we then write $:\tilde{Z}_{i_1}\cdots\tilde{Z}_{i_k}:$ as the limit of $:\tilde{Z}_{i_1,\varepsilon}\cdots\tilde{Z}_{i_k,\varepsilon}:$ as $\varepsilon\to 0$. Here $:\tilde{Z}_{i_1,\varepsilon}\cdots\tilde{Z}_{i_k,\varepsilon}:$ is the canonical Wick product, which in particular is mean zero. More precisely,

$$:\tilde{Z}_i\tilde{Z}_j: \;=\begin{cases}\lim\limits_{\varepsilon\to0}(\tilde{Z}_{i,\varepsilon}^2-a_\varepsilon) & (i=j)\\ \lim\limits_{\varepsilon\to0}\tilde{Z}_{i,\varepsilon}\tilde{Z}_{j,\varepsilon} & (i\neq j)\end{cases}\qquad :\tilde{Z}_i\tilde{Z}_j^2: \;=\begin{cases}\lim\limits_{\varepsilon\to0}(\tilde{Z}_{i,\varepsilon}^3-3a_\varepsilon\tilde{Z}_{i,\varepsilon}) & (i=j)\\ \lim\limits_{\varepsilon\to0}(\tilde{Z}_{i,\varepsilon}\tilde{Z}_{j,\varepsilon}^2-a_\varepsilon\tilde{Z}_{i,\varepsilon}) & (i\neq j)\end{cases}\tag{3.4}$$

where $a_\varepsilon=\mathbf{E}[\tilde{Z}_{i,\varepsilon}^2(0,0)]$ is a diverging constant independent of i and the limits are understood in $C_T\mathbf{C}^{-\kappa}$ for $\kappa>0$. (see [MW17b] for more details).

We now define the Wick products for Z_i by combining the above with the smoothing properties of the heat semi-group S_t associated with $\mathscr{L}$. Defining $\tilde{z}_i\stackrel{\text{def}}{=}z_i-\tilde{Z}_i(0)$, we have the decomposition

$$Z_i=\tilde{Z}_i+S_t\tilde{z}_i.$$

We then overload notation and define the Wick products of Z_i by the binomial formula [6] namely

$$\begin{aligned}:Z_j^2: \;&= :\tilde{Z}_j^2: +2S_t\tilde{z}_j\tilde{Z}_j+(S_t\tilde{z}_j)^2,\\ :Z_j^3: \;&= :\tilde{Z}_j^3: +3S_t\tilde{z}_j :\tilde{Z}_j^2: +3(S_t\tilde{z}_j)^2\tilde{Z}_j+(S_t\tilde{z}_j)^3,\end{aligned}$$

and for $i\neq j$

$$\begin{aligned}:Z_iZ_j: \;&= :\tilde{Z}_i\tilde{Z}_j: +S_t\tilde{z}_i\tilde{Z}_j+S_t\tilde{z}_j\tilde{Z}_i+S_t\tilde{z}_iS_t\tilde{z}_j,\\ :Z_iZ_j^2: \;&= :\tilde{Z}_i\tilde{Z}_j^2: +S_t\tilde{z}_i :\tilde{Z}_j^2: +2S_t\tilde{z}_j :\tilde{Z}_i\tilde{Z}_j: +2S_t\tilde{z}_iS_t\tilde{z}_j\tilde{Z}_j+(S_t\tilde{z}_j)^2\tilde{Z}_i+S_t\tilde{z}_i(S_t\tilde{z}_j)^2.\end{aligned}$$

We caution the reader that this definition is non-canonical, in the sense that these renormalized products are not necessarily mean zero. By the calculation in [MW17b, Corollary 3] (see also [RZZ17, Lemma 3.5]) we have the following estimate:

Lemma 3.1. *For each $\kappa'>\kappa>0$ and all $p\geqslant 1$, we have the following bounds*

$$\begin{aligned}\mathbf{E}\|\tilde{Z}_i\|^p_{C_T\mathbf{C}^{-\kappa}}+\mathbf{E}\|Z_i\|^p_{C_T\mathbf{C}^{-\kappa}}&\lesssim 1.\\ \mathbf{E}\|:\tilde{Z}_i\tilde{Z}_j:\|^p_{C_T\mathbf{C}^{-\kappa}}+\mathbf{E}\|:\tilde{Z}_i\tilde{Z}_j^2:\|^p_{C_T\mathbf{C}^{-\kappa}}&\lesssim 1.\\ \mathbf{E}(\sup_{t\in[0,T]}t^{\kappa'}\|:Z_iZ_j:\|_{\mathbf{C}^{-\kappa}})^p+\mathbf{E}(\sup_{t\in[0,T]}t^{2\kappa'}\|:Z_iZ_j^2:\|_{\mathbf{C}^{-\kappa}})^p&\lesssim 1.\end{aligned}$$

Furthermore, the proportional constants in the inequalities are independent of i,j,N.

By Lemma 3.1, there exists a measurable $\Omega_0\subset\Omega$ with $\mathbf{P}(\Omega_0)=1$ such that for $\omega\in\Omega_0$ and every i,j

$$\|Z_i\|_{C_T\mathbf{C}^{-\kappa}}+\sup_{t\in[0,T]}t^{\kappa'}\|:Z_iZ_j:\|_{\mathbf{C}^{-\kappa}}+\sup_{t\in[0,T]}t^{2\kappa'}\|:Z_iZ_j^2:\|_{\mathbf{C}^{-\kappa}}<\infty.$$

In the following we always consider $\omega\in\Omega_0$. With the above choice of renormalization, classical arguments from [DPD03] can be used to obtain local existence and uniqueness to equation (3.3) by a pathwise fixed point argument. This solution can also be shown to be global, as a simple consequence of a much stronger result, Lemma 3.3, which will be established in detail below. Since the well-posedness arguments for solving equation (3.3) with a fixed number of components is essentially known, we relegate the proof to Appendix A and only state the result here.

[6] This definition is in line with [MW17b, (5.42)], which first considers a linear solution with 0 initial condition rather than a stationary solution as here.

Lemma 3.2. *For each N, there exist unique global solutions (Y_i) to equation (3.3) such that for $1 \leqslant i \leqslant N$, $Y_i \in C_T L^2 \cap L^4_T L^4 \cap L^2_T H^1$.*

3.2. **Uniform in N estimate.** We now turn to our uniform in N bounds on equation (3.3) and note that Y_i itself depends on N, but we omit this throughout. In the following lemma, we show that the empirical averages of the L^2 norms of Y_i can be controlled pathwise in terms of averages of the $C_T \mathbf{C}^{-\kappa}$ norms of Z_i, $:Z_i Z_j:$ and $:Z_i^2 Z_j:$ discussed in Lemma 3.1 .

Lemma 3.3. *Let $s \in [2\kappa, \frac{1}{4})$. There exists a universal constant C such that*

$$\frac{1}{N}\sup_{t\in[0,T]}\sum_{j=1}^N \|Y_j\|_{L^2}^2 + \frac{1}{N}\sum_{j=1}^N \|\nabla Y_j\|_{L^2_T L^2}^2 + \Big\|\frac{1}{N}\sum_{i=1}^N Y_i^2\Big\|_{L^2_T L^2}^2 \leqslant C\int_0^T R_N \mathrm{d}t + \frac{1}{N}\sum_{j=1}^N \|y_j\|_{L^2}^2, \quad (3.5)$$

where

$$\begin{aligned} R_N := 1 &+ \Big(\frac{1}{N}\sum_{j=1}^N \|Z_j\|_{\mathbf{C}^{-s}}^2\Big)^{\frac{2}{1-s}} + \Big(\frac{1}{N}\sum_{j=1}^N \|:Z_j^2:\|_{\mathbf{C}^{-s}}\Big)^{\frac{4}{2-s}} \\ &+ \Big(\frac{1}{N^2}\sum_{i,j=1}^N \|:Z_j Z_i:\|_{\mathbf{C}^{-s}}^2\Big)^{\frac{2}{2-s}} + \Big(\frac{1}{N^2}\sum_{i,j=1}^N \|:Z_j^2 Z_i:\|_{\mathbf{C}^{-s}}^2\Big). \end{aligned}$$

Proof. The proof is based on an energy estimate. In Step 1 we establish the energy identity (3.7) which identifies the coercive quantities and involves three types of terms on the RHS. These are labelled I_N^1, I_N^2, and I_N^3, which are respectively linear, quadratic, and cubic in Y. In Steps 2-4, we estimate each of these quantities, proceeding in order of difficulty, in terms of the coercive terms and the quantities R_N^i for $i=1,2,3$ defined below. The main ingredient is Lemma 2.5, restated here: for $s \in (0,1)$

$$\langle g, f\rangle \lesssim \big(\|\nabla g\|_{L^1}^s \|g\|_{L^1}^{1-s} + \|g\|_{L^1}\big)\|f\|_{\mathbf{C}^{-s}}. \quad (3.6)$$

The final output of Steps 1-4 is that for some universal constant C it holds

$$\begin{aligned} &\frac{1}{N}\sum_{i=1}^N \frac{\mathrm{d}}{\mathrm{d}t}\|Y_i\|_{L^2}^2 + \frac{1}{N}\sum_{i=1}^N \|\nabla Y_i\|_{L^2}^2 + \frac{1}{N^2}\Big\|\sum_{i=1}^N Y_i^2\Big\|_{L^2}^2 + \frac{m}{N}\sum_{j=1}^N \|Y_j\|_{L^2}^2 \\ &\leqslant C\frac{R_N^1}{N} + C\big(R_N^2 + R_N^3\big)\frac{1}{N}\sum_{i=1}^N \|Y_i\|_{L^2}^2. \end{aligned}$$

Noting that by Hölder's inequality

$$\frac{1}{N}\sum_{i=1}^N \|Y_i\|_{L^2}^2 = \frac{1}{N}\Big\|\sum_{i=1}^N Y_i^2\Big\|_{L^1} \leqslant \frac{1}{N}\Big\|\sum_{i=1}^N Y_i^2\Big\|_{L^2},$$

the estimate (3.5) follows from Young's inequality with exponents $(2,2)$ and an integration over $[0,T]$. The condition $s \in [2\kappa, \frac{1}{4})$ ensures that R_N is integrable near the origin, c.f. Lemma 3.1.

STEP 1 (Energy balance)

In this step, we justify the energy identity

$$\frac{1}{2}\sum_{i=1}^N \frac{\mathrm{d}}{\mathrm{d}t}\|Y_i\|_{L^2}^2 + \sum_{i=1}^N \|\nabla Y_i\|_{L^2}^2 + m\sum_{i=1}^N \|Y_i\|_{L^2}^2 + \Big\|\frac{1}{\sqrt{N}}\sum_{i=1}^N Y_i^2\Big\|_{L^2}^2 = I_N^1 + I_N^2 + I_N^3, \quad (3.7)$$

where the quantities I_N^i for $i=1,2,3$ are defined by

$$I_N^1 \stackrel{\text{def}}{=} -\frac{1}{N}\sum_{i,j=1}^N \langle Y_i, :Z_j^2 Z_i:\rangle$$

$$I_N^2 \stackrel{\text{def}}{=} -\frac{1}{N}\sum_{i,j=1}^N 2\langle Y_i Y_j, :Z_j Z_i:\rangle + \langle Y_i^2, :Z_j^2:\rangle$$

$$I_N^3 \stackrel{\text{def}}{=} -\frac{1}{N}\sum_{i,j=1}^N 3\langle Y_i^2 Y_j, Z_j\rangle$$

Notice that I_N^1, I_N^2, and I_N^3 are linear, quadratic, and cubic in Y, respectively. Formally, the identity (3.7) follows from testing (3.3) by Y_i, integrating by parts, summing over $i = 1, \dots, N$, and using symmetry with respect i and j. Since Y_i is not sufficiently smooth in the time variable, some care is required to make this fully rigorous, and we direct the reader to [MW17b, Proposition 6.8] for more details.

STEP 2 (Estimates for I_N^1)

In this step, we show there is a universal constant C such that

$$I_N^1 \leqslant \frac{1}{4}\sum_{i=1}^N \|\nabla Y_i\|_{L^2}^2 + \sum_{i=1}^N \|Y_i\|_{L^2}^2 + CR_N^1, \tag{3.8}$$

where

$$R_N^1 \stackrel{\text{def}}{=} \sum_{i=1}^N \Big\| \frac{1}{N}\sum_{j=1}^N {:}Z_j^2 Z_i{:} \Big\|_{\mathbf{C}^{-s}}^2 .$$

To establish (3.8), we apply (3.6) with Y_i playing the role of g and $\frac{1}{N}\sum_{j=1}^N {:}Z_j^2 Z_i{:}$ playing the role of f to find

$$I_N^1 \lesssim \sum_{i=1}^N \big(\|Y_i\|_{L^1}^{1-s}\|\nabla Y_i\|_{L^1}^s + \|Y_i\|_{L^1}\big)\Big\|\frac{1}{N}\sum_{j=1}^N {:}Z_j^2 Z_i{:}\Big\|_{\mathbf{C}^{-s}} . \tag{3.9}$$

We now use Young's inequality with exponents $(\frac{2}{1-s}, \frac{2}{s}, 2)$ for the first term and $(2,2)$ for the second term and the embedding of L^2 into L^1 to obtain (3.8), considering separately the cases $m = 0$ and $m > 0$.

STEP 3 (Estimates for I_N^2)

In this step, we show there is a universal constant C such that

$$I_N^2 \leqslant \frac{1}{4}\sum_{i=1}^N \|\nabla Y_i\|_{L^2}^2 + C(1+R_N^2)\Big(\sum_{i=1}^N \|Y_i\|_{L^2}^2\Big), \tag{3.10}$$

where

$$R_N^2 \stackrel{\text{def}}{=} \Big(\frac{1}{N^2}\sum_{i,j=1}^N \|{:}Z_jZ_i{:}\|_{\mathbf{C}^{-s}}^2\Big)^{\frac{1}{2-s}} + \Big\|\frac{1}{N}\sum_{j=1}^N {:}Z_j^2{:}\Big\|_{\mathbf{C}^{-s}}^{\frac{2}{2-s}} \stackrel{\text{def}}{=} \big(\widetilde{R_N^2}\big)^{\frac{1}{2-s}} + \big(\overline{R_N^2}\big)^{\frac{2}{2-s}}.$$

Applying (3.6) with Y_iY_j playing the role of g and ${:}Z_jZ_i{:}$ playing the role of f followed by Hölder's inequality in L^2, the product rule and symmetry with respect to i,j we find

$$\begin{aligned}
\frac{1}{N}\sum_{i,j=1}^N \langle Y_iY_j, {:}Z_jZ_i{:}\,\rangle &\lesssim \frac{1}{N}\sum_{i,j=1}^N \big(\|Y_iY_j\|_{L^1}^{1-s}\|\nabla(Y_iY_j)\|_{L^1}^s + \|Y_iY_j\|_{L^1}\big)\|{:}Z_jZ_i{:}\|_{\mathbf{C}^{-s}} \\
&\lesssim \frac{1}{N}\sum_{i,j=1}^N \big(\|Y_j\|_{L^2}\|Y_i\|_{L^2}^{1-s}\|\nabla Y_i\|_{L^2}^s + \|Y_i\|_{L^2}\|Y_j\|_{L^2}\big)\|{:}Z_jZ_i{:}\|_{\mathbf{C}^{-s}} \\
&\lesssim \Big(\sum_{j=1}^N \|Y_j\|_{L^2}^2\Big)^{\frac12}\Big(\sum_{i=1}^N \|Y_i\|_{L^2}^{2(1-s)}\|\nabla Y_i\|_{L^2}^{2s}\Big)^{\frac12}\big(\widetilde{R_N^2}\big)^{\frac12} + \Big(\sum_{i=1}^N \|Y_i\|_{L^2}^2\Big)\big(\widetilde{R_N^2}\big)^{\frac12} \\
&\lesssim \Big(\sum_{j=1}^N \|Y_j\|_{L^2}^2\Big)^{1-\frac{s}{2}}\Big(\sum_{i=1}^N \|\nabla Y_i\|_{L^2}^2\Big)^{\frac{s}{2}}\big(\widetilde{R_N^2}\big)^{\frac12} + \Big(\sum_{i=1}^N \|Y_i\|_{L^2}^2\Big)\big(\widetilde{R_N^2}\big)^{\frac12},
\end{aligned} \tag{3.11}$$

where we used Hölder's inequality for the summation in i,j with exponents $(2,2)$ followed by Hölder's inequality for the summation in i with exponents $(\frac{1}{1-s},\frac{1}{s})$. Finally, applying (3.6) with Y_i^2 playing the role of g and $\frac{1}{N}\sum_{j=1}^N :Z_j^2:$ playing the role of f we find

$$\frac{1}{N}\sum_{i,j=1}^N \langle Y_i^2, :Z_j^2:\rangle \lesssim \sum_{i=1}^N \big(\|Y_i^2\|_{L^1}^{1-s}\|Y_i\nabla Y_i\|_{L^1}^s + \|Y_i^2\|_{L^1}\big)\overline{R_N^2} \lesssim \sum_{i=1}^N \big(\|Y_i\|_{L^2}^{2-s}\|\nabla Y_i\|_{L^2}^s + \|Y_i\|_{L^2}^2\big)\overline{R_N^2}$$
$$\lesssim \bigg[\Big(\sum_{i=1}^N \|Y_i\|_{L^2}^2\Big)^{1-\frac{s}{2}}\Big(\sum_{i=1}^N \|\nabla Y_i\|_{L^2}^2\Big)^{\frac{s}{2}} + \sum_{i=1}^N \|Y_i\|_{L^2}^2\bigg]\overline{R_N^2}, \tag{3.12}$$

where we used Hölder's inequality for the summation in i with exponents $(\frac{2}{2-s},\frac{2}{s})$. The inequality (3.10) now follows from (3.11)-(3.12) by Young's inequality with exponents $(\frac{2}{2-s},\frac{2}{s})$.

Step 4 (Estimates for I_N^3: cubic terms in Y)

In this step, we show there exists a universal constant C such that

$$I_N^3 \leqslant \frac{1}{4}\bigg(\sum_{i=1}^N \|\nabla Y_i\|_{L^2}^2 + \Big\|\frac{1}{\sqrt{N}}\sum_{i=1}^N Y_i^2\Big\|_{L^2}^2\bigg) + C(1+R_N^3)\Big(\sum_{i=1}^N \|Y_i\|_{L^2}^2\Big), \tag{3.13}$$

where

$$R_N^3 \stackrel{\text{def}}{=} \bigg(\frac{1}{N}\sum_{j=1}^N \|Z_j\|_{\mathbf{C}^{-s}}^2\bigg)^{\frac{1}{1-s}} = (\frac{1}{N}\mathscr{Z}_N)^{\frac{1}{1-s}} \qquad \text{with} \quad \mathscr{Z}_N \stackrel{\text{def}}{=} \sum_{j=1}^N \|Z_j\|_{\mathbf{C}^{-s}}^2 \tag{3.14}$$

Appealing again to (3.6), we find

$$I_N^3 \lesssim \frac{1}{N}\sum_{j=1}^N \Big(\Big\|\sum_{i=1}^N Y_i^2Y_j\Big\|_{L^1}^{1-s}\Big\|\nabla\Big(\sum_{i=1}^N Y_i^2Y_j\Big)\Big\|_{L^1}^s + \Big\|\sum_{i=1}^N Y_i^2Y_j\Big\|_{L^1}\Big)\|Z_j\|_{\mathbf{C}^{-s}} \tag{3.15}$$
$$\lesssim \frac{1}{N}\Big(\sum_{j=1}^N \Big\|\sum_{i=1}^N Y_i^2Y_j\Big\|_{L^1}^{2(1-s)}\Big\|\nabla\Big(\sum_{i=1}^N Y_i^2Y_j\Big)\Big\|_{L^1}^{2s}\Big)^{\frac{1}{2}}\mathscr{Z}_N^{\frac{1}{2}} + \frac{1}{N}\Big(\sum_{j=1}^N \Big\|\sum_{i=1}^N Y_i^2Y_j\Big\|_{L^1}^2\Big)^{\frac{1}{2}}\mathscr{Z}_N^{\frac{1}{2}}.$$

By Hölder's inequality, it holds that

$$\Big\|\sum_{i=1}^N Y_i^2Y_j\Big\|_{L^1} \lesssim \Big\|\sum_{i=1}^N Y_i^2\Big\|_{L^2}\|Y_j\|_{L^2}. \tag{3.16}$$

Furthermore, we find that

$$\Big\|\nabla\Big(\sum_{i=1}^N Y_i^2Y_j\Big)\Big\|_{L^1} \lesssim \Big\|\sum_{i=1}^N Y_i^2\nabla Y_j\Big\|_{L^1} + \Big\|\sum_{i=1}^N \nabla Y_iY_iY_j\Big\|_{L^1}$$
$$\lesssim \Big\|\sum_{i=1}^N Y_i^2\Big\|_{L^2}\|\nabla Y_j\|_{L^2} + \Big(\sum_{i=1}^N \|\nabla Y_i\|_{L^2}^2\Big)^{1/2}\Big\|\sum_{i=1}^N Y_i^2Y_j^2\Big\|_{L^1}^{1/2} \tag{3.17}$$

Hence, we find that

$$\sum_{j=1}^N \Big\|\sum_{i=1}^N Y_i^2Y_j\Big\|_{L^1}^{2(1-s)}\Big\|\nabla\Big(\sum_{i=1}^N Y_i^2Y_j\Big)\Big\|_{L^1}^{2s} \lesssim \Big\|\sum_{i=1}^N Y_i^2\Big\|_{L^2}^2\Big(\sum_{j=1}^N \|Y_j\|_{L^2}^{2(1-s)}\|\nabla Y_j\|_{L^2}^{2s}\Big)$$
$$+ \Big\|\sum_{i=1}^N Y_i^2\Big\|_{L^2}^{2(1-s)}\Big(\sum_{i=1}^N \|\nabla Y_i\|_{L^2}^2\Big)^s\Big(\sum_{j=1}^N \Big\|\sum_{i=1}^N Y_i^2Y_j^2\Big\|_{L^1}^s\|Y_j\|_{L^2}^{2(1-s)}\Big)$$
$$\lesssim \Big\|\sum_{i=1}^N Y_i^2\Big\|_{L^2}^2\Big(\sum_{j=1}^N \|Y_j\|_{L^2}^2\Big)^{1-s}\Big(\sum_{j=1}^N \|\nabla Y_j\|_{L^2}^2\Big)^s$$

$$+\Big\|\sum_{i=1}^N Y_i^2\Big\|_{L^2}^{2(1-s)}\Big(\sum_{i=1}^N\|\nabla Y_i\|_{L^2}^2\Big)^s\Big(\sum_{j=1}^N\Big\|\sum_{i=1}^N Y_i^2Y_j^2\Big\|_{L^1}\Big)^s\Big(\sum_{j=1}^N\|Y_j\|_{L^2}^2\Big)^{1-s}$$
$$\lesssim\Big\|\sum_{i=1}^N Y_i^2\Big\|_{L^2}^2\Big(\sum_{j=1}^N\|Y_j\|_{L^2}^2\Big)^{1-s}\Big(\sum_{j=1}^N\|\nabla Y_j\|_{L^2}^2\Big)^s.$$

Inserting this into (3.15), taking the square root, and using (3.16) we find

$$I_N^3\lesssim\frac{1}{N}\Big\|\sum_{i=1}^N Y_i^2\Big\|_{L^2}\Big(\sum_{j=1}^N\|Y_j\|_{L^2}^2\Big)^{\frac{1-s}{2}}\Big(\sum_{j=1}^N\|\nabla Y_j\|_{L^2}^2\Big)^{\frac{s}{2}}\mathscr{L}_N^{\frac12}+\frac{1}{N}\Big\|\sum_{i=1}^N Y_i^2\Big\|_{L^2}\Big(\sum_{j=1}^N\|Y_j\|_{L^2}^2\Big)^{\frac12}\mathscr{L}_N^{\frac12}. \quad (3.18)$$

Applying Young's inequality with exponent $(2,\frac{2}{1-s},\frac{2}{s})$ we arrive at (3.13). $\square$

Corollary 3.4. *Let $q\geqslant 1$, $s\in[2\kappa,\frac{2}{q+1})$. There exists a universal constant C such that*

$$\sup_{t\in[0,T]}\Big(\frac1N\sum_{j=1}^N\|Y_j\|_{L^2}^2\Big)^q+\int_0^T\Big(\frac1N\sum_{j=1}^N\|Y_j\|_{L^2}^2\Big)^{q-1}\Big[\frac1N\sum_{j=1}^N\|\nabla Y_j\|_{L^2}^2+\Big\|\frac1N\sum_{i=1}^N Y_i^2\Big\|_{L^2}^2\Big]\mathrm{d}t$$
$$\leqslant C\int_0^T R_N^{\frac{q+1}{2}}\mathrm{d}t+\Big(\frac1N\sum_{j=1}^N\|y_j\|_{L^2}^2\Big)^q,$$

with R_N introduced in Lemma 3.3.

Proof. Set $V=\frac1N\sum_{i=1}^N\|Y_i\|_{L^2}^2$ and $G=\frac1N\sum_{j=1}^N\|\nabla Y_j\|_{L^2}^2+\big\|\frac1N\sum_{i=1}^N Y_i^2\big\|_{L^2}^2$. By Lemma 3.3 we deduce for $q\geqslant 1$

$$\frac{\mathrm{d}}{\mathrm{d}t}V^q+GV^{q-1}\leqslant CR_NV^{q-1}\leqslant CR_N^{\frac{q+1}{2}}+\frac12V^{q+1}.$$

Note that $G\geqslant V^2$ since $\|\sum_{i=1}^N Y_i^2\|_{L^1}=\sum_{i=1}^N\|Y_i\|_{L^2}^2$, which implies the result. $\square$

Lemma 3.5. *Let $s\in[2\kappa,1/4)$. There exists a universal constant C such that*

$$\sup_{t\in[0,T]}\sum_{j=1}^N\|Y_j\|_{L^2}^2+\sum_{j=1}^N\|\nabla Y_j\|_{L_T^2L^2}^2+\frac1N\Big\|\sum_{i=1}^N Y_i^2\Big\|_{L_T^2L^2}^2$$
$$\leqslant C(\|R_N^0\|_{L_T^1}+\sum_{j=1}^N\|y_j\|_{L^2}^2)\exp\Big\{\int_0^T(1+R_N^2+R_N^3)\mathrm{d}t\Big\}, \quad (3.19)$$

where R_N^2, R_N^3 given in the proof of Lemma 3.3 and

$$R_N^0=\frac{1}{N^2}\sum_{i=1}^N\Big\|\sum_{j=1}^N\Lambda^{-s}(:Z_j^2Z_i:)\Big\|_{L^2}^2.$$

Proof. The proof is almost the same as Lemma 3.3. We appeal to Steps 1,3, and 4 of Lemma 3.3 and only modify Step 2. To estimate I_N^1 we write

$$I_N^1\leqslant\frac1N\sum_{i=1}^N\|\Lambda^sY_i\|_{L^2}\Big\|\sum_{j=1}^N\Lambda^{-s}(:Z_j^2Z_i:)\Big\|_{L^2}$$
$$\leqslant\frac18\sum_{i=1}^N\|Y_i\|_{L^2}^2+\frac18\sum_{i=1}^N\|\nabla Y_i\|_{L^2}^2+\frac{C}{N^2}\sum_{i=1}^N\Big\|\sum_{j=1}^N\Lambda^{-s}(:Z_j^2Z_i:)\Big\|_{L^2}^2, \quad (3.20)$$

where, in the last step, we applied Young's inequality for products, and then interpolation. Combining (3.10), and (3.13) (3.12) with (3.20) and inserting these inequalities into the energy identity (3.7), we obtain

$$\sum_{j=1}^{N}\frac{\mathrm{d}}{\mathrm{d}t}\|Y_j\|_{L^2}^2+\sum_{j=1}^{N}\|\nabla Y_j\|_{L^2}^2+\frac{1}{N}\Big\|\sum_{i=1}^{N}Y_i^2\Big\|_{L^2}^2+m\sum_{j=1}^{N}\|Y_j\|_{L^2}^2$$
$$\leqslant CR_N^0+\sum_{j=1}^{N}\|Y_j\|_{L^2}C(1+R_N^2+R_N^3). \tag{3.21}$$

The estimate (3.19) now follows from Gronwall's inequality. □

Remark 3.6. *For the estimate of $\frac{1}{N}\sum_{i=1}^N\|Y_i\|_{L^2}^2$ in Lemma 3.3, the dissipation term $\big\|\frac{1}{N}\sum_{i=1}^N Y_i^2\big\|_{L^2}^2$ could be used to avoid Gronwall's Lemma. However, for $\sum_{i=1}^N\|Y_i\|_{L^2}^2$ or $\frac{1}{N}\sum_{i=1}^N\|Y_i\|_{L^p}^p$ for $p>2$ it is less clear how to exploit the corresponding dissipation term and we need to use Gronwall's inequality to derive a uniform estimate. Since the R_N^2, R_N^3 appear in the exponential, this makes it unclear how to obtain moment estimates directly.*

4. Global solvability of the mean-field SPDE

In this section, we develop a solution theory for the mean-field SPDE (1.6), the renormalized version of the formal equation (1.5). In two dimensions this is a singular SPDE where the ill-defined non-linearity depends on the law of the solution. As a result, we cannot proceed via path-wise arguments alone as in [DPD03] and [MW17b] and we need to develop a few new tricks for both the local and global well-posedness.

We begin by explaining our assumptions on the initial data and our notion of solution to (1.6). The initial datum ψ_i decompose as $\psi_i=z_i+\eta_i$, where $\mathbf{E}\|z_i\|_{\mathbf{C}^{-\kappa}}^p\lesssim 1$ for $\kappa>0$ and every $p>1$, and $\mathbf{E}\|\eta_i\|_{L^4}^4<\infty$ (except for Lemma 4.4 which is an L^p estimate). We define Ψ_i to be a solution to the renormalized, mean-field SPDE (1.6) starting from ψ_i provided that $\Psi_i=Z_i+X_i$ holds, where Z_i is the solution to (3.2) with $Z_i(0)=z_i$ as in Section 3 and X_i is a random process satisfying

$$\mathscr{L}X_i=-\mu(X_i+Z_i),\quad X_i(0)=\eta_i. \tag{4.1}$$

Here, μ depends on the law of X_i and is defined as

$$\mu\stackrel{\text{def}}{=}\mathbf{E}[X_i^2]+2\mathbf{E}[X_iZ_i]+\mathbf{E}[\,{:}Z_i^2{:}\,].$$

We now comment on the meaning of the non-linearity in equation (4.1). Recall from Section 3.1 that $Z_i\in\mathbf{C}^{-\kappa}$ (Lemma 3.1), while $\mathbf{E}[\,{:}Z_i^2{:}\,]=\mathbf{E}[(S_t\tilde z_i)^2]$ with $\tilde z_i=z_i-\tilde Z_i(0)$, so by Schauder theory we expect that X_i is Hölder continuous. Hence, we anticipate that $\mathbf{E}[X_i^2]$ is a well-defined function, while $\mathbf{E}[X_iZ_i]$ is a distribution satisfying for $t>0$ and $\beta>\kappa$

$$\|\mathbf{E}[X_iZ_i](t)\|_{\mathbf{C}^{-\kappa}}\lesssim\mathbf{E}[\|X_i(t)\|_{\mathbf{C}^\beta}\|Z_i(t)\|_{\mathbf{C}^{-\kappa}}].$$

We immediately find that all terms in $\mu(X_i+Z_i)$ are classically defined in the sense of distributions except for $\mathbf{E}[X_iZ_i]Z_i$, which requires more care and a suitable probabilistic argument. The idea used to overcome this difficulty, which is repeated in different ways throughout the section, is to view the expectation μ as coming from a suitable independent copy of (X_i,Z_i). To avoid notational confusion, we now comment further on our convention throughout this section. We consider equation (1.6) for a *fixed* i and when we write (η_j,z_j) for $j\neq i$ we mean an independent copy of (η_i,z_i), and we then write (Z_j,X_j) for the solution driven by white noise ξ_j, which is independent of ξ_i, from initial data (z_j,η_j).

4.1. **Local well-posedness.**

Lemma 4.1. *For $p \in [1,\infty]$ and $0<\kappa<s$ it holds*

$$\|Z_i \mathbf{E}[Z_i X_i]\|_{B^{-\kappa}_{p,\infty}} \lesssim \big(\mathbf{E}\|X_i\|^2_{B^s_{p,\infty}}\big)^{\frac12} \big(\mathbf{E}[\| :Z_iZ_j: \|^2_{\mathbf{C}^{-\kappa}} \mid Z_i]\big)^{\frac12}. \tag{4.2}$$

Here the conditional expectation is on the σ-algebra generated by the stochastic process Z_i.

Proof. Letting (X_j, Z_j) be an independent copy of (X_i, Z_i) we have

$$Z_i\mathbf{E}[Z_iX_i] = Z_i\mathbf{E}[Z_jX_j] = \mathbf{E}[:Z_iZ_j: X_j \mid Z_i].$$

We then use Jensen's inequality to find

$$\|Z_i\mathbf{E}[Z_jX_j]\|_{B^{-\kappa}_{p,\infty}} \leqslant \mathbf{E}\big[\| :Z_iZ_j: X_j\|_{B^{-\kappa}_{p,\infty}} \mid Z_i\big] \leqslant \mathbf{E}\big[\| :Z_iZ_j: \|_{\mathbf{C}^{-\kappa}} \|X_j\|_{B^s_{p,\infty}} \mid Z_i\big],$$

where we used Lemma 2.3 in the last line. The claim now follows from conditional Hölder's inequality and the independence of X_j from Z_i. $\square$

We now apply the above result to obtain a local well-posedness result for (4.1), which yields in turn a local well-posedness result for (1.6).

Lemma 4.2. *There exists $T^*>0$ small enough such that (4.1) has a unique mild solution $X_i \in L^2(\Omega; C_{T^*}L^4 \cap C((0,T^*];\mathbf{C}^\beta))$ and for $\beta > 3\kappa$ small enough, $\gamma = \beta + \frac12$, one has*

$$\mathbf{E}[\sup_{t\in[0,T^*]} t^\gamma \|X_i\|^2_{\mathbf{C}^\beta}] \leqslant 1.$$

Proof. For $T>0$ define the ball

$$\mathcal{B}_T \stackrel{\text{def}}{=} \big\{X_i \in L^2(\Omega, C((0,T];\mathbf{C}^\beta)) \mid \mathbf{E}[\sup_{t\in[0,T]} t^\gamma \|X_i\|^2_{\mathbf{C}^\beta}] \leqslant 1, \quad X_i(0) = \eta_i\big\}.$$

Here we endow the space $C((0,T];\mathbf{C}^\beta)$ with norm $(\sup_{t\in[0,T]} t^\gamma \|f(t)\|^2_{\mathbf{C}^\beta})^{1/2}$.

