Near-isometries of hyperbolic graphs and bi-Lipschitz embeddings of their boundaries

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Abstract

In [KLW], we formulated a class of Gromov hyperbolic graphs (expansive hyperbolic graphs) arising from iterated function systems (IFS) in fractal geometry, and studied the relations of the hyperbolic boundaries and the attractors in a wide setup beyond IFS. In this paper, we extend the scope of our study to general hyperbolic graphs via some near-isometries (that is, the graph distances are altered by at most some additive constants). Using the properties of expansive hyperbolic graphs, we investigate the connection of the near-isometries between hyperbolic graphs and the Lipschitz equivalences between their boundaries, and provide a combinatorial characterization of all bi-Lipschitz embeddings of hyperbolic boundaries. We further apply the hyperbolic techique developed to produce some "good distances" on spaces of homogeneous type.

Contents

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1 Introduction

The notion of hyperbolic graphs, along with hyperbolic groups, was invented and remarkably developed by M. Gromov in geometric group theory [Gr]. Every infinite hyperbolic graph or group X possesses a boundary ∂X, known as the *(Gromov) hyperbolic boundary*, which consists of all "ends at infinity". This boundary has a very rich structure, and plays an extremely important role in the study of hyperbolic groups (see the survey [KaB]).

Without reference to any group structure, there are also interesting hyperbolic graphs arising from various mathematical subjects. One of the examples is the *Elek's cone* C_K [E] associated to a compact set K in a Euclidean cube I: the vertices are the dyadic subcubes of I that meet K; hierarchically they constitute an infinite tree, and each intersected pair of subcubes in the same size are jointed by an extra "horizontal" edge. It has been proved that such C_K is a hyperbolic graph, and the boundary ∂C_K is homeomorphic to K. This construction was extended by Bourdon and Pajot [BP], and further adapted by Carrasco Piaggio [P] for studying the conformal gauges and dimensions of compact metric spaces.

In fractal geometry, a counterpart is the *augmented tree* associated to a contractive iterated function system (IFS), where the tree structure is naturally from the symbolic space of IFS, and each additional horizontal edge represents a pair of neighboring cells in the attractor with approximately equal sizes. This notion was initiated by Kaimanovich [Ka] on the Sierpinski gasket, and was carried out by the authors [LW1, LW3] on general IFS. In most common cases, the hyperbolicity of augmented trees has been proved to be valid, and it derives a Hölder equivalence of the hyperbolic boundary and the attractor. It has also found important applications in the study of the bi-Lipschitz classification as well as the probabilistic potential theory of fractal sets [LL, DLL, KLW1, KL, KLLW].

Motivated by this, in a recent paper [KLW], we formulated a broad class of hyperbolic graphs, called *expansive hyperbolic graphs*, to capture the essential properties which were used extensively in the previous study. This class embraces not only the graphs mentioned (Elek's cones, Bourdon-Pajot's and Carrasco Piaggio's graphs, augmented trees), but also the ones that the vertical parts are not trees (e.g. the quotients of augmented trees [Wa]). Based on some properties abstracted from the IFS and refinement systems, we introduced a more flexible setup of *index maps*, and investigated two types of expansive graphs, *AI_∞-graphs* and *AI_a-graphs*, to carry out the idea of augmentation much further. Our results involved the hyperbolicity of these graphs, the identification of the hyperbolic boundaries with the attractors, and the relation of the bounded degree property versus the separation conditions on the index maps. Moreover, using this hyperbolic technique, we constructed some new metrics closely related to the self-similar energy forms on p.c.f. fractals.

In this paper, we broaden the scope of our investigation to general hyperbolic graphs. For this, the key concept is the *near-isometry* φ between two graphs X and X' (endowed with graph distances d, d'), that is, a map from X to X' that satisfies $\sup_{x,y\in X}|d(x,y) - d'(\varphi(x),\varphi(y))| < \infty$ and $\sup_{z\in X'} d'(z,\varphi(X)) < \infty$ (without the latter φ is called a *near-isometric embedding*); it is also known as the *quasi-isometry* in [Gr] with multiplicative constant 1. Associated to a hyperbolic graph (X, E), we define the *auxiliary graph* $(X, \widehat{E})$ to be the "minimal" expansive graph with $E \subset \widehat{E}$ (Definition 3.1 and Proposition 3.2). We conclude that (Theorem 3.5)

Theorem 1.1. *For a hyperbolic graph (X, E), the identity map $\mathrm{id}_X : (X, E) \to (X, \widehat{E})$ is a near-isometry, and $(X, \widehat{E})$ is an expansive hyperbolic graph.*

The result yields a classification of hyperbolic graphs according to the (m, k)-departing property of the auxiliary graphs ((m, k)-*hyperbolic graphs*, see Corollary 3.6). We also have another near-isometry $\mathrm{id}_X : (X, \widehat{E}) \to (X, \widehat{E}^k)$, where $(X, \widehat{E}^k)$ enlarges $(X, \widehat{E})$ by adding horizontal edges to all the pairs with horizontal distances of at most k (Definition 3.8 and Corollary 3.9). By these transforms, we are able to associate every hyperbolic graph with an expansive $(m, 1)$-hyperbolic graph.

An important fact is that near-isometrically transforming a hyperbolic graph (X, E) distort the boundary ∂X bi-Lipschitzly (Theorem 4.1); here ∂X is always equipped with a quasi-metric ϱ_a, $a > 0$ (*Gromov distance*, see Definition 2.4). Recall that a *quasi-metric* ρ on a set M is a symmetic function from $M \times M$ to $[0, \infty)$ that vanishes if and only if $\xi = \eta$, and satisfies the quasi-triangle inequality with some $C_\rho \geq 1$: $\rho(\xi, \eta) \leq C_\rho(\rho(\xi, \zeta) + \rho(\zeta, \eta))$ for all $\xi, \eta, \zeta \in M$ (it is a metric if $C_\rho = 1$). Such (M, ρ) is called a *quasi-metric space*, and it is endowed with a canonical topology

$$\mathcal{T}_\rho := \{\Omega \subset M : \forall\ \xi \in \Omega,\ \exists\ r > 0 \text{ such that } B_\rho(\xi, r) \subset \Omega\},$$

where $B_\rho(\xi, r) := \{\eta \in M : \rho(\xi, \eta) < r\}$ denotes the *ball* centered at ξ with radius r; the compactness of (M, ρ) is equivalent to the one of $(M, \mathcal{T}_\rho)$. We say that (M, ρ) is *doubling* if there exists a positive integer ℓ such that for any $\xi \in M$ and $r > 0$, each ball $B_\rho(\xi, r)$ can be covered by a union of at most ℓ balls of radii $r/2$. It is known that the hyperbolic boundary $(\partial X, \varrho_a)$ is a compact quasi-metric space. Using the near-isometry in Theorem 1.1, we improve a result in [KLW] by (Theorem 3.11)

Theorem 1.2. *For a hyperbolic graph (X, E) with bounded degree, $(\partial X, \varrho_a)$ is doubling.*

A map τ from a quasi-metric space (M, ρ) to another quasi-metric space (M', ρ') is said to be a *bi-Lipschitz embedding* if $\rho'(\tau(\xi), \tau(\eta)) \asymp \rho(\xi, \eta)$ for all $\xi, \eta \in M$ (here by $f \asymp g$ we mean that there exists $C \geq 1$ such that $C^{-1} f(x) \leq g(x) \leq C f(x)$ for all variables x in a given domain). Furthermore if τ is bijective, we call τ a *Lipschitz equivalence*. For the relation of the near-isometries between hyperbolic graphs and the Lipschitz equivalences between their boundaries, we provide a detailed result through the technique on expansive hyperbolic graphs (Theorems 4.1 and 4.3).

Theorem 1.3. *Let $(X, E), (X', E')$ be two hyperbolic graphs. Then*

(i) every near-isometric embedding (or near-isometry) $\varphi : (X, E) \to (X', E')$ induces a unique bi-Lipschitz embedding (or Lipschitz equivalence resp.) $\widehat{\varphi} : (\partial X, \varrho_a) \to (\partial X', \varrho'_a)$;

(ii) for a bi-Lipschitz embedding (or a Lipschitz equivalence) $\tau : (\partial X, \varrho_a) \to (\partial X', \varrho'_a)$, there exists a near-isometric embedding (or a near-isometry resp.) $\varphi : (X, E) \to (X', E')$ such that $\widehat{\varphi} = \tau$.

We remark that previously, Bonk and Schramm [BS] have obtained some similar results on hyperbolic boundaries in the setting of Gromov hyperbolic spaces. Compared to their methods, our combinatorial proofs on hyperbolic graphs are more straightforward.

With the intension to study the embeddings of hyperbolic boundaries, we restate the index map and the admissibility introduced in [KLW] as purely topological concepts: an *index triple* (X, E, Φ) over a Hausdorff space M is a rooted graph (X, E) with a map Φ (*index map*) on X such that (i) each $\Phi(x)$ is a nonempty compact subsets of M; (ii) for any geodesic ray $\mathbf{x} = [x_i]_{i=0}^{\infty}$ from the root ϑ, the sequence $\{\Phi(x_i)\}_{i=0}^{\infty}$ is decreasing and the intersection is a singleton $\{\kappa_0(\mathbf{x})\}$. We call this triple *admissible* if (X, E) is hyperbolic, and for two rays $\mathbf{x}, \mathbf{y}$ as in (ii), $\kappa_0(\mathbf{x}) = \kappa_0(\mathbf{y})$ if and only if $\mathbf{x}, \mathbf{y}$ have the same limit in ∂X. Every admissible index triple (X, E, Φ) defines a *boundary map* $\kappa : \partial X \to M$ as the quotient of κ_0, which turns out to be a *(topological) embedding*, i.e., a homeomorphism from ∂X to the attractor $K = \kappa(\partial X)$ (Proposition 5.5). Using the intrinsic index map $\mathcal{J}_\partial$ (*G-cell*, see (2.7)) on a hyperbolic graph (X, E), we also obtain a converse result which characterizes all embeddings of ∂X (Theorem 5.6):

Theorem 1.4. *Let (X, E) be a hyperbolic graph. A map τ from ∂X to a Hausdorff space M is an embedding if and only if (X, E, Φ) is an admissible index triple over M, where $\Phi := \tau \circ \mathcal{J}_\partial$. In this case, τ is the boundary map of Φ.*

When the underlying space is a quasi-metric space (M, ρ), we define the AI_a-triple (*index triple of augmented type-(a)*, $a > 0$) to be an index triple (X, E, Φ) over M that satisfies the diameter $|\Phi(x)|_\rho = O(e^{-a|x|})$ as $|x| \to \infty$, and for some $\gamma_2 \geq \gamma_1 > 0$,

$$|x| = |y|,\ \mathrm{dist}_\rho(\Phi(x), \Phi(y)) \leq \gamma_1 e^{-a|x|} \ \Rightarrow\ (x, y) \in E_h \ \Rightarrow\ \mathrm{dist}_\rho(\Phi(x), \Phi(y)) \leq \gamma_2 e^{-a|x|},$$

where $|x|$ is the graph distance from the root ϑ to $x \in X$, and E_h is the set of horizontal edges in (X, E). This notion is extended from the AI_a-graphs in [KLW] so that a similar result still holds (Theorem 6.3), and it also provides a characterization of all bi-Lipschitz embeddings of ∂X (Theorem 6.4, here $\Phi := \tau \circ \mathcal{J}_\partial$ as above):

Theorem 1.5. *Every AI_a-triple over a quasi-metric space (M, ρ) is admissible, and the boundary map $\kappa : (\partial X, \varrho_a) \to (M, \rho)$ is a bi-Lipschitz embedding.*

Moreover for an (m, k)-hyperbolic graph (X, E), a map τ from $(\partial X, \varrho_a)$ to (M, ρ) is a bi-Lipschitz embedding if and only if $(X, \widehat{E}^k, \Phi)$ is an AI_a-triple over (M, ρ).

From a construction inspired by *Christ's dyadic cubes* [Ch], we see that every compact quasi-metric space (K, ρ) can be the attractor of some AI_a-triple (Example 6.8). Moreover, as a consequence of Theorem 1.2, 1.5 and the *Assouad's theorem* [A] on bi-Lipschitz embeddings, we can associate any (m,k)-hyperbolic graph (X,E) of bounded degree to an AI_a-triple $(X,\widehat{E}^k,\Phi)$ over some Euclidean space (with small $a>0$, Proposition 6.9).

We also continue to investigate the AI_∞-triples (*augmented index triples of intersection type*, Definition 5.7) as in [KLW], and present an application on the *space of homogeneous type* [CW], i.e., a quasi-metric space (M,ρ) that carries a volume doubling (VD) measure μ: for an index triple (X,E,Φ) over (M,ρ), by regrouping the vertices in X according to the μ-volume of $\Phi(x)$, we obtain a new coding space $X(\mu)$. For the associated AI_∞-triple $(X(\mu),E^{(\infty)},\Phi)$, we have (Theorem 7.2)

Theorem 1.6. *Suppose the index triple (X,E,Φ) over (M,ρ) satisfies some mild assumptions, and μ is a (VD)-measure on (M,ρ). Then the AI_∞-triple $(X(\mu),E^{(\infty)},\Phi)$ is admissible.*

Via the boundary map on $\partial(X(\mu),E^{(\infty)})$, the Gromov distance ϱ_a defines a new quasi-metric $\tilde{\varrho}_a$ on the attractor K. Under a separation condition on μ, it turns out that the graph $(X(\mu),E^{(\infty)})$ has bounded degree, and $(K,\tilde{\varrho}_a,\mu)$ is Ahlfors regular (Theorem 7.3).

For the organization of the paper, we recall some definitions and preliminary results on expansive hyperbolic graphs in Section 2. We investigate the notion of near-isometry with two transforms on hyperbolic graphs in Section 3, and prove Theorems 1.1 and 1.2. The relation of near-isometries on graphs and Lipschitz equvalences on boundaries (Theorem 1.3) is detailed in Section 4. In Section 5, we present the topological setup of admissible index triples as well as a proof of Theorem 1.4. We show the duality of AI_a-triples and bi-Lipschiz embeddings of boundaries (Theorem 1.5), and revisit some conditions on the admissibility of AI_∞-triples in Section 6. This technique is applied in Section 7 to prove Theorem 1.6 and produce some "good" quasi-metrics on spaces of homogeneous type.

2 Preliminaries

We will briefly summerize some notions and background results on *expansive hyperbolic graphs*; in the case of unexplained notations, the reader can refer to [KLW] for details. Let (X,E) be a locally finite connected (undirected simple) graph. We use a vertex $\vartheta\in X$ as the *root*, and call such (X,E) a *rooted graph*. We denote by $d(\cdot,\cdot)$ the *graph distance* of (X,E); write $|x|:=d(\vartheta,x)$ for $x\in X$, and $X_n=\{x\in X:|x|=n\}$ for the n-th level set.

For $x\in X$ and an integer $m\geq 0$, let $\mathcal{J}_m(x)=\{y\in X:|y|-|x|=d(x,y)=m\}$ and $\mathcal{J}_{-m}(x)=\{z\in X:x\in\mathcal{J}_m(z)\}$ be the m-th *descendant set* and the m-th *predecessor set* of x respectively. We also let $\mathcal{J}_*(x)=\bigcup_{m=0}^\infty\mathcal{J}_m(x)$. Throughout this paper, we only consider the rooted graph (X,E) that satisfies

$$\mathcal{J}_1(x)\neq\emptyset,\qquad\forall\, x\in X. \tag{2.1}$$

We can decompose the edge set E into the *vertical edge set* $E_v = \{(x,y) \in E : |x| = |y| \pm 1\}$ and the *horizontal edge set* $E_h = \{(x,y) \in E : |x| = |y|\}$. A *vertical graph* is a rooted graph (X,E) satisfying $E = E_v$, denoted by (X,E_v) usually. The *horizontal distance* $d_h(\cdot,\cdot)$ is the graph distance of the subgraph (X,E_h) (by convention $d_h(x,y) = \infty$ if x,y are not connected by paths in (X,E_h)). A geodesic in (X,E) is called a *horizontal geodesic* if it lies entirely in (X,E_h).

The *Gromov product* of $x,y \in X$ is defined as

$$(x|y) := \frac{1}{2}(|x| + |y| - d(x,y)).$$

Definition 2.1. [Gr] *A rooted graph (X,E) is said to be* (Gromov) hyperbolic *if there is a constant $\delta \geq 0$ such that*

$$(x|y) \geq \min\{(x|z),(z|y)\} - \delta, \qquad \forall\ x,y,z \in X. \tag{2.2}$$

On a hyperbolic graph (X,E), for fixed $a > 0$, we define $\varrho_a(x,y) = e^{-a(x|y)}, x \neq y \in X$ and $= 0$ if $x = y$. It is direct to check that

$$\varrho_a(x,y) \leq e^{a\delta} \max\{\varrho_a(x,z), \varrho_a(z,y)\}, \qquad \forall\ x,y,z \in X.$$

Definition 2.2. [KLW] *We call a rooted graph (X,E)* expansive *if for $x,y \in X$,*

$$d_h(x,y) > 1\ \Rightarrow\ d_h(u,v) > 1, \quad \forall\ u \in \mathcal{J}_1(x),\ v \in \mathcal{J}_1(y), \tag{2.3}$$

and call it (m,k)-departing *with two integers $m,k \geq 1$ if*

$$d_h(x,y) > k\ \Rightarrow\ d_h(u,v) > 2k, \quad \forall\ u \in \mathcal{J}_m(x),\ v \in \mathcal{J}_m(y). \tag{2.4}$$

One important property of an expansive graph (X,E) is that any two vertices $x,y \in X$ can be connected by a *convex geodesic* $\pi(x,u,v,y)$ [KLW, Proposition 2.3], which consists of two vertical geodesics $\pi(x,u), \pi(v,y)$ and a horizontal geodesic $\pi(u,v)$. This simple form allows us to effectively handle the geodesics in (X,E). We also proved that $(1,1)$-departing implies $(m,1)$-departing, and $(m,1)$-departing implies (m,k)-departing for all $m,k \geq 1$. A simple example of a $(1,1)$-departing graph is the SG-graph (X,E) where X is the symbolic space representing the Sierpinski gasket K, E_v is the natural tree structure on X, and E_h consists of all pairs (x,y) that $|x| = |y|$ and the corresponding cells K_x, K_y meet (cf. [Ka] and [KLW, Example 2.8]).

