MAXIMUM PRINCIPLE FOR NON-UNIFORMLY PARABOLIC EQUATIONS AND APPLICATIONS

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Abstract. In this paper we study the global boundedness for the solutions to a class of possibly degenerate parabolic equations by De-Giorgi's iteration. As applications, we show the existence of weak solutions for possibly degenerate stochastic differential equations with singular diffusion and drift coefficients. Moreover, by the Markov selection theorem of Krylov [8], we also establish the existence of the associated strong Markov family.

Keywords: Maximum principle, De-Giorgi's iteration, Stochastic differential equation, Krylov's estimate, Markov selection.

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1. Introduction

Consider the following elliptic equation of divergence form in $\mathbb{R}^d$ $(d \geqslant 2)$:

$$\mathrm{div}(a \cdot \nabla u) = 0, \tag{1.1}$$

where $a : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is a Borel measurable function and $\nabla := (\partial_{x_1}, \cdots, \partial_{x_d})$. We introduce the following two functions:

$$\lambda_0(x) := \inf_{|\xi|=1} \xi \cdot a(x)\xi, \quad \mu_0(x) := \sup_{|\xi|=1} \frac{|a(x)\xi|^2}{\xi \cdot a(x)\xi}. \tag{1.2}$$

Suppose that λ_0 and μ_0 are nonnegative measurable functions. If λ_0^{-1} and μ_0 are essentially bounded, that is, a is uniformly elliptic, then the celebrated works of De-Giorgi [4] and Nash [13] said that any weak solutions of elliptic equation (1.1) are bounded and Hölder continuous. Moreover, Moser [12] showed that any weak solutions of (1.1) satisfy the Harnack inequality.

In [17], Trudinger considered the non-uniformly elliptic equation (1.1) under the following integrability assumptions:

$$\lambda_0^{-1} \in L^{p_0},\ \mu_0 \in L^{p_1} \text{ with } p_0, p_1 \in (1, \infty] \text{ satisfying } \tfrac{1}{p_0} + \tfrac{1}{p_1} < \tfrac{2}{d},$$

and showed that any generalized solutions of (1.1) are locally bounded and weak Harnack inequality holds. Recently, Bella and Schäffner [3] showed the same results under the following *sharp* condition on p_0, p_1,

$$\tfrac{1}{p_0} + \tfrac{1}{p_1} < \tfrac{2}{d-1}, \quad p_0, p_1 \in [1, \infty], \tag{1.3}$$

where the key point is a new Sobolev embedding inequality of variational type. In this paper we are interested in a parabolic version of [3], and aim to establish the global boundedness for the solutions of non-uniformly parabolic equations. More

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precisely, we shall consider the following parabolic equation of divergence form in $\mathbb{R}^{d+1}$:

$$\partial_t u = \mathrm{div}(a \cdot \nabla u) + b \cdot \nabla u + f, \tag{1.4}$$

where

$$a : \mathbb{R}^{d+1} \to \mathbb{R}^{d\times d},\ b : \mathbb{R}^{d+1} \to \mathbb{R}^d,\ f : \mathbb{R}^{d+1} \to \mathbb{R}$$

are Borel measurable functions. As in (1.2), we introduce

$$\lambda(x) := \inf_{t\geqslant 0, |\xi|=1} \xi \cdot a(t,x)\xi, \quad \mu(x) := \sup_{t\geqslant 0,|\xi|=1} \frac{|a(t,x)\xi|^2}{\xi \cdot a(t,x)\xi}, \tag{1.5}$$

and suppose that λ and μ are nonnegative Borel measurable functions.

First of all we introduce the following notion of weak solutions to PDE (1.4).

Definition 1.1. *A continuous function $u : \mathbb{R}^{d+1} \to \mathbb{R}$ is called a Lipschitz weak (super/sub)-solution of PDE* (1.4) *if ∇u is locally bounded and for any nonnegative Lipschitz function φ on $\mathbb{R}^{d+1}$ with compact support,*

$$-\langle\!\langle u, \partial_t\varphi\rangle\!\rangle = (\geqslant / \leqslant) - \langle\!\langle a \cdot \nabla u, \nabla\varphi\rangle\!\rangle + \langle\!\langle b \cdot \nabla u, \varphi\rangle\!\rangle + \langle\!\langle f, \varphi\rangle\!\rangle, \tag{1.6}$$

where $\langle\!\langle f, g\rangle\!\rangle := \int_{\mathbb{R}} \int_{\mathbb{R}^d} f(t,x)g(t,x)\mathrm{d}x\mathrm{d}t$.

Throughout this paper, we fix $p_0 \in (\frac{d}{2}, \infty]$ and $p_1 \in [1,\infty]$ with

$$\tfrac{1}{p_0} + \tfrac{1}{p_1} < \tfrac{2}{d-1}, \tag{1.7}$$

and introduce the index set

$$\mathbb{I}^d_{p_0} := \Big\{(p,q) \in [1,\infty]^2 : \tfrac{1}{p} < (1 - \tfrac{1}{q})(\tfrac{2}{d} - \tfrac{1}{p_0})\Big\}.$$

Using the localized space introduced in (2.3) and (2.4) below, we make the following assumptions about a and b:

(H^a) $|\!|\!|\lambda^{-1}|\!|\!|_{p_0} + |\!|\!|\mu|\!|\!|_{p_1} < \infty$, where λ, μ are defined by (1.5).

(H^b) $b = b_1 + b_2$, where if $p_0 \in (\frac{d}{2}, d]$, $b_1 \equiv 0$, and if $p_0 > d$, $b_1 \in \widetilde{\mathbb{L}}^{q_2,p_2}_{t,x}$ for some $(p_2, q_2) \in [1,\infty]^2$ with

$$\tfrac{1}{2p_0} + \tfrac{1}{p_2} < (\tfrac{1}{2} - \tfrac{1}{q_2})(\tfrac{2}{d} - \tfrac{1}{p_0}), \tag{1.8}$$

and $b_2 \in \widetilde{\mathbb{L}}^{p_1,\infty}_{x,t}$ and $(\mathrm{div} b_2)^- \in \widetilde{\mathbb{L}}^{q_3,p_3}_{t,x}$ for some $(p_3, q_3) \in \mathbb{I}^d_{p_0}$.

Remark 1.2. Note that condition (1.8) is satisfied for $p_2 = q_2 = \infty$ if and only if $p_0 > d$. This is why we need to put $b_1 \equiv 0$ for $p_0 \leqslant d$. If $p_0 = \infty$, i.e., a has a lower bound, condition (1.8) reduces to the usual one $\frac{d}{p_2} + \frac{2}{q_2} < 1$, and

$$\mathbb{I}^d_\infty = \Big\{(p,q) \in [1,\infty]^2 : \tfrac{d}{p} + \tfrac{2}{q} < 2\Big\}.$$

For simplicity of notations, we introduce the following parameter set

$$\Theta := \Big(d, p_i, q_i, |\!|\!|\lambda^{-1}|\!|\!|_{p_0}, |\!|\!|\mu|\!|\!|_{p_1}, |\!|\!|b_1|\!|\!|_{\widetilde{\mathbb{L}}^{q_2,p_2}_{t,x}}, |\!|\!|b_2|\!|\!|_{\widetilde{\mathbb{L}}^{p_1,\infty}_{x,t}}, |\!|\!|(\mathrm{div} b_2)^-|\!|\!|_{\widetilde{\mathbb{L}}^{q_3,p_3}_{t,x}}\Big). \tag{1.9}$$

The main aim of this paper is to prove the following apriori estimate.

Theorem 1.3. *Under* **(H^a)** *and* **(H^b)**, *for any $f \in \widetilde{\mathbb{L}}^{q_4,p_4}_{t,x}$ with $(p_4, q_4) \in \mathbb{I}^d_{p_0}$ and for any $T > 0$, there exists a constant $C = C(T, \Theta, p_4, q_4) > 0$ such that for any Lipschitz weak solution u of PDE* (1.4) *in $\mathbb{R}^{d+1}$ with $u(t)|_{t\leqslant 0} \equiv 0$,*

$$\|u\|_{L^\infty([0,T]\times\mathbb{R}^d)} + |\!|\!|u\mathbf{1}_{[0,T]}|\!|\!|_{\widetilde{\mathscr{V}}} \leqslant C|\!|\!|f\mathbf{1}_{[0,T]}|\!|\!|_{\widetilde{\mathbb{L}}^{q_4,p_4}_{t,x}}, \tag{1.10}$$

where $\widetilde{\mathscr{V}}$ is defined by (3.2) *below.*

Consider the following heat equation with divergence free drift b:

$$\partial_t u = \Delta u + b\cdot\nabla u + f,\ u(t)|_{t\leqslant 0} = 0. \tag{1.11}$$

The following apriori global boundedness estimate is an easy consequence of Theorem 1.3, which seems to be new.

Corollary 1.4. *Let $b \in \widetilde{\mathbb{L}}^{p,\infty}_{x,t}$ with $\mathrm{div} b = 0$, where $p \in [1,\infty] \cap (\frac{d-1}{2},\infty]$. For any $T > 0$ and $f \in \widetilde{\mathbb{L}}^{q',p'}_{t,x}$, where $p', q' \in [1,\infty]$ satisfy $\frac{d}{p'} + \frac{2}{q'} < 2$, there exists a constant $C > 0$ only depending on T, d, p, p', q' and $\|b\|_{\widetilde{\mathbb{L}}^{p,\infty}_{x,t}}$ such that for any Lipschitz weak solution u of* (1.11)*,*

$$\|u\|_{L^\infty([0,T]\times\mathbb{R}^d)} \leqslant C|\!|\!| f\mathbf{1}_{[0,T]}|\!|\!|_{\widetilde{\mathbb{L}}^{q',p'}_{t,x}}. \tag{1.12}$$

Remark 1.5. Note that when $\frac{d}{p} + \frac{2}{q} < 2$ and $b \in \widetilde{\mathbb{L}}^{q,p}_{t,x}$ with $\mathrm{div} b = 0$, it is well known that (1.12) holds (cf. [14], [19]). When b does not depend on t, the current condition $p > \frac{d-1}{2}$ in Corollary 1.4 is clearly better than $p > \frac{d}{2}$.

In [3], the local boundedness of generalized solutions of elliptic equations is used to establish the L^∞-sublinearity of the corrector in stochastic homogenization in non-uniformly case, which is a key step of proving quenched invariance principle for random walks [1]. As in [3] and [2], Theorem 1.3 could be used to showing a quenched invariance principle for random walks in time-dependent ergodic environment. This is not the purpose of the present paper, and will be studied in future.

As one application of the global boundedness estimate (1.10), we shall establish the existence of weak solutions to possibly degenerate SDEs with singular diffusion and drift coefficients in this paper. Consider the following SDE:

$$\mathrm{d}X_t = \sqrt{2}\sigma(t,X_t)\mathrm{d}W_t + b(t,X_t)\mathrm{d}t,\quad X_0 = x, \tag{1.13}$$

where W is a d-dimensional standard Brownian motion on some stochastic basis $(\Omega,\mathscr{F},\mathbb{P};(\mathscr{F}_t)_{t\geqslant 0})$ and $\sigma : \mathbb{R}_+\times\mathbb{R}^d \to \mathbb{R}^d\otimes\mathbb{R}^d$ and $b : \mathbb{R}_+\times\mathbb{R}^d \to \mathbb{R}^d$ are Borel measurable functions. Note that the generator of SDE (1.13) is given by

$$\mathscr{L}^{\sigma,b}_t f(x) = (\sigma^{ik}\sigma^{jk})(t,x)\partial_i\partial_j f(x) + b^j(t,x)\partial_j f(x).$$

Here and after we shall use the usual Einstein convention for summation: an index appearing twice in a product will be summed automatically.

It is well known that if σ and b are Lipschitz continuous in x uniformly in t, then SDE (1.13) admits a unique strong solution. When σ is bounded measurable and uniformly elliptic and $b \in L^{d+1}(\mathbb{R}_+\times\mathbb{R}^d)$, recently, Krylov [11] showed the existence of weak solutions to SDE (1.13) (see [10] for bounded measurable drift b). When σ is the identity matrix and b is divergence free and belongs to $\widetilde{\mathbb{L}}^{q,p}_{t,x}$ for some $p, q \in [1,\infty]$ with $\frac{d}{p} + \frac{2}{q} < 2$, utilizing the like-estimate (1.12), in a joint work [19] with G. Zhao, we showed the existence of weak solutions to SDE (1.13). In particular, the stochastic Lagrangian trajectories associated with Leray's solutions of 3D-Navier-Stokes equations are constructed. However, when diffusion coefficient σ is possibly degenerate or singular, and b is irregular (saying only bounded measurable), to the author's knowledge, it seems that there are few results about the existence of solutions to SDE (1.13) except for [18]. To show the existence of weak solutions, the key step is to prove the following estimate of Krylov's type: for any $(p,q) \in \mathbb{I}^d_{p_0}$,

$$\mathbb{E}\left(\int_0^t f(s,X_s)\mathrm{d}s\right) \leqslant C|\!|\!| f|\!|\!|_{\widetilde{\mathbb{L}}^{q,p}_{t,x}}. \tag{1.14}$$

Note that if we let $a = \sigma\sigma^*$, then $\mathscr{L}_t^{\sigma,b}$ can be written as the divergence form:

$$\mathscr{L}_t^{\sigma,b} f(x) = \partial_i(a^{ij}(t,\cdot)\partial_j f)(x) + (b^j - \partial_i a^{ij})(t,x)\partial_j f(t,x).$$

Under suitable conditions, (1.14) will be a consequence of Itô's formula and (1.10) (see Theorem 4.3 below).

Although we can show the existence of weak solutions for SDE (1.13) with singular coefficients, in many cases, the uniqueness is not easily obtained and even does not hold for SDEs with measurable coefficients. In 1973, N.V. Krylov [9] proved a Markov selection theorem from the family of solutions of SDE (1.13) when b and σ are bounded continuous. His method was presented in a different way in [16, Chapter 12]. For applications in SPDEs, we refer to [5] and [6]. Here we shall follow Stroock and Varadhan's method [16] to select a strong Markovian solution for SDEs (1.13) with singular coefficients when the uniqueness is not applicable.

We would like to mention the following examples to illustrate our main results obtained in Sections 4 and 5.

Example 1.6. Let $d = 3$ and $\mathbf{u}(t,x)$ be any Leray solutions of 3D-Navier-Stokes equations. Consider the following SDEs:

$$\mathrm{d}X_{t,s} = \sqrt{2}\mathrm{d}W_t + \mathbf{u}(t, X_{t,s})\mathrm{d}t,\ t \geqslant s \geqslant 0,\ X_{s,s} = x \in \mathbb{R}^3.$$

In [19], the existence of weak solutions is obtained to the above SDE. By [19, Theorem 1.1] and Theorem 5.5 below, one can select a family of probability measures $(\mathbb{P}_{s,x})_{(s,x)\in\mathbb{R}_+\times\mathbb{R}^3}$ on the continuous function space $\mathbb{C}$ so that for each $(s,x) \in \mathbb{R}_+ \times \mathbb{R}^3$, $\mathbb{P}_{s,x}$ solves the martingale problem associated to the above SDE, and $(\mathbb{P}_{s,x})_{(s,x)\in\mathbb{R}_+\times\mathbb{R}^3}$ forms a time-inhomoegenous strong Markovian family.

Example 1.7. Let $d \geqslant 3$ and $\alpha \in (0, (\frac{d}{2} - 1) \wedge (\frac{1}{2} + \frac{1}{d-1}))$, $\beta \in (0, 2\alpha)$. For any $\lambda \geqslant 0$ and $x \in \mathbb{R}^d$, the following SDE admits a unique strong solution (see Proposition 6.2 below):

$$\mathrm{d}X_t = |X_t|^{-\alpha}\mathrm{d}W_t + \lambda X_t|X_t|^{-\beta-1}\mathrm{d}t,\quad X_0 = x.$$

Note that the starting point can be zero.

This paper is organized as follows: In Section 2, we prove a time-dependent variational embedding theorem, which in particular extends the result obtained in [3]. In Section 3, we prove our main Theorem 1.3 by De-Giorgi's iteration (cf. [4]). In Section 4, we apply our main result to SDEs with rough coefficients. In Section 5, we use Krylov's Markov selection theorem to select a strong Markov family from the weak solution family. In Section 6, we present two examples to illustrate our result. In the appendix, we recall some results about the regular conditional probability distribution (abbreviated as r.c.p.d.) as well as the abstract time-inhomoegenous strong Markov selection theorem.

Throughout this paper, we use the following conventions: The letter $C = C(\cdots)$ denotes a constant, whose value may change in different places, and which is increasing with respect to its argument. We also use $A \lesssim B$ to denote $A \leqslant CB$ for some unimportant constant $C > 0$.

2. Preliminaries

Let $\mathscr{D} := C_c^\infty(\mathbb{R}^{d+1})$ be the space of all smooth functions in $\mathbb{R}^{d+1}$ with compact supports and $\mathscr{D}'$ the dual space of $\mathscr{D}$, which is also called the distribution space.

The duality between $\mathscr{D}'$ and $\mathscr{D}$ is denoted by $\langle\!\langle \cdot, \cdot \rangle\!\rangle$. In particular, if $f \in \mathscr{D}'$ is locally integrable and $g \in \mathscr{D}$, then

$$\langle\!\langle f, g \rangle\!\rangle = \int_{\mathbb{R}} \langle f(t), g(t) \rangle \mathrm{d}t \ \text{ with } \ \langle f(t), g(t) \rangle := \int_{\mathbb{R}^d} f(t,x) g(t,x) \mathrm{d}x. \tag{2.1}$$

For $p, q \in [1, \infty]$, let $\mathbb{L}^{q,p}_{t,x} := L^q(\mathbb{R}; L^p(\mathbb{R}^d))$ and $\mathbb{L}^{p,q}_{x,t} := L^p(\mathbb{R}^d; L^q(\mathbb{R}))$ be the space of spatial-time functions with norms, respectively,

$$\|f\|_{\mathbb{L}^{q,p}_{t,x}} := \left(\int_{\mathbb{R}} \|f(t, \cdot)\|_p^q \mathrm{d}t \right)^{1/q}, \ \|f\|_{\mathbb{L}^{p,q}_{x,t}} := \left(\int_{\mathbb{R}^d} \|f(\cdot, x)\|_q^p \mathrm{d}x \right)^{1/p},$$

where $\| \cdot \|_p$ stands for the usual L^p-norm. By Minkowskii's inequality,

$$\|f\|_{\mathbb{L}^{q,p}_{t,x}} \leqslant \|f\|_{\mathbb{L}^{p,q}_{x,t}} \text{ if } q \geqslant p; \quad \|f\|_{\mathbb{L}^{p,q}_{x,t}} \leqslant \|f\|_{\mathbb{L}^{q,p}_{t,x}} \text{ if } q \leqslant p. \tag{2.2}$$

For $r > 0$ and $(s, z) \in \mathbb{R}^{d+1}$, we define

$$Q_r := [-r^2, r^2] \times B_r \subset \mathbb{R}^{d+1}, \ Q^{s,z}_r := Q_r + (s, z), \quad B^z_r := B_r + z,$$

and for $p \in [1, \infty]$, introduce the following localized L^p-space:

$$\widetilde{L}^p := \left\{ f \in L^1_{loc}(\mathbb{R}^d) : |\!|\!| f |\!|\!|_p := \sup_z \|\mathbf{1}_{B^z_1} f\|_p < \infty \right\}, \tag{2.3}$$

and for $p, q \in [1, \infty]$,

$$\widetilde{\mathbb{L}}^{q,p}_{t,x} := \left\{ f \in L^1_{loc}(\mathbb{R}^{d+1}) : |\!|\!| f |\!|\!|_{\widetilde{\mathbb{L}}^{q,p}_{t,x}} := \sup_{s,z} \|\mathbf{1}_{Q^{s,z}_1} f\|_{\mathbb{L}^{q,p}_{t,x}} < \infty \right\}, \tag{2.4}$$

and similarly for $\widetilde{\mathbb{L}}^{p,q}_{x,t}$. Clearly, for $p \leqslant p'$ and $q \leqslant q'$,

$$\widetilde{\mathbb{L}}^{q',p'}_{t,x} \subset \widetilde{\mathbb{L}}^{q,p}_{t,x}, \ \ \widetilde{\mathbb{L}}^{p',q'}_{x,t} \subset \widetilde{\mathbb{L}}^{p,q}_{x,t}.$$

By a finite covering technique, it is easy to see that for any $T, r > 0$ (see [19]),

$$|\!|\!| \mathbf{1}_{[0,T]} f |\!|\!|_{\widetilde{\mathbb{L}}^{q,p}_{t,x}} \asymp \sup_z \|\mathbf{1}_{[0,T] \times B^z_r} f\|_{\mathbb{L}^{q,p}_{t,x}}. \tag{2.5}$$

First of all, we have the following Gagliado-Nirenberge's interpolation estimate.