For $X_i \in \mathcal{B}_T$, define $\mathcal{M}_T X_i : (0,T] \mapsto \mathbf{C}^\beta$ via

$$\mathcal{M}_T X_i(t) := \int_0^t S_{t-s}\mathbf{E}[X_i^2 + 2X_iZ_i + :Z_i^2:](X_i+Z_i)\mathrm{d}s + S_t\eta_i.$$

Using Lemma 2.4 and Lemma 2.3 noting $\beta > \kappa$, we find that

$$\begin{aligned}
\|\mathcal{M}_T X_i(t) - S_t\eta_i\|_{\mathbf{C}^\beta} &\lesssim \int_0^t (t-s)^{-\frac{\beta+\kappa}{2}} \|\mathbf{E}[X_iZ_i]Z_i\|_{\mathbf{C}^{-\kappa}}\mathrm{d}s \\
&+ \int_0^t \big(\|\mathbf{E}X_i^2\|_{\mathbf{C}^\beta} + \|\mathbf{E} :Z_i^2: \|_{\mathbf{C}^\beta} + (t-s)^{-\frac{\beta+\kappa}{2}} \|\mathbf{E}[X_iZ_i]\|_{\mathbf{C}^{-\kappa}}\big)\|X_i\|_{\mathbf{C}^\beta}\mathrm{d}s \\
&+ \int_0^t (t-s)^{-\frac{\beta+\kappa}{2}} \big(\|\mathbf{E}X_i^2\|_{\mathbf{C}^\beta} + \|\mathbf{E} :Z_i^2: \|_{\mathbf{C}^\beta}\big)\|Z_i\|_{\mathbf{C}^{-\kappa}}\mathrm{d}s \quad \stackrel{\text{def}}{=} \quad \sum_{i=1}^3 J_i(t).
\end{aligned}$$

We start by applying Lemma 4.1 to obtain the pathwise bound

$$\begin{aligned}
J_1(t) &\lesssim \int_0^t (t-s)^{-\frac{\beta+\kappa}{2}} \big(\mathbf{E}\|X_i\|^2_{\mathbf{C}^\beta}\big)^{\frac12} \big(\mathbf{E}[\| :Z_iZ_j: \|^2_{\mathbf{C}^{-\kappa}} \mid Z_i]\big)^{\frac12}\mathrm{d}s \\
&\lesssim \big(\mathbf{E}[\sup_{r\in[0,t]} (r^{\kappa'}\| :Z_iZ_j: (r)\|_{\mathbf{C}^{-\kappa}})^2 \mid Z_i]\big)^{\frac12} \int_0^t (t-s)^{-\frac{\beta+\kappa}{2}} s^{-\frac{\gamma+2\kappa'}{2}}\mathrm{d}s,
\end{aligned}$$

for $\kappa' > \kappa > 0$, which is integrable provided $\beta < 2-\kappa$ and $\gamma < 2-2\kappa'$. We may now apply Lemma 3.1 to find that

$$\mathbf{E}|J_1(t)|^2 \lesssim \mathbf{E}[\sup_{r\in[0,t]} (r^{\kappa'}\| :Z_iZ_j: (r)\|_{\mathbf{C}^{-\kappa}})^2] t^{2-(\beta+\kappa+2\kappa'+\gamma)} \lesssim t^{2-(\beta+\kappa+2\kappa'+\gamma)}.$$

Before estimating $J_2(t)$ and $J_3(t)$, we make three observations. First note that by Jensen's inequality and Lemma 2.3 it holds

$$\|\mathbf{E}X_i^2\|_{\mathbf{C}^\beta} \lesssim \mathbf{E}[\|X_i\|_{\mathbf{C}^\beta}^2] \lesssim s^{-\gamma}.$$

Furthermore, using again Lemma 2.4 we find

$$\|\mathbf{E} :Z_i^2: \|_{\mathbf{C}^\beta} \leqslant \mathbf{E}\big\|(S_t\tilde{z}_i)^2\big\|_{\mathbf{C}^\beta} \leqslant \mathbf{E}\big\|S_t\tilde{z}_i\big\|_{\mathbf{C}^\beta}^2 \lesssim s^{-(\beta+\kappa)}.$$

Finally, note that by Lemma 2.3

$$\|\mathbf{E}[X_iZ_i]\|_{\mathbf{C}^{-\kappa}} \lesssim \mathbf{E}[\|X_i\|_{\mathbf{C}^\beta}\|Z_i\|_{\mathbf{C}^{-\kappa}}] \lesssim \big(\mathbf{E}[\|X_i\|_{\mathbf{C}^\beta}^2]\big)^{\frac{1}{2}}\big(\mathbf{E}[\|Z_i\|_{\mathbf{C}^{-\kappa}}^2]\big)^{\frac{1}{2}} \lesssim s^{-\frac{\gamma}{2}}.$$

Inserting these three bounds, we find the inequalities

$$J_2(t) \lesssim \int_0^t \big(s^{-\gamma} + s^{-(\beta+\kappa)} + (t-s)^{-\frac{\beta+\kappa}{2}}s^{-\frac{\gamma}{2}}\big)\|X_i\|_{\mathbf{C}^\beta}\mathrm{d}s.$$
$$J_3(t) \lesssim \int_0^t (t-s)^{-\frac{\beta+\kappa}{2}}\big(s^{-\gamma} + s^{-(\beta+\kappa)}\big)\|Z_i\|_{\mathbf{C}^{-\kappa}}\mathrm{d}s.$$

Squaring and taking expectation, we find

$$\mathbf{E}|J_2(t)|^2 \lesssim t^{-\gamma}\big(t^{2-2\gamma} + t^{2-2(\beta+\kappa)} + t^{2-(\beta+\kappa+\gamma)}\big).$$
$$\mathbf{E}|J_3(t)|^2 \lesssim \big(t^{2-(\beta+\kappa+2\gamma)} + t^{2-3(\beta+\kappa)}\big).$$

Under our assumption on β these are all bounded by $t^{-\gamma}$. Finally, note that by Lemma 2.4 and the embedding $L^4 \subset \mathbf{C}^{-\frac{1}{2}}$, we obtain

$$\|S_t\eta\|_{\mathbf{C}^\beta} \lesssim t^{-\frac{1+2\beta}{4}}\|\eta\|_{L^4}. \tag{4.3}$$

Combining the above estimates we can find T^* small enough to have

$$\mathbf{E}[\sup_{t\in[0,T^*]} t^\gamma\|\mathcal{M}_{T^*}X(t)\|_{\mathbf{C}^\beta}^2] \leqslant 1,$$

which implies that for T^* small enough $\mathcal{M}_{T^*}$ maps $\mathcal{B}_{T^*}$ into itself. The contraction property follows similarly. Now the local existence and uniqueness in $L^2(\Omega; C((0,T],\mathbf{C}^\beta))$ follows. Furthermore, we know $\int_0^t S_{t-s}\mu(X(s)+Z(s))\mathrm{d}s$ is continuous in $\mathbf{C}^\beta$ and $S_t\eta_i \in C_TL^4$. The result follows. $\square$

4.2. **Global well-posedness.** We now extend our local solution to a global solution through a series of a priori bounds, starting with a uniform in time on the $L^2(\Omega; L^2)$ norm of X_i together with an $L^2(\Omega; L_T^2H^1)$ bound.

Lemma 4.3. *There exists a universal constant C such that*

$$\sup_{t\in[0,T]}\mathbf{E}\|X_i\|_{L^2}^2 + \mathbf{E}\|\nabla X_i\|_{L_T^2L^2}^2 + \|\mathbf{E}X_i^2\|_{L_T^2L^2}^2 + m\mathbf{E}\|X_i\|_{L_T^2L^2}^2 \leqslant C\int_0^T R\mathrm{d}t + \mathbf{E}\|\eta_i\|_{L^2}^2, \tag{4.4}$$

where, for $i \neq j$ we define

$$R \stackrel{\text{def}}{=} 1 + \big(\mathbf{E}\|Z_i\|_{\mathbf{C}^{-s}}^2\big)^{\frac{2}{1-s}} + \mathbf{E}\| :Z_j^2Z_i: \|_{\mathbf{C}^{-s}}^2 + C\big(\mathbf{E}\| :Z_jZ_i: \|_{\mathbf{C}^{-s}}^2\big)^2 + C\big(\mathbf{E}\| :Z_i^2: \|_{\mathbf{C}^{-s}}\big)^4.$$

Proof. The proof is similar in spirit to the proof of Lemma 3.3, proceeding by energy estimates.

STEP 1 (Expected energy balance)

In this step, we establish the following identity

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}\|X_i\|_{L^2}^2 + \mathbf{E}\|\nabla X_i\|_{L^2}^2 + \|\mathbf{E}X_i^2\|_{L^2}^2 + m\mathbf{E}\|X_i\|_{L^2}^2 = I^1 + I^2 + I^3, \tag{4.5}$$

where

$$I^1 \stackrel{\text{def}}{=} \mathbf{E}\langle X_i, Z_i :Z_j^2: \rangle, \qquad I^2 \stackrel{\text{def}}{=} \mathbf{E}\langle X_i^2, :Z_j^2: \rangle + 2\mathbf{E}\langle X_iX_j, Z_iZ_j\rangle, \qquad I^3 \stackrel{\text{def}}{=} 3\mathbf{E}\langle X_i^2X_j, Z_j\rangle.$$

Formally, testing (4.1) with X_i, integrating by parts and using that X_i, X_iZ_i, and $:Z_i^2:$ are respectively equal in law to X_j, X_jZ_j, and $:Z_j^2:$ we find

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|X_i\|_{L^2}^2+\|\nabla X_i\|_{L^2}^2+m\|X_i\|_{L^2}^2+\|X_i^2\mathbf{E}X_i^2\|_{L^1}\\&\qquad=-\langle X_i,Z_i\mathbf{E}(:Z_j^2:)\rangle-\langle X_i^2,\mathbf{E}(:Z_j^2:)\rangle-2\langle X_i,Z_i\mathbf{E}(X_jZ_j)\rangle-2\langle X_i^2,\mathbf{E}(X_jZ_j)\rangle-\langle X_i\mathbf{E}(X_j^2),Z_i\rangle.\end{aligned}$$

Taking expectation on both sides, using independence, and the fact that $X_i^2X_jZ_j$ has the same law as $X_j^2X_iZ_i$ we obtain (4.5).

STEP 2 (Estimates for I^1)

In this step, we show there is a universal constant C such that

$$I^1\leqslant\frac{1}{4}\big(\|\mathbf{E}X_i^2\|_{L^2}^2+\mathbf{E}\|\nabla X_i\|_{L^2}^2\big)+C\big(1+\mathbf{E}\|:Z_j^2:Z_i\|_{\mathbf{C}^{-s}}^2\big).\tag{4.6}$$

To prove the claim, we apply (3.6) to have

$$I^1\lesssim\mathbf{E}\big[\big(\|X_i\|_{L^1}^{1-s}\|\nabla X_i\|_{L^1}^s+\|X_i\|_{L^1}\big)\|:Z_j^2Z_i:\|_{\mathbf{C}^{-s}}\big]$$

Hence, (4.6) follows from the inequality $\mathbf{E}\|X_i\|_{L^1}^2\leqslant\|\mathbf{E}X_i^2\|_{L^2}$ and Young's inequality with exponents $(\frac{2}{1-s},\frac{2}{s},2)$ and $(2,2)$.

STEP 3 (Estimates for I^2)

In this step, we show that there is a universal constant C such that

$$I^2\leqslant\frac{1}{4}\big(\mathbf{E}\|\nabla X_i\|_{L^2}^2+\|\mathbf{E}X_i^2\|_{L^2}^2\big)+C+C\big(\mathbf{E}\|:Z_jZ_i:\|_{\mathbf{C}^{-s}}^2\big)^2+C\big(\mathbf{E}\|:Z_j^2:\|_{\mathbf{C}^{-s}}\big)^4.\tag{4.7}$$

Using again (3.6), Young's inequality, Hölder's inequality and the independence of X_i and X_j we obtain

$$\begin{aligned}\mathbf{E}\langle X_iX_j,:Z_jZ_i:\rangle&\lesssim\mathbf{E}\big(\|X_iX_j\|_{L^1}+\|\nabla X_iX_j\|_{L^1}+\|X_i\nabla X_j\|_{L^1}\big)\|:Z_jZ_i:\|_{\mathbf{C}^{-s}}\\&\lesssim\mathbf{E}\big(\|X_i\|_{L^2}\|X_j\|_{L^2}+\|\nabla X_i\|_{L^2}\|X_j\|_{L^2}\big)\|:Z_jZ_i:\|_{\mathbf{C}^{-s}}\\&\lesssim\big(\mathbf{E}\|X_i\|_{L^2}^2\mathbf{E}\|X_j\|_{L^2}^2+\mathbf{E}\|\nabla X_i\|_{L^2}^2\mathbf{E}\|X_j\|_{L^2}^2\big)^{1/2}\big(\mathbf{E}\|:Z_jZ_i:\|_{\mathbf{C}^{-s}}^2\big)^{1/2}.\\&\lesssim\big(\|\mathbf{E}X_i^2\|_{L^1}^2+\mathbf{E}\|\nabla X_i\|_{L^2}^2\|\mathbf{E}X_j^2\|_{L^1}\big)^{1/2}\big(\mathbf{E}\|:Z_jZ_i:\|_{\mathbf{C}^{-s}}^2\big)^{1/2}.\end{aligned}\tag{4.8}$$

Similarly, using this time independence of X_i^2 and $:Z_j^2:$ we obtain

$$\begin{aligned}\mathbf{E}\langle X_i^2,:Z_j^2:\rangle&\lesssim\mathbf{E}\big(\|X_i^2\|_{L^1}+\|\nabla X_i\|_{L^2}\|X_i\|_{L^2}\big)\|:Z_j^2:\|_{\mathbf{C}^{-s}}\\&=\big(\|\mathbf{E}X_i^2\|_{L^1}+\mathbf{E}\|\nabla X_i\|_{L^2}\|X_i\|_{L^2}\big)\,\mathbf{E}\|:Z_j^2:\|_{\mathbf{C}^{-s}}\\&\lesssim\big(\|\mathbf{E}X_i^2\|_{L^1}+\big(\mathbf{E}\|\nabla X_i\|_{L^2}^2\big)^{1/2}\|\mathbf{E}X_i^2\|_{L^1}^{1/2}\big)\,\mathbf{E}\|:Z_j^2:\|_{\mathbf{C}^{-s}}.\end{aligned}\tag{4.9}$$

To obtain (4.7) we use Young's inequality with exponents $(2,2)$ and $(2,4,4)$ for both (4.8) (4.9).

STEP 4 (Estimates for I^3)

In this step, we show there is a universal constant C such that

$$I^3\leqslant\frac{1}{4}\big(\mathbf{E}\|\nabla X_i\|_{L^2}^2+\|\mathbf{E}X_i^2\|_{L^2}^2\big)+C\big(\big(\mathbf{E}\|Z_j\|_{\mathbf{C}^{-s}}^2\big)^{\frac{2}{1-s}}+1\big).\tag{4.10}$$

To this end, we write

$$\begin{aligned}I^3&\lesssim\mathbf{E}\big(\|X_i^2X_j\|_{L^1}^{1-s}\|\nabla(X_i^2X_j)\|_{L^1}^s+\|X_i^2X_j\|_{L^1}\big)\|Z_j\|_{\mathbf{C}^{-s}}\\&\lesssim(\mathbf{E}\|X_i^2X_j\|_{L^1}\|Z_j\|_{\mathbf{C}^{-s}})^{1-s}(\mathbf{E}\|\nabla(X_i^2X_j)\|_{L^1}\|Z_j\|_{\mathbf{C}^{-s}})^s+\mathbf{E}\|X_i^2X_j\|_{L^1}\|Z_j\|_{\mathbf{C}^{-s}}.\end{aligned}$$

By independence and Hölder's inequality, it holds that

$$\begin{aligned}\mathbf{E}\|X_i^2X_j\|_{L^1}\|Z_j\|_{\mathbf{C}^{-s}}&\lesssim\big\|\mathbf{E}X_i^2\mathbf{E}[|X_j|\|Z_j\|_{\mathbf{C}^{-s}}]\big\|_{L^1}\lesssim\|\mathbf{E}X_i^2\|_{L^2}\big\|(\mathbf{E}X_j^2)^{1/2}(\mathbf{E}\|Z_j\|_{\mathbf{C}^{-s}}^2)^{1/2}\big\|_{L^2}\\&\lesssim\|\mathbf{E}X_i^2\|_{L^2}\big(\mathbf{E}\|X_j\|_{L^2}^2\big)^{\frac{1}{2}}\big(\mathbf{E}\|Z_j\|_{\mathbf{C}^{-s}}^2\big)^{1/2},\end{aligned}\tag{4.11}$$

where we used that $\|(\mathbf{E}X_j^2)^{1/2}\|_{L^2} = \|\mathbf{E}X_j^2\|_{L^1}^{\frac12} = \big(\mathbf{E}\|X_j\|_{L^2}^2\big)^{\frac12}$. Furthermore,

$$\begin{aligned}\mathbf{E}\|X_i^2\nabla X_j\|_{L^1}\|Z_j\|_{\mathbf{C}^{-s}} &= \big\|\mathbf{E}X_i^2\mathbf{E}\big(|\nabla X_j|\|Z_j\|_{\mathbf{C}^{-s}}\big)\big\|_{L^1}\\ &\leqslant \|\mathbf{E}X_i^2\|_{L^2}\big\|\mathbf{E}\big(|\nabla X_j|\|Z_j\|_{\mathbf{C}^{-s}}\big)\big\|_{L^2} \leqslant \|\mathbf{E}X_i^2\|_{L^2}\big\|(\mathbf{E}|\nabla X_j|^2)^{\frac12}(\mathbf{E}\|Z_j\|^2_{\mathbf{C}^{-s}})^{\frac12}\big\|_{L^2}\\ &\lesssim \|\mathbf{E}X_i^2\|_{L^2}\big(\mathbf{E}\|\nabla X_j\|_{L^2}^2\big)^{1/2}\big(\mathbf{E}\|Z_j\|^2_{\mathbf{C}^{-s}}\big)^{\frac12}. \end{aligned} \tag{4.12}$$

Similarly, note that

$$\begin{aligned}\mathbf{E}\|X_iX_j\nabla X_i\|_{L^1}\|Z_j\|_{\mathbf{C}^{-s}} &\lesssim \mathbf{E}\|X_iX_j\|_{L^2}\big(\|\nabla X_i\|_{L^2}\|Z_j\|_{\mathbf{C}^{-s}}\big)\\ &\lesssim \big(\mathbf{E}\|X_iX_j\|_{L^2}^2\big)^{1/2}\big(\mathbf{E}\|\nabla X_i\|^2_{L^2}\mathbf{E}\|Z_j\|^2_{\mathbf{C}^{-s}}\big)^{1/2}\\ &\lesssim \|\mathbf{E}X_i^2\|_{L^2}\big(\mathbf{E}\|\nabla X_i\|^2_{L^2}\big)^{1/2}\big(\mathbf{E}\|Z_j\|^2_{\mathbf{C}^{-s}}\big)^{1/2}. \end{aligned} \tag{4.13}$$

Combining the above estimate we arrive at

$$\begin{aligned} I^3 &\lesssim \|\mathbf{E}X_i^2\|_{L^2}\big(\mathbf{E}\|\nabla X_i\|^2_{L^2}\big)^{\frac{s}{2}}\big(\mathbf{E}\|X_i\|^2_{L^2}\big)^{\frac{1-s}{2}}\big(\mathbf{E}\|Z_j\|^2_{\mathbf{C}^{-s}}\big)^{\frac12} + \|\mathbf{E}X_i^2\|_{L^2}\big(\mathbf{E}\|X_j\|^2_{L^2}\big)^{\frac12}\big(\mathbf{E}\|Z_j\|^2_{\mathbf{C}^{-s}}\big)^{\frac12}\\ &\lesssim \|\mathbf{E}X_i^2\|_{L^2}^{\frac{3-s}{2}}\big(\mathbf{E}\|\nabla X_i\|^2_{L^2}\big)^{\frac{s}{2}}\big(\mathbf{E}\|Z_j\|^2_{\mathbf{C}^{-s}}\big)^{\frac12} + \|\mathbf{E}X_i^2\|_{L^2}^{\frac32}\big(\mathbf{E}\|Z_j\|^2_{\mathbf{C}^{-s}}\big)^{\frac12}. \end{aligned} \tag{4.14}$$

Applying Young's inequality with exponents $(\frac{4}{3-s}, \frac{2}{s}, \frac{4}{1-s})$ we arrive at (4.10). □

In Section 5 we will study the large N limit of (3.1) by comparing the dynamics of each component to the corresponding mean-field evolution. To control the equation for the difference, we will need a stronger control on X_i than the L^2 type bound obtained above. In the following lemma, we show that L^p bounds can be propagated in time, which will turn out to be a necessary ingredient in Section 5.

Lemma 4.4. *Let $p > 2$ and assume that $\mathbf{E}\|\eta_i\|^p_{L^p} \lesssim 1$. Then, we have*

$$\sup_{t\in[0,T]} \mathbf{E}\|X_i\|^p_{L^p} + \mathbf{E}\||X_i|^{\frac{p-2}{2}}\nabla X_i\|^2_{L^2_TL^2} + \|\mathbf{E}|X_i|^p\mathbf{E}X_i^2\|_{L^1_TL^1} \lesssim 1,$$

where the implicit constant is independent of i.

Proof. Given $p > 2$, we fix $s > 0$ sufficiently small such that $sp < \frac12$ and $\frac{2}{p} + s < 1$. We will perform an L^p estimate: integrating (4.1) against $|X_i|^{p-2}X_i$ we get

$$\begin{aligned} &\frac1p\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}\|X_i\|^p_{L^p} + (p-1)\mathbf{E}\||X_i|^{p-2}|\nabla X_i|^2\|_{L^1} + \mathbf{E}\||X_i|^pX_j^2\|_{L^1} + m\mathbf{E}\|X_i\|^p_{L^p}\\ &= -2\mathbf{E}\Big\langle \mathbf{E}[X_jZ_j], |X_i|^p\Big\rangle - \mathbf{E}\Big\langle \mathbf{E}[X_j^2]|X_i|^{p-2}X_i, Z_i\Big\rangle - 2\mathbf{E}\Big\langle \mathbf{E}[X_jZ_j]Z_i, |X_i|^{p-2}X_i\Big\rangle\\ &\quad + \mathbf{E}\Big\langle \mathbf{E}[\,{:}Z_j^2{:}\,], |X_i|^p\Big\rangle + \mathbf{E}\Big\langle \mathbf{E}[\,{:}Z_j^2{:}\,]X_i|X_i|^{p-2}, Z_i\Big\rangle =: \sum_{k=1}^5 I_k. \end{aligned} \tag{4.15}$$

Set $D \stackrel{\text{def}}{=} \|X_i^{p-2}|\nabla X_i|^2\|_{L^1}$ and $A \stackrel{\text{def}}{=} \|X_i^pX_j^2\|_{L^1}$. We claim that there is some R so that

$$I_k \leqslant \frac14\mathbf{E}A + \frac14\mathbf{E}D + (\mathbf{E}[\|X_i^p\|_{L^1}] + C)R, \qquad \text{with } \int_0^T R \lesssim 1. \tag{4.16}$$

Step 1 (Estimate of I_1)

Using Lemma 2.5, we have

$$I_1 \lesssim \mathbf{E}\Big[\|X_jX_i^p\|_{L^1}^{1-s}\|\nabla(X_j|X_i|^p)\|^s_{L^1}\|Z_j\|_{\mathbf{C}^{-s}}\Big] + \mathbf{E}\Big[\|X_jX_i^p\|_{L^1}\|Z_j\|_{\mathbf{C}^{-s}}\Big] =: I_1^{(1)} + I_1^{(2)}.$$

Using

$$\|X_jX_i^p\|_{L^1} \lesssim A^{1/2}\|X_i^p\|_{L^1}^{1/2} \tag{4.17}$$

and independence, one has

$$I_1^{(2)} \leqslant \mathbf{E}\Big[A^{1/2}\|X_i^p\|^{1/2}_{L^1}\|Z_j\|_{\mathbf{C}^{-s}}\Big] \leqslant \frac{1}{10}\mathbf{E}A + C\mathbf{E}\|X_i^p\|_{L^1}\mathbf{E}\|Z_j\|^2_{\mathbf{C}^{-s}}.$$

Regarding $I_1^{(1)}$, using Hölder inequality and then Gagliardo-Nirenberg with $(s,q,r,\alpha)=(0,4,2,\frac12)$,

$$\begin{aligned}\|\nabla(X_j|X_i|^p)\|_{L^1}&\leqslant \|\nabla X_j\|_{L^2}\||X_i|^{\frac p2}\|_{L^4}^2+2\||X_i|^{\frac p2}X_j\|_{L^2}\|\nabla|X_i|^{\frac p2}\|_{L^2}\\&\lesssim \||X_i|^{\frac p2}\|_{H^1}\||X_i|^{\frac p2}\|_{L^2}\|X_j\|_{H^1}+\sqrt{AD}\end{aligned}$$

Since $\||X_i|^{\frac p2}\|_{H^1}\lesssim D^{\frac12}+\||X_i|^{\frac p2}\|_{L^2}$, together with (4.17) one has

$$\begin{aligned}I_1^{(1)}&\lesssim \mathbf{E}\Big[A^{\frac{1-s}{2}}\|X_i^p\|_{L^1}^{\frac{1-s}{2}}\Big(D^{\frac s2}\|X_i^p\|_{L^1}^{\frac s2}\|X_j\|_{H^1}^s+\|X_i^p\|_{L^1}^s\|X_j\|_{H^1}^s+A^{\frac s2}D^{\frac s2}\Big)\|Z_j\|_{\mathbf{C}^{-s}}\Big]\\&\leqslant \frac1{10}\mathbf{E}A+\frac1{10}\mathbf{E}D+C\mathbf{E}\|X_i^p\|_{L^1}\Big(\mathbf{E}\|X_j\|_{H^1}^{2s}\|Z_j\|_{\mathbf{C}^{-s}}^2+\mathbf{E}\|X_j\|_{H^1}^{\frac{2s}{1+s}}\|Z_j\|_{\mathbf{C}^{-s}}^{\frac{2}{1+s}}+\mathbf{E}\|Z_j\|_{\mathbf{C}^{-s}}^{2/(1-s)}\Big),\end{aligned}$$

where in the last inequality we used independence and Young's inequality for products with exponents $(\frac{2}{1-s},2,\frac2s)$ for the first and third term, and exponents $(\frac{2}{1-s},\frac{2}{1+s})$ for the second term. Therefore invoking Lemma 4.3 we have obtained a bound of the form (4.16).

STEP 2 (Estimates for I_2)

For the second term on the right hand side of (4.15) we use Lemma 2.2 to have

$$\begin{aligned}I_2&=\mathbf{E}\langle\Lambda^s(X_j^2X_i|X_i|^{p-2}),\Lambda^{-s}Z_i\rangle\\&\lesssim \mathbf{E}\Big[\|\Lambda^s(X_j^2)\|_{L^p}\|X_i^{p-1}\|_{L^{\frac{p}{p-1}}}\|\Lambda^{-s}Z_i\|_{L^\infty}\Big]+\mathbf{E}\Big[\|X_j^2\|_{L^p}\|\Lambda^s(X_i|X_i|^{p-2})\|_{L^{\frac{p}{p-1}}}\|\Lambda^{-s}Z_i\|_{L^\infty}\Big]\\&=: I_2^{(1)}+I_2^{(2)}.\end{aligned}$$

Using independence, $\|\Lambda^s(X_j^2)\|_{L^p}\lesssim\|\Lambda^sX_j\|_{L^{2p}}\|X_j\|_{L^{2p}}$ by (2.1), and Lemma 2.1(iii) with $sp<1$,

$$I_2^{(1)}\lesssim \mathbf{E}\Big[\|X_i\|_{L^p}^{p-1}\|\Lambda^{-s}Z_i\|_{L^\infty}\Big]\mathbf{E}\Big[\|\Lambda^s(X_j^2)\|_{L^p}\Big]\lesssim(\mathbf{E}[\|X_i\|_{L^p}^p]+1)\mathbf{E}[\|X_j\|_{H^1}^2].$$

Regarding $I_2^{(2)}$, by the interpolation Lemma 2.2 followed by Hölder's inequality,

$$\begin{aligned}\|\Lambda^s(X_i|X_i|^{p-2})\|_{L^{\frac{p}{p-1}}}&\lesssim\|\nabla(X_i|X_i|^{p-2})\|_{L^{\frac{p}{p-1}}}^s\|X_i^{p-1}\|_{L^{\frac{p}{p-1}}}^{1-s}+\|X_i^{p-1}\|_{L^{\frac{p}{p-1}}}\\&\lesssim D^{s/2}\|X_i^p\|_{L^1}^{(1-s)\frac{p-1}{p}+\frac{s(p-2)}{2p}}+\|X_i^p\|_{L^1}^{\frac{p-1}{p}}.\end{aligned}\tag{4.18}$$

By (2.2) with $(q,s,\alpha,r)=(2p,0,\beta,2)$ with $\beta\stackrel{\text{def}}{=}1-\frac1p$, and then Hölder inequality

$$\mathbf{E}\|X_j^2\|_{L^p}\lesssim\mathbf{E}\Big[\|X_j\|_{H^1}^{2\beta}\|X_j\|_{L^2}^{2(1-\beta)}\Big]\lesssim(\mathbf{E}\|X_j\|_{H^1}^2)^\beta(\mathbf{E}\|X_j\|_{L^2}^2)^{1-\beta}.\tag{4.19}$$

Recall from Lemma 4.3 that $\mathbf{E}[\|X_j\|_{L^2}^2]\lesssim 1$. With (4.18)-(4.19), using again independence, and Hölder's inequality with exponents $(\frac2s,\frac{2}{2-s})$, together with Lemma 3.1, we obtain

$$I_2^{(2)}\lesssim\big(\mathbf{E}\|X_j\|_{H^1}^2\big)^\beta\big(\mathbf{E}D\big)^{\frac s2}\mathbf{E}\Big[\|X_i^p\|_{L^1}^\eta\|\Lambda^{-s}Z_i\|_{L^\infty}^{\frac{2}{2-s}}\Big]^{1-\frac s2}+\mathbf{E}[\|X_j\|_{H^1}^2]^\beta(\mathbf{E}\|X_i^p\|_{L^1}\mathbf{E}\|X_j\|_{L^2}^2+1)$$

where $\eta\stackrel{\text{def}}{=}(1-\frac1p-\frac s2)\frac{2}{2-s}$ and clearly $\eta<1$. The first term on the RHS can be bounded by, using Young's inequality with exponents $(\frac2s,\frac{2}{2-s})$ and then with exponents $(\frac1\eta,\frac1{1-\eta})$,

$$\begin{aligned}&\frac1{10}\mathbf{E}[D]+C\mathbf{E}\Big[\|X_i^p\|_{L^1}^\eta\|\Lambda^{-s}Z_i\|_{L^\infty}^{\frac{2}{2-s}}\Big]\mathbf{E}\Big[\|X_j\|_{H^1}^2\Big]^{\frac{2\beta}{2-s}}\\&\qquad\leqslant\frac1{10}\mathbf{E}[D]+C\Big(\mathbf{E}\|X_i^p\|_{L^1}+\mathbf{E}\|\Lambda^{-s}Z_i\|_{L^\infty}^{\frac{2}{2-s}\frac{1}{1-\eta}}\Big)\mathbf{E}\Big[\|X_j\|_{H^1}^2\Big]^{\frac{2\beta}{2-s}}\end{aligned}$$

Using Lemma 3.1, and Lemma 4.3 noting that $2\beta/(2-s)<1$ by our smallness assumption on $s>0$, we obtain a bound of the form (4.16) for I_2.

STEP 3 (Estimate of I_3-I_5)

Using Lemma 2.2 we obtain

$$I_3=\mathbf{E}\langle\Lambda^s(X_jX_i|X_i|^{p-2}),\Lambda^{-s}(\,{:}Z_iZ_j{:}\,)\rangle$$

$$\begin{aligned}&\lesssim \mathbf{E}\Big[\|\Lambda^{-s}(\,{:}Z_iZ_j{:}\,)\|_{L^\infty}\|\Lambda^s X_j\|_{L^p}\|X_i^p\|_{L^1}^{\frac{p-1}{p}}\Big]\\&\quad+\mathbf{E}\Big[\|\Lambda^{-s}(\,{:}Z_iZ_j{:}\,)\|_{L^\infty}\|X_j\|_{L^p}\|\Lambda^s(X_i|X_i|^{p-2})\|_{L^{\frac{p}{p-1}}}\Big]\quad=: I_3^{(1)}+I_3^{(2)}.\end{aligned}$$

For $I_3^{(1)}$ we use Sobolev embedding $H^1\subset H^s_p$ and Young's inequality and independence to have

$$I_3^{(1)}\lesssim \mathbf{E}[\|\Lambda^{-s}(\,{:}Z_iZ_j{:}\,)\|_{L^\infty}^p]+\mathbf{E}[\|X_j\|_{H^1}^{\frac{p}{p-1}}]\,\mathbf{E}[\|X_i^p\|_{L^1}].$$

For $I_3^{(2)}$ we plug in (4.18):

$$I_3^{(2)}\lesssim \mathbf{E}\Big[\Big(D^{s/2}\|X_i^p\|_{L^1}^{(2p-2-sp)/(2p)}+\|X_i^p\|_{L^1}^{\frac{p-1}{p}}\Big)\|X_j\|_{L^p}\|\Lambda^{-s}(\,{:}Z_iZ_j{:}\,)\|_{L^\infty}\Big]$$

Using Young's inequality with $(\frac{2}{s},\frac{2p}{2p-2-sp},p)$ and $(\frac{p}{p-1},p)$, and Sobolev embedding,

$$I_3^{(2)}\leqslant \frac{1}{10}\mathbf{E}D+C\mathbf{E}\|X_i^p\|_{L^1}\mathbf{E}\|X_j\|_{H^1}^{\frac{2p}{2p-2-sp}}+\mathbf{E}\|X_j\|_{H^1}^{\frac{p}{p-1}}\mathbf{E}\|X_i^p\|_{L^1}+C\mathbf{E}\|\Lambda^{-s}(\,{:}Z_iZ_j{:}\,)\|_{L^\infty}^p.$$

For $s>0$ small enough $\frac{2}{p}+s<1$ so that $\frac{2p}{2p-2-sp}<2$, so Lemma 4.3 applies.

