# On interpolation of reflexive variable Lebesgue spaces on which the Hardy-Littlewood maximal operator is bounded

Lars Diening, Alexei Karlovich, and Eugene Shargorodsky

*To Professor Stefan Samko on the occasion of his 80th birthday*

Abstract. We show that if the Hardy-Littewood maximal operator $M$ is bounded on a reflexive variable exponent space $L^{p(\cdot)}(\mathbb{R}^d)$, then for every $q \in (1,\infty)$, the exponent $p(\cdot)$ admits, for all sufficiently small $\theta > 0$, the representation $1/p(x) = \theta/q + (1-\theta)/r(x)$, $x \in \mathbb{R}^d$ such that the operator $M$ is bounded on the variable Lebesgue space $L^{r(\cdot)}(\mathbb{R}^d)$. This result can be applied for transferring properties like compactness of linear operators from standard Lebesgue spaces to variable Lebesgue spaces by using interpolation techniques.

## 1. Introduction

Let $L^0(\mathbb{R}^d)$ denote the space of all (equivalence classes of) Lebesgue measurable complex-valued functions on $\mathbb{R}^d$ with the topology of convergence in measure on sets of finite measure. Let $p(\cdot) : \mathbb{R}^d \to [1,\infty]$ be a measurable a.e. finite function. By $L^{p(\cdot)}(\mathbb{R}^d)$ we denote the set of all functions $f \in L^0(\mathbb{R}^d)$ such that

$$I_{p(\cdot)}(f/\lambda) := \int_{\mathbb{R}^d} |f(x)/\lambda|^{p(x)} dx < \infty$$

for some $\lambda > 0$. This set becomes a Banach space when equipped with the Luxemburg-Nakano norm

$$\|f\|_{p(\cdot)} := \inf \big\{ \lambda > 0 : I_{p(\cdot)}(f/\lambda) \le 1 \big\}.$$

It is easy to see that if $p(\cdot) = p$ is constant, then $L^{p(\cdot)}(\mathbb{R}^d)$ is nothing but the standard Lebesgue space $L^p(\mathbb{R}^d)$. The space $L^{p(\cdot)}(\mathbb{R}^d)$ is referred to as a *variable Lebesgue space*.

Let $1 \le q < \infty$. Given $f \in L^q_{\rm loc}(\mathbb{R}^d)$, the $q$-th maximal operator is defined by

$$(M_q f)(x) := \sup_{Q \ni x} \left( \frac{1}{|Q|} \int_Q |f(y)|^q dy \right)^{1/q},$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^d$ containing $x$ (here, and throughout, cubes will be assumed to have their sides parallel to the coordinate axes). Note that $M := M_1$ is the usual Hardy-Littlewood maximal operator. By

2020 *Mathematics Subject Classification.* 46E30, 42B25.

$\mathcal{B}_M(\mathbb{R}^d)$ denote the set of all measurable a.e. finite functions $p(\cdot) : \mathbb{R}^d \to [1,\infty]$ such that the Hardy-Littlewood maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^d)$.

We will use the following standard notation:

$$p_- := \operatorname*{ess\,inf}_{x\in\mathbb{R}^d} p(x), \quad p_+ := \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} p(x).$$

It is well known that the space $L^{p(\cdot)}(\mathbb{R}^d)$ is reflexive if and only if $1 < p_- \le p_+ < \infty$. In this case, its dual space is isomorphic to $L^{p'(\cdot)}(\mathbb{R}^d)$, where

$$1/p(x) + 1/p'(x) = 1, \quad x \in \mathbb{R}^d$$

(see, e.g., [**5**, Chap. 3]).

Suppose that $1 < p_- \le p_+ < \infty$ and there exist constants $c_0, c_\infty \in (0,\infty)$ and $p_\infty \in (1,\infty)$ such that

$$|p(x) - p(y)| \le \frac{c_0}{\log(e + 1/|x-y|)}, \quad x, y \in \mathbb{R}^d, \tag{1.1}$$

$$|p(x) - p_\infty| \le \frac{c_\infty}{\log(e + |x|)}, \quad x \in \mathbb{R}^d. \tag{1.2}$$

Then $p(\cdot) \in \mathcal{B}_M(\mathbb{R}^d)$ (see [**3**, Theorem 3.16] or [**5**, Theorem 4.3.8]). Following [**5**, Section 4.1] or [**3**, Section 2.1], we will say that $p(\cdot)$ is *globally log-Hölder continuous* if conditions (1.1)–(1.2) are satisfied. The class of all globally log-Hölder continuous exponents will be denoted by $\mathcal{P}^{\log}(\mathbb{R}^d)$.

