LARGE DEVIATIONS AND A PHASE TRANSITION IN THE BLOCK SPIN POTTS MODELS

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Abstract. We introduce and analyze a generalization of the blocks spin Ising (Curie-Weiss) models that were discussed in a number of recent articles. In these block spin models each spin in one of $s$ blocks can take one of a finite number of $q \geq 3$ values, hence the name block spin Potts model. The values a spin can take are called colors. We prove a large deviation principle for the percentage of spins of a certain color in a certain block. These values are represented in an $s \times q$ matrix. We show that for uniform block sizes and appropriately chosen interaction strength there is a phase transition. In some regime the only equilibrium is the uniform distribution of all colors in all blocks, while in other parameter regimes there is one predominant color, and this is the same color with the same frequency for all blocks. Finally, we establish log-Sobolev-type inequalities for the block spin Potts model.

1. Introduction

Mean-field models as the Curie–Weiss model are a first order approximation of lattice models, yet they often show qualitatively interesting results (see [13] for a classic survey). In particular, mean-field block models have been proposed as an approximation of lattice models for meta-magnets, see e.g. [24]. To describe them, assume that we have $N$ interacting particles that carry a spin. Also assume that we can group these particles into several groups. The interaction is such that particles within the same group interact with one interaction strength, while particles in different groups have another, usually smaller, interaction strength.

In a sequence of papers the statistical mechanics of such models was studied from various points of view, see [17], [15], [9], [30], [25], [27], [26]. In particular they were also discussed as models for social interactions between several groups, e.g. in [16], [2], [34], [32] (the latter paper studies a combination of sparse Ising models on Erdős–Rényi graphs as in [7], [21], or [22] and block models). Recently, block models have also been studied in a statistical context (see [3], [31]). Here the task is to
exactly recover the block structure from a given number of realizations of the model
and it turns out that this can be done surprisingly effectively.
However, all the literature cited above deals with Ising spins, i.e. the spins take two
values (usually $\pm 1$). Of course, the physics literature knows many more spin models
than just the Ising model, in particular models with a continuous spin as Heisenberg
models and XY-models.
On the discrete side Potts models (cf. e.g. [38, 23, 14, 10]) are the most natural
generalization of Ising models. For them each particle carries a spin from a finite set
(of cardinality 3 or larger).
The aim of the present note is to investigate block spin Potts models as a natural
generalization of block spin Ising models. We will basically concentrate on models
where the blocks have approximately identical size and where the interaction is purely
ferromagnetic, i.e. particles tend to have the same spin, no matter, whether they
are in the same block or in different ones.
Similar to [30] and parts of [27] our main tool are large deviation techniques. Indeed,
as we will see in Section 3, it is not too difficult to establish a large deviation principle
for the “block magnetizations”. However, to derive a limit theorem with an explicit
limit law from there turns out to be more complicated than in the case of Ising spins
(which is quite a common feature in Potts models).
The rest of this note is organized in the following way. In the next section we will
describe the block spin Potts model. Section 3 contains a large deviation analysis of
this model. In Section 4, we will concentrate on a version with blocks of asymptotically
equal size and compute the possible limit laws for such models. Finally, in Section 5
we prove (modified) logarithmic Sobolev inequalities for the block spin Potts model
and discuss some background and possible applications.
Let us mention at this point that, while we were finishing the current manuscript
we learned that in [29] the author studies a very similar model: Here the number of
blocks is restricted to two, but they may be of different size. His techniques, however
are different from ours. Moreover, extending his results, we prove a large deviation
principle, are able to locate the minima of the rate functions and show logarithmic
Sobolev inequalities.

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2. The model

In the sequel we will consider the following model. Take the set $S = \{1, \ldots, N\}$ and
partition $S$ into $s$ sets $S_1, \ldots, S_s$. These sets will, of course, depend on $N$ and we
assume that the limits $\gamma_k := \lim_{N \to \infty} \frac{|S_k|}{N} \in (0, 1)$ exist (and, of course, $\sum_{k=1}^s \gamma_k = 1$).
Moreover, take an integer $q \geq 3$ and for $\omega \in \{1, \ldots, q\}^S$ and $0 < \alpha < \beta$ introduce the Hamiltonian

$$H_{N,\alpha,\beta}(\omega) := H_N(\omega) := -\frac{\beta}{2N} \sum_{i \sim j} 1_{\omega_i = \omega_j} - \frac{\alpha}{2N} \sum_{i \not\sim j} 1_{\omega_i = \omega_j}.$$
Here $i \sim j$ means that the indices $i$ and $j$ belong to the same block $S_k$ (where the case $i = j$ is included) for some $k \in \{1, \ldots, s\}$, while we write $i \not\sim j$, if this is not the case. With $H_{N,\alpha,\beta}$ we will associate the Gibbs measure
\[
\mu_{N,\alpha,\beta}(\omega) := \mu_N(\omega) := \frac{\exp(-H_N(\omega))}{Z_{N,\alpha,\beta}}
\]
where, of course,
\[
Z_{N,\alpha,\beta} := Z_N := \sum_{\omega'} \exp(-H_N(\omega')).
\]
For $k \in \{1, \ldots, s\}$ and $c \in \{1, \ldots, q\}$ denote by $m_{k,c}$ the relative number of spins of “color” $c$ in the block $S_k$, i.e.
\[
m_{k,c} := m_{k,c}(\omega) := \frac{1}{|S_k|} \sum_{i \in S_k} 1_{\omega_i = c},
\]
and set $M_N := (m_{k,c}) \in M(s \times q)$. Note that $M_N$ is an order parameter of the model in the sense that the Hamiltonian is a function of $M_N$ rather than $\omega$.