Concerning the hyperbolicity of expansive graphs, we have the following useful criteria.

Theorem 2.3. [KLW, Theorem 2.11] *For an expansive graph (X,E), the following are equivalent.*

(i) *(X,E) is hyperbolic;*

(ii) *$\exists\ L < \infty$ such that the lengths of all horizontal geodesics are bounded by L;*

(iii) *(X,E) is (m,k)-departing for some integer $m,k \geq 1$.*

Properties (ii) and (iii) give us constructive ways to check the hyperbolicity, and will be used throughout the paper.

Let $\mathcal{R}_v := \{[x_i]_{i=0}^\infty : x_0 = \vartheta, \text{ and } x_{i+1} \in \mathcal{J}_1(x_i), \forall\, i \geq 0\}$ denote the family of *(geodesic) rays* starting from the root ϑ, and write $\mathbf{x}, \mathbf{y}, \mathbf{z}, \cdots$ for the rays $[x_i]_i, [y_i]_i, [z_i]_i \cdots$ in $\mathcal{R}_v$ respectively. For $x \in X$, we also denote by $\mathcal{R}_v[x]$ the subclass of rays in $\mathcal{R}_v$ that pass through x (which is not empty by the assumption (2.1)).

By the triangle inequality of the graph distance $d(\cdot,\cdot)$, it is seen that for $\mathbf{x}, \mathbf{y} \in \mathcal{R}_v$, the Gromov product $(x_i|y_i)$ is increasing in i. We set $(\mathbf{x}|\mathbf{y}) := \lim_{i\to\infty}(x_i|y_i) = \sup_{i\geq 0}(x_i|y_i)$, and write $\mathbf{x} \sim \mathbf{y}$ if and only if $(\mathbf{x}|\mathbf{y}) = \infty$. When (X, E) is hyperbolic, by (2.2), we see that this $\sim$ is an equivalence relation in $\mathcal{R}_v$.

Definition 2.4. *For a hyperbolic graph (X, E), the* hyperbolic boundary *is defined as the quotient set $\partial X = \partial(X, E) := \mathcal{R}_v/\sim$. For $a > 0$, define the* Gromov distance *on ∂X by*

$$\varrho_a(\xi, \eta) := e^{-a(\xi|\eta)}, \qquad \xi, \eta \in \partial X$$

($e^{-\infty} = 0$ by convention), where $(\xi|\eta)$ is the extended Gromov product *on ∂X given by*

$$(\xi|\eta) := \sup\{(\mathbf{x}|\mathbf{y}) : \mathbf{x} \in \xi,\ \mathbf{y} \in \eta\}. \tag{2.5}$$

By using (2.2), it is easy to check that $(\xi|\eta) \leq (\mathbf{x}|\mathbf{y}) + 2\delta$ for all $\mathbf{x} \in \xi$ and $\mathbf{y} \in \eta$, and the extended Gromov product satisfies $(\xi|\eta) \geq \min\{(\xi|\zeta), (\zeta|\eta)\} - 3\delta$ for all $\xi, \eta, \zeta \in \partial X$. This implies

$$\varrho_a(\xi, \eta) \leq e^{3a\delta} \max\{\varrho_a(\xi, \zeta), \varrho_a(\zeta, \eta)\}, \qquad \forall\, \xi, \eta, \zeta \in \partial X, \tag{2.6}$$

hence ϱ_a is a quasi-metric on ∂X; moreover, the quasi-metric space $(\partial X, \varrho_a)$ is compact.

For $x \in X$ and an integer $k \geq 0$, we call

$$\mathcal{J}_\partial(x) := \{\xi \in \partial X : \xi \cap \mathcal{R}_v[x] \neq \emptyset\}, \qquad \mathcal{J}_\partial^k(x) := \bigcup\{\mathcal{J}_\partial(y) : d_h(x, y) \leq k\} \tag{2.7}$$

the *G-cell* and *k-shadow* of x in ∂X respectively. Under the assumption (2.1), it is known that every G-cell (or k-shadow) is a nonempty compact subset of ∂X. Intuitively, $\mathcal{J}_\partial(x)$ is the quotient of the descendants of x run to infinity. Also note that for $y, z \in \mathcal{J}_*(x)$, $(y|z) \geq |x|$. By Definition 2.4, for $\xi, \eta \in \mathcal{J}_\partial(x)$, we have $\varrho_a(\xi, \eta) = e^{-a(\xi|\eta)} \leq e^{-a|x|}$. Hence we have the following estimate for the diameter of a G-cell:

$$|\mathcal{J}_\partial(x)|_{\varrho_a} := \sup\{\rho(\xi, \eta) : \xi, \eta \in \mathcal{J}_\partial(x)\} \leq e^{-a|x|}.$$

There are some alternative definitions of hyperbolic boundary (all are equivalent); the reader can refer to [CDP, Gr, GH, Wo]. In [KLW], we used $\partial X = \widehat{X} \setminus X$, where $\widehat{X}$ is the completion of X under another equivalent metric θ_a on X. The ϱ_a fulfills the same estimates as θ_a in [KLW]. Consequently, we have

Proposition 2.5. [KLW, Propositions 3.2, 3.3] *Let (X, E) be an (m, k)-departing expansive graph. Then there exist constants $\gamma = \gamma(m, k, a) > 0$ and $C = C(m, k, a) \geq 1$ such that*

(i) $\mathrm{dist}_{\varrho_a}(\mathcal{J}_\partial(x), \mathcal{J}_\partial(y)) > \gamma e^{-a|x|}$ *for all $x, y \in X$ with $|x| = |y|$ and $d_h(x, y) > k$;*

(ii) $B_{\varrho_a}(\xi, C^{-1}e^{-a|x|}) \subset \mathcal{J}^k_\partial(x) \subset B_{\varrho_a}(\xi, Ce^{-a|x|})$ *for all $x \in X$ and $\xi \in \mathcal{J}_\partial(x)$.*

It follows from (ii) that $|\mathcal{J}^k_\partial(x)|_{\varrho_a} \asymp e^{-a|x|}$ for all $x \in X$, where the involved constant in $\asymp$ depends on m, k, a only.

In Section 4, the family of G-cells will play an important role in the Lipschitz equivalence of the hyperbolic boundaries. Let (X, E) be a hyperbolic graph, and let $c > 0$ be fixed. Under the Gromov distance ϱ_a on ∂X, the G-cells induce a new horizontal edge set

$$E^{(c)}_h = \bigcup\nolimits_{n=1}^{\infty} \{(x, y) \in X_n \times X_n \setminus \Delta : \ \mathrm{dist}_{\varrho_a}(\mathcal{J}_\partial(x), \mathcal{J}_\partial(y)) \leq ce^{-a|x|}\} \tag{2.8}$$

as well as a new edge set $E^{(c)} = E_v \cup E^{(c)}_h$ on X. It was shown in [KLW, Theorem 4.5] that $(X, E^{(c)})$ is expansive and $(m, 1)$-departing for some $m \geq 1$, hence is also hyperbolic.

Lemma 2.6. *Let (X, E) be an $(m, 1)$-departing expansive graph. Then $E^{(\gamma)}_h \subset E_h \subset E^{(C)}_h$ where the constants γ and C are as in Proposition 2.5.*

Proof. Let $x, y \in X$ with $|x| = |y|$ and $d_h(x, y) > 1$, by Proposition 2.5(i) with $k = 1$, we have $\mathrm{dist}_{\varrho_a}(\mathcal{J}_\partial(x), \mathcal{J}_\partial(y)) > \gamma e^{-a|x|}$ for some constant $\gamma > 0$, this yields $E^{(\gamma)}_h \subset E_h$. On the other hand, for $(x, y) \in E_h$, using Proposition 2.5(ii) with $k = 1$,

$$\mathcal{J}_\partial(y) \subset \mathcal{J}^1_\partial(x) \subset B_{\varrho_a}(\xi, Ce^{-a|x|}), \qquad \forall\ \xi \in \mathcal{J}_\partial(x),$$

which implies $\mathrm{dist}_{\varrho_a}(\mathcal{J}_\partial(x), \mathcal{J}_\partial(y)) \leq Ce^{-a|x|}$, and hence $E_h \subset E^{(C)}_h$. □

3 Expansion of graphs and near-isometries

In this section we aim to extend the scope of our study from expansive hyperbolic graphs to the more general ones, and to investigate the notion of near-isometry. First we provide a simple way to "expand" an arbitrary rooted graph (X, E) to possess the expansive property. For nonempty sets $F, G \subset X$, we shall write $d(F, G) := \inf\{d(x, y) : x \in F,\ y \in G\}$, and the same for d_h.

Definition 3.1. *For a rooted graph (X, E) with graph distance d, we define a new horizontal edge set by*

$$\widehat{E}_h := \bigcup\nolimits_{n=1}^{\infty} \{(x, y) \in X_n \times X_n \setminus \Delta : d(\mathcal{J}_*(x), \mathcal{J}_*(y)) \leq 1\}. \tag{3.1}$$

We call $(X, \widehat{E})$ with $\widehat{E} = E_v \cup \widehat{E}_h$ the auxiliary graph *of (X, E) (see Figure 1).*

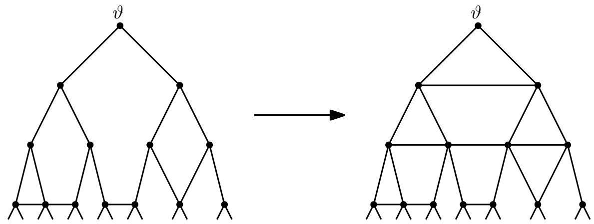

Figure 1: A rooted graph and its auxiliary graph.

Clearly $E_h \subset \widehat{E}_h$. The following is the main purpose of the auxiliary graph $(X, \widehat{E})$.

Proposition 3.2. *For a rooted graph (X, E), its auxiliary graph $(X, \widehat{E})$ is an expansive graph. Moreover, $\widehat{E}$ is "minimal", in the sense that if an expansive graph (X, E') satisfies $E_v = E'_v$ and $E_h \subset E'_h$, then $\widehat{E} \subset E'$.*

Proof. For $x, y \in X$, $u \in \mathcal{J}_1(x), v \in \mathcal{J}_1(y)$ with $\widehat{d}_h(u, v) \leq 1$ (where $\widehat{d}_h$ is the horizontal distance of $(X, \widehat{E})$), we have

$$d(\mathcal{J}_*(x), \mathcal{J}_*(y)) \leq d(\mathcal{J}_*(u), \mathcal{J}_*(v)) \leq 1 \quad \text{in} \quad (X, E).$$

It follows that $\widehat{d}_h(x, y) \leq 1$. This proves the expansive property of $(X, \widehat{E})$.

For the minimality, it follows from $E_v = E'_v$ that every vertex x has the same descendant set $\mathcal{J}_*(x)$ in (X, E) as in (X, E'). For $(x, y) \in \widehat{E}_h$, by the definition in (3.1), there exist $u \in \mathcal{J}_*(x)$ and $v \in \mathcal{J}_*(y)$ satisfying $(u, v) \in E_h \cup \Delta \subset E'_h \cup \Delta$. Hence $d'_h(u, v) \leq 1$. By the expansive property of (X, E'), we have $(x, y) \in E'_h$. Hence $\widehat{E} \subset E'$. $\square$

We recall the notion of quasi-isometry, which was originated in geometric group theory (cf. [CDP, Gr, GH, Ha]): let $(X, d), (X', d')$ be two metric spaces. A map $\varphi : X \to X'$ is called a *quasi-isometry* if there are constants $C \geq 1$ and $D \geq 0$ such that

$$C^{-1} d(x, y) - D \leq d'(\varphi(x), \varphi(y)) \leq C d(x, y) + D, \qquad \forall\ x, y \in X,$$

and $\sup_{z \in X'} d'(z, \varphi(X)) < \infty$. In this case we say that (X, d) is *quasi-isometric* to (X', d'). Here we concern only that $(X, E), (X', E')$ are two rooted graphs endowed with graph distances d, d' respectively. We shall denote such φ by $\varphi : (X, E) \to (X', E')$, and study the following sharper notion of near-isometry.

Definition 3.3. *A map $\varphi : (X, E) \to (X', E')$ is called a* near-isometric embedding *if there is a constant $D \geq 0$ such that*

$$d(x, y) - D \leq d'(\varphi(x), \varphi(y)) \leq d(x, y) + D, \qquad \forall\ x, y \in X. \tag{3.2}$$

If further $D' := \sup_{z \in X'} d'(z, \varphi(X)) < \infty$, then we call φ a near-isometry*. Particularly if $D = D' = 0$, i.e., φ is a bijection such that $d(x, y) = d'(\varphi(x), \varphi(y))$ for all $x, y \in X$, we call φ a* (graph) isomorphism.

Furthermore, φ is called v-invariant *if $\varphi(\vartheta) = \vartheta'$ (here $\vartheta \in X, \vartheta' \in X'$ are roots), and φ is an isomorphism from the vertical part (X, E_v) to another vertical part (X', E'_v).*

It is well known that the hyperbolicity is preserved by quasi-isometry (cf. [GH, Section 5.2]). Note that the composition of quasi-isometries is also a quasi-isometry, and for every quasi-isometry $\varphi : X \to X'$, a quasi-isometry $\bar{\varphi} : X' \to X$ exists as its *rough inverse* [Wo, p.28]. Therefore, to be quasi-isometric is an equivalence relation between rooted graphs. The same statements remain true if we replace "quasi-" by "near-" in the above.

In the rest of this section, we start the investigation for the v-invariant near-isometries of hyperbolic graphs. First, we prove a lemma to strengthen v-invariant quasi-isometries to near-isometries by using the convex geodesic property of expansive hyperbolic graphs [KLW, Proposition 2.3].

Lemma 3.4. *Let $(X, E), (X', E')$ be two rooted graphs. Suppose that (X, E) is expansive and hyperbolic, and there is a v-invariant map $\varphi : (X, E) \to (X', E')$ such that*

$$d'(\varphi(x), \varphi(y)) \le Cd(x, y) + D, \qquad \forall\ x, y \in X$$

for some $C, D > 0$. Then there is a constant $D' \ge 0$ such that $d'(\varphi(x), \varphi(y)) \le d(x, y) + D'$ for all $x, y \in X$.

Proof. By Theorem 2.3(ii), let $L < \infty$ be the maximal length of horizontal geodesics in (X, E). For $x, y \in X$, let $\pi(x, u, v, y)$ be a convex geodesic connecting x and y in (X, E). As $x \in \mathcal{J}_*(u)$ and φ is v-invariant, it follows that $d(x, u) = d'(\varphi(x), \varphi(u))$. Similarly, we have $d(v, y) = d'(\varphi(v), \varphi(y))$. Hence

$$\begin{aligned} d'(\varphi(x), \varphi(y)) &\le d'(\varphi(x), \varphi(u)) + d'(\varphi(u), \varphi(v)) + d'(\varphi(v), \varphi(y)) \\ &\le d(x, u) + (Cd(u, v) + D) + d(v, y) \\ &= d(x, y) + (C-1)d(u, v) + D \le d(x, y) + (C-1)L + D. \end{aligned}$$

This proves the statement with $D' = (C-1)L + D$. $\square$

Our first main result in this section is

Theorem 3.5. *For a hyperbolic graph (X, E), the identity map $\mathrm{id}_X : (X, E) \to (X, \widehat{E})$ is a near-isometry, and $(X, \widehat{E})$ is an expansive hyperbolic graph.*

Proof. The expansive property of $(X, \widehat{E})$ is proved in Proposition 3.2. For $(x, y) \in \widehat{E}_h$, by (3.1) there exist $u \in \mathcal{J}_*(x)$ and $v \in \mathcal{J}_*(y)$ such that $d(u, v) \le 1$. Note that $(x|u) = |x| = |y|$, and

$$(u|y) = \frac{1}{2}(|u| + |y| - d(u, y)) \ge \frac{1}{2}(|v| + |y| - d(v, y) - 2) = (v|y) - 1 = |y| - 1.$$

Using (2.2), we have

$$|y| - \frac{1}{2}d(x, y) = (x|y) \ge \min\{(x|u), (u|y)\} - \delta \ge |y| - 1 - \delta,$$

which implies that $d(x,y) \le 2(1+\delta)$. This provides an estimate on the graph distance $\widehat{d}$ of $(X,\widehat{E})$ by

$$\widehat{d}(x,y) \le d(x,y) \le 2(1+\delta)\cdot \widehat{d}(x,y), \qquad \forall\, x,y \in X,$$

which says that $\mathrm{id}_X : (X,E) \to (X,\widehat{E})$ is a quasi-isometry, hence $(X,\widehat{E})$ is also hyperbolic.

