MEAN-FIELD ANALYSIS OF INDUSTRY DYNAMICS UNDER FINANCIAL CONSTRAINTS

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Abstract. We analyze the stochastic dynamics of an industry populated by competing firms which may increase their product quality by investing in product innovation but face the risk of bankruptcy induced market exit in case their liquidity becomes negative. The industry dynamics, which also incorporates stochastic firm entry, is described as a system of interacting entities. Performing the classical mean-field limit for the part of the dynamics which preserves the number of firms (conservative part) we identify the limiting process with a solution to a nonlinear martingale problem associated with a McKean-Vlasov stochastic equation. Moreover, we estimate the rate of convergence in mean field limit for the conservative part of dynamics. Combining the results obtained for the conservative part of the dynamics with the Vlasov scaling approach developed in [5] we obtain the kinetic equation for the whole model which includes exit and entrance mechanism of firms.

1. Introduction

1.1. **General introduction.** Successful product innovations are a decisive factor for the long term success of firms in many industry environments. At the same time, innovation projects are associated with substantial technological and demand uncertainty (e.g. [1]) and unsuccessful externally financed innovation investments might jeopardize the financial standing of incumbent firms to an extent that they become bankrupt and have to leave the market ([2]). In [3] the inter-temporally optimal investment strategy of a single (monopolistic) firm facing both technological uncertainty and bankruptcy risk has been analyzed. It has been shown that the firm's optimal investment has a U-shaped relationship with its liquidity, where the lowest investment occurs if liquidity is close to zero. In this paper we build on this insight and analyze the dynamics that emerges in an industry populated by competing firms which use a U-shaped innovation investment function and face technological uncertainty and bankruptcy risk. More precisely, each firm can improve its market profits by increasing its technology level relative to that of its competitors. The innovation projects leading to technology improvements have stochastic completion times where the arrival rate of the new technology depends positively on the firm's innovation investment. If the innovation investments cannot be covered by current market profits plus savings the firm can use external financing and go into debt (i.e. it accumulates negative liquidity). While in debt the firm faces a bankruptcy risk, where the arrival rate of bankruptcy is an increasing function of the absolute value of its negative liquidity. This formulation captures that credit lines might be withdrawn by banks or exogenous financial shocks might make it impossible for the firm the meet its debt obligations. Hence, we consider the evolution of an industry with frictions on the credit market, where the amount of debt directly influences a firm's bankruptcy risk. In this way we deviate from standard industry dynamics models where exit is determined by the firm's value, but its current liquidity does not play any role (e.g. [4, 8]). Apart from firm exit due to bankruptcy we also allow for the entry of new firms, where we consider different entry

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mechanisms determining the entry rate as well as the distribution of the technology level of entering firms.

1.2. **Heuristic description of the microscopic model.** Consider first a scenario with a fixed number of firms $n \in \mathbb{N}$. Each firm is described by its liquidity $e \in \mathbb{R}$ and its technology level $\alpha \in \mathbb{N}_0$. Hence, the state space of all firms is $(\mathbb{R} \times \mathbb{N}_0)^n$. Let $\mathcal{P}(\mathbb{N}_0)$ be the space of Borel probability measures over $\mathbb{N}_0$. The time evolution of the liquidity dynamics for the firms, that is $(e_1(t), \dots, e_n(t))$, is given by

$$\frac{d}{dt} e_j(t) = r e_j(t) + \pi\left(\alpha_j(t) \mid \mu_t^\alpha\right) - c\left(\phi\left(e_j(t), \alpha_j(t) \mid \mu_t^\alpha\right)\right) - d(e_j(t)), \tag{1.1}$$

where $(\alpha_1(t), \dots, \alpha_n(t))$ describe the technology levels of the firms at time $t \geq 0$, and

$$\mu_t^\alpha = \frac{1}{n} \sum_{k=1}^{n} \delta_{\alpha_k(t)}. \tag{1.2}$$

denotes the 'empirical distribution' of the technology levels. Here $r > 0$ is the interest rate, $\pi : \mathbb{N}_0 \times \mathcal{P}(\mathbb{N}_0) \longrightarrow \mathbb{R}$ describes the firm profit for given technology levels, $c : \mathbb{R} \to \mathbb{R}$ denotes the cost of innovation activities of the firm and $d : \mathbb{R} \longrightarrow [0, \infty)$ is the firm's dividend strategy. A firm's innovation activities are determined by its innovation strategy $\phi : (\mathbb{R} \times \mathbb{N}_0) \times \mathcal{P}(\mathbb{N}_0) \longrightarrow \mathbb{R}_+$. We assume that ϕ depends on the current technology levels as well as on the liquidity of the firm, where the dependence on liquidity e_j is U-shaped.

Competition between the firms is captured by the dependence of a firm's market profit π on the average quality on the market μ_t^α, where it is assumed that π increases with respect to α_j and decreases with respect to μ_t^α (in the sense of first order stochastic dominance). This reflects that competition of firms is incorporated through their technology levels.

Furthermore, it is worthwhile to mention that we assume that all firms share the same innovation and dividend strategies. In light of the fact that there are no structural differences between the firms this is a natural assumption, which allows to perform the mean-field limit leading to an effective equation for a single firm when the number of firm n is large.

The time evolution of the technology levels $\alpha_1(t), \dots, \alpha_n(t)$ is random and described by a pure jump process with the jump intensity determined by the firm's innovation activity $\phi(e, \alpha \mid \nu)$. In other words

$$\mathbb{P}\left[\alpha_j(t + \Delta) = \alpha + 1 \mid \alpha_j(t) = \alpha\right] = \phi(e_j(t), \alpha \mid \mu_t^\alpha)\, \Delta + o(\Delta), \quad \Delta \to 0. \tag{1.3}$$

Note that ϕ can also be interpreted as the firm's innovation rate.

To incorporate exit and entry, we assume that a firm with negative liquidity goes bankrupt and has to exit the market with a rate $q = -\min\{0, \gamma^{EX} e\}$. The parameter γ^{EX} captures the strength of the credit market, respectively the firm's access to that market. A large value of γ^{EX} corresponds to a weak credit market with large frictions inducing a high bankruptcy risk of indebted firms. To avoid a systematic concentration of the industry we also incorporate entry of firms with a constant rate $p^{EN} > 0$. Clearly, under entry and exit the number of firms in the industry evolves stochastically over time.

1.3. **Mean-field limit of microscopic model.** One goal of this paper is to derive the evolution of one dimensional distribution for the "typical" firm on the market with arbitrary large number of interacting firms evolving according to the mechanism of industry dynamics under financial constraints. First, assuming that the number of firms $n \in \mathbb{N}$ is conserved, we construct in Section 3 the microscopic dynamics described in (1.1) and (1.3) by means of a solution to a coupled system of stochastic differential equations. Afterward, we study in Section 4 the classical mean-field approach obtaining in the limit $n \to \infty$ the corresponding

McKean-Vlasov stochastic equation. Such stochastic equation describes the stochastic evolution $(e(t), \alpha(t))_{t\geq 0}$ of the liquidity/technology level dynamics of a single firm when the total number of firms is sufficiently large. The resulting mean-field liquidity equation is

$$\frac{d}{dt}e(t) = re(t) + \pi\left(\alpha(t) \mid \mu_t^{\alpha}\right) - c\left(\phi\left(e(t), \alpha(t) \mid \mu_t^{\alpha}\right)\right) - d(e(t)), \tag{1.4}$$

where μ_t denotes the distribution of $(e(t), \alpha(t))$ on $\mathbb{R} \times \mathbb{N}_0$ at time $t \geq 0$, μ_t^{α} is its marginal on $\mathbb{N}_0$, and the changes in the technology levels are governed by

$$\mathbb{P}\left[\alpha(t+\Delta) = \alpha + 1 \mid \alpha(t) = \alpha\right] = \phi(e(t), \alpha \mid \mu_t^{\alpha})\,\Delta + o(\Delta), \quad \Delta \to 0. \tag{1.5}$$

In our analysis performed in Section 4 we first show that the above equations have a unique solution and then prove that the convergence rate when $n \to \infty$ passing from (1.1) and (1.3) to (1.4) and (1.5) is given by $\log(1+n)n^{-1/2}$. Note that, since π and ϕ also depend on the law of the solution to under investigation, equations (1.4) and (1.5) are of mean-field type.