Indeed, since $1_{\omega_i = \omega_j} = \sum_{c=1}^q 1_{\omega_i = c} 1_{\omega_j = c}$ we have
\[
-2NH_N(\omega) = \beta \sum_{c=1}^q \sum_{k=1}^s \sum_{j \in S_k} 1_{\omega_i = c} 1_{\omega_j = c} + \alpha \sum_{c=1}^q \sum_{k \neq k'} \sum_{i \in S_k} \sum_{j \in S_{k'}} 1_{\omega_i = c} 1_{\omega_j = c}
\]
(2.1)
\[
= \operatorname{tr}(B^tAB)
\]
where $A_{\alpha,\beta} := A \in M(s \times s)$ is the symmetric matrix with entries $\beta$ on and $\alpha$ off the diagonal (the block interaction matrix) and $B \in M(s \times q)$ has entries
\[
b_{k,c} = \sum_{i \in S_k} 1_{\omega_i = c} = |S_k| m_{k,c}.
\]

Now $A$ is positive definite for $0 < \alpha < \beta$ due to $x^t Ax = (\beta - \alpha)x^2 + \alpha(\sum_k x_k)^2$ for $x \in \mathbb{R}^s$. So using the (unique) positive definite symmetric matrix $\sqrt{A}$ we see that
\[
\operatorname{tr}(B^tAB) = \operatorname{tr} \left( (\sqrt{A})^t (\sqrt{A}) \right) = |\sqrt{A}, \sqrt{A}|
\]
denoting by $[\cdot, \cdot]$ the Frobenius scalar product. Hence the Hamiltonian is a positive definite quadratic form of the matrix $B$ and we will write $\operatorname{tr}(B^tAB) = \langle B, B \rangle_A$. Now introducing the diagonal matrix $\Gamma_N \in M(s \times s)$ given by $(\Gamma_N)_{k,k} = |S_k|$ we finally rewrite (2.1) as
\[
H_N(\omega) = -\frac{1}{2N} \langle \Gamma_N M_N, \Gamma_N M_N \rangle_A.
\]
(2.3)

It is therefore natural to study the distribution of $M_N$ under the Gibbs measure $\mu_N$.

3. A LARGE DEVIATION PRINCIPLE FOR $M_N$

In this section we prove a Large Deviation Principle (LDP) for the matrix $M_N$. The analysis of the corresponding rate function will help us to determine the limiting behavior of $M_N$ and to prove the existence of a phase transition.

Let us briefly recall the definition of a large deviation principle (cf. \[12\] and \[11\] for a rich survey of many large deviation results): For a Polish space $\mathcal{X}$ and an increasing sequence of non-negative real numbers $(a_n)_{n \in \mathbb{N}}$ a sequence of probability
measures \((\nu_n)_n\) on \(\mathcal{X}\) is said to satisfy a large deviation principle with speed \(a_n\) and rate function \(I : \mathcal{X} \to \mathbb{R}\) (by which we mean a lower semi-continuous function with compact level sets \(\{x : I(x) \leq L\}\) for all \(L > 0\)), if for all Borel sets \(B \in \mathcal{B}(\mathcal{X})\) we have
\[
- \inf_{x \in \text{int}(B)} I(x) \leq \liminf_{n \to \infty} \frac{\log \nu_n(B)}{a_n} \leq \limsup_{n \to \infty} \frac{\log \nu_n(B)}{a_n} \leq - \inf_{x \in \text{cl}(B)} I(x).
\]

Here \(\text{int}(B)\) and \(\text{cl}(B)\) denote the topological interior and closure of a set \(B\), respectively.

We say that a sequence of random variables \(X_n : \Omega \to \mathcal{X}\) satisfies an LDP with speed \(a_n\) and rate function \(I : \mathcal{X} \to \mathbb{R}\) under a sequence of measures \(\mu_n\) if the push-forward sequence \(\nu_n := \mu_n \circ X_n^{-1}\) satisfies an LDP with speed \(a_n\) and rate function \(I\).

Now if \(M_N(k)\) denotes the row \(k\) of \(M_N\) for a fixed \(k\), then under the uniform measure \(\rho\) on \(\{1,\ldots,q\}\) the vector \(M_N(k)\) is the empirical vector of a \(|S_k|\)-fold drawing from the alphabet \(\{1,\ldots,q\}\). Thus, under the uniform measure \(\rho^{\lfloor S_k\rfloor}\) the vector \(M_N(k)\) obeys an LDP with speed \(|S_k|\) and a rate function that is given by the relative entropy of a probability measure \(\nu \in \mathcal{M}^\Gamma(\{1,\ldots,q\})\) with respect to \(\rho\)
\[
H(\nu|\rho) := \frac{1}{q} \sum_{c=1}^q \nu(c) \log \frac{\nu(c)}{\rho(c)}
\]
(see e.g. [11, Theorem 2.1.10] for a reference). Note that
\[
H(\nu) = \sum_{c=1}^q \nu(c) \log \nu(c) + \sum_{c=1}^q \nu(c) \log q =: H(\nu) + \log q.
\]

where \(H(\nu)\) is the entropy of \(\nu\) and we adopt the convention that \(0 \log 0 = 0\).

Now \(M_N(k)\) are independent random vectors for \(k = 1,\ldots,s\). Consequently, for \(\bar{\nu}_N := \otimes_{k=1}^s \rho^{\lfloor S_k\rfloor}\) we have
\[
\frac{1}{N} \log \bar{\nu}_N(M_N \in B) = \frac{1}{N} \sum_{k=1}^s \log \rho^{\lfloor S_k\rfloor}(M_N(k) \in B_k).
\]

for any set \(B = \prod_{k=1}^s B_k\) with Borel sets \(B_k \subseteq \mathbb{R}^q\) (here we associate probabilities \(\nu\) on the set \(\{1,\ldots,q\}\) with vectors in \(\mathbb{R}^q\) and define \(H(\nu|\rho) = H(\nu) = \infty\), if \(\nu \in \mathbb{R}^q\) does not have non-negative components summing to 1). Together with the above mentioned LDP for the components \(M_N(k)\) and the assumption that \(|S_k|/N\) converges to \(\gamma_k\) as \(N \to \infty\), this observation implies that the matrix \(M_N\) under \(\bar{\nu}_N\) obeys an LDP with speed \(N\) and rate function
\[
I(\nu) := \sum_{k=1}^s \gamma_k H(\nu_k|\rho) = \log q + \sum_{k=1}^s \gamma_k H(\nu_k)
\]

Here \(\nu := (\nu_k)_{1 \leq k \leq s} \in M(s \times q)\) and the \(\nu_k\) are probabilities on \(\{1,\ldots,q\}\), otherwise \(I(\nu)\) is defined to be \(\infty\). Thus we have seen

**Proposition 3.1.** Under the measure \(\bar{\nu}_N\) the matrix valued random variable \(M_N\) obeys an LDP with speed \(N\) and rate function \(I\) given by (3.1).