Lemma 2.1. *Fix $\varkappa \in [2d/(d+2), 2]$ and $\theta \in [0, 1]$ with exception $\theta = 1$ and $\varkappa = d$. For any $r \geqslant 2$ and $s \geqslant 1$ with*

$$\tfrac{1}{2} - \tfrac{1}{r} = \tfrac{\theta}{2}\Big(\tfrac{2}{d} + 1 - \tfrac{2}{\varkappa} \Big), \quad s\theta \leqslant 2,$$

there is a constant $C = C(\varkappa, d, r, \theta) > 0$ such that

$$\|f\|_{\mathbb{L}^{s,r}_{t,x}} \leqslant C \|\nabla f\|^{\theta}_{\mathbb{L}^{2,\varkappa}_{t,x}} \|f\|^{1-\theta}_{\mathbb{L}^{2(1-\theta)s/(2-s\theta),2}_{t,x}}. \tag{2.6}$$

Proof. By Gagliado-Nirenberge's interpolation inequality, we have

$$\|f\|_r \leqslant C \|\nabla f\|^{\theta}_{\varkappa} \|f\|^{1-\theta}_2.$$

Since $s\theta \leqslant 2$, by Hölder's inequality we further have

$$\|f\|_{\mathbb{L}^{s,r}_{t,x}} \leqslant C \|\nabla f\|^{\theta}_{\mathbb{L}^{2,\varkappa}_{t,x}} \|f\|^{1-\theta}_{\mathbb{L}^{2(1-\theta)s/(2-s\theta),2}_{t,x}}.$$

The proof is complete. □

Next for fixed $\varkappa \in [1, 2]$, we introduce the following index set

$$\mathscr{I}_\varkappa := \left\{ (r, s) \in [2, \infty) \times [1, \infty) : \tfrac{1}{2} - \tfrac{1}{r} < \tfrac{1}{s}\big(\tfrac{2}{d} + 1 - \tfrac{2}{\varkappa} \big) \right\}.$$

The following lemma is an easy consequence of (2.6).

Lemma 2.2. *For any $(r,s)\in\mathscr{I}_\varkappa$ and $\varepsilon\in(0,1)$, there are $\beta\in(1,\infty)$ and constant $C_\varepsilon=C_\varepsilon(r,s,\varkappa,d)>0$ such that for any $1\leqslant\tau_1<\tau_2\leqslant 2$,*

$$\|\mathbf{1}_{Q_{\tau_1}}f\|_{\mathbb{L}^{s,r}_{t,x}}\leqslant\varepsilon\|\mathbf{1}_{Q_{\tau_2}}\nabla f\|_{\mathbb{L}^{2,\varkappa}_{t,x}}+C_\varepsilon(\tau_2-\tau_1)^{-1}\|\mathbf{1}_{Q_{\tau_2}}f\|_{\mathbb{L}^{\beta,2}_{t,x}}. \tag{2.7}$$

Proof. Let $\eta\in C^\infty_c(Q_{\tau_2};[0,1])$ with

$$\eta|_{Q_{\tau_1}}=1,\quad |\nabla\eta|\leqslant 2(\tau_2-\tau_1)^{-1}.$$

Since $(r,s)\in\mathscr{I}_\varkappa$, by (2.6), there are $\theta\in[0,\frac{2}{s}\wedge 1)$ such that

$$\|\mathbf{1}_{Q_{\tau_1}}f\|_{\mathbb{L}^{s,r}_{t,x}}\leqslant\|\eta f\|_{\mathbb{L}^{s,r}_{t,x}}\lesssim\|\nabla(\eta f)\|^\theta_{\mathbb{L}^{2,\varkappa}_{t,x}}\|\eta f\|^{1-\theta}_{\mathbb{L}^{2(1-\theta)s/(2-s\theta),2}_{t,x}}.$$

Moreover, we have

$$\|\nabla(\eta f)\|_{\mathbb{L}^{2,\varkappa}_{t,x}}\leqslant\|\nabla\eta f\|_{\mathbb{L}^{2,\varkappa}_{t,x}}+\|\eta\nabla f\|_{\mathbb{L}^{2,\varkappa}_{t,x}}\lesssim(\tau_2-\tau_1)^{-1}\|\mathbf{1}_{Q_{\tau_2}}f\|_{\mathbb{L}^{2,\varkappa}_{t,x}}+\|\mathbf{1}_{Q_{\tau_2}}\nabla f\|_{\mathbb{L}^{2,\varkappa}_{t,x}}.$$

Since $\theta\in[0,1)$ and $s\theta<2$, the desired estimate follows by Young's inequality. □

We need the following simple variational inequality.

Lemma 2.3. *Let $\alpha>1$. For any $\tau<\delta$ and $\gamma>0$, we have for $f\in L^1([\tau,\sigma])$,*

$$\inf_{\ell\in C^1([\tau,\delta])}\left\{\int_\tau^\delta|\ell'(r)|^\alpha|f(r)|\mathrm{d}r:\ell(\tau)=1,\ell(\delta)=0\right\}$$
$$\leqslant(\delta-\tau)^{1-\alpha-\frac{1}{\gamma}}\left(\int_\tau^\delta|f(r)|^\gamma\mathrm{d}r\right)^{\frac{1}{\gamma}}.$$

Proof. The one dimensional variational problem in the lemma is clearly less than

$$\mathcal{J}_\varepsilon(f):=\inf_{\ell\in C^1([\tau,\delta])}\left\{\int_\tau^\delta|\ell'(r)|^\alpha(|f(r)|+\varepsilon)\mathrm{d}r:\ell(\tau)=1,\ell(\delta)=0\right\},\quad\varepsilon>0.$$

Let f_n be the mollifying approximation of f and let us choose

$$\ell(r)=\int_r^\delta(|f_n(s)|+\varepsilon)^{-\frac{1}{\alpha-1}}\mathrm{d}s\left(\int_\tau^\delta(|f_n(s)|+\varepsilon)^{-\frac{1}{\alpha-1}}\mathrm{d}s\right)^{-1}$$

so that

$$\mathcal{J}_\varepsilon(f)\leqslant\left(\int_\tau^\delta(|f_n(s)|+\varepsilon)^{-\frac{\alpha}{\alpha-1}}(|f(s)|+\varepsilon)\mathrm{d}s\right)\left(\int_\tau^\delta(|f_n(s)|+\varepsilon)^{-\frac{1}{\alpha-1}}\mathrm{d}s\right)^{-\alpha}.$$

Taking limits $n\to\infty$, by the dominated convergence theorem, we get

$$\mathcal{J}_\varepsilon(f)\leqslant\left(\int_\tau^\delta(|f(s)|+\varepsilon)^{-\frac{1}{\alpha-1}}\mathrm{d}s\right)^{1-\alpha}.$$

By the inverse Hölder inequality, we obtain that for any $\beta>1$,

$$\mathcal{J}_\varepsilon(f)\leqslant(\delta-\tau)^{-\frac{\beta(\alpha-1)}{\beta-1}}\left(\int_\tau^\delta(|f(r)|+\varepsilon)^{\frac{\beta-1}{\alpha-1}}\mathrm{d}r\right)^{\frac{\alpha-1}{\beta-1}}.$$

The desired estimate follows by letting $\gamma=\frac{\beta-1}{\alpha-1}$ and $\varepsilon\downarrow 0$. □

The following lemma is a time-dependent version of [3, Lemma 2.1].

Lemma 2.4. *Let $w : \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$ be a measurable function with support in $I \times B_2$, where $I \subset \mathbb{R}$ is a finite time interval. Let $p, q \geqslant 1$ and $\alpha > 1/p$. For given $1 \leqslant \tau < \delta \leqslant 2$, consider the following variational problem*

$$\mathcal{J}(w) := \inf\left\{\|w|\nabla\eta|^\alpha\|_{\mathbb{L}^{p,q}_{x,t}} : \eta \in C^1_0(B_\delta), \eta \geqslant 0, \eta = 1 \text{ on } B_\tau\right\}.$$

For $\beta \geqslant 1$ with $\frac{1}{\beta} = \frac{1}{p} + \frac{\theta}{d-1}$, where $\theta \in [0,1]$, there is a constant $C = C(\beta, p, q, d) > 0$ such that

$$\mathcal{J}(w) \leqslant C(\delta-\tau)^{-\alpha-\frac{1}{d-1}}\Big(\|\mathbf{1}_Q\nabla w\|^\theta_{\mathbb{L}^{\beta,q}_{x,t}}\|\mathbf{1}_Q w\|^{1-\theta}_{\mathbb{L}^{\beta,q}_{x,t}} + \|\mathbf{1}_Q w\|_{\mathbb{L}^{\beta,q}_{x,t}}\Big),$$

where $Q := I \times (B_\delta \setminus B_\tau)$.

Proof. Let $p, q \geqslant 1$ and $\alpha > 1/p$. Let $F(x) := \left(\int_I |w(t,x)|^q \mathrm{d}t\right)^{1/q}$. For given radial test function $\eta(x) = \ell(|x|)$, by Fubini's theorem and the transform of polar coordinates, we have

$$\|w|\nabla\eta|^\alpha\|^p_{\mathbb{L}^{p,q}_{x,t}} = \int_{\mathbb{R}^d} F^p|\nabla\eta|^{\alpha p} = \int_\tau^\delta |\ell'(s)|^{\alpha p}\left(\int_{\mathbb{S}_s} F^p\right)\mathrm{d}s,$$

where $\mathbb{S}_s := \{x \in \mathbb{R}^d : |x| = s\}$. Thus, for any $\gamma > 0$, by Lemma 2.3 we have

$$\begin{aligned}\mathcal{J}(w) &\leqslant \inf\left\{\int_\tau^\delta |\ell'(s)|^{\alpha p}\left(\int_{\mathbb{S}_s} F^p\right)\mathrm{d}s : \ell(\tau) = 1, \ell(\delta) = 0\right\}^{1/p}\\ &\leqslant (\delta-\tau)^{\frac{1}{p}-\frac{1}{p\gamma}-\alpha}\left(\int_\tau^\delta\left(\int_{\mathbb{S}_s} F^p\right)^\gamma \mathrm{d}s\right)^{\frac{1}{p\gamma}}. \end{aligned}\tag{2.8}$$

Now for $\beta \geqslant 1$ with $\frac{1}{\beta} = \frac{1}{p} + \frac{\theta}{d-1}$, by the Sobolev embedding in sphere $\mathbb{S}_s$, we have

$$\|F\|_{L^p(\mathbb{S}_s)} \lesssim \|\nabla F\|^\theta_{L^\beta(\mathbb{S}_s)}\|F\|^{1-\theta}_{L^\beta(\mathbb{S}_s)} + \|F\|_{L^\beta(\mathbb{S}_s)}, \quad s \in [1,2].$$

Substituting this into (2.8) and taking $\gamma = \beta/p$, we obtain

$$\begin{aligned}\mathcal{J}(w) &\lesssim (\delta-\tau)^{\frac{1}{p}-\frac{1}{\beta}-\alpha}\left(\int_\tau^\delta\left(\|\nabla F\|^\theta_{L^\beta(\mathbb{S}_s)}\|F\|^{1-\theta}_{L^\beta(\mathbb{S}_s)} + \|F\|_{L^\beta(\mathbb{S}_s)}\right)^\beta \mathrm{d}s\right)^{\frac{1}{\beta}}\\ &\lesssim (\delta-\tau)^{-\frac{\theta}{d-1}-\alpha}\left(\|\nabla F\|^\theta_{L^\beta(B_\delta\setminus B_\tau)}\|F\|^{1-\theta}_{L^\beta(B_\delta\setminus B_\tau)} + \|F\|_{L^\beta(B_\delta\setminus B_\tau)}\right).\end{aligned}\tag{2.9}$$

On the other hand, let

$$F_\varepsilon(x) := \left(\int_I (|w(t,x)|^q + \varepsilon)\mathrm{d}t\right)^{1/q}.$$

By the chain rule and Hölder's inequality, we have

$$|\nabla F_\varepsilon(x)| \leqslant \left(\int_I (|w(t,x)|^q + \varepsilon)\mathrm{d}t\right)^{\frac{1-q}{q}}\int_I |w(t,x)|^{q-1}|\nabla w(t,x)|\mathrm{d}t \leqslant \|\nabla w(\cdot,x)\|_{L^q(I)}.$$

Letting $\varepsilon \downarrow 0$, we obtain

$$|\nabla F(x)| \leqslant \|\nabla w(\cdot,x)\|_{L^q(I)}.$$

Substituting this into (2.9), we obtain the desired estimate. □

Remark 2.5. Suppose that $w : \mathbb{R}^d \to \mathbb{R}$ is time-independent and $\frac{1}{\beta} \leqslant \frac{1}{p} + \frac{1}{d}$. Directly using Sobolev's embedding, we have

$$\mathcal{J}(w) \leqslant (\delta-\tau)^{-\alpha}\|w\|_{L^p(B_\delta)} \leqslant C(\delta-\tau)^{-\alpha}\Big(\|\nabla w\|_{L^\beta(B_\delta)} + \|w\|_{L^\beta(B_\delta)}\Big).$$

However, by Lemma 2.2, we have for $\frac{1}{\beta}\leqslant\frac{1}{p}+\frac{1}{d-1}$,

$$\mathcal{J}(w)\leqslant C(\delta-\tau)^{-\frac{1}{d-1}-\alpha}\Big(\|\nabla w\|_{L^\beta(B_\delta)}+\|w\|_{L^\beta(B_\delta)}\Big),$$

which clearly has better β than the above estimate.

We also need the following iteration lemma (cf. [7, Lemma 4.3]).

Lemma 2.6. *Let $h(\tau)\geqslant 0$ be bounded in $[\tau_1,\tau_2]$ with $\tau_1\geqslant 0$. Let $A,B>0$. Suppose that for some $\alpha\geqslant 0$, $\theta\in(0,1)$ and any $\tau_1\leqslant\tau<\tau'\leqslant\tau_2$,*

$$h(\tau)\leqslant\theta h(\tau')+(\tau'-\tau)^{-\alpha}A+B.$$

Then there is a $C=C(\alpha,\theta)>0$ such that

$$h(\tau_1)\leqslant C((\tau_2-\tau_1)^{-\alpha}A+B).$$

3. Maximum principle for linear parabolic equations

Let $p_0>\frac{d}{2}$ be as in $(\mathbf{H}^a)$. We define $\varkappa\in[1,2]$ by

$$\frac{2}{\varkappa}=\frac{1}{p_0}+1. \tag{3.1}$$

For a set $Q\subset\mathbb{R}^{d+1}$, we also introduce

$$\mathscr{V}_Q:=\Big\{f\in\mathbb{L}^1_{loc}:\|f\|_{\mathscr{V}_Q}:=\|\mathbf{1}_Qf\|_{\mathbb{L}^{\infty,2}_{t,x}}+\|\mathbf{1}_Q\nabla_xf\|_{\mathbb{L}^{\varkappa,2}_{x,t}}<\infty\Big\}$$

and

$$\widetilde{\mathscr{V}}:=\Big\{f\in\mathbb{L}^1_{loc}:|\!|\!|f|\!|\!|_{\widetilde{\mathscr{V}}}:=|\!|\!|f|\!|\!|_{\widetilde{\mathbb{L}}^{\infty,2}_{t,x}}+|\!|\!|\nabla_xf|\!|\!|_{\widetilde{\mathbb{L}}^{\varkappa,2}_{x,t}}<\infty\Big\}. \tag{3.2}$$

3.1. **Energy type estimate.** In this subsection we fix $1\leqslant\tau_1<\tau_2\leqslant 2$ and

$$Q_i:=Q_{\tau_i}=[-\tau_i^2,\tau_i^2]\times B_{\tau_i},\ i=1,2.$$

Let $\mathscr{C}$ be the class of all functions $\eta\in C_c^\infty(Q_2;[0,1])$ with

$$\eta|_{Q_1}=1,\ \ \eta|_{Q_2^c}=0,\ \|\nabla\eta\|_\infty+\|\partial_t\eta\|_\infty\leqslant 4(\tau_2-\tau_1)^{-1}. \tag{3.3}$$

We first prepare the following important variational estimate.