While the solution to (1.4) and (1.5) provides information about all possible paths of the process, in some cases one is only interested in distributional properties of a typical firm in the market (e.g. mean technology level, mean liqidity, etc.). Such properties can be effectively studied in terms of the time marginals $(\mu_t)_{t\geq 0}$, i.e., the distribution of $(e(t), \alpha(t))_{t\geq 0}$. This time marginals allows us to compute averaged characteristics

$$\int_{\mathbb{R}\times\mathbb{N}_0} f(e,\alpha)\mu_t(de, d\alpha) = \mathbb{E}[f(e(t), \alpha(t))],$$

where $f : \mathbb{R} \times \mathbb{N}_0 \longrightarrow \mathbb{R}$ is integrable with respect to μ_t. This includes, as special cases, the mean liquidity $f(e, \alpha) = e$ and mean technology level $f(e, \alpha) = \alpha$ of the market. However, also more sophisticated expectations could be studied by this approach. In Section 4 it is shown that the time marginals $(\mu_t)_{t\geq 0}$ satisfy the following nonlinear Fokker-Planck equation in weak form

$$\langle f, \mu_t\rangle = \langle f, \mu_0\rangle + \int_0^t \langle L(\mu_s^{\alpha})f, \mu_s\rangle ds, \qquad f \in C_c^2(\mathbb{R} \times \mathbb{N}_0), \tag{1.6}$$

where $\langle f, \nu\rangle := \int_{\mathbb{R}\times\mathbb{N}_0} f(e, \alpha)\nu(de, d\alpha)$ denotes the expected value of ν with respect to f and $L(\mu^{\alpha})f$ is given by

$$\begin{aligned} L(\mu^{\alpha})f(e,\alpha) = &\left[re + \pi\left(\alpha \mid \mu^{\alpha}\right) - c\left(\phi\left(e, \alpha \mid \mu^{\alpha}\right)\right) - d(e)\right]\frac{\partial f(e,\alpha)}{\partial e} \\ &+ \phi\left(e, \alpha \mid \mu^{\alpha}\right)\left(f(e, \alpha + \mathbb{1}) - f(e,\alpha)\right). \end{aligned}$$

1.4. **Extension of model to random exit and entry of firms.** So far we have only studied industry dynamics under financial constraints where the total number of firms is fixed but may be arbitrary large (the mean-field limit). Below we briefly discuss an extension of this model where new firms may appear (entry) and existent firms may disappear (exit) from the market. Both random events are assumed to happen independently of each other, but depending on the current state of the system. Unfortunately, neither the stochastic description provided in (1.1), (1.3), nor the description given in (1.4), (1.5) is sufficient to take this random events into account. In this work we, therefore, study both effects on the level of kinetic equations.

Afterward, we combine the above equation (1.6) with the kinetic equation obtained for the entry and exit process with the help of Vlasov approach for non-conservative systems to find the correct kinetic equation for the industry dynamics under financial constraints with random exit and entry. The resulting equation for the distribution of the liquidity/technology level of a firm, that is μ_t, is given by

$$\langle f, \mu_t\rangle = \langle f, \mu_0\rangle + \int_0^t \langle L(\mu_s^{\alpha})f, \mu_s\rangle ds - \int_0^t \langle q\cdot f, \mu_s\rangle ds + \int_0^t \langle f, M(\mu_s, \cdot)\rangle ds, \tag{1.7}$$

where again $q(e,\alpha) = -\min\{0, \gamma^{EX} e\}$ with constant $\gamma^{EX} > 0$ is the exit rate of firms leaving the market, while $M(\mu_s, de, d\alpha)$ describes the total mass and distribution of firms entering the market at time s. Particular examples of this measure are discussed in the last section of this work.

1.5. **Structure of the work.** In Section 2 we first formulate conditions imposed in this work and then briefly discuss our main guiding example. Section 3 is devoted to the construction of the microscopic dynamics of firms without exit and entry. Such construction is based on solving the associated coupled system of stochastic differential equations. Afterward, in Section 4, we rigorously perform the mean-field limit for the microscopic dynamics without exit and entry. At this point, we also establish a convergence rate. Finally, Section 5 is devoted to a formal derivation of the kinetic equation with exit and entry of firms.

2. Assumptions and Examples

2.1. **Wasserstein distances.** Let $\mathcal{P}(\mathbb{R}\times\mathbb{N}_0)$ be the space of Borel probability measures over $\mathbb{R}\times\mathbb{N}_0$ and let $\mathcal{P}_1(\mathbb{R}\times\mathbb{N}_0)$ be the subspace of probability measures with finite first moment. The Wasserstein-1 distance on $\mathcal{P}_1(\mathbb{R}\times\mathbb{N}_0)$ is defined by

$$W_1(\mu,\nu) = \sup_{\|f\|_{\mathrm{Lip}(\mathbb{R}\times\mathbb{N}_0)}\leq 1} \left| \int_{\mathbb{R}\times\mathbb{N}_0} f(e,\alpha)\mu(de,d\alpha) - \int_{\mathbb{R}\times\mathbb{N}_0} f(e,\alpha)\nu(de,d\alpha) \right|,$$

where $\|f\|_{\mathrm{Lip}(\mathbb{R}\times\mathbb{N}_0)}$ denotes the Lipschitz semi-norm for a function $f : \mathbb{R}\times\mathbb{N}_0 \longrightarrow \mathbb{R}$ defined by

$$\|f\|_{\mathrm{Lip}(\mathbb{R}\times\mathbb{N}_0)} = \sup_{(e,\alpha)\neq(e',\alpha')} \frac{|f(e,\alpha) - f(e',\alpha')|}{|e-e'| + |\alpha-\alpha'|}.$$

Note that $\mathcal{P}_1(\mathbb{R}\times\mathbb{N}_0)$ equipped with W_1 is a complete separable metric space. For further results and additional details on Wasserstein distances we refer to the monograph by C. Villani [9].

For $\mu \in \mathcal{P}(\mathbb{R}\times\mathbb{N}_0)$ we let μ^α be the marginal on the $\mathbb{N}_0$ component, i.e. $\mu^\alpha(A) := \mu(\mathbb{R}\times A)$. Then μ^α is a Borel probability measure on $\mathbb{N}_0$. Let $\mathcal{P}(\mathbb{N}_0)$ be the space of all Borel probability measures and let $\mathcal{P}_1(\mathbb{N}_0)$ be the subspace of all probability measures with finite first moment. Analogously, we define the Wasserstein-1 distance on $\mathcal{P}_1(\mathbb{N}_0)$ by

$$W_1^\alpha(\mu,\nu) = \sup_{\|f\|_{\mathrm{Lip}(\mathbb{N}_0)}\leq 1} \left| \int_{\mathbb{N}_0} f(\alpha)\mu(d\alpha) - \int_{\mathbb{N}_0} f(\alpha)\nu(d\alpha) \right|,$$

where the Lipschitz semi-norm $\|f\|_{\mathrm{Lip}(\mathbb{N}_0)}$ is given by

$$\|f\|_{\mathrm{Lip}(\mathbb{N}_0)} = \sup_{\alpha\neq\alpha'} \frac{|f(\alpha) - f(\alpha')|}{|\alpha-\alpha'|}.$$

The following simple observation will be frequently used in our calculations.

Lemma 2.1. *For $\mu,\nu \in \mathcal{P}_1(\mathbb{R}\times\mathbb{N}_0)$ one has $\mu^\alpha, \nu^\alpha \in \mathcal{P}_1(\mathbb{N}_0)$ and*

$$W_1^\alpha(\mu^\alpha,\nu^\alpha) \leq W_1(\mu,\nu). \tag{2.1}$$

Proof. The first assertion immediately follows from

$$\int_{\mathbb{N}_0} \alpha\nu^\alpha(d\alpha) = \int_{\mathbb{R}\times\mathbb{N}_0} \alpha\nu(de,d\alpha) \leq \int_{\mathbb{R}\times\mathbb{N}_0} (|e|+\alpha)\nu(de,d\alpha) < \infty.$$

For the second assertion observe that

$$\begin{aligned} W_1^\alpha(\mu^\alpha, \nu^\alpha) &= \sup_{\|f\|_{\mathrm{Lip}(\mathbb{N}_0)} \leq 1} \left| \int_{\mathbb{N}_0} f(\alpha)\mu(d\alpha) - \int_{\mathbb{N}_0} f(\alpha)\nu(d\alpha) \right| \\ &= \sup_{\|\widetilde{f}\|_{\mathrm{Lip}(\mathbb{R}\times\mathbb{N}_0)} \leq 1} \left| \int_{\mathbb{R}\times\mathbb{N}_0} \widetilde{f}(e,\alpha)\mu(de,d\alpha) - \int_{\mathbb{R}\times\mathbb{N}_0} \widetilde{f}(e,\alpha)\nu(de,d\alpha) \right| \\ &\leq W_1(\mu,\nu), \end{aligned}$$

where we have used the fact that a function $f : \mathbb{N}_0 \longrightarrow \mathbb{R}$ can be extended by $\widetilde{f}(e,\alpha) = f(\alpha)$ to a function $\widetilde{f} : \mathbb{R} \times \mathbb{N}_0 \longrightarrow \mathbb{R}$ still satisfying $\|\widetilde{f}\|_{\mathrm{Lip}(\mathbb{R}\times\mathbb{N}_0)} \leq 1$. □