Clearly id_X is υ-invariant. Applying Lemma 3.4 to $\mathrm{id}_X : (X,\widehat{E}) \to (X,E)$, the above inequality can be enhanced to $d(x,y) \le \widehat{d}(x,y) + D$ for some constant $D \ge 0$. Hence id_X is a near-isometry from (X,E) to $(X,\widehat{E})$. $\square$

Remark. A rooted graph may not be hyperbolic even when its auxiliary graph is hyperbolic. For example, we consider the *two-dimensional lattice* (X,E), where $X = \mathbb{Z}^2$ with root $\vartheta = (0,0)$, and $E = \{(\mathbf{x},\mathbf{y}) \in X \times X : \mathbf{x} - \mathbf{y} = (\pm 1, 0) \text{ or } (0, \pm 1)\}$. Then it is direct to check that $E = E_v$, and the n-th level set X_n consists of all boundary vertices of the square $\bar{\mathcal{A}}_n$ as the convex hull of $\mathcal{A}_n := \{(0,\pm n), (\pm n, 0)\}$.

For $n \ge 1$ and $\mathbf{x} \in X_n \setminus \mathcal{A}_n$ (or $\mathbf{x} \in \mathcal{A}_n$), let $C(\mathbf{x})$ be the quadrant (or the half-plane respectively) translated from ϑ to $\mathbf{x}$ that intersects X_n only at $\mathbf{x}$ (see Figure 2(a)). Then the descendant set $\mathcal{J}_*(\mathbf{x})$ consists of all vertices lying in $C(\mathbf{x})$. It follows that for distinct $\mathbf{x},\mathbf{y} \in X_n$, $\mathcal{J}_*(\mathbf{x}) \cap \mathcal{J}_*(\mathbf{y}) \ne \emptyset$ (i.e., $(\mathbf{x},\mathbf{y}) \in \widehat{E}_h$) if and only if $\mathbf{x},\mathbf{y}$ lies on the same side of the square $\bar{\mathcal{A}}_n$. This shows that $\widehat{d}_h(\mathbf{x},\mathbf{y}) \le 3$ whenever $|\mathbf{x}| = |\mathbf{y}|$ (see Figure 2(b)). By Theorem 2.3(ii), the auxiliary graph $(X,\widehat{E})$ is hyperbolic. Moreover, all geodesic rays are equivalent in $(X,\widehat{E})$, thus the hyperbolic boundary is a singleton.

On the other hand, it is well-known that (X,E) is not hyperbolic. A direct check of the Gromov hyperbolic condition (2.2) is: for $\mathbf{x} = (n,0), \mathbf{y} = (0,n), \mathbf{z} = (n,n)$ with $n \ge 1$,

$$0 = (\mathbf{x}|\mathbf{y}) \ge \min\{(\mathbf{x}|\mathbf{z}),\, (\mathbf{z}|\mathbf{y})\} - \delta = n - \delta.$$

Hence the δ in the definition does not exist as n is arbitrary.

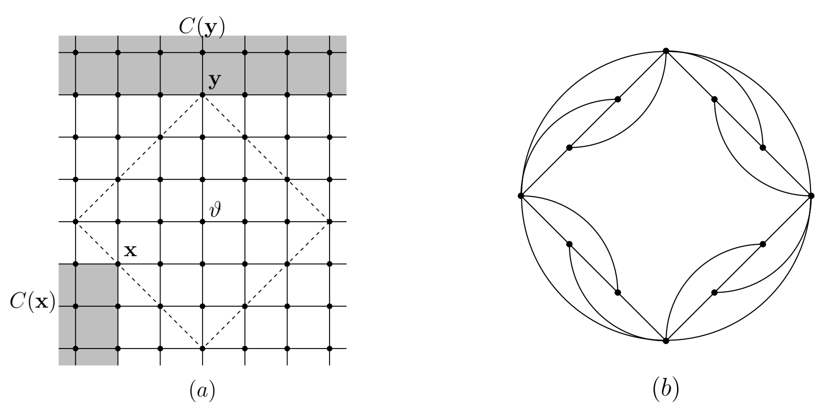

Figure 2: (a) The lattice and the descendant sets; (b) the subgraph $(X_3, \widehat{E}_h|_{X_3\times X_3})$.

A direct corollary of Theorems 2.3, 3.5 and Proposition 3.2 is

Corollary 3.6. *For a rooted graph (X, E), the following are equivalent.*

(i) (X, E) is hyperbolic;

(ii) the auxiliary graph $(X, \widehat{E})$ is near-isometric to (X, E), and is (m, k)-departing for some integer $m, k \geq 1$.

We shall call the graph (X, E) in (ii) an (m, k)**-hyperbolic graph**. In particular for an expansive graph, to be (m, k)-hyperbolic is the same as to be (m, k)-departing.

As another consequence of above, we have the following improvement of v-invariant quasi-isometries to near-isometries.

Proposition 3.7. *Let $(X, E), (X', E')$ be two hyperbolic graphs. If $\varphi : (X, E) \to (X', E')$ is a v-invariant quasi-isometry, then it is a near-isometry.*

Proof. By Theorem 3.5, the identity maps $\mathrm{id}_X : (X, E) \to (X, \widehat{E})$ and $\mathrm{id}_{X'} : (X', E') \to (X', \widehat{E'})$ are v-invariant near-isometries.

$$\begin{array}{ccc} (X, E) & \xrightarrow{\varphi} & (X', E') \\ {\scriptstyle \mathrm{id}_X}\updownarrow & & {\scriptstyle \mathrm{id}_{X'}}\updownarrow \\ (X, \widehat{E}) & \xrightarrow{\varphi} & (X', \widehat{E'}) \end{array}$$

By composition, $\varphi : (X, \widehat{E}) \to (X', \widehat{E'})$ is a v-invariant quasi-isometry. As both auxiliary graphs $(X, \widehat{E})$ and $(X, \widehat{E'})$ are expansive (Proposition 3.2) and hyperbolic, applying Lemma 3.4 to φ and its inverse, it follows that $\varphi : (X, \widehat{E}) \to (X', \widehat{E'})$ is a near-isometry, so is the one from (X, E) to (X', E') by composition. $\square$

We present another useful enlargement of a graph for which the near-isometry applies.

Definition 3.8. *For a rooted graph (X, E) and an interger $k \geq 1$, we enlarge the horizontal edge set by letting*

$$E_h^k := \{(x, y) \in X \times X \setminus \Delta : d_h(x, y) \leq k\} \tag{3.3}$$

and call (X, E^k) with the edge set $E^k = E_v \cup E_h^k$ the (horizontal) k-fuzz *of (X, E)* (cf. [Wo, Section 3.A]).

It is straightforward to check that the k-fuzz of an expansive graph is still expansive, and a rooted graph is (m, k)-departing if and only if the k-fuzz is $(m, 1)$-departing. We shall denote the graph distance of (X, E^k) by $d^{(k)}$. As $d^{(k)}(x, y) \leq d(x, y) \leq kd^{(k)}(x, y)$, the identity map id_X from (X, E) to (X, E^k) is a v-invariant quasi-isometry. Applying Proposition 3.7, we have

Corollary 3.9. *Let (X, E) be a hyperbolic graph. Then for any $k \geq 1$, the identity map $\mathrm{id}_X : (X, E) \to (X, E^k)$ is a near-isometry.*

Next we turn to consider the relations of hyperbolic boundaries. It is known that (cf. [Gr, Theorem 7.2.H], [BS, Theorem 6.5]) if φ is a quasi-isometry between two hyperbolic graphs $(X,E),(X',E')$, then φ induces a homeomorphism $\widehat{\varphi}$ from ∂X to $\partial X'$. Here we consider the case that φ is a v-invariant near-isometry.

Proposition 3.10. *Suppose $(X,E),(X',E')$ are hyperbolic graphs, and $\varphi:(X,E)\to(X',E')$ is a near-isometry and is v-invariant. Then the induced map $\widehat{\varphi}:(\partial X,\varrho_a)\to(\partial X',\varrho_a')$ is a Lipschitz equivalence, i.e., $\widehat{\varphi}(\partial X)=\partial X'$, and*

$$\varrho_a'(\widehat{\varphi}(\xi),\widehat{\varphi}(\eta))\asymp\varrho_a(\xi,\eta),\qquad\forall\ \xi,\eta\in\partial X. \tag{3.4}$$

Consequently, $\partial(X,E)=\partial(X,\widehat{E})=\partial(X,E^k)$, on which the corresponding Gromov distances $\varrho_a,\widehat{\varrho}_a,\varrho_a^{(k)}$ are Lipschitz equivalent.

Proof. For $\mathbf{x}\in\mathcal{R}_v$, the sequence $\varphi(\mathbf{x}):=[\varphi(x_i)]_{i=0}^\infty$ is also a ray in $\mathcal{R}_v'$. By near-isometry, we have $|(\varphi(x)|\varphi(y))'-(x|y)|=\frac{1}{2}|d'(\varphi(x),\varphi(y))-d(x,y)|\le\frac{D}{2}$ for all $x,y\in X$. Taking limit, we have $|(\varphi(\mathbf{x})|\varphi(\mathbf{y}))'-(\mathbf{x}|\mathbf{y})|\le\frac{D}{2}$ for all $\mathbf{x},\mathbf{y}\in\mathcal{R}_v$. This implies that $\mathbf{x}\sim\mathbf{y}$ in $\mathcal{R}_v$ if and only if $\varphi(\mathbf{x})\sim\varphi(\mathbf{y})$ in $\mathcal{R}_v'$, thus φ naturally induces a bijection $\widehat{\varphi}:\partial X\to\partial X'$. Moreover, by taking the supremum (as in (2.5)), it follows that $|(\widehat{\varphi}(\xi)|\widehat{\varphi}(\eta))'-(\xi|\eta)|\le\frac{D}{2}$ for all $\xi,\eta\in\partial X$. This implies

$$\varrho_a'(\widehat{\varphi}(\xi),\widehat{\varphi}(\eta))\asymp e^{-a((\widehat{\varphi}(\xi)|\widehat{\varphi}(\eta))')}\asymp e^{-a(\xi|\eta)}\asymp\varrho_a(\xi,\eta). \tag{3.5}$$

Taking $\varphi=\mathrm{id}_X$, the last statement follows from Theorem 3.5 and Corollary 3.9. $\square$

Remark. In the next section, we will show by more elaborate proofs that the v-invariant assumption on φ can be removed (Theorem 4.1), and the existence of the near-isometry is necessary and sufficient for the Lipschitz equivalence of boundaries (Theorem 4.3).

It has been proved in [KLW, Theorem 3.6] that an expansive hyperbolic graph (X,E) with *bounded degree* (i.e., $\sup_{x\in X}\deg(x)<\infty$) has a doubling hyperbolic boundary. Now in view of Theorem 3.5 and Proposition 3.10, we can extend this result for the hyperbolic graphs without assuming the expansive property.

Theorem 3.11. *Suppose (X,E) is a hyperbolic graph of bounded degree. Then $(\partial X,\varrho_a)$ is a doubling quasi-metric space.*

Proof. By Theorem 3.5, the auxiliary graph $(X,\widehat{E})$ is near-isometric to (X,E), hence is an expansive hyperbolic graph with bounded degree. This yields that $(\partial(X,\widehat{E}),\widehat{\varrho}_a)$ is doubling [KLW, Theorem 3.6]. As the doubling property is preserved by Lipschitz equivalence, the boundary $(\partial X,\varrho_a)$ of (X,E) is also doubling (by Proposition 3.10). $\square$

To conclude this section, we establish a near-isometry of (X,E) and the graph $(X,E^{(c)})$ introduced in (2.8), which will be applied in the next section (Theorem 4.3).

Proposition 3.12. *Suppose (X, E) is a hyperbolic graph. Then for any $c > 0$, the identity map $\mathrm{id}_X : (X, E) \to (X, E^{(c)})$ is a near-isometry.*

Consequently, there exists a constant $\widetilde{D}_c \geq 0$ (depends on c) such that for $x, y \in X$,

$$|x| = |y| \quad \textit{and} \quad \mathrm{dist}_{\varrho_a}\big(\mathcal{J}\partial(x), \mathcal{J}\partial(y)\big) \leq ce^{-a|x|} \quad \Rightarrow \quad d(x, y) \leq \widetilde{D}_c. \tag{3.6}$$

We first prove for the case that (X, E) is expansive and $(m, 1)$-departing.

Lemma 3.13. *Suppose (X, E) is an expansive $(m, 1)$-departing graph. Then the identity map $\mathrm{id}_X : (X, E) \to (X, E^{(c)})$ is a near-isometry.*

Proof. Denote the graph distances of $(X, E), (X, E^{(c)})$ by $d, d^{(c)}$ respectively. By Lemma 2.6, we have $E_h^{(\gamma)} \subset E_h \subset E_h^{(C)}$ for some constants $\gamma, C > 0$. Let n be the smallest nonnegative integer such that $e^{-an} \leq c^{-1}\gamma$. Suppose $(x, y) \in E_h^{(c)}$. If $|x| = |y| \leq n$, then $d(x, y) \leq d(x, \vartheta) + d(\vartheta, y) = 2n$; if $|x| = |y| > n$, taking $u \in \mathcal{J}_{-n}(x)$ and $v \in \mathcal{J}_{-n}(y)$, it follows that

$$\mathrm{dist}_{\varrho_a}(\mathcal{J}\partial(u), \mathcal{J}\partial(v)) \leq \mathrm{dist}_{\varrho_a}(\mathcal{J}\partial(x), \mathcal{J}\partial(y)) \leq ce^{-a|x|} \leq \gamma e^{an} e^{-a|x|} = \gamma e^{-a|u|},$$

so that $d_h(u, v) \leq 1$ in (X, E), and $d(x, y) \leq d(x, u) + d_h(u, v) + d(v, y) \leq 2n + 1$. This yields that $d(x, y) \leq (2n + 1)d^{(c)}(x, y)$ for all $x, y \in X$. Similarly we use $E_h^{(C)}$ to obtain $d^{(c)}(x, y) \leq (2n' + 1)d(x, y)$ for all $x, y \in X$, where n' is a nonnegative integer that depends on a, c, C only. Therefore, $\mathrm{id}_X : (X, E) \to (X, E^{(c)})$ is a quasi-isometry.

Note that the graph $(X, E^{(c)})$ is hyperbolic (by [KLW, Theorem 4.5]). As id_X is v-invariant, it follows from Proposition 3.7 that id_X is actually a near-isometry. □

Proof of Proposition 3.12. By Corollary 3.6, (X, E) is (m, k)-hyperbolic for some integers $m, k \geq 1$, and it follows that the k-fuzz $(X, \widehat{E}^k)$ of the auxiliary graph $(X, \widehat{E})$ is expansive and $(m, 1)$-departing. For $c > 0$, define the horizontal edge set $\widetilde{E}_h^{(c)}$ (and $\widetilde{E}^{(c)}$) as in (2.8) with ϱ_a replaced by $\widehat{\varrho}_a^{(k)}$, the Gromov distance on $\partial(X, \widehat{E}^k)$. Applying Lemma 3.13, the identity map $\mathrm{id}_X : (X, \widehat{E}^k) \to (X, \widetilde{E}^{(c)})$ is a near-isometry. So is $\mathrm{id}_X : (X, E) \to (X, \widetilde{E}^{(c)})$ by Theorem 3.5 and Corollary 3.9.

By Proposition 3.10, $\varrho_a, \widehat{\varrho}_a^{(k)}$ are Lipschitz equivalent, i.e., there is $C_0 \geq 1$ such that

$$C_0^{-1}\varrho_a(\xi, \eta) \leq \widehat{\varrho}_a^{(k)}(\xi, \eta) \leq C_0\varrho_a(\xi, \eta), \qquad \forall\, \xi, \eta \in \partial(X, E) = \partial(X, \widehat{E}^k).$$

Thus we have $\widetilde{E}_h^{(C_0^{-1}c)} \subset E_h^{(c)} \subset \widetilde{E}_h^{(C_0 c)}$, and the corresponding graph distances satisfy

$$\widetilde{d}^{(C_0 c)}(x, y) \leq d^{(c)}(x, y) \leq \widetilde{d}^{(C_0^{-1}c)}(x, y), \qquad \forall\, x, y \in X.$$

Then it follows that

$$\begin{aligned} D &:= \sup\nolimits_{x,y\in X}|d(x, y) - d^{(c)}(x, y)| \\ &\leq \max\Big\{\sup\nolimits_{x,y\in X}|d(x, y) - \widetilde{d}^{(C_0 c)}(x, y)|, \sup\nolimits_{x,y\in X}|d(x, y) - \widetilde{d}^{(C_0^{-1}c)}(x, y)|\Big\}, \end{aligned}$$

which is finite since both $\mathrm{id}_X : (X, E) \to (X, \widetilde{E}^{(C_0 c)})$ and $\mathrm{id}_X : (X, E) \to (X, \widetilde{E}^{(C_0^{-1} c)})$ are near-isometries. This proves that $\mathrm{id}_X : (X, E) \to (X, E^{(c)})$ is also a near-isometry.

For the last statement, the condition in (3.6) means $(x, y) \in E_h^{(c)}$, and it follows that $d(x, y) \le d^{(c)}(x, y) + D = 1 + D =: \widetilde{D}_c$. $\square$

4 On Lipschitz equivalences

In this section, we investigate more closely the connection of the near-isometries between two hyperbolic graphs and the Lipschitz equivalences between the hyperbolic boundaries. First we give an improvement of Proposition 3.10 by eliminating the v-invariant assumption on the near-isometry $\varphi : (X, E) \to (X', E')$.