Lemma 3.1. *For any $p\in[1,\infty]$ with $\frac{1}{p_0}+\frac{1}{p}<\frac{2}{d-1}$, there are $\gamma_0,\gamma_1,\gamma_2>0$ and constant $C>0$ only depending on p,d,p_0 such that for any $\varepsilon\in(0,1)$, $w\in\mathscr{V}_{Q_2}$ and $g\in\mathbb{L}^{p,\infty}_{x,t}(Q_2)$,*

$$\inf_{\eta\in\mathscr{C}}\|gw^2\nabla\eta\|_{\mathbb{L}^{1,1}_{t,x}}\lesssim_C\varepsilon\|w\|^2_{\mathscr{V}_{Q_2}}+\frac{\|\mathbf{1}_{Q_2}g\|^{\gamma_2}_{\mathbb{L}^{p,\infty}_{x,t}}+1}{\varepsilon^{\gamma_0}(\tau_2-\tau_1)^{\gamma_1}}\|\mathbf{1}_{Q_2}w\|^2_{\mathbb{L}^{2,2}_{t,x}}. \tag{3.4}$$

Proof. Let $\bar p=\frac{p}{p-1}$. By Hölder's inequality, we have

$$\|gw^2\nabla\eta\|_{\mathbb{L}^{1,1}_{t,x}}\leqslant\|\mathbf{1}_{Q_2}g\|_{\mathbb{L}^{p,\infty}_{x,t}}\|w^2\nabla\eta\|_{\mathbb{L}^{\bar p,1}_{x,t}}=\|\mathbf{1}_{Q_2}g\|_{\mathbb{L}^{p,\infty}_{x,t}}\|w|\nabla\eta|^{\frac12}\|^2_{\mathbb{L}^{2\bar p,2}_{x,t}}. \tag{3.5}$$

Since $\frac{1}{p_0}+\frac{1}{p}<\frac{2}{d-1}$, we have for some $\theta\in(0,1)$,

$$\frac{1}{2p_0}+\frac12=\frac1\chi=\frac{1}{2\bar p}+\frac{\theta}{d-1}.$$

Thus by Lemma 2.4, we have

$$\inf_{\eta\in\mathscr{C}}\|w|\nabla\eta|^{\frac12}\|_{\mathbb{L}^{2\bar p,2}_{x,t}}\lesssim(\tau_2-\tau_1)^{-\frac{d+1}{2(d-1)}}\Big(\|\mathbf{1}_{Q_2}\nabla w\|^\theta_{\mathbb{L}^{\chi,2}_{x,t}}\|\mathbf{1}_{Q_2}w\|^{1-\theta}_{\mathbb{L}^{\chi,2}_{x,t}}+\|\mathbf{1}_{Q_2}w\|_{\mathbb{L}^{\chi,2}_{x,t}}\Big).$$

Substituting this into (3.5), we obtain

$$\begin{aligned}\inf_{\eta\in\mathscr{C}}\|gw^2\nabla\eta\|_{\mathbb{L}^{1,1}_{t,x}}&\lesssim\|\mathbf{1}_{Q_2}g\|_{\mathbb{L}^{p,\infty}_{x,t}}(\tau_2-\tau_1)^{-\frac{d+1}{d-1}}\|\mathbf{1}_{Q_2}\nabla w\|^{2\theta}_{\mathbb{L}^{\chi,2}_{x,t}}\|\mathbf{1}_{Q_2}w\|^{2(1-\theta)}_{\mathbb{L}^{\chi,2}_{x,t}}\\&\quad+\|\mathbf{1}_{Q_2}g\|_{\mathbb{L}^{p,\infty}_{x,t}}(\tau_2-\tau_1)^{-\frac{d+1}{d-1}}\|\mathbf{1}_{Q_2}w\|^2_{\mathbb{L}^{\chi,2}_{x,t}}.\end{aligned}$$

By Young's inequality and $\|\mathbf{1}_{Q_2}w\|_{\mathbb{L}^{\chi,2}_{x,t}} \leqslant C_{d,\chi}\|\mathbf{1}_{Q_2}w\|_{\mathbb{L}^{2,2}_{t,x}}$, we obtain (3.4). □

Recall Θ being the parameter set (1.9). Now we prove the following local energy estimate by Lemma 3.1.

Lemma 3.2. *Under* $(\mathbf{H}^a)$ *and* $(\mathbf{H}^b)$*, for any* $f \in \mathbb{L}^{q_4,p_4}_{t,x}(Q_2)$ *with* $(p_4, q_4) \in \mathbb{I}^d_{p_0}$*, there are* $(r_i, s_i) \in \mathscr{I}_\varkappa$*,* $i = 1, 2, 3, 4$*,* $\gamma = \gamma(p_0, p_1, d) \geqslant 1$ *and constant* $C = C(\Theta) > 0$ *such that for any Lipschitz weak subsolution* u *of PDE* (1.4) *and* $t \geqslant 0$*,*

$$\|w\mathcal{I}^t\|^2_{\mathscr{V}_{Q_1}} \lesssim_C (\tau_2 - \tau_1)^{-\gamma} \sum_{i=1,2,3} \|\mathbf{1}_{Q_2}w\mathcal{I}^t\|^2_{\mathbb{L}^{s_i,r_i}_{t,x}} + \|f\mathbf{1}_{Q_2}\|^2_{\mathbb{L}^{q_4,p_4}_{t,x}}\|\mathbf{1}_{\{w\neq 0\}\cap Q_2}\mathcal{I}^t\|^2_{\mathbb{L}^{s_4,r_4}_{t,x}},$$

where $\mathcal{I}^t(\cdot) := \mathbf{1}_{(-\infty,t]}(\cdot)$ *and* $w := (u - \kappa)^+$ *and* $\kappa \geqslant 0$.

Proof. We divide the proof into three steps.

(i) Fix $\eta \in \mathscr{C}$ (see (3.3)). In this step we show that for all $t \in \mathbb{R}$,

$$\begin{aligned}\|(\eta w)(t)\|_2^2 &\leqslant \langle\!\langle \partial_s\eta^2, w^2\mathcal{I}^t\rangle\!\rangle - 2\langle\!\langle a\cdot\nabla u, \nabla(\eta^2 w)\mathcal{I}^t\rangle\!\rangle \\ &\quad + 2\langle\!\langle b\cdot\nabla u, \eta^2 w\mathcal{I}^t\rangle\!\rangle + 2\langle\!\langle f, \eta^2 w\mathcal{I}^t\rangle\!\rangle.\end{aligned} \tag{3.6}$$

Since we want to take the test function $\varphi = w\eta^2$ in (1.6), and $\partial_s u$ only makes sense in the distributional sense, we shall first approximate u by its Steklov mean:

$$S_h u(t,x) := \frac{1}{h}\int_0^h u(t+s,x)\mathrm{d}s = \frac{1}{h}\int_t^{t+h} u(s,x)\mathrm{d}s, \quad h \in (0,1). \tag{3.7}$$

Let $u_h := S_h u$ and S_h^* be the adjoint operator of S_h. Let φ be a nonnegative Lipschitz function in $\mathbb{R}^{d+1}$ with compact support in Q_2. By Definition 1.1 with test function $S_h^*\varphi$, using integration by parts and Fubini's theorem, one sees that

$$\langle\!\langle \partial_s u_h, \varphi\rangle\!\rangle \leqslant -\langle\!\langle S_h(a\cdot\nabla u), \nabla\varphi\rangle\!\rangle + \langle\!\langle S_h(b\cdot\nabla u), \varphi\rangle\!\rangle + \langle\!\langle f_h, \varphi\rangle\!\rangle. \tag{3.8}$$

Now fix $t \in \mathbb{R}$ and define

$$\zeta^\varepsilon_t(s) = \mathbf{1}_{(-\infty,t]}(s) + (1 - \varepsilon^{-1}(s-t))\mathbf{1}_{(t,t+\varepsilon]}(s), \ \varepsilon \in (0,1).$$

Let $w_h := (u_h - k)^+$. Note that

$$\begin{aligned}2\langle \partial_s u_h, w_h\eta^2\zeta^\varepsilon_t\rangle &= 2\langle \partial_s w_h, w_h\eta^2\zeta^\varepsilon_t\rangle \\ &= \partial_s\langle w_h^2, \eta^2\zeta^\varepsilon_t\rangle - \int_{\mathbb{R}^d} w_h^2\eta^2(\zeta^\varepsilon_t)' - \int_{\mathbb{R}^d} w_h^2(\partial_s\eta^2\zeta^\varepsilon_t).\end{aligned}$$

By (3.8) with $\varphi = w_h\eta^2\zeta_{t,\varepsilon}$ and $\int_{\mathbb{R}^{d+1}} \partial_s\langle w_h^2, \eta^2\zeta^\varepsilon_t\rangle = 0$, we get

$$\begin{aligned}-\int_{\mathbb{R}^{d+1}} \eta^2 w_h^2(\zeta^\varepsilon_t)' &\leqslant \int_{\mathbb{R}^{d+1}} w_h^2(\partial_s\eta^2\zeta^\varepsilon_t) - 2\langle\!\langle S_h(a\cdot\nabla u), \nabla(w_h\eta^2\zeta^\varepsilon_t)\rangle\!\rangle \\ &\quad + 2\langle\!\langle S_h(b\cdot\nabla u), w_h\eta^2\zeta^\varepsilon_t\rangle\!\rangle + 2\langle\!\langle f_h, w_h\eta^2\zeta^\varepsilon_t\rangle\!\rangle.\end{aligned}$$

Letting $h \downarrow 0$ and by the dominated convergence theorem, we obtain

$$\begin{aligned}-\int_{\mathbb{R}^{d+1}} (\eta w)^2(\zeta^\varepsilon_t)' &\leqslant \int_{\mathbb{R}^{d+1}} w^2(\partial_t\eta^2\zeta^\varepsilon_t) - 2\langle\!\langle a\cdot\nabla u, \nabla(w\eta^2\zeta^\varepsilon_t)\rangle\!\rangle \\ &\quad + 2\langle\!\langle b\cdot\nabla u, w\eta^2\zeta^\varepsilon_t\rangle\!\rangle + 2\langle\!\langle f, w\eta^2\zeta^\varepsilon_t\rangle\!\rangle.\end{aligned}$$

Since $\lim_{\varepsilon\downarrow 0}\zeta^\varepsilon_t(s) = \mathcal{I}^t(s)$ for each $s \in \mathbb{R}$, the right hand side of the above inequality converges to the right hand side of (3.6) as $\varepsilon \downarrow 0$. On the other hand, by the Lebesgue differential theorem, we also have

$$-\int_{\mathbb{R}^{d+1}} (\eta w)^2(\zeta^\varepsilon_t)' = \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \|(\eta w)(s)\|_2^2\mathrm{d}s \overset{\varepsilon\downarrow 0}{\to} \|(\eta w)(t)\|_2^2.$$

Thus, we obtain (3.6).

(ii) Recalling $w=(u-\kappa)^+$ and noting that

$$\nabla u\cdot\nabla w=|\nabla w|^2,\ (\nabla u)w=(\nabla w)w=\nabla w^2/2, \tag{3.9}$$

by the chain rule and Young's inequality, we have

$$\begin{aligned}-\langle a\cdot\nabla u,\nabla(\eta^2 w)\rangle&=-\int_{\mathbb{R}^d}\eta^2(\nabla w)^*a\nabla w-2\int_{\mathbb{R}^d}\eta w(\nabla\eta)^*a\nabla w\\&\leqslant-\frac12\int_{\mathbb{R}^d}\eta^2(\nabla w)^*a\nabla w+4\int_{\mathbb{R}^d}w^2|\nabla\eta|^2\frac{|a\nabla w|^2}{(\nabla w)^*a\nabla w}\\&\overset{(1.5)}{\leqslant}-\frac12\int_{\mathbb{R}^d}\eta^2|\nabla w|^2\lambda+4\int_{\mathbb{R}^d}w^2|\nabla\eta|^2\mu,\end{aligned}$$

which in turn gives that

$$-\langle\!\langle a\cdot\nabla u,\nabla(\eta^2 w)\mathcal{I}^t\rangle\!\rangle\leqslant-\tfrac12\|\eta\nabla w\lambda^{\frac12}\mathcal{I}^t\|^2_{\mathbb{L}^{2,2}_{x,t}}+4\|w\nabla\eta\mu^{\frac12}\mathcal{I}^t\|^2_{\mathbb{L}^{2,2}_{t,x}}. \tag{3.10}$$

Due to $b=b_1+b_2$, by (3.9) and the integration by parts, we have

$$\begin{aligned}\langle\!\langle b\cdot\nabla u,\eta^2 w\mathcal{I}^t\rangle\!\rangle&=\langle\!\langle \eta b_1\cdot\nabla w\lambda^{\frac12},\lambda^{-\frac12}\eta w\mathcal{I}^t\rangle\!\rangle+\tfrac12\langle\!\langle b_2\cdot\nabla w^2,\eta^2\mathcal{I}^t\rangle\!\rangle\\&\leqslant\tfrac14\|\eta\nabla w\lambda^{\frac12}\mathcal{I}^t\|^2_{\mathbb{L}^{2,2}_{t,x}}+4\|\lambda^{-\frac12}b_1\eta w\mathcal{I}^t\|^2_{\mathbb{L}^{2,2}_{t,x}}\\&\quad-\langle\!\langle\eta b_2\cdot\nabla\eta,w^2\mathcal{I}^t\rangle\!\rangle-\tfrac12\langle\!\langle \mathrm{div} b_2 w^2,\eta^2\mathcal{I}^t\rangle\!\rangle.\end{aligned}$$

Let (r_2,s_2), (r_3,s_3) and (r_4,s_4) be defined by

$$\tfrac{1}{2p_0}+\tfrac{1}{p_2}+\tfrac{1}{r_2}=\tfrac12,\quad\tfrac{1}{q_2}+\tfrac{1}{s_2}=\tfrac12$$

and

$$\tfrac{1}{p_3}+\tfrac{2}{r_3}=1,\quad\tfrac{1}{q_3}+\tfrac{2}{s_3}=1,\quad\tfrac{1}{p_4}+\tfrac{1}{r_4}=\tfrac12,\quad\tfrac{1}{q_4}+\tfrac{1}{s_4}=1.$$

Since (p_2,q_2) satisfies (1.8) and $(p_3,q_3),(p_4,q_4)\in\mathbb{I}^d_{p_0}$, one sees that

$$(r_2,s_2),(r_3,s_3),(r_4,s_4)\in\mathscr{I}_\varkappa.$$

Thus by Hölder's inequality and Young's inequality, we further have

$$\begin{aligned}\langle\!\langle b\cdot\nabla u,\eta^2 w\mathcal{I}^t\rangle\!\rangle&\leqslant\tfrac14\|\eta\nabla w\lambda^{\frac12}\mathcal{I}^t\|^2_{\mathbb{L}^{2,2}_{t,x}}+4\|\lambda^{-1}\mathbf{1}_{B_2}\|_{p_0}\|b_1\mathbf{1}_{Q_2}\|^2_{\mathbb{L}^{q_2,p_2}_{t,x}}\|\eta w\mathcal{I}^t\|^2_{\mathbb{L}^{s_2,r_2}_{t,x}}\\&\quad+\|b_2\nabla\eta w^2\mathcal{I}^t\|_{\mathbb{L}^{1,1}_{t,x}}+\|(\mathrm{div} b_2)^-\mathbf{1}_{Q_2}\|_{\mathbb{L}^{q_3,p_3}_{t,x}}\|\eta w\mathcal{I}^t\|^2_{\mathbb{L}^{s_3,r_3}_{t,x}}\\&\leqslant\tfrac14\|\eta\nabla w\lambda^{\frac12}\mathcal{I}^t\|^2_{\mathbb{L}^{2,2}_{t,x}}+4|\!|\!|\lambda^{-1}|\!|\!|_{p_0}|\!|\!|b_1|\!|\!|^2_{\widetilde{\mathbb{L}}^{q_2,p_2}_{t,x}}\|\eta w\mathcal{I}^t\|^2_{\mathbb{L}^{s_2,r_2}_{t,x}}\\&\quad+\|b_2\nabla\eta w^2\mathcal{I}^t\|_{\mathbb{L}^{1,1}_{t,x}}+\tfrac12|\!|\!|(\mathrm{div} b_2)^-|\!|\!|_{\widetilde{\mathbb{L}}^{q_3,p_3}_{t,x}}\|\eta w\mathcal{I}^t\|^2_{\mathbb{L}^{s_3,r_3}_{t,x}},\end{aligned} \tag{3.11}$$

and also,

$$\begin{aligned}\langle\!\langle f,\eta^2 w\mathcal{I}^t\rangle\!\rangle&\leqslant\|f\eta\|_{\mathbb{L}^{q_4,p_4}_{t,x}}\|\eta w\mathcal{I}^t\|_{\mathbb{L}^{\infty,2}_{t,x}}\|\mathbf{1}_{\{w\neq0\}\cap Q_2}\mathcal{I}^t\|_{\mathbb{L}^{s_4,r_4}_{t,x}}\\&\leqslant\tfrac14\|\eta w\mathcal{I}^t\|^2_{\mathbb{L}^{\infty,2}_{t,x}}+4\|f\eta\|^2_{\mathbb{L}^{q_4,p_4}_{t,x}}\|\mathbf{1}_{\{w\neq0\}\cap Q_2}\mathcal{I}^t\|^2_{\mathbb{L}^{s_4,r_4}_{t,x}}.\end{aligned} \tag{3.12}$$

(iii) Combining (3.6) and (3.10)-(3.12), we obtain

$$\begin{aligned}\|(\eta w)(t)\|_2^2+\tfrac12\|\eta\nabla w\lambda^{\frac12}\mathcal{I}^t\|^2_{\mathbb{L}^{2,2}_{t,x}}&\leqslant2\|\partial_s\eta w^2\mathcal{I}^t\|_{\mathbb{L}^{1,1}_{t,x}}+8\|w\nabla\eta\mu^{\frac12}\mathcal{I}^t\|^2_{\mathbb{L}^{2,2}_{t,x}}\\&\quad+C\|\eta w\mathcal{I}^t\|^2_{\mathbb{L}^{s_2,r_2}_{t,x}}+C\|\eta w\mathcal{I}^t\|^2_{\mathbb{L}^{s_3,r_3}_{t,x}}+2\|b_2\nabla\eta w^2\mathcal{I}^t\|_{\mathbb{L}^{1,1}_{t,x}}\\&\quad+\tfrac12\|\eta w\mathcal{I}^t\|^2_{\mathbb{L}^{\infty,2}_{t,x}}+8\|f\eta\|^2_{\mathbb{L}^{q_4,p_4}_{t,x}}\|\mathbf{1}_{\{w\neq0\}\cap Q_2}\mathcal{I}^t\|^2_{\mathbb{L}^{s_4,r_4}_{t,x}}.\end{aligned}$$

Furthermore, we have for some $C=C(\Theta)>0$,

$$\begin{aligned}\|\eta w\mathcal{I}^t\|^2_{\mathbb{L}^{\infty,2}_{t,x}}+\|\eta\nabla w\lambda^{\frac12}\mathcal{I}^t\|^2_{\mathbb{L}^{2,2}_{t,x}}&\lesssim_C\|\partial_s\eta w^2\mathcal{I}^t\|_{\mathbb{L}^{1,1}_{t,x}}+\|w\nabla\eta\mu^{\frac12}\mathcal{I}^t\|^2_{\mathbb{L}^{2,2}_{t,x}}\\&\quad+\|b_2\nabla\eta w^2\mathcal{I}^t\|_{\mathbb{L}^{1,1}_{t,x}}+\|\eta w\mathcal{I}^t\|^2_{\mathbb{L}^{s_2,r_2}_{t,x}}+\|\eta w\mathcal{I}^t\|^2_{\mathbb{L}^{s_3,r_3}_{t,x}}\end{aligned}$$

$$+\|f\eta\|^2_{\mathbb{L}^{q_4,p_4}_{t,x}}\|\mathbf{1}_{\{w\neq0\}\cap Q_2}\mathcal{I}^t\|^2_{\mathbb{L}^{s_4,r_4}_{t,x}}. \tag{3.13}$$