By (4.18) we have for $\epsilon>0$ small enough

$$\begin{aligned}&I_4+I_5\\&\lesssim \|\mathbf{E}[\,{:}Z_j^2{:}\,]\|_{L^\infty}\mathbf{E}[\|X_i\|_{L^p}^p]+\|\mathbf{E}[\,{:}Z_j^2{:}\,]\|_{\mathbf{C}^{s+\epsilon}}\mathbf{E}[\|\Lambda^s(X_i|X_i|^{p-2})\|_{L^1}\|\Lambda^{-s}Z_i\|_{L^\infty}]\\&\lesssim \|\mathbf{E}[\,{:}Z_j^2{:}\,]\|_{L^\infty}\mathbf{E}[\|X_i\|_{L^p}^p]+\|\mathbf{E}[\,{:}Z_j^2{:}\,]\|_{\mathbf{C}^{s+\epsilon}}\mathbf{E}[D^{s/2}\|X_i^p\|_{L^1}^{(1-s)\frac{p-1}{p}+\frac{s(p-2)}{2p}}\|\Lambda^{-s}Z_i\|_{L^\infty}]\\&\qquad+\|\mathbf{E}[\,{:}Z_j^2{:}\,]\|_{\mathbf{C}^{s+\epsilon}}\mathbf{E}[\|X_i^p\|_{L^1}^{\frac{p-1}{p}}\|\Lambda^{-s}Z_i\|_{L^\infty}]\\&\lesssim \mathbf{E}[D]^{s/2}\mathbf{E}[\|X_i^p\|_{L^1}^\eta\|\Lambda^{-s}Z_i\|_{L^\infty}^{\frac{2}{2-s}}]^{1-s/2}\|\mathbf{E}[\,{:}Z_j^2{:}\,]\|_{\mathbf{C}^{s+\epsilon}}+\|\mathbf{E}[\,{:}Z_j^2{:}\,]\|_{\mathbf{C}^{s+\epsilon}}(\mathbf{E}[\|X_i\|_{L^p}^p]+1)\\&\leqslant \frac{1}{10}\mathbf{E}[D]+C\mathbf{E}[\|X_i^p\|_{L^1}]+C\|\mathbf{E}[\,{:}Z_j^2{:}\,]\|_{\mathbf{C}^{s+\epsilon}}^{\frac{2q}{2-s}}\mathbf{E}[\|\Lambda^{-s}Z_i\|_{L^\infty}^{\frac{2q}{2-s}}]+C\|\mathbf{E}[\,{:}Z_j^2{:}\,]\|_{\mathbf{C}^{s+\epsilon}}(\mathbf{E}[\|X_i\|_{L^p}^p]+1),\end{aligned}$$

for $q=1/(1-\eta)$. Combining all the above estimates and using Gronwall's inequality, we obtain the claimed bound. □

We now conclude this section by combining our energy estimates with Schauder theory to obtain a global Hölder bound on X_i.

Lemma 4.5. *Assume that $\mathbf{E}\|\eta_i\|_{L^4}^4\lesssim 1$. For $\beta>\kappa$ sufficiently small, $\gamma=\beta+\frac{1}{2}$, we have*

$$\mathbf{E}[\sup_{t\in[0,T]}t^\gamma\|X_i\|_{\mathbf{C}^\beta}^2]\lesssim 1.$$

Proof. Recall that X_i satisfies the mild formulation of (4.1), which we write using our independent copy (X_j,Z_j) as

$$X_i(t)=S_t\eta_i+\int_0^t S_{t-s}\mathbf{E}[X_j^2+2X_jZ_j+\,{:}Z_j^2{:}\,](X_i+Z_i)\mathrm{d}s.$$

We start by applying the Schauder estimate, Lemma 2.4, with δ playing the role $\beta+1$, $\beta+\kappa$, and $\beta+\kappa+\frac{2}{3}$ respectively to bound

$$\begin{aligned}&\|X_i(t)-S_t\eta_i\|_{\mathbf{C}^\beta}\lesssim \int_0^t(t-s)^{-\frac{\beta+1}{2}}\|\mathbf{E}[X_j^2]X_i\|_{\mathbf{C}^{-1}}\mathrm{d}s\\&\quad+\int_0^t(t-s)^{-\frac{\beta+\kappa}{2}}\|\mathbf{E}[\,{:}Z_j^2{:}\,](X_i+Z_i)\|_{\mathbf{C}^{-\kappa}}\mathrm{d}s+\int_0^t(t-s)^{-\frac{\beta+2/3+\kappa}{2}}\|\mathbf{E}[X_jZ_j]Z_i\|_{\mathbf{C}^{-\kappa-\frac{2}{3}}}\mathrm{d}s\\&\quad+\int_0^t(t-s)^{-\frac{\beta+2/3+\kappa}{2}}\Big(\|\mathbf{E}[X_jZ_j]X_i\|_{\mathbf{C}^{-\kappa-\frac{2}{3}}}+\|\mathbf{E}[X_j^2]Z_i\|_{\mathbf{C}^{-\kappa-\frac{2}{3}}}\Big)\mathrm{d}s\quad\stackrel{\text{def}}{=}\sum_{i=1}^4 J_i.\end{aligned}$$

To estimate J_1, first recall Lemma 4.4 implies that

$$\sup_{t\in[0,T]}\|\mathbf{E}X_j^2\|_{L^2}^2=\sup_{t\in[0,T]}\|\mathbf{E}X_i^2\|_{L^2}^2\lesssim\sup_{t\in[0,T]}\mathbf{E}\|X_i\|_{L^4}^4\lesssim 1, \tag{4.20}$$

which can be combined with the Sobolev embedding $L^2\hookrightarrow\mathbf{C}^{-1}$ in $d=2$ corresponding to Lemma 2.1 with $\alpha=0$, $p_1=q_1=2$ and $p_2=q_2=\infty$ to find

$$J_1\lesssim\int_0^t(t-s)^{-\frac{\beta+1}{2}}\|\mathbf{E}[X_j^2]\|_{L^2}\|X_i\|_{\mathbf{C}^\beta}\mathrm{d}s\lesssim\int_0^t(t-s)^{-\frac{\beta+1}{2}}\|X_i\|_{\mathbf{C}^\beta}\mathrm{d}s.$$

We now turn to J_2 and apply Lemma 2.3 to find for $\beta>\kappa$ and $\kappa'>\kappa$

$$\begin{aligned}J_2&\lesssim\int_0^t(t-s)^{-\frac{\beta+\kappa}{2}}[\|\mathbf{E}[\,{:}Z_j^2{:}\,]\|_{\mathbf{C}^{-\kappa}}\|X_i\|_{\mathbf{C}^\beta}+\|\mathbf{E}[\,{:}Z_j^2{:}\,]\|_{\mathbf{C}^{2\kappa}}\|Z_i\|_{\mathbf{C}^{-\kappa}}\mathrm{d}s\\&\lesssim\int_0^t(t-s)^{-\frac{\beta+\kappa}{2}}s^{-\frac{\kappa'}{2}}\|X_i\|_{\mathbf{C}^\beta}\mathrm{d}s+\|Z_i\|_{C_T\mathbf{C}^{-\kappa}}.\end{aligned}$$

We now turn to J_3 and J_4 and use the Besov embedding $B^{-\kappa}_{3,\infty}\hookrightarrow\mathbf{C}^{-\kappa-\frac{2}{3}}$ in $d=2$ in Lemma 2.1. Let's begin with J_3 which is simpler. Using Lemma 4.4 and Lemma 2.1 and Lemma 2.2 we have

$$\begin{aligned}&\int_0^T\|\mathbf{E}[X_i^2]\|^2_{B^{2\kappa}_{3,\infty}}\mathrm{d}s\lesssim\int_0^T[\mathbf{E}\|X_i^2\|_{B^{2\kappa}_{3,\infty}}]^2\mathrm{d}s\lesssim\int_0^T[\mathbf{E}\|\Lambda^{2\kappa}(X_i^2)\|_{L^3}]^2\mathrm{d}s\lesssim\int_0^T[\mathbf{E}\|\Lambda(X_i^2)\|_{L^{4/3}}]^2\mathrm{d}s\\&\lesssim\int_0^T[\mathbf{E}\|X_i\|_{H^1}\|X_i\|_{L^4}]^2\mathrm{d}s\lesssim\int_0^T\mathbf{E}[\|X_i\|_{H^1}^2]\mathbf{E}[\|X_i\|_{L^4}^2]\mathrm{d}s\lesssim1,\end{aligned}\tag{4.21}$$

where we used (2.1) in the fourth inequality and Hölder inequality in the fifth inequality. Note that by Hölder's inequality in time which exponents $(\frac{3}{2},3)$ and taking into account that $\frac{3}{4}(\beta+\frac{2}{3}+\kappa)<1$ for β small enough and using Lemma 2.3 we find

$$\begin{aligned}J_3&\lesssim\int_0^T\|\mathbf{E}[X_jZ_j]Z_i\|^3_{B^{-\kappa}_{3,\infty}}\mathrm{d}s\lesssim\int_0^T(\mathbf{E}[\|X_j\|^2_{B^{2\kappa}_{3,\infty}}])^{\frac{3}{2}}(\mathbf{E}[\|\,{:}Z_iZ_j{:}\,\|^2_{\mathbf{C}^{-\kappa}}\mid Z_i])^{\frac{3}{2}}\mathrm{d}s\\&\lesssim1+\int_0^T(\mathbf{E}[\|\,{:}Z_iZ_j{:}\,\|^2_{\mathbf{C}^{-\kappa}}\mid Z_i])^6\mathrm{d}s.\end{aligned}\tag{4.22}$$

Here we used Lemma 4.1 and (4.21). Finally, we turn to J_4. By Lemma 2.3 and Lemma 2.2 we deduce

$$\|X_i\|_{B^{2\kappa}_{3,\infty}}\lesssim\|X_i\|_{B^{2\kappa}_{\frac{4}{1+2\kappa},\infty}}\lesssim\|X_i\|^{2\kappa}_{B^1_{2,\infty}}\|X_i\|^{1-2\kappa}_{B^0_{4,\infty}}\lesssim\|X_i\|^{2\kappa}_{H^1}\|X_i\|^{1-2\kappa}_{L^4},$$

which implies that

$$\begin{aligned}&\int_0^T\|\mathbf{E}(X_iZ_i)\|^3_{B^{-\kappa}_{3,\infty}}\mathrm{d}s\lesssim\int_0^T\mathbf{E}[\|X_i\|^3_{B^{2\kappa}_{3,\infty}}\|Z_i\|^3_{\mathbf{C}^{-\kappa}}]\lesssim\int_0^T\mathbf{E}[\|X_i\|^{6\kappa}_{H^1}\|X_i\|^{3(1-2\kappa)}_{L^4}\|Z_i\|^3_{\mathbf{C}^{-\kappa}}]\\&\lesssim\int_0^T\mathbf{E}[\|X_i\|^2_{H^1}]+\int_0^T\mathbf{E}[\|X_i\|^4_{L^4}]+\int_0^T\mathbf{E}[\|Z_i\|^l_{\mathbf{C}^{-\kappa}}]\lesssim1,\end{aligned}\tag{4.23}$$

for some $l>1$. This combined with (4.21) and Lemma 2.3 implies that

$$\begin{aligned}J_4&\lesssim\int_0^t(t-s)^{-\frac{\beta+2/3+\kappa}{2}}\Big(\|\mathbf{E}[X_jZ_j]\|_{B^{-\kappa}_{3,\infty}}\|X_i\|_{\mathbf{C}^\beta}+\|\mathbf{E}[X_j^2]\|_{B^{2\kappa}_{3,\infty}}\|Z_i\|_{\mathbf{C}^{-\kappa}}\Big)\mathrm{d}s\\&\lesssim\int_0^t(t-s)^{-\frac{\beta+2/3+\kappa}{2}}\|\mathbf{E}[X_jZ_j]\|_{B^{-\kappa}_{3,\infty}}\|X_i\|_{\mathbf{C}^\beta}\mathrm{d}s+\|Z_i\|_{C_T\mathbf{C}^{-\kappa}}\end{aligned}$$

For $S_t\eta_i$ we use (4.3) to have the desired bound. Combining the above estimates, using (4.23), Hölder's inequality and Gronwall's inequality, the result follows. □

Combining the local well-posedness result and the uniform estimate Lemma 4.5 we conclude the following result:

Theorem 4.6. *For given Z_i as the solution to (3.2) and $\mathbf{E}\|\eta_i\|_{L^4}^4 \lesssim 1$, there exists a unique solution $X_i \in L^2(\Omega; C((0,T]; \mathbf{C}^\beta) \cap C_T L^4)$ to (4.1) such that*

$$\mathbf{E}[\sup_{t\in[0,T]} t^\gamma \|X_i\|_{\mathbf{C}^\beta}^2] + \sup_{t\in[0,T]} \mathbf{E}\|X_i\|_{L^4}^4 + \mathbf{E}\|X_i\|_{L_T^2 H^1}^2 \lesssim 1.$$

In particular, for every $\psi_i \in \mathbf{C}^{-\kappa}$ with $\mathbf{E}\|\psi_i\|_{\mathbf{C}^{-\kappa}}^p \lesssim 1$, $p > 1$, there exists a unique solution $\Psi_i \in L^2(\Omega; C_T \mathbf{C}^{-\kappa})$ to (1.6) such that

$$\mathbf{E}[\sup_{t\in[0,T]} t^\gamma \|\Psi_i - Z_i\|_{\mathbf{C}^\beta}^2] \lesssim 1,$$

for $\beta > 3\kappa > 0$ small enough and $\gamma = \frac{1}{2} + \beta$.

5. Large N limit of the dynamics

In this section we study the large N behavior of a fixed component Φ_i^N satisfying (3.1) with initial condition $\phi_i^N = y_i^N + z_i^N$. Namely, under suitable assumptions on the initial conditions, we show that as $N \to \infty$, the component converges to the corresponding solution Ψ_i to (1.6) with initial condition $\psi_i = \eta_i + z_i$. Recall that by definition, $\Phi_i^N = Y_i^N + Z_i^N$, where Y_i^N satisfies (3.3) and Z_i^N satisfies (3.2) with initial conditions y_i^N and z_i^N respectively. Similarly, $\Psi_i = X_i + Z_i$, where X_i satisfies (4.1) and Z_i satisfies (3.2) with initial conditions y_i and η_i respectively. We now define

$$v_i^N \stackrel{\text{def}}{=} Y_i^N - X_i.$$

For future reference, we note that in light of the decomposition, c.f. Section 3.1 for the definition of $\tilde{Z}_i$,

$$Z_i^N = \tilde{Z}_i + S_t(z_i^N - \tilde{Z}_i(0)), \quad Z_i = \tilde{Z}_i + S_t(z_i - \tilde{Z}_i(0)),$$

it follows that

$$\Phi_i^N - \Psi_i = Y_i^N - X_i + Z_i^N - Z_i = v_i^N + e^{t\Delta}(z_i^N - z_i).$$

Hence, our main task is to study v_i^N and this will occupy the bulk of the proof. We now give our assumptions on the initial conditions.

Assumption 5.1. *Suppose the following assumptions:*

- *The random variables $(z_i, \eta_i)_{i=1}^N$ are i.i.d.*
- *For every $p > 1$, and every i,*

$$\mathbf{E}[\|z_i^N - z_i\|_{\mathbf{C}^{-\kappa}}^p] \to 0, \quad \mathbf{E}[\|y_i^N - \eta_i\|_{L^2}^2] \to 0, \qquad \text{as } N \to \infty.$$

$$\frac{1}{N}\sum_{i=1}^N \|z_i^N - z_i\|_{\mathbf{C}^{-\kappa}}^p \to^{\mathbf{P}} 0, \quad \frac{1}{N}\sum_{i=1}^N \|y_i^N - \eta_i\|_{L^2}^2 \to^{\mathbf{P}} 0, \qquad \text{as } N \to \infty,$$

  where $\to^{\mathbf{P}}$ means the convergence in probability.
- *For some $q > 1$, $p_0 > 4/(1-4\kappa)$, and every $p > 1$*

$$\mathbf{E}[\|z_i^N\|_{\mathbf{C}^{-\kappa}}^p + \|z_i\|_{\mathbf{C}^{-\kappa}}^p] \lesssim 1, \quad \mathbf{E}\|\eta_i\|_{L^{p_0}}^{p_0} \lesssim 1, \quad \mathbf{E}[\frac{1}{N}\sum_{i=1}^N \|y_i^N\|_{L^2}^2]^q \lesssim 1,$$

  where the implicit constant is independent of i, N.

The following theorem is our main convergence result - which in particular implies Theorem 1.1. The proof is inspired by mean field theory for *SDE* systems such as Sznitman's article [Szn91], which as the general philosophy starts by directly subtracting the two dynamics and thereby cancelling the white noises, and then controls the difference. To this end we establish energy estimates for the difference v_i^N below, c.f. (5.3) from Step 1. The key to the proof is that for the terms collected in I_1^N, I_2^N below we interpolate with $\mathbf{C}^{-s}$ and $B_{1,1}^s$ spaces and leverage various a-priori estimates obtained in the previous sections; but for terms collected in I_3^N which are suitably *centered*, we interpolate with *Hilbert spaces* and invoke the fact (2.3), which in certain sense gives us a crucial "factor of $1/N$".

Theorem 5.1. *If the initial datum $(z_i^N, y_i^N, z_i, \eta_i)_i$ satisfy Assumption 5.1, then for every i and every $T > 0$, $\|v_i^N\|_{C_T L^2}$ converges to zero in probability, as $N \to \infty$. Moreover, under the additional hypothesis that $(z_i^N, y_i^N, z_i, \eta_i)_{i=1}^N$ are exchangeable, for all $t > 0$ it holds*

$$\lim_{N\to\infty} \mathbf{E}\|\Phi_i^N(t) - \Psi_i(t)\|_{L^2}^2 = 0. \tag{5.1}$$

Proof. The proof has a similar flavor to the Lemma 3.3, and in fact we will continue to use the notation R_N^i for $i = 1, 2, 3$ for the same quantities. One additional ingredient required is the following instance of the Gagliardo-Nirenberg inequality (a special case of Lemma 2.2),

$$\|g\|_{L^4} \leqslant C\|g\|_{H^1}^{1/2}\|g\|_{L^2}^{1/2}. \tag{5.2}$$

In the proof we omit the superscript N and simply write v_i for v_i^N throughout. Furthermore, in Steps 1-5 we work under the simplifying assumption that $z_i^N = z_i$, so that also $Z_i^N = Z_i$. In Step 7, we sketch the argument in the more general case.

Step 1 (Energy balance)

In this step, we justify the following energy identity

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^N \|v_i\|_{L^2}^2 + \sum_{i=1}^N \|\nabla v_i\|_{L^2}^2 + m\sum_{i=1}^N \|v_i\|_{L^2}^2 + \frac{1}{N}\sum_{i,j=1}^N \|Y_j v_i\|_{L^2}^2 + \frac{1}{N}\Big\|\sum_{j=1}^N X_j v_j\Big\|_{L^2}^2 = \sum_{k=1}^3 I_k^N \tag{5.3}$$

where

$$I_1^N \stackrel{\text{def}}{=} -\frac{1}{N}\sum_{i,j=1}^N \big(2\langle v_i v_j,\, {:}Z_j Z_i{:}\,\rangle + \langle v_i^2,\, {:}Z_j^2{:}\,\rangle + 2\langle v_i^2 Y_j, Z_j\rangle\big)\,,$$

$$I_2^N \stackrel{\text{def}}{=} -\frac{1}{N}\sum_{i,j=1}^N \langle v_i v_j, \big(X_i Y_j + (3X_j + Y_j)Z_i\big)\rangle\,,$$

$$I_3^N \stackrel{\text{def}}{=} -\frac{1}{N}\sum_{i,j=1}^N \Big\langle \big[\,{:}Z_j^2{:}\, - \mathbf{E}\,{:}Z_j^2{:}\, + X_j(X_j + 2Z_j) - \mathbf{E}X_j(X_j + 2Z_j)\big](X_i + Z_i)\,,\, v_i\Big\rangle\,, \tag{5.4}$$

In the definition of I_3^N, to have a compact formula, we slightly abuse notation for the contribution of the diagonal part $i = j$, where we understand $Z_i Z_j$ to be ${:}Z_i^2{:}$ and ${:}Z_j^2{:}\,Z_i$ to be ${:}Z_i^3{:}$.

We now turn the justification of this identity, and for the convenience of the reader, we write the equations for Y_i and X_i side by side as

$$\mathscr{L}Y_i = -\frac{1}{N}\sum_{j=1}^N \Big(Y_j^2 Y_i + Y_j^2 Z_i + 2Y_j Z_j Y_i + 2Y_j\,{:}Z_i Z_j{:}\, + Y_i\,{:}Z_j^2{:} + {:}Z_i Z_j^2{:}\,\Big) \tag{5.5}$$

$$\begin{aligned}\mathscr{L}X_i = -\frac{1}{N}\sum_{j=1}^N \Big(&\mathbf{E}(X_j^2)X_i + \mathbf{E}(X_j^2)Z_i + 2\mathbf{E}(X_j Z_j)X_i + 2\mathbf{E}(X_j Z_j)Z_i + X_i\mathbf{E}\,{:}Z_j^2{:}\\ &+ Z_i\mathbf{E}\,{:}Z_j^2{:}\,\Big)\end{aligned} \tag{5.6}$$

where we used that X_j and X_i are equal in law. We now compare each of the first 4 terms in (5.5) to the corresponding terms in (5.6). Note first that

$$\begin{aligned}Y_j^2 Y_i - \mathbf{E}(X_j^2)X_i &= Y_j^2 Y_i - X_j^2 X_i + \big(X_j^2 - \mathbf{E}(X_j^2)\big)X_i\\ &= Y_j^2 v_i + v_j(Y_j + X_j)X_i + \big(X_j^2 - \mathbf{E}(X_j^2)\big)X_i.\end{aligned}$$

Similarly, we find

$$\begin{aligned}\big(Y_j^2 - \mathbf{E}(X_j^2)\big)Z_i &= v_j(Y_j + X_j)Z_i + \big(X_j^2 - \mathbf{E}(X_j^2)\big)Z_i.\\ 2Y_j Z_j Y_i - 2\mathbf{E}(X_j Z_j)X_i &= 2\big(v_i Y_j + v_j X_i\big)Z_j + 2\big(X_j Z_j - \mathbf{E}(X_j Z_j)\big)X_i.\\ 2Y_j\,{:}Z_i Z_j{:}\, - 2\mathbf{E}(X_j Z_j)Z_i &= 2v_j\,{:}Z_i Z_j{:}\, + 2\big(X_j\,{:}Z_i Z_j{:}\, - \mathbf{E}(X_j Z_j)Z_i\big).\end{aligned}$$

$$Y_i :Z_j^2: \;= v_i :Z_j^2: \;+ X_i :Z_j^2: .$$

Taking the difference of (5.5) and (5.6), using the identities above, multiplying by v_i, integrating by parts, and summing over i leads to (5.3). Indeed, notice that each equality gives a sum of two pieces, one with a factor of v and one without any factor of v, but with a re-centering. The terms which have a factor of v lead to I_1^N and I_2^N, except for $Y_j^2 v_i$ and $v_j X_j X_i$, which lead to the two coercive quantities on the LHS of (5.3). The terms which have been re-centered lead to I_3^N.

STEP 2 (Estimates for I_1^N)

In this step, we show there is a universal constant C such that

$$I_1^N \leqslant \frac{1}{8}\Big(\sum_{i=1}^N \|\nabla v_i\|_{L^2}^2 + \frac{1}{N}\sum_{i,j=1}^N \|Y_j v_i\|_{L^2}^2\Big) + C(1+R_N^2+R_N^3+R_N^5)\sum_{i=1}^N \|v_i\|_{L^2}^2, \tag{5.7}$$

where R_N^2 and R_N^3 are defined in terms of Z in the same way as in Lemma 3.3 and

$$R_N^5 \stackrel{\text{def}}{=} \Big(1+\frac{1}{N}\sum_{j=1}^N \|\nabla Y_j\|_{L^2}^2\Big)^s \Big(\frac{1}{N}\sum_{i=1}^N \|Z_i\|_{\mathbf{C}^{-s}}^2\Big),$$

with $1 > s \geqslant 2\kappa$ and s small enough. Indeed, (5.7) follows from arguments identical to the ones leading to (3.10) and (3.13) in Lemma 3.3, but with a different labelling of the integrands which we now explain. There are three contributions to I_1^N and each can be treated separately. For the contribution of $v_i v_j :Z_j Z_i:$ we argue exactly as for (3.11) but with $v_i v_j$ in place of $Y_i Y_j$. For the contribution of $v_i^2 :Z_j^2:$, we argue exactly as in (3.12), but with v_i^2 in place of Y_i^2. This leads to the inequality

$$\frac{1}{N}\sum_{i,j=1}^N \big(2\langle v_i v_j,\ :Z_i Z_j:\,\rangle + \langle v_i^2,\ :Z_j^2:\,\rangle\big) \leqslant \frac{1}{16}\Big(\sum_{i=1}^N \|\nabla v_i\|_{L^2}^2\Big) + C(1+R_N^2)\sum_{i=1}^N \|v_i\|_{L^2}^2. \tag{5.8}$$

Finally, for the contribution of $v_i^2 Y_j Z_j$ the argument is similar as for (3.13). This leads to the estimate

$$\begin{aligned}
-\frac{1}{N}\sum_{j=1}^N \Big\langle \sum_{i=1}^N v_i^2 Y_j, Z_j\Big\rangle &\lesssim \frac{1}{N}\sum_{j=1}^N \Big(\Big\|\sum_{i=1}^N v_i^2 Y_j\Big\|_{L^1}^{1-s}\Big\|\nabla\Big(\sum_{i=1}^N v_i^2 Y_j\Big)\Big\|_{L^1}^s + \Big\|\sum_{i=1}^N v_i^2 Y_j\Big\|_{L^1}\Big)\|Z_j\|_{\mathbf{C}^{-s}} \qquad (5.9)\\
&\lesssim \frac{1}{N}\Big(\sum_{j=1}^N \Big\|\sum_{i=1}^N v_i^2 Y_j\Big\|_{L^1}^{2(1-s)}\Big\|\nabla\Big(\sum_{i=1}^N v_i^2 Y_j\Big)\Big\|_{L^1}^{2s}\Big)^{1/2}\mathscr{Z}_N^{\frac12} + \frac{1}{N}\Big(\sum_{j=1}^N \Big\|\sum_{i=1}^N v_i^2 Y_j\Big\|_{L^1}^2\Big)^{1/2}\mathscr{Z}_N^{\frac12}
\end{aligned}$$

where $\mathscr{Z}_N \stackrel{\text{def}}{=} \sum_{j=1}^N \|Z_j\|_{\mathbf{C}^{-s}}^2$ as in (3.14). By Hölder's inequality, it holds that

$$\Big\|\sum_{i=1}^N v_i^2 Y_j\Big\|_{L^1} \lesssim \Big(\sum_{i=1}^N \|v_i Y_j\|_{L^2}^2\Big)^{1/2}\Big(\sum_{i=1}^N \|v_i\|_{L^2}^2\Big)^{1/2}. \tag{5.10}$$

Furthermore, we find that

$$\begin{aligned}
&\Big\|\nabla\Big(\sum_{i=1}^N v_i^2 Y_j\Big)\Big\|_{L^1} \lesssim \Big\|\sum_{i=1}^N v_i^2 \nabla Y_j\Big\|_{L^1} + \Big\|\sum_{i=1}^N \nabla v_i v_i Y_j\Big\|_{L^1}\\
&\lesssim \sum_{i=1}^N \|v_i\|_{L^4}^2\|\nabla Y_j\|_{L^2} + \Big(\sum_{i=1}^N \|\nabla v_i\|_{L^2}^2\Big)^{1/2}\Big(\sum_{i=1}^N \|v_i Y_j\|_{L^2}^2\Big)^{1/2}\\
&\lesssim \Big(\sum_{i=1}^N \|v_i\|_{H^1}^2\Big)^{1/2}\Big(\sum_{i=1}^N \|v_i\|_{L^2}^2\Big)^{1/2}\|\nabla Y_j\|_{L^2} + \Big(\sum_{i=1}^N \|\nabla v_i\|_{L^2}^2\Big)^{1/2}\Big(\sum_{i=1}^N \|v_i Y_j\|_{L^2}^2\Big)^{1/2},
\end{aligned}$$

where we used (5.2) in the last step. Hence, we find that

$$\begin{aligned}
&\sum_{j=1}^N \Big\|\sum_{i=1}^N v_i^2 Y_j\Big\|_{L^1}^{2(1-s)} \Big\|\nabla\Big(\sum_{i=1}^N v_i^2 Y_j\Big)\Big\|_{L^1}^{2s} \\
&\lesssim \Big(\sum_{i,j=1}^N \|v_i Y_j\|_{L^2}^2\Big)^{1-s}\Big(\sum_{i=1}^N \|v_i\|_{H^1}^2\Big)^s\Big(\sum_{i=1}^N \|v_i\|_{L^2}^2\Big)\Big(\sum_{j=1}^N \|\nabla Y_j\|_{L^2}^2\Big)^s \\
&+\Big(\sum_{i=1}^N \|\nabla v_i\|_{L^2}^2\Big)^s\Big(\sum_{i=1}^N \|v_i\|_{L^2}^2\Big)^{1-s}\Big(\sum_{i,j=1}^N \|v_i Y_j\|_{L^2}^2\Big).
\end{aligned}$$

Inserting this into (5.9), taking the square root, and using (5.10) and Young's inequality with exponent $(\frac{2}{1-s}, \frac{2}{s}, 2)$ we arrive at

$$-\frac{1}{N}\sum_{i,j=1}^N \langle 2v_i^2 Y_j, Z_j\rangle \leqslant \frac{1}{16}\Big(\sum_{i=1}^N \|\nabla v_i\|_{L^2}^2 + \frac{1}{N}\sum_{i,j=1}^N \|Y_j v_i\|_{L^2}^2\Big) + C(1+R_N^3+R_N^5)\sum_{i=1}^N \|v_i\|_{L^2}^2. \quad (5.11)$$

Combining (5.8) and (5.11) and recalling the definition of I_1^N we obtain (5.7).

STEP 3 (Estimates for I_2^N)

In this step, we show there is a universal constant C such that

$$\begin{aligned}
I_2^N &\leqslant \frac{1}{4}\Big(\sum_{i=1}^N \|\nabla v_i\|_{L^2}^2 + \frac{1}{N}\sum_{i,j=1}^N \|v_i Y_j\|_{L^2}^2 + \Big\|\frac{1}{\sqrt{N}}\sum_{j=1}^N v_j X_j\Big\|_{L^2}^2\Big) \\
&+ C\Big(1+R_N^3+R_N^4+R_N^5+R_N^6+\Big(\frac{1}{N}\sum_{i=1}^N \|X_i\|_{L^4}^2\Big)^2\Big)\Big(\sum_{i=1}^N \|v_i\|_{L^2}^2\Big), \quad (5.12)
\end{aligned}$$

where R_N^3 is defined as in Lemma 3.3 and R_N^4 and R_N^6 are defined by

$$R_N^4 \stackrel{\text{def}}{=} \Big(1+\frac{1}{N}\sum_{j=1}^N \|X_j\|_{L^4}^2\Big)^{\frac{2s}{2-s}}\Big(\frac{1}{N}\sum_{i=1}^N \|Z_i\|_{\mathbf{C}^{-s}}^2\Big)^{\frac{2}{2-s}} + \Big(\frac{1}{N}\sum_{j=1}^N \|\nabla X_j\|_{L^2}^2\Big)^s\Big(\frac{1}{N}\sum_{i=1}^N \|Z_i\|_{\mathbf{C}^{-s}}^2\Big).$$

$$R_N^6 \stackrel{\text{def}}{=} \Big(\frac{1}{N}\sum_{j=1}^N \|Y_j\|_{L^4}^2\Big)^{\frac{s}{1-s}} R_N^3$$

We break I_2^N into the separate contributions where v_iv_j multiplies X_iY_j, X_jZ_i, and Y_jZ_i respectively. For the first contribution, Cauchy's inequality yields

$$\begin{aligned}
&\frac{1}{N}\sum_{i,j=1}^N \int v_i v_j X_i Y_j \mathrm{d}x \leqslant \frac{1}{8}\frac{1}{N}\sum_{i,j=1}^N \int v_i^2 Y_j^2 \mathrm{d}x + C\frac{1}{N}\sum_{i,j=1}^N \int v_j^2 X_i^2 \mathrm{d}x. \\
&\leqslant \frac{1}{8}\frac{1}{N}\sum_{i,j=1}^N \int v_i^2 Y_j^2 \mathrm{d}x + C\Big(\sum_{i=1}^N \|v_i\|_{L^4}^2\Big)\Big(\frac{1}{N}\sum_{i=1}^N \|X_i\|_{L^4}^2\Big), \\
&\leqslant \frac{1}{8}\frac{1}{N}\sum_{i,j=1}^N \|v_i Y_j\|_{L^2}^2 + C\Big(\sum_{i=1}^N \|v_i\|_{H^1}^2\Big)^{1/2}\Big(\sum_{i=1}^N \|v_i\|_{L^2}^2\Big)^{1/2}\Big(\frac{1}{N}\sum_{i=1}^N \|X_i\|_{L^4}^2\Big),
\end{aligned}$$

where we used (5.2). Using Young's inequality with exponents $(2,2)$ leads to the last contribution to (5.12). The remaining contributions to I_2^N are more involved to estimate. Our next claim is that

$$\frac{3}{N}\sum_{i=1}^N \Big\langle v_i \sum_{j=1}^N v_j X_j, Z_i\Big\rangle \leqslant \frac{1}{8}\Big(\sum_{i=1}^N \|\nabla v_i\|_{L^2}^2 + \Big\|\frac{1}{\sqrt{N}}\sum_{j=1}^N v_j X_j\Big\|_{L^2}^2\Big) + C(1+R_N^3+R_N^4)\sum_{i=1}^N \|v_i\|_{L^2}^2. \quad (5.13)$$

The basic setup is the same as the bound leading to (3.15) via the inequality (3.6) with $v_i \sum_{j=1}^N v_j X_j$ playing the role of g and Z_i playing the role of f, followed by an application of the Cauchy-Schwartz inequality for the summation in i. The l.h.s. of (5.13) is then bounded by

$$\sum_{s' \in \{s,0\}} \frac{1}{\sqrt{N}} \Big(\sum_{i=1}^N \Big\| v_i \sum_{j=1}^N v_j X_j \Big\|_{L^1}^{2(1-s')} \Big\| \nabla \Big(v_i \sum_{j=1}^N v_j X_j \Big) \Big\|_{L^1}^{2s'} \Big)^{\frac{1}{2}} \Big(\frac{1}{N} \mathscr{Z}_N \Big)^{\frac{1}{2}}. \tag{5.14}$$

Using Hölder's inequality in the form $\big\| v_i \sum_{j=1}^N v_j X_j \big\|_{L^1} \leqslant \|v_i\|_{L^2} \big\| \sum_{j=1}^N v_j X_j \big\|_{L^2}$ together with

$$\begin{aligned}\Big\| \nabla \Big(v_i \sum_{j=1}^N v_j X_j \Big) \Big\|_{L^1} \leqslant{}& \|\nabla v_i\|_{L^2} \Big\| \sum_{j=1}^N v_j X_j \Big\|_{L^2} + \|v_i\|_{L^2}^{1/2} \|v_i\|_{H^1}^{1/2} \Big(\sum_{j=1}^N \|\nabla v_j\|_{L^2}^2 \Big)^{\frac{1}{2}} \Big(\sum_{j=1}^N \|X_j\|_{L^4}^2 \Big)^{\frac{1}{2}} \\ &+ \|v_i\|_{L^2}^{1/2} \|v_i\|_{H^1}^{1/2} \Big(\sum_{j=1}^N \|v_j\|_{L^2}^2 \Big)^{1/4} \Big(\sum_{j=1}^N \|v_j\|_{H^1}^2 \Big)^{1/4} \Big(\sum_{j=1}^N \|\nabla X_j\|_{L^2}^2 \Big)^{1/2}\end{aligned}$$

where we used (5.2), and inserting this into (5.14) and applying Hölder's inequality for the summation in i together with

$$\Big(\sum_{i=1}^N \|v_i\|_{L^2}^{2(1-s)} \|v_i\|_{L^2}^s \|v_i\|_{H^1}^s \Big)^{\frac{1}{2}} \leqslant \Big(\sum_{i=1}^N \|v_i\|_{L^2}^2 \Big)^{\frac{2-s}{4}} \Big(\sum_{i=1}^N \|v_i\|_{H^1}^2 \Big)^{\frac{s}{4}},$$

we obtain a majorization by

$$\begin{aligned}&\Big\| \frac{1}{\sqrt{N}} \sum_{j=1}^N v_j X_j \Big\|_{L^2} \Big(\sum_{i=1}^N \|v_i\|_{L^2}^2 \Big)^{\frac{1-s}{2}} \Big(\sum_{i=1}^N \|v_i\|_{H^1}^2 \Big)^{\frac{s}{2}} \Big(\frac{1}{N} \mathscr{Z}_N \Big)^{1/2} \\ &+ \Big\| \frac{1}{\sqrt{N}} \sum_{j=1}^N v_j X_j \Big\|_{L^2}^{1-s} \Big(\sum_{i=1}^N \|v_i\|_{L^2}^2 \Big)^{\frac{1}{2} - \frac{s}{4}} \Big(\sum_{i=1}^N \|v_i\|_{H^1}^2 \Big)^{\frac{3s}{4}} \Big(\frac{1}{N} \sum_{j=1}^N \|X_j\|_{L^4}^2 \Big)^{\frac{s}{2}} \Big(\frac{1}{N} \mathscr{Z}_N \Big)^{1/2} \\ &+ \Big\| \frac{1}{\sqrt{N}} \sum_{j=1}^N v_j X_j \Big\|_{L^2}^{1-s} \Big(\sum_{i=1}^N \|v_i\|_{L^2}^2 \Big)^{\frac{1}{2}} \Big(\sum_{i=1}^N \|v_i\|_{H^1}^2 \Big)^{\frac{s}{2}} \Big(\frac{1}{N} \sum_{j=1}^N \|\nabla X_j\|_{L^2}^2 \Big)^{\frac{s}{2}} \Big(\frac{1}{N} \mathscr{Z}_N \Big)^{1/2}.\end{aligned}$$

Finally, we apply Young's inequality with exponents $(2, \frac{2}{1-s}, \frac{2}{s})$ for the first term, $(\frac{2}{1-s}, \frac{4}{2-s}, \frac{4}{3s})$ for the second term, and $(\frac{2}{1-s}, 2, \frac{2}{s})$ for the third term which leads to (5.13).