Proposition 3.1 together with the representation of our Hamiltonian in terms of the matrix \(M_N\) (2.3) immediately yields an LDP for \(M_N\) under the Gibbs measure \(\mu_N\).
Theorem 3.2. Under the Gibbs measure $\mu_{\alpha,\beta,N}$ the matrix valued random variable $M_N$ obeys an LDP with speed $N$ and rate function $J = J_{\alpha,\beta}$

$$J(\nu) := - \frac{1}{2} (\nu, \nu)_A - I(\nu) + \sup_{\mu} \left[ \frac{1}{2} (\mu, \mu)_A - I(\mu) \right]$$

(3.2)

$$= - \left[ \frac{\beta}{2} \sum_{c=1}^q \sum_{k=1}^s \gamma_{k,c}^2 \nu_{k,c}^2 + \frac{\alpha}{2} \sum_{c=1}^q \sum_{k\neq k'} \gamma_{k,c} \gamma_{k',c} \nu_{k,c} \nu_{k',c} - I(\nu) \right]$$

$$+ \sup_{\mu} \left[ \frac{\beta}{2} \sum_{c=1}^q \sum_{k=1}^s \gamma_{k,c}^2 \nu_{k,c}^2 + \frac{\alpha}{2} \sum_{c=1}^q \sum_{k\neq k'} \gamma_{k,c} \nu_{k,c} \nu_{k',c} - I(\mu) \right].$$

Here $\Gamma$ is the $s \times s$ diagonal matrix with $(\Gamma)_{kk} = \gamma_k$ and $\nu := (\nu_k)_k$ and $\mu := (\mu_k)_k$ are $s \times q$–matrices and the $\nu_k$ and $\mu_k$ are probabilities on $\{1, \ldots, q\}$, otherwise $J(\nu)$ is defined to be $\infty$.

Proof. Starting from (2.3) we write

$$H_N(\omega) = - \frac{N}{2} \langle (N^{-1} \Gamma_N) M, (N^{-1} \Gamma_N) M \rangle_A$$

and so the assumption that $|S_k|/N \to \gamma_k$ for each $k$, Proposition 3.1 together with Varadhan’s Lemma ([12, Theorem III.13]) and the tilted LDP ([12, Theorem III.17]) in the version of [27, Lemma 2.1] show the result.

It may be more convenient to reformulate the LDP above in terms of an LDP for the matrix $M'_N$ with entries

$$m'_{k,c} := m'_{k,c}(\omega) := \frac{1}{N} \sum_{i \in S_k} I_{\omega_i = c},$$

so that asymptotically, $M'_N \approx \Gamma M_N$. Then, of course, the relevant matrices are the matrices $\nu' := \Gamma \nu = (\gamma_k \nu_{k,c})_{kc}$ where $\nu$ are the matrices appearing in Theorem 3.2 Using (3.1) and calculating

$$\sum_{k=1}^s \sum_{c=1}^q \nu'_{k,c} \log \nu'_{k,c} = I(\nu) - \log q + H(\gamma)$$

where of course $\gamma = (\gamma_k)$, we can identify the new rate function. Indeed, as the term $H(\gamma) - \log q$ is independent of $\nu'$, we can reformulate the LDP as follows.

Theorem 3.3. Under the Gibbs measure $\mu_{\alpha,\beta,N}$ the matrix valued random variable $M'_N$ obeys an LDP with speed $N$ and rate function

$$J'(\nu) = \begin{cases} - \frac{1}{2} (\nu, \nu)_A - \sum_{c=1}^q \nu_{k,c} \log \nu_{k,c} \\
+ \sup_{\mu \in \mathcal{C}(\gamma)} \left[ \frac{1}{2} (\mu, \mu)_A - \sum_{c=1}^q \mu_{k,c} \log \mu_{k,c} \right] \quad &\nu \in \mathcal{C}(\gamma) \\
\infty \quad &\nu \notin \mathcal{C}(\gamma) \end{cases}$$

where $\mathcal{C}(\gamma) = \{ \mu = (\mu_{k,c}) \in M(s \times q) : \mu_{kc} \geq 0 \text{ and } \sum_{c=1}^q \mu_{k,c} = \gamma_k \text{ for all } k \}$. Note that every matrix $\mu \in \mathcal{C}(\gamma)$ can actually be considered as a probability distribution on $[sq]$ and the term $- \sum_{k \in [s]} \sum_{c \in [q]} \mu_{k,c} \log (\mu_{k,c})$ is its entropy. However, the set $\mathcal{C}(\gamma)$ places restrictions on the mass that can be placed on every block.
4. Equilibria for uniform block sizes

We will now try to find the limit distributions of the matrix valued random variables $M_N$ and $M_N'$ under the sequence of Gibbs measure $\mu_{\alpha, \beta, N}$. An LDP as in Theorem 3.2 or Theorem 3.3 is, in principle, of course able to determine these limit distributions. Indeed, they are given by the minima of the corresponding rate functions.

Corollary 4.1. The weak limit points of the sequence $(M_N)$ under the sequence of Gibbs measures $\mu_{N, \alpha, \beta}$ are given by the minima of $J(\cdot)$ and the weak limit points of the sequence $(M'_N)$ under the sequence of Gibbs measures $\mu_{N, \alpha, \beta}$ are given by the minima of $J'(\cdot)$.

Proof. This is actually folklore in large deviation theory and not difficult to prove, when realizing that the upper bound in an LDP implies that any measurable subset of $\mathbb{R}$ whose closure does not contain a minimum of the rate function has a probability that converges to 0. □

We will, in the sequel, determine the minima of $J'$. From here, of course, it is also obvious, what the minima of $J$ are. We start with the observation that the minimum points of $J'$ are the maximum points of

$$G(\mu) := \frac{1}{2} \langle \mu, \mu \rangle_A - \sum_{k=1}^{s} \sum_{c=1}^{q} \mu_{k,c} \log \mu_{k,c}$$

(4.1)

$$= \sum_{c=1}^{q} \sum_{k=1}^{s} \frac{\beta}{2} \mu_{k,c}^2 + \sum_{k=1}^{s} \left( \sum_{c=1}^{q} \mu_{k,c} \log \mu_{k,c} \right) - \sum_{k=1}^{s} \sum_{c=1}^{q} \mu_{k,c} \log \mu_{k,c},$$

where $\mu = (\mu_{k,c}) \in C(\gamma)$ and we have set $0 \log 0 := 0$.