Now since $\eta|_{Q_1}=1$ and $\eta|_{Q_2^c}=0$, by (3.1) and Hölder's inequality, we have

$$\|\mathbf{1}_{Q_1}\nabla w\mathcal{I}^t\|^2_{\mathbb{L}^{\varkappa,2}_{x,t}}\leqslant\|\eta\nabla w\mathcal{I}^t\|^2_{\mathbb{L}^{\varkappa,2}_{x,t}}\leqslant\|\lambda^{-1}\|_{L^{p_0}(B_{\tau_2})}\|\eta\nabla w\lambda^{\frac12}\mathcal{I}^t\|^2_{\mathbb{L}^{2,2}_{x,t}},$$

and by (3.3),

$$\|\partial_s\eta w^2\mathcal{I}^t\|_{\mathbb{L}^{1,1}_{t,x}}\leqslant C(\tau_2-\tau_1)^{-1}\|\mathbf{1}_{Q_2}w\mathcal{I}^t\|^2_{\mathbb{L}^{2,2}_{t,x}}.$$

Substituting these into (3.13), we obtain that for any $\eta\in\mathscr{C}$,

$$\begin{aligned}\|\mathbf{1}_{Q_1}w\mathcal{I}^t\|^2_{\mathbb{L}^{\infty,2}_{t,x}}+\|\mathbf{1}_{Q_1}\nabla w\mathcal{I}^t\|^2_{\mathbb{L}^{\varkappa,2}_{x,t}}&\leqslant\|\eta w\mathcal{I}^t\|^2_{\mathbb{L}^{\infty,2}_{t,x}}+\|\lambda^{-1}\|_{p_0}\|\eta\nabla w\lambda^{\frac12}\mathcal{I}^t\|^2_{\mathbb{L}^{2,2}_{t,x}}\\&\lesssim_C(\tau_2-\tau_1)^{-1}\|\mathbf{1}_{Q_2}w\mathcal{I}^t\|^2_{\mathbb{L}^{2,2}_{t,x}}+\|(\mu|\nabla\eta|+|b_2|)w^2|\nabla\eta|\mathcal{I}^t\|_{\mathbb{L}^{1,1}_{t,x}}\\&\quad+\sum_{i=2,3}\|\mathbf{1}_{Q_2}w\mathcal{I}^t\|^2_{\mathbb{L}^{s_i,r_i}_{t,x}}+\|f\mathbf{1}_{Q_2}\|^2_{\mathbb{L}^{q_4,p_4}_{t,x}}\|\mathbf{1}_{\{w\neq0\}\cap Q_2}\mathcal{I}^t\|^2_{\mathbb{L}^{s_4,r_4}_{t,x}}.\end{aligned}\tag{3.14}$$

Note that by (3.3) and the assumptions,

$$\mu|\nabla\eta|+|b_2|\leqslant(\tau_2-\tau_1)^{-1}(2\mu+4|b_2|)=:(\tau_2-\tau_1)^{-1}g\in\widetilde{\mathbb{L}}^{p_1,\infty}_{x,t}. \tag{3.15}$$

By (1.7) and Lemma 3.1, there are $\gamma_0,\gamma_1>0$ such that for all $t\geqslant0$ and $\varepsilon\in(0,1)$,

$$\inf_{\eta\in\mathscr{C}}\|gw^2|\nabla\eta|\mathcal{I}^t\|_{\mathbb{L}^{1,1}_{t,x}}\leqslant\varepsilon\|w\mathcal{I}^t\|^2_{\mathscr{V}_{Q_2}}+C\varepsilon^{-\gamma_0}(\tau_2-\tau_1)^{-\gamma_1}\|\mathbf{1}_{Q_2}w\mathcal{I}^t\|^2_{\mathbb{L}^{2,2}_{t,x}}. \tag{3.16}$$

Let $(r_1,s_1)=(2,2)\in\mathscr{I}_\chi$. By (3.14), (3.15) and (3.16), there are $\gamma=\gamma(p_0,p_1,d)\geqslant1$ and $C>0$ such that for all $1\leqslant\tau_1<\tau_2\leqslant2$,

$$\begin{aligned}\|w\mathcal{I}^t\|^2_{\mathscr{V}_{Q_1}}\leqslant\tfrac12\|w\mathcal{I}^t\|^2_{\mathscr{V}_{Q_2}}+C(\tau_2-\tau_1)^{-\gamma}\sum_{i=1,2,3}\|\mathbf{1}_{Q_2}w\mathcal{I}^t\|^2_{\mathbb{L}^{s_i,r_i}_{t,x}}\\+\|f\mathbf{1}_{Q_2}\|^2_{\mathbb{L}^{q_4,p_4}_{t,x}}\|\mathbf{1}_{\{w\neq0\}\cap Q_2}\mathcal{I}^t\|^2_{\mathbb{L}^{s_4,r_4}_{t,x}}.\end{aligned}$$

Recall $Q_i=Q_{\tau_i}$ for $i=1,2$. If we let $h(\tau):=\|w\mathcal{I}^t\|^2_{\mathscr{V}_{Q_\tau}}$, then the above inequality implies that for any $\tau_1\leqslant\tau<\tau'\leqslant\tau_2$,

$$\begin{aligned}h(\tau)\leqslant\tfrac12h(\tau')+C(\tau'-\tau)^{-\gamma}\sum_{i=1,2,3}\|\mathbf{1}_{Q_{\tau_2}}w\mathcal{I}^t\|^2_{\mathbb{L}^{s_i,r_i}_{t,x}}\\+\|f\mathbf{1}_{Q_{\tau_2}}\|^2_{\mathbb{L}^{q_4,p_4}_{t,x}}\|\mathbf{1}_{\{w\neq0\}\cap Q_{\tau_2}}\mathcal{I}^t\|^2_{\mathbb{L}^{s_4,r_4}_{t,x}}.\end{aligned}$$

The desired estimate now follows by Lemma 2.6. □

3.2. **Local maximum estimate.** The following lemma is easy by Hölder's inequality.

Lemma 3.3. *Let $Q=I\times D\subset\mathbb{R}\times\mathbb{R}^d$ be a bounded domain. For any $p,q\in[1,\infty)$, there are constants $C_1,C_2>0$ only depending on Q,p,q such that for any $A\subset Q$,*

$$\|\mathbf{1}_A\|_{\mathbb{L}^{q,p}_{t,x}}+\|\mathbf{1}_A\|_{\mathbb{L}^{p,q}_{x,t}}\leqslant C_1\|\mathbf{1}_A\|^{1/(p\vee q)}_{\mathbb{L}^{1,1}_{t,x}}\leqslant C_2\big(\|\mathbf{1}_A\|_{\mathbb{L}^{q,p}_{t,x}}+\|\mathbf{1}_A\|_{\mathbb{L}^{p,q}_{x,t}}\big)^{1/(p\vee q)}.$$

We need the following simple De-Giorgi's iteration lemma.

Lemma 3.4. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers. Suppose that for some $C_0,\lambda>1$ and $\delta_j>0$, $j=1,\cdots,m$,*

$$a_{n+1}\leqslant C_0\lambda^na_n\sum_{j=1}^ma_n^{\delta_j},\quad n=1,2,\cdots.$$

If $a_1\leqslant(mC_0\lambda^{(1+\delta)/\delta})^{-1/\delta}$, where $\delta=\delta_1\wedge\cdots\wedge\delta_m$, then

$$\lim_{n\to\infty}a_n=0.$$

Proof. We use induction to prove that if $a_1 \leqslant (mC_0\lambda^{(1+\delta)/\delta})^{-1/\delta} \leqslant 1$, then

$$a_n \leqslant a_1\lambda^{-(n-1)/\delta}, \quad \forall n \in \mathbb{N}.$$

By the induction hypothesis, we have

$$\begin{aligned} a_{n+1} &\leqslant mC_0\lambda^n a_n^{1+\delta} \leqslant mC_0 a_1^{1+\delta}\lambda^{n-(n-1)(1+\delta)/\delta} \\ &= \big(mC_0 a_1^\delta \lambda^{(1+\delta)/\delta}\big) a_1\lambda^{-n/\delta} \leqslant a_1\lambda^{-n/\delta}, \end{aligned}$$

where the last step is due to $mC_0 a_1^\delta\lambda^{(1+\delta)/\delta} \leqslant 1$. □

Lemma 3.5. *Let $1 \leqslant \tau_1 < \tau_0 \leqslant 2$ and $0 < \kappa_0 < \kappa_1$. Define*

$$\Gamma_i := Q_{\tau_i}, \quad w_i := (u - \kappa_i)^+, \quad i = 0, 1.$$

(i) For any $r, s \in [1, \infty]$, we have

$$\|\mathbf{1}_{\{w_1\neq 0\}\cap\Gamma_0}\|_{\mathbb{L}^{s,r}_{t,x}} \leqslant \|w_0\mathbf{1}_{\Gamma_0}\|_{\mathbb{L}^{s,r}_{t,x}}/(\kappa_1 - \kappa_0). \tag{3.17}$$

(ii) For any $r \in [1, \varkappa)$ and $s \in [1, 2)$, there is a universal constant $C > 0$ such that

$$\|\mathbf{1}_{\Gamma_0}\nabla w_1\|_{\mathbb{L}^{r,s}_{x,t}} \leqslant C\|w_1\|_{\mathscr{V}_{\Gamma_0}}\Big(\tfrac{\|w_0\mathbf{1}_{\Gamma_0}\|_{\mathbb{L}^{1,1}_{t,x}}}{\kappa_1-\kappa_0}\Big)^{(\frac{1}{r}-\frac{1}{\varkappa})\wedge(\frac{1}{s}-\frac{1}{2})}. \tag{3.18}$$

(iii) For any $(r, s) \in \mathscr{I}_\varkappa$, there are $\delta \in (0, 1)$ and $C = C(r, s, d, \varkappa) > 0$ such that

$$\|\mathbf{1}_{\Gamma_1} w_1\|_{\mathbb{L}^{s,r}_{t,x}} \leqslant C(\tau_0 - \tau_1)^{-1}\|w_1\|_{\mathscr{V}_{\Gamma_0}} \cdot \Big(\tfrac{\|w_0\mathbf{1}_{\Gamma_0}\|_{\mathbb{L}^{1,1}_{t,x}}}{\kappa_1-\kappa_0}\Big)^\delta. \tag{3.19}$$

Proof. (i) Noting that

$$w_0|_{w_1\neq 0} = (u - \kappa_1 + \kappa_1 - \kappa_0)^+|_{w_1\neq 0} \geqslant \kappa_1 - \kappa_0,$$

for given $r, s \in [1, \infty]$, we have

$$\|w_0\mathbf{1}_{\Gamma_0}\|_{\mathbb{L}^{s,r}_{t,x}} \geqslant \|w_0\mathbf{1}_{\{w_1\neq 0\}\cap\Gamma_0}\|_{\mathbb{L}^{s,r}_{t,x}} \geqslant (\kappa_1 - \kappa_0)\|\mathbf{1}_{\{w_1\neq 0\}\cap\Gamma_0}\|_{\mathbb{L}^{s,r}_{t,x}},$$

which implies (3.17).

(ii) Let $\frac{1}{r} = \frac{1}{\varkappa} + \frac{1}{r'}$ and $\frac{1}{s} = \frac{1}{2} + \frac{1}{s'}$. By Hölder's inequality, we have

$$\begin{aligned} \|\mathbf{1}_{\Gamma_0}\nabla w_1\|_{\mathbb{L}^{r,s}_{x,t}} &= \|\mathbf{1}_{\Gamma_0\cap\{w_1\neq 0\}}\nabla w_1\|_{\mathbb{L}^{r,s}_{x,t}} \leqslant \|\mathbf{1}_{\Gamma_0}\nabla w_1\|_{\mathbb{L}^{\varkappa,2}_{x,t}}\|\mathbf{1}_{\Gamma_0\cap\{w_1\neq 0\}}\|_{\mathbb{L}^{r',s'}_{x,t}} \\ &\lesssim \|w_1\|_{\mathscr{V}_{\Gamma_0}}\|\mathbf{1}_{\Gamma_0\cap\{w_1\neq 0\}}\|^{1/(s'\vee r')}_{\mathbb{L}^{1,1}_{t,x}}, \end{aligned}$$

which implies (3.18) by (3.17).

(iii) Since $\frac{1}{2} - \frac{1}{r} < \frac{1}{s}(\frac{2}{d} - \frac{1}{p_0})$, we can choose $r', \beta > r$ and $s', \theta > s$ so that

$$\tfrac{1}{r'} + \tfrac{1}{\beta} = \tfrac{1}{r}, \quad \tfrac{1}{s'} + \tfrac{1}{\theta} = \tfrac{1}{s}, \quad \tfrac{1}{2} - \tfrac{1}{r'} < \tfrac{1}{s'}(\tfrac{2}{d} - \tfrac{1}{p_0}).$$

By Hölder's inequality, we have

$$\|w_1\mathbf{1}_{\Gamma_1}\|_{\mathbb{L}^{s,r}_{t,x}} \leqslant \|w_1\mathbf{1}_{\Gamma_1}\|_{\mathbb{L}^{s',r'}_{t,x}}\|\mathbf{1}_{\{w_1\neq 0\}\cap\Gamma_1}\|_{\mathbb{L}^{\theta,\beta}_{t,x}},$$

and by Lemma 2.2 and (2.2),

$$\begin{aligned} \|w_1\mathbf{1}_{\Gamma_1}\|_{\mathbb{L}^{s',r'}_{t,x}} &\lesssim_C \|\mathbf{1}_{\Gamma_0}\nabla w_1\|_{\mathbb{L}^{2,\varkappa}_{t,x}} + (\tau_0 - \tau_1)^{-1}\|\mathbf{1}_{\Gamma_0} w_1\|_{\mathbb{L}^{\infty,2}_{t,x}} \\ &\lesssim_C (\tau_0 - \tau_1)^{-1}\|w_1\|_{\mathscr{V}_{\Gamma_0}}, \end{aligned}$$

which in turn yields (3.19) by Lemma 3.3 and (3.17). □

Now we can show the following local maximum principle for PDE (1.4).

Theorem 3.6 (Local maximum estimate)**.** *Under the assumption of Lemma 3.2, for any $p>0$, there is a constant $C=C(p,\Theta)>0$ such that for any Lipschitz weak subsolution u of PDE* (1.4),

$$\|u^+\mathbf{1}_{Q_1}\|_\infty \leqslant C\left(\|u^+\mathbf{1}_{Q_2}\|_{\mathbb{L}^{p,p}_{t,x}}+\|f\mathbf{1}_{Q_2}\|_{\mathbb{L}^{q_4,p_4}_{t,x}}\right), \tag{3.20}$$

where $Q_1:=[-1,1]\times B_1$ and $Q_2:=[-4,4]\times B_2$.

Proof. Fix $1\leqslant\tau<\sigma\leqslant 2$. Let $\kappa>0$, which will be determined below. For $n\in\mathbb{N}$, define

$$\tau_n=\tau+(\sigma-\tau)2^{1-n},\quad \tilde\tau_n:=\tau+3(\sigma-\tau)2^{-n-1},\quad \kappa_n:=\kappa\left(1-2^{1-n}\right)$$

and

$$w_n:=(u-\kappa_n)^+,\quad \Gamma_n:=(-\tau_n^2,\tau_n^2)\times B_{\tau_n},\tilde\Gamma_n:=(-\tilde\tau_n^2,\tilde\tau_n^2)\times B_{\tilde\tau_n}.$$

Clearly,

$$\kappa_n\uparrow\kappa,\quad \Gamma_{n+1}\subset\tilde\Gamma_n\subset\Gamma_n\downarrow[-\tau^2,\tau^2]\times\bar B_\tau=\bar Q_\tau.$$

Since $\kappa_{n+1}-\kappa_n=\kappa 2^{-n}$, for any $r,s\in[1,\infty]$, by (3.17) we have

$$\|\mathbf{1}_{\{w_{n+1}\neq 0\}\cap\Gamma_n}\|_{\mathbb{L}^{s,r}_{t,x}}\leqslant 2^n\kappa^{-1}\|\mathbf{1}_{\Gamma_n}w_n\|_{\mathbb{L}^{s,r}_{t,x}}, \tag{3.21}$$

and by (3.19), for any $(r,s)\in\mathscr{I}_\varkappa$, there is a $\delta\in(0,1)$ such that

$$\|\mathbf{1}_{\Gamma_{n+1}}w_{n+1}\|_{\mathbb{L}^{s,r}_{t,x}}\lesssim \tfrac{2^n\|w_{n+1}\|_{\mathscr{V}_{\tilde\Gamma_n}}}{\sigma-\tau}\cdot\left(2^n\kappa^{-1}\|\mathbf{1}_{\Gamma_n}w_n\|_{\mathbb{L}^{1,1}_{t,x}}\right)^\delta. \tag{3.22}$$

Now let $(r_i,s_i),i=1,2,3,4$ be as in Lemma 3.2. If we define

$$\ell^{(i)}_n:=\|\mathbf{1}_{\Gamma_n}w_n\|_{\mathbb{L}^{s_i,r_i}_{t,x}},\ i=1,2,3,4,$$

then by (3.22), for some $\delta_i=\delta_i(r_i,s_i,\varkappa)\in(0,1)$ and $C=C(r_i,s_i,\varkappa,d)>0$,

$$\ell^{(i)}_{n+1}\lesssim_C \tfrac{2^n\|w_{n+1}\|_{\mathscr{V}_{\tilde\Gamma_n}}}{\sigma-\tau}\cdot\left(2^n\kappa^{-1}\|\mathbf{1}_{\Gamma_n}w_n\|_{\mathbb{L}^{1,1}_{t,x}}\right)^{\delta_i},\ i=1,2,3,4.$$

In particular, we have

$$\begin{aligned}a_{n+1}:=\frac{1}{\kappa}\sum_{i=1}^4\ell^{(i)}_{n+1}&\lesssim\tfrac{2^n\|w_{n+1}\|_{\mathscr{V}_{\tilde\Gamma_n}}}{(\sigma-\tau)\kappa}\sum_{i=1}^4\left(2^n\kappa^{-1}\|\mathbf{1}_{\Gamma_n}w_n\|_{\mathbb{L}^{1,1}_{t,x}}\right)^{\delta_i}\\&\lesssim\tfrac{4^n\|w_{n+1}\|_{\mathscr{V}_{\tilde\Gamma_n}}}{(\sigma-\tau)\kappa}\sum_{i=1}^4\left(\frac{\ell^{(1)}_n}{\kappa}\right)^{\delta_i}\leqslant\tfrac{4^n\|w_{n+1}\|_{\mathscr{V}_{\tilde\Gamma_n}}}{(\sigma-\tau)\kappa}\sum_{i=1}^4 a_n^{\delta_i},\end{aligned} \tag{3.23}$$

where the second inequality is due to $\|\mathbf{1}_{\Gamma_n}w_n\|_{\mathbb{L}^{1,1}_{t,x}}\leqslant C\|\mathbf{1}_{\Gamma_n}w_n\|_{\mathbb{L}^{s_1,r_1}_{t,x}}$.