Similar to (5.13), we now claim that

$$\frac{1}{N} \sum_{i=1}^N \Big\langle \sum_{j=1}^N v_i v_j Y_j, Z_i \Big\rangle \leqslant \frac{1}{8} \Big(\sum_{i=1}^N \|\nabla v_i\|_{L^2}^2 + \frac{1}{N} \sum_{i,j=1}^N \|v_i Y_j\|_{L^2}^2 \Big) + C \Big(1 + \sum_{k \in \{3,5,6\}} R_N^k \Big) \sum_{i=1}^N \|v_i\|_{L^2}^2. \tag{5.15}$$

The basic setup is again similar to the bound leading to (3.15) via the inequality (3.6) with $\sum_{j=1}^N v_i v_j Y_j$ playing the role of g and Z_i playing the role of f, followed by an application of the Cauchy-Schwartz inequality for the summation in i. The l.h.s. of (5.15) is then bounded by

$$\sum_{s' \in \{s,0\}} \frac{1}{\sqrt{N}} \Big(\sum_{i=1}^N \Big\| \sum_{j=1}^N v_i v_j Y_j \Big\|_{L^1}^{2(1-s')} \Big\| \sum_{j=1}^N \nabla (v_i v_j Y_j) \Big\|_{L^1}^{2s'} \Big)^{\frac{1}{2}} \Big(\frac{1}{N} \mathscr{Z}_N \Big)^{\frac{1}{2}} \tag{5.16}$$

By Hölder's inequality and the Cauchy-Schwartz inequality we find

$$\Big\| \sum_{j=1}^N v_i v_j Y_j \Big\|_{L^1} \leqslant \Big(\sum_{j=1}^N \|v_i Y_j\|_{L^2}^2 \Big)^{1/2} \Big(\sum_{j=1}^N \|v_j\|_{L^2}^2 \Big)^{1/2},$$

together with (5.2) to have

$$\Big\|\sum_{j=1}^{N}\nabla(v_iv_jY_j)\Big\|_{L^1}\leqslant\Big(\sum_{j=1}^{N}\|v_iY_j\|_{L^2}^2\Big)^{1/2}\Big(\sum_{j=1}^{N}\|\nabla v_j\|_{L^2}^2\Big)^{1/2}$$
$$+\|\nabla v_i\|_{L^2}\Big(\sum_{j=1}^{N}\|v_j\|_{L^4}^2\Big)^{1/2}\Big(\sum_{j=1}^{N}\|Y_j\|_{L^4}^2\Big)^{1/2}+\|v_i\|_{L^4}\Big(\sum_{j=1}^{N}\|v_j\|_{L^4}^2\Big)^{1/2}\Big(\sum_{j=1}^{N}\|\nabla Y_j\|_{L^2}^2\Big)^{1/2}.$$

Inserting this into (5.16) and applying Hölder's inequality for the summation in i with exponents $(\frac{1}{1-s},\frac{1}{s})$ leads to a majorization by

$$\Big(\frac{1}{N}\sum_{i,j=1}^{N}\|v_iY_j\|_{L^2}^2\Big)^{\frac12}\Big(\sum_{i=1}^{N}\|v_i\|_{L^2}^2\Big)^{\frac{1-s}{2}}\Big(\sum_{i=1}^{N}\|\nabla v_i\|_{L^2}^2\Big)^{\frac{s}{2}}\Big(\frac{1}{N}\mathscr{Z}_N\Big)^{1/2}$$
$$+\Big(\frac{1}{N}\sum_{i,j=1}^{N}\|v_iY_j\|_{L^2}^2\Big)^{\frac{1-s}{2}}\Big(\sum_{j=1}^{N}\|v_j\|_{L^2}^2\Big)^{\frac{1-s}{2}}\Big(\sum_{j=1}^{N}\|v_j\|_{L^4}^2\Big)^{\frac{s}{2}}\Big(\sum_{i=1}^{N}\|\nabla v_i\|_{L^2}^2\Big)^{\frac{s}{2}}\Big(\frac{1}{N}\sum_{j=1}^{N}\|Y_j\|_{L^4}^2\Big)^{\frac{s}{2}}\Big(\frac{1}{N}\mathscr{Z}_N\Big)^{\frac12}$$
$$+\Big(\frac{1}{N}\sum_{i,j=1}^{N}\|v_iY_j\|_{L^2}^2\Big)^{\frac{1-s}{2}}\Big(\sum_{i=1}^{N}\|v_i\|_{H^1}^2\Big)^{\frac{s}{2}}\Big(\sum_{j=1}^{N}\|v_j\|_{L^2}^2\Big)^{\frac12}\Big(\frac{1}{N}\sum_{j=1}^{N}\|\nabla Y_j\|_{L^2}^2\Big)^{\frac{s}{2}}\Big(\frac{1}{N}\mathscr{Z}_N\Big)^{1/2}.$$

Note that for the third term, we also took advantage of (5.2). We now apply Young's inequality with exponents $(2,\frac{2}{1-s},\frac{2}{s})$ for the first term, $(\frac{2}{1-s},\frac{2}{1-s},\frac{2}{s},\frac{2}{s})$ for the second term, and $(\frac{2}{1-s},\frac{2}{s},2)$ for the third term, which leads to (5.15). Finally, combining (5.13) and (5.15) we obtain (5.12).

STEP 4 (Law of large numbers type bounds: estimates for I_3^N)

For I_3^N we obtain a bound in expectation in the spirit of the law of large numbers in a Hilbert space to generate cancellations. To this end, we define

$$G_j\stackrel{\text{def}}{=}(X_j^2-\mathbf{E}X_j^2)+2(X_jZ_j-\mathbf{E}X_jZ_j)+(\,{:}Z_j^2{:}\,-\mathbf{E}\,{:}Z_j^2{:}\,)\stackrel{\text{def}}{=}G_j^{(1)}+G_j^{(2)}+G_j^{(3)}.$$

We show there is a universal constant C such that

$$I_3^N\leqslant C(\bar R_N+\bar R_N')+\frac18\frac1N\Big\|\sum_{i=1}^{N}X_iv_i\Big\|_{L^2}^2+\frac14\sum_{i=1}^{N}\|v_i\|_{H^1}^2 \tag{5.17}$$
$$+C\Big(\sum_{i=1}^{N}\|v_i\|_{L^2}^2\Big)\Big[R_N^7+1+\frac1N\sum_{i=1}^{N}\Big(\|X_i\|_{L^4}^{4/(1-2s)}+\|\Lambda^sX_i\|_{L^4}^4\Big)\Big],$$

with

$$\bar R_N\stackrel{\text{def}}{=}\frac1N\Big\|\sum_{j=1}^{N}G_j^{(1)}\Big\|_{H^s}^2+\sum_{k\in\{2,3\}}\frac1N\Big\|\sum_{j=1}^{N}G_j^{(k)}\Big\|_{H^{-s}}^2,\qquad \bar R_N'\stackrel{\text{def}}{=}\sum_{k\in\{2,3\}}\frac{1}{N^2}\sum_{i=1}^{N}\|\sum_{j=1}^{N}G_j^{(k)}Z_i\|_{H^{-s}}^2,$$
$$R_N^7\stackrel{\text{def}}{=}\Big(\frac1N\sum_{i=1}^{N}\|\Lambda^{-s}Z_i\|_{L^\infty}^2\Big)^{\frac{1}{1-s}}.$$

We write $I_3^N=\sum_{k=1}^3(I_{3,k}^N+J_{3,k}^N)$ with

$$I_{3,k}^N\stackrel{\text{def}}{=}\frac1N\sum_{i=1}^{N}\Big\langle\sum_{j=1}^{N}G_j^{(k)}X_i,v_i\Big\rangle,\qquad J_{3,k}^N\stackrel{\text{def}}{=}\frac1N\sum_{i=1}^{N}\Big\langle\sum_{j=1}^{N}G_j^{(k)}Z_i,v_i\Big\rangle.$$

We consider each term separately: For $I_{3,1}^N$ we have the following

$$I_{3,1}^N\leqslant\frac1N\Big\|\sum_{j=1}^{N}G_j^{(1)}\Big\|_{L^2}\Big\|\sum_{i=1}^{N}X_iv_i\Big\|_{L^2}\leqslant C\frac1N\Big\|\sum_{j=1}^{N}G_j^{(1)}\Big\|_{L^2}^2+\frac18\frac1N\Big\|\sum_{i=1}^{N}X_iv_i\Big\|_{L^2}^2.$$

For $J_{3,1}^N$ we use (2.1), the interpolation Lemma 2.2 and Young's inequality to obtain

$$
\begin{aligned}
J_{3,1}^N &= \frac{1}{N}\sum_{i=1}^N \Big\langle \Lambda^s\Big(\sum_{j=1}^N G_j^{(1)} v_i\Big), \Lambda^{-s} Z_i\Big\rangle \\
&\leqslant \frac{1}{N}\sum_{i=1}^N \Big[\Big\|\Lambda^s \sum_{j=1}^N G_j^{(1)}\Big\|_{L^2}\|v_i\|_{L^2} + \Big\|\sum_{j=1}^N G_j^{(1)}\Big\|_{L^2}\|\Lambda^s v_i\|_{L^2}\Big]\|\Lambda^{-s}Z_i\|_{L^\infty} \\
&\leqslant \frac{C}{N}\Big\|\sum_{j=1}^N G_j^{(1)}\Big\|_{H^s}^2 + \frac{1}{20}\sum_{i=1}^N \|v_i\|_{H^1}^2 + C\sum_{s'\in\{0,s\}}\Big(\sum_{i=1}^N\|v_i\|_{L^2}^2\Big)\Big(\frac{1}{N}\sum_{i=1}^N\|\Lambda^{-s}Z_i\|_{L^\infty}^2\Big)^{\frac{1}{1-s'}}.
\end{aligned}
$$

For $I_{3,2}^N$ we have

$$
\begin{aligned}
I_{3,2}^N &= \frac{1}{N}\sum_{i=1}^N\Big\langle \Lambda^{-s}\sum_{j=1}^N G_j^{(2)}, \Lambda^s(X_i v_i)\Big\rangle \leqslant \frac{C}{N}\Big\|\sum_{j=1}^N G_j^{(2)}\Big\|_{H^{-s}}^2 + \frac{C}{N}\Big(\sum_{i=1}^N\|\Lambda^s(X_iv_i)\|_{L^2}\Big)^2 \\
&\leqslant \frac{C}{N}\Big\|\sum_{j=1}^N G_j^{(2)}\Big\|_{H^{-s}}^2 + \frac{1}{20}\sum_{i=1}^N\|v_i\|_{H^1}^2 + C\Big(\sum_{i=1}^N\|v_i\|_{L^2}^2\Big)\Big(\frac{1}{N}\sum_{i=1}^N(\|X_i\|_{L^4}^{4/(1-2s)} + \|\Lambda^s X_i\|_{L^4}^4)\Big),
\end{aligned}
$$

where we used (2.1), (5.2) to have

$$
\begin{aligned}
\frac{1}{N}\Big(\sum_{i=1}^N\|\Lambda^s(X_iv_i)\|_{L^2}\Big)^2 &\lesssim \frac{1}{N}\Big(\sum_{i=1}^N\|\Lambda^s X_i\|_{L^4}\|v_i\|_{L^4} + \|\Lambda^s v_i\|_{L^4}\|X_i\|_{L^4}\Big)^2 \\
&\lesssim \frac{1}{N}\Big(\sum_{i=1}^N\|v_i\|_{H^1}^{1/2}\|v_i\|_{L^2}^{1/2}\|\Lambda^s X_i\|_{L^4} + \sum_{i=1}^N\|v_i\|_{H^1}^{\frac12+s}\|v_i\|_{L^2}^{\frac12-s}\|X_i\|_{L^4}\Big)^2,
\end{aligned}
\tag{5.18}
$$

followed by Hölder inequality with exponents $(4,4,2)$, $(\frac{4}{1+2s},\frac{4}{1-2s},2)$, Young's inequality and finally Jensen's inequalities for the terms with X_i in the last inequality. For $J_{3,2}^N$ we have

$$
J_{3,2}^N \lesssim \frac{1}{N^2}\sum_{i=1}^N\|\sum_{j=1}^N G_j^{(2)} Z_i\|_{H^{-1}}^2 + \frac{1}{20}\sum_{i=1}^N\|v_i\|_{H^1}^2.
$$

For $I_{3,3}^N$ we have

$$
\begin{aligned}
I_{3,3}^N &\lesssim \frac{1}{N}\Big\|\sum_{j=1}^N G_j^{(3)}\Big\|_{H^{-s}}^2 + \frac{1}{N}\Big(\sum_{i=1}^N\|\Lambda^s(X_iv_i)\|_{L^2}\Big)^2 \\
&\lesssim \frac{1}{N}\Big\|\sum_{j=1}^N G_j^{(3)}\Big\|_{H^{-s}}^2 + \frac{1}{20}\sum_{i=1}^N\|v_i\|_{H^1}^2 + \Big(\sum_{i=1}^N\|v_i\|_{L^2}^2\Big)\Big(\frac{1}{N}\sum_{i=1}^N(\|X_i\|_{L^4}^{4/(1-2s)} + \|\Lambda^s X_i\|_{L^4}^4)\Big),
\end{aligned}
$$

where we used (5.18) in the last inequality. For the last term we have

$$
J_{3,3}^N \lesssim \frac{1}{N^2}\sum_{i=1}^N\|\sum_{j=1}^N G_j^{(3)} Z_i\|_{H^{-1}}^2 + \frac{1}{20}\sum_{i=1}^N\|v_i\|_{H^1}^2.
$$

Combining all the estimates for $I_{3,k}^N$ and $J_{3,k}^N$ we arrive at (5.17). In the following we calculate $\mathbf{E}\|\bar{R}_N\|_{L_T^1} + \mathbf{E}\|\bar{R}'_N\|_{L_T^1}$. To this end, we recall the general fact (2.3) for centered independent Hilbert space-valued random variables. Applying (2.3) we obtain

$$
\mathbf{E}\|\bar{R}_N\|_{L_T^1} \lesssim \mathbf{E}\|G_1^{(1)}\|_{L_T^2H^s}^2 + \mathbf{E}\|G_1^{(2)}\|_{L_T^2H^{-s}}^2 + \mathbf{E}\|G_1^{(3)}\|_{L_T^2H^{-s}}^2.
$$

It is obvious that $\mathbf{E}\|G_1^{(3)}\|^2_{L^2_T H^{-s}} \lesssim 1$. By Lemma 4.4 we know

$$\mathbf{E}\|G_1^{(1)}\|^2_{L^2_T H^s} \lesssim \int_0^T \mathbf{E}(\|X_1\nabla X_1\|^2_{L^2} + \|X_1\|^4_{L^4})\mathrm{d}t \lesssim 1,$$
$$\mathbf{E}\|G_1^{(2)}\|^2_{L^2_T H^{-s}} \lesssim \int_0^T \mathbf{E}[\|Z_1\|^2_{\mathbf{C}^{-s/2}}\|X_1\|^2_{H^s}]\mathrm{d}t \lesssim \int_0^T (\mathbf{E}[\|X_1\|^2_{H^1} + \|X_1\|^4_{L^4}] + 1)\mathrm{d}t \lesssim 1, \tag{5.19}$$

where we used Lemma 2.2 and Lemma 2.3. Therefore $\mathbf{E}\|\bar{R}_N\|_{L^1_T} \lesssim 1$. For $\bar{R}'_N$ we have

$$\begin{aligned}\mathbf{E}\frac{1}{N^2}\sum_{i=1}^N \|\sum_{j=1}^N G_j^{(2)} Z_i\|^2_{H^{-s}} &= \frac{1}{N^2}\sum_{i,j,\ell=1}^N \mathbf{E}\langle G_j^{(2)} Z_i, G_\ell^{(2)} Z_i\rangle_{H^{-s}} \\ &= \frac{1}{N^2}\Big[\sum_{i=j=\ell} + 2\sum_{i=j\neq\ell} + \sum_{\ell=j\neq i}\Big] \lesssim \mathbf{E}\|X_1 :Z_1^2:\|^2_{H^{-s}} + \mathbf{E}\|X_1 :Z_1Z_2:\|^2_{H^{-s}}\end{aligned}$$

where we used independence to have $\sum_{i\neq j\neq\ell} = 0$. Similarly, we have

$$\mathbf{E}\frac{1}{N^2}\sum_{i=1}^N \|\sum_{j=1}^N G_j^{(3)} Z_i\|^2_{H^{-s}} \lesssim \mathbf{E}\| :Z_1^3:\|^2_{H^{-s}} + \mathbf{E}\| :Z_1^2 Z_2:\|^2_{H^{-s}}.$$

Combining the above two estimates and using Lemma 3.1 and the same argument as in (5.19) with Z_1 replaced by $:Z_1Z_2:$ and $:Z_1^2:$, we obtain $\mathbf{E}\|\bar{R}'_N\|_{L^1_T} \lesssim 1$.

STEP 5 (Convergence of v_i to zero in $L^2(\Omega)$)

We now combine our estimates and conclude the proof of (5.1). Namely, we insert the estimates (5.7) and (5.12) into (5.3) and also appeal to our bounds from Step 4 to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^N \|v_i\|^2_{L^2} \leqslant C(\bar{R}_N + \bar{R}'_N) + C\Big(1 + \sum_{i=2}^7 R_N^i + \frac{1}{N}\sum_{i=1}^N \big(\|X_i\|^{4/(1-2s)}_{L^4} + \|\Lambda^s X_i\|^4_{L^4}\big)\Big)\sum_{i=1}^N \|v_i\|^2_{L^2}, \tag{5.20}$$

where $\bar{R}_N + \bar{R}^1_N$ is uniformly bounded in $L^1(\Omega\times[0,T])$. Furthermore, by Lemma 3.1, Lemma 3.3, Lemma 4.4 and (5.2) we deduce also that R_N^i is uniformly bounded in $L^1(\Omega\times[0,T])$ for each $i = 2,\dots,7$. By the Gagliardo-Nirenberg inequality in Lemma 2.2 we have for $s \geqslant 2\kappa$, $r > 4$, $\frac{1}{4} = \frac{s}{2} + \frac{1-s}{r}$, $\|\Lambda^s X_i\|^4_{L^4} \lesssim \|X_i\|^{4s}_{H^1}\|X_i\|^{4(1-s)}_{L^r}$, which combined with Lemma 4.4 implies that

$$\mathbf{E}\int_0^T \frac{1}{N}\sum_{i=1}^N \big(\|X_i\|^{\frac{4}{1-2s}}_{L^4} + \|\Lambda^s X_i\|^4_{L^4}\big)\mathrm{d}t \lesssim \mathbf{E}\int_0^T \frac{1}{N}\sum_{i=1}^N \big(\|X_i\|^2_{H^1} + \|X_i\|^{\frac{4}{1-2s}}_{L^{\frac{4}{1-2s}}} + 1\big)\mathrm{d}t < \infty.$$

We now divide (5.20) by N and use the above observations together with Gronwall's inequality. Note that in light of Assumption 5.1, it holds that

$$\frac{1}{N}\sum_{i=1}^N \|v_i(0)\|^2_{L^2} \to^{\mathbf{P}} 0. \tag{5.21}$$

It now follows that

$$\sup_{t\in[0,T]}\frac{1}{N}\sum_{i=1}^N \|v_i\|^2_{L^2} + \frac{1}{N}\sum_{i=1}^N \|v_i\|^2_{L^2_T H^1} + \frac{1}{N^2}\sum_{i,j=1}^N \|Y_j v_i\|^2_{L^2_T L^2} + \frac{1}{N^2}\|\sum_j X_j v_j\|^2_{L^2_T L^2} \tag{5.22}$$

converges to zero in probability by Lemma 5.2 below. We now upgrade this from convergence in probability to convergence in $L^1(\Omega)$ by bounding higher moments and applying Vitali's convergence theorem. Only in this part we use the condition that the initial conditions $(z_i^N, y_i^N, z_i, \eta_i)_{i=1}^N$ are exchangeable, which implies that the law of $v_i(t)$ and $v_j(t)$, $i \neq j$ are the same.

Indeed, first note that $\sup_{t\in[0,T]}\frac{1}{N}\sum_{i=1}^N\|Y_i\|_{L^2}^2$ is uniformly bounded in $L^q(\Omega)$ for q in Assumption 5.1 by Lemma 3.3. Additionally, by Lemma 4.4 , Jensen's inequality, and the fact that X_i and X_j are identically distributed (which follows from the i.i.d. hypothesis in Assumption 5.1) it holds

$$\sup_{N\geqslant 1}\sup_{t\in[0,T]}\mathbf{E}\Big(\frac{1}{N}\sum_{i=1}^N\|X_i(t)\|_{L^2}^2\Big)^2\leqslant\sup_{t\in[0,T]}\mathbf{E}\|X_1(t)\|_{L^2}^4<\infty.$$

Notice that at this stage we are appealing to the assumption $\mathbf{E}\|\eta_i\|_{L^{p_0}}^{p_0}\lesssim 1$ in order to meet the hypotheses of Lemma 4.4 and deduce the final step above. Hence, by the triangle inequality we find that

$$\sup_{N\geqslant 1}\sup_{t\in[0,T]}\mathbf{E}\Big(\frac{1}{N}\sum_{i=1}^N\|v_i(t)\|_{L^2}^2\Big)^q<\infty,$$

which implies the following convergence upgrade: $\frac{1}{N}\sum_{i=1}^N\|v_i(t)\|_{L^2}^2$ converges to zero in $L^1(\Omega)$ for each $t\in[0,T]$. Finally, we appeal once more to the first bullet point in Assumption 5.1 which is designed to ensure that v_i and v_j have the same law. As a consequence we can now pass from empirical averages to components in light of

$$\mathbf{E}\|v_i(t)\|_{L^2}^2=\frac{1}{N}\sum_{i=1}^N\mathbf{E}\|v_i(t)\|_{L^2}^2\to 0. \tag{5.23}$$

Step 6 (Convergence as a stochastic process)

The proof is largely the same as above, except that we do not estimate v_i by an average over i as in (5.23), since a supremum over time would not commute with a sum over i. Instead we deduce the following bound

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|v_i\|_{L^2}^2+\frac{1}{2}\|v_i\|_{H^1}^2\leqslant C\Big(\frac{\bar{R}_N}{N}+\tilde{\bar{R}}_N'\Big)+\frac{1}{N}\sum_{i=1}^N\|v_i\|_{H^1}^2+\frac{1}{N^2}\Big\|\sum_j X_jv_j\Big\|_{L^2}^2 &(5.24)\\ &+C\Big(1+\tilde{R}_N^{21}+\sum_{k\in\{3,5\}}R_N^k+\sum_{k\in\{3,4,7\}}\tilde{R}_N^k+\frac{1}{N}\sum_{j=1}^N\|X_j\|_{L^4}^4+\Big(\|X_i\|_{L^4}^{4/(1-2s)}+\|\Lambda^sX_i\|_{L^4}^4\Big)\Big)\|v_i\|_{L^2}^2\\ &+C\Big(1+\tilde{R}_N^{22}+\tilde{R}_N^3+\tilde{R}_N^5+\tilde{R}_N^6+\|X_i\|_{L^4}^4\Big)\frac{1}{N}\sum_{j=1}^N\|v_j\|_{L^2}^2, &(5.25)\end{aligned}$$

where all the "tilde R-terms" are defined analogously to their "un-tilde" counterparts with slight tweaks:

$$\tilde{\bar{R}}_N'\stackrel{\text{def}}{=}\frac{1}{N^2}\Big\|\sum_{j=1}^N G_j^2Z_i\Big\|_{H^{-s}}^2+\frac{1}{N^2}\Big\|\sum_{j=1}^N G_j^3Z_i\Big\|_{H^{-s}}^2,$$

$$\tilde{R}_N^{21}\stackrel{\text{def}}{=}\frac{1}{N}\sum_{j=1}^N\|:Z_j^2:\|_{\mathbf{C}^{-s}}^{2/(2-s)},\qquad \tilde{R}_N^{22}\stackrel{\text{def}}{=}\frac{1}{N}\sum_{j=1}^N\|:Z_iZ_j:\|_{\mathbf{C}^{-s}}^2,$$

$$\tilde{R}_N^3\stackrel{\text{def}}{=}\|Z_i\|_{\mathbf{C}^{-s}}^{2/(1-s)},\qquad \tilde{R}_N^5\stackrel{\text{def}}{=}\Big(1+\frac{1}{N}\sum_{j=1}^N\|\nabla Y_j\|_{L^2}^2\Big)^{\frac{s}{1-s}}\|Z_i\|_{\mathbf{C}^{-s}}^{\frac{2}{1-s}},$$

$$\tilde{R}_N^4\stackrel{\text{def}}{=}\Big(1+\frac{1}{N}\sum_{j=1}^N\|X_j\|_{H^1}^2\Big)^{\frac{2s}{2-s}}(\|Z_i\|_{\mathbf{C}^{-s}}^{\frac{4}{2-s}}+1)+\Big(\frac{1}{N}\mathscr{Z}_N\Big)^{\frac{2}{2-s}}\|X_i\|_{H^1}^{\frac{4s}{2-s}}+\Big(\frac{1}{N}\mathscr{Z}_N\Big)^2\|X_i\|_{L^4}^2,$$

$$\tilde{R}_N^6\stackrel{\text{def}}{=}\Big(\frac{1}{N}\sum_{j=1}^N\|Y_j\|_{L^4}^2\Big)^{\frac{s}{1-s}}\|Z_i\|_{\mathbf{C}^{-s}}^{\frac{2}{1-s}},\qquad \tilde{R}_N^7\stackrel{\text{def}}{=}\|\Lambda^{-s}Z_i\|_{L^\infty}^{2/(1-s)},$$

where $\mathscr{Z}_N$ is as in (3.14). In fact all the terms are similar as above except the following two terms:

$$-\frac{1}{N}\sum_{j=1}^{N}\int X_iX_jv_jv_i\mathrm{d}x-\frac{2}{N}\sum_{j=1}^{N}\langle X_iv_jv_i,Z_j\rangle:=J_1+J_2.$$

The term J_1 is treated differently than above, since without the sum over i we could not move it to the l.h.s. as a coercive quantity. We have

$$\begin{aligned}J_1&\leqslant\frac{1}{N}\sum_{j=1}^{N}\int v_j^2X_i^2\mathrm{d}x+\frac{1}{N}\sum_{j=1}^{N}\int v_i^2X_j^2\mathrm{d}x\\&\leqslant C\Big(\frac{1}{N}\sum_{j=1}^{N}\|v_j\|_{L^4}^2\Big)\|X_i\|_{L^4}^2+C\|v_i\|_{L^4}^2\frac{1}{N}\sum_{j=1}^{N}\|X_j\|_{L^4}^2\\&\leqslant C\Big(\frac{1}{N}\sum_{j=1}^{N}\|v_j\|_{H^1}^2\Big)^{1/2}\Big(\frac{1}{N}\sum_{j=1}^{N}\|v_j\|_{L^2}^2\Big)^{1/2}\|X_i\|_{L^4}^2+\frac{1}{8}\|v_i\|_{H^1}^2+C\|v_i\|_{L^2}^2\Big(\frac{1}{N}\sum_{j=1}^{N}\|X_j\|_{L^4}^2\Big)^2,\end{aligned}$$

which by Young's inequality deduce one contribution to (5.25). For the second term we have

$$J_2\lesssim\sum_{s'\in\{0,s\}}\Big(\frac{1}{N}\sum_{j=1}^{N}\|v_iv_jX_i\|_{L^1}^{2(1-s')}\|\nabla(v_iv_jX_i)\|_{L^1}^{2s'}\Big)^{\frac12}\Big(\frac{1}{N}\mathscr{Z}_N\Big)^{\frac12}.\tag{5.26}$$

Using Hölder's inequality in the form $\|v_iv_jX_i\|_{L^1}\leqslant\|v_i\|_{L^2}\|v_jX_i\|_{L^2}$ together with the bound for the first term in J_1 we obtain the estimate for $s'=0$ in J_2, which corresponds to the last term in $\tilde{R}_N^4$. Moreover, we have

$$\begin{aligned}\|\nabla(v_iv_jX_i)\|_{L^1}\leqslant{}&\|\nabla v_i\|_{L^2}\|v_jX_i\|_{L^2}+\|v_i\|_{L^2}^{1/2}\|v_i\|_{H^1}^{1/2}\|\nabla v_j\|_{L^2}\|X_i\|_{L^4}\\&+\|v_i\|_{L^2}^{1/2}\|v_i\|_{H^1}^{1/2}\|v_j\|_{L^2}^{1/2}\|v_j\|_{H^1}^{1/2}\|\nabla X_i\|_{L^2},\end{aligned}$$

and inserting this into the term $s'=s$ in (5.26) and applying Hölder's inequality for the summation in j leads to the following

$$\begin{aligned}&\Big(\frac{1}{N}\sum_{j=1}^{N}\|v_jX_i\|_{L^2}^2\Big)^{1/2}\|v_i\|_{L^2}^{1-s}\|v_i\|_{H^1}^{s}\Big(\frac{1}{N}\mathscr{Z}_N\Big)^{1/2}\\&+\Big(\frac{1}{N}\sum_{j=1}^{N}\|v_jX_i\|_{L^2}^{2(1-s)}\|v_j\|_{H^1}^{2s}\Big)^{1/2}\|v_i\|_{L^2}^{1-s/2}\|v_i\|_{H^1}^{s/2}\|X_i\|_{L^4}^{s}\Big(\frac{1}{N}\mathscr{Z}_N\Big)^{1/2}\\&+\Big(\frac{1}{N}\sum_{j=1}^{N}\|v_jX_i\|_{L^2}^{2(1-s)}\|v_j\|_{H^1}^{s}\|v_j\|_{L^2}^{s}\Big)^{1/2}\|\nabla X_i\|_{L^2}^{s}\|v_i\|_{L^2}^{1-\frac{s}{2}}\|v_i\|_{H^1}^{\frac{s}{2}}\Big(\frac{1}{N}\mathscr{Z}_N\Big)^{1/2}.\end{aligned}$$

Finally, we apply Young's inequality and obtain the contribution of $\tilde{R}_N^4$ in the estimate (5.25).

Using the fact that (5.22) converges to zero in probability, we deduce the $L^1(0,T)$ norm of (5.25) and the right hand side of (5.24) converges to zero in probability. Then by Gronwall's inequality and Lemma 5.2 imply $\sup_{t\in[0,T]}\|v_i(t)\|_{L^2}^2\to0$ in probability, as $N\to\infty$. In this step we see that we don't use the condition that the initial conditions $(z_i^N,y_i^N,z_i,\eta_i)_{i=1}^N$ are exchangeable.

Step 7 (General initial datum)

In this final and independent step, we sketch how to modify the arguments above to treat the case that $z_i^N\neq z_i$. Since the proof follows a similar line of argument as in Steps 2–3, we place the details in a supplementary file. To this end, define $u_i\overset{\text{def}}{=}S_t(z_i^N-z_i)$ and note that we have the following extra terms:

$$\bar{I}^N := -\frac{1}{N^2}\sum_{i,j=1}^N \Big[\langle Y_j^2, v_i u_i\rangle + 2\langle Y_j Y_i u_j, v_i\rangle + 2\langle Y_j v_i,\ {:}Z_i^N Z_j^N{:} - {:}Z_i Z_j{:}\ \rangle$$
$$+ \langle Y_i v_i,\ {:}Z_j^{N,2}{:} - {:}Z_j^2{:}\ \rangle + \langle v_i,\ {:}Z_i^N Z_j^{N,2}{:} - {:}Z_i Z_j^2{:}\ \rangle\Big]$$

These terms could also be estimated similarly as that for I_1^N and I_2^N by using $\|u_i(t)\|_{\mathbf{C}^s} \lesssim t^{-(s+\kappa)/2}\|z_i^N - z_i\|_{\mathbf{C}^{-\kappa}}$. □

We recall from classical probability theory which can be deduced with elementary arguments.

Lemma 5.2. *Let $\{U_N\}_{N=1}^\infty$ be a non-negative sequence of 1d random variables converging to zero in probability. Let $\{V_N\}_{N=1}^\infty$ be a non-negative sequence of random variables with tight laws. Then the sequence $\{U_N V_N\}_{N=1}^\infty$ converges to zero in probability.*

6. Invariant measure and observables

We now study the invariant measure for the equation

$$\mathscr{L}\Psi = -\mathbf{E}[\Psi^2 - Z^2]\Psi + \xi, \tag{6.1}$$

with $\mathbf{E}[\Psi^2 - Z^2] = \mathbf{E}[X^2] + 2\mathbf{E}[XZ]$ for $X = \Psi - Z$ and ξ space-time white noise. Here, since we are only interested in the stationary setting in this section, we overload the notation in the previous sections and simply write Z for the stationary solution to the linear equation

$$\mathscr{L}Z = \xi\ , \tag{6.2}$$

and we consider the decomposition (slightly different from Section 4) $X \stackrel{\text{def}}{=} \Psi - Z$, so that

$$\mathscr{L}X = -\mathbf{E}[X^2 + 2XZ](X + Z), \quad X(0) = \Psi(0) - Z(0). \tag{6.3}$$

For the case that $m = 0$ we restrict the solutions Ψ and Z satisfying $\langle \Psi, 1\rangle = \langle Z, 1\rangle = 0$.