Theorem 4.1. *Suppose $(X, E), (X', E')$ are hyperbolic graphs, and $\varphi : (X, E) \to (X', E')$ is a near-isometric embedding. Then for any $\xi \in \partial X$, there is a unique $\zeta \in \partial X'$ such that*

$$\lim_{n\to\infty} (\varphi(x_n)|y_n)' = \infty, \qquad \forall\ \mathbf{x} \in \xi,\ \mathbf{y} \in \zeta. \tag{4.1}$$

Hence we can define $\widehat{\varphi}(\xi) = \zeta$, and this $\widehat{\varphi} : (\partial X, \varrho_a) \to (\partial X', \varrho'_a)$ is a bi-Lipschitz embedding. Furthermore if φ is a near-isometry, then $\widehat{\varphi}$ is a Lipschitz equivalence.

Proof. Let $D \ge 0$ be the constant such that $|d'(\varphi(x), \varphi(y)) - d(x, y)| \le D$ for all $x, y \in X$, as in Definition 3.3. Denote $\ell := |\varphi(\vartheta)|'$. Then

$$\big||\varphi(x)|' - |x|\big| \le \big|d'(\varphi(\vartheta), \varphi(x)) - d(\vartheta, x)\big| + |\varphi(\vartheta)|' \le D + \ell, \qquad \forall\ x \in X. \tag{4.2}$$

It follows that for any $x, y \in X$,

$$\begin{aligned} |(\varphi(x)|\varphi(y))' - (x|y)| &= \frac{1}{2}\left|\big(|\varphi(x)|' - |x|\big) + \big(|\varphi(y)|' - |y|\big) - \big(d'(\varphi(x), \varphi(y)) - d(x, y)\big)\right| \\ &\le \frac{1}{2}\big((D + \ell) + (D + \ell) + D\big) = \frac{3}{2}D + \ell := \bar{D}. \end{aligned} \tag{4.3}$$

To prove (4.1), we first fix $\xi \in \partial X$ and $\mathbf{x} \in \xi$. Note that the sequence $\{\varphi(x_i)\}_i$ may not be a ray in (X', E'). However, it follows from (4.2) that $|\varphi(x_i)|' \ge |x_i| - (D + \ell) \to \infty$ as $i \to \infty$; using the local finiteness of (X', E'_v) and a diagonal argument, we can choose a ray $\mathbf{y} \in \mathcal{R}'_v$ and an infinite subsequence $\{\varphi(x_{j_n})\}_n$ such that $\varphi(x_{j_n}) \in \mathcal{J}'_*(y_n)$ for all $n \ge 0$. Clearly $j_n \ge n$. Using (2.2) and (4.3), we have for $n \ge 0$,

$$\begin{aligned} (\varphi(x_n)|y_n)' &\ge \min\{(\varphi(x_n)|\varphi(x_{j_n}))', (\varphi(x_{j_n})|y_n)'\} - \delta' \\ &\ge \min\{(x_n|x_{j_n}) - \bar{D}, n\} - \delta' = n - \bar{D} - \delta'. \end{aligned} \tag{4.4}$$

Taking $n \to \infty$, we get $\lim_{n\to\infty} (\varphi(x_n)|y_n)' = \infty$.

Next let $\zeta \in \partial X'$ be the equivalence class of $\mathbf{y}$. Then for any $\mathbf{x}' \in \xi$ and $\mathbf{y}' \in \zeta$, it follows from the same technique as in (4.4) (with intermediate terms $\varphi(x_n)$ and y_n) that $\lim_{n\to\infty} (\varphi(x'_n)|y'_n)' = \infty$. This proves (4.1).

To prove that ζ is unique, we take a ray $\mathbf{y}'' \in \mathcal{R}_v$ satisfying $\lim_{n\to\infty}(\varphi(x_n)|y''_n) = \infty$. Similarly we have

$$(y_n|y''_n)' \geq \min\{(y_n|\varphi(x_n))', (\varphi(x_n)|y''_n)'\} - \delta' \to \infty \qquad \text{as } n \to \infty.$$

Hence $\mathbf{y}''$ also belongs to the equivalence class ζ, and this proves the uniqueness. Therefore $\widehat{\varphi}$ is well-defined.

We prove that $\widehat{\varphi}$ is a bi-Lipschitz embedding. Suppose $\xi, \eta \in \partial X$ with $\xi \neq \eta$. For rays $\mathbf{v} \in \xi$, $\mathbf{t} \in \eta$, and $\mathbf{v}' \in \widehat{\varphi}(\xi)$, $\mathbf{t}' \in \widehat{\varphi}(\eta)$, using (4.3) and (2.2) again, we have

$$\begin{aligned}(v_n|t_n) &\geq (\varphi(v_n)|\varphi(t_n))' - \bar{D} \\ &\geq \min\{(\varphi(v_n)|v'_n)', (v'_n|t'_n)', (t'_n|\varphi(t_n))'\} - 2\delta' - \bar{D}.\end{aligned}$$

As $\lim_{n\to\infty}(\varphi(v_n)|v'_n)' = \lim_{n\to\infty}(t'_n|\varphi(t_n))' = \infty$ (by (4.1)), we have $(\mathbf{v}|\mathbf{t}) \geq (\mathbf{v}'|\mathbf{t}')' - 2\delta' - \bar{D}$. Similarly, we also have $(\mathbf{v}'|\mathbf{t}')' \geq (\mathbf{v}|\mathbf{t}) - 2\delta' - \bar{D}$. Taking supremums over $\mathbf{v}, \mathbf{t}$, and $\mathbf{v}', \mathbf{t}'$ as in (2.5), it follows that

$$|(\widehat{\varphi}(\xi)|\widehat{\varphi}(\eta))' - (\xi|\eta)| \leq 2\delta' + \bar{D},$$

and this implies the bi-Lipschitzness of $\widehat{\varphi}$ as in (3.5).

For the last statement, we suppose $D' = \sup_{z\in X'} d'(z, \varphi(X)) < \infty$ and show that $\widehat{\varphi}$ is surjective. Let $\mathbf{z} \in \mathcal{R}'_v$. Then there exists a sequence $\{u_i\}_i$ in X such that $d'(z_i, \varphi(u_i)) \leq D'$ for all $i \geq 0$ (by the second assumption on near-isometry). Using (4.2), we have

$$|u_i| \geq |\varphi(u_i)|' - (D + \ell) \geq |z_i|' - (D + D' + \ell) = i - (D + D' + \ell) \to \infty \qquad \text{as } i \to \infty.$$

As in the second paragraph, we can choose a ray $\mathbf{w} \in \mathcal{R}_v$ and an infinite subsequence $\{u_{k_n}\}_n$ such that $u_{k_n} \in \mathcal{J}_*(w_n)$ for all $n \geq 0$. Using a similar technique as in (4.4) (with intermediate terms $\varphi(u_{k_n})$ and z_{k_n}), we have $\lim_{n\to\infty}(\varphi(w_n)|z_n)' = \infty$. By the uniqueness of $\widehat{\varphi}(\cdot)$ just proved, the ray $\mathbf{z}$ belongs to the class $\widehat{\varphi}(\eta)$, where $\eta \in \partial X$ is the equivalence class of $\mathbf{w}$. This shows that $\widehat{\varphi}(\partial X) = \partial X'$. $\square$

Corollary 4.2. *With the same assumption as in Theorem 4.1, and let $\widehat{\varphi}$ be the induced map defined there. Then there is $\gamma_0 > 0$ such that*

$$\operatorname{dist}_{\varrho'_a}\big(\mathcal{J}'_\partial(\varphi(x)), \widehat{\varphi}(\mathcal{J}_\partial(x))\big) \leq \gamma_0 e^{-a|x|}, \qquad \forall\, x \in X.$$

Proof. We use the same notations as in the proof of Theorem 4.1. Let $x \in X_n$, and denote $n' := |\varphi(x)|'$. Then $n' = |\varphi(x)| \geq |x| - (D + \ell) = n - (D + \ell)$ by (4.2).

We choose two rays $\mathbf{x} \in \mathcal{R}_v[x]$, $\mathbf{y} \in \mathcal{R}'_v[\varphi(x)]$ (i.e., $x_n = x$ and $y_{n'} = \varphi(x)$), and let $\xi \in \partial X, \eta \in \partial X'$ be the equivalence classes of $\mathbf{x}, \mathbf{y}$ respectively. Then $\xi \in \mathcal{J}_\partial(x)$ and

$\eta \in \mathcal{J}'_\partial(\varphi(x))$. For a ray $\mathbf{z} \in \widehat{\varphi}(\xi)$, note that $\lim_{i\to\infty}(z_i|\varphi(x_i))' = \infty$ (by (4.1)). Together with (2.2) and (4.3), we have for sufficiently large i,

$$\begin{aligned}(z_{n'}|\varphi(x))' &\geq \min\{(z_{n'}|z_i)', (z_i|\varphi(x_i))', (\varphi(x_i)|\varphi(x_n))'\} - 2\delta' \\ &\geq \min\left\{(z_{n'}|z_i)', (z_i|\varphi(x_i))', (x_i|x_n) - \bar{D}\right\} - 2\delta' \\ &\geq \min\left\{n', n - \bar{D}\right\} - 2\delta \geq n - \bar{D} - 2\delta'.\end{aligned}$$

Hence it follows from (2.5) that

$$(\widehat{\varphi}(\xi)|\eta)' \geq (\mathbf{z}|\mathbf{y})' \geq (z_{n'}|y_{n'})' = (z_{n'}|\varphi(x))' \geq n - \bar{D} - 2\delta'.$$

This implies

$$\operatorname{dist}_{\varrho'_a}\left(\widehat{\varphi}(\mathcal{J}_\partial(x)), \mathcal{J}'_\partial(\varphi(x))\right) \leq \varrho'_a(\widehat{\varphi}(\xi), \eta) = e^{-a(\widehat{\varphi}(\xi)|\eta)'} \leq \gamma_0 e^{-an},$$

where $\gamma_0 := e^{a(\bar{D}+2\delta')}$. □

In the following we show that the converse of Theorem 4.1 is also true.

Theorem 4.3. *Let $(X,E), (X',E')$ be two hyperbolic graphs. Then a map $\tau : (\partial X, \varrho_a) \to (\partial X', \varrho'_a)$ is a bi-Lipschitz embedding (or a Lipschitz equivalence) if and only if there exists a near-isometric embedding (or a near-isometry, respectively) $\varphi : (X,E) \to (X',E')$ such that $\widehat{\varphi} = \tau$.*

We need a lemma to estimate the distance of the G-cells in the hyperbolic boundaries.

Lemma 4.4. *Let (X,E) be a hyperbolic graph, and let A, B be two subsets in ∂X. Then for a sequence of "intermidate sets" $A_1, A_2, \cdots, A_n \subset \partial X$ $(n \geq 1)$, we have*

$$\operatorname{dist}_{\varrho_a}(A,B) \leq e^{6na\delta} \max\{N_1, N_2\}, \tag{4.5}$$

where $N_1 := \max_{1\leq i\leq n+1}\{\operatorname{dist}_{\varrho_a}(A_{i-1}, A_i)\}$(here $A_0 := A$ and $A_{n+1} := B$), and $N_2 := \max_{1\leq i\leq n}\{|A_i|_{\varrho_a}\}$.

Proof. Let $\xi \in A$, $\eta \in B$ and $\xi_i, \eta_i \in A_i$ for $1 \leq i \leq n$. By applying (2.6) repeatedly to the sequence $\{\xi =: \xi_0, \eta_1, \xi_1, \eta_2, \cdots, \xi_{n-1}, \eta_n, \xi_n, \eta_{n+1} := \eta\}$, we have

$$\begin{aligned}\operatorname{dist}_{\varrho_a}(A,B) \leq \varrho_a(\xi,\eta) &\leq e^{6na\delta} \max\left\{\max_{1\leq i\leq n+1}\{\varrho_a(\xi_{i-1}, \eta_i)\}, \max_{1\leq i\leq n}\{\varrho_a(\eta_i, \xi_i)\}\right\} \\ &\leq e^{6na\delta} \max\left\{\max_{1\leq i\leq n+1}\{\varrho_a(\xi_{i-1}, \eta_i)\}, N_2\right\}.\end{aligned}$$

By taking infimum on the variables $\xi_{i-1} \in A_{i-1}, \eta_i \in A_i$, $1 \leq i \leq n+1$, (4.5) follows. □

Proof of Theorem 4.3. For simplicity, we write $\boldsymbol{\delta}(\cdot,\cdot) := \mathrm{dist}_{\varrho_a}(\cdot,\cdot)$, $\boldsymbol{\delta}'(\cdot,\cdot) := \mathrm{dist}_{\varrho'_a}(\cdot,\cdot)$, $\mathcal{J} := \mathcal{J}_\partial$ and $\mathcal{J}' := \mathcal{J}'_\partial$. In view of Theorem 4.1, we need only the proof of the necessity.

For $\tau : \partial X \to \partial X'$, we can choose a map $\varphi : X \to X'$ that satisfies

$$|\varphi(x)|' = |x|, \quad \text{and} \quad \mathcal{J}'(\varphi(x)) \cap \tau(\mathcal{J}(x)) \neq \emptyset, \quad \forall\, x \in X. \tag{4.6}$$

In fact, for $x \in X_n$, since $\tau(\mathcal{J}(x)) \subset \partial X' = \mathcal{J}'(X'_n)$, there exists $y =: \varphi(x) \in X'_n$ (the choice is not unique) such that $\mathcal{J}'(y) \cap \tau(\mathcal{J}(x)) \neq \emptyset$, and this $\varphi : X \to X'$ satisfies (4.6). The main proof is to show that such φ is a near-isometric embedding (or a near-isometry when τ is bijective). It will be done by three steps, and an additional step for $\widehat{\varphi} = \tau$.

Let $C_\tau \geq 1$ be the constant for the bi-Lipschitzness of τ, i.e.,

$$C_\tau^{-1} \varrho_a(\xi,\eta) \leq \varrho'_a(\tau(\xi),\tau(\eta)) \leq C_\tau \varrho_a(\xi,\eta), \qquad \forall\, \xi,\eta \in \partial X.$$

Step I. We show that $\sup_{x,y\in X}\{d'(\varphi(x),\varphi(y)) - d(x,y)\} < \infty$. Without loss of generality, we assume that (X,E) is expansive (otherwise we can prove the statement for the auxiliary graph $(X,\widehat{E})$ first, and then use $\widehat{d}(x,y) \leq d(x,y)$ to make the conclusion). For $x,y \in X$, we consider the convex geodesic $\pi(x,u,v,y)$ in (X,E). Since (X,E) is an expansive hyperbolic graph, the lengths of horizontal geodesics in (X,E) are bounded by some constant $L < \infty$ (Theorem 2.3). In particular, $d(u,v) \leq L$, and this yields

$$\boldsymbol{\delta}\big(\mathcal{J}(u),\mathcal{J}(v)\big) \leq e^{-a(u|v)} = e^{-a(|u|-d(u,v)/2)} \leq e^{aL/2}e^{-a|u|} \tag{4.7}$$

(the first inequality holds since $\varrho_a(\xi,\eta) = e^{-a(\xi|\eta)} \leq e^{-a(u|v)}$ for all $\xi \in \mathcal{J}(u)$, $\eta \in \mathcal{J}(v)$).

Let $n := |u| = |\varphi(u)|'$. Note that $\mathcal{J}(x) \subset \mathcal{J}(u)$, thus $\tau(\mathcal{J}(u))$ intersects $\mathcal{J}'(\varphi(x))$ as $\tau(\mathcal{J}(x))$ does. Also, we have $\tau(\mathcal{J}(u)) \cap \mathcal{J}'(\varphi(u)) \neq \emptyset$ by the choice of φ in (4.6). Hence

$$\boldsymbol{\delta}'\big(\mathcal{J}'(\varphi(x)),\mathcal{J}'(\varphi(u))\big) \leq |\tau(\mathcal{J}(u))|_{\varrho'_a} \leq C_\tau|\mathcal{J}(u)|_{\varrho_a} \leq C_\tau e^{-an}. \tag{4.8}$$

We choose $u' \in X'_n$ such that $\varphi(x) \in \mathcal{J}'_*(u')$. Then $\mathcal{J}'(\varphi(x)) \subset \mathcal{J}'(u')$. Similarly, we choose $v' \in X'_n$ such that $\varphi(y) \in \mathcal{J}'_*(v')$ (See Figure 3). Making use of Lemma 4.4 on $\partial X'$, we obtain the following estimates:

(i) $\boldsymbol{\delta}'\big(\mathcal{J}'(u'),\mathcal{J}'(\varphi(u))\big) \leq c_1 e^{-an}$ by using an intermediate set $\mathcal{J}'(\varphi(x))$ and (4.8);

(ii) $\boldsymbol{\delta}'\big(\mathcal{J}'(\varphi(u)),\mathcal{J}'(\varphi(v))\big) \leq c_2 e^{-an}$ by using intermediate sets $\tau(\mathcal{J}(u)), \tau(\mathcal{J}(v))$, the bi-Lipschitzness of τ and (4.7).