On the other hand, note that

$$0\leqslant w_{n+1}\leqslant w_n\Rightarrow|\nabla w_{n+1}|=|\nabla u|\mathbf{1}_{\{w_{n+1}\neq 0\}}\leqslant|\nabla u|\mathbf{1}_{\{w_n\neq 0\}}=|\nabla w_n|.$$

By Lemma 3.2 with $w=w_{n+1}$, $Q_1=\tilde\Gamma_n$, $Q_2=\Gamma_n$ and (3.17), we have for some $\gamma\geqslant 1$,

$$\begin{aligned}\|w_{n+1}\|_{\mathscr{V}_{\tilde\Gamma_n}}&\lesssim_C 2^{\gamma n}\sum_{i=1,2,3}\|\mathbf{1}_{\Gamma_n}w_{n+1}\|_{\mathbb{L}^{s_i,r_i}_{t,x}}+\|f\mathbf{1}_{\Gamma_n}\|_{\mathbb{L}^{q_4,p_4}_{t,x}}\|\mathbf{1}_{\{w_{n+1}\neq 0\}\cap\Gamma_n}\|_{\mathbb{L}^{s_4,r_4}_{t,x}}\\&\lesssim_C 2^{\gamma n}(\ell^{(1)}_n+\ell^{(2)}_n+\ell^{(3)}_n)+\|f\mathbf{1}_{Q_2}\|_{\mathbb{L}^{q_4,p_4}_{t,x}}(2^n\kappa^{-1}\ell^{(4)}_n),\end{aligned}$$

where $C=C(\Theta)>0$. This implies that for $\kappa\geqslant\|f\mathbf{1}_{Q_2}\|_{\mathbb{L}^{q_4,p_4}_{t,x}}$,

$$\|w_{n+1}\|_{\mathscr{V}_{\tilde\Gamma_n}}\lesssim_C 2^{\gamma n}(\ell^{(1)}_n+\ell^{(2)}_n+\ell^{(3)}_n)+2^n\ell^{(4)}_n\leqslant 2^{\gamma n}\sum_{i=1}^4\ell^{(i)}_n=2^{\gamma n}a_n\kappa.$$

Substituting this into (3.23), we obtain that for some $C_0, \gamma_0 > 1$,

$$a_{n+1} \leqslant \frac{C_0 2^{\gamma_0 n} a_n}{\sigma - \tau} \sum_{i=1}^4 a_n^{\delta_i}, \quad \forall n \in \mathbb{N}.$$

Let $\delta := \delta_1 \wedge \cdots \wedge \delta_4$. Suppose that

$$\kappa \geqslant \left(\left[\frac{4C_0 2^{\gamma_0(1+\delta)/\delta}}{\sigma - \tau} \right]^{\frac{1}{\delta}} \sum_{i=1,2,3,4} \|u^+ \mathbf{1}_{Q_\sigma}\|_{\mathbb{L}^{s_i,r_i}_{t,x}} \right) \vee \|f \mathbf{1}_{Q_2}\|_{\mathbb{L}^{q_4,p_4}_{t,x}}.$$

Then $a_1 \leqslant (4C_0 2^{\gamma_0(1+\delta)/\delta})^{-\frac{1}{\delta}}$, and by Fatou's lemma and Lemma 3.4,

$$\|(u-\kappa)^+ \mathbf{1}_{Q_\tau}\|_{\mathbb{L}^{s_1,r_1}_{t,x}} \leqslant \liminf_{n\to\infty} \|w_n \mathbf{1}_{\Gamma_n}\|_{\mathbb{L}^{s_1,r_1}_{t,x}} = \liminf_{n\to\infty} \ell_n^{(1)} \leqslant \kappa \cdot \limsup_{n\to\infty} a_n = 0,$$

which in turn implies that

$$\|u^+ \mathbf{1}_{Q_\tau}\|_\infty \leqslant \left(\left[\frac{4C_0 2^{\gamma_0(1+\delta)/\delta}}{\sigma - \tau} \right]^{\frac{1}{\delta}} \sum_{i=1,2,3,4} \|u^+ \mathbf{1}_{Q_\sigma}\|_{\mathbb{L}^{s_i,r_i}_{t,x}} \right) \vee \|f \mathbf{1}_{Q_2}\|_{\mathbb{L}^{q_4,p_4}_{t,x}}.$$

To show (3.20), without loss of generality, we may assume

$$p \leqslant \gamma/2, \quad \gamma := \max_{i=1,2,3,4} \{s_i, r_i\}.$$

Thus by Hölder's inequality and Young's inequality, we have

$$\begin{aligned} \|u^+ \mathbf{1}_{Q_\tau}\|_\infty &\leqslant C(\sigma-\tau)^{-\frac{1}{\delta}} \|u^+ \mathbf{1}_{Q_\sigma}\|_{\mathbb{L}^{\gamma,\gamma}_{t,x}} + \|f \mathbf{1}_{Q_2}\|_{\mathbb{L}^{q_4,p_4}_{t,x}} \\ &\leqslant C(\sigma-\tau)^{-\frac{1}{\delta}} \|u^+ \mathbf{1}_{Q_2}\|_\infty^{1-\frac{p}{\gamma}} \|u^+ \mathbf{1}_{Q_\sigma}\|_{\mathbb{L}^{p,p}_{t,x}}^{\frac{p}{\gamma}} + \|f \mathbf{1}_{Q_2}\|_{\mathbb{L}^{q_4,p_4}_{t,x}} \\ &\leqslant \tfrac{1}{2} \|u^+ \mathbf{1}_{Q_\sigma}\|_\infty + C(\sigma-\tau)^{-\frac{\gamma}{p\delta}} \|u^+ \mathbf{1}_{Q_2}\|_{\mathbb{L}^{p,p}_{t,x}} + \|f \mathbf{1}_{Q_2}\|_{\mathbb{L}^{q_4,p_4}_{t,x}}, \end{aligned}$$

where $C = C(\Theta)$ is independent of σ, τ. By Lemma 2.6, we conclude the proof. □

3.3. **Proof of Theorem 1.3.** Without loss of generality, we assume $T = 1$ and

$$u(t,x) = f(t,x) \equiv 0, \quad \forall t \leqslant 0.$$

For $z \in \mathbb{R}^d$, we write

$$Q_i^z := Q_i^{0,z}, \quad i = 1,2,3.$$

Let u be a Lipschitz weak solution of PDE (1.4) in the sense of Definition 1.1. By translation and Lemma 3.2 with $w = u^+, u^-$, there is a constant $C = C(\Theta) > 0$ such that for all $t \in [0,1]$,

$$\|u\mathcal{I}^t\|_{\mathscr{V}_{Q_1^z}} \lesssim_C \sum_{i=1,2,3} \|\mathbf{1}_{Q_2^z} u \mathcal{I}^t\|_{\mathbb{L}^{s_i,r_i}_{t,x}} + \|f \mathbf{1}_{Q_2^z} \mathcal{I}^t\|_{\mathbb{L}^{q_4,p_4}_{t,x}},$$

where (r_i, s_i) are as in Lemma 3.2. By Lemma 2.2, we further have for some $\beta \in (1,\infty)$ and any $\varepsilon \in (0,1)$,

$$\|u\mathcal{I}^t\|_{\mathscr{V}_{Q_1^z}} \leqslant \varepsilon \|\mathbf{1}_{Q_3^z} \nabla u \mathcal{I}^t\|_{\mathbb{L}^{\varkappa,2}_{x,t}} + C_\varepsilon \|\mathbf{1}_{Q_3^z} u \mathcal{I}^t\|_{\mathbb{L}^{\beta,2}_{t,x}} + C \|f \mathbf{1}_{Q_2^z} \mathcal{I}^t\|_{\mathbb{L}^{q_4,p_4}_{t,x}}.$$

Taking supremum in $z \in \mathbb{R}^d$ for both sides and by (2.5), we obtain

$$|\!|\!| u\mathcal{I}^t |\!|\!|_{\widetilde{\mathscr{V}}} \lesssim \sup_z \|u\mathcal{I}^t\|_{\mathscr{V}_{Q_1^z}} \lesssim \varepsilon |\!|\!| \nabla u \mathcal{I}^t |\!|\!|_{\widetilde{\mathbb{L}}^{\varkappa,2}_{x,t}} + |\!|\!| u \mathcal{I}^t |\!|\!|_{\widetilde{\mathbb{L}}^{\beta,2}_{t,x}} + |\!|\!| f \mathcal{I}^t |\!|\!|_{\mathbb{L}^{q_4,p_4}_{t,x}}.$$

Since $|\!|\!| \nabla u \mathcal{I}^t |\!|\!|_{\widetilde{\mathbb{L}}^{\varkappa,2}_{x,t}} \leqslant |\!|\!| u\mathcal{I}^t |\!|\!|_{\widetilde{\mathscr{V}}}$, choosing ε small enough, we obtain

$$|\!|\!| u\mathcal{I}^t |\!|\!|_{\widetilde{\mathscr{V}}} \lesssim_C |\!|\!| u\mathcal{I}^t |\!|\!|_{\widetilde{\mathbb{L}}^{\beta,2}_{t,x}} + |\!|\!| f\mathcal{I}^t |\!|\!|_{\mathbb{L}^{q_4,p_4}_{t,x}}. \tag{3.24}$$

Since $\beta < \infty$ and $u(t) \equiv 0$ for $t \leqslant 0$, the above inequality implies that for any $t \in [0,1]$,

$$\|u(t)\|_2 \lesssim_C \left(\int_0^t \|u(s)\|_2^\beta \mathrm{d}s\right)^{1/\beta} + \|f\mathcal{I}^t\|_{\mathbb{L}^{q_4,p_4}_{t,x}}.$$

By Gronwall's inequality we obtain

$$\|u\mathcal{I}^t\|_{\widetilde{\mathbb{L}}^{\infty,2}_{t,x}} \leqslant \sup_{t\in[0,1]} \|u(t)\|_2 \lesssim \|f\mathcal{I}^t\|_{\mathbb{L}^{q_4,p_4}_{t,x}}.$$

which together with (3.24) yields

$$\|u\mathbf{1}_{[0,1]}\|_{\widetilde{\mathscr{V}}} \lesssim_C \|f\mathbf{1}_{[0,1]}\|_{\mathbb{L}^{q_4,p_4}_{t,x}}. \tag{3.25}$$

Finally, by (3.20) and (3.25), we also have

$$\begin{aligned}\|u\|_{L^\infty([0,1]\times\mathbb{R}^d)} &\leqslant \sup_z \|(u^+ + u^-)\mathbf{1}_{[0,1]\times B_1^z}\|_\infty \\ &\lesssim \|u\mathbf{1}_{[0,1]}\|_{\widetilde{\mathbb{L}}^{2,2}_{t,x}} + \|f\mathbf{1}_{[0,1]}\|_{\mathbb{L}^{q_4,p_4}_{t,x}} \lesssim \|f\mathbf{1}_{[0,1]}\|_{\mathbb{L}^{q_4,p_4}_{t,x}}.\end{aligned}$$

The proof is complete.

4. Weak solutions of SDEs with rough coefficients

In this section we present an application of the global boundedness (1.10) in SDEs, and show the existence of weak solutions to SDE (1.13) with rough coefficients. First of all, we recall the following notion of weak solutions to SDE (1.13).

Definition 4.1. *Let $\mathfrak{F} := (\Omega, \mathscr{F}, \mathbf{P}; (\mathscr{F}_t)_{t\geqslant 0})$ be a stochastic basis and (X, W) a pair of $\mathscr{F}_t$-adapted processes defined thereon. Given $(s,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, we call triple $(\mathfrak{F}, X, W)$ a weak solution of SDE (1.13) with starting point $x \in \mathbb{R}^d$ at time s if*

(i) $\mathbf{P}(X_t = x, t \in [0,s]) = 1$ *and W is an $\mathscr{F}_t$-Brownian motion;*

(ii) for all $t \geqslant s$, it holds that $\mathbf{P}$-a.s.,

$$\int_s^t \Big(|\sigma(r,X_r)|^2 + |b(r,X_r)|\Big)\mathrm{d}r < \infty,$$

and

$$X_t = x + \sqrt{2}\int_s^t \sigma(r,X_r)\mathrm{d}W_r + \int_s^t b(r,X_r)\mathrm{d}r.$$

Recall p_0, p_1 from (1.7) and the convention that the repeated indices will be summed automatically, for instances,

$$\partial_i a^{ij} = \sum_{i=1}^d \partial_i a^{ij}, \quad \partial_i\partial_j a^{ij} = \sum_{i,j=1}^d \partial_i\partial_j a^{ij}.$$

We introduce the following assumptions on σ and b:

($\widetilde{\mathbf{H}}^\sigma$) Suppose that there are a sequence of $d\times d$-matrix functions $\sigma_n \in L^\infty(\mathbb{R}_+; C_b^\infty)$, $(p_2, q_2) \in \mathbb{I}^d_{p_0}$ and $\kappa_0 > 0$ such that for all $n \in \mathbb{N}$,

$$\|\lambda_n^{-1}\|_{p_0} + \|\mu_n\|_{p_1} + \|\partial_i a_n^{ij}\|_{\widetilde{\mathbb{L}}^{p_1,\infty}_{x,t}} + \|(\partial_i\partial_j a_n^{ij})^+\|_{\widetilde{\mathbb{L}}^{q_2,p_2}_{t,x}} \leqslant \kappa_0, \tag{4.1}$$

where $a_n := \sigma_n\sigma_n^*$, λ_n and μ_n are defined as in (1.5) by a_n. Moreover, for some $p_3, q_3 \in [2,\infty]$ with $(\frac{p_3}{2}, \frac{q_3}{2}) \in \mathbb{I}^d_{p_0}$ and for any $T, R > 0$,

$$\sup_n \|\sigma_n\|_{\widetilde{\mathbb{L}}^{q_3,p_3}_{t,x}} =: \kappa_1 < \infty, \quad \lim_{n\to\infty} \|(\sigma_n - \sigma)\mathbf{1}_{[0,T]\times B_R}\|_{\mathbb{L}^{q_3,p_3}_{t,x}} = 0. \tag{4.2}$$

($\widetilde{\mathbf{H}}^b$) Let $b = b_1 + b_2$ satisfy **($\mathbf{H}^b$)** and belong to $\widetilde{\mathbb{L}}^{q_4,p_4}_{t,x}$ for some $(p_4, q_4) \in \mathbb{I}^d_{p_0}$.

Let Θ be defined by (1.9). Below we shall write

$$\widetilde{\Theta} := \Big(\Theta, p_i, q_i, \kappa_0, \kappa_1, \|b\|_{\widetilde{\mathbb{L}}^{q_4,p_4}_{t,x}}\Big).$$

We have the following existence result.

Theorem 4.2. *Under* $(\widetilde{\mathbf{H}}^\sigma)$ *and* $(\widetilde{\mathbf{H}}^b)$, *for any* $(s,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, *there is a weak solution* $(\mathfrak{F}, X, W)$ *for SDE* (1.13) *starting from* x *at time* s. *Moreover, for any* $(p,q) \in \mathbb{I}^d_{p_0}$ *and* $T > s$, *there are* $\theta \in (0,1)$ *and constant* $C = C(T, \widetilde{\Theta}, p, q) > 0$ *such that for any* $s \leqslant t_0 < t_1 \leqslant T$ *and* $f \in \widetilde{\mathbb{L}}^{q,p}_{t,x}$,

$$\mathbf{E}\left(\int_{t_0}^{t_1} f(r, X_r)\mathrm{d}r\middle|\mathscr{F}_{t_0}\right) \leqslant C(t_1 - t_0)^\theta \|f\|_{\widetilde{\mathbb{L}}^{q,p}_{t,x}}. \tag{4.3}$$

In the following proof, we assume $s = 0$ and $x \in \mathbb{R}^d$. Let σ_n be as in $(\widetilde{\mathbf{H}}^\sigma)$ and $b_n(t,x) = b * \rho_n(t,x)$ be the mollifying approximation of b. In particular,

$$\sigma_n, b_n \in L^\infty([0,T]; C^\infty_b(\mathbb{R}^d)), \tag{4.4}$$

and the following SDE admits a unique strong solution (cf. [16]):

$$\mathrm{d}X^n_t = b_n(t, X^n_t)\mathrm{d}t + \sqrt{2}\sigma_n(t, X^n_t)\mathrm{d}W_t, \quad X^n_0 = x. \tag{4.5}$$

Note that the generator of SDE (6.1) is given by

$$\begin{aligned}\mathscr{L}^{\sigma_n,b_n}_t f(x) &= (\sigma^{ik}_n\sigma^{jk}_n)(t,x)\partial_i\partial_j f(x) + b^j_n(t,x)\partial_j f(x)\\ &= \partial_i(a^{ij}_n(t,\cdot)\partial_j f)(x) + \tilde{b}^j_n(t,x)\partial_j f(x),\end{aligned}$$

where

$$a^{ij}_n := \sigma^{ik}_n\sigma^{jk}_n, \quad \tilde{b}^j_n := b^j_n - \partial_i a^{ij}_n.$$

In particular, by $(\widetilde{\mathbf{H}}^\sigma)$, one sees that $(\mathbf{H}^a)$ holds for a_n uniformly in n, and $(\mathbf{H}^b)$ holds for $\tilde{b}_n = b_{1,n} + (b_{2,n} - \partial_i a^{i\cdot}_n)$ uniformly in n, where $b_{i,n} := b_i * \rho_n$.

We first show the following key Krylov estimate (see [19]).

Theorem 4.3. *Under* $(\widetilde{\mathbf{H}}^\sigma)$ *and* $(\widetilde{\mathbf{H}}^b)$, *for any* $(p,q) \in \mathbb{I}^d_{p_0}$ *and* $T > 0$, *there are* $\theta = \theta(p,q,d,p_0) \in (0,1)$ *and* $C = C(T, \widetilde{\Theta}, p, q) > 0$ *independent of starting point* x *such that for any* $0 \leqslant t_0 < t_1 \leqslant T$ *and* $f \in \widetilde{\mathbb{L}}^{q,p}_{t,x}$,

$$\sup_n \mathbb{E}\left(\int_{t_0}^{t_1} f(s, X^n_s)\mathrm{d}s\middle|\mathscr{F}_{t_0}\right) \leqslant C(t_1 - t_0)^\theta \|f\|_{\widetilde{\mathbb{L}}^{q,p}_{t,x}}. \tag{4.6}$$

Proof. Below we drop the super and subscripts n for simplicity. Without loss of generality, we may assume $f \in C^\infty_0(\mathbb{R}^{d+1})$. Fix $t_1 \in (0,T]$ and consider the following backward PDEs:

$$\partial_t u + \mathscr{L}^{\sigma,b}_t u = f, \quad u(t_1) = 0.$$

Under (4.4), it is well known that there is a unique solution $u \in L^\infty_{loc}([0,t_1]; C^\infty_b(\mathbb{R}^d))$ so that (cf. [16])

$$u(t,x) = \int_t^{t_1} (\mathscr{L}^{\sigma,b}_s u - f)(s,x)\mathrm{d}s, \quad t \in [0,t_1].$$

By Itô's formula, for any $t_0 \leqslant t_1$, we have

$$u(t_1, X_{t_1}) - u(t_0, X_{t_0}) = \int_{t_0}^{t_1} f(s, X_s)\mathrm{d}s + \sqrt{2}\int_{t_0}^{t_1} (\sigma^*\nabla u)(s, X_s)\mathrm{d}W_s.$$

Taking conditional expectations with respect to $\mathscr{F}_{t_0}$, we obtain

$$\mathbb{E}\left(\int_{t_0}^{t_1} f(s, X_s)\mathrm{d}s\middle|\mathscr{F}_{t_0}\right) \leqslant \|u\|_{L^\infty([t_0,t_1]\times\mathbb{R}^d)}. \tag{4.7}$$

On the other hand, since $(p,q)\in\mathbb{I}^d_{p_0}$, we can choose $q'<q$ so that

$$\tfrac{1}{p}<\big(1-\tfrac{1}{q'}\big)\big(\tfrac{2}{d}-\tfrac{1}{p_0}\big).$$

Thus by the assumptions, (1.10) of Theorem 1.3, there exists a constant $C=C(T,\widetilde{\theta},p,q)>0$ independent of n such that

$$\|u\|_{L^\infty([t_0,t_1]\times\mathbb{R}^d)}\lesssim_C \|f\mathbf{1}_{[t_0,t_1]}\|_{\mathbb{L}^{q',p}_{t,x}}\lesssim_C (t_1-t_0)^\theta\|f\|_{\mathbb{L}^{q,p}_{t,x}},$$

where $\theta=\frac{1}{q'}-\frac{1}{q}$ and the second inequality is due to Hölder's inequality. Substituting it into (4.7), we obtain (4.6). □

We need the following simple lemma.