Finally, given $\mu^\alpha, \nu^\alpha \in \mathcal{P}(\mathbb{N}_0)$, a coupling is a Borel probability measure H on $\mathbb{N}_0 \times \mathbb{N}_0$ whose marginals are given by μ^α and ν^α, respectively. Let $\mathcal{H}(\mu^\alpha, \nu^\alpha)$ be the set of all couplings for μ^α, ν^α. Note that the product measure $\mu^\alpha \otimes \nu^\alpha$ is a coupling, so that $\mathcal{H}(\mu^\alpha, \nu^\alpha)$ is not empty. The Kantorovich duality implies that

$$W_1^\alpha(\mu^\alpha, \nu^\alpha) = \inf_{H \in \mathcal{H}(\mu^\alpha,\nu^\alpha)} \int_{\mathbb{N}_0\times\mathbb{N}_0} |\alpha - \alpha'| H(d\alpha, d\alpha').$$

Moreover, it can be shown that this infimum is attained, i.e., there exists $H_* \in \mathcal{H}(\mu^\alpha, \nu^\alpha)$ such that

$$W_1^\alpha(\mu^\alpha, \nu^\alpha) = \int_{\mathbb{N}_0\times\mathbb{N}_0} |\alpha - \alpha'| H_*(d\alpha, d\alpha').$$

The measure H_* is called optimal coupling of μ^α and ν^α. Again, for additional details we refer to the monograph by C. Villani [9].

2.2. **Assumptions on the coefficients.** In this work we will use the following assumptions on the coefficients of the model:

(A1) The Firm profit function $\pi : \mathbb{N}_0 \times \mathcal{P}_1(\mathbb{N}_0) \longrightarrow \mathbb{R}$ is such that there exists a constant $C_\pi > 0$ with
$$|\pi(\alpha \mid \nu) - \pi(\alpha' \mid \nu')| \leq C_\pi \left(|\alpha - \alpha'| + W_1^\alpha(\nu, \nu') \right)$$
for all $\alpha, \alpha' \in \mathbb{N}_0$ and $\nu, \nu' \in \mathcal{P}_1(\mathbb{N}_0)$.

(A2) The Research and Development function $\phi : \mathbb{R} \times \mathbb{N}_0 \times \mathcal{P}_1(\mathbb{N}_0) \longrightarrow \mathbb{R}_+$ is such that there exists a constant $C_\phi > 0$ with
$$|\phi(e, \alpha \mid \nu) - \phi(e', \alpha' \mid \nu')| \leq C_\phi \left(|e - e'| + |\alpha - \alpha'| + W_1^\alpha(\nu, \nu') \right)$$
for all $(e, \alpha), (e', \alpha') \in \mathbb{R} \times \mathbb{N}_0$ and $\nu, \nu' \in \mathcal{P}_1(\mathbb{N}_0)$.

(A3) The costs of R & D activities $c : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are such that there exists a constant $C_c > 0$ with
$$|c(\phi(e, \alpha \mid \nu)) - c(\phi(e', \alpha' \mid \nu'))| \leq C_c \left(|e - e'| + |\alpha - \alpha'| + W_1^\alpha(\nu, \nu') \right)$$
for all $(e, \alpha), (e', \alpha') \in \mathbb{R} \times \mathbb{N}_0$ and $\nu, \nu' \in \mathcal{P}_1(\mathbb{N}_0)$.

(A4) The dividend payout function $d : \mathbb{R} \longrightarrow \mathbb{R}_+$ is such that there exists a constant $C_c > 0$ with
$$|d(e) - d(e')| \leq C_d |e - e'|, \qquad e, e' \in \mathbb{R}.$$

Thereafter, we seek to rewrite (1.1) into a coupled system of stochastic differential equations. For this purpose, we define a new function $B : \mathbb{R} \times \mathbb{N}_0 \times \mathcal{P}_1(\mathbb{N}_0) \longrightarrow \mathbb{R}$ by

$$B(e, \alpha \mid \nu) = re + \pi(\alpha \mid \nu) - c(\phi(e, \alpha \mid \nu)) - d(e).$$

The next lemma shows that this function is globally Lipschitz continuous. Its proof follows immediately form the triangle inequality combined with conditions (A1) – (A4).

Lemma 2.2. *Suppose that conditions (A1) – (A4) are satisfied. Then*

$$|B(e,\alpha \mid \nu) - B(e',\alpha' \mid \nu')| \leq K\left(|e-e'| + |\alpha - \alpha'| + W_1^{\alpha}(\nu,\nu')\right)$$

holds for all $(e,\alpha,\nu) \in \mathbb{R} \times \mathbb{N}_0 \times \mathcal{P}_1(\mathbb{N}_0)$ *where* $K := |r| + C_\pi + C_c C_\phi + C_d$.

2.3. **Main example.** Below we briefly explain our main example for coefficients π, ϕ, c, d. The firm profit function π is assumed to consist of a part being independent of the technology level and another part which depends on the distribution of the technology level on the market. Namely,

(a) the firm profit π is given by

$$\pi(\alpha \mid \nu) = \beta + \int_{\mathbb{N}_0} g(\alpha - \overline{\alpha})\nu(d\overline{\alpha}), \qquad \alpha \in \mathbb{N}_0, \quad \nu \in \mathcal{P}_1(\mathbb{N}_0),$$

where $\beta \in \mathbb{R}$, and $g : \mathbb{Z} \longrightarrow \mathbb{R}$ is globally Lipschitz continuous, and increasing.[1]

The costs for research and development depend on the firm's innovation activity and therefore on its innovation strategy ϕ. To account for the well known fact that innovation is a cumulative process and therefore it is hard to speed it up substantially (i.e., doubling the research and development budget does not necessary lead to a doubled innovation rate.), we assume for low innovation activities these costs are proportional to ϕ, while for large innovation rates the corresponding costs typically grow super-linear. In order to model this effect, we suppose that the function c satisfies the following more general condition:

(b) The function $c : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is bijective with inverse c^{-1}, c is locally Lipschitz continuous, and c^{-1} is globally Lipschitz continuous.

Note that this condition includes cases where c is a smooth function with smooth inverse c^{-1} which satisfy for some $p > 1$

$$c(x) \simeq \begin{cases} x, & x \ll 1 \\ x^p, & x \gg 1 \end{cases}, \qquad c^{-1}(x) \simeq \begin{cases} x, & x \ll 1 \\ x^{1/p}, & x \gg 1. \end{cases}$$

The considered form of the firm's innovation strategy ϕ is based on two considerations. First, it is assumed that the firm's innovation activities increase with the expected return of a successful innovation. This is captured by assuming that without considerations of financial constraints the level of its innovative activities would be $\gamma^{RD}\left(\pi(\alpha+1 \mid \nu) - \pi(\alpha \mid \nu)\right)$. Second, based on [3] it is assumed that, for a firm who has to (partially) finance its innovation activities externally, the size of the firm's innovation activities have a U-shaped relationship with its liquidity level. As discussed in detail in [3], this dependence reflects the firm's tradeoff between speeding up a profitable innovation and running the risk of market exit due to bankruptcy. To capture this effect, we note that if $c\left(\gamma^{RD}\left(\pi(\alpha+1 \mid \nu) - \pi(\alpha \mid \nu)\right)\right) \leq \pi(\alpha \mid \nu)$ the firm can finance its planned innovation activities via current profits, and it is assumed that it does not deviate from this investment level. If $\gamma^{RD}(\pi(\alpha+1 \mid \nu) - \pi(\alpha \mid \nu)) - c^{-1}(\pi(\alpha \mid \nu)) > 0$,

[1] Such a profit function arises for example if it is assumed that in every small time interval Δt every firm i with probability $\tilde{\beta}\Delta t$ is matched with one other firm and one consumer, where the matching probability across all other firms is uniform. All firms have identical production costs and identical mark-ups, such that the profit per sold unit is the same for all firms and is normalized to 1. The consumer at each match buys one unit of the good. If the consumer is matched with firm i and j the probability to buy the product from firm j is given by $0.5 + v(\alpha_j - \alpha_i)$ with $v(0) = 0, v' > 0, v(x) \in [-0.5, 0.5]$ for all x. The expected profit flow for firm j is

$$\pi(\alpha_j|\nu) = \tilde{\beta}\left(0.5 + \frac{1}{n_t - 1}\sum_{i \neq j} v(\alpha_j - \alpha_i)\right) = \beta + \sum_{\bar{\alpha} \in \mathbb{N}_0} g(\alpha_j - \bar{\alpha})\nu(\bar{\alpha})$$

with $\beta = 0.5\tilde{\beta}, g(x) = \tilde{\beta}v(x)$ and $\nu(\bar{\alpha})$ denoting the frequency of firms with technology $\bar{\alpha}$.

then the firm has to take on new debt or reduce its savings in order to finance the unconstrained level of innovative activities. Referring to the findings in [3] about the intertemporally optimal innovation strategy in such a case we assume that the firm reduces its innovative activity relative to the unconstrained level if its liquidity is in the interval $(-\bar{e}, \bar{e})$ with the strongest reduction at a liquidity level of zero. These considerations lead to the following functional form.