Lemma 4.2. $G$ attains its maximum on the set

$$C^+(\gamma) := \{ \mu \in C(\gamma) : 0 < \mu_{k,c} < 1 \text{ for all } 1 \leq k \leq s, 1 \leq c \leq q \}.$$

Proof. Suppose one of $\mu$’s entries equals 0, without loss of generality $\mu_{11} = 0$. Then, there is $2 \leq i \leq q$ such that $\mu_{i1} \geq \gamma_i / (q - 1)$. Note that $G$ is the sum of a polynomial of degree two in the $\mu_{k,c}$’s and $-\sum_{k=1}^{s} \sum_{c=1}^{q} \mu_{k,c} \log \mu_{k,c}$. Now $-t \log t$ has derivative infinity at 0. Hence, for $\varepsilon > 0$ small enough, we have $G(\mu) < G(\mu')$ where $\mu'$ is the matrix that we obtain from $\mu$, if we replace $\mu_{11}$ by $\mu'_{11} = \varepsilon$, $\mu_{i1}$ by $\mu'_{i1} = \mu_{i1} - \varepsilon$ and leave the other entries unaltered. This shows the claim. □

Let us now apply the method of Lagrange multipliers to find the maximum points of $G$. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$. We then need to find the critical points of

$$L(\mu, \lambda) = G(\mu) - \sum_{k=1}^{s} \lambda_k \left( \sum_{c=1}^{q} \mu_{k,c} - \gamma_k \right).$$

Differentiating with respect to the $\mu_{k,c}$, $1 \leq k \leq s$, $1 \leq c \leq q$ gives the following set of equations

$$0 = \partial_{\mu_{k,c}} G(\mu) - \lambda_k = \beta \mu_{k,c} + \alpha \sum_{k' \neq k}^{s} \mu_{k',c} - \log \mu_{k,c} - 1 - \lambda_k.$$

(4.2)
In particular, for \( \alpha > 0 \) and \( \sigma \) for every two permutations \( (4.5) \)

\[
\sum_{C} \left( \mu_{k \sigma} - \frac{\gamma_{k}}{q} \right) + \alpha \sum_{k' = 1, k' \neq k}^{s} \left( \mu_{k' \sigma'} - \frac{\gamma_{k'}}{q} \right) = \log \frac{\mu_{kc}}{\sqrt[\gamma]{\prod d \mu_{kd}}}. 
\]

that any maximum point has to solve. Let us rephrase the value of \( G \) in critical points by multiplying \((4.3)\) by \( \mu_{kc} \) and summing over \( c \):

\[
\sum_{c} (\beta \mu_{kc}^{2} + \alpha \sum_{k' \neq k}^{s} \mu_{kc} \mu_{k'c} - 2 \mu_{kc} \log \mu_{kc}) = \frac{\gamma_{k}}{q} (\beta \gamma_{k} + \alpha (1 - \gamma_{k})) - \sum_{c} \left( \frac{\gamma_{k}}{q} + \mu_{kc} \right) \log \mu_{kc}. 
\]

Hence, if \( \mu^{\text{crit}} \) is a critical point of \( G \), we can write \( G(\mu^{\text{crit}}) \) as

\[
G(\mu^{\text{crit}}) = \frac{1}{2q} ((\beta - \alpha) ||\gamma||^{2} + \alpha) - \frac{1}{2q} \sum_{k=1}^{s} \sum_{c=1}^{q} (\gamma_{k} + q \mu^{\text{crit}}_{kc}) \log \mu^{\text{crit}}_{kc}. 
\]

Next we will see that of all critical points only those where all rows of \( \mu \) have the same, e.g. an increasing order, are relevant.

**Lemma 4.3.** For \( \mu \in C^{+}(\gamma) \) with increasing rows \( \mu_{k1} \leq \ldots \leq \mu_{kq} \), \( 1 \leq k \leq s \), we have

\[
(4.5) \quad \sum_{k' \neq k}^{q} \mu_{kc} \mu_{k'c} \geq \sum_{k' \neq k}^{q} \mu_{k \sigma_{k}(c)} \mu_{k' \sigma_{k'}(c)} 
\]

for all \( s \)-tuples \((\sigma_{k})_{k}\) of permutations \( \sigma_{k} \in S_{q} \).

In particular, for \( \alpha > 0 \), the function \( G \) can only be maximal in a point \( \mu' \), if the rows of \( \mu' \) are ordered in the same way, i.e. if there is a \( \sigma \in S_{q} \) such that

\[
\mu_{k \sigma(1)}^{l} \leq \ldots \leq \mu_{k \sigma(q)}^{l}
\]

for all \( 1 \leq k \leq s \).

**Proof.** Recall the rearrangement inequality [20]. Theorem 368: When \( x_{1} \leq \ldots \leq x_{n} \) and \( y_{1} \leq \ldots \leq y_{n} \) are sequences of real numbers, then for every permutation \( \pi \in S_{n} \) one has

\[
\sum_{i=1}^{n} x_{i} y_{\pi(i)} \leq x_{1} y_{1} + x_{2} y_{2} + \ldots x_{n} y_{n}
\]

and the inequality is strict if there are indices \( j < j' \) with \( x_{j} < x_{j'} \) and \( y_{\pi(j)} > y_{\pi(j')} \). Applying this to row \( k \) and \( k' \) of \( \mu \) gives

\[
\sum_{c=1}^{q} \mu_{kc} \mu_{k'c} \geq \sum_{c=1}^{q} \mu_{k \sigma_{k}(c)} \mu_{k' \sigma_{k'}(c)}
\]

for every two permutations \( \sigma_{k}, \sigma_{k'} \in S_{q} \). Summing over all \( k' \neq k \) yields \((4.5)\).

In particular, \( \mu' \) is not a maximum point, if we have two rows \( k \neq k' \) and indices \( j < j' \) with \( \mu_{k j} < \mu_{k j}^{l} \) and \( \mu_{k' j} > \mu_{k' j}^{l} \). \( \square \)
So we can and will assume in the following, that all rows of a critical point \( \mu \) of \( G \) are increasing. The next lemma determines the structure of a critical \( \mu \).

**Lemma 4.4.** Let \( \mu \) be a critical point of \( G \).

1. If \( \mu_{kc} = \mu_{kc'} \) for a \( k \) and \( c \neq c' \) then \( \mu_{kc'} = \mu_{kc} \) for all \( 1 \leq k' \leq s \).
2. Each row of \( \mu \) has at most two different entries.

**Proof.** Substracting (4.3) for \( c \) from the equation for \( c' \) yields

\[
\alpha \sum_{k' \neq k} \mu_{k'c} = \alpha \sum_{k' \neq k} \mu_{k'c'}
\]

and thus, by increasing order, \( \mu_{kc} = \mu_{kc'} \) for all \( 1 \leq k' \leq s \). This is the first claim.