By Theorem 4.6 for every initial data

$$\Psi(0) = \psi \in \mathbf{C}^{-\kappa}$$

with $\mathbf{E}\|\psi\|_{\mathbf{C}^{-\kappa}}^p \lesssim 1$ there exists a unique global solution Ψ to (6.1). We immediately find that Z is a stationary solution to (6.1). This follows since the unique solution to (6.3) starting from zero is identically zero. Furthermore, we define a semigroup $P_t^*\nu$ to denote the law of $\Psi(t)$ with the initial condition distributed according to a measure ν. By uniqueness of the solutions to (6.1), we have $P_t^* = P_{t-s}^* P_s^*$ for $t \geqslant s \geqslant 0$. By direct probabilistic calculation we easily obtain the following result, which implies that the implicit constant in Lemma 3.1 is independent of m.

Lemma 6.1. *For $\kappa' > \kappa > 0$ and $p \geqslant 1$, it holds that*

$$\sup_{m\geqslant 0}\mathbf{E}[\|Z_i\|_{C_T\mathbf{C}^{-\kappa}}^p] + \sup_{m\geqslant 0}\mathbf{E}[\|\ {:}Z_i Z_j{:}\ \|_{C_T\mathbf{C}^{-\kappa}}^p] + \sup_{m\geqslant 0}\mathbf{E}[\|\ {:}Z_i Z_j^2{:}\ \|_{C_T\mathbf{C}^{-\kappa}}^p] \lesssim 1,$$

where the proportional constants are independent of i, j, N.

Proof. By a standard technique (c.f. [GP17]), it is sufficient to calculate

$$\mathbf{E}|\Delta_q Z_i(t)|^2 \lesssim \sum_{k\in\mathbb{Z}^2}\int_{-\infty}^t \theta(2^{-q}k)^2 |e^{-2(t-s)(|k|^2+m)}|\mathrm{d}s \lesssim \sum_{k\in\mathbb{Z}^2} 2^{q\kappa}\frac{1}{|k|^\kappa(|k|^2+m)},$$

where Δ_q is a Littlewood-Paley block and θ is the Fourier multiplier associated with Δ_q. From here we see the bound is independent of m. Other terms can be bounded in a similar way. □

For R_N^0 defined in Lemma 3.5 with Z_i stationary, we have the following result.

Lemma 6.2. *For every $q \geqslant 1$ it holds that*

$$\mathbf{E}[(R_N^0)^q] \lesssim 1. \tag{6.4}$$

Proof. Since we will have several similar calculations in the sequel, we first demonstrate such calculation in the case $q=1$. We have

$$\mathbf{E}\frac{1}{N^2}\sum_{i=1}^N\Big\|\sum_{j=1}^N\Lambda^{-s}(\,{:}Z_j^2Z_i{:}\,)\Big\|_{L^2}^2=\frac{1}{N^2}\sum_{i,j_1,j_2=1}^N\mathbf{E}\Big\langle\Lambda^{-s}\,{:}Z_{j_1}^2Z_i{:}\,,\Lambda^{-s}\,{:}Z_{j_2}^2Z_i{:}\,\Big\rangle.$$

We have 3 summation indices and a factor $1/N^2$. The contribution to the sum from the cases $j_1=i$ or $j_2=i$ or $j_1=j_2$ is bounded by a constant in light of Lemma 6.1. If i,j_1,j_2 are all different, by independence and the fact that Wick products are mean zero, the terms are zero.

For general $q\geqslant 1$, by Gaussian hypercontractivity and the fact that R_N^0 is a random variable with finite Wiener chaos decomposition, we have

$$\mathbf{E}[(R_N^0)^q]\lesssim\mathbf{E}[(R_N^0)^2]^{q/2}.$$

For the case that $q=2$ we write it as

$$\frac{1}{N^4}\sum_{\substack{i_1,i_2,j_k=1\\k=1\dots4}}^N\mathbf{E}\Big\langle\Lambda^{-s}\,{:}Z_{j_1}^2Z_{i_1}{:}\,,\Lambda^{-s}\,{:}Z_{j_2}^2Z_{i_1}{:}\,\Big\rangle\Big\langle\Lambda^{-s}\,{:}Z_{j_3}^2Z_{i_2}{:}\,,\Lambda^{-s}\,{:}Z_{j_4}^2Z_{i_2}{:}\,\Big\rangle.$$

We have 6 indices $i_1,i_2,j_k,k=1,...,4$ summing from 1 to N and an overall factor $1/N^4$. Using again Lemma 6.1, we reduce the problem to the cases where five or six of the indices are different. However, in these two cases, by independence the expectation is zero, so (6.4) follows. ☐

6.1. **Uniqueness the of invariant measure.** We now turn to the question of uniqueness for the invariant measure of (6.1). Since the non-linearity in the SPDE (6.1) involves the law of the solution, the associated semigroup P_t^* is generally nonlinear i.e.

$$P_t^*\nu\neq\int(P_t^*\delta_\psi)\nu(\mathrm{d}\psi),$$

for a non-trivial distribution ν (see e.g. [Wan18]). As a result, its unclear if the general ergodic theory for Markov processes (see e.g. [DPZ96], [HMS11]) can be applied directly in our setting. Fortunately, (6.3) has a strong damping property in the mean-square sense, which comes to our rescue and allows us to proceed directly by a priori estimates.

Lemma 6.3. *There exists $C_0>0$ such that for all*

$$m\geqslant 2C_0(\mathbf{E}\|\,{:}Z_2Z_1{:}\,\|_{\mathbf{C}^{-s}}^2+(\mathbf{E}\|Z_1\|_{\mathbf{C}^{-s}}^2)^{\frac{1}{1-s}}+1):=m_0,$$

there exists a universal C with the following property: for every solution Ψ to (6.1) with $\Psi(0)\in\mathbf{C}^{-\kappa}$,

$$\sup_{t\geqslant 1}e^{\frac{mt}{2}}\mathbf{E}\|\Psi(t)-Z(t)\|_{L^2}^2\leqslant C. \tag{6.5}$$

Proof. The proof relies heavily on several computations performed in Lemma 4.3 where we used slightly different notation, so we will write X_i instead of X and Z_i instead of Z for the remainder of this proof. Revisiting the first step of Lemma 4.3 where we established (4.5), we find that I_1 and the first contribution to I_2 vanishes in light of $\mathbf{E}(\,{:}Z_j^2{:}\,)=0$. It follows that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}\|X_i\|_{L^2}^2+\mathbf{E}\|\nabla X_i\|_{L^2}^2+m\mathbf{E}\|X_i\|_{L^2}^2+\|\mathbf{E}X_i^2\|_{L^2}^2\\&=-2\mathbf{E}\langle X_iX_j,Z_iZ_j\rangle-3\mathbf{E}\langle X_iX_j^2,Z_i\rangle.\end{aligned}$$

Furthermore, in light of (4.8) and (4.14), we obtain

$$\mathbf{E}\langle X_iX_j,Z_iZ_j\rangle\lesssim\big(\|\mathbf{E}X_i^2\|_{L^1}^2+\mathbf{E}\|\nabla X_i\|_{L^2}^2\|\mathbf{E}X_j^2\|_{L^1}\big)^{1/2}\big(\mathbf{E}\|\,{:}Z_jZ_i{:}\,\|_{\mathbf{C}^{-s}}^2\big)^{\frac{1}{2}}$$

$$\mathbf{E}\langle X_i X_j^2, Z_i\rangle \lesssim \|\mathbf{E}X_i^2\|_{L^2}\big(\mathbf{E}\|\nabla X_i\|_{L^2}^2\big)^{\frac{s}{2}}\big(\mathbf{E}\|X_i\|_{L^2}^2\big)^{\frac{1-s}{2}}\big(\mathbf{E}\|Z_i\|_{\mathbf{C}^{-s}}^2\big)^{\frac12}$$
$$+\|\mathbf{E}X_i^2\|_{L^2}\big(\mathbf{E}\|X_j\|_{L^2}^2\big)^{\frac12}\big(\mathbf{E}\|Z_i\|_{\mathbf{C}^{-s}}^2\big)^{\frac12}.$$

We will use these estimates in two different ways. On one hand, using Young's inequality with respective exponents $(2,2)$ and $(2,\frac{2}{s},\frac{2}{1-s})$ followed by Lemma 6.1, we find that

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}\|X_i\|_{L^2}^2+\frac12\mathbf{E}\|\nabla X_i\|_{L^2}^2+m\mathbf{E}\|X_i\|_{L^2}^2+\|\mathbf{E}X_i^2\|_{L^2}^2\lesssim 1. \tag{6.6}$$

As a consequence, noting that $\|\mathbf{E}X_i^2\|_{L^2}^2\geqslant\big(\mathbf{E}\|X_i\|_{L^2}^2\big)^2$, applying Lemma 2.6 it holds

$$\sup_{t>0}(t\wedge 1)\mathbf{E}[\|X_i(t)\|_{L^2}^2]\lesssim 1, \tag{6.7}$$

where the implicit constant is independent of the initial data. On the other hand, Young's inequality also yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}\|X_i\|_{L^2}^2+m\mathbf{E}\|X_i\|_{L^2}^2\leqslant C_0(\mathbf{E}\|:Z_2Z_1:\|_{\mathbf{C}^{-s}}^2+(\mathbf{E}\|Z_1\|_{\mathbf{C}^{-s}}^2)^{\frac{1}{1-s}}+1)\mathbf{E}\|X_i\|_{L^2}^2. \tag{6.8}$$

Applying Gronwall's inequality over $[1,t]$ leads to

$$e^{(m-\frac{m_0}{2})t}\mathbf{E}\|X_i(t)\|_{L^2}^2\lesssim\mathbf{E}\|X_i(1)\|_{L^2}^2,$$

so choosing $m\geqslant m_0$, using (6.7), and taking the supremum over $t\geqslant 1$, we arrive at (6.5). □

We now apply the above result to show that for sufficiently large mass, the unique invariant measure to (6.1) is Gaussian. To this end, define the $\mathbf{C}^{-1}$-Wasserstein distance

$$\mathbb{W}_p'(\nu_1,\nu_2):=\inf_{\pi\in\mathscr{C}(\nu_1,\nu_2)}\left(\int\|\phi-\psi\|_{\mathbf{C}^{-1}}^p\pi(\mathrm{d}\phi,\mathrm{d}\psi)\right)^{1/p},$$

where $\mathscr{C}(\nu_1,\nu_2)$ denotes the collection of all couplings of ν_1,ν_2 satisfying $\int\|\phi\|_{\mathbf{C}^{-1}}^p\nu_i(\mathrm{d}\phi)<\infty$ for $i=1,2$.

Theorem 6.4. *For m_0 as in Lemma 6.3 and $m\geqslant m_0$ the unique invariant measure to (6.1) supported on $\mathbf{C}^{-\kappa}$ is $\mathcal{N}(0,(-\Delta+m)^{-1})$, the law of the Gaussian free field.*

Proof. Recall that Z is a stationary solution to (6.1). Indeed, by definition, $\Psi=X+Z$, where X solves (6.3). However, since $X(0)=0$, the identically zero process is the unique solution to (6.3). Hence, the law of Z, which we now denote by ν, is invariant under P_t^*. We now claim that for $m\geqslant m_0$, this is the only invariant measure supported on $\mathbf{C}^{-\kappa}$. Indeed, let ν_1 be another such measure, then modifying the stochastic basis if needed, we may assume there exists $\psi\in\mathbf{C}^{-\kappa}$ on it such that $\psi\sim\nu_1$. By similar arguments as in Theorem 4.6 we may construct a solution Ψ to (6.1) with $\Psi(0)=\psi$. By invariance of ν_1 and ν and the embedding $L^2\hookrightarrow\mathbf{C}^{-1}$, c.f. Lemma 2.1, it follows that

$$\mathbb{W}_2'(\nu,\nu_1)^2=\mathbb{W}_2'(P_t^*\nu,P_t^*\nu_1)^2\leqslant\mathbf{E}\|\Psi(t)-Z(t)\|_{\mathbf{C}^{-1}}^2\lesssim e^{-\frac{mt}{2}},$$

for $t\geqslant 1$ by Lemma 6.3. Letting $t\to\infty$ we obtain $\nu=\nu_1$. □

Remark 6.5. *Note that for the limiting equation $\mathscr{L}\Psi=-\mu\Psi+\xi$, if we assume that μ is simply a constant, it has a Gaussian invariant measure $\mathcal{N}(0,(-\Delta+m+\mu)^{-1})$. Assuming $\Psi\sim\mathcal{N}(0,(-\Delta+m+\mu)^{-1})$ and $Z\sim\mathcal{N}(0,(-\Delta+m)^{-1})$, the self-consistent requirement $\mathbf{E}[\Psi^2-Z^2]=\mu$ then yields*

$$\sum_{k\in\mathbb{Z}^2}\Big(\frac{1}{|k|^2+m+\mu}-\frac{1}{|k|^2+m}\Big)=\mu$$

and for $\mu+m\geqslant 0$ we only have one solution $\mu=0$, since the LHS is monotonically decreasing in μ.

Remark 6.6. *Changing the renormalization constant in (6.1) will alter the mass of the Gaussian invariant measure. For instance, if we replace Z in (6.1) by the stationary solution to $(\partial_t-(\Delta-a))Z_i=\xi_i$ with $a>0$, one invariant measure is Gaussian $\bar\nu \stackrel{def}{=} \mathcal{N}(0,(-\Delta+m+\mu_0)^{-1})$ with μ_0 satisfying*

$$\sum_{k\in\mathbb{Z}^2}\Big(\frac{1}{|k|^2+m+\mu_0}-\frac{1}{|k|^2+a}\Big)=\mu_0.$$

Moreover by the same proof of Lemma 6.3 and Theorem 6.4, for $m+\mu_0$ large enough, $\bar\nu$ is the unique invariant measure. Indeed, let $\Psi=\bar X+\bar Z$ with $\bar Z$ the stationary solution to $\mathscr{L}\bar Z=-\mu_0\bar Z+\xi$, then $\bar X$ satisfies the following equation:

$$\mathscr{L}\bar X=-\mu_0\bar X-\mathbf{E}[\bar X^2+2\bar X\bar Z](\bar X+\bar Z),$$

which is the same case as (4.1) with m replaced by $m+\mu_0$.

6.2. **Convergence of the invariant measures.** As a consequence of Lemma 3.2, the solutions $(\Phi_i)_{1\leqslant i\leqslant N}$ to (3.1) form a Markov process on $(\mathbf{C}^{-\kappa})^N$ which, by strong Feller property in [HM18b] and irreducibility in [HS19], will turn out to admit a unique invariant measure, henceforth denoted by ν^N. Our goal in this section is to study the large N behavior of ν^N and show that for sufficiently large mass, as $N\to\infty$, it's marginals are simply products of the Gaussian invariant measure for Ψ identified in Theorem 6.4. For this we rely heavily on the computations from Section 3.2 for the remainder Y, but we leverage these estimates with consequences of stationarity. To this end, it will be convenient to have a stationary coupling of the linear and non-linear dynamics (3.2) and (3.1) respectively, which is the focus of the following lemma.

Lemma 6.7. *There exists a unique invariant measure ν^N on $(\mathbf{C}^{-\kappa})^N$ to (3.1). Furthermore, there exists a stationary process $(\Phi_i^N,Z_i)_{1\leqslant i\leqslant N}$ such that the components Φ_i^N,Z_i are stationary solutions to (3.1) and (3.2), respectively. Moreover, $\mathbf{E}\|\Phi_i^N(0)-Z_i(0)\|_{H^1}^2\lesssim 1$ and for every $q>1$*

$$\mathbf{E}\Big(\frac{1}{N}\sum_{i=1}^N\|\Phi_i^N(0)-Z_i(0)\|_{L^2}^2\Big)^q\lesssim 1 \tag{6.9}$$

Proof. In the proof we fix N. Let Φ_i and Z_i be solutions to (3.1) and (3.2), respectively. By the general results of [HM18b], it follows that $(\Phi_i,Z_i)_{1\leqslant i\leqslant N}$ is a Markov process on $(\mathbf{C}^{-\kappa})^{2N}$, and we denote by $(P_t^N)_{t\geqslant 0}$ the associated Markov semigroup. To derive the desired structural properties about the limiting measure, we will follow the Krylov-Bogoliubov construction with a specific choice of initial condition that allows to exploit Lemma 3.3. Namely, we denote by Φ_i the solution to (3.1) starting from the stationary solution $\tilde Z_i(0)$, so that the process $Y_i=\Phi_i-Z_i$ starts from the origin. Using Lemma 3.3 and Corollary 3.4 with $y_j=0$ together with Lemma 6.1 to obtain a uniform bound on $\mathbf{E}R_N$, we find for every $T\geqslant 1$

$$\int_0^T\mathbf{E}\Big(\frac{1}{N}\sum_{i=1}^N\|Y_i(t)\|_{H^1}^2\Big)\mathrm{d}t\lesssim T, \tag{6.10}$$

$$\mathbf{E}\int_0^T\Big(\frac{1}{N}\sum_{i=1}^N\|Y_i(t)\|_{L^2}^2\Big)^q\mathrm{d}t\lesssim T, \tag{6.11}$$

where the implicit constant is independent of T and for $m=0$ we used the Poincaré inequality. By (6.10) we obtain

$$\int_0^T\mathbf{E}\Big(\frac{1}{N}\sum_{i=1}^N\|\Phi_i(t)\|_{\mathbf{C}^{-\kappa/2}}^2\Big)+\int_0^T\mathbf{E}\Big(\frac{1}{N}\sum_{i=1}^N\|Z_i(t)\|_{\mathbf{C}^{-\kappa/2}}^2\Big)\lesssim T.$$

Defining $R_t^N:=\frac{1}{t}\int_0^t P_r^N\mathrm{d}r$, the above estimates and the compactness of the embedding $\mathbf{C}^{-\kappa/2}\hookrightarrow\mathbf{C}^{-\kappa}$ imply the induced laws of $\{R_t^N\}_{t\geqslant 0}$ started from $(\tilde Z(0),\tilde Z(0))$ are tight on $(\mathbf{C}^{-\kappa})^{2N}$. Furthermore, by the continuity with respect to initial data, it is easy to check that $(P_t^N)_{t\geqslant 0}$ is Feller on $(\mathbf{C}^{-\kappa})^{2N}$. By the Krylov-Bogoliubov existence theorem (see [DPZ96, Corollary 3.1.2]) , these laws converge weakly in $(\mathbf{C}^{-\kappa})^{2N}$ along a subsequence $t_k \to \infty$ to an invariant measure π_N for $(P_t^N)_{t\geqslant 0}$. The desired stationary process $(\Phi_i^N, Z_i)_{1\leqslant i\leqslant N}$ is defined to be the unique solution to (3.1) and (3.2) obtained by sampling the initial datum $(\phi_i, z_i)_i$ from π_N. By (6.10) we deduce

$$\mathbf{E}^{\pi_N}\|\Phi_i(0)-Z_i(0)\|_{H^1}^2 = \mathbf{E}^{\pi_N}\sup_{\varphi}\langle \Phi_i(0)-Z_i(0),\varphi\rangle^2$$
$$=\mathbf{E}\sup_{\varphi}\lim_{T\to\infty}\left[\frac{1}{T}\int_0^T\langle Y_i(t),\varphi\rangle \mathrm{d}t\right]^2 \leqslant \lim_{T\to\infty}\frac{1}{T}\int_0^T\mathbf{E}\|Y_i(t)\|_{H^1}^2\mathrm{d}t \lesssim 1,$$

where $\sup_\varphi$ is over smooth functions φ with $\|\varphi\|_{H^{-1}}\leqslant 1$. Similarly using (6.11), (6.9) follows. Finally, we project onto the first component and consider the map $\bar{\Pi}_1 : \mathcal{S}'(\mathbb{T}^2)^{2N} \to \mathcal{S}'(\mathbb{T}^2)^N$ defined through $\bar{\Pi}_1(\Phi, Z) = \Phi$. Letting $\nu^N = \pi_N \circ \bar{\Pi}_1^{-1}$ yields an invariant measure to (3.1), and uniqueness follows from the general results of strong Feller property in [HM18b] and irreducibility in [HS19]. □

Remark 6.8. *Using a lattice approximation (see e.g. [GH18], [HM18a, ZZ18]) one can show that the measure $\nu^N(\mathrm{d}\Phi)$ indeed has the form (1.2) (with Wick renormalization).*

The next step is to study tightness of the marginal laws of ν^N over $\mathcal{S}'(\mathbb{T}^2)^N$. To this end, consider the projection $\Pi_i : \mathcal{S}'(\mathbb{T}^2)^N \to \mathcal{S}'(\mathbb{T}^2)$ defined by $\Pi_i(\Phi) = \Phi_i$ and let $\nu^{N,i} \stackrel{\mathrm{def}}{=} \nu^N \circ \Pi_i^{-1}$ be the marginal law of the i^{th}. Furthermore, for $k \leqslant N$, define the map $\Pi^{(k)} : \mathcal{S}'(\mathbb{T}^2)^N \to \mathcal{S}'(\mathbb{T}^2)^k$ via $\Pi^{(k)}(\Phi) = (\Phi_i)_{1\leqslant i\leqslant k}$ and the let $\nu_k^N \stackrel{\mathrm{def}}{=} \nu^N \circ (\Pi^{(k)})^{-1}$ be the marginal law of the first k components. We have the following result:

Theorem 6.9. *$\{\nu^{N,i}\}_{N\geqslant 1}$ form a tight set of probability measures on $\mathbf{C}^{-\kappa}$ for $\kappa > 0$.*

Proof. Let $(\Phi_i^N, Z_i)_{1\leqslant i\leqslant N}$ be the jointly stationary solutions to (3.1) and (3.2) constructed in Lemma 6.7. To prove the result, in light of the compact embedding of $\mathbf{C}^{-\kappa/2} \hookrightarrow \mathbf{C}^{-\kappa}$ and the stationarity of Φ_i^N, it suffices to show that the second moment of $\|\Phi_i^N(0)\|_{\mathbf{C}^{-\kappa/2}}$ is bounded uniformly in N. To this end, let $Y_i^N = \Phi_i^N - Z_i$, which is also stationary and note that

$$\mathbf{E}\|\Phi_i^N(0)\|_{\mathbf{C}^{-\kappa/2}}^2 = \frac{2}{T}\int_{T/2}^T \mathbf{E}\|\Phi_i^N(s)\|_{\mathbf{C}^{-\kappa/2}}^2\mathrm{d}s$$
$$\leqslant \frac{4}{T}\int_{T/2}^T \mathbf{E}\|Z_i(s)\|_{\mathbf{C}^{-\kappa/2}}^2\mathrm{d}s + \frac{4}{T}\int_{T/2}^T \mathbf{E}\|Y_i^N(s)\|_{H^1}^2\mathrm{d}s.$$

The first term is controlled by Lemma 6.1. For the second term, symmetry yields Y_i^N and Y_j^N are identical in law, which combined with Lemma 3.3 implies that

$$\frac{2}{T}\int_{T/2}^T \mathbf{E}\|Y_i^N(s)\|_{H^1}^2\mathrm{d}s = \frac{2}{T}\int_{T/2}^T \frac{1}{N}\sum_{i=1}^N \mathbf{E}\|Y_i^N(s)\|_{H^1}^2\mathrm{d}s \leqslant \frac{C}{T}\mathbf{E}[\int_0^T R_N \mathrm{d}t] \leqslant C,$$

where we used that by stationarity $\sum_{i=1}^N \mathbf{E}\|Y_i^N(T)\|_{L^2}^2 = \sum_{i=1}^N \mathbf{E}\|Y_i^N(T/2)\|_{L^2}^2$, with both being finite in view of Lemma 6.7. For $m = 0$ we also used the Poincaré inequality. □

Remark 6.10. *It is reasonable to expect that any limiting measure obtained in Theorem 6.9 is an invariant measure for (1.6) assuming only $m \geqslant 0$. However, this cannot be directly deduced from our main result in Section 5 because we do not know a-priori that any limiting measure of ν^N is a product measure. This is problematic because the initial conditions for each component of Ψ_i are assumed to be independent in Section 5. Nevertheless, we can prove below that this is indeed true if m is large.*

In the following we prove the convergence of the measure to the unique invariant measure by using the estimate in Lemma 3.5, which requires m large enough.

Define the $\mathbf{C}^{-\kappa}$-Wasserstein distance

$$\mathbb{W}_2(\nu_1,\nu_2) := \inf_{\pi\in\mathscr{C}(\nu_1,\nu_2)} \left(\int \|\phi-\psi\|^2_{\mathbf{C}^{-\kappa}}\pi(\mathrm{d}\phi,\mathrm{d}\psi)\right)^{1/2},$$

where $\mathscr{C}(\nu_1,\nu_2)$ denotes the set of all couplings of ν_1,ν_2 satisfying $\int\|\phi\|^2_{\mathbf{C}^{-\kappa}}\nu_i(\mathrm{d}\phi)<\infty$ for $i=1,2$.

Theorem 6.11. *Let $\nu=\mathcal{N}(0,(m-\Delta)^{-1})$. There exist $C_0>0$ such that for all $m\geqslant m_1$ where*

$$m_1 \stackrel{def}{=} C_0(\mathbf{E}\|Z_1\|^{\frac{2}{2-s}}_{\mathbf{C}^{-s}}+\mathbf{E}\|:Z_1^2:\|^{\frac{2}{2-s}}_{\mathbf{C}^{-s}}+E\|:Z_2Z_1:\|^2_{\mathbf{C}^{-s}}+1)$$

one has

$$\mathbb{W}_2(\nu^{N,i},\nu)\leqslant CN^{-\frac{1}{2}}. \tag{6.12}$$

Furthermore, ν^N_k converges to $\nu\times\ldots\times\nu$, as $N\to\infty$.

Proof. By Lemma 6.7 we may construct a stationary coupling (Φ^N_i,Z_i) of ν_N and ν whose components satisfy (1.1) and (3.2), respectively. The stationarity of the joint law of (Φ^N_i,Z_i) implies that also $Y^N_i=\Phi^N_i-Z_i$ is stationary. In the following we freely omit the time argument of expectations of stationary quantities. We now claim that

$$\mathbf{E}\|Y^N_i\|^2_{H^1}\leqslant CN^{-1}, \tag{6.13}$$

which implies (6.12) by definition of the Wasserstein metric and the embedding $H^1\hookrightarrow\mathbf{C}^{-\kappa}$ in $d=2$, c.f. Lemma 2.1. To ease notation, we write $Y_i=Y^N_i$ in the following. By (3.21) combined with the stationarity of $(Y_j)_j$ and $(Z_j)_j$, we find

$$\begin{aligned}&\sum_{j=1}^N\mathbf{E}\|\nabla Y_j\|^2_{L^2}+m\sum_{j=1}^N\mathbf{E}\|Y_j\|^2_{L^2}+\frac{1}{N}\mathbf{E}\Big\|\sum_{i=1}^N Y_i^2\Big\|^2_{L^2}\\&\leqslant C\mathbf{E}R^0_N+\mathbf{E}\Big(\sum_{j=1}^N\|Y_j\|^2_{L^2}(D_N+D^1_N)\Big),\end{aligned}$$

where R^0_N is defined in Lemma 3.5 and

$$D_N=C\Big(\frac{1}{N}\sum_{j=1}^N\|Z_j\|^{\frac{2}{2-s}}_{\mathbf{C}^{-s}}+\frac{1}{N}\sum_{j=1}^N\|:Z_j^2:\|^{\frac{2}{2-s}}_{\mathbf{C}^{-s}}+1\Big).$$

$$D^1_N=C\Big(\frac{1}{N^2}\sum_{i,j=1}^N\|:Z_jZ_i:\|^2_{\mathbf{C}^{-s}}\Big).$$

Setting $A\stackrel{\text{def}}{=}\mathbf{E}D_N$ and $A_1\stackrel{\text{def}}{=}\mathbf{E}\|:Z_2Z_1:\|^2_{\mathbf{C}^{-s}}$ we may re-center D_N and D^1_N above and divide by N to obtain

$$\begin{aligned}&\frac{1}{N}\sum_{j=1}^N\mathbf{E}\|\nabla Y_j(t)\|^2_{L^2}+(m-A-A_1)\frac{1}{N}\sum_{j=1}^N\mathbf{E}\|Y_j(t)\|^2_{L^2}+\frac{1}{N^2}\mathbf{E}\Big\|\sum_{i=1}^N Y_i^2(t)\Big\|^2_{L^2}\\&\leqslant C\frac{1}{N}\mathbf{E}R^0_N+\frac{1}{N}\mathbf{E}\Big(\sum_{j=1}^N\|Y_j(t)\|^2_{L^2}(|D_N-A|+|D^1_N-A_1|)\Big)\\&\leqslant C\frac{1}{N}\mathbf{E}R^0_N+\frac{1}{2}\frac{1}{N^2}\mathbf{E}\Big(\sum_{j=1}^N\|Y_j\|^2_{L^2}\Big)^2+\mathbf{E}|D_N-A|^2+\mathbf{E}|D^1_N-A_1|^2.\end{aligned}$$

For $m\geqslant A+A_1+1$, using that Y_i and Y_j are equal in law, we obtain

$$\mathbf{E}\|Y_i\|^2_{H^1}\leqslant\frac{1}{N}\sum_{j=1}^N\mathbf{E}\|\nabla Y_j(t)\|^2_{L^2}+(m-A-A_1)\frac{1}{N}\sum_{j=1}^N\mathbf{E}\|Y_j\|^2_{L^2}$$

$$\leqslant C\frac{1}{N}\mathbf{E}R_N^0+\mathbf{E}|D_N-A|^2+\mathbf{E}|D_N^1-A_1|^2. \tag{6.14}$$

Using independence, we find

$$\mathbf{E}|D_N(t)-A|^2\leqslant\frac{1}{N}\mathrm{Var}\Big(\|Z_1\|_{\mathbf{C}^{-s}}^{\frac{2}{2-s}}+\|:Z_1^2:\|_{\mathbf{C}^{-s}}^{\frac{2}{2-s}}\Big)\leqslant\frac{C}{N}. \tag{6.15}$$

To estimate D_N^1, we write $M_{i,j}=\|:Z_jZ_i:\|_{\mathbf{C}^{-s}}^2-A_1$ for $i\neq j$ and $M_{i,i}=\|:Z_i^2:\|_{\mathbf{C}^{-s}}^2-A_1$ and have

$$\begin{aligned}\mathbf{E}\Big(\frac{1}{N^2}\sum_{i,j=1}^N M_{i,j}\Big)^2&\leqslant\mathbf{E}\Big(\frac{1}{N^2}\sum_{i=1}^N\sum_{j\neq i}M_{i,j}+\frac{1}{N^2}\sum_{i=1}^N M_{i,i}\Big)^2\\&\leqslant\frac{2}{N^4}\sum_{i_1\neq j_1,i\neq j}\mathbf{E}(M_{i,j}M_{i_1,j_1})+\frac{2}{N^2}\mathbf{E}(M_{1,1}^2)\lesssim\frac{1}{N}+\frac{2}{N^2}\mathbf{E}(M_{1,1}^2)\lesssim\frac{1}{N},\end{aligned}$$

where we used that for the case that (i,j,i_1,j_1) are different, $\mathbf{E}(M_{i,j}M_{i_1,j_1})=\mathbf{E}M_{i,j}\mathbf{E}M_{i_1,j_1}=0$.

Then we have

$$\mathbf{E}\big|D_N^1-A_1\big|^2\lesssim\frac{1}{N}. \tag{6.16}$$

Inserting the estimates (6.15), and (6.16) into (6.14) and using (6.4), we obtain (6.13), completing the proof. □

Remark 6.12. *Instead of assuming m large, one could alternatively consider arbitrary $m>0$ and assume small nonlinearity. Namely, we could consider a nonlinearity $-\frac{\lambda}{N}\sum_{j=1}^N :\Phi_j^2\Phi_i:$ instead of that of (3.1), and $-\lambda\mathbf{E}[\Psi^2-Z^2]\Psi$ instead of that of (6.1), for $\lambda>0$. By tracing the proofs of Lemma 6.3 and Theorem 6.11, we can easily see that given any $m>0$, there exists a constant $\lambda_0>0$, so that the statements of Lemma 6.3 and Theorem 6.11 hold for any $\lambda\in(0,\lambda_0)$.*

Remark 6.13. *Following Remark 6.6, with a change of renormalization constant therein, we can write $\Phi_i=\bar Y_i+\bar Z_i$ with $\bar Z_i$ the stationary solution to $\mathscr{L}\bar Z_i=-\mu_0\bar Z_i+\xi_i$. Then $\bar Y_i$ satisfies*

$$\mathscr{L}\bar Y_i=-\mu_0\bar Y_i-\frac{1}{N}\sum_{j=1}^N(\bar Y_j^2\bar Y_i+\bar Y_j^2\bar Z_i+2\bar Y_j\bar Y_i\bar Z_j+2\bar Y_j:\bar Z_i\bar Z_j:+:\bar Z_j^2:\bar Y_i+:\bar Z_i\bar Z_j^2:)-\frac{2\mu_0}{N}(\bar Y_i+\bar Z_i),$$

which is the same case as (3.3) with m replaced by $m+\mu_0$ and an extra term $\frac{2\mu_0}{N}(\bar Y_i+\bar Z_i)$. Here the Wick product of $\bar Z_j$ is defined similarly as in section 3.1. By the same proof of Theorem 6.11 we know for $m+\mu_0$ large enough, $\nu^{N,i}$ (renormalized as in Remark 6.6) converges to $\bar\nu$ and the other results in Theorem 6.11 also hold in this case.

6.3. **Observables.** In quantum field theories with symmetries, quantities that are invariant under action of the symmetry group are of particular interest; examples of such quantities in the SPDE setting include gauge invariant observables e.g. [She18, Section 2.4]. The model we study here exhibits $O(N)$ rotation symmetry and formally, functions of the squared "norm" $\sum_i\Phi_i^2$ are quantities that are $O(N)$ invariant. Of course, such observables need to be suitably renormalized to be well-defined and suitably scaled by factors of N to have nontrivial limit as $N\to\infty$.