(c) The research and development function is given by

$$\begin{aligned}&\phi(e, \alpha \mid \nu) = \gamma^{RD}(\pi(\alpha+1 \mid \nu) - \pi(\alpha \mid \nu))\\&\quad - \max\{0, \xi(\bar{e}^2 - e^2)\} \max\left\{0, \gamma^{RD}(\pi(\alpha+1 \mid \nu) - \pi(\alpha \mid \nu)) - c^{-1}(\pi(\alpha \mid \nu))\right\},\end{aligned}$$

where $\bar{e} \in \mathbb{R}$, $\xi > 0$, and $\gamma^{RD} \geq 0$ are some constants. We suppose that $\xi\bar{e}^2 \leq 1$.

With respect to the firm's dividend strategy we rely on simple linear rule. We suppose that a firm pays no dividend, if its liquidity is negative, but in case of positive liquidity pays out a given fraction $\varkappa \in (0, 1]$ of that liquidity. Both assumptions are covered by the following condition:

(d) The dividend payout function is given by

$$d(e) = \max\{0, \varkappa e\}$$

where $\varkappa \in (0, 1]$ denotes the fraction of profit to be paid as dividend.

The next lemma shows that conditions (a) – (d) imply that our assumptions (A1) – (A4) are satisfied.

Lemma 2.3. *Suppose that the parameters (π, c, ϕ, d) are given as in (a) – (d). Then conditions (A1) – (A4) are satisfied.*

Proof. To prove condition (A1) we let $\alpha, \alpha' \in \mathbb{N}_0$, $\mu, \nu \in \mathcal{P}_1(\mathbb{N}_0)$ and let H be any coupling of (μ, ν). Then condition (A1) follows from

$$\begin{aligned}|\pi(\alpha|\mu) - \pi(\alpha'|\nu)| &= \left|\int_{\mathbb{N}_0} g(\alpha - \bar{\alpha})\mu(d\bar{\alpha}) - \int_{\mathbb{N}_0} g(\alpha' - \bar{\alpha})\nu(d\bar{\alpha})\right|\\&= \left|\int_{\mathbb{N}_0\times\mathbb{N}_0} \big(g(\alpha - \bar{\alpha}) - g(\alpha' - \bar{\alpha}')\big)\, H(d\bar{\alpha}, d\bar{\alpha}')\right|\\&\leq \|g\|_{\mathrm{Lip}(\mathbb{Z}_0)} \int_{\mathbb{N}_0\times\mathbb{N}_0} \big(|\alpha - \alpha'| + |\bar{\alpha} - \bar{\alpha}'|\big)\, H(d\bar{\alpha}, d\bar{\alpha}')\\&= \|g\|_{\mathrm{Lip}(\mathbb{Z}_0)} \left(|\alpha - \alpha'| + \int_{\mathbb{N}_0\times\mathbb{N}_0} |\bar{\alpha} - \bar{\alpha}'| H(d\bar{\alpha}, d\bar{\alpha}')\right),\end{aligned}$$

where $\|g\|_{\mathrm{Lip}(\mathbb{Z}_0)}$ denotes the Lipschitz constant of g. Indeed, if we choose H to be the optimal coupling of (μ, ν), then we arrive at (A1) with $C_\pi = \|g\|_{\mathrm{Lip}(\mathbb{Z}_0)}$. Finally note that, since g is increasing, also $\alpha \longmapsto \pi(\alpha \mid \nu)$ is increasing and hence

$$\pi(\alpha+1 \mid \nu) - \pi(\alpha \mid \nu) \geq 0.$$

This inequality will be used below. To prove that ϕ satisfies (A2) we first observe that due to $\xi\bar{e}^2 \leq 1$ one has

$$\begin{aligned}&\max\{0, \xi(\bar{e}^2 - e^2)\} \max\left\{0, \gamma^{RD}(\pi(\alpha+1 \mid \nu) - \pi(\alpha \mid \nu)) - c^{-1}(\pi(\alpha \mid \nu))\right\}\\&\leq \gamma^{RD}(\pi(\alpha+1 \mid \nu) - \pi(\alpha \mid \nu))\end{aligned}$$

and hence $\phi \geq 0$. To show that ϕ is globally Lipschitz continuous, let us write

$$|\phi(e, \alpha \mid \nu) - \phi(e', \alpha' \mid \nu')| \leq I_1 + I_2 + I_3$$

where

$$\begin{aligned}
I_1 &= \left|\gamma^{RD}(\pi(\alpha+1 \mid \nu) - \pi(\alpha \mid \nu)) - \gamma^{RD}(\pi(\alpha'+1 \mid \nu') - \pi(\alpha' \mid \nu'))\right| \\
I_2 &= |\max\{0, \xi(\overline{e}^2 - e^2)\} - \max\{0, \xi(\overline{e}^2 - e'^2)\}| \\
&\qquad \cdot \max\left\{0, \gamma^{RD}(\pi(\alpha+1 \mid \nu) - \pi(\alpha \mid \nu)) - c^{-1}(\pi(\alpha \mid \nu))\right\} \\
I_3 &= \max\{0, \xi(\overline{e}^2 - e'^2)\} \cdot \Bigg| \max\left\{0, \gamma^{RD}(\pi(\alpha+1 \mid \nu) - \pi(\alpha \mid \nu)) - c^{-1}(\pi(\alpha \mid \nu))\right\} \\
&\qquad - \max\left\{0, \gamma^{RD}(\pi(\alpha'+1 \mid \nu') - \pi(\alpha' \mid \nu')) - c^{-1}(\pi(\alpha' \mid \nu'))\right\} \Bigg|.
\end{aligned}$$

The first term can be estimated by the Lipschitz continuity of π, i.e.,

$$I_1 \leq 2\gamma^{RD}\|g\|_{\mathrm{Lip}(\mathbb{Z}_0)}\left(|\alpha-\alpha'| + W_1^{\alpha}(\nu,\nu')\right).$$

To estimate the second term, let us first observe that $\max\{0, \xi(\overline{e}^2 - e^2)\} = 0$ whenever $|e| > |\overline{e}|$. Hence, if $|e|, |e'| < |\overline{e}|$ we find that

$$\begin{aligned}
|\max\{0, \xi(\overline{e}^2 - e^2)\} - \max\{0, \xi(\overline{e}^2 - e'^2)\}| &= |\xi(\overline{e}^2 - e^2) - \xi(\overline{e}^2 - e'^2)| \\
&\leq \xi|e^2 - e'^2| \\
&\leq \xi(|e| + |e'|)|e - e'| \\
&\leq 2\xi|\overline{e}||e - e'|.
\end{aligned}$$

If $|e| \leq |\overline{e}|$ and $|e'| > |\overline{e}|$, we obtain

$$\begin{aligned}
|\max\{0, \xi(\overline{e}^2 - e^2)\} - \max\{0, \xi(\overline{e}^2 - e'^2)\}| &= \xi(\overline{e}^2 - e^2) \\
&= \xi(|\overline{e}| + |e|)(|\overline{e}| - |e|) \\
&\leq 2\xi\overline{e}|e' - e|.
\end{aligned}$$

Similarly, we estimate the case $|e| > |\overline{e}|$ and $|e'| \leq |\overline{e}|$. Altogether, we arrive at

$$|\max\{0, \xi(\overline{e}^2 - e^2)\} - \max\{0, \xi(\overline{e}^2 - e'^2)\}| \leq 2\xi|\overline{e}||e' - e|.$$

Since $c^{-1} \geq 0$ we arrive at

$$\begin{aligned}
&\max\left\{0, \gamma^{RD}(\pi(\alpha+1 \mid \nu) - \pi(\alpha \mid \nu)) - c^{-1}(\pi(\alpha \mid \nu))\right\} \\
&\leq \gamma^{RD}(\pi(\alpha+1 \mid \nu) - \pi(\alpha \mid \nu)) \\
&\leq \gamma^{RD}\|g\|_{\mathrm{Lip}(\mathbb{Z})}.
\end{aligned}$$

This implies that

$$I_2 \leq 2\xi\overline{e}\gamma^{RD}\|g\|_{\mathrm{Lip}(\mathbb{Z})}|e' - e|.$$