Lemma 4.4. *Let $(X_t)_{t\geqslant 0}$ be a right continuous stochastic process on a filtered probability space $(\Omega,\mathscr{F},\mathbb{P};(\mathscr{F}_t)_{t\geqslant 0})$. Suppose that for some $Y\in L^1(\Omega)$ and $A>0$,*

$$|X_t|\leqslant Y,\quad \mathbb{E}(X_t|\mathscr{F}_t)\leqslant A,\quad \mathbb{P}-a.s.$$

Then for any finite stopping time τ, it holds that

$$\mathbb{E}(X_\tau|\mathscr{F}_\tau)\leqslant A,\quad \mathbb{P}-a.s.$$

Proof. Let τ_n be a sequence of decreasing stopping times with values in $\mathbb{D}:=\{k\cdot 2^{-n}:k,n\in\mathbb{N}\}$ and so that $\tau_n\to\tau$ as $n\to\infty$. Note that for each $n\in\mathbb{N}$,

$$\mathbb{E}\left(X_{\tau_n}|\mathscr{F}_{\tau_n}\right)=\mathbb{E}\left(\sum_{t\in\mathbb{D}}\mathbf{1}_{\{\tau_n=t\}}X_t|\mathscr{F}_{\tau_n}\right)=\sum_{t\in\mathbb{D}}\mathbf{1}_{\{\tau_n=t\}}\mathbb{E}\left(X_t|\mathscr{F}_t\right)\leqslant A.$$

By the dominated convergence theorem and $\mathscr{F}_\tau\subset\mathscr{F}_{\tau_n}$, we have

$$\mathbb{E}\left(X_\tau|\mathscr{F}_\tau\right)=\lim_{n\to\infty}\mathbb{E}\left(X_{\tau_n}|\mathscr{F}_\tau\right)=\lim_{n\to\infty}\mathbb{E}\left(X_{\tau_n}|\mathscr{F}_{\tau_n}|\mathscr{F}_\tau\right)\leqslant A.$$

The proof is complete. □

Remark 4.5. By this lemma, one sees that (4.6) is equivalent to that for any stopping time $\tau\leqslant T$, $\delta\in(0,1)$ and $f\in\widetilde{\mathbb{L}}^{q,p}_{t,x}$,

$$\sup_n\mathbb{E}\left(\int_\tau^{\tau+\delta}f(s,X^n_s)\mathrm{d}s\Big|\mathscr{F}_\tau\right)\leqslant C\delta^\theta\|f\|_{\widetilde{\mathbb{L}}^{q,p}_{t,x}}.\tag{4.8}$$

Lemma 4.6. *Under* $(\widetilde{\mathbf{H}}^\sigma)$ *and* $(\widetilde{\mathbf{H}}^b)$*, for any $T>0$, there are $\theta\in(0,1)$ and constant $C=C(T,\widetilde{\Theta})>0$ such that for all $\delta\in(0,1)$,*

$$\sup_n\mathbb{E}\left(\sup_{t\in[0,T]}\sup_{s\in[0,\delta]}|X^n_{t+s}-X^n_t|^{1/2}\right)\leqslant C\delta^{\theta/2}\tag{4.9}$$

and

$$\sup_n\mathbb{E}\left(\sup_{t\in[0,T]}|X^n_t|\right)\leqslant C.\tag{4.10}$$

Proof. We only prove (4.9). Let τ be any stopping time less than T. Notice that

$$X^n_{\tau+t}-X^n_\tau=\int_\tau^{\tau+t}b_n(s,X^n_s)\mathrm{d}s+\sqrt{2}\int_\tau^{\tau+t}\sigma_n(s,X^n_s)\mathrm{d}W_s,\quad t>0.$$

By Burkholder's inequality and (4.8), we have

$$\begin{aligned}\mathbb{E}\left(\sup_{t\in[0,\delta]}|X^n_{\tau+t}-X^n_\tau|\right)&\lesssim\mathbb{E}\int_\tau^{\tau+\delta}|b_n|(s,X^n_s)\mathrm{d}s+\left(\mathbb{E}\int_\tau^{\tau+\delta}|\sigma_n(s,X^n_s)|^2\mathrm{d}s\right)^{1/2}\\&\leqslant C\delta^\theta\|b_n\|_{\widetilde{\mathbb{L}}^{q_4,p_4}_{t,x}}+C\delta^\theta\|\sigma_n\|_{\widetilde{\mathbb{L}}^{q_3,p_3}_{t,x}}\leqslant C\delta^\theta,\end{aligned}$$

where C is independent of n and δ. Thus by [20, Lemma 2.7], we obtain (4.9). $\square$

Let $\mathbb{C}$ be the space of all continuous functions from $\mathbb{R}_+$ to $\mathbb{R}^d$, which is endowed with the locally uniformly metric so that $\mathbb{C}$ becomes a Polish space. Let $\mathbb{Q}_n$ be the law of $(X^n_\cdot, W_\cdot)$ in product space $\mathbb{C}\times\mathbb{C}$. For each $x\in\mathbb{R}^d$, by Lemma 4.6 and [16, Theorem 1.3.2], the law of $X^n_\cdot$ is tight in $\mathbb{C}$. By Lemma 4.6 and Prokhorov's theorem, there are a subsequence still denoted by n and $\mathbb{Q}\in\mathcal{P}(\mathbb{C}\times\mathbb{C})$ so that

$$\mathbb{Q}_n\to\mathbb{Q} \text{ weakly.}$$

Now, by Skorokhod's representation theorem, there are a probability space $(\tilde\Omega,\tilde{\mathscr{F}},\tilde{\mathbb{P}})$ and random variables $(\tilde X^n,\tilde W^n)$ and $(\tilde X,\tilde W)$ defined on it such that

$$(\tilde X^n,\tilde W^n)\to(\tilde X,\tilde W),\quad \tilde{\mathbb{P}}-a.s. \tag{4.11}$$

and

$$\tilde{\mathbb{P}}\circ(\tilde X^n,\tilde W^n)^{-1}=\mathbb{Q}_n=\mathbb{P}\circ(X^n,W)^{-1},\quad \tilde{\mathbb{P}}\circ(\tilde X,\tilde W)^{-1}=\mathbb{Q}. \tag{4.12}$$

Define $\tilde{\mathscr{F}}^n_t:=\sigma(\tilde W^n_s,\tilde X^n_s;s\leqslant t)$. Notice that

$$\mathbb{P}(W_t-W_s\in\cdot|\mathscr{F}_s)=\mathbb{P}(W_t-W_s\in\cdot)$$

implies that

$$\tilde{\mathbb{P}}(\tilde W^n_t-\tilde W^n_s\in\cdot|\tilde{\mathscr{F}}^n_s)=\tilde{\mathbb{P}}(\tilde W^n_t-\tilde W^n_s\in\cdot).$$

In other words, $\tilde W^n_t$ is an $\tilde{\mathscr{F}}^n_t$-Brownian motion. Thus, by (6.1) and (4.12) we have

$$\tilde X^n_t=x+\int_0^t b_n\big(s,\tilde X^n_s\big)\mathrm{d}s+\int_0^t\sigma_n\big(s,\tilde X^n_s\big)\mathrm{d}\tilde W^n_s. \tag{4.13}$$

Moreover, by (4.6), we also have

$$\sup_n\tilde{\mathbb{E}}\left(\int_{t_0}^{t_1}f(s,\tilde X^n_s)\mathrm{d}s\Big|\tilde{\mathscr{F}}_{t_0}\right)\leqslant C(t_1-t_0)^\theta|\!|\!|f|\!|\!|_{\widetilde{\mathbb{L}}^{q,p}_{t,x}}. \tag{4.14}$$

In order to take the limits, we recall a result of Skorokhod [15, p.32].

Lemma 4.7. *Let $\{f_n(t),t\geqslant0,n\in\mathbb{N}\}$ be a sequence of measurable $\tilde{\mathscr{F}}^n_t$-adapted processes. Suppose that for every $T,\varepsilon>0$, there is an $M_\varepsilon>0$ such that*

$$\sup_n\tilde{\mathbb{P}}\left\{\sup_{t\in[0,T]}|f_n(t)|>M_\varepsilon\right\}\leqslant\varepsilon,$$

and also,

$$\lim_{n\to\infty}\tilde{\mathbb{P}}\left\{\sup_{t\in[0,T]}|f_n(t)-f(t)|>\varepsilon\right\}=0.$$

Then it holds that for every $T>0$,

$$\int_0^T f_n(t)\mathrm{d}\tilde W^n_t\to\int_0^T f(t)\mathrm{d}\tilde W_t \text{ in probability as } n\to\infty.$$

Lemma 4.8. *For each $t>0$, the following limits hold in probability as $n\to\infty$,*

$$\int_0^t b_n\big(s,\tilde X^n_s\big)\mathrm{d}s\to\int_0^t b\big(s,\tilde X_s\big)\mathrm{d}s, \tag{4.15}$$

$$\int_0^t\sigma_n\big(s,\tilde X^n_s\big)\mathrm{d}\tilde W^n_s\to\int_0^t\sigma\big(s,\tilde X_s\big)\mathrm{d}\tilde W_s. \tag{4.16}$$

Proof. We only prove (4.16). For simplicity, we shall write $\sigma_\infty := \sigma$ and drop the tilde. For each $n \in \mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$, let $\sigma^\varepsilon_n(t,x) := \sigma_n * \rho_\varepsilon(t,x)$ be the mollifying approximation of σ_n. It suffices to show the following two limits: for fixed $\varepsilon > 0$,

$$\int_0^t \sigma^\varepsilon_n\big(s, X^n_s\big)\mathrm{d}W^n_s \to \int_0^t \sigma^\varepsilon_\infty\big(s, X_s\big)\mathrm{d}W_s \text{ in probability } n \to \infty, \tag{4.17}$$

and

$$\lim_{\varepsilon\to 0}\sup_{n\in\mathbb{N}_\infty} \mathbb{E}\left|\int_0^t (\sigma^\varepsilon_n\big(s,X^n_s\big) - \sigma_n\big(s,X^n_s\big))\mathrm{d}W^n_s\right|^2 = 0. \tag{4.18}$$

Clearly, limit (4.17) follows by Lemma 4.7. We look at (4.18). For $R > 0$, we define

$$\tau^n_R := \inf\{t > 0 : |X^n_t| \geqslant R\}.$$

By (4.10), we have

$$\lim_{R\to\infty}\sup_n \mathbb{P}(\tau^n_R \leqslant t) \leqslant \lim_{R\to\infty}\sup_n \frac{1}{R}\mathbb{E}\left(\sup_{s\in[0,t]}|X^n_s|\right) = 0. \tag{4.19}$$

For (4.18), by Itô's isometry we have

$$\begin{aligned}
&\mathbb{E}\left|\int_0^t (\sigma^\varepsilon_n\big(s,X^n_s\big) - \sigma_n\big(s,X^n_s\big))\mathrm{d}W^n_s\right|^2 \\
&\qquad \leqslant \mathbb{E}\int_0^t |\sigma^\varepsilon_n\big(s,X^n_s\big) - \sigma_n\big(s,X^n_s\big)|^2\mathrm{d}s =: I^n_R(\varepsilon) + J^n_R(\varepsilon),
\end{aligned} \tag{4.20}$$

where

$$I^n_R(\varepsilon) := \mathbb{E}\left(\mathbf{1}_{\{\tau^n_R > t\}}\int_0^t |\sigma^\varepsilon_n\big(s,X^n_s\big) - \sigma_n\big(s,X^n_s\big)|^2\mathrm{d}s\right),$$

$$J^n_R(\varepsilon) := \mathbb{E}\left(\mathbf{1}_{\{\tau^n_R \leqslant t\}}\int_0^t |\sigma^\varepsilon_n\big(s,X^n_s\big) - \sigma_n\big(s,X^n_s\big)|^2\mathrm{d}s\right).$$

For $I^n_R(\varepsilon)$, since $(\frac{p_3}{2}, \frac{q_3}{2}) \in \mathbb{I}^d_{p_0}$, by (4.14) we have

$$\begin{aligned}
I^n_R(\varepsilon) &\leqslant \mathbb{E}\left(\int_0^t \mathbf{1}_{|X^n_s|\leqslant R}|\sigma^\varepsilon_n\big(s,X^n_s\big) - \sigma_n\big(s,X^n_s\big)|^2\mathrm{d}s\right) \\
&\lesssim \|\mathbf{1}_{[0,t]\times B_R}(\sigma^\varepsilon_n - \sigma_n)\|^2_{\mathbb{L}^{q_3,p_3}_{t,x}},
\end{aligned}$$

where the implicit constant is independent of n, ε, R. For each $R > 0$, since

$$\lim_{n\to\infty}\sup_{\varepsilon\in(0,1)} \|\mathbf{1}_{[0,t]\times B_R}(\sigma^\varepsilon_n - \sigma^\varepsilon)\|_{\mathbb{L}^{q_3,p_3}_{t,x}} \leqslant \lim_{n\to\infty}\|\mathbf{1}_{[0,t]\times B_{2R}}(\sigma_n - \sigma)\|_{\mathbb{L}^{q_3,p_3}_{t,x}} = 0,$$

and for each $n \in \mathbb{N}_\infty$,

$$\lim_{\varepsilon\to 0}\|\mathbf{1}_{[0,t]\times B_R}(\sigma^\varepsilon_n - \sigma_n)\|_{\mathbb{L}^{q_3,p_3}_{t,x}} = 0,$$

it follows that for each $R > 0$,

$$\lim_{\varepsilon\to 0}\sup_n I^n_R(\varepsilon) \lesssim \lim_{\varepsilon\to 0}\sup_n \|\mathbf{1}_{[0,t]\times B_R}(\sigma^\varepsilon_n - \sigma_n)\|^2_{\mathbb{L}^{q_3,p_3}_{t,x}} = 0. \tag{4.21}$$

For $J^n_R(\varepsilon)$, since $(\frac{p_3}{2}, \frac{q_3}{2}) \in \mathbb{I}^d_{p_0}$, one can choose $\gamma > 1$ being close to 1 so that $(\frac{p_3}{2\gamma}, \frac{q_3}{2\gamma}) \in \mathbb{I}^d_{p_0}$. By Hölder's inequality and (4.6) we have

$$\begin{aligned}
J^n_R(\varepsilon) &\leqslant (\mathbb{P}(\tau^n_R \leqslant t))^{\frac{\gamma-1}{\gamma}}\left(\mathbb{E}\int_0^t |\sigma^\varepsilon_n\big(s,X^n_s\big) - \sigma_n\big(s,X^n_s\big)|^{2\gamma}\mathrm{d}s\right)^{\frac{1}{\gamma}} \\
&\lesssim (\mathbb{P}(\tau^n_R \leqslant t))^{\frac{\gamma-1}{\gamma}}\|\sigma^\varepsilon_n - \sigma_n\|^{2\gamma}_{\mathbb{L}^{q_3,p_3}_{t,x}} \lesssim (\mathbb{P}(\tau^n_R \leqslant t))^{\frac{\gamma-1}{\gamma}},
\end{aligned}$$

where the implicit constant is independent of ε, n, R. By (4.19), we have

$$\lim_{R\to\infty}\sup_n\sup_\varepsilon J^n_R(\varepsilon)=0. \tag{4.22}$$

Combining (4.20), (4.21) and (4.22), we obtain (4.18). The proof is complete. □

Proof of Theorem 4.2. It follows by taking limits for both sides of (4.13) and Lemma 4.8. As for (4.3), it follows by taking limits for (4.14) with $f\in C_0(\mathbb{R}_+\times\mathbb{R}^d)$. □

5. Strong Markov selection

In this section we use Krylov's Markov selection theorem to show the existence of a strong Markov solution under $(\widetilde{\mathbf{H}}^\sigma)$ and $(\widetilde{\mathbf{H}}^b)$. Let ω_t be the coordinate process on the continuous function space $\mathbb{C}$ and $\mathcal{B}_t:=\sigma\{\omega_s: s\leqslant t\}$ the natural σ-filtration. We first recall the following notion of local martingale solutions to SDE (1.13).

Definition 5.1. *Let $(s,x)\in\mathbb{R}_+\times\mathbb{R}^d$. A probability measure $\mathbb{P}\in\mathcal{P}(\mathbb{C})$ is called a local martingale solution of SDE* (1.13) *starting from x at time s if*

(i) $\mathbb{P}(\omega_t=x, t\in[0,s])=1$ and for each $t>s$,

$$\mathbb{P}\left(\omega:\int_s^t(|b(r,\omega_r)|+|(\sigma\sigma^*)(r,\omega_r)|)\mathrm{d}s<\infty\right)=1.$$

(ii) For any $f\in C^\infty(\mathbb{R}^d)$, the process

$$M^f_t(\omega):=f(\omega_t)-f(\omega_s)-\int_s^t\mathscr{L}^{\sigma,b}_r f(\omega_r)\mathrm{d}r$$

is a continuous local $\mathcal{B}_t$-martingale after time s.

The set of all the local martingale solutions of (1.13) *is denoted by $\mathcal{M}^{\sigma,b}_{s,x}\subset\mathcal{P}(\mathbb{C})$.*

By Itô's formula, it is easy to see that the law of a weak solution in Definition 4.1 is a local martingale solution. Moreover, we also have the following opposite conclusion (see [8, p314, Proposition 4.11]).

Theorem 5.2. *For any $\mathbb{P}\in\mathcal{M}^{\sigma,b}_{s,x}$, there is a weak solution $(\mathfrak{F},X,W)$ starting from x at time s, where $\mathfrak{F}=(\Omega,\mathscr{F},\mathbf{P};(\mathscr{F}_t)_{t\geqslant 0})$ is a stochastic basis, and so that*

$$\mathbb{P}=\mathbf{P}\circ X^{-1}.$$

One also needs the following notion about Krylov's estimate (see Theorem 4.3).