In this section we study the following two observables:

$$\frac{1}{N^{1/2}}\sum_{i=1}^N:\Phi_i^2:\ ,\qquad\frac{1}{N}:\Big(\sum_{i=1}^N\Phi_i^2\Big)^2:\ , \tag{6.17}$$

with $\Phi=(\Phi_i)_{1\leqslant i\leqslant N}\sim\nu^N$. In this section we omit the superscript N for simplicity. These are defined as follows. By Lemma 6.7 we decompose $\Phi_i=Y_i+Z_i$ with (Y_i,Z_i) stationary. With this we define

$$\frac{1}{\sqrt N}\sum_{i=1}^N:\Phi_i^2:\stackrel{\text{def}}{=}\frac{1}{\sqrt N}\sum_{i=1}^N(Y_i^2+2Y_iZ_i+:Z_i^2:), \tag{6.18}$$

$$\frac{1}{N}:\Big(\sum_{i=1}^N \Phi_i^2\Big)^2: \stackrel{\text{def}}{=} \frac{1}{N}\sum_{i,j=1}^N \Big(Y_i^2Y_j^2 + 4Y_i^2Y_jZ_j + 2Y_i^2 :Z_j^2: \tag{6.19}$$

$$+ :Z_i^2Z_j^2: + 4Y_i :Z_iZ_j^2: + 4Y_iY_j :Z_iZ_j: \Big). \tag{6.20}$$

Here the Wick products are canonically defined as in (3.4) with $a_\varepsilon = \mathbf{E}[Z_{i,\varepsilon}^2(0,0)]$, in particular

$$:Z_i^2Z_j^2: = \begin{cases} \lim_{\varepsilon\to 0}(Z_{i,\varepsilon}^4 - 6a_\varepsilon Z_{i,\varepsilon}^2 + 3a_\varepsilon^2) & (i=j) \\ \lim_{\varepsilon\to 0}(Z_{i,\varepsilon}^2 - a_\varepsilon)(Z_{j,\varepsilon}^2 - a_\varepsilon) & (i\neq j). \end{cases} \tag{6.21}$$

Remark 6.14. *One could also define* (6.17) *in $L^p(\nu^N)$ directly without using the decomposition $\Phi_i = Y_i + Z_i$. In fact, by similar argument as in* [GJ87, Section 8.6] *or* [Sim74], *one can show that ν^N is absolutely continuous with respect to the corresponding Gaussian free field $\tilde\nu$ with a density in $L^p(\tilde\nu)$ for $p\in(1,\infty)$. Since* (6.17) *with each Φ_i replaced by Z_i can be defined via $L^p(\tilde\nu)$ limit of mollification, using argument along the line of* [RZZ17, Lemma 3.6] *we know that* (6.17) *can be also defined as $L^p(\nu^N)$ limit of mollification (essentially Hölder inequality), and they have the same law as the right hand side of* (6.18) *and* (6.19)+(6.20).

In this section we also consider Y_i, Z_i as stationary process with Z_i as the stationary solution of (6.2) and Y_i as the solution of (3.3).

Lemma 6.15. *There exists an m_0 such that for $m \geqslant m_0$ and $q\geqslant 1$*

$$\mathbf{E}\Big[\Big(\sum_{i=1}^N \|Y_i\|_{L^2}^2\Big)^q\Big] + \mathbf{E}\Big[\Big(\sum_{i=1}^N \|Y_i\|_{L^2}^2 + 1\Big)^q\Big(\sum_{i=1}^N \|\nabla Y_i\|_{L^2}^2\Big)\Big] \lesssim 1, \tag{6.22}$$

$$\mathbf{E}\Big[\Big(\sum_{i=1}^N \|Y_i\|_{L^2}^2 + 1\Big)^q\Big\|\sum_{i=1}^N Y_i^2\Big\|_{L^2}^2\Big] \lesssim 1, \tag{6.23}$$

where the implicit constant is independent of N.

Proof. First we observe that (6.23) may be quickly deduced from (6.22) with the help of the inequality

$$\Big\|\sum_{i=1}^N Y_i^2\Big\|_{L^2}^2 \lesssim \Big(\sum_{i=1}^N \|Y_i\|_{H^1}^2\Big)\Big(\sum_{i=1}^N \|Y_i\|_{L^2}^2\Big). \tag{6.24}$$

To obtain (6.24), note first that

$$\Big\|\sum_{i=1}^N Y_i^2\Big\|_{L^2}^2 = \sum_{i,j=1}^N \|Y_iY_j\|_{L^2}^2.$$

Furthermore, by Hölder's inequality, (5.2), and Young's inequality

$$\begin{aligned}\|Y_iY_j\|_{L^2}^2 \leqslant \|Y_i\|_{L^4}^2\|Y_j\|_{L^4}^2 &\lesssim \|Y_i\|_{H^1}\|Y_j\|_{L^2}\|Y_j\|_{H^1}\|Y_i\|_{L^2} \\ &\lesssim \|Y_i\|_{H^1}^2\|Y_j\|_{L^2}^2 + \|Y_i\|_{H^1}^2\|Y_j\|_{L^2}^2.\end{aligned}$$

Summing both sides over i,j and using symmetry with respect to the roles of i and j, we obtain (6.24). The remainder of the proof is devoted to (6.22).

To shorten the expressions that follow, we introduce the quantities $F \stackrel{\text{def}}{=} \sum_{i=1}^N \|\nabla Y_i\|_{L^2}^2 + \frac{1}{N}\|\sum_{i=1}^N Y_i^2\|_{L^2}^2$ and $U \stackrel{\text{def}}{=} \sum_{i=1}^N \|Y_i\|_{L^2}^2$. Note that F and U are stationary, so we will freely omit the time argument below. Our starting point is the key inequality (3.21), which may be recast in terms of U and F as

$$\frac{\mathrm{d}}{\mathrm{d}t}U + F + mU \leqslant CR_N^0 + C\big(D_N + D_N^1\big)U.$$

Muliplying the above by U^{q-1} we find that for $q \geqslant 1$ it holds

$$\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}U^q + U^{q-1}F + mU^q \leqslant CR_N^0 U^{q-1} + C\big(D_N + D_N^1\big)U^q.$$

As in the proof of Theorem 6.11, we now define $A \stackrel{\text{def}}{=} \mathbf{E}(D_N)$ and $A_1 \stackrel{\text{def}}{=} \mathbf{E}\| :Z_1 Z_2: \|^2_{\mathbf{C}^{-s}}$. Subtract the mean from $D_N + D_N^1$ and take expectation on both sides to find

$$\begin{aligned}&\mathbf{E}\big[U^{q-1}F\big] + (m - A - A_1)\mathbf{E}\big[U^q\big]\\ &\leqslant C\mathbf{E}\big[R_N^0 U^{q-1}\big] + C\mathbf{E}\big[\big(D_N + D_N^1 - A - A_1\big)U^q\big].\\ &\leqslant C\|R_N^0\|_{L^q(\Omega)}\big(\mathbf{E}U^q\big)^{\frac{q-1}{q}} + C\|D_N - A + D_N^1 - A_1\|_{L^{q+1}(\Omega)}\big(\mathbf{E}U^{q+1}\big)^{\frac{q}{q+1}}\\ &\leqslant C\big(\mathbf{E}U^q\big)^{\frac{q-1}{q}} + CN^{-\frac{1}{2}}\big(\mathbf{E}U^{q+1}\big)^{\frac{q}{q+1}},\end{aligned}$$

where we used $\mathbf{E}U^q(t) = \mathbf{E}U^q(0)$ in the first inequality and we used a Gaussian hypercontractivity upgrade of (6.15) and (6.16) in the last line. Using Young's inequality, we may absorb the first term to the left and obtain

$$\mathbf{E}\big[U^{q-1}F\big] + (m - A - A_1 - 1)\mathbf{E}\big[U^q\big] \leqslant C + CN^{-\frac{1}{2}}\big(\mathbf{E}U^{q+1}\big)^{\frac{q}{q+1}}. \tag{6.25}$$

The strategy now is to first use the dissipative quantity on the LHS of (6.25) to obtain $\mathbf{E}(U^q) \leqslant CN^{\frac{q-1}{2}}$, and then use the massive term on the LHS of (6.25) to iteratively decrease the power of N and eventually arrive at $\mathbf{E}(U^q) \leqslant C$. Once this is established, plugging the bound back into (6.25) completes the proof.

Indeed, first observe that $F \geqslant N^{-1}U^2$ so that $\mathbf{E}(U^{q-1}F) \geqslant N^{-1}\mathbf{E}(U^{q+1})$. Hence, Young's inequality with exponents $(q+1, \frac{q+1}{q})$ leads to $\mathbf{E}(U^q) \leqslant CN^{\frac{q-1}{2}}$. Defining $A_q \stackrel{\text{def}}{=} \mathbf{E}U^q$ and discarding the dissipative term, (6.25) implies

$$A_q \lesssim A_{q+1}^{\frac{q}{q+1}} N^{-1/2} + 1. \tag{6.26}$$

We have $A_q \lesssim N^{\frac{q-1}{2}}$, which gives

$$A_q \lesssim N^{\frac{q^2}{2(q+1)} - \frac{1}{2}} + 1.$$

Substituting into (6.26) and use induction we have for $m \geqslant 1$

$$A_q \lesssim N^{a_{m,q}} + 1, \tag{6.27}$$

with $a_{m,q} = \frac{q(q+m-1)}{2(q+m)} - \frac{q}{2}(\sum_{k=1}^{m-1}\frac{1}{q+k}) - \frac{1}{2}$. Here $\sum_{k=1}^{0} = 0$. In fact, we could check (6.27) by

$$A_q \lesssim (N^{a_{m,q+1}})^{\frac{q}{q+1}} N^{-\frac{1}{2}} + 1 \lesssim N^{a_{m+1,q}} + 1.$$

For fixed $q \geqslant 1$ we could always find m large enough such that $a_{m,q} < 0$, which implies that $A_q \lesssim 1$ and the result follows. □

Theorem 6.16. *Let $\Phi = (\Phi_i)_{1\leqslant i\leqslant N} \sim \nu^N$ and m be given as in Lemma 6.15, then the laws of $\frac{1}{\sqrt{N}}\sum_{i=1}^N :\Phi_i^2:$ are tight on $B^{-2\kappa}_{2,2}$, and the laws of $\frac{1}{N} :\Big(\sum_{i=1}^N \Phi_i^2\Big)^2:$ are tight on $B^{-3\kappa}_{1,1}$.*

Proof. Note that the first term $\frac{1}{\sqrt{N}}\sum_{i=1}^N Y_i^2$ on RHS of (6.18) converges to zero in $L^2(\Omega; L^2)$ as an immediate consequence of (6.23); so we can actually prove a stronger result than stated, namely the subseqential limits can be identified with those of the last two terms in (6.18). We will now show that the other two quantities induce tight laws on $B^{-3\kappa}_{2,2}$, which implies the first part of the theorem. The second sum in (6.18) can be estimated using Lemma 2.3 and Lemma 2.1 to find for $s \in (\kappa, 2\kappa)$

$$\Big\|\frac{1}{\sqrt{N}}\sum_{i=1}^N Y_i Z_i\Big\|_{B^{-s}_{2,2}} \lesssim \frac{1}{\sqrt{N}}\sum_{i=1}^N \|Y_i\|_{B^s_{2,2}}\|Z_i\|_{\mathbf{C}^{-\kappa}}$$

$$\lesssim \sum_{i=1}^{N} \|Y_i\|_{B^s_{2,2}}^2 + \frac{1}{N}\sum_{i=1}^{N} \|Z_i\|_{\mathbf{C}^{-\kappa}}^2 \lesssim \sum_{i=1}^{N} \|Y_i\|_{H^1}^{2s}\|Y_i\|_{L^2}^{2(1-s)} + \frac{1}{N}\sum_{i=1}^{N} \|Z_i\|_{\mathbf{C}^{-\kappa}}^2$$
$$\lesssim \sum_{i=1}^{N} \|Y_i\|_{H^1}^2 + \sum_{i=1}^{N} \|Y_i\|_{L^2}^2 + \frac{1}{N}\sum_{i=1}^{N} \|Z_i\|_{\mathbf{C}^{-\kappa}}^2,$$

which is bounded by a constant using Lemma 6.15 for $q = 1$ and Lemma 6.1. For the third sum in (6.18) we use independence to find for $s \in (\kappa, 2\kappa)$

$$\mathbf{E}\left\|\frac{1}{\sqrt{N}}\sum_{i=1}^{N} {:}Z_i^2{:}\right\|_{B^{-s}_{2,2}}^2 = \mathbf{E}\frac{1}{N}\left\langle \Lambda^{-s}\sum_{i=1}^{N} {:}Z_i^2{:}\,, \Lambda^{-s}\sum_{i=1}^{N} {:}Z_i^2{:}\right\rangle = \mathbf{E}\|\,{:}Z_i^2{:}\,\|_{B^{-s}_{2,2}}^2 \lesssim 1.$$

By the triangle inequality and the embedding $L^2 \hookrightarrow B^{-\kappa}_{2,2}$ we find that $\frac{1}{\sqrt{N}}\sum_{i=1}^{N} {:}\Phi_i^2{:}$ is bounded in $L^1(\Omega; B^{-s}_{2,2})$. In light of the compact embedding $B^{-s}_{2,2} \subset B^{-2\kappa}_{2,2}$, the tightness claim follows.

For the second observable, we will also show a stronger result: the subseqential limits can be identified with those of the last three terms in (6.20). We start with the first 3 terms in (6.19), which will be shown to converge to zero in $L^1(\Omega; B^{-2s}_{1,1})$ for $s > \kappa$. For the first term of (6.19) we use Lemma 6.15 to obtain

$$\mathbf{E}\left\|\frac{1}{N}\sum_{i,j=1}^{N} Y_i^2 Y_j^2\right\|_{L^1} = \frac{1}{N}\mathbf{E}\left\|\sum_{i=1}^{N} Y_i^2\right\|_{L^2}^2 \lesssim \frac{1}{N},$$

so this term converges to zero in $L^1(\Omega; L^1)$. For the second term of (6.19), using (3.18) of Lemma 3.3 with φZ_j in place of Z_j we obtain

$$\sup_{\|\varphi\|_{\mathbf{C}^{2s}} \leqslant 1}\left|\frac{1}{N}\sum_{i,j=1}^{N}\langle Y_i^2 Y_j Z_j, \varphi\rangle\right|$$
$$\lesssim \frac{1}{N}\left\|\sum_{i=1}^{N} Y_i^2\right\|_{L^2}\left(\sum_{j=1}^{N}\|Y_j\|_{L^2}^2\right)^{\frac{1-s}{2}}\left(\sum_{j=1}^{N}\|\nabla Y_j\|_{L^2}^2\right)^{\frac{s}{2}}\left(\sum_{j=1}^{N}\|Z_j\|_{\mathbf{C}^{-s}}^2\right)^{1/2}$$
$$+ \frac{1}{N}\left\|\sum_{i=1}^{N} Y_i^2\right\|_{L^2}\left(\sum_{j=1}^{N}\|Y_j\|_{L^2}^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{N}\|Z_j\|_{\mathbf{C}^{-s}}^2\right)^{1/2}.$$

Hence, using Young's inequality we find for $\delta > 0$ small enough

$$\left\|\frac{1}{N}\sum_{i,j=1}^{N} Y_i^2 Y_j Z_j\right\|_{B^{-2s}_{1,1}} \leqslant N^{-\delta}\left\|\sum_{i=1}^{N} Y_i^2\right\|_{L^2}^2 + N^{-\delta}\left(\sum_{j=1}^{N}\|\nabla Y_j\|_{L^2}^2\right)$$
$$+ N^{-\delta}\left(\sum_{j=1}^{N}\|Y_j\|_{L^2}^2\right)\left(\frac{1}{N}\sum_{j=1}^{N}\|Z_j\|_{\mathbf{C}^{-s}}^2 + 1\right)^{\frac{1}{1-s}}.$$

Both terms above converge to zero in $L^1(\Omega)$ as a consequence of (6.22), (6.23) and Lemma 6.1. For the third term in (6.19), a calculation similar to (3.12) using Lemma 2.3 with Y_i replaced by φY_i with $\varphi \in \mathbf{C}^{2s}$ yields

$$\sup_{\|\varphi\|_{\mathbf{C}^{2s}} \leqslant 1}\left|\frac{1}{N}\sum_{i,j=1}^{N}\langle Y_i^2\,{:}Z_j^2{:}\,, \varphi\rangle\right| \lesssim \frac{1}{N}\sum_{i=1}^{N}\|\Lambda^s(Y_i^2)\|_{L^2}\left\|\sum_{j=1}^{N}\Lambda^{-s}(\,{:}Z_j^2{:}\,)\right\|_{L^2}$$
$$\lesssim \left(\sum_{i=1}^{N}\|\nabla Y_i\|_{L^2}^{1+s}\|Y_i\|_{L^2}^{1-s} + \|Y_i^2\|_{L^2}\right)\left\|\frac{1}{N}\sum_{j=1}^{N}\Lambda^{-s}(\,{:}Z_j^2{:}\,)\right\|_{L^2}$$
$$\lesssim \left(\sum_{i=1}^{N}\|Y_i\|_{H^1}^2\right)^{\frac{1+s}{2}}\left(\sum_{i=1}^{N}\|Y_i\|_{L^2}^2\right)^{\frac{1-s}{2}}\left\|\frac{1}{N}\sum_{j=1}^{N}\Lambda^{-s}(\,{:}Z_j^2{:}\,)\right\|_{L^2}, \tag{6.28}$$

where we used (2.1) and Lemma 2.2 to have

$$\|\Lambda^s(Y_i^2)\|_{L^2} \lesssim \|\Lambda^s Y_i\|_{L^4}\|Y_i\|_{L^4} \lesssim \|\nabla Y_i\|_{L^2}^{1+s}\|Y_i\|_{L^2}^{1-s} + \|Y_i\|_{L^4}^2.$$

The first part of the product in (6.28) is bounded in $L^1(\Omega)$ by (6.22). For the second part of the product, we use independence to obtain

$$\mathbf{E}\Big\|\frac{1}{N}\sum_{j=1}^N \Lambda^{-s}(\,{:}Z_j^2{:}\,)\Big\|_{L^2}^2 \lesssim \frac{1}{N^2}\sum_{j=1}^N \mathbf{E}\|\Lambda^{-s}(\,{:}Z_j^2{:}\,)\|_{L^2}^2 \lesssim \frac{1}{N},$$

so together we find $\mathbf{E}\|\frac{1}{N}\sum_{i,j} Y_i^2\,{:}Z_j^2{:}\,\|_{B^{-2s}_{1,1}}$ converges to 0.

We now turn to terms in (6.20) and derive suitable moment bounds. For the first of these terms we have

$$\begin{aligned}\mathbf{E}\Big\|\frac{1}{N}\sum_{i,j=1}^N {:}Z_i^2Z_j^2{:}\,\Big\|_{B^{-\kappa}_{2,2}}^2 &= \mathbf{E}\frac{1}{N^2}\Big\langle \Lambda^{-\kappa}\sum_{i,j=1}^N {:}Z_i^2Z_j^2{:}\,,\Lambda^{-\kappa}\sum_{i,j=1}^N {:}Z_i^2Z_j^2{:}\,\Big\rangle \\ &\lesssim \mathbf{E}\|\,{:}Z_1^2Z_2^2{:}\,\|_{B^{-\kappa}_{2,2}}^2 + \frac{1}{N}\mathbf{E}\|\,{:}Z_1^4{:}\,\|_{B^{-\kappa}_{2,2}}^2 \lesssim 1.\end{aligned}$$

For the next term, using (3.20) with φY_i in place of Y_i we find

$$\begin{aligned}&\sup_{\|\varphi\|_{\mathbf{C}^{2s}}\leqslant 1}\Big|\frac{1}{N}\sum_{i,j=1}^N \langle Y_i\,{:}Z_iZ_j^2{:}\,,\varphi\rangle\Big| \\ &\lesssim \Big(\sum_{i=1}^N\|\Lambda^s Y_i\|_{L^2}^2\Big)^{1/2}\Big(\frac{1}{N^2}\sum_{i=1}^N\Big\|\sum_{j=1}^N\Lambda^{-s}(\,{:}Z_j^2Z_i{:}\,)\Big\|_{L^2}^2\Big)^{1/2} \\ &\lesssim \Big(\sum_{i=1}^N\|Y_i\|_{H^1}^2\Big)^{s/2}\Big(\sum_{i=1}^N\|Y_i\|_{L^2}^2\Big)^{(1-s)/2}\Big(\frac{1}{N^2}\sum_{i=1}^N\Big\|\sum_{j=1}^N\Lambda^{-s}(\,{:}Z_j^2Z_i{:}\,)\Big\|_{L^2}^2\Big)^{1/2}.\end{aligned}$$

Using (6.4) and Lemma 6.15 we deduce for some p satisfying $sp<2$

$$\mathbf{E}\Big\|\frac{1}{N}\sum_{i,j=1}^N Y_i,\ {:}Z_iZ_j^2{:}\,\Big\|_{B^{-2s}_{1,1}}^p \lesssim 1.$$

For the last term, we argue similarly to (3.10) but ${:}Z_iZ_j{:}$ replace by $\varphi\,{:}Z_iZ_j{:}$ for $\varphi\in\mathbf{C}^s$ to deduce boundedness in $L^1(\Omega;B^{-2s}_{1,1})$.

Combining the above observations with the triangle inequality, we find that the second observable is uniformly bounded in probability as a $B^{-2s}_{1,1}$ valued random variable. By compactness of the embedding of $B^{-2s}_{1,1}$ into $B^{-2s-\delta}_{1,1}$ for $\delta>0$, we obtain the result. □

7. One space dimension

In this section we turn to one space dimension $d=1$ and state the analogous results obtained in the previous sections. Moreover we prove a result (Theorem 7.5) which states that in the large N limit, certain $O(N)$ invariant observables have nontrivial law and we derive explicit formulae for certain correlations of them. We will put the proofs which follow the same way as (or much simpler than) the 2D case in the appendix.

7.1. **Convergence of the dynamics.** Consider the following equation on $\mathbb{R}^+\times\mathbb{T}$:

$$\mathscr{L}\Phi_i = -\frac{1}{N}\sum_{j=1}^N \Phi_j^2\Phi_i + \xi_i,\quad \Phi_i(0)=\phi_i, \tag{7.1}$$

with $\mathscr{L} = \partial_t - \Delta + m$ for $m \geqslant 0$, $1 \leqslant i \leqslant N$. We assume $\mathbf{E}\|\phi_i\|_{L^2}^2 \lesssim 1$ uniformly in i, N. Now we decompose (7.1) to the following two equations:

$$\mathscr{L} Z_i = \xi_i, \tag{7.2}$$

where Z_i is the stationary solution to (7.2), and

$$\mathscr{L} Y_i = -\frac{1}{N}\sum_{j=1}^{N}(Y_j + Z_j)^2(Y_i + Z_i), \quad Y_i(0) = y_i \stackrel{\text{def}}{=} \phi_i - Z_i(0). \tag{7.3}$$

For $d = 1$ it is well-known that

$$\sup_{m\geqslant 0} \mathbf{E}\|Z_i\|^p_{C_T\mathbf{C}^{\frac{1}{2}-\kappa}} \lesssim 1, \tag{7.4}$$

for $\kappa > 0, p > 1$ and, of course, no renormalization is required to define products of Z_i, Z_j. By similar arguments as the two dimensional case (see Lemma 3.2), we can show existence and uniqueness of solutions $Y_i \in C_T L^2 \cap L^4_T L^4 \cap L^2_T H^1$ to (7.3) $\mathbf{P}$-a.s.. We give uniform in N a-priori estimates in Appendix B.

In $d = 1$, we will show that the limiting equation as $N \to \infty$ is given by

$$\mathscr{L}\Psi_i = -\mu\Psi_i + \xi_i, \qquad \text{with} \qquad \mu(t, x) = \mathbf{E}[\Psi_i(t, x)^2] \tag{7.5}$$

where $(\xi_i)_{i\in\mathbb{N}}$ are the same collection of noises as in (7.1). We assume that the initial conditions $\Psi_i(0) = \psi_i$ are such that $\mathbf{E}\|\psi_i\|^2_{L^\infty}$ is bounded. Set $\Psi_i = Z_i + X_i$ so that

$$\mathscr{L} X_i = -\mu(X_i + Z_i), \qquad X_i(0) = \psi_i - Z_i(0). \tag{7.6}$$

Existence and uniqueness of solutions to (7.6) could be obtained by similar arguments as in the two dimensional case and one has $X_i \in L^2(\Omega; C_T L^\infty \cap L^2_T H^1)$, but we skip the details and put the a-priori estimates in Appendix B.

We now state the convergence of the dynamics. Recall that Φ_i^N is the solution to (7.1) with initial condition ϕ_i^N. Ψ_i is the solution to (1.6) with initial condition ψ_i. We set $v_i^N = \Phi_i^N - \Psi_i$ and have

$$\begin{aligned}\mathscr{L} v_i &= -\frac{1}{N}\sum_{j=1}^{N}\Phi_j^2\Phi_i + \mu\Psi_i \\ &= -\frac{1}{N}\sum_{j=1}^{N}\Phi_j^2(\Phi_i - \Psi_i) - \frac{1}{N}\sum_{j=1}^{N}(\Phi_j^2 - \Psi_j^2)\Psi_i - (\frac{1}{N}\sum_{j=1}^{N}\Psi_j^2 - \mu)\Psi_i. \end{aligned}\tag{7.7}$$

Here we write $\Phi_i = \Phi_i^N, v_i = v_i^N$ for notation's simplicity.

Assumption 7.1. *Suppose the following assumptions:*

- *The random variables $\{(\phi_i^N, \psi_i)\}_{i=1}^N$ are exchangeable and $(\psi_i)_i$ are independent.*
- *For every i,*
$$\mathbf{E}[\|\phi_i^N - \psi_i\|^2_{L^2}] \to 0, \qquad \text{as } N \to \infty.$$
- *For some $q > 1$,*
$$\mathbf{E}[\|\psi_i\|^4_{L^2}] \lesssim 1, \qquad \mathbf{E}[\|\phi_i^N\|^{2q}_{L^2}] \lesssim 1, \tag{7.8}$$
where the implicit constant is independent of N. For $1 \leqslant i \leqslant N$, $\mathbf{E}\|\psi_i\|^2_{L^\infty} < \infty$.

Theorem 7.1. *Under Assumption 7.1, for every i and every $T > 0$, $\mathbf{E}\|v_i^N\|^2_{C_T L^2}$ converges to zero as $N \to \infty$.*

We give the proof in Appendix B.

7.2. **Invariant measure.** Now we study the invariant measure to (7.5). We first note that (7.5) has a Gaussian invariant measure with mean zero and covariance $(-\Delta+m+\mu)^{-1}$ with μ being a constant. Indeed if μ is constant, the stationary solution is given by $\int_{-\infty}^t e^{(t-s)(\Delta-m-\mu)}\mathrm{d}W_s$ with W being an $L^2(\mathbb{T})$-cylindrical Wiener process, and by definition of μ it should satisfy the following self-consistent equation

$$\sum_{k\in\mathbb{Z}}\frac{1}{k^2+m+\mu}=\mu\ .$$

By monotonicity, there is a unique solution $\mu>0$. Define the L^2-Wasserstein distance

$$\overline{\mathbb{W}}_2(\nu_1,\nu_2):=\inf_{\pi\in\mathscr{C}(\nu_1,\nu_2)}\left(\int\|\phi-\psi\|^2_{L^2(\mathbb{T})}\pi(\mathrm{d}\phi,\mathrm{d}\psi)\right)^{1/2},$$

where $\mathscr{C}(\nu_1,\nu_2)$ is the set of all couplings for $\nu_1,\nu_2\in\mathscr{P}_2$ with $\mathscr{P}_2=\{\nu:\int\|\phi\|^2_{L^2(\mathbb{T})}\mathrm{d}\nu<\infty\}$. Similar as 2d case, we define a semigroup $P_t^*\nu$ to denote the law of $\Psi(t)$ with initial condition distributed by ν which satisfies $\int\|\psi\|^2_{L^\infty}\mathrm{d}\nu<\infty$. By uniqueness of the solutions to (7.5), we have $P_t^*=P_{t-s}^*P_s^*$, for $t\geqslant s\geqslant 0$. We have the following result, whose proof is given in Appendix B.

Theorem 7.2. *There exists $m_0\geqslant 0$ such that for $m\geqslant m_0$, there exist a unique invariant measure ν satisfying $\int\|\psi\|^2_{L^\infty}\mathrm{d}\nu<\infty$ to (7.5).*

Now we turn to the question of convergence of invariant measures. Consider the measure ν^N in (1.2) with $d=1$ defined on $\mathcal{S}'(\mathbb{T})^N$, and recall the notation Π_i, $\nu^{N,i}$ and $\Pi^{(k)}$, ν_k^N from (1.9) and (1.10).

One can show that ν^N is the unique invariant measure to (7.1), by similar argument as the two dimensional case, and for any $m\geqslant 0$, $\nu^{N,i}$ are tight (as in Theorem 6.9 for $d=2$ case). Here we do not repeat these details.

The proof for the following result on convergence of measures as $N\to\infty$ is given in Appendix B.

Theorem 7.3. *There exist $m_0>0$ such that for $m\geqslant m_0$, $\nu^{N,i}$ converges to the law ν of the unique invariant measure to (7.5). Furthermore, ν_k^N converges to $\nu\times\ldots\times\nu$.*

7.3. **Observables.** In this section we study $O(N)$ invariant observables of our SPDE system. We consider the following model, where m is a sufficiently large constant

$$\partial_t\Phi_i=(\Delta-m)\Phi_i-\frac{1}{N}\sum_{j=1}^N(\Phi_j^2-C_{\mathrm{w}})\Phi_i+\xi_i, \tag{7.9}$$
$$\partial_t Z_i=(\Delta-m)Z_i+\xi_i,\qquad C_{\mathrm{w}}\stackrel{\mathrm{def}}{=}\mathbf{E}(Z(t,x)^2)$$

where Z is the stationary solution. Here, C_{w} is finite, and we incorporate this constant for convenience (namely it will be easier to compare with the two dimensional case, certain products below will be centered, and the limiting SPDE will have simpler form). Of course, (7.9) is equivalent with (7.1) via a finite shift of C_W.

In particular, for m large enough, by Theorem 7.3, the unique invariant measure $\nu^{N,i}$ to (7.9) converge to the unique invariant measure of (7.2), as $N\to\infty$. In fact, it is easy to see that the stationary solution Ψ to the limiting equation in this case is simply given as the stationary solution Z.

We consider the stationary setting, namely, suppose $\Phi=(\Phi_i)_{1\leqslant i\leqslant N}\sim\nu^N$ (where ν^N is as in (1.2) but with m therein replaced by $m-C_{\mathrm{w}}$). Similar as in Section 6.3 the stationary solution Φ to (7.9) has decomposition $\Phi_i=Y_i+Z_i$ with Z_i stationary, and Y_i solves (7.3) (with m therein replaced by $m-C_{\mathrm{w}}$). We then write for shorthand

$$\mathbf{\Phi}^2\stackrel{\mathrm{def}}{=}\sum_{i=1}^N\Phi_i^2,\quad {:}\mathbf{\Phi}^2{:}\stackrel{\mathrm{def}}{=}\sum_{i=1}^N(\Phi_i^2-C_{\mathrm{w}}),\quad \mathbf{Z}^2\stackrel{\mathrm{def}}{=}\sum_{i=1}^N Z_i^2,\quad {:}\mathbf{Z}^2{:}\stackrel{\mathrm{def}}{=}\sum_{i=1}^N(Z_i^2-C_{\mathrm{w}}) \tag{7.10}$$

Consider the observables

$$\frac{1}{\sqrt{N}} :\mathbf{\Phi}^2: \qquad \frac{1}{N} :(\mathbf{\Phi}^2)^2:$$

Here the precise definition of $:(\mathbf{\Phi}^2)^2:$ is given by

$$:(\mathbf{\Phi}^2)^2: = \sum_{i=1}^N (\Phi_i^4 - 6C_{\mathrm{w}}\Phi_i^2 + 3C_{\mathrm{w}}^2) + \sum_{i\neq j}(\Phi_i^2 - C_{\mathrm{w}})(\Phi_j^2 - C_{\mathrm{w}}).$$

We can decompose them as in (6.18), (6.20), where the "Wick product" notation for $(Z_i)_i$ is as in two dimension (3.4), (6.21), with C_{w} in place of a_ε and without mollification, for instance $:Z_i^2 Z_j^2: = Z_i^4 - 6C_{\mathrm{w}}Z_i^2 + 3C_{\mathrm{w}}^2$ if $i = j$.

Analogous results as in Lemma 6.15, Theorem 6.16 (tightness of these observables) could also be proved in the same way. More precisely, we have

Theorem 7.4. *Suppose that $\Phi = (\Phi_i)_{1\leqslant i\leqslant N} \sim \nu^N$. For m large enough, the following result holds:*
(1) for $q \geqslant 1$ the following bound holds uniformly in N

$$\mathbf{E}\Big[\Big(\sum_{i=1}^N \|Y_i\|_{L^2}^2\Big)^q\Big] + \mathbf{E}\Big[\Big(\sum_{i=1}^N \|Y_i\|_{L^2}^2 + 1\Big)^q \Big(\sum_{i=1}^N \|\nabla Y_i\|_{L^2}^2 + \Big\|\sum_{i=1}^N Y_i^2\Big\|_{L^2}^2\Big)\Big] \lesssim 1. \tag{7.11}$$

(2) $\frac{1}{\sqrt{N}} :\mathbf{\Phi}^2:$ is tight in L^2 and $\frac{1}{N} :(\mathbf{\Phi}^2)^2:$ is tight in L^1.

We will skip the proof details of the above statement, and turn to study the statistical property of the limiting observable, namely, we show that the limiting observables have *nontrivial* laws, in the sense that although Φ_i converges to the (trivial) stationary solution Z_i (and $\Phi_i^2 - C_{\mathrm{w}} \to Z_i^2 - C_{\mathrm{w}}$ as $N \to \infty$ for each i), the observables do not converge to the ones with Φ_i replaced by Z_i.

To state such "nontriviality" results, we define

$$G_N(x-z) \stackrel{\text{def}}{=} \mathbf{E}[\frac{1}{\sqrt{N}} :\mathbf{\Phi}^2: (x) \frac{1}{\sqrt{N}} :\mathbf{\Phi}^2: (z)].$$

Here we use the translation invariance of ν^N. For comparison, we first note that

$$\mathbf{E}[\frac{1}{\sqrt{N}} :\mathbf{Z}_i^2: (x) \frac{1}{\sqrt{N}} :\mathbf{Z}_i^2: (z)] = 2C(x-z)^2$$

for any N and $x, z \in \mathbb{T}$, where $C = (m - \Delta)^{-1}$, which follows from definition of $:\mathbf{Z}_i^2:$ and Wick's theorem. Also, $\mathbf{E} :(\mathbf{Z}^2)^2: = 0$ for any N.

We denote $\widehat{f}$ the Fourier transform of a function f.

Theorem 7.5. *Under the same setting as in Theorem 7.4, it holds that*

$$\lim_{N\to\infty} \widehat{G_N} = 2\widehat{C^2}/(1+\widehat{C^2}), \tag{7.12}$$

$$\lim_{N\to\infty} \mathbf{E}\frac{1}{N} :(\mathbf{\Phi}^2)^2: (x) = -2\sum_{k\in\mathbb{Z}} \widehat{C^2}(k)^2/(1+\widehat{C^2}(k)).$$

In particular, in view of the discussion above the theorem, as $N \to \infty$ the limiting law of $\frac{1}{\sqrt{N}} :\mathbf{\Phi}^2:$ and $\frac{1}{N} :(\mathbf{\Phi}^2)^2:$ are different from that of $\frac{1}{\sqrt{N}} :\mathbf{Z}^2:$ and $\frac{1}{N} :(\mathbf{Z}^2)^2:$.