To estimate the last term, let us first check that

$$\begin{aligned}
&\Bigg| \max\left\{0, \gamma^{RD}(\pi(\alpha+1 \mid \nu) - \pi(\alpha \mid \nu)) - c^{-1}(\pi(\alpha \mid \nu))\right\} \\
&\quad - \max\left\{0, \gamma^{RD}(\pi(\alpha'+1 \mid \nu') - \pi(\alpha' \mid \nu')) - c^{-1}(\pi(\alpha' \mid \nu'))\right\} \Bigg| \\
&\leq \gamma^{RD}|\pi(\alpha+1 \mid \nu) - \pi(\alpha'+1 \mid \nu')| + \gamma^{RD}|\pi(\alpha \mid \nu) - \pi(\alpha' \mid \nu')| \\
&\quad + |c^{-1}(\pi(\alpha \mid \nu)) - c^{-1}(\pi(\alpha' \mid \nu'))| \\
&\leq 2\gamma^{RD}\|g\|_{\mathrm{Lip}(\mathbb{Z})}\left(|\alpha - \alpha'| + W_1^{\alpha}(\nu,\nu')\right) + \|c^{-1}\|_{\mathrm{Lip}(\mathbb{R}_+)}|\pi(\alpha \mid \nu) - \pi(\alpha' \mid \nu')| \\
&\leq \left(2\gamma^{RD} + \|c^{-1}\|_{\mathrm{Lip}(\mathbb{R}_+)}\right)\|g\|_{\mathrm{Lip}(\mathbb{Z})}\left(|\alpha - \alpha'| + W_1^{\alpha}(\nu,\nu')\right).
\end{aligned}$$

This yields

$$I_3 \leq \xi \bar{e}^2 \left(2\gamma^{RD} + \|c^{-1}\|_{\mathrm{Lip}(\mathbb{R}_+)}\right) \|g\|_{\mathrm{Lip}(\mathbb{Z})} \left(|\alpha - \alpha'| + W_1^{\alpha}(\nu, \nu')\right)$$

and hence shows that (A2) is satisfied. To prove condition (A3), let us first observe that ϕ is bounded due to

$$\phi(e, \alpha \mid \mu) \leq \gamma^{RD}(\pi(\alpha + 1 \mid \mu) - \pi(\alpha \mid \mu)) \leq \gamma^{RD} \|g\|_{\mathrm{Lip}(\mathbb{Z}_0)}.$$

Since c is locally Lipschitz continuous, it is globally Lipschitz continuous on the closed interval $R := [0, \gamma^{RD}\|g\|_{\mathrm{Lip}(\mathbb{Z}_0)}]$. Hence,

$$|c(\phi(e, \alpha \mid \nu)) - c(\phi(e', \alpha' \mid \nu'))| \leq \|c\|_{\mathrm{Lip}(R)} |\phi(e, \alpha \mid \nu) - \phi(e', \alpha' \mid \nu')|.$$

Property (A3) is now a consequence of (A2). Finally, condition (A4) is clearly satisfied with $C_d = \varkappa$. □

3. Construction of the particle dynamics

Below we construct the liquidity dynamics in terms of a solution to a system of stochastic differential equations. Namely, the liquidity dynamics and the technology levels can be described by the following $(\mathbb{R} \times \mathbb{N}_0)^n$-valued SDE:

$$\begin{cases} e_j(t) &= e_j(0) + \int_0^t B\left(e_j(s), \alpha_j(s) \;\middle|\; \frac{1}{n}\sum_{k=1}^n \delta_{\alpha_k(s)}\right) ds, \\ \alpha_j(t) &= \alpha_j(0) + \int_0^t \int_0^\infty \mathbb{1}_{\{u \leq \phi\left(e_j(s), \alpha_j(s-) \mid \frac{1}{n}\sum_{k=1}^n \delta_{\alpha_k(s-)}\right)\}} N_j(ds, du), \end{cases} \tag{3.1}$$

where $j = 1, \dots, n$, and $N_1, \dots, N_n$ are independent Poisson random measures on $\mathbb{R}_+ \times \mathbb{R}_+$ with compensators $dsdu$. Indeed, the first equation is exactly (1.1), while the second equation is a formulation of (1.3) in terms of Poisson random measures and a stochastic equation. The next theorem states that under conditions (A1) – (A4) equation (3.1) has a unique strong solution.

Theorem 3.1. *Suppose that conditions (A1) – (A4) are satisfied. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a stochastic basis with the usual conditions and let $N_1, \dots, N_n$ be independent $(\mathcal{F}_t)_{t\geq 0}$-Poisson random measures on $\mathbb{R}_+ \times \mathbb{R}_+$ with compensators dsdu. Then for each $\mathcal{F}_0$-measurable random variable $(e(0), \alpha(0)) \in (\mathbb{R} \times \mathbb{N}_0)^n$ with finite first moments there exists a unique solution to* (3.1) *with finite first moments.*

Proof. Define $\widetilde{B} : (\mathbb{R} \times \mathbb{N}_0)^n \longrightarrow \mathbb{R}^n$ and $\widetilde{\phi} : (\mathbb{R} \times \mathbb{N}_0)^n \longrightarrow [0, \infty)$ by

$$\widetilde{B}_j\left(e_1, \alpha_1, \dots, e_n, \alpha_n\right) = B\left(e_j, \alpha_j \mid \frac{1}{n}\sum_{k=1}^n \delta_{\alpha_k}\right)$$

$$\widetilde{\phi}\left(e_1, \alpha_1, \dots, e_n, \alpha_n\right) = \phi\left(e_j, \alpha_j \mid \frac{1}{n}\sum_{k=1}^n \delta_{\alpha_k}\right).$$

Then (3.1) takes the form

$$\begin{cases} e_j(t) &= e_j(0) + \int_0^t \widetilde{B}_j\left(e_1(s), \alpha_1(s), \dots, e_n(s), \alpha_n(s)\right) ds, \\ \alpha_j(t) &= \alpha_j(0) + \int_0^t \int_0^\infty \mathbb{1}_{\{u \leq \widetilde{\phi}_j(e_1(s), \alpha_1(s-), \dots, e_n(s), \alpha_n(s-))\}} N_j(ds, du). \end{cases} \tag{3.2}$$

Using Lemma 2.2 combined with the elementary inequality

$$W_1^{\alpha}\left(\frac{1}{n}\sum_{k=1}^n \delta_{\alpha_k}, \frac{1}{n}\sum_{k=1}^n \delta_{\alpha'_k}\right) \leq \frac{1}{n}\sum_{k=1}^n |\alpha_k - \alpha'_k| \tag{3.3}$$

we find that

$$|\widetilde{B}_j(e_1, \alpha_1, \dots, e_n, \alpha_n) - \widetilde{B}_j(e'_1, \alpha'_1, \dots, e'_n, \alpha'_n)|$$

$$
\begin{aligned}
&= \left| B\left(e_j, \alpha_j \mid \frac{1}{n}\sum_{k=1}^n \delta_{\alpha_k}\right) - B\left(e_j', \alpha_j' \mid \frac{1}{n}\sum_{k=1}^n \delta_{\alpha_k'}\right)\right| \\
&\leq K\left(|e_j - e_j'| + |\alpha_j - \alpha_j'| + W_1^{\alpha}\left(\frac{1}{n}\sum_{k=1}^n \delta_{\alpha_k}, \frac{1}{n}\sum_{k=1}^n \delta_{\alpha_k'}\right)\right) \\
&\leq K\left(|e_j - e_j'| + |\alpha_j - \alpha_j'| + \frac{1}{n}\sum_{k=1}^n |\alpha_k - \alpha_k'|\right) \\
&\leq 2K \sum_{k=1}^n \left(|e_k - e_k'| + |\alpha_k - \alpha_k'|\right).
\end{aligned}
$$

Analogously we find by (A2)

$$
\begin{aligned}
&|\widetilde{\phi}_j(e_1, \alpha_1, \dots, e_n, \alpha_n) - \widetilde{\phi}_j(e_1', \alpha_1', \dots, e_n', \alpha_n')| \\
&\leq C_\phi\left(|e_j - e_j'| + |\alpha_j - \alpha_j'| + W_1^{\alpha}\left(\frac{1}{n}\sum_{k=1}^n \delta_{\alpha_k}, \frac{1}{n}\sum_{k=1}^n \delta_{\alpha_k'}\right)\right) \\
&\leq 2C_\phi \sum_{k=1}^n \left(|e_k - e_k'| + |\alpha_k - \alpha_k'|\right).
\end{aligned}
$$