It follows from (i) and Proposition 3.12 that there exists a constant $D_1 := \widetilde{D}_{c_1}$ such that $d'(u',\varphi(u)) \leq D_1$. Therefore

$$\begin{aligned} d'(\varphi(x),\varphi(u)) &\leq d'(\varphi(x),u') + d'(u',\varphi(u)) \leq \big||\varphi(x)|' - |\varphi(u)|'\big| + D_1 \\ &= \big||x| - |u|\big| + D_1 = d(x,u) + D_1. \end{aligned} \tag{4.9}$$

By symmetry, we also have

$$d'(\varphi(y),\varphi(v)) \leq d(y,v) + D_1. \tag{4.10}$$

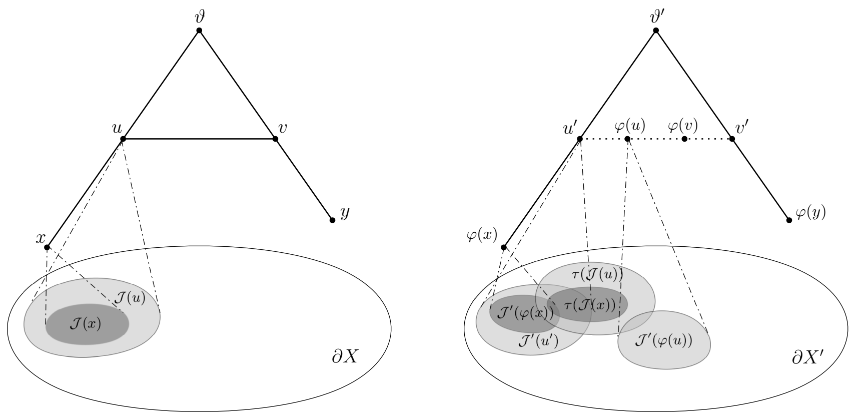

Figure 3: The near-isometry $\varphi : (X, E) \to (X', E')$.

By (ii) and Proposition 3.12 with $D_2 := \widetilde{D}_{c_2}$, we have

$$d'(\varphi(u), \varphi(v)) \le D_2. \tag{4.11}$$

Combining (4.9), (4.10) and (4.11), we have

$$\begin{aligned} d'(\varphi(x), \varphi(y)) &\le d'(\varphi(x), \varphi(u)) + d'(\varphi(u), \varphi(v)) + d'(\varphi(v), \varphi(y)) \\ &\le d(x, u) + d(v, y) + 2D_1 + D_2 \le d(x, y) + 2D_1 + D_2 \end{aligned}$$

(the last inequality follows as $d(x, y)$ is the length of the convex geodesic $\pi(x, u, v, y)$).

Step II. We prove the other direction: $\sup_{x,y\in X}\{d(x, y) - d'(\varphi(x), \varphi(y))\} < \infty$. The proof uses a similar idea as in Step I. Without loss of generality, we assume that (X', E') is expansive (otherwise we use $\widehat{E'}$ and observe that $\widehat{d'}(\varphi(x), \varphi(y)) \le d'(\varphi(x), \varphi(y))$). For $x, y \in X$, consider the convex geodesic $\pi(\varphi(x), \tilde{u}, \tilde{v}, \varphi(y))$ in (X', E'). Let L' be the upper bound of the lengths of horizontal geodesics in (X', E') (which is finite by Theorem 2.3). Analogous to (4.7), we obtain

$$\boldsymbol{\delta}'\big(\mathcal{J}'(\tilde{u}), \mathcal{J}'(\tilde{v})\big) \le e^{aL'/2} e^{-an'}, \tag{4.12}$$

where $n' = |\tilde{u}| = |\tilde{v}|$. We choose $\bar{u}, \bar{v} \in X_{n'}$ such that $x \in \mathcal{J}_*(\bar{u})$ and $y \in \mathcal{J}_*(\bar{v})$. Note that

$$\tau(\mathcal{J}(\bar{u})) \cap \mathcal{J}'(\tilde{u}) \supset \tau(\mathcal{J}(x)) \cap \mathcal{J}'(\varphi(x)) \ne \emptyset.$$

Using the same argument as in (4.8), we obtain

$$\boldsymbol{\delta}'\big(\mathcal{J}'(\tilde{u}),\, \mathcal{J}'(\varphi(\bar{u}))\big) \le |\tau(\mathcal{J}(\bar{u}))|_{\varrho'_a} \le C_\tau e^{-an'}. \tag{4.13}$$

Applying Lemma 4.4 with intermediate sets $\mathcal{J}'(\varphi(\bar{u})), \mathcal{J}'(\tilde{u}), \mathcal{J}'(\tilde{v}), \mathcal{J}'(\varphi(\bar{v}))$ and (4.12), (4.13), we have $\boldsymbol{\delta}'\big(\tau(\mathcal{J}(\bar{u})),\, \tau(\mathcal{J}(\bar{v}))\big) \le c_3 e^{-an'}$. This and the bi-Lipschitzness of τ imply

$$\boldsymbol{\delta}(\mathcal{J}(\bar{u}), \mathcal{J}(\bar{v})) \le C_\tau c_3 e^{-an'}.$$

By Proposition 3.12, there is a constant $D_3 := \widetilde{D}_{C_\tau c_3}$ such that $d(\bar{u}, \bar{v}) \leq D_3$. Hence

$$
\begin{aligned}
d(x, y) &\leq d(x, \bar{u}) + d(\bar{u}, \bar{v}) + d(\bar{v}, y) \leq \big||x| - |\bar{u}|\big| + D_3 + \big||y| - |\bar{v}|\big| \\
&= \big||\varphi(x)|' - |\tilde{u}|'\big| + D_3 + \big||\varphi(y)|' - |\tilde{v}|'\big| \leq d'(\varphi(x), \varphi(y)) + D_3
\end{aligned}
$$

(the last inequality follows from the convex geodesic $\pi(\varphi(x), \tilde{u}, \tilde{v}, \varphi(y))$). Therefore, φ is a near-isometric embedding.

Step III. We suppose τ is bijective in this step. To show that $\varphi : (X, E) \to (X, E')$ is a near-isometry, it remains to verify that $\sup_{z \in X'}\{d'(z, \varphi(X))\} < \infty$. For $z \in X'_n$, as $\tau(\mathcal{J}(X_n)) = \tau(\partial X) = \partial X'$, there exists $x \in X_n$ such that $\tau(\mathcal{J}(x))$ intersects $\mathcal{J}'(z)$. By the same argument as in (4.8), we have

$$
\boldsymbol{\delta}'\big(\mathcal{J}'(\varphi(x)), \mathcal{J}'(z)\big) \leq |\tau(\mathcal{J}(x))|_{\varrho'_a} \leq C_\tau e^{-an}.
$$

It follows from Proposition 3.12 that $d'(\varphi(x), z) \leq D_4 := \widetilde{D}_{C_\tau}$. Hence φ is a near-isometry.

Step IV. Note that $\widehat{\varphi}$ is bi-Lipschitz by Theorem 4.1. We need to show that $\widehat{\varphi} = \tau$. By Corollary 4.2, we have

$$
\boldsymbol{\delta}'\big(\mathcal{J}'(\varphi(x)), \widehat{\varphi}(\mathcal{J}(x))\big) \leq \gamma_0 e^{-a|x|}, \qquad x \in X. \tag{4.14}
$$

Using Lemma 4.4 with an intermediate set $\mathcal{J}'(\varphi(x))$ and (4.14), we have

$$
\boldsymbol{\delta}'\big(\widehat{\varphi}(\mathcal{J}(x)), \tau(\mathcal{J}(x))\big) \leq c_4 e^{-a|x|}. \tag{4.15}
$$

For $\xi \in \partial X$, let $\mathbf{x} \in \mathcal{R}_v$ be a ray in the equivalence class ξ. As $\xi \in \mathcal{J}(x_n)$, it follows from Lemma 4.4 with an intermediate set $\tau(\mathcal{J}(x_n))$ and (4.15) that

$$
\begin{aligned}
\varrho'_a\big(\widehat{\varphi}(\xi), \tau(\xi)\big) &\leq e^{6a\delta} \max\big\{|\widehat{\varphi}(\mathcal{J}(x_n))|_{\varrho'_a}, \boldsymbol{\delta}'\big(\widehat{\varphi}(\mathcal{J}(x_n)), \tau(\mathcal{J}(x_n))\big), |\tau(\mathcal{J}(x_n))|_{\varrho'_a}\big\} \\
&\leq e^{6a\delta} \max\big\{C_{\widehat{\varphi}}, c_4, C_\tau\big\} e^{-an},
\end{aligned}
$$

where $C_{\widehat{\varphi}} \geq 1$ is the bi-Lipschitz constant of the map $\widehat{\varphi}$ (Theorem 4.1). By taking $n \to \infty$, we obtain $\widehat{\varphi}(\xi) = \tau(\xi)$, and this completes the proof. $\square$

For a Lipschitz equivalence τ of two hyperbolic boundaries, the above proof constructs a near-isometry φ of two hyperbolic graphs so that $\tau = \widehat{\varphi}$. In the following, we show that the proof can be modified slightly to characterize all such φ.

Proposition 4.5. *Suppose $(X, E), (X', E')$ are two hyperbolic graphs, and $\tau : (\partial X, \varrho_a) \to (\partial X', \varrho'_a)$ is a Lipschitz equivalence. For a mapping $\varphi : X \to X'$, the following assertions are equivalent.*

(i) φ is a near-isometry from (X, E) to (X', E'), and the induced mapping $\widehat{\varphi}$ (as in Theorem 4.1) equals τ.

(ii) There exist $D_0, \gamma_0 \in [0, \infty)$ such that for any $x \in X$,

$$
\big||\varphi(x)|' - |x|\big| \leq D_0, \tag{4.16}
$$

$$
\text{and} \qquad \operatorname{dist}_{\varrho'_a}\big(\mathcal{J}'_\partial(\varphi(x)), \tau(\mathcal{J}_\partial(x))\big) \leq \gamma_0 e^{-a|x|}. \tag{4.17}
$$

Remark. Note that the φ in the proof of Theorem 4.3 satisfies $D_0 = \gamma_0 = 0$. If we replace "Lipschitz equivalence" and "near-isometry" in the above by "bi-Lipschitz embedding" and "near-isometric embedding" respectively, the conclusion still holds.

Proof. (i) $\Rightarrow$ (ii): Suppose φ is a near-isometry with the constant $D \geq 0$ as in Definition 3.3, and denote $\ell := |\varphi(\vartheta)|'$. Then the inequality (4.16) is given by (4.2) with $D_0 = D + \ell$. Moreover, (4.17) follows from Corollary 4.2.

(ii) $\Rightarrow$ (i): The proof will be done by the same four steps as in Theorem 4.3. We only provide the main adjustments in Step I here.

Note that for the convex geodesic $\pi(x, u, v, y)$ in (X, E), unlike Theorem 4.3, $|\varphi(u)|'$ may not equal $|u|$, and it is possible that $|\varphi(x)|' < |\varphi(u)|'$. Let $n := |\varphi(u)|'$. We choose $u' \in X_n'$ such that $\varphi(x) \in \mathcal{J}_*'(u')$ if $|\varphi(x)|' \geq n$ and $u' \in \mathcal{J}_*'(\varphi(x))$ otherwise. Using (4.16), we have

$$|\varphi(x)|' \geq |x| - D_0 \geq |u| - D_0 \geq |\varphi(u)|' - 2D_0 = n - 2D_0, \tag{4.18}$$

$$\text{and} \quad d'(\varphi(x), u') = \big||\varphi(x)|' - |\varphi(u)|'\big| \leq \big||x| - |u|\big| + 2D_0 = d(x, u) + 2D_0. \tag{4.19}$$

The estimate of $d'(\varphi(x), \varphi(u))$ (similar to (4.9)) follows from (4.19) and the estimate of $\boldsymbol{\delta}'\big(\mathcal{J}'(u'), \mathcal{J}'(\varphi(u))\big)$ (using Lemma 4.4 with intermediate sets $\mathcal{J}'(\varphi(x), \tau(\mathcal{J}(x)), \tau(\mathcal{J}(u))$ and (4.17), (4.18)). The same is for $d'(\varphi(v), \varphi(y))$.

It remains to estimate $d'(\varphi(u), \varphi(v))$. Note that $\varphi(u), \varphi(v)$ may not be on the same level as in Theorem 4.3. Similarly, we choose $v' \in X_n'$ such that $\varphi(v) \in \mathcal{J}_*'(v')$ if $|\varphi(v)|' \geq n$ and $v' \in \mathcal{J}_*'(\varphi(v)$ otherwise. Also, we have

$$|\varphi(v)|' \geq n - 2D_0, \quad \text{and} \quad d'(v', \varphi(v)) \leq \big||u| - |v|\big| + 2D_0 = 2D_0.$$

These together with the estimate of $\boldsymbol{\delta}'(\mathcal{J}'(\varphi(u)), \mathcal{J}'(v'))$ (applying Lemma 4.4 with intermediate sets $\tau(\mathcal{J}(u)), \tau(\mathcal{J}(v)), \mathcal{J}'(\varphi(v))$ and (4.17)) imply that $d'(\varphi(u), \varphi(v))$ is bounded by some constant. Combining these three estimates, Step I is completed.

The similar adjustments can be applied to other steps, and we omit the details. $\square$

5 Index maps and admissibility

In this section, we will present the notion of index maps, introduced in [LW2, KLW], into a purely topological framework. Let M be a Hausdorff (topological) space, and let $\mathcal{C}_M$ denote the family of nonempty compact subsets of M.

Definition 5.1. *Let (X, E) be a rooted graph. A map $\Phi : X \to \mathcal{C}_M$ is called an* index map *on (X, E_v) over M if it satisfies*

(i) $\Phi(y) \subset \Phi(x)$ for all $x \in X$ and $y \in \mathcal{J}_1(x)$;

(ii) $\bigcap_{i=0}^{\infty} \Phi(x_i)$ is a singleton for all $\mathbf{x} = [x_i]_i \in \mathcal{R}_v$.

We call such (X, E, Φ) *an* index triple *(over* M*). In particular, such* Φ *(or* (X, E, Φ)*) is called* saturated *if the above (i) is strengthened to* $\Phi(x) = \bigcup_{y \in \mathcal{J}_1(x)} \Phi(y)$.

The motivation of this definition is from fractal consideration. For a contractive iterated functions system (IFS) $\{S_i\}_{i=1}^N$ with attractor K, let (X, E_v) be the symbolic space (coding space) used as indices. For $x \in X$, take $\Phi(x) = S_x(F)$, where F is a nonempty compact set such that $\bigcup_{i=1}^N S_i(F) \subset F$. This Φ satisfies (i) and (ii), hence (X, E_v, Φ) is an index triple. In particular, if $F = K$, then $\Phi(x)$ equals K_x, the x-cell of the attractor K. As K_x is the union of its offspring cells, the index triple is saturated [Ki1, Ka, LW1].

Example 5.2. (A fundamental intrinsic index triple) Let (X, E) be a hyperbolic graph, and let $M = \partial X$ be equipped with the Gromov distance ϱ_a. Consider the map $\mathcal{J}_\partial : X \to \mathcal{C}_{\partial X}$ defined by the G-cells $\mathcal{J}_\partial(x)$ as in (2.7). Clearly, $\mathcal{J}_\partial(x) = \bigcup_{y \in \mathcal{J}_1(x)} \mathcal{J}_\partial(y)$, and for a ray $\mathbf{x}$ in an equivalence class $\xi \in \partial X$, the intersection $\bigcap_{i=0}^\infty \mathcal{J}_\partial(x_i) = \{\xi\}$. It follows that $(X, E, \mathcal{J}_\partial)$ is a saturated index triple.

The index map Φ defines a map $\kappa_0^\Phi : \mathcal{R}_v \to M$ by $\{\kappa_0^\Phi(\mathbf{x})\} = \bigcap_{i=0}^\infty \Phi(x_i)$ for $\mathbf{x} = [x_i]_i \in \mathcal{R}_v$. We denote the image of κ_0^Φ by K^Φ and call it the *attractor* of Φ. Clearly $K^\Phi \in \mathcal{C}_M$. We say that two index maps Φ and Ψ *behave the same at infinity* if $\kappa_0^\Phi = \kappa_0^\Psi$. For an index map Φ, we define

$$\widetilde{\Phi}(x) := \bigcap_{n=0}^\infty \Big(\bigcup_{y \in \mathcal{J}_n(x)} \Phi(y) \Big), \qquad x \in X.$$

Then $\widetilde{\Phi} : X \to \mathcal{C}_{K^\Phi}$ is a saturated index map; we call it the *saturated part* of Φ. It is clear that $\widetilde{\Phi}(x) \subset \Phi(x)$, and $\widetilde{\Phi}$ behaves the same as Φ at infinity.

Recall that when (X, E) is hyperbolic, the hyperbolic boundary ∂X is defined as the quotient set $\mathcal{R}_v / \sim$ (Definition 2.4). In this case, if an index map Φ on (X, E_v) satisfies

$$\kappa_0^\Phi(\mathbf{x}) = \kappa_0^\Phi(\mathbf{y}) \quad \Leftrightarrow \quad \mathbf{x} \sim \mathbf{y}, \tag{5.1}$$

then it induces an injection $\kappa^\Phi : \partial X \to M$ via the quotient; we call κ^Φ the *boundary map* of Φ. For simplicity we shall omit the superscript Φ in $\kappa_0^\Phi, \kappa^\Phi, K^\Phi$ if no confusion occurs.