Proof. Integration by parts formula gives us the following identities (derivation can be found in Appendix C)

$$-\frac{1}{4N}\mathbf{E} :(\mathbf{\Phi}^2)^2: (x) = \frac{(N+2)}{4N}\int C(x-z)^2 G_N(x-z)\,\mathrm{d}z + R_N, \tag{7.13}$$

$$G_N(x-z) = 2C(x-z)\mathbf{E}[\Phi_1(x)\Phi_1(z)] - \frac{N+2}{N}\int C(x-y)^2 G_N(y-z)\mathrm{d}y + \frac{1}{N}Q_N(x-z), \tag{7.14}$$

with

$$R_N = -\frac{1}{4N^2}\int C(x-z_1)C(x-z_2)\mathbf{E}\Big[\, {:}\Phi_1(z_1)\mathbf{\Phi}^2(z_1){:}\ \ {:}\Phi_1(z_2)\mathbf{\Phi}^2(z_2){:}\ \ {:}\mathbf{\Phi}^2(x){:}\ \Big]\mathrm{d}z_1\mathrm{d}z_2,$$

$$\begin{aligned}Q_N(x-z) &= -2\int C(x-y)C(x-z)\mathbf{E}\Big[\, {:}\Phi_1(y)\mathbf{\Phi}^2(y){:}\ \Phi_1(z)\Big]\,\mathrm{d}y\\ &\quad+\frac{1}{N}\int C(x-z_1)C(x-z_2)\mathbf{E}\Big[\, {:}\Phi_1(z_1)\mathbf{\Phi}^2(z_1){:}\ \ {:}\Phi_1(z_2)\mathbf{\Phi}^2(z_2){:}\ \ {:}\mathbf{\Phi}^2(z){:}\ \Big]\mathrm{d}z_1\mathrm{d}z_2\\ &\stackrel{\text{def}}{=} Q_1^N(x-z)+Q_2^N(x-z).\end{aligned}$$

One can rewrite (7.14) as

$$(1+\frac{N+2}{N}\widehat{C^2})\widehat{G_N} = 2\widehat{CC_N}+\widehat{Q_N}/N$$

where $C_N(x-z)=\mathbf{E}[\Phi_1(x)\Phi_1(z)]\to C(x-z)$ as $N\to\infty$. Here, since C is positive definite, $\widehat{C}$ is a positive function, and so is $\widehat{C^2}$; this allows us to divide both sides by $1+\widehat{C^2}$. In Lemmas 7.6 and 7.7 below we show that Q_N/N vanishes in L^1 as $N\to\infty$, which implies that $\widehat{Q_N}/N$ vanishes in L^∞ in the limit. We therefore has (7.12). In particular, $\lim_{N\to\infty}\widehat{G_N}\neq 2\widehat{C^2}$, showing that the random fields $\frac{1}{\sqrt{N}}\,{:}\mathbf{\Phi}^2{:}$ and $\frac{1}{\sqrt{N}}\,{:}\mathbf{Z}^2{:}$ have different limiting laws.

Furthermore, note that R_N is independent of x, by spatial translation invariance. By Lemma 7.7

$$\begin{aligned}\lim_{N\to\infty}\mathbf{E}\frac{1}{N}\,{:}(\mathbf{\Phi}^2)^2{:}\ (x) &= -\int C(x-z)^2\lim_{N\to\infty}G_N(x-z)\,\mathrm{d}z\\ &= -\,2\sum_{k\in\mathbb{Z}}\widehat{C^2}(k)\cdot\widehat{C^2}(k)/(1+\widehat{C^2}(k))\end{aligned}$$

where the sum is over integers (i.e. Fourier variables). This is non-zero, showing that the limiting law of $\frac{1}{N}\,{:}(\mathbf{\Phi}^2)^2{:}$ is different from that of $\frac{1}{N}\,{:}(\mathbf{Z}^2)^2{:}$. □

Lemma 7.6. *It holds that*

$$\|Q_1^N\|_{L^1}\lesssim N^{1/2}.$$

Proof. We first expand the term inside the expectation part in Q_1^N as

$$\mathbf{E}\Big[\, {:}\Phi_1(y)\mathbf{\Phi}^2(y){:}\ \Phi_1(z)\Big] = \mathbf{E}\Big[(Y_1+Z_1)\sum_{i=1}^N (Y_i^2+2Y_iZ_i+\,{:}Z_i^2{:}\ )(y)(Y_1+Z_1)(z)\Big].$$

The term involving only $(Z_i)_i$ here can be computed explicitly with Wick's theorem

$$\begin{aligned}&\mathbf{E}\Big[Z_1(z)\ {:}Z_1(y)\sum_{i=1}^N Z_i^2(y){:}\ \Big]\\ &\quad= \mathbf{E}\Big[Z_1(z)\Big(Z_1(y)^3-3C_{\mathrm{w}}Z_1(y)\Big)\Big]+(N-1)\mathbf{E}\Big[Z_1(z)Z_1(y)\Big(Z_2(y)^2-C_{\mathrm{w}}\Big)\Big]=0.\end{aligned}$$

So we only need to bound the other terms. Since C is a continuous kernel in $d=1$, we have

$$\mathbf{E}\iint |C(x-y)C(x-z)f(y)g(z)|\,\mathrm{d}x\mathrm{d}y\lesssim \mathbf{E}\|f\|_{L^1}\|g\|_{L^1},$$

which implies that $\|Q_1^N\|_{L^1}$ is bounded by

$$\begin{aligned}&\mathbf{E}\Big[\Big\|(Y_1+Z_1)\sum_{i=1}^N(Y_i^2+2Y_iZ_i+\,{:}Z_i^2{:}\ )\Big\|_{L^1}\|Y_1\|_{L^1}\\ &\quad+\Big(\Big\|Y_1\sum_{i=1}^N(Y_i^2+2Y_iZ_i+\,{:}Z_i^2{:}\ )\Big\|_{L^1}+\Big\|Z_1\sum_{i=1}^N(Y_i^2+2Y_iZ_i)\Big\|_{L^1}\Big)\|Z_1\|_{L^1}\Big].\end{aligned}$$

We will expand the above multiplication and bound each them separately. In what follows, we will frequently exploit symmetry to rewrite certain quantity of Y_1 in terms of an average, e.g. replacing $\|Y_1\|_{L^2}^2$ by $\frac{1}{N}\sum_i \|Y_i\|_{L^2}^2$ under expectation; this will give us nice factor $1/N$ since we have moment bounds on $\sum_i \|Y_i\|_{L^2}^2$ thanks to Theorem 7.4. We will also often invoke moment bounds (7.4) on Z_i without explicitly mentioning.

For each of the terms, we will turn them into an expression of the form (7.11) and then apply Theorem 7.4. First consider terms with $\sum_{i=1}^N Y_i^2$ for which we have

$$\begin{aligned}
\mathbf{E}\Big[\Big\|Y_1\sum_{i=1}^N Y_i^2\Big\|_{L^1}\|Y_1\|_{L^1}\Big] &\lesssim \mathbf{E}\Big[\|Y_1\|_{L^2}^2\Big\|\sum_{i=1}^N Y_i^2\Big\|_{L^2}\Big],\\
\mathbf{E}\Big[\Big\|Y_1\sum_{i=1}^N Y_i^2\Big\|_{L^1}\|Z_1\|_{L^1}\Big] &\lesssim \mathbf{E}\Big[\|Y_1\|_{L^2}\Big\|\sum_{i=1}^N Y_i^2\Big\|_{L^2}\times\|Z_1\|_{L^\infty}\Big],\\
\mathbf{E}\Big[\Big\|Z_1\sum_{i=1}^N Y_i^2\Big\|_{L^1}\|Y_1\|_{L^1}\Big] &\lesssim \mathbf{E}\Big[\Big\|\sum_{i=1}^N Y_i^2\Big\|_{L^2}\|Y_1\|_{L^2}\times\|Z_1\|_{L^\infty}\Big],\\
\mathbf{E}\Big[\Big\|Z_1\sum_{i=1}^N Y_i^2\Big\|_{L^1}\|Z_1\|_{L^1}\Big] &\lesssim \mathbf{E}\Big[\Big\|\sum_{i=1}^N Y_i^2\Big\|_{L^2}\times\|Z_1\|_{L^\infty}^2\Big].
\end{aligned}$$

Applying Hölder with exponents $(2,2)$ to the products $\times$, and replacing $\|Y_1\|_{L^2}^2$ by $\frac{1}{N}\sum_i\|Y_i\|_{L^2}^2$ under expectation, and applying Theorem 7.4, these are bounded by $\frac{1}{N}$, $\frac{1}{\sqrt{N}}$, $\frac{1}{\sqrt{N}}$ and 1 respectively.

Next, consider terms involving $\sum_i Y_i Z_i$. Noting that

$$\Big\|\sum_i Y_iZ_i\Big\|_{L^2}\lesssim\Big(\sum_i\|Y_i\|_{L^2}^2\Big)^{\frac12}\Big(\sum_i\|Z_i\|_{L^\infty}^2\Big)^{\frac12}\tag{7.15}$$

we have

$$\begin{aligned}
\mathbf{E}\Big[\Big\|Y_1\sum_{i=1}^N Y_iZ_i\Big\|_{L^1}\|Y_1\|_{L^1}\Big] &\lesssim \mathbf{E}\Big[\|Y_1\|_{L^2}^2\Big(\sum_{i=1}^N\|Y_i\|_{L^2}^2\Big)^{\frac12}\times\Big(\sum_{i=1}^N\|Z_i\|_{L^\infty}^2\Big)^{\frac12}\Big]\\
\mathbf{E}\Big[\Big\|Y_1\sum_{i=1}^N Y_iZ_i\Big\|_{L^1}\|Z_1\|_{L^1}\Big] &\lesssim \mathbf{E}\Big[\|Y_1\|_{L^2}\Big(\sum_{i=1}^N\|Y_i\|_{L^2}^2\Big)^{\frac12}\times\|Z_1\|_{L^\infty}\Big(\sum_{i=1}^N\|Z_i\|_{L^\infty}^2\Big)^{\frac12}\Big]\\
\mathbf{E}\Big[\Big\|Z_1\sum_{i=1}^N Y_iZ_i\Big\|_{L^1}\|Y_1\|_{L^1}\Big] &\lesssim \mathbf{E}\Big[\|Y_1\|_{L^2}\Big(\sum_{i=1}^N\|Y_i\|_{L^2}^2\Big)^{\frac12}\times\|Z_1\|_{L^\infty}\Big(\sum_{i=1}^N\|Z_i\|_{L^\infty}^2\Big)^{\frac12}\Big]\\
\mathbf{E}\Big[\Big\|Z_1\sum_{i=1}^N Y_iZ_i\Big\|_{L^1}\|Z_1\|_{L^1}\Big] &\lesssim \mathbf{E}\Big[\Big(\sum_{i=1}^N\|Y_i\|_{L^2}^2\Big)^{\frac12}\times\Big(\sum_{i=1}^N\|Z_i\|_{L^\infty}^2\Big)^{\frac12}\times\|Z_1\|_{L^\infty}^2\Big]
\end{aligned}$$

Replacing $\|Y_1\|_{L^2}^2$ by $\frac{1}{N}\sum_i\|Y_i\|_{L^2}^2$ as above gains a factor $\frac{1}{N}$ (and $\|Y_1\|_{L^2}$ gains a factor $\frac{1}{\sqrt{N}}$), and we can throw this factor into $\sum_i\|Z_i\|_{L^\infty}^2$. Applying Hölder with exponents $(2,2)$ except the last one with $(4,2,4)$ to the products $\times$, and applying Theorem 7.4, these are bounded by $\frac{1}{\sqrt{N}}$, 1, 1 and $\sqrt{N}$ respectively.

Next, by Young and Hölder inequality and symmetry,

$$\begin{aligned}
&\mathbf{E}\Big[\Big\|Y_1\sum_{i=1}^N :Z_i^2:\Big\|_{L^1}(\|Y_1\|_{L^1}+\|Z_1\|_{L^1})\Big]\\
&\lesssim\frac{1}{N}\mathbf{E}\Big[\Big(\sum_{i=1}^N\|Y_i\|_{L^2}^2\Big)\Big(\Big\|\sum_{i=1}^N :Z_i^2:\Big\|_{L^2}^2\Big)\Big]+\mathbf{E}\Big[\|Y_1\|_{L^1}^2+\|Z_1\|_{L^\infty}^2\Big]
\end{aligned}$$

which is bounded by 1 since $\mathbf{E}\big[\big(\frac{1}{N}\|\sum_i :Z_i^2: \|_{L^2}^2\big)^2\big] \lesssim 1$ by independence. Next,

$$\mathbf{E}\Big[\Big\|Z_1\sum_{i=1}^N :Z_i^2: \Big\|_{L^2}\|Y_1\|_{L^1}\Big] \lesssim \mathbf{E}[\|Y_1\|_{L^2}^2]^{\frac12}\mathbf{E}\Big[\Big\|\sum_{i=1}^N Z_1 :Z_i^2: \Big\|_{L^2}^2\Big]^{\frac12}$$

$$\lesssim \mathbf{E}\Big[\sum_{i=1}^N \|Y_i\|_{L^2}^2\Big]^{\frac12}\mathbf{E}\Big[\frac{1}{N}\Big\|\sum_{i=1}^N Z_1 :Z_i^2: \Big\|_{L^2}^2\Big]^{\frac12}$$

which is bounded by 1. Here we also used independence to have

$$\mathbf{E}\Big[\frac{1}{N}\Big\|\sum_{i=1}^N Z_1 :Z_i^2: \Big\|_{L^2}^2\Big] = \frac{1}{N}\sum_{i,i_1=1}^N \mathbf{E}[\langle Z_1 :Z_i^2: , Z_1 :Z_{i_1}^2: \rangle] \lesssim 1.$$

in the last inequality. Summarizing all the estimate the result follows. □

Lemma 7.7. *It holds that*

$$\|Q_2^N\|_{L^1} \lesssim N^{1/2}, \qquad \|R_N\|_{L^1} \lesssim N^{-1/2}.$$

Proof. First the term involving only Z

$$\frac{1}{N}\int C(x-z_1)C(x-z_2)\mathbf{E}\Big[:Z_1(z_1)\mathbf{Z}^2(z_1): \ :Z_1(z_2)\mathbf{Z}^2(z_2): \ :\mathbf{Z}^2(z): \Big]\mathrm{d}z_1\mathrm{d}z_2$$

can be computed explicitly using Wick's theorem. Indeed, the above expression equals (up to a constant which is finite as $N\to\infty$)

$$\int C(x-z_1)C(x-z_2)C(z_1-z_2)^2C(z-z_1)C(z-z_2)\mathrm{d}z_1\mathrm{d}z_2$$

which as a function of $x-z$ is bounded in L^∞ since C is continuous in $d=1$.

In the following we only consider the other terms. Since C is continuous in $d=1$, $\|Q_2^N\|_{L^1}$ is bounded by a constant times the following

$$\frac{1}{N}\mathbf{E}\Big[\Big\|(Y_1+Z_1)\sum_{i=1}^N(Y_i^2+2Y_iZ_i+ :Z_i^2:)\Big\|_{L^1}^2\Big\|\sum_{i=1}^N(Y_i^2+2Y_iZ_i)\Big\|_{L^1}\Big]$$

$$+\frac{1}{N}\mathbf{E}\Big[\Big(\Big\|(Y_1+Z_1)\sum_{i=1}^N(Y_i^2+2Y_iZ_i)\Big\|_{L^1}^2+\Big\|Y_1\sum_{i=1}^N :Z_i^2: \Big\|_{L^1}^2\Big)\Big\|\sum_{i=1}^N :Z_i^2: \Big\|_{L^1}\Big]+1,$$

and $\|NR_N\|_{L^1}$ satisfies exactly the same bound. We expand the above expression. We then proceed as in the proof of Lemma 7.6, by repeatedly using symmetry and Theorem 7.4. As in the proof of Lemma 7.6 for the most terms we will actually have better bound than what's claimed.

Using the fact that $\|\sum_i Y_i^2\|_{L^1} \lesssim \sum_i \|Y_i\|_{L^2}^2$ and (7.15), we have

$$\frac{1}{N}\mathbf{E}\Big[\Big\|(Y_1+Z_1)\sum_{i=1}^N Y_i^2\Big\|_{L^1}^2\Big\|\sum_{i=1}^N Y_i^2\Big\|_{L^1}\Big] \lesssim \frac{1}{N}\mathbf{E}\Big[\|Y_1+Z_1\|_{L^\infty}^2\Big(\sum_{i=1}^N\|Y_i\|_{L^2}^2\Big)^3\Big],$$

$$\frac{1}{N}\mathbf{E}\Big[\Big\|(Y_1+Z_1)\sum_{i=1}^N Y_i^2\Big\|_{L^1}^2\Big\|\sum_{i=1}^N Y_iZ_i\Big\|_{L^1}\Big] \lesssim \frac{1}{N}\mathbf{E}\Big[\|Y_1+Z_1\|_{L^\infty}^2\Big(\sum_{i=1}^N\|Y_i\|_{L^2}^2\Big)^{\frac52}\Big(\sum_{i=1}^N\|Z_i\|_{L^2}^2\Big)^{\frac12}\Big],$$

$$\frac{1}{N}\mathbf{E}\Big[\Big\|(Y_1+Z_1)\sum_{i=1}^N Y_i^2\Big\|_{L^1}^2\Big\|\sum_{i=1}^N :Z_i^2: \Big\|_{L^1}\Big] \lesssim \frac{1}{N}\mathbf{E}\Big[\|Y_1+Z_1\|_{L^\infty}^2\Big(\sum_{i=1}^N\|Y_i\|_{L^2}^2\Big)^2\Big\|\sum_{i=1}^N :Z_i^2: \Big\|_{L^1}\Big].$$

Noting that for $s>1/2$

$$\|Y_1+Z_1\|_{L^\infty}^2 \lesssim \|Y_1\|_{H^1}^{2s}\|Y_1\|_{L^2}^{2(1-s)}+\|Z_1\|_{L^\infty}^2,$$

using symmetry, Hölder's inequality and Theorem 7.4, these are bounded by $\frac{1}{N}$, 1 and 1 respectively.

Next, again using $\|\sum_i Y_i^2\|_{L^1} \lesssim \sum_i \|Y_i\|_{L^2}^2$ and (7.15),

$$\frac{1}{N}\mathbf{E}\Big[\Big\|(Y_1+Z_1)\sum_{i=1}^N Y_iZ_i\Big\|_{L^1}^2\Big\|\sum_{i=1}^N Y_i^2\Big\|_{L^1}\Big] \lesssim \frac{1}{N}\mathbf{E}\Big[\|Y_1+Z_1\|_{L^2}^2\Big(\sum_{i=1}^N\|Y_i\|_{L^2}^2\Big)^2\Big(\sum_{i=1}^N\|Z_i\|_{L^\infty}^2\Big)\Big]$$

$$\frac{1}{N}\mathbf{E}\Big[\Big\|(Y_1+Z_1)\sum_{i=1}^N Y_iZ_i\Big\|_{L^1}^2\Big\|\sum_{i=1}^N Y_iZ_i\Big\|_{L^1}\Big] \lesssim \sqrt{N}\mathbf{E}\Big[\|Y_1+Z_1\|_{L^2}^2\Big(\sum_{i=1}^N\|Y_i\|_{L^2}^2\Big)^{\frac32}\Big(\frac{1}{N}\sum_{i=1}^N\|Z_i\|_{L^\infty}^2\Big)^{\frac32}\Big]$$

By the same arguments as above, these are bounded by 1, $\sqrt{N}$ respectively.

For the rest of the terms, we need the following fact

$$\mathbf{E}\Big[\Big\|\frac{1}{\sqrt{N}}\sum_{j=1}^N :Z_j^2:\Big\|_{L^2}^{2q}\Big] \lesssim 1 \qquad \forall q \geqslant 1 \tag{7.16}$$

by independence (in fact the l.h.s can be explicitly calculated using Wick theorem). Thus we have

$$\frac{1}{N}\mathbf{E}\Big[\|(Y_1+Z_1)\sum_{i=1}^N Y_iZ_i\Big\|_{L^1}^2\Big\|\sum_{i=1}^N :Z_i^2:\Big\|_{L^1}\Big] \lesssim \frac{1}{N}\mathbf{E}\Big[\|Y_1+Z_1\|_{L^2}^2\sum_{i=1}^N\|Y_i\|_{L^2}^2\sum_{i=1}^N\|Z_i\|_{L^\infty}^2\Big\|\sum_{i=1}^N :Z_i^2:\Big\|_{L^2}\Big]$$

$$\frac{1}{N}\mathbf{E}\Big[\Big\|(Y_1+Z_1)\sum_{i=1}^N :Z_i^2:\Big\|_{L^1}^2\Big\|\sum_{i=1}^N Y_i^2\Big\|_{L^1}\Big] \lesssim \frac{1}{N}\mathbf{E}\Big[\|Y_1+Z_1\|_{L^2}^2\Big\|\sum_{i=1}^N :Z_i^2:\Big\|_{L^2}^2\sum_{i=1}^N\|Y_i\|_{L^2}^2\Big]$$

$$\frac{1}{N}\mathbf{E}\Big[\Big\|(Y_1+Z_1)\sum_{i=1}^N :Z_i^2:\Big\|_{L^1}^2\Big\|\sum_{i=1}^N Y_iZ_i\Big\|_{L^1}\Big] \lesssim \frac{1}{N}\mathbf{E}\Big[\|Y_1+Z_1\|_{L^2}^2\Big\|\sum_{i=1}^N :Z_i^2:\Big\|_{L^2}^2\Big(\sum_{i=1}^N\|Y_i\|_{L^2}^2\Big)^{\frac12}\Big(\sum_{i=1}^N\|Z_i\|_{L^2}^2\Big)^{\frac12}\Big]$$

$$\frac{1}{N}\mathbf{E}\Big[\Big\|Y_1\sum_{i=1}^N :Z_i^2:\Big\|_{L^1}^2\Big\|\sum_{i=1}^N :Z_i^2:\Big\|_{L^1}\Big] \lesssim \frac{1}{N}\mathbf{E}\Big[\|Y_1\|_{L^2}^2\Big\|\sum_{i=1}^N :Z_i^2:\Big\|_{L^2}^3\Big].$$

Using (7.16), symmetry and Hölder inequality, these are bounded by $\sqrt{N}$, 1, $\sqrt{N}$, $\frac{1}{\sqrt{N}}$ respectively. Note that before Hölder inequality one needs to assign suitable powers of N to various factors on the r.h.s., for instance in the third line, we assign $1/N$ to $\|\sum_i :Z_i^2:\|_{L^2}^2$ in view of (7.16) and $\frac{1}{\sqrt{N}}$ to $\big(\sum_i\|Z_i\|_{L^2}^2\big)^{\frac12}$ and thus we're left with $\sqrt{N}$. In the last line we use symmetry to get extra $\frac{1}{N}$. Combining all the above estimate the result follows. □

Appendix A. Proof of Lemma 3.2

Proof. For initial value $y_i \in \mathbf{C}^\beta(\mathbb{T}^2), \beta \in (1,2)$ we could use similar argument as in [MW17b, Theorem 6.1] to obtain global solutions (Y_i) to (3.3) with each $Y_i \in C_T\mathbf{C}^\beta$. In fact, we use mild solutions and fixed point argument to obtain unique local solutions. Furthermore, for fixed N we obtain a global in time L^p-estimate, $p > 1$, which gives the required global solutions.

Moreover, for general initial data $y_i \in L^2$, we consider smooth approximation (y_i^ε) to initial data y_i. For (y_i^ε) we construct solutions $Y_i^\varepsilon \in C_T\mathbf{C}^\beta$ by the above argument. For Y_i^ε we could do the uniform estimate as in Lemma 3.3 and obtain

$$\frac{1}{N}\sup_{t\in[0,T]}\sum_{j=1}^N\|Y_j^\varepsilon\|_{L^2}^2 + \frac{1}{N}\sum_{j=1}^N\|\nabla Y_j^\varepsilon\|_{L^2(0,T;L^2)}^2 + \Big\|\frac{1}{N}\sum_{i=1}^N(Y_i^\varepsilon)^2\Big\|_{L^2(0,T;L^2)}^2 \leqslant C,$$

where C is independent of ε. By standard compactness argument we deduce that there exist a sequence $\{\varepsilon_k\}$ and $Y_i \in L_T^\infty L^2 \cap L_T^2 H^1 \cap L_T^4 L^4$ such that $Y_i^{\varepsilon_k} \to Y$ in $L_T^2 H^\delta \cap C_T H^{-1}$, $\delta < 1$. Furthermore, by similar argument as in the proof of [RYZ18, Theorem 4.3] we obtain $Y_i \in C_T L^2 \cap L_T^4 L^4 \cap L_T^2 H^1$. For the uniqueness part we could do similar estimate as I_1^N and I_2^N for the difference v_i in Section 5. From the estimate in Section 5 the regularity for Y_i is enough for the uniqueness.

□

Appendix B. Proof in one dimensional case

In this section we give the proof of Theorem 7.1 and Theorem 7.2. In the following we first give a uniform in N estimate for Y_i.

Lemma B.1. *There exists a universal constant C such that*

$$\begin{aligned}&\frac{1}{N}\sup_{t\in[0,T]}\sum_{j=1}^N\|Y_j(t)\|_{L^2}^2+\frac{1}{N}\sum_{j=1}^N\|\nabla Y_j\|_{L_T^2L^2}^2+\frac{m}{N}\sum_{j=1}^N\|Y_j\|_{L_T^2L^2}^2+\Big\|\frac{1}{N}\sum_{i=1}^N Y_i^2\Big\|_{L_T^2L^2}^2\\&\leqslant C\int_0^T R_N\mathrm{d}t+\frac{1}{N}\sum_{j=1}^N\|y_j\|_{L^2}^2,\end{aligned}\tag{B.1}$$

where

$$R_N:=\Big(\frac{1}{N}\sum_{i=1}^N\|Z_i\|_{L^\infty}^2\Big)^2+\int\frac{1}{N^2}\sum_{i,j=1}^N Z_j^2Z_i^2\mathrm{d}x.$$

Proof. Taking L^2-inner product with Y_i in (7.3), one has

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|Y_i\|_{L^2}^2+\|\nabla Y_i\|_{L^2}^2+m\|Y_i\|_{L^2}^2+\int\frac{1}{N}\sum_{j=1}^N(Y_j+Z_j)^2Y_i^2\,\mathrm{d}x\\&=-\int\frac{1}{N}\sum_{j=1}^N(Y_j+Z_j)^2(Y_iZ_i)\,\mathrm{d}x\\&\leqslant\frac{1}{2}\int\frac{1}{N}\sum_{j=1}^N(Y_j+Z_j)^2Y_i^2\,\mathrm{d}x+\frac{1}{2}\int\frac{1}{N}\sum_{j=1}^N(Y_j+Z_j)^2Z_i^2\,\mathrm{d}x.\end{aligned}\tag{B.2}$$

Half of the third term on the LHS is canceled by the first term on the RHS, and for the rest half, since $\frac{1}{2}Y_i^2Y_j^2+2Y_i^2Y_jZ_j+2Y_i^2Z_j^2\geqslant 0$ one has

$$\int\frac{1}{2N}\sum_{j=1}^N(Y_j+Z_j)^2Y_i^2\,\mathrm{d}x\geqslant\frac{1}{2N}\sum_{j=1}^N\int\frac{1}{2}Y_j^2Y_i^2-Z_j^2Y_i^2\,\mathrm{d}x.$$

Now we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|Y_i\|_{L^2}^2+\|\nabla Y_i\|_{L^2}^2+m\|Y_i\|_{L^2}^2+\int\frac{1}{4N}\sum_{j=1}^N Y_j^2Y_i^2\,\mathrm{d}x\\&\leqslant\int\frac{1}{2N}\sum_{j=1}^N Y_i^2Z_j^2\,\mathrm{d}x+\frac{1}{2}\int\frac{1}{N}\sum_{j=1}^N(Y_j+Z_j)^2Z_i^2\,\mathrm{d}x.\end{aligned}$$

Multiplying $1/N$ and taking sum over i we obtain

$$\begin{aligned}&\frac{1}{2N}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{j=1}^N\|Y_j\|_{L^2}^2+\frac{1}{N}\sum_{j=1}^N\|\nabla Y_j\|_{L^2}^2\,\mathrm{d}x+\frac{m}{N}\sum_{j=1}^N\|Y_j\|_{L^2}^2+\int\frac{1}{4N^2}\sum_{i,j=1}^N Y_j^2Y_i^2\,\mathrm{d}x\\&\leqslant\int\frac{3}{2N^2}\sum_{i,j=1}^N Y_j^2Z_i^2\,\mathrm{d}x+\int\frac{1}{N^2}\sum_{i,j=1}^N Z_j^2Z_i^2\,\mathrm{d}x.\end{aligned}$$

Using $(\frac{1}{N}\sum_{j=1}^N Y_j^2)^2=\frac{1}{N^2}\sum_{i,j=1}^N Y_i^2Y_j^2$, we obtain

$$\frac{1}{2N}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{j=1}^N\|Y_j\|_{L^2}^2+\frac{1}{2N}\sum_{j=1}^N\|\nabla Y_j\|_{L^2}^2+\int\frac{1}{8N^2}\sum_{i,j=1}^N Y_j^2Y_i^2$$

$$\leqslant\Big(\frac{3C}{2N}\sum_{i=1}^N\|Z_i\|_{L^\infty}^2\Big)^2+\int\frac{1}{N^2}\sum_{i,j=1}^N Z_j^2Z_i^2, \tag{B.3}$$

which implies (B.1). Here we use Young's inequality for products,

$$\frac{3}{2}\int\Big(\frac{1}{N}\sum_j Y_j^2\Big)\Big(\frac{1}{N}\sum_i Z_i^2\Big)\,\mathrm{d}x\leqslant\int\frac{1}{8}\Big(\frac{1}{N}\sum_j Y_j^2\Big)^2+C\Big(\frac{1}{N}\sum_i Z_i^2\Big)^2\,\mathrm{d}x.$$

□

Corollary B.2. *For every $p>1$ it holds that*

$$\mathbf{E}\Big[\Big(\frac{1}{N}\sum_{j=1}^N\|Y_j(t)\|_{L^2}^2\Big)^p\Big]\lesssim 1+\mathbf{E}\Big[\Big(\frac{1}{N}\sum_{j=1}^N\|\phi_j\|_{L^2}^2\Big)^p\Big], \tag{B.4}$$

$$\mathbf{E}\Big[\|Y_i(t)\|_{L_T^\infty L^2}^{2p}\Big]\lesssim 1+\mathbf{E}\Big[\Big(\frac{1}{N}\sum_{j=1}^N\|\phi_j\|_{L^2}^2\Big)^{p'}\Big]+\mathbf{E}\left[\|\phi_i\|_{L^2}^{2p}\right], \tag{B.5}$$

where $p'>p$ and the proportionality constant is independent of N and i. Furthermore,

$$\sup_{t>0}(t\wedge 1)\mathbf{E}\Big[\frac{1}{N}\sum_{j=1}^N\|Y_j(t)\|_{L^2}^2\Big]\lesssim 1, \tag{B.6}$$

where the proportionality constant is independent of initial data.

Proof. It is easy to deduce (B.4) by using (B.1) and (7.4). By (B.2) we know

$$\frac{\mathrm{d}}{\mathrm{d}t}\|Y_i\|_{L^2}^2\lesssim\int\frac{1}{N}\sum_{j=1}^N(Y_j^2+Z_j^2)Z_i^2\lesssim\|Z_i\|_{L^\infty}^2\frac{1}{N}\sum_{j=1}^N\|Y_j(t)\|_{L^2}^2+\int\frac{1}{N}\sum_{j=1}^N Z_j^2Z_i^2.$$

Using (B.4), (7.4) and Young's inequality we know (B.5) follows. The last bound (B.6) follows from Lemma 2.6 and (B.3). □

In what follows, we will focus on the a priori estimate for the equation (7.5).

Lemma B.3. *There exists a universal constant C_T such that*

$$\sup_{t\in[0,T]}\mathbf{E}[\|X_i\|_{L^2}^2]+\mathbf{E}[\|\nabla X_i\|_{L_T^2L^2}^2]+\|\mathbf{E}X_i^2\|_{L_T^2L^2}^2\leqslant C_T+\mathbf{E}\|\psi_i\|_{L^2}^2, \tag{B.7}$$

$$\|X_i\|_{L_T^\infty L^2}^2+\|\nabla X_i\|_{L_T^2L^2}^2\leqslant C_T\|Z_i\|_{L_T^\infty L^\infty}^2+\|\psi_i\|_{L^2}^2, \tag{B.8}$$

$$\|X_i\|_{L_T^\infty L^2}^4+\int_0^T\|\nabla X_i\|_{L^2}^2\|X_i\|_{L^2}^2\mathrm{d}t\leqslant C_T\|Z_i\|_{L_T^\infty L^\infty}^4+\|\psi_i\|_{L^2}^4. \tag{B.9}$$

Proof. Taking inner product with X_i for (7.6) we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|X_i\|_{L^2}^2+\|\nabla X_i\|_{L^2}^2+m\|X_i\|_{L^2}^2+\int\mu X_i^2\,\mathrm{d}x\\ =-\int\mu X_iZ_i\,\mathrm{d}x\leqslant\frac{1}{2}\int\mu X_i^2\mathrm{d}x+\frac{1}{2}\int\mu Z_i^2\mathrm{d}x.\end{aligned} \tag{B.10}$$

The first term on the RHS can be absorbed to the LHS. Write X_j, Z_j $(i\neq j)$ for independent copies of X_i, Z_i respectively. Note that

$$\frac{1}{2}\mathbf{E}[X_j(t,x)^2]-\mathbf{E}[Z_j(t,x)^2]\leqslant\mu(t,x)\leqslant 2\mathbf{E}[X_j(t,x)^2]+2\mathbf{E}[Z_j(t,x)^2].$$

Taking expectation, one then has

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}[\|X_i\|_{L^2}^2]+\mathbf{E}[\|\nabla X_i\|_{L^2}^2]+\frac{1}{4}\int\mathbf{E}[X_i^2X_j^2]\,\mathrm{d}x\leqslant\int\mathbf{E}[X_j^2Z_i^2]\,\mathrm{d}x+\int\mathbf{E}[Z_j^2Z_i^2]\,\mathrm{d}x$$

$$\leqslant \mathbf{E}[\|X_j\|_{L^2}^2]\mathbf{E}[\|Z_i\|_{L^\infty}^2] + \mathbf{E}[\|Z_i\|_{L^\infty}^2]^2 \leqslant \frac{1}{8}\mathbf{E}[\|X_i\|_{L^2}^2]^2 + C\mathbf{E}[\|Z_i\|_{L^\infty}^2]^2 \tag{B.11}$$

where we applied Young's inequality in the last step. Now the claimed bound (B.7) follows by integrating time.