For two different columns \( c \neq c' \) we obtain from (4.3)

\[
(\beta - \alpha) (\mu_{kc'} - \mu_{kc}) + \alpha \sum_{k' = 1}^{s} (\mu_{k'c'} - \mu_{k'c}) = \log \mu_{kc'} - \log \mu_{kc}.
\]

Now if we had three columns \( c < c' < c'' \) with \( \mu_{kc} < \mu_{kc'} < \mu_{kc''} \) for one row \( k \) (and hence for all due to the first part of this lemma) we would have

\[
\frac{1}{\alpha} = \sum_{k = 1}^{s} \frac{\mu_{kc'} - \mu_{kc}}{\alpha \sum_{k' = 1}^{s} (\mu_{k'c'} - \mu_{k'c})} = \sum_{k = 1}^{s} \left( \frac{\log \mu_{kc'} - \log \mu_{kc}}{\mu_{kc'} - \mu_{kc}} - (\beta - \alpha) \right)^{-1}
\]

and the same equation for the pair \( c, c'' \). However, every summand on the right of that latter equation would be larger (and positive) than the corresponding summand in (4.6) by concavity of the logarithm. Hence we have a contradiction. \( \square \)

So far we have proved, that according to Lemma 4.2, Lemma 4.3 and Lemma 4.4 we can constrain our search for maximum points of \( G \) to matrices \( \mu \) with positive entries, increasingly ordered rows and having at most two different columns. Taking into account that the entries in row \( k \) sum up to \( \gamma_k \) we see that the largest column \( \mu^+ \) (component by component) of \( \mu \) together with the number \( 1 \leq r \leq q \) of columns equal to \( \mu^+ \) is all the information we need to build up \( \mu \). So either \( \mu \) has \( q \) identical columns \( \gamma/q \) or (the increasingly ordered) \( \mu \) reads

\[
\mu = (\mu^- \ldots \mu^- \mu^+ \ldots \mu^+)_{q-r}
\]

with \( \mu^- = (\gamma - r\mu^+)/ (q-r) \) for some \( 1 \leq r \leq q-1 \) and \( \mu^-_k > \gamma_k/q > \mu^-_k \) for all \( 1 \leq k \leq s \) is readily accomplished.

**Proposition 4.5.** Let \( \gamma_k = 1/s \) for all \( 1 \leq k \leq s \) and \( Q \in M(s \times q) \) with identical entries \( 1/sq \). If a critical \( \mu \) with \( G(Q) \leq G(\mu) \) does not have identical rows then \( \mu \) is not a maximum point.

**Proof.** Recall that the system of critical equations (4.3) reads

\[
\alpha \sum_{k' = 1}^{s} \left( \mu^+_{k'} - 1/sq \right) = \frac{q-r}{q} \log \frac{s(q-r)\mu^+_k}{1-sr\mu^+_k} - (\beta - \alpha) \left( \mu^+_k - 1/sq \right)
\]
and regard the right hand side of this equation as a function of $\mu_k^+$, say $\psi(\mu_k^+)$. Now if the largest entry of the vector $\mu^+$ occurs in line $K$ (and perhaps somewhere else) but not in every line, then

$$\psi(\mu_K^+) - \alpha \left( s\mu_K^+ - \frac{1}{q} \right) < 0.$$  

Since $\psi(t)$ diverges to $+\infty$ when $t$ approaches $1/sr$ from below we can find a $t_0 > \mu_K^+$ with $\psi(t_0) = \alpha(st_0 - 1/q)$. Now building a matrix $\nu$ by taking instead of $r$ columns equal to $\mu^+$ just $r$ columns with identical entries $t_0$ and completing the matrix with $q-r$ columns with identical entries $(1-srt_0)/(s(q-r))$ then clearly $\nu$ is (well defined and) a critical point. So we just have to prove that $G(\mu) < G(\nu)$.

To that end observe that according to (4.4) the value of $G$ in critical points $p$ is given up to constants as $\sum_k w(p_k)$ with $w : (0, 1/(sr)) \to \mathbb{R}$

$$w(x) = -((q-r) + q(1-srx)) \log \frac{1-srx}{s(q-r)} - r(1+sqx) \log x.$$  

Calculating

$$w'(x) = srq \log \frac{1-srx}{s(q-r)x} + r \frac{sqx-1}{x(1-srx)}$$  

and

$$w''(x) = \frac{r(sqx-1)(2srx-1)}{x^2(srx-1)^2}$$  

we see that if $q > 2r$ then the graph of $w$, coming from $+\infty$ at the vertical asymptotic line $x = 0$, has a saddle point at $x = 1/sq$ changing from bending to the left to bending to the right. It decreases to the second inflection point at $x = 1/(2sr)$, passes, now bending to the left again, the unique minimum point at say $\xi$ and disappears to $+\infty$ approaching the vertical line at $x = 1/(sr)$. In this case clearly $w(\mu_k^+) \leq \max(w(\mu_K^+), w(1/sq))$ for all $k$ since $1/sq < \mu_k^+ \leq \mu_K^+$. Now $w(\mu_K^+) < w(1/sq)$ would imply the contradiction

$$G(\mu) = \sum_k w(\mu_k^+) < sw(1/sq) = G(Q)$$  

so we have $w(1/sq) \leq w(\mu_K^+)$ and therefore $\mu_K^+ > \xi$ which means $w(t_0) > w(\mu_K^+)$ and so

$$G(\nu) = sw(t_0) > sw(\mu_K^+) \geq G(\mu).$$  

If $q \leq 2r$ then $w$ is strictly increasing on $[1/sq, 1/sr)$ so that

$$G(\nu) = sw(t_0) > \sum_k w(\mu_k^+) = G(\mu).$$  

\[\square\]

Wrapping up what we have seen, we state

**Proposition 4.6.** Let $\gamma \in \mathbb{R}^s$ have identical entries $1/s$. The function $G := G^{bP}$ in the block spin Potts model is maximal on $C(\gamma)$ if its rows are identical and equal to a maximizer of the corresponding target function

$$(4.7) \quad G^P(\nu) = \frac{1}{2s}(\beta + (s-1)\alpha) \sum_{c=1}^q v_c^2 - \sum_{c=1}^q v_c \log v_c$$

\[9\]
on the set \( V := \{v = (v_1, \ldots, v_q) \mid \sum_{c} v_c = 1, v_c > 0\} \) in the Potts model.

**Proof.** The maximum of \( G_{bP} \) on \( C(\gamma) \) is attained on the subset of matrices with identical rows taken from \( \{v/s \mid v \in V\} \) and the value is equal to

\[
\frac{\beta + (s - 1)\alpha}{2s} \sum_{c=1}^{q} v_c^2 - \sum_{c=1}^{q} v_c \log \frac{v_c}{s}.
\]

However, up to a minus sign and ignoring the summand \( \log s \) this is the free energy functional in a Potts model at inverse temperature \((\beta + (s - 1)\alpha)/s\). \( \square \)

The following theorem hence follows from the results in [23, 14], where the critical temperature and the behaviour of the Potts model is computed.