Definition 5.3. *Let $p,q\in[1,\infty)$ and $s\geqslant 0$. We call a probability measure $\mathbb{P}\in\mathcal{P}(\mathbb{C})$ satisfy the Krylov estimate with indices p,q and starting from s if for any $T>s$, there are constants $\kappa,\theta>0$ such that for any $s\leqslant t_0<t_1\leqslant T$ and $f\in C_0(\mathbb{R}_+\times\mathbb{R}^d)$,*

$$\mathbb{E}^{\mathbb{P}}\left(\int_{t_0}^{t_1}f(r,\omega_r)\mathrm{d}r\Big|\mathcal{B}_{t_0}\right)\leqslant\kappa(t_1-t_0)^\theta\|f\|_{\mathbb{L}^{q,p}_{t,x}}. \tag{5.1}$$

We shall denote by $\mathcal{K}^{p,q}_{s,T}$ the set of all the above $\mathbb{P}$.

Remark 5.4. By a standard approximation, (5.1) holds for all $f\in\mathbb{L}^{q,p}_{t,x}$.

Now we show the following main result of this section.

Theorem 5.5. *Assume $(\widetilde{\mathbf{H}}^\sigma)$ and $(\widetilde{\mathbf{H}}^b)$. For given $(s,x)\in\mathbb{R}_+\times\mathbb{R}^d$, let*

$$\mathscr{C}(s,x):=\cap_{(p,q)\in\mathbb{I}^d_{p_0},T>s}\mathcal{K}^{p,q}_{s,T}\cap\mathcal{M}^{\sigma,b}_{s,x}.$$

Then $\mathscr{C}(s,x)$ is a non-empty convex subset of $\mathcal{P}(\mathbb{C})$ and satisfies **(C1)**, **(C2)** *and* **(C3)** *in appendix. In particular, there is a measurable mapping*

$$\mathbb{R}_+\times\mathbb{R}^d\ni(s,x)\mapsto\mathbb{P}_{s,x}\in\mathscr{C}(s,x)$$

so that for each fixed $(s,x)\in\mathbb{R}_+\times\mathbb{R}^d$ and finite stopping time $\tau\geqslant s$, there is a $\mathbb{P}_{s,x}$-null set $N\in\mathcal{B}_\tau$ such that for all $\omega\notin N$,

$$\mathbb{P}_{s,x}(\cdot|\mathcal{B}_\tau)(\omega)=\mathbb{P}_{\tau(\omega),\omega_{\tau(\omega)}}(\cdot).$$

Proof. First of all, by Theorem 4.3, for each (s,x), $\mathscr{C}(s,x)$ is non-empty and convex, and for given $(p,q)\in\mathbb{I}^d_{p_0}$, the constants κ,θ appearing in (5.1) are independent of s,x.

Verification of (C1) Let (s_n,x_n) converge to (s,x) and $\mathbb{P}^n\in\mathscr{C}(s_n,x_n)$. We want to show that $(\mathbb{P}^n)_{n\in\mathbb{N}}$ is tight. By the equivalence between martingale solutions and weak solutions, for each $n\in\mathbb{N}$, there exists a weak solution $(\mathfrak{F}^n,X^n,W^n)$ starting from x_n at time s_n, where $\mathfrak{F}_n:=(\Omega^n,\mathscr{F}^n,\mathbf{P}^n;(\mathscr{F}^n_t)_{t\geqslant0})$, so that

$$\mathbb{P}^n=\mathbf{P}^n\circ(X^n)^{-1}.$$

Note that

$$X^n_t=x_n+\sqrt{2}\int_{s_n}^t\sigma(r,X^n_r)\mathrm{d}W^n_r+\int_{s_n}^t b(r,X^n_r)\mathrm{d}r,\quad t>s_n.$$

Since $\mathbb{P}^n\in\cap_{(p,q)\in\mathbb{I}_{p_0},T>s_n}\mathcal{K}^{p,q}_{s_n,T}$, and the constants κ,θ appearing in (5.1) are independent of n, as in Lemma 4.6, one can show that for each $T>\sup s_n+1$,

$$\sup_n\mathbf{E}_n\left(\sup_{t\in[0,T]}\sup_{s\in[0,\delta]}|X^n_{t+s}-X^n_t|^{1/2}\right)\leqslant C\delta^{\theta/2},\quad\delta\in(0,1),$$

where $\mathbf{E}_n$ stands for the expectation with respect to $\mathbf{P}^n$. Thus $(\mathbb{P}^n)_{n\in\mathbb{N}}$ is tight. Let $\mathbb{P}$ be any accumulation point of $\mathbb{P}^n$. If necessary, by substracting a subsequence, without loss of generality we assume $\mathbb{P}^n$ weakly converges to $\mathbb{P}$. For given compact support continuous function f, by taking weak limits for

$$\mathbb{E}^{\mathbb{P}_n}\left(\int_{t_0}^{t_1}f(r,\omega_r)\mathrm{d}r\Big|\mathcal{B}_{t_0}\right)\leqslant\kappa(t_1-t_0)^\theta\|f\|_{\mathbb{L}^{q,p}_{t,x}},$$

one sees that

$$\mathbb{P}\in\cap_{(p,q)\in\mathbb{I}_{p_0},T>s}\mathcal{K}^{p,q}_{s,T}.$$

Moreover, as in the proof in Section 4, one can show that $\mathbb{P}\in\mathcal{M}^{\sigma,b}_{s,x}$.

Verification of (C2) Let $\mathbb{P}\in\mathscr{C}(s,x)$ and $\tau\geqslant s$ be a finite stopping time. Let Q_ω be a r.c.p.d. of $\mathbb{P}(\cdot|\mathcal{B}_\tau)$. By [16, Theorem 6.1.3], there is a $\mathbb{P}$-null set $N_1\in\mathcal{B}_\tau$ such that for all $\omega\notin N_1$,

$$Q_\omega\in\mathcal{M}^{\sigma,b}_{\tau(\omega),\omega_{\tau(\omega)}}.\tag{5.2}$$

On the other hand, for fixed $p,q\in\mathbb{I}^d_{p_0}$, $\delta\in(0,1)$ and $T>s+\delta$, since $\mathbb{P}\in\mathcal{K}^{p,q}_{s,T}$, we have for all $t\in[s,T-\delta]$ and $f\in C_0(\mathbb{R}_+\times\mathbb{R}^d)$,

$$\mathbb{E}^{\mathbb{P}}\left(\int_t^{t+\delta}f(r,\omega_r)\mathrm{d}r\Big|\mathcal{B}_t\right)\leqslant\kappa\delta^\theta\|f\|_{\mathbb{L}^{q,p}_{t,x}},\ \mathbb{P}-a.s.$$

By Lemma 7.4, there is a $\mathbb{P}$-null set $N=N(p,q,f,T)\in\mathcal{B}_\tau$ such that for all $\omega\notin N$ and $T>\tau(\omega)+\delta$, $t\in[\tau(\omega),T-\delta]$,

$$\mathbb{E}^{Q_\omega}\left(\int_t^{t+\delta}f(r,\omega_r)\mathrm{d}r\Big|\mathcal{B}_t\right)\leqslant\kappa\delta^\theta\|f\|_{\mathbb{L}^{q,p}_{t,x}},\ Q_\omega-a.s.,$$

Since $C_0(\mathbb{R}_+\times\mathbb{R}^d)$ and $\mathbb{I}^d_{p_0}$ are separable, one can find a common $\mathbb{P}$-null set N_2 such that for all $\omega\notin N_2$ and $(p,q)\in\mathbb{I}^d_{p_0}$, $\delta\in(0,1)$, $T>\tau(\omega)+\delta$, $t\in[\tau(\omega),T-\delta)$, $f\in C_0(\mathbb{R}_+\times\mathbb{R}^d)$,

$$\mathbb{E}^{Q_\omega}\left(\int_t^{t+\delta} f(r,\omega_r)\mathrm{d}r\Big|\mathcal{B}_t\right)\leqslant\kappa\delta^\theta\|f\|_{\mathbb{L}^{q,p}_{t,x}},\ Q_\omega-a.s.$$

In other words,

$$Q_\omega\in\cap_{(p,q)\in\mathbb{I}_{p_0},T>\tau(\omega)}\mathcal{K}^{p,q}_{\tau(\omega),T},$$

which together with (5.2) yields that there is a $\mathbb{P}$-null set N such that for all $\omega\notin N$,

$$Q_\omega\in\mathscr{C}(\tau(\omega),\omega_{\tau(\omega)}).$$

Verification of (C3) Let $\mathbb{P}\in\mathscr{C}(s,x)$ and $\tau\geqslant s$ be a finite stopping time. For any $\mathcal{B}_\tau$-measurable kernel $\mathbb{C}\ni\omega\mapsto Q_\omega\in\mathcal{P}(\mathbb{C})$ with

$$Q_\omega\in\mathscr{C}(\tau(\omega),\omega_{\tau(\omega)}),\quad\forall\omega\in\mathbb{C}.$$

By [16, Theorem 6.1.2], one knows that

$$\mathbb{P}\otimes_\tau Q\in\mathcal{M}^{\sigma,b}_{s,x}. \tag{5.3}$$

For fixed $p,q\in\mathbb{I}^d_{p_0}$ and $T>s$, we want to show that there are κ,θ independent of (s,x) such that for any $s\leqslant t_0<t_1\leqslant T$,

$$\mathbb{E}^{\mathbb{P}\otimes_\tau Q}\left(\int_{t_0}^{t_1} f(r,\omega_r)\mathrm{d}r\Big|\mathcal{B}_{t_0}\right)\leqslant\kappa(t_1-t_0)^\theta\|f\|_{\mathbb{L}^{q,p}_{t,x}},\ \mathbb{P}\otimes_\tau Q-a.s., \tag{5.4}$$

which means that

$$\mathbb{P}\otimes_\tau Q\in\cap_{(p,q)\in\mathbb{I}_{p_0},T>s}\mathcal{K}^{p,q}_{s,T}.$$

We make the following decomposition:

$$\mathbb{E}^{\mathbb{P}\otimes_\tau Q}\left(\int_{t_0}^{t_1} f(r,\omega_r)\mathrm{d}r\Big|\mathcal{B}_{t_0}\right)=I_1+I_2+I_3+I_4,$$

where

$$\begin{aligned}
I_1&:=\mathbf{1}_{\{\tau\leqslant t_0\}}\mathbb{E}^{\mathbb{P}\otimes_\tau Q}\left(\int_{t_0}^{t_1} f(r,\omega_r)\mathrm{d}r\Big|\mathcal{B}_{t_0}\right),\\
I_2&:=\mathbf{1}_{\{t_0<\tau\leqslant t_1\}}\mathbb{E}^{\mathbb{P}\otimes_\tau Q}\left(\int_{t_0}^{\tau} f(r,\omega_r)\mathrm{d}r\Big|\mathcal{B}_{t_0}\right),\\
I_3&:=\mathbf{1}_{\{t_0<\tau\leqslant t_1\}}\mathbb{E}^{\mathbb{P}\otimes_\tau Q}\left(\int_{\tau}^{t_1} f(r,\omega_r)\mathrm{d}r\Big|\mathcal{B}_{t_0}\right),\\
I_4&:=\mathbf{1}_{\{t_1<\tau\}}\mathbb{E}^{\mathbb{P}\otimes_\tau Q}\left(\int_{t_0}^{t_1} f(r,\omega_r)\mathrm{d}r\Big|\mathcal{B}_{t_0}\right).
\end{aligned}$$

For I_1, noting that

$$I_1=\mathbf{1}_{\{\tau\leqslant t_0\}}\mathbb{E}^{\mathbb{P}\otimes_\tau Q}\left(\int_{t_0\vee\tau}^{t_1\vee\tau} f(r,\omega_r)\mathrm{d}r\Big|\mathcal{B}_{t_0\vee\tau}\right),$$

by Lemma 7.3 below, there is a $\mathbb{P}\otimes_\tau Q$-null set $N\in\mathcal{B}_\tau$ so that for all $\omega\notin N$,

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}\otimes_\tau Q}\left(\int_{t_0\vee\tau}^{t_1\vee\tau} f(r,\omega_r)\mathrm{d}r\Big|\mathcal{B}_{t_0\vee\tau}\right)&=\mathbb{E}^{Q_\omega}\left(\int_{t_0\vee\tau}^{t_1\vee\tau} f(r,\omega_r)\mathrm{d}r\Big|\mathcal{B}_{t_0\vee\tau}\right)\\
&\leqslant\kappa(t_1-t_0)^\theta\|f\|_{\mathbb{L}^{q,p}_{t,x}},\ Q_\omega-a.s.
\end{aligned}$$

Hence,

$$I_1\leqslant\kappa(t_1-t_0)^\theta\|f\|_{\mathbb{L}^{q,p}_{t,x}},\ \mathbb{P}\otimes_\tau Q-a.s.$$

For I_2, since $\mathbb{P}\otimes_\tau Q|_{\mathcal{B}_\tau}=\mathbb{P}|_{\mathcal{B}_\tau}$, we have

$$I_2=\mathbf{1}_{\{t_0<\tau\leqslant t_1\}}\mathbb{E}^{\mathbb{P}}\left(\int_{t_0}^{\tau} f(r,\omega_r)\mathrm{d}r\Big|\mathcal{B}_{t_0}\right)\leqslant \kappa(t_1-t_0)^\theta\|f\|_{\mathbb{L}^{q,p}_{t,x}}.$$

For I_3, since $(\mathbb{P}\otimes_\tau Q)(\cdot|\mathcal{B}_\tau)(\omega)=Q_\omega$, we have

$$\begin{aligned}I_3&=\mathbf{1}_{\{t_0<\tau\leqslant t_1\}}\mathbb{E}^{\mathbb{P}\otimes_\tau Q}\left(\int_{\tau}^{t_1} f(r,\omega_r)\mathrm{d}r\Big|\mathcal{B}_{t_0\wedge\tau}\right)\\&=\mathbf{1}_{\{t_0<\tau\leqslant t_1\}}\mathbb{E}^{\mathbb{P}}\left(\mathbb{E}^{Q_\cdot}\left(\int_{t_0\wedge\tau}^{\tau} f(r,\omega_r)\mathrm{d}r\right)\Big|\mathcal{B}_{t_0\wedge\tau}\right)\\&\leqslant \kappa(t_1-t_0)^\theta\|f\|_{\mathbb{L}^{q,p}_{t,x}}.\end{aligned}$$

Lastly, for I_4 we have

$$I_4=\mathbf{1}_{\{t_1<\tau\}}\mathbb{E}^{\mathbb{P}}\left(\int_{t_0}^{t_1} f(r,\omega_r)\mathrm{d}r\Big|\mathcal{B}_{t_0}\right)\leqslant \kappa(t_1-t_0)^\theta\|f\|_{\mathbb{L}^{q,p}_{t,x}}.$$

Combining the above calculations, we obtain (5.4). The proof is competed by Theorem 7.2 below. □

6. Examples

For $R\geqslant 1$, let $\phi_R:[0,\infty)\to[0,\infty)$ be a smooth increasing function with

$$\phi_R(r)=r,\quad r\leqslant R;\quad \phi_R(r)=R+1,\quad r\geqslant 2R.$$

For $\alpha\in\mathbb{R}$ and $n\in\mathbb{N}$, define

$$f^{(\alpha)}_R(r):=(\phi_R(r))^\alpha,\quad f^{(\alpha)}_{R,n}(r):=(\phi_R(r+\tfrac{1}{n}))^\alpha.$$

Clearly,

$$f^{(\alpha)}_R(r)=r^\alpha\ \text{ for } r<R \ \text{ and }\ f^{(\alpha)}_{R,n}(r)=(r+\tfrac{1}{n})^\alpha\ \text{ for } r+\tfrac{1}{n}<R.$$

Below we provide two examples to illustrate the assumption $(\widetilde{\mathbf{H}}^\sigma)$.

Example 6.1. Let $d\geqslant 3$ and $0<\alpha<(\frac{d}{2}-1)\wedge(\frac12+\frac{1}{d-1})$. Let

$$\sigma(x)=f^{(-\frac{\alpha}{2})}_R(|x|^2)\mathbb{I}_{d\times d}.$$

We verify $(\widetilde{\mathbf{H}}^\sigma)$ for $\sigma_n(x)=f^{(-\frac{\alpha}{2})}_{R,n}(|x|^2)\mathbb{I}_{d\times d}$. Note that

$$a_n(x)=(\sigma_n\sigma_n^*)(x)=f^{(-\alpha)}_{R,n}(|x|^2)\mathbb{I}_{d\times d}.$$

Thus,

$$\lambda_n(x)=\mu_n(x)=f^{(-\alpha)}_{R,n}(|x|^2).$$

In particular, we have

$$\lambda_n^{-1}(x)\leqslant \phi_R^\alpha(|x|^2+1)\in\widetilde{\mathbb{L}}^\infty(\mathbb{R}^d),$$

and for $p_1<\frac{d}{2\alpha}$,

$$\mu_n(x)\leqslant \phi_R^{-\alpha}(|x|^2)\in\widetilde{\mathbb{L}}^{p_1}(\mathbb{R}^d).$$

On the other hand, by the chain rule, we have

$$\partial_i a_n^{ij}(x)=2x_j(f^{(-\alpha)}_{R,n})'(|x|^2)$$

and

$$\partial_i\partial_j a_n^{jj}(x)=\Delta f^{(-\alpha)}_{R,n}(|\cdot|^2)(x)=2d(f^{(-\alpha)}_{R,n})'(|x|^2)+4|x|^2(f^{(-\alpha)}_{R,n})''(|x|^2).$$

Note that

$$(f^{(-\alpha)}_{R,n})'(r)=-\alpha\phi_R(r+\tfrac{1}{n})^{-\alpha-1}\phi_R'(r+\tfrac{1}{n})$$

and

$$(f_{R,n}^{(-\alpha)})''(r) = -\alpha\phi_R(r+\tfrac{1}{n})^{-\alpha-1}(\phi_R'(r+\tfrac{1}{n}) + \phi_R''(r+\tfrac{1}{n}))$$
$$+\alpha(\alpha+1)\phi_R(r+\tfrac{1}{n})^{-\alpha-2}(\phi_R'(r+\tfrac{1}{n}))^2.$$

It is easy to see that for $p_1 < \frac{d}{2\alpha+1}$,

$$|\partial_i a_n^{ij}(x)| \leqslant 2\alpha|x|^{-2\alpha-1}\mathbf{1}_{\{|x|^2\leqslant 2R\}} \in \widetilde{\mathbb{L}}^{p_1}(\mathbb{R}^d),$$

and due to $\alpha < \frac{d}{2} - 1$,

$$\partial_i\partial_j a_n^{jj}(x) \leqslant C_{\alpha,R}.$$

Hence, (4.1) holds for $p_0 = \infty$, $p_1 \in (\frac{d-1}{2}, \frac{d}{2\alpha+1})$ and $p_2 = q_2 = \infty$. Moreover, if $p_3 < \frac{d}{\alpha}$, then

$$|\sigma_n(x)| \leqslant \phi_R^{-\frac{\alpha}{2}}(|x|^2) \in \widetilde{\mathbb{L}}^{p_3}(\mathbb{R}^d).$$

Thus, (4.2) holds for $p_3 \in (d, \frac{d}{\alpha})$ and $q_3 = \infty$. Therefore, $(\widetilde{\mathbf{H}}^\sigma)$ is satisfied for the above $\sigma_n(x)$. In particular, by Theorem 4.3, there exists at least one solution for the following singular SDE:

$$\mathrm{d}X_t = \phi_R(|X_t|^2)^{-\alpha/2}\mathrm{d}W_t + b(X_t)\mathrm{d}t, \quad X_0 = x,$$

where $\alpha \in (0, (\frac{d}{2}-1)\wedge(\frac{1}{2}+\frac{1}{d-1}))$ and $b \in \widetilde{L}^p$ for some $p > \frac{d}{2}$ satisfies $(\mathrm{div} b)^- = 0$.