Definition 5.3. *An index triple* (X, E, Φ) *is said to be* admissible *if it satisfies*

(i) the graph (X, E) *is hyperbolic;*

(ii) the boundary map $\kappa : \partial X \to M$ *is a well-defined injection, i.e.,* (5.1) *holds.*

Remark. In Example 5.2, it is clear that the intrinsic index triple $(X, E, \mathcal{J}_\partial)$ is admissible with $\kappa = \mathrm{id}_{\partial X}$.

We will see in Proposition 5.5 that the κ in (ii) is actually a (topological) embedding, i.e., a homeomorphism from ∂X to the image K. For an admissible index triple (X, E, Φ), it is easy to check that

$$\kappa(\mathcal{J}_\partial(x)) = \kappa_0(\mathcal{R}_v[x]) = \widetilde{\Phi}(x) \subset \Phi(x), \qquad \forall\ x \in X, \tag{5.2}$$

and the inclusion is an "=" for all $x \in X$ if and only if (X, E, Φ) is saturated. If an index map Ψ on (X, E_v) behaves the same as Φ at infinity, we know from (5.1) that the index triple (X, E, Ψ) is also admissible.

For an integer $k \geq 1$, define

$$\Phi^k(x) := \bigcup \{\Phi(y) : d_h(x, y) \leq k\}, \qquad x \in X. \tag{5.3}$$

Note that by (5.2), $\Phi^k(x)$ contains the κ-image of the k-shadow of x (see (2.7)).

Lemma 5.4. *Let (X, E, Φ) be an admissible index triple. Suppose (X, E) is expansive. Then for an integer $k \geq 1$, the Φ^k in (5.3) is an index map that behaves the same as Φ, hence (X, E, Φ^k) is also an admissible index triple.*

Proof. For $x \in X$ and $u \in \mathcal{J}_1(x)$, using the expansive property (2.3), each vertex v with $d_h(u, v) \leq k$ satisfies $d_h(x, y) \leq k$ for all $y \in \mathcal{J}_{-1}(v)$. This shows that $\Phi^k(u) \subset \Phi^k(x)$.

For a ray $\mathbf{x} \in \mathcal{R}_v$, we have

$$\{\kappa_0(\mathbf{x})\} = \bigcap_{i=0}^{\infty} \Phi(x_i) \subset \bigcap_{i=0}^{\infty} \Phi^k(x_i).$$

Let $\xi \in \bigcap_{i=0}^{\infty} \Phi^k(x_i)$. Then there is a sequence $\{y_i\}_{i=0}^{\infty}$ such that $\xi \in \bigcap_{i=0}^{\infty} \Phi(y_i)$ and $d_h(x_i, y_i) \leq k$ for all $i \geq 0$. This $\{y_i\}_{i=0}^{\infty}$ may not be a ray; however, using the local finiteness of (X, E_v) and a diagonal argument, we can choose a ray $\mathbf{z} \in \mathcal{R}_v$ and a subsequence $\{y_{i_n}\}_{n=0}^{\infty}$ with $y_{i_n} \in \mathcal{J}_*(z_n)$ for all $n \geq 0$. As $\xi \in \Phi(y_{i_n}) \subset \Phi(z_n)$, we get $\xi = \kappa_0(\mathbf{z})$. Using the expansive property (2.3) we have

$$d_h(x_n, z_n) \leq d_h(x_{i_n}, y_{i_n}) \leq k, \qquad \forall\, n \geq 0.$$

Therefore $\mathbf{x} \sim \mathbf{z}$, and it follows from (5.1) that $\kappa_0(\mathbf{x}) = \kappa_0(\mathbf{z}) = \xi$. Hence the intersection $\bigcap_{i=0}^{\infty} \Phi^k(x_i)$ is the singleton $\{\kappa_0(\mathbf{x})\}$, i.e., $\Phi^k : X \to \mathcal{C}_M$ is an index map with $\kappa_0^{\Phi^k} = \kappa_0$, and this completes the proof. $\square$

Proposition 5.5. *Let M be a Hausdorff space. For an admissible index triple (X, E, Φ) over M, the boundary map $\kappa : \partial X \to M$ is a (topological) embedding.*

Proof. First we assume that (X, E) is expansive. Then by Theorem 2.3, (X, E) is (m, k)-departing for some integers $m, k \geq 1$. Let $C \geq 1$ be as in Proposition 2.5(ii). Suppose $\xi \in \partial X$ is the equivalence class of a ray $\mathbf{x} = [x_i]_{i=0}^{\infty} \in \mathcal{R}_v$. For a sequence $\{\eta_n\}_{n=1}^{\infty}$ in ∂X with $\varrho_a(\eta_n, \xi) < C^{-1} e^{-an}$, it follows from Proposition 2.5(ii) that $\eta_n \in \mathcal{J}_\partial^k(x_n)$. Using the definition of k-shadow (2.7), (5.2) and (5.3), we have

$$\kappa(\eta_n) \in \kappa(\mathcal{J}_\partial^k(x_n)) = \kappa\Big(\bigcup \{\mathcal{J}_\partial(y) : d_h(x_n, y) \leq k\}\Big) \subset \Phi^k(x_n).$$

By Lemma 5.4, the sequence $\{\kappa(\eta_n)\}_{n=1}^{\infty}$ converges to $\kappa_0^{\Phi^k}(\mathbf{x}) = \kappa_0(\mathbf{x}) = \kappa(\xi)$. This shows that $\kappa : \partial X \to M$ is continuous, hence is closed by the compactness of ∂X and the

Hausdorff property of M. Since κ is injective by definition, it is a topological embedding, i.e, a homeomorphism from ∂X to the image K.

For the (X, E) that is not expansive, by Proposition 3.2, Theorem 3.5, the auxiliary graph $(X, \widehat{E})$ is expansive and hyperbolic, and by Corollary 3.10, the hyperbolic boundary $\partial(X, \widehat{E}) = \partial(X, E)$ (where the identity map is a homeomorphism). Hence the embedding κ from $\partial(X, \widehat{E})$ to M is also an embedding from $\partial(X, E)$ to M. $\square$

As a consequence, for an admissible index triple (X, E, Φ), the attractor $K \subset M$ is *metrizable*. More precisely, the Gromov distance ϱ_a on ∂X defines a quasi-metric $\tilde{\varrho}_a$ on K by

$$\tilde{\varrho}_a(\xi, \eta) = \varrho_a(\kappa^{-1}(\xi), \kappa^{-1}(\eta)), \qquad \xi, \eta \in K, \tag{5.4}$$

and the induced topology equals the one inherited from M.

The following is the main conclusion of this section. We use the above considerations to characterize all embeddings of hyperbolic boundaries by admissible index triples.

Theorem 5.6. *Let (X, E) be a hyperbolic graph, and let M be a Hausdorff space. Then a map $\tau : \partial X \to M$ is an embedding if and only if $(X, E, \tau \circ \mathcal{J}_\partial)$ is an admissible index triple over M. Moreover in this case, we have $\kappa^{\tau \circ \mathcal{J}_\partial} = \tau$.*

Proof. Let $\Phi := \tau \circ \mathcal{J}_\partial$. For a ray $\mathbf{x} = [x_i]_i$ that belongs to an equivalence class $\xi \in \partial X$, we have

$$\{\tau(\xi)\} = \tau\big(\bigcap\nolimits_{i=0}^{\infty} \mathcal{J}_\partial(x_i)\big) \ \subset \ \bigcap\nolimits_{i=0}^{\infty} \tau(\mathcal{J}_\partial(x_i)) = \bigcap\nolimits_{i=0}^{\infty} \Phi(x_i). \tag{5.5}$$

For the necessity, as $\mathcal{J}_\partial(x) = \bigcup_{y \in \mathcal{J}_1(x)} \mathcal{J}_\partial(y)$, we have $\Phi(x) = \bigcup_{y \in \mathcal{J}_1(x)} \Phi(y)$. On the other hand, note that τ is an injection, the inclusion in (5.5) is actually an "=". Hence Φ is a saturated index map, and the boundary map κ^Φ is well-defined and equals τ. It follows that (X, E, Φ) is an admissible index triple over M.

For the sufficiency, as $\Phi = \tau \circ \mathcal{J}_\partial$ is an index map, it follows from (5.5) that $\bigcap_{i=0}^{\infty} \Phi(x_i) = \{\tau(\xi)\}$ whenever $\mathbf{x} \in \mathcal{R}_v$ belongs to $\xi \in \partial X$, hence κ_0^Φ satisfies (5.1), and the boundary map $\kappa^\Phi = \tau$. By Proposition 5.5, such τ is an embedding from ∂X to M. $\square$

For an index map Φ fixed on a vertical graph (X, E_v) (e.g. the family of cells indexed by the symbolic spaces of a contractive IFS), if we can *augment* the graph by adding a set E_h of horizontal edges on each level X_n such that the index triple (X, E, Φ) with $E = E_v \cup E_h$ is admissible, then the attractor K and the hyperbolic boundary ∂X will be (topologically) identified (Proposition 5.5). The following is a natural choice of augmentation which uses the intersecting pairs in $\{\Phi(x)\}_{x \in X_n}$, $n \geq 1$ [Ka, LW1].

Definition 5.7. *An AI_∞-triple (*augmented index triple of type-∞*, or* intersection type*) is an index triple (X, E, Φ) in which the horizontal edge set E_h equals*

$$E_h^{(\infty)} := \bigcup\nolimits_{n=1}^{\infty} \big\{(x, y) \in X_n \times X_n \setminus \Delta : \Phi(x) \cap \Phi(y) \neq \emptyset\big\}. \tag{5.6}$$

Such (X, E) is called an AI_∞-graph (associated to Φ) [LW1, LW2, KLW].

Clearly every AI_∞-graph is expansive, and it was proved in [KLW, Proposition 4.6] that the hyperbolicity of AI_∞-graph is sufficient for the admissibility of AI_∞-triple. However, AI_∞-graphs are not always hyperbolic (e.g. [KLW, Example 6.1]). In Section 6, we give more discussion of this type of triples when the underlying spaces are quasi-metric spaces.

6 AI_a-triples and bi-Lipschitz embeddings

In this section, we consider some specific index triples for which the underlying space M is equipped with a quasi-metric ρ, and explore the bi-Lipschitz property of the boundary maps.

Let Φ be an index map on a vertical graph (X, E_v) over (M, ρ). For $a \in (0, \infty)$, we say that Φ is of *exponential type-*(a) (under ρ) if $|\Phi(x)|_\rho = O(e^{-a|x|})$ as $|x| \to \infty$. As we will see in Example 6.8, every $K \in \mathcal{C}_M$ is the attractor of some index map of exponential type-(a).

Similar to (2.8), for $\gamma \in (0, \infty)$ and fixed $a > 0$, we define a horizontal edge set by

$$E_h^{(\gamma)}(= E_h^{(a,\gamma)}) := \bigcup_{n=1}^{\infty} \big\{(x, y) \in X_n \times X_n \setminus \Delta : \mathrm{dist}_\rho(\Phi(x), \Phi(y)) \le \gamma e^{-an}\big\}, \qquad (6.1)$$

and let $E^{(\gamma)} := E_v \cup E_h^{(\gamma)}$. It is known that $(X, E^{(\gamma)})$ is expansive $(m, 1)$-hyperbolic for some integer $m > 0$ [KLW, Theorem 4.5].

Definition 6.1. *For $a \in (0, \infty)$, we say that an index triple (X, E, Φ) is of* augmented type-(a) *(AI_a-*triple*) over (M, ρ) if it satisfies*

(i) Φ is of exponential type-(a) under ρ;

(ii) there exist $\gamma_1, \gamma_2 \in (0, \infty)$ such that $E_h^{(a,\gamma_1)} \subset E_h \subset E_h^{(a,\gamma_2)}$.

*In this case, we call (X, E) an AI_a-*graph *(associated to Φ).*

Remark 1. In [KLW], we called only the $(X, E^{(\gamma)})$ defined by (6.1) an AI_a-graph, which augments (X, E_v) by explicit horizontal edges in $E_h^{(\gamma)}$. This is generalized to the present definition that brings more flexible choices of E_h. By a slight abuse of notation, we still use AI_a to denote such graph (X, E). Unlike $(X, E^{(\gamma)})$, this (X, E) is not always expansive (unless $\gamma_2/\gamma_1 \le e^a$). Nevertheless, by using a near-isometry in the following proposition, we are still able to prove that (X, E) is $(m, 1)$-hyperbolic.

Remark 2. Suppose (X, E) is an expansive $(m, 1)$-hyperbolic graph. In view of Lemma 2.6, the intrinsic index triple $(X, E, \mathcal{J}_\partial)$ (see Example 5.2) is an AI_a-triple. This is also a motivation to extend the definition of AI_a-graph into the present setting.

Proposition 6.2. *Let (X, E, Φ) be an AI_a-triple. Then*

(i) the identity map $\mathrm{id}_X : (X, E) \to (X, E^{(1)})$ is a near-isometry;

(ii) the AI_a-graph (X, E) is $(m, 1)$-hyperbolic for some integer $m \ge 1$.

Proof. The assertion (i) follows from the same argument as in Lemma 3.13 (in which we replace the Gromov distance ϱ_a by ρ, and the constants c, γ, C by $1, \gamma_1, \gamma_2$ respectively).

For (ii), note that $(X, E^{(1)})$ is hyperbolic, so is (X, E) via the near-isometry in (i). It remains to show that the auxiliary graph $(X, \widehat{E})$ is $(m, 1)$-departing for some $m \geq 1$.

We claim that $E_h^{(\gamma_1)} \subset \widehat{E}_h \subset E_h^{(\gamma_2)}$, thus $(X, \widehat{E})$ is also an AI_a-graph. Indeed, since $E_h \subset E_h^{(\gamma_2)}$, and $(X, E^{(\gamma_2)})$ is expansive, by using the minimality of $\widehat{E}$ (Proposition 3.2), we have $\widehat{E}_h \subset E_h^{(\gamma_2)}$. This together with $E_h^{(\gamma_1)} \subset E_h \subset \widehat{E}_h$ proves the claim.

The remaining proof is similar to the one in [KLW, Theorem 4.5], so we omit it. □

Theorem 6.3. *Suppose (X, E, Φ) is an AI_a-triple over a quasi-metric space (M, ρ). Then (X, E, Φ) is an admissible index triple, and the boundary map $\kappa : (\partial X, \varrho_a) \to (M, \rho)$ is a bi-Lipschitz embedding, i.e.,*

$$\rho(\kappa(\xi), \kappa(\eta)) \asymp \varrho_a(\xi, \eta), \qquad \forall\ \xi, \eta \in \partial X. \tag{6.2}$$

Proof. The hyperbolicity of (X, E) is proved in Proposition 6.2. From the near-isometry in Proposition 6.2(i), we see that $\mathbf{x} \sim \mathbf{y}$ in (X, E) if and only if it holds in $(X, E^{(1)})$. Hence the boundary map κ is well-defined, so that (X, E, Φ) is admissible. Also we have $\partial(X, E) = \partial(X, E^{(1)})$, on which the corresponding Gromov distances $\varrho_a, \varrho_a^{(1)}$ are Lipschitz equivalent by Proposition 3.10. Now applying [KLW, Theorem 4.5] to $(X, E^{(1)})$, we see that the boundary map κ is a bi-Lipschitz embedding from $\partial(X, E^{(1)})$ to (M, ρ), so is the one from $\partial(X, E)$. □

It is straightforward to check that to be an AI_a-triple is preserved by bi-Lipschitz maps: for an AI_a-triple (X, E, Φ) over (M, ρ) and a bi-Lipschitz embedding $\tau : (M, \rho) \to (M', \rho')$, $(X, E, \tau \circ \Phi)$ is an AI_a-triple over (M', ρ').

Recall that a hyperbolic graph (X, E) is (m, k)-hyperbolic for some integers $m, k \geq 1$ (Corollary 3.6). This means that the auxiliary graph $(X, \widehat{E})$ is (m, k)-departing, and the k-fuzz $(X, \widehat{E}^k)$ is $(m, 1)$-departing. Analogous to Theorem 5.6, we are able to characterize all bi-Lipschitz embeddings of hyperbolic boundaries by AI_a-triples.

Theorem 6.4. *Let (X, E) be an (m, k)-hyperbolic graph, and let (M, ρ) be a quasi-metric space. Then a map $\tau : (\partial X, \varrho_a) \to (M, \rho)$ is a bi-Lipschitz embedding if and only if $(X, \widehat{E}^k, \tau \circ \mathcal{J}_\partial)$ is an AI_a-triple over (M, ρ).*

Proof. By Theorem 3.5, Corollary 3.9 and Proposition 3.10, $(X, \widehat{E}^k)$ is a hyperbolic graph with the boundary $\partial(X, \widehat{E}^k) = \partial(X, E)$, and the corresponding Gromov distance $\widehat{\varrho}_a^{(k)}$ is Lipschitz equivalent to ϱ_a on ∂X.

Let $\Phi := \tau \circ \mathcal{J}_\partial$. Suppose $\tau : (\partial X, \varrho_a) \to (M, \rho)$ is a bi-Lipschitz embedding, then it is also bi-Lipschitz when ϱ_a is replaced by $\widehat{\varrho}_a^{(k)}$. It follows from Remark 2 that $(X, \widehat{E}^k, \mathcal{J}_\partial)$ is an AI_a-triple over $(\partial X, \widehat{\varrho}_a^{(k)})$, and so is $(X, \widehat{E}^k, \Phi)$ over (M, ρ). This proves the necessity.