Furthermore, by (B.10) we have

$$\frac{1}{2}\|X_i\|_{L_T^\infty L^2}^2 + \|\nabla X_i\|_{L_T^2 L^2}^2 \lesssim \|Z_i\|_{L_T^\infty L^\infty}^2(\|\mu\|_{L_T^1 L^1} + 1) + \|\psi_i\|_{L^2}^2.$$

Now we apply the bound (B.7), which we have just proved; this yields

$$\|\mu\|_{L_T^1 L^1} \lesssim \mathbf{E}\|X_i\|_{L_T^2 L^2}^2 + \mathbf{E}\|Z_i\|_{L_T^2 L^2}^2 \lesssim 1, \tag{B.12}$$

which implies the second result (B.8). Moreover, using (B.10) we deduce

$$\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\|X_i\|_{L^2}^4 + \|\nabla X_i\|_{L^2}^2\|X_i\|_{L^2}^2 \leqslant \frac{1}{2}\|X_i\|_{L^2}^2 \int \mu Z_i^2 \mathrm{d}x.$$

Taking integration with respect to time and using (B.8) and (B.12), (B.9) follows.

□

Corollary B.4. *The following hold with implicit constants independent of the initial data and m:*

$$\sup_{t>0}(t\wedge 1)\mathbf{E}[\|X_i(t)\|_{L^2}^2] \lesssim 1, \tag{B.13}$$

$$\sup_{t\geqslant 1}\frac{1}{t}\int_1^t \mathbf{E}\|\nabla X_i(t)\|_{L^2}^2 \mathrm{d}t \lesssim 1. \tag{B.14}$$

Proof. Combining (B.11) and (7.4) one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}[\|X_i\|_{L^2}^2] + \frac{1}{4}\mathbf{E}[\|X_i\|_{L^2}^2]^2 \leqslant C\mathbf{E}[\|Z_i\|_{L^\infty}^2]^2$$

Using Lemma 2.6 we obtain (B.13). Similarly, integrating (B.11) over time interval $[1,t]$,

$$\int_1^t \mathbf{E}\|\nabla X_i(t)\|_{L^2}^2 \mathrm{d}t \lesssim \mathbf{E}[\|X_i(1)\|_{L^2}^2] + t \lesssim t$$

which proves (B.14). □

We also also establish an L^∞ bound, as stated in the following Lemma.

Lemma B.5. *It holds that*

$$\|\Psi_i\|_{L_T^\infty L^\infty} \lesssim \|Z_i\|_{L_T^\infty L^\infty} + \|\psi_i\|_{L^\infty} + 1. \tag{B.15}$$

Proof. Using (B.7) we obtain for $\delta > 0$

$$\begin{aligned}\|\mu\|_{L^\infty} \lesssim & \mathbf{E}\|X_i\|_{L^\infty}^2 + \mathbf{E}\|Z_i\|_{L^\infty}^2 \lesssim \mathbf{E}\|X_i\|_{H^1}^{1+\delta}\|X_i\|_{L^2}^{1-\delta} + \mathbf{E}\|Z_i\|_{L^\infty}^2 \\ \lesssim & (\mathbf{E}\|X_i\|_{H^1}^2)^{\frac{1+\delta}{2}}(\mathbf{E}\|X_i\|_{L^2}^2)^{\frac{1-\delta}{2}} + \mathbf{E}\|Z_i\|_{L^\infty}^2,\end{aligned}$$

where we used Sobolev embedding $H^{\frac{1+\delta}{2}} \subset L^\infty$ and interpolation in the second inequality and we used Hölder's inequality in the last inequality. This combined with (B.7) and (7.4) implies that

$$\|\mu\|_{L_T^{\frac{2}{1+\delta}} L^\infty} \lesssim 1. \tag{B.16}$$

By using mild formulation we have

$$\Psi_i(t) = S_t\psi_i + Z_i(t) - \int_0^t S_{t-s}\mu(s)\Psi_i(s)\mathrm{d}s,$$

which implies that

$$\|\Psi_i(t)\|_{L^\infty} \lesssim \|\psi_i\|_{L^\infty} + \|Z_i(t)\|_{L^\infty} + \int_0^T \|\mu(s)\|_{L^\infty}\|X_i(s)\|_{L^\infty}\mathrm{d}s + \int_0^T \|\mu(s)\|_{L^\infty}\|Z_i(s)\|_{L^\infty}\mathrm{d}s.$$

For the third term on the right hand side we use Sobolev embedding and interpolation as above

$$\begin{aligned}&\int_0^T \|\mu(s)\|_{L^\infty}\|X_i(s)\|_{L^\infty}\mathrm{d}s \lesssim \int_0^T \|\mu(s)\|_{L^\infty}\|X_i(s)\|_{H^1}^{\frac{1+\delta}{2}}\|X_i(s)\|_{L^2}^{\frac{1-\delta}{2}}\mathrm{d}s\\ &\lesssim \Big(\int_0^T \|\mu(s)\|_{L^\infty}^{\frac{2}{1+\delta}}\mathrm{d}s\Big)^{\frac{1+\delta}{2}}\Big(\int_0^T \|X_i(s)\|_{H^1}^2\mathrm{d}s\Big)^{\frac{1+\delta}{4}}\sup_{s\in[0,T]}\|X_i(s)\|_{L^2}^{\frac{1-\delta}{2}}\\ &\lesssim \|Z_i\|_{L^\infty_T L^\infty}+\|\psi_i\|_{L^\infty}+1,\end{aligned}$$

where we used (B.16) and (B.8) in the last inequality. Similarly, the last term could be estimated by

$$\int_0^T \|\mu(s)\|_{L^\infty}\|Z_i(s)\|_{L^\infty}\mathrm{d}s \lesssim \|Z_i\|_{L^\infty_T L^\infty}.$$

Combining all the above estimates the result follows. □

Proof of Theorem 7.1. In the proof we omit the superscript N for simplicity. Taking L^2-inner product with v_i on both side of (7.7) we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v_i\|_{L^2}^2+\|\nabla v_i\|_{L^2}^2+m\|v_i\|_{L^2}^2+\int\frac{1}{N}\sum_{j=1}^N\Phi_j^2v_i^2\,\mathrm{d}x+\frac{1}{N}\sum_{j=1}^N\int\Psi_jv_j\Psi_iv_i\,\mathrm{d}x\\ &=-\frac{1}{N}\sum_{j=1}^N\int\Phi_jv_j\Psi_iv_i\,\mathrm{d}x-\int(\frac{1}{N}\sum_{j=1}^N\Psi_j^2-\mu)\Psi_iv_i\,\mathrm{d}x.\end{aligned}\tag{B.17}$$

Summing over i, we obtain

$$\begin{aligned}&\frac{1}{2}\sum_{i=1}^N\frac{\mathrm{d}}{\mathrm{d}t}\|v_i\|_{L^2}^2+\sum_{i=1}^N\|\nabla v_i\|_{L^2}^2+\frac{1}{N}\sum_{i,j=1}^N\|\Phi_jv_i\|_{L^2}^2+\frac{1}{N}\Big\|\sum_{i=1}^N\Psi_iv_i\Big\|_{L^2}^2\\ &\leqslant\frac{1}{N}\sum_{i,j=1}^N\int\Big(\frac{1}{2}\Phi_j^2v_i^2+\frac{1}{2}v_j^2\Psi_i^2\Big)\,\mathrm{d}x+\frac{1}{2N}\Big\|\sum_{i=1}^N\Psi_iv_i\Big\|_{L^2}^2+\frac{1}{N}\Big\|\sum_{j=1}^N(\Psi_j^2-\mu)\Big\|_{L^2}^2.\end{aligned}$$

Therefore we have

$$\begin{aligned}&\frac{1}{2}\sum_{i=1}^N\frac{\mathrm{d}}{\mathrm{d}t}\|v_i\|_{L^2}^2+\sum_{i=1}^N\|\nabla v_i\|_{L^2}^2+\frac{1}{2N}\sum_{i,j=1}^N\|\Phi_jv_i\|_{L^2}^2+\frac{1}{2N}\Big\|\sum_{i=1}^N\Psi_iv_i\Big\|_{L^2}^2\\ &\leqslant\Big(\frac{1}{N}\sum_{i=1}^N\|\Psi_i\|_{L^\infty}^2\Big)\Big(\sum_{j=1}^N\|v_j\|_{L^2}^2\Big)+\frac{1}{N}\Big\|\sum_{j=1}^N(\Psi_j^2-\mu)\Big\|_{L^2}^2.\end{aligned}\tag{B.18}$$

Using (2.3) we obtain

$$\mathbf{E}\frac{1}{N}\Big\|\sum_{j=1}^N(\Psi_j^2-\mu)\Big\|_{L^2}^2=\mathbf{E}\|\Psi_1^2-\mu\|_{L^2}^2\lesssim\mathbf{E}\|\Psi_1\|_{L^4}^4\lesssim\mathbf{E}\|X_1\|_{L^2}^3\|X_1\|_{H^1}+\mathbf{E}\|Z_1\|_{L^4}^4.$$

Combining this, (B.18), (B.9) and Gronwall's inequality, we obtain that $\sup_{t\leqslant T}\sum_i\|v_i\|_{L^2}^2$ is uniformly bounded in probability. Hence,

$$\lim_{N\to\infty}\sup_{t\in[0,T]}\frac{1}{N}\sum_{i=1}^N\|v_i\|_{L^2}^2=0\qquad\text{in probability.}$$

Furthermore, using the bound (B.4) on moments of $\frac{1}{N}\sum_j\|Y_j(t)\|_{L^2}^2$ and the bound (B.8) on X_i and triangular inequality, we obtain that the above convergence can be upgraded to convergence in $L^1(\Omega)$:

$$\mathbf{E}\sup_{t\in[0,T]}\frac{1}{N}\sum_{i=1}^N\|v_i\|_{L^2}^2\text{ converges to zero .}\tag{B.19}$$

On the other hand, we use (B.17) to have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v_i\|_{L^2}^2+\|\nabla v_i\|_{L^2}^2+\int\frac{1}{2N}\sum_{j=1}^N\Phi_j^2v_i^2\,\mathrm{d}x$$
$$\leqslant\|\Psi_i\|_{L^\infty}^2\frac{1}{N}\sum_{j=1}^N\int v_j^2\,\mathrm{d}x+\frac{1}{2N}\sum_{j=1}^N\|\Psi_j\|_{L^\infty}^2\int v_i^2\,\mathrm{d}x+\|v_i\|_{L^2}^2\|\Psi_i\|_{L^\infty}^2+\frac{1}{N^2}\Big\|\sum_{j=1}^N(\Psi_j^2-\mu)\Big\|_{L^2}^2,$$

which combined with Gronwall's inequality and (B.19), (B.7) implies that $\sup_{t\in[0,T]}\|v_i\|_{L^2}^2$ converges to zero in probability. By using (B.5) and (B.8), and assumption (7.8), we deduce the result. □

Proof of Theorem 7.2. Let Ψ_1 and Ψ_2 be two solutions to (7.5) with the initial distributions given by ν_1 and ν_2 satisfying $\int\|\psi\|_{L^\infty}^2\mathrm{d}\nu_i<\infty$ for $i=1,2$. Set $u=\Psi_2-\Psi_1$. We have

$$\partial_t u=(\Delta-m)u-\mathbf{E}[\Psi_2^2]u+\mathbf{E}[\Psi_1^2-\Psi_2^2]\Psi_1.$$

Now we take L^2-inner product with u and obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^2}^2+m\|u\|_{L^2}^2+\|\nabla u\|_{L^2}^2+\int\mathbf{E}[\Psi_2^2]u^2\,\mathrm{d}x&=\int\mathbf{E}[\Psi_1^2-\Psi_2^2]\Psi_1u\,\mathrm{d}x\\&\leqslant2\int(\mathbf{E}[u^2])^{1/2}\left((\mathbf{E}[\Psi_2^2])^{1/2}+(\mathbf{E}[\Psi_1^2])^{1/2}\right)|\Psi_1u|\,\mathrm{d}x\\&\leqslant\frac{1}{2}\int\mathbf{E}[\Psi_2^2]u^2\,\mathrm{d}x+\int\mathbf{E}[u^2]\Psi_1^2\,\mathrm{d}x+\int\mathbf{E}[\Psi_1^2]u^2\,\mathrm{d}x.\end{aligned}$$

Now we take expectation and obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}\|u\|_{L^2}^2+m\mathbf{E}\|u\|_{L^2}^2+\mathbf{E}\|\nabla u\|_{L^2}^2+\frac{1}{2}\int\mathbf{E}[\Psi_2^2]\mathbf{E}u^2\,\mathrm{d}x\\&\lesssim\int\mathbf{E}[u^2]\mathbf{E}[\Psi_1^2]\,\mathrm{d}x\lesssim\mathbf{E}[\|u\|_{L^2}^2]\mathbf{E}[\|\Psi_1\|_{L^\infty}^2].\end{aligned}\tag{B.20}$$

By Gronwall's inequality we have

$$\mathbf{E}[\|u(t)\|_{L^2}^2]\lesssim\mathbf{E}[\|u(1)\|_{L^2}^2]\exp\left(-m(t-1)+C\int_1^t\mathbf{E}[\|\Psi_1\|_{L^\infty}^2]\mathrm{d}s\right).$$

Choosing m large enough and invoking Corollary B.4 we obtain

$$\mathbf{E}[\|u(t)\|_{L^2}^2]\lesssim\exp\left(-mt/2\right).$$

Now we take two different invariant measures ν_1, ν_2 supported on L^∞ to equation (7.5). We could construct a stochastic basis $(\Omega,\mathcal{F},\mathbf{P})$ and two random variables $\psi_1,\psi_2\in L^\infty$ on it such that $\psi_i\sim\nu_i$. For ψ_i we could construct two solutions Ψ_1,Ψ_2 to equation (7.5) with $\Psi_i(0)=\psi_i$. Since ν_i is invariant measure we have

$$\overline{\mathbb{W}}_2(\nu_1,\nu_2)^2=\overline{\mathbb{W}}_2(P_t^*\nu_1,P_t^*\nu_2)^2\leqslant\mathbf{E}[\|\Psi_1(t)-\Psi_2(t)\|_{L^2}^2]\lesssim e^{-mt/2}.$$

Letting $t\to\infty$ we obtain $\psi_1=\psi_2$ $\mathbf{P}-a.s.$, which implies that $\nu_1=\nu_2$ and the result follows. □

Proof of Theorem 7.3. Similar as in Lemma 6.7 we could find stationary process $(\Phi_i,\Psi_i)_{1\leqslant i\leqslant N}$ such that Φ_i, Ψ_i are the stationary solution to (7.1) and (7.5), respectively. Set $v_i=\Phi_i-\Psi_i$. By using (B.18) for the difference v_i, we obtain

$$\frac{1}{2N}\sum_{i=1}^N\frac{\mathrm{d}}{\mathrm{d}t}\|v_i\|_{L^2}^2+\frac{1}{N}\sum_{i=1}^N\|\nabla v_i\|_{L^2}^2+m\frac{1}{N}\sum_{i=1}^N\|v_i\|_{L^2}^2\lesssim\frac{1}{N}\sum_{i=1}^N\|v_i\|_{L^2}^2D_N+R_N^0,$$

with

$$D_N = \frac{1}{N}\sum_{j=1}^N \|\Psi_j\|_{L^\infty}^2 + 1, \qquad R_N^0 = \frac{1}{N^2}\|\sum_{j=1}^N (\Psi_j^2 - \mu)\|_{L^2}^2.$$

Then the rest of the proof is the same as 2d case (see Theorem 6.11). □

Appendix C. Consequences of Dyson-Schwinger equations

Dyson-Schwinger equations are relations between correlation functions of different orders. Here we derive the identities (7.13) and (7.14) using Dyson-Schwinger equations; these are essentially in [Kup80b] (Eq (7)(8) therein), but since we're in slightly different setting, we give some details here to be self-contained. Dyson-Schwinger equations are direct consequences of integration by parts formula, for instance, for the Φ^4 model, we refer to [GH18, Theorem 6.7]. In the case of the N-component Φ^4 model (i.e. linear sigma model), for a fixed N, it is easy to derive the following integration by parts formula

$$\mathbf{E}\Big(\frac{\delta F(\Phi)}{\delta\Phi_1(x)}\Big) = \mathbf{E}\Big((m-\Delta_x)\Phi_1(x)F(\Phi)\Big) + \frac{1}{N}\mathbf{E}\Big(F(\Phi) :\Phi_1(x)\mathbf{\Phi}(x)^2: \Big)$$

where $\Phi \sim \nu^N$ and $\mathbf{\Phi}^2 \stackrel{\text{def}}{=} \sum_{i=1}^N \Phi_i^2$ following the notation in Section 7.3; or in terms of Green's function $C(x-y) = (m-\Delta_x)^{-1}(x-y)$

$$\int C(x-z)\mathbf{E}\Big(\frac{\delta F(\Phi)}{\delta\Phi_1(z)}\Big)\mathrm{d}z = \mathbf{E}\Big(\Phi_1(x)F(\Phi)\Big) + \frac{1}{N}\int C(x-z)\mathbf{E}\Big(F(\Phi) :\Phi_1(z)\mathbf{\Phi}(z)^2: \Big)\mathrm{d}z \tag{C.1}$$

Here we will assume $d=1$ and the Wick products are understood as Sec 7.

The proof of (C.1) is standard by using Gaussian integration by parts. Here we apply (C.1) to prove (7.13). Recall that $C_{\mathrm{w}} = C(0)$. Taking $F(\Phi) = :\Phi_1(x)\mathbf{\Phi}^2(x):$ one has

$$\begin{aligned}
&\frac{1}{N}\int C(x-z)\mathbf{E}\Big(:\Phi_1(x)\mathbf{\Phi}^2(x): \; :\Phi_1(z)\mathbf{\Phi}^2(z): \Big)\mathrm{d}z \\
&\quad = C_{\mathrm{w}}\mathbf{E}\Big(\frac{\delta :\Phi_1(x)\mathbf{\Phi}^2(x):}{\delta\Phi_1(x)}\Big) - \mathbf{E}[\Phi_1(x) :\Phi_1(x)\mathbf{\Phi}^2(x):] = -\frac{1}{N}\mathbf{E} :(\mathbf{\Phi}^2)^2: (x)
\end{aligned}$$

where the last step can be checked straightforwardly [7] using the definition of Wick products and the symmetry (i.e. exchangeability of $(\Phi_i)_i$).

Next, taking $F = :\mathbf{\Phi}^2(x): \; :\Phi_1\mathbf{\Phi}^2(z):$ one has

$$\begin{aligned}
&\frac{1}{N}\int C(x-z_1)\mathbf{E}\Big(:\mathbf{\Phi}^2(x): \; :\Phi_1\mathbf{\Phi}^2(z_1): \; :\Phi_1\mathbf{\Phi}^2(z): \Big)\mathrm{d}z_1 \\
&\quad = C(x-z)\mathbf{E}\Big(:\mathbf{\Phi}^2(x): \frac{\delta :\Phi_1\mathbf{\Phi}^2(z):}{\delta\Phi_1(z)}\Big) + C_{\mathrm{w}}\mathbf{E}\Big(\frac{\delta :\mathbf{\Phi}^2(x):}{\delta\Phi_1(x)} \; :\Phi_1\mathbf{\Phi}^2(z): \Big) \\
&\qquad\qquad - \mathbf{E}\Big(\Phi_1(x) :\mathbf{\Phi}^2(x): \; :\Phi_1\mathbf{\Phi}^2(z): \Big) \\
&\quad = \frac{N+2}{N}C(x-z)\mathbf{E}[:\mathbf{\Phi}^2(x): \; :\mathbf{\Phi}^2(z):] - \mathbf{E}\Big(:\Phi_1(x)\mathbf{\Phi}^2(x): \; :\Phi_1(z)\mathbf{\Phi}^2(z): \Big)
\end{aligned}$$

where we again used symmetry, for instance we replaced $\frac{\delta :\Phi_1\mathbf{\Phi}^2(z):}{\delta\Phi_1(z)}$ by $\frac{N+2}{N} :\mathbf{\Phi}^2(z):$ under expectation, as well as a simple relation

$$2C_{\mathrm{w}}\Phi_1(x) - \Phi_1(x) :\mathbf{\Phi}(x)^2: \; = - :\Phi_1(x)\mathbf{\Phi}^2(x):$$

From the above two correlation identities, we cancel out the 6th order correlation term and then obtain the first equation in (7.13).

[7] This could be viewed as an N dimensional generalization of the well-known relation $H_{n+1}(x) = xH_n(x) - H_n'(x)$ for Hermite polynomials H_n.

Now take $F(\Phi)=\int C(x-y)\,{:}\Phi_1\mathbf{\Phi}^2(y){:}\ \ {:}\mathbf{\Phi}^2(z){:}\,\mathrm{d}y$. We have, by (C.1),

$$
\begin{aligned}
&\frac{1}{N}\int C(x-y_1)C(x-y_2)\mathbf{E}\Big(\,{:}\Phi_1\mathbf{\Phi}^2(y_1){:}\ \ {:}\Phi_1\mathbf{\Phi}^2(y_2){:}\ \ {:}\mathbf{\Phi}^2(z){:}\,\Big)\mathrm{d}y_1\mathrm{d}y_2\\
&=\int C(x-z)C(x-y)\mathbf{E}\Big(\frac{\delta\,{:}\mathbf{\Phi}^2(z){:}}{\delta\Phi_1(z)}\ {:}\Phi_1\mathbf{\Phi}^2(y){:}\,\Big)\mathrm{d}y\\
&\quad+\int C(x-y)^2\mathbf{E}\Big(\,{:}\mathbf{\Phi}^2(z){:}\ \frac{\delta\,{:}\Phi_1\mathbf{\Phi}^2(y){:}}{\delta\Phi_1(y)}\Big)\mathrm{d}y\\
&\quad-\int C(x-y)\mathbf{E}\Big(\Phi_1(x)\,{:}\Phi_1\mathbf{\Phi}^2(y){:}\ \ {:}\mathbf{\Phi}^2(z){:}\,\Big)\mathrm{d}y.
\end{aligned}
$$

Note that the LHS is precisely Q_2^N and the first term on the RHS is just Q_1^N. The second term on the RHS equals

$$
\frac{N+2}{N}\int C(x-y)^2\mathbf{E}\Big(\,{:}\mathbf{\Phi}^2(z){:}\ \ {:}\mathbf{\Phi}^2(y){:}\,\Big)\mathrm{d}y.
$$

To deal with the last term above, taking $F(\Phi)=\Phi_1(x)\,{:}\mathbf{\Phi}^2(z){:}$ and applying (C.1), one has

$$
\begin{aligned}
&\int C(x-y)\mathbf{E}\Big(\Phi_1(x)\,{:}\Phi_1\mathbf{\Phi}^2(y){:}\ \ {:}\mathbf{\Phi}^2(z){:}\,\Big)\mathrm{d}y\\
&=NC_{\mathrm{w}}\mathbf{E}\,{:}\mathbf{\Phi}(z)^2{:}\ +NC(x-z)\mathbf{E}\Big(\Phi_1(x)\frac{\delta\,{:}\mathbf{\Phi}^2(z){:}}{\delta\Phi_1(z)}\Big)-N\mathbf{E}[\Phi_1(x)^2\,{:}\mathbf{\Phi}^2(z){:}\,]\\
&=-\mathbf{E}[\,{:}\mathbf{\Phi}^2(x){:}\ \ {:}\mathbf{\Phi}^2(z){:}\,]+2NC(x-z)\mathbf{E}[\Phi_1(x)\Phi_1(z)].
\end{aligned}
$$

We then obtain (7.14).

References

[AK17] S. Albeverio and S. Kusuoka. The invariant measure and the flow associated to the ϕ^4_3-quantum field model. *arXiv:1711.07108*, 2017.

[Alf83] J. Alfaro. Stochastic quantization and the large-N reduction of U(N) gauge theory. *Physical Review D*, 28(4):1001, 1983.

[AR91] S. Albeverio and M. Röckner. Stochastic differential equations in infinite dimensions: solutions via Dirichlet forms. *Probab. Theory Related Fields*, 89(3):347–386, 1991.

[AS83] J. Alfaro and B. Sakita. Derivation of quenched momentum prescription by means of stochastic quantization. *Physics Letters B*, 121(5):339–344, 1983.

[AS12] M. Anshelevich and A. N. Sengupta. Quantum free Yang-Mills on the plane. *J. Geom. Phys.*, 62(2):330–343, 2012.

[BCD18] I. Bailleul, R. Catellier, and F. Delarue. Mean field rough differential equations. *arXiv preprint arXiv:1802.05882*, 2018.

[BGHZ19] Y. Bruned, F. Gabriel, M. Hairer, and L. Zambotti. Geometric stochastic heat equations. *arXiv:1902.02884*, 2019.

[BR82] C. Billionnet and P. Renouard. Analytic interpolation and Borel summability of the $(\frac{\lambda}{N}\Phi^{:4}_{N})_2$ models. I. Finite volume approximation. *Comm. Math. Phys.*, 84(2):257–295, 1982.

[BW93] E. Brezin and S. R. Wadia. *The large N expansion in quantum field theory and statistical physics: from spin systems to 2-dimensional gravity*. World scientific, 1993.

[CC18] R. Catellier and K. Chouk. Paracontrolled distributions and the 3-dimensional stochastic quantization equation. *Ann. Probab.*, 46(5):2621–2679, 2018.

[CCHS20] A. Chandra, I. Chevyrev, M. Hairer, and H. Shen. Langevin dynamic for the 2D Yang-Mills measure. *arXiv preprint arXiv:2006.04987*, 2020.

[CDFM18] M. Coghi, J.-D. Deuschel, P. Friz, and M. Maurelli. Pathwise Mckean-Vlasov theory with additive noise. *arXiv preprint arXiv:1812.11773*, 2018.

[Cha19] S. Chatterjee. Rigorous solution of strongly coupled $SO(N)$ lattice gauge theory in the large N limit. *Comm. Math. Phys.*, 366(1):203–268, 2019.

[CJ16] S. Chatterjee and J. Jafarov. The $1/N$ expansion for SO(N) lattice gauge theory at strong coupling. *arXiv preprint arXiv:1604.04777*, 2016.

[CJP74] S. Coleman, R. Jackiw, and H. Politzer. Spontaneous symmetry breaking in the O(N) model for large N. *Physical Review D*, 10(8):2491, 1974.

[CL15] T. Cass and T. Lyons. Evolving communities with individual preferences. *Proc. Lond. Math. Soc. (3)*, 110(1):83–107, 2015.

[CMW19] A. Chandra, A. Moinat, and H. Weber. A priori bounds for the ϕ^4 equation in the full sub-critical regime. *arXiv preprint arXiv:1910.13854*, 2019.
[Col88] S. Coleman. *Aspects of symmetry: selected Erice lectures*. Cambridge University Press, 1988.
[CWZZ18] X. Chen, B. Wu, R. Zhu, and X. Zhu. Stochastic heat equations for infinite strings with values in a manifold. *arXiv preprint arXiv:1812.01942*, 2018.
[DH87] P. H. Damgaard and H. Hüffel. Stochastic quantization. *Physics Reports*, 152(5-6):227–398, 1987.
[DPD03] G. Da Prato and A. Debussche. Strong solutions to the stochastic quantization equations. *Ann. Probab.*, 31(4):1900–1916, 2003.
[DPZ96] G. Da Prato and J. Zabczyk. *Ergodicity for infinite-dimensional systems*, volume 229 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1996.
[ES13] W. E and H. Shen. Mean field limit of a dynamical model of polymer systems. *Sci. China Math.*, 56(12):2591–2598, 2013.
[ESY10] L. Erdős, B. Schlein, and H.-T. Yau. Derivation of the Gross-Pitaevskii equation for the dynamics of Bose-Einstein condensate. *Ann. of Math. (2)*, 172(1):291–370, 2010.
[FH17] T. Funaki and M. Hoshino. A coupled KPZ equation, its two types of approximations and existence of global solutions. *J. Funct. Anal.*, 273(3):1165–1204, 2017.
[FMR82] J. Fröhlich, A. Mardin, and V. Rivasseau. Borel summability of the $1/N$ expansion for the N-vector [O(N) nonlinear σ] models. *Comm. Math. Phys.*, 86(1):87–110, 1982.
[GH18] M. Gubinelli and M. Hofmanová. A PDE construction of the Euclidean Φ^4_3 quantum field theory. *arXiv:1810.01700*, 2018.
[GH19] M. Gubinelli and M. Hofmanová. Global solutions to elliptic and parabolic Φ^4 models in Euclidean space. *Comm. Math. Phys.*, 368(3):1201–1266, 2019.
[GIP15] M. Gubinelli, P. Imkeller, and N. Perkowski. Paracontrolled distributions and singular PDEs. *Forum Math. Pi*, 3:e6, 75, 2015.
[GJ87] J. Glimm and A. Jaffe. *Quantum physics*. Springer-Verlag, New York, second edition, 1987. A functional integral point of view.
[GN74] D. J. Gross and A. Neveu. Dynamical symmetry breaking in asymptotically free field theories. *Physical Review D*, 10(10):3235, 1974.
[GO14] L. Grafakos and S. Oh. The Kato-Ponce inequality. *Comm. Partial Differential Equations*, 39(6):1128–1157, 2014.
[Gol16] F. Golse. On the dynamics of large particle systems in the mean field limit. In *Macroscopic and large scale phenomena: coarse graining, mean field limits and ergodicity*, volume 3 of *Lect. Notes Appl. Math. Mech.*, pages 1–144. Springer, [Cham], 2016.
[GP17] M. Gubinelli and N. Perkowski. KPZ reloaded. *Commun. Math. Phys.*, 349:165–269, 2017.
[Hai14] M. Hairer. A theory of regularity structures. *Invent. Math.*, 198(2):269–504, 2014.
[Hai16] M. Hairer. The motion of a random string. *arXiv:1605.02192*, 2016.
[HM18a] M. Hairer and K. Matetski. Discretisations of rough stochastic PDEs. *Ann. Probab.*, 46(3):1651–1709, 2018.
[HM18b] M. Hairer and J. Mattingly. The strong Feller property for singular stochastic PDEs. *Ann. Inst. Henri Poincaré Probab. Stat.*, 54(3):1314–1340, 2018.
[HMS11] M. Hairer, J. C. Mattingly, and M. Scheutzow. Asymptotic coupling and a general form of Harris' theorem with applications to stochastic delay equations. *Probab. Theory Related Fields*, 149(1-2):223–259, 2011.
[HS19] M. Hairer and P. Schönbauer. The support of singular stochastic PDEs. *arXiv preprint arXiv:1909.05526*, 2019.
[Jab14] P.-E. Jabin. A review of the mean field limits for Vlasov equations. *Kinet. Relat. Models*, 7(4):661–711, 2014.
[JW18] P.-E. Jabin and Z. Wang. Quantitative estimates of propagation of chaos for stochastic systems with $W^{-1,\infty}$ kernels. *Invent. Math.*, 214(1):523–591, 2018.
[Kup80a] A. Kupiainen. $1/n$ expansion—some rigorous results. In *Mathematical problems in theoretical physics (Proc. Internat. Conf. Math. Phys., Lausanne, 1979)*, volume 116 of *Lecture Notes in Phys.*, pages 208–210. Springer, Berlin-New York, 1980.
[Kup80b] A. J. Kupiainen. $1/n$ expansion for a quantum field model. *Comm. Math. Phys.*, 74(3):199–222, 1980.
[Kup80c] A. J. Kupiainen. On the $1/n$ expansion. *Comm. Math. Phys.*, 73(3):273–294, 1980.
[KX95] G. Kallianpur and J. Xiong. *Stochastic differential equations in infinite-dimensional spaces*, volume 26 of *Institute of Mathematical Statistics Lecture Notes—Monograph Series*. Institute of Mathematical Statistics, Hayward, CA, 1995.
[Lév11] T. Lévy. The master field on the plane. *arXiv preprint arXiv:1112.2452*, 2011.
[LL07] J.-M. Lasry and P.-L. Lions. Mean field games. *Jpn. J. Math.*, 2(1):229–260, 2007.
[McK67] H. P. McKean, Jr. Propagation of chaos for a class of non-linear parabolic equations. In *Stochastic Differential Equations (Lecture Series in Differential Equations, Session 7, Catholic Univ., 1967)*, pages 41–57. Air Force Office Sci. Res., Arlington, Va., 1967.
[MW17a] J.-C. Mourrat and H. Weber. The dynamic Φ^4_3 model comes down from infinity. *Comm. Math. Phys.*, 356(3):673–753, 2017.

[MW17b] J.-C. Mourrat and H. Weber. Global well-posedness of the dynamic Φ^4 model in the plane. *Ann. Probab.*, 45(4):2398–2476, 2017.

[MW18] A. Moinat and H. Weber. Space-time localisation for the dynamic ϕ^4_3 model. *arXiv:1811.05764*, 2018.

[MZJ03] M. Moshe and J. Zinn-Justin. Quantum field theory in the large N limit: A review. *Physics Reports*, 385(3-6):69–228, 2003.

[RWZZ20] M. Röckner, B. Wu, R. Zhu, and X. Zhu. Stochastic Heat Equations with Values in a Manifold via Dirichlet Forms. *SIAM J. Math. Anal.*, 52(3):2237–2274, 2020.

[RYZ18] M. Röckner, H. Yang, and R. Zhu. Conservative stochastic 2-dimensional Cahn-Hilliard equation. *arXiv preprint arXiv:1802.04141*, 2018.

[RZZ17] M. Röckner, R. Zhu, and X. Zhu. Restricted Markov uniqueness for the stochastic quantization of $P(\Phi)_2$ and its applications. *J. Funct. Anal.*, 272(10):4263–4303, 2017.

[She18] H. Shen. Stochastic quantization of an Abelian gauge theory. *arXiv:1801.04596*, 2018.

[Sim74] B. Simon. *The $P(\phi)_2$ Euclidean (quantum) field theory.* Princeton University Press, Princeton, N.J., 1974. Princeton Series in Physics.

[Spo91] H. Spohn. *Large Scale Dynamics of Interacting Particles.* Texts and Monographs in Physics. Springer, 1991.

[Sta68] H. E. Stanley. Spherical model as the limit of infinite spin dimensionality. *Physical Review*, 176(2):718, 1968.

[Sym77] K. Symanzik. 1/N expansion in $P(\varphi^2)_{4-\epsilon}$ theory I. massless theory $0 < \epsilon < 2$. *preprint DESY*, 77(05), 1977.

[Szn91] A.-S. Sznitman. Topics in propagation of chaos. In *École d'Été de Probabilités de Saint-Flour XIX—1989*, volume 1464 of *Lecture Notes in Math.*, pages 165–251. Springer, Berlin, 1991.

[t'H74] G. t'Hooft. A planar diagram theory for strong interactions. *Nuclear Physics. B*, 72(3):461–473, 1974.

[Tri78] H. Triebel. *Interpolation theory, function spaces, differential operators*, volume 18 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam-New York, 1978.

[TW18] P. Tsatsoulis and H. Weber. Spectral gap for the stochastic quantization equation on the 2-dimensional torus. *Ann. Inst. Henri Poincaré Probab. Stat.*, 54(3):1204–1249, 2018.

[Wan18] F.-Y. Wang. Distribution dependent SDEs for Landau type equations. *Stochastic Process. Appl.*, 128(2):595–621, 2018.

[Wil73] K. G. Wilson. Quantum field-theory models in less than 4 dimensions. *Physical Review D*, 7(10):2911, 1973.

[Wit80] E. Witten. The 1/N expansion in atomic and particle physics. In *Recent developments in gauge theories*, pages 403–419. Springer, 1980.

[ZZ18] R. Zhu and X. Zhu. Lattice approximation to the dynamical Φ^4_3 model. *Ann. Probab.*, 46(1):397–455, 2018.

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