Proposition 6.2. *Let $d \geqslant 3$, $\alpha \in (0, (\frac{d}{2}-1)\wedge(\frac{1}{2}+\frac{1}{d-1}))$, $\beta \in (0, 2\alpha)$ and $\lambda \geqslant 0$. For each $x \in \mathbb{R}^d$, the following SDEs admits a unique strong solution:*

$$\mathrm{d}X_t = |X_t|^{-\alpha}\mathrm{d}W_t + \lambda X_t|X_t|^{-\beta-1}\mathrm{d}t, \quad X_0 = x. \tag{6.1}$$

Proof. Let $b(x) := \lambda x|x|^{-\beta-1}$, and for $R \in \mathbb{N}$, let $\sigma_R(x) = \phi_R(|x|^2)^{-\alpha/2}\mathbb{I}$. Since $\lambda \geqslant 0$ and $\beta < 2$, it is easy to see that $b \in \widetilde{L}^p$ for any $p \in (\frac{d}{2}, \frac{d}{\beta})$ and $(\mathrm{div} b)^- \equiv 0$. Let X_t^R solve the following SDE:

$$X_t^R = x + \int_0^t \sigma_R(X_s^R)\mathrm{d}W_s + \int_0^t b(X_s^R)\mathrm{d}s.$$

Let $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a smooth function with $\Phi(r) = 1$ for $|r| \leqslant 1$ and $\Phi(r) = r$ for $r > 2$. By Itô's formula, it is easy to see that

$$\sup_{R\in\mathbb{N}} \mathbf{E}\left(\sup_{t\in[0,T]} \Phi(|X_t^R|^2)\right) \leqslant C.$$

From this, by Chebyshev's inequality, we derive that

$$\lim_{R\to\infty} \mathbf{P}\left(\sup_{t\in[s,T]} |X_t^R| > R\right) = 0,$$

which together with Theorem 5.5 implies that the assumptions of Theorem 7.5 is satisfied. So, there exists a solution to SDE (6.1). To show the pathwise uniqueness, note that

$$|\nabla b(x)| \leqslant C|x|^{-\beta-1},$$

and for any $R > 0$,

$$\int_{B_R} |x|^{-(\beta+1)d}\det(\sigma\sigma^*)^{-1}(x)\mathrm{d}x = \int_{B_R} |x|^{-(\beta+1)d+2\alpha d}\mathrm{d}x < \infty.$$

Thus by [18, Theorem 1.1] and the computations in Example 1 of [18], we obtain the uniqueness. □

Example 6.3. Let $d = 2$ and $\alpha \in (0, \frac{1}{2})$. Consider the following diffusion matrix:

$$\sigma(x) = \begin{pmatrix} f_R^{(\frac{\alpha}{2})}(|x_2|^2), & 0 \\ 0, & f_R^{(\frac{\alpha}{2})}(|x_1|^2) \end{pmatrix}.$$

Let us define

$$\sigma_n(x) := \begin{pmatrix} f_{R,n}^{(\frac{\alpha}{2})}(|x_2|^2), & 0 \\ 0, & f_{R,n}^{(\frac{\alpha}{2})}(|x_1|^2) \end{pmatrix}, \ a_n(x) := \begin{pmatrix} f_{R,n}^{(\alpha)}(|x_2|^2), & 0 \\ 0, & f_{R,n}^{(\alpha)}(|x_1|^2) \end{pmatrix}.$$

Then

$$\lambda_n(x) = f_{R,n}^{(\alpha)}(|x_2|^2) \wedge f_{R,n}^{(\alpha)}(|x_1|^2), \quad \mu_n(x) = f_{R,n}^{(\alpha)}(|x_2|^2) \vee f_{R,n}^{(\alpha)}(|x_1|^2).$$

Clearly, we have

$$\lambda_n^{-1}(x) = f_{R,n}^{(-\alpha)}(|x_2|^2) \vee f_{R,n}^{(-\alpha)}(|x_1|^2)$$

and

$$\partial_i a_n^{ij}(x) = \partial_i \partial_j a_n^{ij}(x) = 0.$$

Thus, (4.1) holds for $p_0 \in (1, \frac{1}{2\alpha})$, $p_1 = \infty$ and $p_2 = q_2 = \infty$. Moreover, it is easy to see that (4.2) holds for $q_3 = \infty$ and any $p_3 \in (\frac{2p_0}{p_0-1}, \infty)$. Therefore, $(\widetilde{\mathbf{H}}^\sigma)$ holds for the above $\sigma_n(x)$. As in Proposition 6.2, by Theorems 5.5 and 7.5, for any starting point $X_0 = x \in \mathbb{R}^2$, there exists at least one solution for the following two dimensional degenerate SDE:

$$\begin{cases} \mathrm{d}X_t^1 = |X_t^2|^\alpha \mathrm{d}W_t^1 + b^1(X_t)\mathrm{d}t, \\ \mathrm{d}X_t^2 = |X_t^1|^\alpha \mathrm{d}W_t^2 + b^2(X_t)\mathrm{d}t, \end{cases}$$

where $\alpha \in (0, \frac{1}{2})$ and $b = (b^1, b^2) \in \widetilde{L}^p(\mathbb{R}^2)$ for some $p > \frac{1}{1-2\alpha}$, and for some $K \in \mathbb{N}$,

$$|b(x)| \leqslant C|x|, \quad |x| > K.$$

We would like to say some words about the range of p. Intuitively, when X_t moves to the unit ball, smaller α means stronger noise and so the drift b could be more singular. While, the uniqueness for the above example is left open, even for $b = 0$.

7. Appendix

We first recall the following lemma (cf. [16, Theorem 6.1.2]).

Lemma 7.1. *Let τ be a finite stopping time and $\mathbb{C} \ni \omega \mapsto Q_\omega \in \mathcal{P}(\mathbb{C})$ be a $\mathcal{B}_\tau$-measurable probability kernel. Given a probability measure $\mathbb{P} \in \mathcal{P}(\mathbb{C})$, there exists a unique probability measure $\mathbb{P} \otimes_\tau Q \in \mathcal{P}(\mathbb{C})$ so that*

$$(\mathbb{P} \otimes_\tau Q)|_{\mathcal{B}_\tau} = \mathbb{P}|_{\mathcal{B}_\tau}, \quad (\mathbb{P} \otimes_\tau Q)(\cdot|\mathcal{B}_\tau)(\omega) = Q_\omega(\cdot).$$

In particular,

$$(\mathbb{P} \otimes_\tau Q)(\Gamma) = \int_{\mathbb{C}} Q_\omega(\Gamma)\mathbb{P}(\mathrm{d}\omega), \quad \forall \Gamma \in \mathcal{B} := \vee_{t \geqslant 0} \mathcal{B}_t.$$

For each $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, let $\mathscr{C}(s, x)$ be a non-empty convex subset of $\mathcal{P}(\mathbb{C})$ with

$$\mathbb{P}\{\omega : \omega_s = x\} = 1.$$

We suppose that $\{\mathscr{C}(s, x) : (s, x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ satisfies

(C1) Let (s_n, x_n) converge to (s, x). For any sequence $\mathbb{P}_n \in \mathscr{C}(s_n, x_n)$, there is a subsequence n_k and $\mathbb{P} \in \mathscr{C}(s, x)$ so that $\mathbb{P}_{n_k}$ converges to $\mathbb{P}$.

(C2) (Disintegration) Let $\mathbb{P} \in \mathscr{C}(s,x)$ and $\tau \geqslant s$ be a finite stopping time. For any r.c.p.d. $(\mathbb{P}_\omega)_{\omega\in\mathbb{C}}$ of $\mathbb{E}^{\mathbb{P}}(\cdot|\mathcal{B}_\tau)$, there is a $\mathbb{P}$-null set $N \in \mathcal{B}_\tau$ so that

$$\mathbb{P}_\omega \in \mathscr{C}(\tau(\omega), \omega_{\tau(\omega)}),\ \omega \notin N.$$

(C3) (Reconstruction) Let $\mathbb{P} \in \mathscr{C}(s,x)$ and $\tau \geqslant s$ be a finite stopping time. For any $\mathcal{B}_\tau$-measurable kernel $\mathbb{C} \ni \omega \mapsto Q_\omega \in \mathcal{P}(\mathbb{C})$ with

$$Q_\omega \in \mathscr{C}(\tau(\omega), \omega_{\tau(\omega)}), \forall \omega \in \mathbb{C},$$

it holds that

$$\mathbb{P} \otimes_\tau Q \in \mathscr{C}(s,x).$$

We have the following strong Markov selection theorem, whose proofs are completely the same as in [16, Theorem 12.2.3] (see also [5] and [6, Theorem 2.7]). We omit the details.

Theorem 7.2. *Under* **(C1)**, **(C2)** *and* **(C3)**, *there is a measurable selection*

$$\mathbb{R}_+ \times \mathbb{R}^d \ni (s,x) \mapsto \mathbb{P}_{s,x} \in \mathscr{C}(s,x)$$

so that for any $(s,x) \in \mathbb{R}_+ \times \mathbb{R}^d$ and finite stopping time $\tau \geqslant s$, $\omega \mapsto \mathbb{P}_{\tau(\omega), x(\tau(\omega),\omega)}$ is a r.c.p.d. of $\mathbb{P}_{s,x}$ with respect to $\mathcal{B}_\tau$. More precisely, there is a $\mathbb{P}_{s,x}$-null set $N \in \mathcal{B}_\tau$ such that for all $\omega \notin N$,

$$\mathbb{P}_{s,x}(\cdot|\mathcal{B}_\tau)(\omega) = \mathbb{P}_{\tau(\omega),\omega_{\tau(\omega)}}(\cdot).$$

The following two simple lemmas are used in the proof of Theorem 5.5 (see [6]).

Lemma 7.3. *Let $\mathscr{G} \subset \mathscr{C}$ be two countably generated sub σ-algebras of $\mathcal{B}$. Given $\mathbb{P} \in \mathcal{P}(\mathbb{C})$, let Q_ω be a r.c.p.d. of $\mathbb{P}$ with respect to $\mathscr{G}$. Then there is a $\mathbb{P}$-null set $N \in \mathscr{G}$ depending on $\mathscr{C}$ and ξ such that for all $\omega \notin N$,*

$$\mathbb{E}^{\mathbb{P}}(\xi|\mathscr{C}) = \mathbb{E}^{Q_\omega}(\xi|\mathscr{C}), \quad Q_\omega - a.s.$$

Proof. Let $A \in \mathscr{G}$ and $B \in \mathscr{C}$. By definition, we have

$$\begin{aligned}\int_A \mathbb{E}^{Q_\omega}(1_B\mathbb{E}^{\mathbb{P}}(\xi|\mathscr{C}))\mathbb{P}(\mathrm{d}\omega) &= \int_A \mathbb{E}^{\mathbb{P}}(1_B\mathbb{E}^{\mathbb{P}}(\xi|\mathscr{C})|\mathscr{G})(\omega)\mathbb{P}(\mathrm{d}\omega)\\ &= \mathbb{E}^{\mathbb{P}}(1_A 1_B \xi) = \int_A \mathbb{E}^{Q_\omega}(1_B\xi)\mathbb{P}(\mathrm{d}\omega)\\ &= \int_A \mathbb{E}^{Q_\omega}(1_B\mathbb{E}^{Q_\omega}(\xi|\mathscr{C}))\mathbb{P}(\mathrm{d}\omega).\end{aligned}$$

Hence, for each $B \in \mathscr{C}$, there is a $\mathbb{P}$-null set $N_B \in \mathscr{G}$ so that for all $\omega \notin N$,

$$\mathbb{E}^{Q_\omega}(1_B\mathbb{E}^{\mathbb{P}}(\xi|\mathscr{C})) = \mathbb{E}^{Q_\omega}(1_B\mathbb{E}^{Q_\omega}(\xi|\mathscr{C})).$$

Since $\mathscr{C}$ is countably generated, one can find a common null set $N_{\xi,\mathscr{C}}$ so that for all $\omega \notin N$ and $B \in \mathscr{C}$,

$$\mathbb{E}^{Q_\omega}(1_B\mathbb{E}^{\mathbb{P}}(\xi|\mathscr{C})) = \mathbb{E}^{Q_\omega}(1_B\mathbb{E}^{Q_\omega}(\xi|\mathscr{C})),$$

which in turn yields the desired result. □

Lemma 7.4. *Let τ be a finite stopping time and Q_ω be a r.c.p.d. of $\mathbb{P}$ with respect to $\mathcal{B}_\tau$. Let X_t be a bounded continuous process. Suppose that for any $t \geqslant 0$,*

$$\mathbb{E}^{\mathbb{P}}(X_t|\mathcal{B}_t) \leqslant A, \quad \mathbb{P} - a.s.$$

Then there is a $\mathbb{P}$-null set $N \in \mathcal{B}_\tau$ such that for all $\omega \notin N$ and $t \geqslant \tau(\omega)$,

$$\mathbb{E}^{Q_\omega}(X_t|\mathcal{B}_t) \leqslant A, \quad Q_\omega - a.s.$$

Proof. By Lemma 4.4, we have

$$\mathbb{E}^{\mathbb{P}}(X_{t\vee\tau}|\mathcal{B}_{t\vee\tau}) \leqslant A, \quad t \geqslant 0.$$

By Lemma 7.3, there is a $\mathbb{P}$-null set N such that for all $\omega \notin N$ and all rational number $t > 0$,

$$\mathbb{E}^{Q_\omega}(X_{t\vee\tau}|\mathcal{B}_{t\vee\tau}) \leqslant A, \quad Q_\omega - a.s.$$

For fixed $\omega \notin N$, since (cf. [16, p34. (3.15)])

$$Q_\omega\{\omega' : \tau(\omega') = \tau(\omega)\} = 1,$$

we have for all rational number $t \geqslant \tau(\omega)$,

$$\mathbb{E}^{Q_\omega}(X_t|\mathcal{B}_t) = \mathbb{E}^{Q_\omega}(X_{t\vee\tau}|\mathcal{B}_{t\vee\tau}) \leqslant A, \quad Q_\omega - a.s.$$

Now for general $t \geqslant \tau(\omega)$, let $t_n \downarrow t$ be rational numbers. By the dominated convergence theorem, we have

$$\mathbb{E}^{Q_\omega}(X_t|\mathcal{B}_t) = \lim_{t_n\downarrow t}\mathbb{E}^{Q_\omega}(X_{t_n}|\mathcal{B}_t) = \lim_{t_n\downarrow t}\mathbb{E}^{Q_\omega}(X_{t_n}|\mathcal{B}_{t_n}|\mathcal{B}_t) \leqslant A.$$

The proof is complete. □

The following result provides a way of constructing a global solution from local solutions.

Theorem 7.5. *Suppose that for each $R \in \mathbb{N}$ and $(s,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, there is at least one local martingale solution $\mathbb{P}^R_{s,x} \in \mathcal{M}^{\sigma_R,b_R}_{s,x}$ so that $(s,x) \mapsto \mathbb{P}^R_{s,x}$ is Borel measurable, where*

$$\sigma_R(t,x) := \sigma(t,\chi_R(x)x), \quad b_R(t,x) := b(t,\chi_R(x)x),$$

and

$$\chi_R(x) = 1,\ |x| \leqslant 2^{R-1}, \quad \chi_R(x) = 0,\ |x| > 2^R.$$

Fix $(s_0,x_0) \in \mathbb{R}_+ \times \mathbb{R}^d$. If for each $T > s_0$ and any choice of $\mathbb{P}^R_{s_0,x_0}$ from $\mathcal{M}^{\sigma_R,b_R}_{s_0,x_0}$,

$$\lim_{R\to\infty}\mathbb{P}^R_{s_0,x_0}\left(\sup_{t\in[s,T]}|\omega_t| > R\right) = 0, \tag{7.1}$$

then there is at least one local martingale solution $\mathbb{P} \in \mathcal{M}^{\sigma,b}_{s_0,x_0}$. In particular, there is a global weak solution $(\mathfrak{F}, X, W)$ for SDE (1.13).

Proof. Without loss of generality, we assume $(s_0,x_0) = (0,0)$. Let $\tau_0 = 0$. We define a sequence of stopping times recursively by

$$\tau_n := \inf\{t > \tau_{n-1} : |\omega_t| > 2^{n-1}\} = \inf\{t > 0 : |\omega_t| > 2^{n-1}\}, \quad n \in \mathbb{N}.$$

Let $\mathbb{P}^n_{s,x} \in \mathcal{M}^{\sigma_n,b_n}_{s,x}$ be as in the assumptions. Define for $n \in \mathbb{N}$,

$$Q^n_\omega := \mathbb{P}^{n+1}_{\tau_n(\omega),\omega_{\tau_n(\omega)}},\ \omega \in \mathbb{C}.$$

Since $(s,x) \mapsto \mathbb{P}^{n+1}_{s,x}$ is measurable, $\omega \mapsto Q^n_\omega$ is a $\mathcal{B}_{\tau_n}$-measurable probability kernel on $\mathbb{C} \times \mathcal{B}$, i.e., for each $\Gamma \in \mathcal{B}$, $\omega \mapsto Q^n_\omega(\Gamma)$ is $\mathcal{B}_{\tau_n}$-measurable, and for each $\omega \in \mathbb{C}$, $Q_\omega \in \mathcal{P}(\mathbb{C})$. Let $\widetilde{\mathbb{P}}_1 \in \mathcal{M}^{\sigma_1,b_1}_{0,0}$. Define for $n \geqslant 2$,

$$\widetilde{\mathbb{P}}_{n+1} := \widetilde{\mathbb{P}}_1 \otimes_{\tau_1} \mathbb{Q}^1 \otimes_{\tau_2} \cdots \otimes_{\tau_n} \mathbb{Q}^n.$$

By the construction and Lemma 7.1, one sees that

$$\widetilde{\mathbb{P}}_{n+1}|_{\mathcal{B}_{\tau_n}} = (\widetilde{\mathbb{P}}_n \otimes_{\tau_n} \mathbb{Q}^n)|_{\mathcal{B}_{\tau_n}} = \widetilde{\mathbb{P}}_n|_{\mathcal{B}_{\tau_n}},$$

and by [16, Theorem 1.2.10],

$$\widetilde{\mathbb{P}}_n \in \mathcal{M}^{\sigma_n,b_n}_{0,0}.$$

Moreover, by (7.1), for each $T > 0$,

$$\lim_{n\to\infty} \widetilde{\mathbb{P}}_n(\tau_n < T) = 0.$$

Finally, by [16, Theorem 1.3.5], there is a unique $\mathbb{P} \in \mathcal{P}(\mathbb{C})$ so that for each $n \in \mathbb{N}$,

$$\mathbb{P}|_{\mathcal{B}_{\tau_n}} = \widetilde{\mathbb{P}}_n|_{\mathcal{B}_{\tau_n}}.$$

The proof is complete. □

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