For the sufficiency, note that the boundary map $\kappa^\Phi = \tau$ (by Theorem 5.6). Applying Theorem 6.3, the map $\tau : (\partial X, \widehat{\varrho}_a^{(k)}) \to (M, \rho)$ is a bi-Lipschitz embedding. As $\varrho_a, \widehat{\varrho}_a^{(k)}$ are Lipschitz equivalent on ∂X, the assertion in the theorem follows. $\square$

As a consequence, we have that in some sense, every admissible index triple can be viewed as an AI_a-triple.

Corollary 6.5. *Suppose (X, E, Φ) is an admissible index triple, and (X, E) is (m, k)-hyperbolic. Then $(X, \widehat{E}^k, \widetilde{\Phi})$ is an AI_a-triple over $(K, \tilde{\varrho}_a)$, where $\widetilde{\Phi}$ is the saturated part of Φ, and $\tilde{\varrho}_a$ is the quasi-metric defined by* (5.4).

Proof. We observe from (5.2) and (5.4) that $\widetilde{\Phi} = \kappa \circ \mathcal{J}_\partial$, and $\kappa : (\partial X, \varrho_a) \to (K, \tilde{\varrho}_a)$ is an isometry. The conclusion follows from Theorem 6.4 with $\tau = \kappa$. $\square$

It follows from Theorems 4.3 and 6.3 that two AI_a-triples possess Lipschitz equivalent attractors if and only if the AI_a-graphs are near-isometric. More precisely, we have

Proposition 6.6. *Suppose $(X, E, \Phi), (X', E', \Phi')$ are two AI_a-triples over quasi-metric spaces $(M, \rho), (M', \rho')$ with attractors K, K' respectively. Then a map $\tau : (K, \rho) \to (K', \rho')$ is a Lipschitz equivalence if and only if there is a near-isometry $\varphi : (X, E) \to (X, E')$ that satisfies*

$$\kappa^{\Phi'} \circ \widehat{\varphi} = \tau \circ \kappa^\Phi.$$

Proof. By Theorem 6.3, both $\kappa^\Phi : (\partial X, \varrho_a) \to (K, \rho)$ and $\kappa^{\Phi'} : (\partial X', \varrho'_a) \to (K', \rho')$ are Lipschitz equivalences.

For the sufficiency, it follows from Theorem 4.1 that the map $\widehat{\varphi} : (\partial X, \varrho_a) \to (\partial X', \varrho'_a)$ induced by the near-isometry φ is a Lipschitz equivalence, and so is $\tau = \kappa^{\Phi'} \circ \widehat{\varphi} \circ (\kappa^\Phi)^{-1} : (K, \rho) \to (K', \rho')$ by composition.

$$\begin{array}{ccc}
(\partial X, \varrho_a) & \xrightarrow{\kappa^\Phi} & (K, \rho) \\
\widehat{\varphi}\Big\downarrow & & \tau\Big\downarrow \\
(\partial X', \varrho'_a) & \xrightarrow{\kappa^{\Phi'}} & (K', \rho')
\end{array}$$

To prove the necessity, note that $\tau' := (\kappa^{\Phi'})^{-1} \circ \tau \circ \kappa^\Phi : (\partial X, \varrho_a) \to (\partial X', \varrho'_a)$ is a Lipschitz equivalence by composition. By applying Theorem 4.3 to this τ', we obtain the desired near-isometry $\varphi : (X, E) \to (X', E')$. $\square$

Now we turn to revisit the AI_∞-triples (X, E, Φ) (Definition 5.7) over a quasi-metric space (M, ρ) (in which (X, E) may not be hyperbolic). Compared with AI_a-triples, it is clear that $E_h^{(\infty)} \subset E_h^{(a,\gamma)}$ for all $a, \gamma > 0$. Consequently if the index map Φ is of exponential type-(a) and the associated AI_∞-graph is hyperbolic, then the boundary map κ is a (one-sided) Lipschitz embedding [KLW, Corollary 4.7], but may not be bi-Lipschitz (e.g. [KLW, Example 6.2]). Particularly for saturated Φ, we have the following characterization of the (m, k)-hyperbolicity of AI_∞-graph together with the bi-Lipschitzness of κ.

Theorem 6.7. *Suppose (X, E, Φ) is a saturated AI_∞-triple over a quasi-metric (M, ρ). Then for $a \in (0, \infty)$ and an integer $k \geq 1$, the following assertions are equivalent.*

(i) The AI_∞-graph (X, E) is (m, k)-hyperbolic for some $m \geq 1$, and the boundary map $\kappa : (\partial X, \varrho_a) \to (M, \rho)$ is a bi-Lipschitz embedding;

(ii) Φ is of exponential type-(a) under ρ, and there exists $\gamma > 0$ such that (X, E) satisfies for $x, y \in X$,

$$|x| = |y|, \quad \textit{and} \quad \mathrm{dist}_\rho(\Phi(x), \Phi(y)) \leq \gamma e^{-a|x|} \quad \Rightarrow \quad d_h(x, y) \leq k; \tag{6.3}$$

(iii) (X, E^k, Φ) is an AI_a-triple over (M, ρ).

Proof. (i) $\Leftrightarrow$ (ii) has been proved in [KLW, Theorem 4.8].

(ii) $\Rightarrow$ (iii): It suffices to show that $E_h^{(\gamma)} \subset E_h^k \subset E_h^{(\gamma_2)}$ for some $\gamma_2 \geq \gamma$. The first inclusion is straightforward by (6.3). For the second inclusion, let $\delta_0 := \sup_{z \in X} e^{a|z|} |\Phi(z)|$, which is finite as Φ is of exponential type-(a). For $(x, y) \in E_h^k$, there is a horizontal path $[x = x_0, x_1, \cdots, x_{\ell-1}, x_\ell = y]$ in (X, E) with $\ell \leq k$. Using the quasi-triangle inequality,

$$\mathrm{dist}(\Phi(x), \Phi(y)) \leq C_\rho^{\ell-2} \sum_{j=1}^{\ell-1} |\Phi(x_j)| \leq C_\rho^{k-2}(k-1)\delta_0 e^{-a|x|}.$$

Hence $(x, y) \in E_h^{(a,\gamma_2)}$ with $\gamma_2 := \max\{C_\rho^{k-2}(k-1)\delta_0, \gamma\}$.

(iii) $\Rightarrow$ (i): By Proposition 6.2(ii), the k-fuzz (X, E^k) is $(m, 1)$-hyperbolic for some $m \geq 1$. This implies that (X, E) is (m, k)-hyperbolic. Moreover, we have $\partial(X, E) = \partial(X, E^k)$ on which two Gromov distances ϱ_a and $\varrho_a^{(k)}$ are Lipschitz equivalent (Proposition 3.10). It follows from Theorem 6.3 that $\kappa : (\partial(X, E^k), \varrho_a^{(k)}) \to (M, \rho)$ is a bi-Lipschitz embedding, and so is the one from $(\partial(X, E), \varrho_a)$. This completes the proof. $\square$

We conclude this section with two statements on the existence of AI_a-triples. The first one is under a given attractor, for which we provide an example posted in the beginning of the section. The construction arises from the *dyadic cubes in spaces of homogeneous type* introduced by Christ [Ch] (see also Example 7.1), where the following concept is needed: for $F \subset M$ and $r > 0$, a discrete subset $\Xi \subset F$ is called an *r-net* on F if $\rho(\xi, \eta) \geq r$ whenever $\xi, \eta \in \Xi$ are distinct, and $F \subset \bigcup_{\xi \in \Xi} B_\rho(\xi, r)$. By Zorn's lemma, one can easily show that such Ξ always exists. Clearly, any r-net on F is a finite set provided that (F, ρ) is compact.

We need to use the following result proved by Macías and Segovia [MS] (see also [He, Proposition 14.5]): *for a quasi-metric space (M, ρ), there exist $\epsilon > 0$ and a metric θ on M such that $\theta(\xi, \eta) \asymp \rho(\xi, \eta)^\epsilon$ for all $\xi, \eta \in M$.*

Example 6.8. (The existence of AI_a-triples under a given attractor) *Let (M, ρ) be a quasi-metric space. For arbitrary $K \in \mathcal{C}_M$ and $a \in (0, \infty)$, there exists an index map Φ of exponential type-(a) with attractor $K^\Phi = K$. Furthermore, the associated AI_a-graph (or AI_∞-graph) has bounded degree provided that (K, ρ) is doubling.*

Suppose $K \in \mathcal{C}_M$. Without loss of generality, we assume that $|K|_\rho = 1$. Fix $a, c_0 \in (0, \infty)$. For $n \geq 0$, let Ξ_n be an e^{-an}*-net* on (K, ρ), and let $X_n = \Xi_n \times \{n\}$ (by convention $X_0 = \{\vartheta\}$). Let $X = \bigcup_{n=0}^\infty X_n$, and define $\iota : X \to K$ to be the natural projection of each X_n onto Ξ_n. Then by the definition of e^{-an}-net,

$$K \subset \bigcup\nolimits_{z \in X_n} B_\rho(\iota(z), e^{-an}) \quad \text{and} \quad \rho(\iota(x), \iota(y)) \geq e^{-an}, \quad x \neq y \in X_n. \tag{6.4}$$

We choose the vertical edge set E_v on X with root ϑ to be a subset of

$$\bigcup\nolimits_{n=0}^\infty \{(x, y),\, (y, x) : x \in X_n,\, y \in X_{n+1},\, \rho(\iota(x), \iota(y)) < e^{-an}\} \tag{6.5}$$

such that $\mathcal{J}_1(x) \neq \emptyset$ for all $x \in X$. Define

$$\Phi^o(x) = \bigcup\nolimits_{z \in \mathcal{J}_*(x)} B_\rho(\iota(z), c_0 e^{-a|z|}), \;\text{ and }\; \Phi(x) = (\Phi^o(x))^-, \qquad x \in X. \tag{6.6}$$

We claim that Φ is an index map on (X, E_v) of exponential type-(a).

Clearly, $\Phi(y) \subset \Phi(x)$ for all $y \in \mathcal{J}_1(x)$. We show that $|\Phi(x)|_\rho = O(e^{-a|x|})$. From the above, we have a metric θ on K satisfying $C^{-1}\rho(\cdot, \cdot)^\epsilon \leq \theta(\cdot, \cdot) \leq C\rho(\cdot, \cdot)^\epsilon$ for some $\epsilon > 0$ and $C \geq 1$. Let $x \in X_n$ and $\zeta \in \Phi^o(x)$. By (6.6), there is a ray $\mathbf{x} = [x_i]_{i=0}^\infty \in \mathcal{R}_v$ such that $x_n = x$ and $\zeta \in B_\rho(\iota(x_{n+m}), c_0 e^{-a(n+m)})$. It follows from (6.5) that

$$\begin{aligned} C^{-1}\rho(\iota(x), \zeta)^\epsilon \leq \theta(\iota(x), \zeta) &\leq \sum\nolimits_{i=1}^m \theta(\iota(x_{n+i-1}), \iota(x_{n+i})) + \theta(\iota(x_{n+m}), \zeta) \\ &< \sum\nolimits_{i=1}^m C e^{-a\epsilon(n+i-1)} + C c_0^\epsilon e^{-a\epsilon(n+m)} < \frac{C(1 + c_0^\epsilon)}{1 - e^{-a\epsilon}} e^{-an\epsilon}, \end{aligned}$$

that is, $\rho(\iota(x), \zeta) < \left(\frac{C^2(1+c_0^\epsilon)}{1-e^{-a\epsilon}}\right)^{1/\epsilon} e^{-an} =: C_1 e^{-an}$. Hence $\Phi(x)$ is covered by the closure of the ball $B_\rho(\iota(x), C_1 e^{-an})$, and this proves the claim.

Moreover, the attractor of Φ equals K. Indeed, for a ray $\mathbf{y} \in \mathcal{R}_v$, we have $\kappa_0^\Phi(\mathbf{y}) \in \Phi(y_n) \subset B_\rho(\iota(y_n), C_1 e^{-an})^-$ for all $n \geq 0$, thus the sequence $\{\iota(y_n)\}_{n=0}^\infty$ converges to $\kappa_0^\Phi(\mathbf{y})$. By the compactness of K, the limit $\kappa_0^\Phi(\mathbf{y})$ belongs to K, and hence $K^\Phi = \kappa_0^\Phi(\mathcal{R}_v) \subset K$. On the other hand, for $\xi \in K$ and $i \geq 0$, by (6.4) there exists $z_i \in X_i$ such that $\xi \in B_\rho(\iota(z_i), e^{-ai})$. This $\{z_i\}_{i=0}^\infty$ may not be a ray; using the local finiteness of (X, E_v) (by the compactness of K) and a diagonal argument, we can choose a ray $\mathbf{w} \in \mathcal{R}_v$ and a subsequence $\{z_{i_n}\}_{n=0}^\infty$ such that $z_{i_n} \in \mathcal{J}_*(w_n)$ for all $n \geq 0$. Let $\eta := \kappa_0^\Phi(\mathbf{w})$. Using the quasi-triangle inequality, we have

$$\begin{aligned} \rho(\xi, \eta) &\leq C_\rho\big(\rho(\xi, \Phi(w_n)) + |\Phi(w_n)|_\rho\big) \\ &\leq C_\rho\big(\rho(\xi, \iota(z_{i_n})) + |\Phi(w_n)|_\rho\big) \leq C_\rho(1 + \delta_0)e^{-an}, \end{aligned}$$

where $\delta_0 := \sup_{z \in X} e^{a|z|}|\Phi(z)|_\rho$ is finite as Φ is of exponential type-(a). Taking $n \to \infty$, $\xi = \eta$ follows. Therefore, $K^\Phi = K$.

If (K, ρ) is doubling, then Φ satisfies the separation condition (S_a) in [KLW, Definition 5.1]. In this case, both the AI_a- and AI_∞-triples are admissible, and the corresponding graphs are of bounded degree (by Theorems 6.3 and [KLW, Theorems 5.4, 5.5]). $\square$

Another issue is on the existence of AI_a-triples over a Euclidean space. We say that an index map Φ is *Euclidean* if the underlying space $M = \mathbb{R}^n$ for some $n \geq 1$. In metric geometry, a related question is to characterize the metric spaces that can be bi-Lipschitz embedded into some Euclidean space. An elegant and well-known result for this is the *Assouad's theorem* [A] (see also [He, Theorem 12.2]), which states that *for any doubling metric space (M, ρ) and $\epsilon \in (0,1)$, there is a bi-Lipschitz embedding τ from $(M, \rho(\cdot,\cdot)^\epsilon)$ to some $\mathbb{R}^n$.* From this we conclude that

Proposition 6.9. *Let (X, E) be an (m,k)-hyperbolic graph with bounded degree. Then there exists a set of Euclidean index maps $\{\Phi^{(a)}\}_{a\in(0,a_0)}$ with some $a_0 > 0$ such that each $(X, \widehat{E}^k, \Phi^{(a)})$ is an AI_a-triple.*

Proof. As (X, E) is of bounded degree, it follows from Theorem 3.11 that the hyperbolic boundary $(\partial X, \varrho_a)$ is doubling. Note that $\varrho_a(\cdot,\cdot)^{b/a} = \varrho_b(\cdot,\cdot)$ for $a, b > 0$. Using the result by Macías and Segovia [MS], there exists $a_0 > 0$ such that ϱ_{a_0} is Lipschitz equivalent to some metric θ on ∂X, and the metric space $(\partial X, \theta)$ is also doubling.

Applying the Assouad's theorem [A] to this $(\partial X, \theta)$, we have a bi-Lipschitz embedding $\tau_\epsilon : (\partial X, \theta(\cdot,\cdot)^\epsilon) \to \mathbb{R}^n$ for each $\epsilon \in (0,1)$ (here n depends on ϵ). Let $\Phi^{(a)} := \tau_\epsilon \circ \mathcal{J}_\partial$ with $a = a_0\epsilon$. By Theorem 6.4, the statement follows. □

7 On spaces of homogeneous type

Let μ be a nonnegative measure on a quasi-metric space (M, ρ) that is regular Borel with respect to the canonical topology $\mathcal{T}_\rho$. We say that μ is *volume doubling* (VD) if there exists $C > 0$ such that for any $\xi \in M$ and $r > 0$,

$$0 < \mu(B_\rho(\xi, 2r)) \leq C\mu(B_\rho(\xi, r)) < \infty.$$

It was observed by Coifman and Weiss [CW] that the existence of (VD)-measure μ implies the doubling property of (M, ρ). Conversely, Luukkainen and Saksman [LuS] proved that every complete doubling metric space carries a (VD)-measure; in view of the result by Macías and Segovia [MS], the same holds true for a complete doubling quasi-metric space.

In this section, we assume that (M, ρ, μ) is a *space of homogeneous type* [CW], i.e., a quasi-metric space (M, ρ) equipped with a *volume doubling* (VD) measure μ. In [Ch], Christ showed that such space admits a partition system (dyadic cubes) that can be represented by a tree. In the following we give a brief outline of his construction, slightly adapted to compact sets. Much of the basic setup is in Example 6.8.

Example 7.1. (Christ's dyadic cubes) Let (M, ρ, μ) be a space of homogeneous type. We fix a nonempty compact set $K \subset M$, and two constants $a, c_0 \in (0, \infty)$ such that

$$\frac{C_\rho^3 e^{-a}}{1 - C_\rho e^{-a}} + C_\rho^2 c_0 \leq \frac{1}{2}. \tag{7.1}$$

where $C_\rho \geq 1$ is the constant in the quasi-triangle inequality.