Since the coefficients $\widetilde{B}, \widetilde{\phi}$ are globally Lipschitz continuous (and hence satisfy the linear growth conditions), it follows from the classical theory of stochastic equations that equation (3.2) has a unique strong solution, see [7, Theorem 1.2], with the desired properties. This also proves that (3.1) has a unique strong solution with the desired properties. □

Note that (3.1) determines a Markov process on state space $(\mathbb{R}\times\mathbb{N}_0)^n$. To find its generator, let us take $F \in C_c^2((\mathbb{R}\times\mathbb{N}_0)^n)$ and apply Ito's formula to the process $F((e_j(t), \alpha_j(t))_{j=1}^n$. A short computation shows that

$$
F((e_j(t), \alpha_j(t) - F((e_j(0), \alpha_j(0))_{j=1}^n) - \int_0^t LF((e_j(s), \alpha_j(s))_{j=1}^n) ds
$$

is an $(\mathcal{F}_t)_{t\geq 0}$-martingale, where LF is given by

$$
\begin{aligned}
LF(e,\alpha) = &\sum_{j=1}^n B\left(e_j, \alpha_j \;\middle|\; \frac{1}{n}\sum_{k=1}^n \delta_{\alpha_k}\right)\frac{\partial F(e,\alpha)}{\partial e_j} \\
&+ \sum_{j=1}^n \phi\left(e_j, \alpha_j \mid \frac{1}{n}\sum_{k=1}^n \delta_{\alpha_k}\right)\left(F(e, \alpha + \mathbb{1}_j) - F(e,\alpha)\right),
\end{aligned}
$$

where $e = (e_1, \dots, e_n)$, $\alpha = (\alpha_1, \dots, \alpha_n)$ and $(\mathbb{1}_j)_k := \delta_{kj}$ denotes the Kronecker-Delta symbol.

4. Mean-field market model

In this section we perform the law of large numbers $n \to \infty$ proving that the sequence of empirical measures

$$
\mu^{(n)} = \frac{1}{n}\sum_{k=1}^n \delta_{(e_k, \alpha_k)}
$$

on the Skorokhod space $D(\mathbb{R}_+, \mathbb{R}\times\mathbb{N}_0)$ converges to probability law μ. This law described the time-evolution of a typical firm when the total number of firms is very large. We will see that it corresponds a stochastic differential equation of McKean-Vlasov type and that its one-dimensional distribution is a solution to a non-linear Fokker-Planck equation.

4.1. **The McKean-Vlasov type mean-field equation.** Performing formally the limit $n \to \infty$ in (3.1) we find the following limiting equation of McKean-Vlasov type

$$\begin{cases} e(t) & = e(0) + \int_0^t B\left(e(s), \alpha(s) \;\middle|\; \mu_s^\alpha\right) ds, \\ \alpha(t) & = \alpha(0) + \int_0^t \int_0^\infty \mathbb{1}_{\{u \leq \phi(e(s), \alpha(s-) \mid \mu_s^\alpha)\}} N(ds, du), \end{cases} \tag{4.1}$$

where N is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}_+$ with compensator $dsdu$ and μ_s denotes the law of $(e(s), \alpha(s))$. Hence μ_s^α is the marginal on $\mathbb{N}_0$, i.e., the law of $\alpha(s)$. The next theorem states that under conditions (A1) – (A4) this equation has a unique strong solution.

Theorem 4.1. *Suppose that conditions (A1) – (A4) are satisfied. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a stochastic basis with the usual conditions and let $N(ds, du)$ be an $(\mathcal{F}_t)_{t\geq 0}$-Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}_+$ with compensator dsdu. Then for each $\mathcal{F}_0$-measurable random variable $(e(0), \alpha(0)) \in \mathbb{R} \times \mathbb{N}_0$ with finite first moments there exists a unique strong solution to* (4.1) *with finite first moments.*

Proof. In view of Lemma 2.2 combined with (A2) it follows that the coefficients in (4.1) are globally Lipschitz continuous. The existence and uniqueness of strong solutions to (4.1) follows from the classical theory of stochastic equations, see [7, Theorem 2.1]. □

Let $f \in C_c^2(\mathbb{R} \times \mathbb{N}_0)$. Applying the Ito formula to the process $f(e(t), \alpha(t))$ shows that

$$f(e(t), \alpha(t)) - f(e(0), \alpha(0)) - \int_0^t L(\mu_s^\alpha) f(e(s), \alpha(s)) ds$$

is a martingale, where μ_t denotes the law of $(e(t), \alpha(t))$, μ_t^α its marginal on $\mathbb{N}_0$ (the law of $\alpha(t)$), and $L(\mu^\alpha)f$ is given by

$$\begin{aligned} L(\mu^\alpha) f(e, \alpha) &= B\left(e, \alpha \;\middle|\; \mu^\alpha\right) \frac{\partial f(e, \alpha)}{\partial e} + \phi\left(e, \alpha \mid \mu^\alpha\right)\left(f(e, \alpha + \mathbb{1}) - f(e, \alpha)\right) \\ &= B\left(e, \alpha \;\middle|\; \mu^\alpha\right) \frac{\partial f(e, \alpha)}{\partial e} + \int_{\mathbb{R}} \left(f(e, \alpha + z) - f(e, \alpha)\right) Q_{\mu^\alpha}(e, \alpha, dz) \end{aligned}$$

with kernel

$$Q_{\mu^\alpha}(e, \alpha, dz) = \phi\left(e, \alpha \mid \mu^\alpha\right) \delta_1(dz).$$

Hence taking expectations, we arrive at the following nonlinear Fokker-Planck equation in the weak form

$$\langle f, \mu_t \rangle = \langle f, \mu_0 \rangle + \int_0^t \langle L(\mu_s^\alpha) f, \mu_s \rangle ds$$

with $\langle f, \nu \rangle := \int_{\mathbb{R}\times\mathbb{N}_0} f(e, \alpha) \nu(de, d\alpha)$. This is precisely equation (1.6)

Lemma 4.2. *Suppose that conditions (A1) – (A4) are satisfied. If for some $q > 2$*

$$\mathbb{E}\left[|e(0)|^q + |\alpha(0)|^q\right] < \infty,$$

then for each $T > 0$ there exists a constant $C(T, q) > 0$ such that

$$\sup_{t\in[0,T]} \mathbb{E}\left[|e(t)|^q + |\alpha(t)|^q\right] \leq C(T, q) \mathbb{E}\left[|e(0)|^q + |\alpha(0)|^q\right] < \infty.$$

Proof. This can be shown by standard arguments using the linear growth of the coefficients. □

4.2. **Law of large numbers in the Wasserstein distance.** In this section we provide a convergence rate for the mean-field limit $n \to \infty$ where (3.1) approximates (4.1). More precisely, let $\mu_0 \in \mathcal{P}(\mathbb{R} \times \mathbb{N}_0)$ be the initial distribution and for each $n \in \mathbb{N}$, let $N_1, \dots, N_n$ be a sequence of independent Poisson random measures on $\mathbb{R}_+ \times \mathbb{R}_+$ with compensators $dsdu$. Associated to these random measures let $(e_1, \alpha_1), \dots, (e_n, \alpha_n)$ be the unique strong solution to (3.1) with $(e_j(0), \alpha_j(0)) \sim \mu_0$ for $j = 1, \dots, n$. Moreover, we suppose that $(e_j(0), \alpha_j(0))$, $j = 1, \dots, n$ are all mutually independent. Finally, according to Theorem 4.1 there exists a unique strong solution (e, α) to (4.1) with initial condition $(e(0), \alpha(0)) \sim \mu_0$. Let μ be its law on the Skorokhod space and denote by μ_t its time marginal at time $t \geq 0$. Define the sequence of empirical measures μ_t^n by

$$\mu_t^n := \frac{1}{n} \sum_{j=1}^{n} \delta_{(e_j(t), \alpha_j(t))}.$$

The next theorem provides a convergence rate for $\mu_t^n \longrightarrow \mu_t$ in the W_1 distance and hence rigorously justifies the formal derivation of (4.1).

Theorem 4.3. *Suppose that conditions (A1) – (A4) are satisfied. Take $\mu_0 \in \mathcal{P}(\mathbb{R} \times \mathbb{N}_0)$ and suppose that there exists $q > 2$ with*

$$\int_{\mathbb{R} \times \mathbb{N}_0} (|e| + \alpha)^q \mu_0(de, d\alpha) < \infty.$$

Then for each $T > 0$ there exists a constant $C(q, T) > 0$ such that

$$\mathbb{E}\left[W_1\left(\mu_t^n, \mu_t\right)\right] \leq C(T, q) \left(\int_{\mathbb{R} \times \mathbb{N}_0} (|e'| + |\alpha'|)^q \mu_0(de', d\alpha') \right)^{1/q} \log(1+n) n^{-1/2}$$

holds for each $t \in [0, T]$.