**Theorem 4.7.** Consider the block spin Potts model in the asymptotically uniform case \( \gamma_k = s^{-1} \). Denote by \( \zeta_q := \left[ \frac{2^{q-1}}{q} \log(q - 1) \right] \) the critical inverse temperature in the \( q \)-color Potts model, and let \( g := \frac{\beta + (s - 1)\alpha}{s} \). Then the \( q \)-color block spin Potts model has a phase transition. More precisely, if \( g < \zeta_q \), then the distribution of \( M_N' \) under the Gibbs measure \( \mu_{N,\alpha,\beta} \) concentrates in a unique point, the matrix with all entries identical to \( 1/(sq) \).

To describe the “low temperature” behavior define the function \( \varphi : [0,1] \to \mathbb{R}^q \):

\[
\varphi(t) := \left( \frac{1 + (q - 1)t}{sq}, \frac{1 - t}{sq}, \ldots, \frac{1 - t}{sq} \right)
\]

and let \( u(g) \) be the largest solution of the equation

\[
u = \frac{1 - e^{-gu}}{1 + (q - 1)e^{-gu}}.
\]

Finally let \( n^1(g) := \varphi(u(g)) \) and \( n^i(g) \) be \( n^1(g) \) with the \( i \)th and the first coordinate interchanged, \( i = 2, \ldots, q \). Let \( \nu^i(g) \) be the matrix with all rows identical to \( n^i(g) \) and \( Q \) be the matrix with all entries identical to \( 1/(qs) \).

Then, if \( g > \zeta_q \) the distribution of \( M_N' \) under the Gibbs measure \( \mu_{N,\alpha,\beta} \) concentrates in a (uniform) mixture of the Dirac measures in \( \nu^1(g), \ldots, \nu^q(g) \).

At \( g = \zeta_q \) the limit points of \( M_N' \) under the Gibbs measure \( \mu_{N,\alpha,\beta} \) are \( Q \) and \( \nu^1(g), \ldots, \nu^q(g) \).

## 5. Logarithmic Sobolev Inequalities

In this section, we present logarithmic Sobolev inequalities for block spin Potts models. Logarithmic Sobolev inequalities are frequently used in concentration of measure theory (yielding non-asymptotic fluctuation and deviation results), for example, where they form the core of the well-known entropy method. See the monographs [28] or [6] for an overview on these topics. Recently, logarithmic Sobolev inequalities for various type of finite spin systems have been established, starting with the Ising model in [18], to be followed by exponential random graph, random coloring and hard-core models in [36]. Here we continue this line of research by considering block spin Potts models.
Let us first recall some basic notions. For \( \omega = (\omega_i)_{i \in S} \in \{1, \ldots, q\}^S \) and \( i \in S \), we write \( \omega_i := (\omega_j)_{j \neq i} \). Moreover, for any function \( f: \{1, \ldots, q\}^S \rightarrow \mathbb{R} \), we define a certain “difference operator” by

\[
|\mathcal{D} f(\omega)| = \left( \sum_{i \in S} \left| f(\omega) - f(\omega_{i'}, \omega_i') \right|^2 d \mu_N(\omega_i | \omega_{i'}) \right)^{1/2},
\]

where \( \mu_N(\cdot | \omega_{i'}) \) denotes the regular conditional probability. The integrals may be regarded as a kind of “local variance” in the respective coordinate. The difference operator \( \mathcal{D} \) is a well-known object, and the integral of \( |\mathcal{D} f|^2 \) with respect to \( \mu_N \) can be regarded as a Dirichlet form (cf. [18, Remark 2.2]). Finally, for any non-negative function \( f, \mathrm{Ent}_{\mu_N}(f) := \int f \log(f) d\mu_N - \int f d\mu_N \log(f d\mu_N) \) denotes the entropy. We now have the following log-Sobolev-type inequalities.

**Theorem 5.1.** Assume that \( 2q\beta e^\beta < 1 \).

1. For \( N \) large enough, \( \mu_N \) satisfies a logarithmic Sobolev inequality with respect to \( \mathcal{D} \). That is, for any function \( f: \{1, \ldots, q\}^S \rightarrow \mathbb{R} \) we have

\[
\mathrm{Ent}_{\mu_N}(f^2) \leq 2\sigma_1^2 \int |\mathcal{D} f|^2 d\mu_N. \tag{5.1}
\]

2. For \( N \) large enough and any function \( f: \{1, \ldots, q\}^S \rightarrow \mathbb{R} \) we have

\[
\mathrm{Ent}_{\mu_N}(e^f) \leq 2\sigma_2^2 \sum_{i \in S} \mathrm{Cov}_{\mu_N} f(\omega_{i'}, \cdot), e^{f(\omega_{i'}, \cdot)} d\mu(\omega). \tag{5.2}
\]

3. For \( N \) large enough and any function \( f: \{1, \ldots, q\}^S \rightarrow \mathbb{R} \) we have

\[
\mathrm{Ent}_{\mu_N}(e^f) \leq \frac{\sigma_3^2}{2} \int |\mathcal{D} f|^2 e^f d\mu_N. \tag{5.3}
\]

Here, \( \sigma_1, \sigma_2, \sigma_3 > 0 \) are constants which depend on \( \beta \) and \( q \) only.

It is possible to give explicit values for \( \sigma_1, \sigma_2, \sigma_3 \) which depend on quantities which stem from the conditional probabilities \( \mu_N(\cdot | \omega_{i'}) \). For details, see Remark 5.3, where we also comment on the condition on \( N \).

Both (5.2) and (5.3) are also known as modified logarithmic Sobolev inequalities. In fact, if \( \mathcal{D} \) is replaced by another difference operator which satisfies the chain rule (for instance, the ordinary Euclidean gradient), then (5.1) and (5.3) are equivalent, but obviously, this is not true for \( \mathcal{D} f \).

Let us briefly discuss the results from Theorem 5.1 including possible applications. From (5.1) and (5.3), we may derive concentration of measure bounds by various techniques. For instance, based on (5.1) we obtain \( L^p \) norm inequalities for any function \( f \in L^\infty(\mu_N) \) and subsequently concentration results, cf. [18, Theorem 1.5]. Moreover, (5.3) gives rise to subgaussian tail bounds for Lipschitz-type functions \( f \) (in the sense of \( |\mathcal{D} f| \leq 1 \)) by applying the Herbst argument, see e.g. [35] (where also slightly more advanced situations are discussed, cf. Section 2.4).