Define the vertex set X and the projection $\iota : X \to K$ as in Example 6.8. For $n \geq 0$ and $x \in X_{n+1}$, we choose $x^- \in X_n$ such that $\iota(x^-)$ is the nearest point from $\iota(x)$ among all points in $\Xi_n = \iota(X_n)$ (if there are two or more nearest points, we select an arbitrary one from them). Let $E_v = \{(x, x^-), (x^-, x) : x \in X,\, x \neq \vartheta\}$. We claim that all assumptions in Example 6.8 are fulfilled by (X, E_v).

Indeed, from the choice of x^-, we see that $\rho(\iota(x), \iota(x^-)) \leq e^{-an}$, thus E_v is a subset of the set in (6.5). Clearly (X, E_v) is a tree with root ϑ. It remains to show that $\mathcal{J}_1(x) \neq \emptyset$ for any $x \in X_n$. For this, as Ξ_{n+1} is an $e^{-a(n+1)}$-net on K, the point $\iota(x)$ must lie in some ball $B_\rho(\iota(y), e^{-a(n+1)})$ with $y \in X_{n+1}$. For this y and any $x' \in X_n \setminus \{x\}$, the quasi-triangle inequality implies that

$$\begin{aligned}
\rho(\iota(x'), \iota(y)) &\geq C_\rho^{-1} \rho(\iota(x'), \iota(x)) - \rho(\iota(x), \iota(y)) \\
&> C_\rho^{-1} e^{-an} - e^{-a(n+1)} = (C_\rho^{-1} e^a - 1) e^{-a(n+1)} \\
&> 2e^{-a(n+1)} > \rho(\iota(x), \iota(y))
\end{aligned}$$

(the third inequality holds since $(C_\rho^{-1} e^a - 1)^{-1} = \frac{C_\rho e^{-a}}{1 - C_\rho e^{-a}} \leq \frac{C_\rho^3 e^{-a}}{1 - C_\rho e^{-a}} < \frac{1}{2}$ by (7.1)). This means that $x = y^-$, and complete the proof for $\mathcal{J}_1(x) \neq \emptyset$ for any $x \in X_n$.

We define Φ^o and Φ as in (6.6), and call $\{\Phi(x)\}_{x \in X}$ a set of *(Christ's) dyadic cubes*. The assumption (7.1) implies that $\Phi^o(x)$ and $\Phi^o(y)$ are disjoint for any distinct $x, y \in X$ with $|x| = |y|$ [Ch, Lemma 15]. As (M, ρ) is doubling, so is the attractor (K, ρ). It follows that both the AI_a- and AI_∞-triples are admissible, and the corresponding graphs are of bounded degree.

Finally we remark that by using the Lebesgue differentiation theorem on metric spaces with (VD)-measures, it was proved in [Ch] that $\mu(\bigcup_{x \in X_n} \Phi^o(x)) = \mu(K)$ for all n. This implies that the $\{\Phi(x)\}_{x \in X}$ is *μ-separated*, i.e., $\mu(\Phi(x) \cap \Phi(y)) = 0$ for any distinct $x, y \in X$ with $|x| = |y|$; such μ-separation property was used in [KLW2] to investigate the random walks on a class of AI_a-graphs over compact spaces of homogenous type. $\square$

In the study of fractals through the augmented trees, we initially started with a modified symbolic space X of the IFS, where each level set X_n consists of indices x that the corresponding cells K_x are of approximately equal sizes, then added the horizonal edges on each X_n to form the augmented tree [LW1, LW3]. In [KLW], we considered the IFS associated with weights, and we formulated the augmented tree by regrouping the indices such that the weights are approximately equal on each level. This works fine for the class of *post critical finite* (p.c.f.) sets equipped with self-similar measures.

To extend this consideration, the family of (VD)-measures can provide a broad class of examples. Note that for self-similar sets satisfying the open set condition, Yung [Y] gave a necessary and sufficient condition for a self-similar measure $\mu_{\mathbf{s}}$ to be volume doubling; in particular, for the IFS $\{S_j\}_{j=1}^8$ of two-dimentional Sierpinski carpet, where the S_j's are arranged in the counterclockwise direction starting from one of the four corners, the condition is that the weight $\mathbf{s}$ satisfies $s_1 = s_3 = s_5 = s_7$, $s_2 = s_6$ and $s_4 = s_8$.

Motivated by this, we consider a general counterpart on spaces of homogeneous type. Let (X, E_v) be a rooted tree such that $\mathcal{J}_1(x) \neq \emptyset$ for all $x \in X$. Write $\mathcal{J}_{-1}(x) = \{x^-\}$. Let Φ be an index map on (X, E_v) over (M, ρ), and let K be the attractor. For convenience of notions, we set $\Phi(x) = K_x (= \Phi(x) \cap K)$, and denote the ball $B_\rho(\xi, r) \cap K$ simply by $B_\rho(\xi, r)$ (that is, consider the case that $M = K$ without loss of generality). Let ν be a probability measure on K with $\text{supp}(\nu) = K$ and $\nu(\{\xi\}) = 0$ for all $\xi \in K$. Suppose

$$c_*(= c_*(\nu)) := \inf\{\nu(K_x)/\nu(K_{x^-}) : x \in X,\, x \neq \vartheta\} > 0. \tag{7.2}$$

Consider a regrouping of vertices in X by setting $X_0(\nu) := \{\vartheta\}$, and for $n \geq 1$,

$$X_n(\nu) := \{x \in X : \nu(K_x) \leq c_*^n < \nu(K_{x^-})\}. \tag{7.3}$$

Let $X(\nu) := \bigcup_{n=0}^\infty X_n(\nu)$ be the new coding space. The initial rooted tree (X, E_v) naturally induces a vertical edge set on $X(\nu)$ as

$$\bigcup_{n=0}^\infty \{(x,y),\, (y,x) : x \in X_n(\nu),\, y \in X_{n+1}(\nu),\, y \in \mathcal{J}_*(x) \text{ in } (X, E_v)\},$$

and we denote it by the same E_v for simplicity. Then it is easy to check that $(X(\nu), E_v)$ is a tree with root ϑ that satisfies $\mathcal{J}_1(x) \neq \emptyset$ for all $x \in X(\nu)$ (note that $X(\nu) \subset X$, and each ray $[x_i]_i$ in X contains a unique subsequence $[x_{i_n}]_n$ that is a ray in $X(\nu)$ by (7.3)). Moreover, Φ restricted on $X(\nu)$ is an index map on $(X(\nu), E_v)$ over (K, ρ), and K is still the attractor.

In the rest of this section, we will make two assumptions on Φ and (K, ρ):

(A1). The index map Φ is of exponential type-(a) and satisfies condition (B_a) for some $a \in (0, \infty)$, i.e., there exist $0 < c_0 < 1$ and a projection $\iota : X \to K$ such that

$$B_\rho(\iota(x), c_0 e^{-an}) \subset K_x \subset B_\rho(\iota(x), c_0^{-1} e^{-an}), \qquad \forall\, x \in X_n,\ n \geq 1. \tag{7.4}$$

(A2). The attractor (K, ρ) is *uniformly perfect*, i.e., there is a constant $t \geq 1$ such that for $\xi \in K$ and $r > 0$,

$$K \setminus B_\rho(\xi, r) \neq \emptyset \quad \Rightarrow \quad B_\rho(\xi, r) \setminus B_\rho(\xi, r/t) \neq \emptyset.$$

Intuitively, the assumption (A2) implies that K cannot have "arbitrarily thick" empty annulus (i.e., the ratio of two radii is bounded). It is known that the class of uniformly perfect sets includes connected sets, self-affine sets, conformal attractors, and Julia sets of rational maps or rational semigroups with a common Lipschitz constant [HM, MR, St1, St2, XYS].

We consider a (VD)-measure μ on (K, ρ), and assume without loss of generality that $\mu(K) = 1$. Clearly $\text{supp}(\mu) = K$. By the uniform perfectness (assumption (A2)), we see that there are no isolated points in K, and thus $\mu(\{\xi\}) = 0$ for all $\xi \in K$ [KaW, Lemma 2]. Using (7.4), the quasi-triangle inequality and the volume doubling property of μ, for any $x \in X_n(\mu)$ with $n \geq 1$, we have

$$\begin{aligned}\mu(K_{x^-}) &\leq \mu(B_\rho(\iota(x^-), c_0^{-1}e^{-a(n-1)})) \leq \mu(B_\rho(\iota(x), 2C_\rho c_0^{-1}e^{-a(n-1)})) \\ &\leq C \cdot \mu(B_\rho(\iota(x), c_0 e^{-an})) \leq C\mu(K_x).\end{aligned}$$

This implies that the constant c_* in (7.2) is positive, and hence $(X(\mu), E_v)$ is well-defined. Consider the AI_∞-triple $(X(\mu), E^{(\infty)}, \Phi)$ as defined in (5.6), and we have

Theorem 7.2. *Suppose the index map Φ on (X, E_v) over (K, ρ) satisfies (A1) and (A2), and μ is a (VD)-measure on (K, ρ). Then the AI_∞-graph $(X(\mu), E^{(\infty)})$ is hyperbolic, and hence the AI_∞-triple $(X(\mu), E^{(\infty)}, \Phi)$ is admissible.*

Proof. We define a new quasi-metric q_μ on K via μ by setting

$$q_\mu(\xi, \eta) := \mu\big(B_\rho(\xi, \rho(\xi, \eta)) \cup B_\rho(\eta, \rho(\xi, \eta))\big), \qquad \xi, \eta \in K. \tag{7.5}$$

It is easy to check that q_μ is a quasi-metric. By [He, Proposition 14.14] (where the uniform perfectness is used), the identity map $\mathrm{id}_K : (K, \rho) \to (K, q_\mu)$ is a quasisymmetry. As the doubling property is a quasisymmetric invariant [He, Theorem 10.18], we know that the quasi-metric space (K, q_μ) is also doubling.

We will consider the AI_∞-triple $(X(\mu), E^{(\infty)}, \Phi)$ over (K, q_μ). Let $a = |\log c_*|$. First, we claim that Φ is of exponential type-(a) under q_μ. Indeed, for $x \in X_n$ and $\xi, \eta \in K_x$, from the quasi-triangle inequality and (7.4) we have

$$B_\rho(\xi, \rho(\xi, \eta)) \subset B_\rho(\xi, 2C_\rho c_0^{-1}e^{-an}) \subset B_\rho(\iota(x), 3C_\rho^2 c_0^{-1}e^{-an}).$$

Similarly, $B_\rho(\eta, \rho(\xi, \eta)) \subset B_\rho(\iota(x), 3C_\rho^2 c_0^{-1}e^{-an})$. Suppose $x \in X_m(\mu)$. It follows that

$$\begin{aligned}q_\mu(\xi, \eta) &\leq 2\mu(B_\rho(\iota(x), 3C_\rho^2 c_0^{-1}e^{-an})) \\ &\leq C\mu(B_\rho(\iota(x), c_0 e^{-an})) \leq C\mu(K_x) \leq Cc_*^m = Ce^{-am}.\end{aligned}$$

This proves the claim.

On the other hand, for $x \in X_n \cap X_m(\mu)$ and $\zeta \in K \setminus B_\rho(\iota(x), c_0 e^{-an})$, we have

$$\begin{aligned}q_\mu(\iota(x), \zeta) &\geq \mu(B_\rho(\iota(x), \rho(\iota(x), \zeta))) \geq \mu(B_\rho(\iota(x), c_0 e^{-an})) \\ &\geq c_1 \mu(B_\rho(\iota(x), c_0^{-1}e^{-an})) \geq c_1\mu(K_x) \geq c_1 c_*^{m+1} = c_2 e^{-am},\end{aligned}$$

where $c_2 = c_1 c_*$. Therefore $B_{q_\mu}(\iota(x), c_2 e^{-am}) \subset B_\rho(\iota(x), c_0 e^{-an}) \subset K_x$. This shows that the index map Φ on $(X(\mu), E_v)$ satisfies the ball condition (B_a) in [KLW, Definition 5.1] under q_μ. Now by [KLW, Theorem 5.4], the AI_∞-graph $(X(\mu), E^{(\infty)})$ is hyperbolic, and hence the AI_∞-triple is admissible. $\square$

Remark 1. We cannot replace the assumption (A1) by $c_*(\mu) > 0$ in the above theorem. For this, a counterexample is the binary partition on $K = [0, 1]^2$ in [KLW, Example 6.1]: let μ be the Lebesgue measure on K. Then $c_*(\mu) = 1/2$. However, the index map Φ is not of exponential type under the Euclidean metric, and it is known that the associated AI_∞-graph is not hyperbolic.

As in (5.4), via the homeomorphism $\kappa^\Phi : \partial(X(\mu), E^{(\infty)}) \to K$, the Gromov distance ϱ_a defines a new quasi-metric $\tilde{\varrho}_a$ on K.

In the next theorem, we need to use the following result in [KLW, Proposition 5.3]: *for an index map Φ of exponential type-(a) with doubling attractor (K, ρ), condition (S_a) is satisfied if and only if there exist a projection $\iota : X \to K$, $c > 0$ and $\ell > 0$ such that*

$$\#\{x \in X_n : \rho(\iota(x), \iota(y)) < ce^{-an}\} \leq \ell, \qquad \forall\ n \geq 0,\ y \in X_n. \tag{7.6}$$

Theorem 7.3. *With the same assumptions as in Theorem 7.2, suppose there exists an $\ell > 0$ such that*

$$\#\{x \in X_n(\mu) : \mu(K_x \cap K_y) > 0\} \leq \ell, \qquad \forall\ n \geq 0,\ y \in X_n(\mu). \tag{7.7}$$

Then the AI_∞-graph $(X(\mu), E^{(\infty)})$ has bounded degree. Furthermore if Φ is saturated, then for any $a > 0$, the measure μ is Ahlfors-regular with exponent $(-\log c_*/a)$ on $(K, \tilde{\varrho}_a)$, i.e.,

$$\mu(B_{\tilde{\varrho}_a}(\xi, r)) \asymp r^{-\log c_*/a}, \qquad \forall\ \xi \in K,\ r \in (0, 1). \tag{7.8}$$

Proof. We first take $a = |\log c_*|$. In the proof of Theorem 7.2, with the quasi-metric q_μ in (7.5), we have observed that (K, q_μ) is doubling, and the index map Φ on $(X(\mu), E_v)$ over (K, q_μ) is of exponential type-(a) and satisfies the condition (B_a). For $x, y \in X_n(\mu)$ with $q_\mu(\iota(x), \iota(y)) < ce^{-an}$ (here c is the constant in the condition (B_a)), as $\mathrm{supp}(\mu) = K$, we have

$$\mu(K_x \cap K_y) \geq \mu(B_{q_\mu}(\iota(x), ce^{-an}) \cap B_{q_\mu}(\iota(y), ce^{-an})) > 0. \tag{7.9}$$

It follows from (7.7) that

$$\#\{x \in X_n(\mu) : q_\mu(\iota(x), \iota(y)) < ce^{-an}\} \leq \ell, \qquad \forall\ n \geq 0,\ y \in X_n(\mu).$$

This shows that Φ on $(X(\mu), E_v)$ satisfies the condition in (7.6) under q_μ, and hence the separation condition (S_a) in [KLW, Definition 5.1]. As a consequence, the AI_∞-graph has bounded degree (by [KLW, Theorem 5.5]).

Now suppose Φ is saturated, and $a > 0$ is arbitrary. The proof of (7.8) is similar to the one in [KLW, Proposition 6.5]. Using Theorems 7.2 and 2.3, we know that $(X(\mu), E^{(\infty)})$ is (m, k)-hyperbolic for some integers $m, k > 0$. For $x \in X(\mu)$, set

$$\Phi^k(x) := \bigcup \{K_y : d_h(x, y) \leq k \text{ in } (X(\mu), E^{(\infty)})\}$$

as in (5.3). Using Proposition 2.5(ii), (5.4) and (5.2), there is a constant $C_0 \geq 1$ such that

$$B_{\tilde{\varrho}_a}(\xi, C_0^{-1}e^{-a|x|}) \subset \kappa^\Phi(\mathcal{J}_\partial^k(x)) = \Phi^k(x) \subset B_{\tilde{\varrho}_a}(\xi, C_0 e^{-a|x|}), \quad \forall\ x \in X(\mu),\ \xi \in K_x,$$

(here the "=" holds since Φ is saturated). As $c_*^{|x|+1} < \mu(K_x) \le \mu(\Phi^k(x)) \le \eta^k c_*^{|x|}$ where $\eta := \sup_{x\in X(\mu)} \deg(x)$, it follows that

$$\begin{cases} \mu(B_{\tilde{\varrho}_a}(\xi, C_0^{-1}e^{-an})) \le \eta^k c_*^n, \\ \mu(B_{\tilde{\varrho}_a}(\xi, C_0 e^{-an})) \ge c_*^{n+1}, \end{cases} \qquad \forall\, \xi \in K,\ n \ge 0,$$

and this proves (7.8). □

Remark 2. Under the assumption (A1), the separation condition in (7.7) is equivalent to the one in (7.6) (the necessity follows from the same estimate as in (7.9)), and also to condition (S_b) (this sufficiency is straightforward). Also note that if Φ is μ-separated (i.e., $\mu(K_x \cap K_y) = 0$ for all $x \ne y$ with $|x| = |y|$ in (X, E_v)), then (7.7) is satisfied for $\ell = 0$. In the study on self-similar sets, the μ-separation property is satisfied for all self-similar measures where the IFS has the OSC.

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