Proof. To prove this statement, we construct for each $n \in \mathbb{N}$ a coupling of (e, α) and $(e_j, \alpha_j)_{j=1}^n$ in such a way that this coupling can be efficiently estimated. To do so, we let $(\overline{e}_1, \overline{\alpha}_1), \dots (\overline{e}_n, \overline{\alpha}_n)$ be a sequence of processes obtained from (4.1) driven by the same Poisson random measures $N_1, \dots, N_n$ as used in the definition of $(e_j, \alpha_j)_{j=1}^n$ in (3.1) and the same initial conditions $(e_j(0), \alpha_j(0))$, i.e.,

$$\begin{cases} \overline{e}_j(t) = e_j(0) + \int_0^t B\left(\overline{e}_j(s), \overline{\alpha}_j(s) \;\middle|\; \mu_s^{\alpha} \right) ds, \\ \overline{\alpha}_j(t) = e_j(0) + \int_0^t \int_0^{\infty} \mathbb{1}_{\{u \leq \phi(\overline{e}_j(s), \overline{\alpha}_j(s-) \;|\; \mu_s^{\alpha})\}} N_j(ds, du), \end{cases} \qquad j = 1, \dots, n.$$

The strong existence of such processes and is guaranteed by Theorem 4.1. The uniqueness statement there implies that $(\overline{e}_1, \overline{\alpha}_1), \dots (\overline{e}_n, \overline{\alpha}_n)$ are all identically distributed with μ. Since $N_j, (e_j(0), \alpha_j(0))$ for $j = 1, \dots, n$ are independent, it follows that also the $(\overline{e}_1, \overline{\alpha}_1), \dots, (\overline{e}_n, \overline{\alpha}_n)$ are independent. Finally, define another sequence of empirical measures $\widehat{\mu}_t^n$ by

$$\widehat{\mu}_t^n := \frac{1}{n} \sum_{i=1}^{n} \delta_{(\overline{e}_i(t), \overline{\alpha}_i(t))}.$$

Step 1. In this step we prove that, for each $T > 0$, there exists a constant $C(T) > 0$ such that

$$\mathbb{E}\left[\frac{1}{n} \sum_{j=1}^{n} \left(|e_j(t) - \overline{e}_j(t)| + |\alpha_j(t) - \overline{\alpha}_j(t)| \right) \right] \leq C(T) \int_0^T \mathbb{E}\left[W_1(\widehat{\mu}_s^n, \mu_s)\right] ds. \tag{4.2}$$

First observe that

$$\mathbb{E}\left[|e_j(t) - \overline{e}_j(t)|\right] \leq \int_0^t \mathbb{E}\left[|B(e_j(s), \alpha_j(s) \mid \mu_s^{n,\alpha}) - B(\overline{e}_j(s), \overline{\alpha}_j(s) \mid \mu_s^{\alpha})| \right] ds$$

$$\leq K \int_0^t \mathbb{E}\Big[|e_j(s) - \overline{e}_j(s)| + |\alpha_j(s) - \overline{\alpha}_j(s)| + W_1(\mu_s^{n,\alpha}, \mu_s^{\alpha})\Big] ds.$$

where the constant K stems from Lemma 2.2. Similarly we obtain from (A2)

$$\mathbb{E}\left[|\alpha_j(t) - \overline{\alpha}_j(t)|\right] \leq \int_0^t \mathbb{E}\Big[|\phi(e_j(s), \alpha_j(s) \mid \mu_s^{n,\alpha}) - \phi(\overline{e}_j(s), \overline{\alpha}_j(s) \mid \mu_s^{\alpha})|\Big] ds$$
$$\leq C_\phi \int_0^t \mathbb{E}\Big[|e_j(s) - \overline{e}_j(s)| + |\alpha_j(s) - \overline{\alpha}_j(s)| + W_1(\mu_s^{n,\alpha}, \mu_s^{\alpha})\Big] ds.$$

Combining both estimates and using (2.1) we conclude that

$$\mathbb{E}\left[\frac{1}{n}\sum_{j=1}^n \Big(|e_j(t) - \overline{e}_j(t)| + |\alpha_j(t) - \overline{\alpha}_j(t)|\Big)\right]$$
$$\leq (K + C_\phi) \int_0^t \mathbb{E}\left[\frac{1}{n}\sum_{j=1}^n \Big(|e_j(s) - \overline{e}_j(s)| + |\alpha_j(s) - \overline{\alpha}_j(s)|\Big)\right] ds$$
$$+ (K + C_\phi) \int_0^t \mathbb{E}\left[W_1(\mu_s^n, \mu_s)\right] ds.$$

The second integral can be now estimated according to

$$\mathbb{E}\left[W_1(\mu_s^n, \mu_s)\right] \leq \mathbb{E}\left[W_1(\mu_s^n, \widehat{\mu}_s^n)\right] + \mathbb{E}\left[W_1(\widehat{\mu}_s^n, \mu_s)\right]$$
$$\leq \frac{1}{n}\sum_{j=1}^n \mathbb{E}\left[|e_j(s) - \overline{e}_j(s)| + |\alpha_j(s) - \overline{\alpha}_j(s)|\right] + \mathbb{E}\left[W_1(\widehat{\mu}_s^n, \mu_s)\right],$$

where we have used the fact that (e_j, α_j) and $(\overline{e}_j, \overline{\alpha}_j)$ are defined on the same probability space with the same noise terms, so that $\mathbb{E}\left[W_1(\mu_s^n, \widehat{\mu}_s^n)\right]$ makes sense, and a similar inequality to (3.3), i.e.,

$$W_1(\mu_s^n, \widehat{\mu}_s^n) \leq \frac{1}{n}\sum_{j=1}^n \Big(|e_j(s) - \overline{e}_j(s)| + |\alpha_j(s) - \overline{\alpha}_j(s)|\Big), \quad a.s.$$

Hence we arrive at

$$\mathbb{E}\left[\frac{1}{n}\sum_{j=1}^n \Big(|e_j(t) - \overline{e}_j(t)| + |\alpha_j(t) - \overline{\alpha}_j(t)|\Big)\right]$$
$$\leq (K + C_\phi + 1) \int_0^t \mathbb{E}\left[\frac{1}{n}\sum_{j=1}^n \Big(|e_j(s) - \overline{e}_j(s)| + |\alpha_j(s) - \overline{\alpha}_j(s)|\Big)\right] ds$$
$$+ (K + C_\phi) \int_0^T \mathbb{E}\left[W_1(\widehat{\mu}_s^n, \mu_s)\right] ds.$$

Inequality (4.2) now follows from the Gronwall Lemma.

Step 2. In this step we estimate $\mathbb{E}\left[W_1(\mu_s^n, \mu_s)\right]$ directly. Namely, using (4.2) we obtain

$$\mathbb{E}\left[W_1(\mu_t^n, \mu_t)\right] \leq \mathbb{E}\left[W_1(\mu_t^n, \widehat{\mu}_t^n)\right] + \mathbb{E}\left[W_1(\widehat{\mu}_t^n, \mu_t)\right]$$
$$\leq \mathbb{E}\left[\frac{1}{n}\sum_{j=1}^n \left(|e_j(t) - \overline{e}_j(t)| + |\alpha_j(t) - \overline{\alpha}_j(t)|\right)\right] + \mathbb{E}\left[W_1(\widehat{\mu}_t^n, \mu_t)\right]$$
$$\leq C(T) \int_0^T \mathbb{E}\left[W_1(\widehat{\mu}_s^n, \mu_s)\right] ds + \mathbb{E}\left[W_1(\widehat{\mu}_t^n, \mu_t)\right].$$

To estimate $\mathbb{E}[W_1(\widehat{\mu}^n_t, \mu_t)]$ we apply the concentration inequalities for empirical measures with respect to Wasserstein distances from [6, Theorem 1] for $p = 1$, $d = 2$, and $q > 2$ which yields for some constant $C_0 > 0$

$$\begin{aligned}\mathbb{E}[W_1(\widehat{\mu}^n_t, \mu_t)] &\leq C_0 \left(\int_{\mathbb{R}\times\mathbb{N}_0} (|e'| + \alpha')^q \mu_t(de', d\alpha') \right)^{1/q} \left(n^{-1/2}\log(1+n) + n^{-(q-1)/q} \right) \\ &\leq C(T,q) \left(\int_{\mathbb{R}\times\mathbb{N}_0} (|e'| + |\alpha'|)^q \mu_0(de', d\alpha') \right)^{1/q} \left(n^{-1/2}\log(1+n) + n^{-(q-1)/q} \right)\end{aligned}$$

where we have used Lemma 4.2 and $C(T,q) > 0$ is some generic constant. Since $q > 2$ we obtain $n^{-(q-1)/q} = n^{-1/2} n^{-\frac{1}{2}+\frac{1}{q}} \leq \log(2)^{-1} n^{-1/2}\log(1+n)$ which proves the assertion. □

5. Exit and entry of firms to the market

5.1. **Heuristic introduction.** In this section, we consider a market with arbitrary finite number of firms that differ from each other in their liquidity and corresponding technological level. We introduce the entry and exit of new firms to such a market. The corresponding process is considered to be independent process from the microscopic process studied in the previous section. We study the mesoscopic level of it (after mean-field limit) in terms of the corresponding kinetic equation. Combining such kinetic equation with the previously obtained non-linear Fokker-Planck equation we get the equation describing the effective evolution of one dimensional distributions of industry dynamics under financial constraints governed by random entry and exit of firms. In other words we obtain the evolution of one-dimensional distribution of the "typical" firm on the market with arbitrary large number of interacting firms. This kinetic description can give information about the long-time behaviour, invariant and stationary states, asymptotic speed of growth, front wave propagation and several other effects.