Recently, in [1] it has been established that (5.3) moreover gives rise to \( L^p \) bounds (via Beckner inequalities) as well. Especially for the case of spin systems and Glauber dynamics, cf. the discussion in Section 4.3 therein.
To consider a simple example, let

$$T_{k,c}(\omega) := \sum_{i \in S_k} 1_{\omega_i = c}$$

denote the number of vertices in the block $S_k$ which have colour $c$. It is easy to check that $|\partial T_{k,c}|^2 \leq |S_k|$, and therefore, using [35, Equation (1.2)], we immediately obtain that

$$\mu_N(|T_{k,c} - \mu_N(T_{k,c})| \geq t) \leq 2 \exp\left(-\frac{t^2}{2|S_k|}\right),$$

where $\mu_N(T_{k,c}) := \int T_{k,c} d\mu_N$. Note that for $N$ large, this probability approaches $2 \exp(-t^2/(2N\gamma_3 \sigma_3))$.

Furthermore, a special class of non-Lipschitz functions for which concentration bounds based on inequalities of type (5.1) and (5.3) have been established in recent years are so-called multilinear polynomials, i.e. polynomials which are affine with respect to each variable. See for instance [19, Theorem 5] and [1, Corollary 5.4].

Finally, note that (5.2) is frequently used in the context of Markov processes, and it can be shown to be equivalent to exponential decay of the relative entropy along the Glauber semigroup (cf. e.g. [4] or [8]). As shown in [36, Theorem 2.2], it moreover implies that the associated Glauber dynamics is rapidly mixing, i.e. its mixing time is of order $O(N \log N)$. This complements the results from [5], where a different situation was considered (the usual Potts model without blocks but on graphs with fixed maximal degree).

The remaining part of this section is devoted to the proof of Theorem 5.1. To convey the basic idea, recall that for product measures $\otimes_{i=1}^n \mu_i$, the entropy functional tensorizes in the sense that

$$\text{Ent}_{\otimes_{i=1}^n \mu_i}(f) \leq \sum_{i=1}^N \int \text{Ent}_{\mu_i}(f) d\mu,$$

and therefore, proving logarithmic Sobolev inequalities reduces to controlling each coordinate separately, i.e. a “one-dimensional” case. For non-product measures, an inequality of this type no longer holds true, but if the dependencies are sufficiently weak, an analogue can be shown which is known as an approximate tensorization property. For probability spaces with finitely many atoms, a suitable criterion to establish approximate tensorization was introduced in [33], which we will exploit in the sequel.

**Proposition 5.2.** Assume that $2q\beta e^\beta < 1$. For $N$ large enough, the approximate tensorization property of entropy holds, i.e.

$$\text{Ent}_{N}(f) \leq C \sum_{i \in S} \int \text{Ent}_{\mu_N(\cdot|\omega_i)}(f(\omega_i, \cdot)) d\mu_N(\omega)$$

with $C$ depending on $\beta$ and $q$ only.

**Proof.** As pointed out before, the proof of Proposition 5.2 works by applying Marton’s approximate tensorization result [33] in the slightly rewritten and corrected form stated in [36, Theorem 4.1]. Essentially, we need to control the conditional probabilities $\mu_N(\cdot|\omega_i)$, which we rewrite in the sequel, generalizing the case of the
Altogether, we arrive at the representation $H_N(\omega) = \frac{\beta}{2N} \sum_{c=1}^{q} \sum_{k=1}^{s} (b_{k,ij,c} + 1_{i \in S_k, \omega_i = c} + 1_{j \in S_k, \omega_j = c})^2$

Then, the Hamiltonian may be decomposed as follows:

$-H_N(\omega) = \frac{\beta}{2N} \sum_{c=1}^{q} \sum_{k=1}^{s} (b_{k,ij,c} + 1_{i \in S_k, \omega_i = c} + 1_{j \in S_k, \omega_j = c})^2$

$+ \frac{\alpha}{2N} \sum_{c=1}^{q} \sum_{k=1}^{s} \sum_{k',k' \neq k} (b_{k,ij,c} + 1_{i \in S_k, \omega_i = c} + 1_{j \in S_k, \omega_j = c})$

$\cdot (b_{k',ij,c} + 1_{i \in S_k', \omega_i = c} + 1_{j \in S_k', \omega_j = c})$.

Now, writing

$(b_{k,ij,c} + 1_{i \in S_k, \omega_i = c} + 1_{j \in S_k, \omega_j = c})^2 = b_{k,ij,c}^2 + 2b_{k,ij,c}(1_{i \in S_k, \omega_i = c} + 1_{j \in S_k, \omega_j = c})$

$+ 1_{i \in S_k, \omega_i = c} + 1_{j \in S_k, \omega_j = c} + 2 \cdot 1_{i \in S_k, \omega_i = c} 1_{j \in S_k, \omega_j = c}$

and summing over $c$ and $k$, we obtain

$\sum_{c=1}^{q} \sum_{k=1}^{s} (b_{k,ij,c} + 1_{i \in S_k, \omega_i = c} + 1_{j \in S_k, \omega_j = c})^2$

$= \sum_{c=1}^{q} \sum_{k=1}^{s} b_{k,ij,c}^2 + 2 \sum_{k=1}^{s} b_{k,ij,\omega_i} 1_{i \in S_k} + 2 \sum_{k=1}^{s} b_{k,ij,\omega_j} 1_{j \in S_k} + 2 + 2 \cdot 1_{i,\omega_i = j,\omega_j}$.

Similarly,

$(b_{k,ij,c} + 1_{i \in S_k, \omega_i = c} + 1_{j \in S_k, \omega_j = c})(b_{k',ij,c} + 1_{i \in S_{k'}, \omega_i = c} + 1_{j \in S_{k'}, \omega_j = c})$

$= b_{k,ij,c}b_{k',ij,c} + b_{k,ij,c}(1_{i \in S_{k'}, \omega_i = c} + 1_{j \in S_{k'}, \omega_j = c}) + b_{k',ij,c}(1_{i \in S_k, \omega_i = c} + 1_{j \in S_k, \omega_j = c})$

$+ (1_{i \in S_k, \omega_i = c} + 1_{j \in S_k, \omega_j = c})(1_{i \in S_{k'}, \omega_i = c} + 1_{j \in S_{k'}, \omega_j = c})$,

leading to

$\sum_{c=1}^{q} \sum_{k=1}^{s} \sum_{k',k' \neq k} (b_{k,ij,c} + 1_{i \in S_k, \omega_i = c} + 1_{j \in S_k, \omega_j = c})(b_{k',ij,c} + 1_{i \in S_{k'}, \omega_i = c} + 1_{j \in S_{k'}, \omega_j = c})$

$= \sum_{c=1}^{q} \sum_{k=1}^{s} \sum_{k',k' \neq k} b_{k,ij,c}b_{k',ij,c} + 2 \sum_{k=1}^{s} b_{k,ij,\omega_i} 1_{i \in S_k} + 2 \sum_{k=1}^{s} b_{k,ij,\omega_j} 1_{j \in S_k} + 2 + 1_{i,\omega_i = j,\omega_j}$.