The classical mean-field approach mentioned in the previous sections is not applicable to the entry and exit dynamics because the number of firms is changing in time. Therefore, we use the approach proposed in [5] which is based on the proper scaling of the corresponding hierarchical equations for correlations. This scheme also has a clear interpretation in terms of scaled Markov generators. The general idea of this scaling consists in making our system more dense and at the same time suppress all correlations. For details of this approach we refer to [5]. Let us consider the process with initial finite number of firms with 0 entry rate (no entry of new firms) and the following exit rate of the existing firm from the market

$$q(e, \alpha) = -\min\{0, \gamma^{EX} e\}.$$

According to [5], such process after Vlasov scaling (which corresponds to the mean-field limit) leads to the following kinetic equation

$$\langle f, \mu_t \rangle = \langle f, \mu_0 \rangle + \int_0^t \int_{\mathbb{R}\times\mathbb{N}} \min\{0, \gamma^{EX} e\} f(e, \alpha) \mu_s(de, d\alpha) ds,$$

where μ_t is the Vlasov limit of rescaled one dimensional distributions of the microscopic process with 0 entry rate and exit q. In other words it is effective distribution of the "typical" firm on the market with arbitrary finite number of interacting firms with exit mechanism described above.

For different possible entry (birth) rates the evolution operator in the corresponding kinetic equation has, in general, the following form:

$$\int_{\mathbb{R}\times\mathbb{N}} f(e, \alpha) M(\mu_s, de, d\alpha)$$

where $M(\mu_s, de, d\alpha)$ is a kernel on $\mathbb{R}\times\mathbb{N}$. Let us consider some particular cases:

I. Entry of firms, which depends on the firms average. In this case the entry of a new firm to the market with $N \in \mathbb{N}$ existing firms at each fixed moment of time t occurs with intensity which depend on $\frac{1}{N}\sum_{k=1}^{N}\delta_{e_k(t)}$. Namely, the time of entry is exponentially distributed with intensity p^{EN}, the initial liquidity of the new firm upon entry is equal to 0 and its technology level is given by the population average $\frac{1}{N}\sum_{k=1}^{N}\alpha_k(t)$. According to [5], this case corresponds to the following form of M

$$M(\mu, de, d\alpha) = p^{EN}\left(\int_{\mathbb{R}\times\mathbb{N}} \bar{e}\mu(d\bar{e}, d\bar{\alpha})\right)\delta_{\left(0, \int_{\mathbb{R}\times\mathbb{N}}\bar{\alpha}\mu(d\bar{e},d\bar{\alpha})\right)}(de, d\alpha).$$

II. The probability of entry of a new firm with liquidity and technology level $(e, \alpha) \in \mathbb{R}\times\mathbb{N}$ to the market with N already existing firms with parameters $\{(e_i, \alpha_i)\}_{i=1}^{N}$, $N \in \mathbb{N}$ in a time interval Δt is equal to

$$p^{EN}\left(\frac{1}{N}\sum_{k=1}^{N}\delta_{e_k}\right)\frac{1}{2\pi}\exp\left\{-\left(e^2 + \left(\alpha - \frac{1}{N}\sum_{k=1}^{N}\alpha_k\right)^2\right)\right\}\Delta t + o(\Delta t), \quad \Delta t \to 0.$$

Such an entry rate leads to the following form of M in the kinetic equation

$$\begin{aligned} M(\mu, de, d\alpha) = {} & \frac{1}{2\pi}p^{EN}\left(\int_{\mathbb{R}\times\mathbb{N}}\bar{e}\mu(d\bar{e}, d\bar{\alpha})\right) \\ & \times \exp\left\{-\left(e^2 + \left(\alpha - \int_{\mathbb{R}\times\mathbb{N}}\bar{\alpha}\mu(d\bar{e}, d\bar{\alpha})\right)^2\right)\right\} de\, n(d\alpha), \end{aligned}$$

where $n(d\alpha)$ is some measure on $\mathbb{N}$.

III. Entry to the market with establishment mechanism. The probability of entry of a new firm with liquidity and technology level $(e, \alpha) \in \mathbb{R}\times\mathbb{N}$ to the market with N already existing firms with parameters $\{(e_i, \alpha_i)\}_{i=1}^{N}$, $N \in \mathbb{N}$ in time Δt is equal to

$$\exp\left\{-\sum_{i=1}^{N}\phi((e - e_i, \alpha - \alpha_i))\right\}\Delta t + o(\Delta t), \quad \Delta t \to 0.$$

Such an entry rate leads to the following form of M in the kinetic equation

$$M(\mu, de, d\alpha) = e^{-(\phi\star\mu)(e,\alpha)}de\, n(d\alpha).$$

Combining on the mesoscopic level the mechanism of entry and exit with the processes of liquidity and technology level described in the previous section we get the analog of Fokker-Planck equations with the following evolution operators

I.

$$\begin{aligned} L(\mu)f(e, \alpha) = {} & B\left(e, \alpha \;\middle|\; \mu\right)\frac{\partial f(e, \alpha)}{\partial e} + \phi\left(e, \alpha \mid \mu\right)\left(f(e, \alpha + \mathbb{1}) - f(e, \alpha)\right) \\ & + \min\{0, \gamma^{EX}e\}f(e, \alpha) + p^{EN}\left(\int_{\mathbb{R}\times\mathbb{N}}\bar{e}\mu(d\bar{e}, d\bar{\alpha})\right)f\left(0, \int_{\mathbb{R}\times\mathbb{N}}\bar{\alpha}\,\mu(d\bar{e}, d\bar{\alpha})\right). \end{aligned}$$

II.

$$\begin{aligned} & \int_{\mathbb{R}\times\mathbb{N}} L(\mu)f(e, \alpha)\mu(de, d\alpha) \\ & = \int_{\mathbb{R}\times\mathbb{N}}\left[B\left(e, \alpha \;\middle|\; \mu\right)\frac{\partial f(e, \alpha)}{\partial e} + \phi\left(e, \alpha \mid \mu\right)\left(f(e, \alpha + \mathbb{1}) - f(e, \alpha)\right)\right]\mu(de, d\alpha) \\ & + \int_{\mathbb{R}\times\mathbb{N}}\min\left\{0, \gamma^{EX}e\right\}f(e, \alpha)\mu(de, d\alpha) + p^{EN}\left(\int_{\mathbb{R}\times\mathbb{N}}\bar{e}\mu(d\bar{e}, d\bar{\alpha})\right) \end{aligned}$$

$$\times \frac{1}{2\pi}\int_{\mathbb{R}\times\mathbb{N}} f(e,\alpha)\exp\left\{-\left(e^2+\left(\alpha-\int_{\mathbb{R}\times\mathbb{N}}\bar{\alpha}\mu(d\bar{e},d\bar{\alpha})\right)^2\right)\right\}de\, n(d\alpha).$$

III.

$$\begin{aligned}&\int_{\mathbb{R}\times\mathbb{N}} L(\mu)f(e,\alpha)\mu(de,d\alpha)\\ &=\int_{\mathbb{R}\times\mathbb{N}}\left[B\left(e,\alpha\;\middle|\;\mu\right)\frac{\partial f(e,\alpha)}{\partial e}+\phi\left(e,\alpha\mid\mu\right)\left(f(e,\alpha+\mathbb{1})-f(e,\alpha)\right)\right]\mu(de,d\alpha)\\ &+\int_{\mathbb{R}\times\mathbb{N}}\min\left\{0,\gamma^{EX}e\right\}f(e,\alpha)\mu(de,d\alpha)+\int_{\mathbb{R}\times\mathbb{N}} f(e,\alpha)e^{-(\phi\star\mu)(e,\alpha)}de\, n(d\alpha).\end{aligned}$$

The detailed study of these equations we postpone to our future paper.

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