Altogether, we arrive at the representation

$-H_N(\omega) = \frac{\beta}{2N} \sum_{c=1}^{q} \sum_{k=1}^{s} b_{k,ij,c}^2 + \frac{\beta}{N} \sum_{k=1}^{s} b_{k,ij,\omega_i} 1_{i \in S_k} + \frac{\beta}{N} \sum_{k=1}^{s} b_{k,ij,\omega_j} 1_{j \in S_k} + \frac{\beta}{N}$

$+ \frac{\beta}{N} 1_{i,\omega_i = j} + \frac{\alpha}{2N} \sum_{c=1}^{q} \sum_{k=1}^{s} \sum_{k',k' \neq k} b_{k,ij,c}b_{k',ij,c}$

$+ \frac{\alpha}{N} \sum_{k=1}^{s} b_{k,ij,\omega_i} 1_{i \in S_k} + \frac{\alpha}{N} \sum_{k=1}^{s} b_{k,ij,\omega_j} 1_{j \in S_k} + \frac{\alpha}{N} 1_{i,\omega_i = j}$.
In particular, the conditional probabilities given \( \omega_i \) can be written as

\[
\mu_N (c | \omega_i) = \frac{\exp(-H_N(\omega_i, c))}{\sum_{c'} \exp(-H_N(\omega_i, c'))} = \frac{1}{1 + \sum_{c' \neq c} \exp(H_N(\omega_i, c) - H_N(\omega_i, c'))} = h(\sum_{c' \neq c} \exp(H_N(\omega_i, c) - H_N(\omega_i, c'))),
\]

where \( h(x) = 1/(1 + x) \), which is 1-Lipschitz. Here, using the representation derived above, we have

\[
\begin{align*}
H_N(\omega_i, c) &- H_N(\omega_i, c') \\
&= \frac{\beta}{N} \sum_{k=1}^s (b_{k,i,j} - b_{k,i,j}) \mathbb{1}_{i \in S_k} + \frac{\beta}{N} (1 - \mathbb{1}_{\omega_i = c} - 1 - \mathbb{1}_{\omega_i = c'}) \mathbb{1}_{i \sim j} \\
&+ \frac{\alpha}{N} \sum_{k=1}^s (b_{k,i,j} - b_{k,i,j}) \mathbb{1}_{i \notin S_k} + \frac{\alpha}{N} (1 - \mathbb{1}_{\omega_i = c} - 1 - \mathbb{1}_{\omega_i = c'}) \mathbb{1}_{i \sim j} \\
&= \frac{1}{N} \sum_{k=1}^s (b_{k,i,j} - b_{k,i,j}) (\mathbb{1}_{i \in S_k} + \alpha \mathbb{1}_{i \notin S_k}) + \frac{1}{N} \mathbb{1}_{\omega_i = c} - \mathbb{1}_{\omega_i = c'} (\mathbb{1}_{i \sim j} + \alpha \mathbb{1}_{i \sim j}).
\end{align*}
\]

From this representation, we now derive two facts: first,

\[
\min_{i \in S} \min_{\omega \in \{1, \ldots, q\}^s} \mu_N(\omega | \omega_i) \geq \gamma_1
\]

for some \( \gamma_1 > 0 \) which only depends on \( \beta \) but not on \( N \) (here we have used that \( \alpha < \beta \)).

Moreover, we need to control the operator norm \( \| J \|_{2 \rightarrow 2} \) of the \( N \times N \) matrix \( J \) whose entries \( J_{ij} \) are given by

\[
J_{ij} = \sup d_{TV}(\mu_N(\cdot | \omega_i), \mu_N(\cdot | \omega_j)),
\]

where the sup is taken over all configurations \( \omega, \sigma \) which differ at site \( j \) only. Obviously, this can be upper bounded by the \( \ell^\infty \rightarrow \ell^\infty \) norm \( \| J \|_{\infty \rightarrow \infty} \). To bound the latter, we fix two such configurations, i.e. \( \omega_j \neq \sigma_j \) and \( \omega_j = \sigma_j \). It follows that for any \( i \neq j \),

\[
d_{TV}(\mu_N(\cdot | \omega_i), \mu_N(\cdot | \sigma_i)) = \frac{1}{2} \sum_{c=1}^q |\mu_N(c | \omega_i) - \mu_N(c | \sigma_i)|
\]

\[
= \frac{1}{2} \sum_{c=1}^q \left| h\left( \sum_{c' \neq c} \exp(H_N(\omega_i, c) - H_N(\omega_i, c')) \right) - h\left( \sum_{c' \neq c} \exp(H_N(\sigma_i, c) - H_N(\sigma_i, c')) \right) \right|
\]

\[
\leq \frac{1}{2} \sum_{c=1}^q \left| \sum_{c' \neq c} \left( \exp(H_N(\omega_i, c) - H_N(\omega_i, c')) - \exp(H_N(\sigma_i, c) - H_N(\sigma_i, c')) \right) \right|
\]

\[
\leq \frac{1}{2} \sum_{c=1}^q \sum_{c' \neq c} \exp\left( \frac{1}{N} \sum_{k=1}^s (b_{k,i,j} - b_{k,i,j}) (\mathbb{1}_{i \in S_k} + \alpha \mathbb{1}_{i \notin S_k}) \right)
\]

\[
\exp\left( \frac{1}{N} \mathbb{1}_{\omega_j = c} - \mathbb{1}_{\omega_j = c'} (\mathbb{1}_{i \sim j} + \alpha \mathbb{1}_{i \sim j}) \right)
\]
Remark 5.3. By a closer look at the respective proofs, it is possible to give explicit values for the constants appearing in Theorem 5.1 and Proposition 5.2 depending on the quantities \( \gamma_1 \) and \( \gamma_2 \) from the proof of Proposition 5.2. Indeed, we may set 
\[
C = (\gamma_1^2 \gamma_2^3)^{-1}, \quad \sigma^2 = \sigma_2^2 = C \quad \text{and} \quad \sigma_1^2 = \log(\gamma_1^{-1})C/\log(4).
\]
Moreover, requiring \( N \) to be large enough means that \( N \) must be so large that
\[
2q\beta e^\beta + O(\beta(N^{-1}) < 1 \quad \text{in the asymptotics leading to (5.5)}.
\]
References