Conditions (1.1) and (1.2) are optimal for the boundedness of $M$ in the sense of modulus of continuity; the corresponding examples are contained in [**16**] and [**2**]. However, neither (1.1) nor (1.2) is necessary for $p(\cdot) \in \mathcal{B}_M(\mathbb{R}^d)$. Thus

$$\mathcal{P}^{\log}(\mathbb{R}^d) \subsetneqq \mathcal{B}_M(\mathbb{R}^d).$$

Here we mention results by Nekvinda [**14, 15**] and Lerner [**13**] and further discussion in the monographs [**3**, Chap. 4] and [**5**, Chaps. 4–5].

The following result was obtained in a somewhat more complete form by the first author (see [**4**, Theorem 8.1] or [**5**, Theorem 5.7.2]).

THEOREM 1.1. *Let $p(\cdot) : \mathbb{R}^d \to [1,\infty]$ be a measurable function satisfying $1 < p_- \le p_+ < \infty$. The following statements are equivalent:*

(a) *$M$ is bounded on $L^{p(\cdot)}(\mathbb{R}^d)$;*
(b) *$M$ is bounded on $L^{p'(\cdot)}(\mathbb{R}^d)$;*
(c) *there exists an $s \in (1/p_-, 1)$ such that $M$ is bounded on $L^{sp(\cdot)}(\mathbb{R}^d)$;*
(d) *there exists a $q \in (1,\infty)$ such that $M_q$ is bounded on $L^{p(\cdot)}(\mathbb{R}^d)$.*

Rabinovich and Samko [**17**] (see also [**12**, Section 9.1.2]) observed that if a variable exponent $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^d)$ satisfies $1 < p_- \le p_+ < \infty$, then it can be decomposed as

$$\frac{1}{p(x)} = \frac{\theta}{2} + \frac{1-\theta}{r(x)}, \quad x \in \mathbb{R}^d, \tag{1.3}$$

where $\theta \in (0,1)$ and $r(\cdot)$ satisfies $1 < r_- \le r_+ < \infty$ and belongs to $\mathcal{P}^{\log}(\mathbb{R}^d)$. This observation was important in the "transfer of the compactness techniques" from $L^2(\mathbb{R}^d)$ to $L^{p(\cdot)}(\mathbb{R}^d)$ by means of the one-sided interpolation of the compactness property between the spaces $L^{r(\cdot)}(\mathbb{R}^d)$ (where an operator is merely bounded) and $L^2(\mathbb{R}^d)$ (where an operator is compact).

The second author and Spitkovsky [**11**] exploited this idea for more general variable exponents $p(\cdot) \in \mathcal{B}_M(\mathbb{R}^d)$ based on the following result obtained by the first author and published in [**11**, Theorem 4.1].

THEOREM 1.2. *Let* $p(\cdot) : \mathbb{R}^d \to [1, \infty]$ *be a measurable function satisfying* $1 < p_- \le p_+ < \infty$. *If* $p(\cdot) \in \mathcal{B}_M(\mathbb{R}^d)$, *then there exist numbers* $q \in (1, \infty)$ *and* $\theta \in (0,1)$ *such that the variable exponent* $r(\cdot)$ *defined by*

$$\frac{1}{p(x)} = \frac{\theta}{q} + \frac{1-\theta}{r(x)}, \quad x \in \mathbb{R}^d, \tag{1.4}$$

*belongs to* $\mathcal{B}_M(\mathbb{R}^d)$.

The above theorem has the disadvantage that the constant $q$ depends on the variable exponent $p(\cdot)$. It is desirable to avoid such a dependence and to find, for a given $p(\cdot) \in \mathcal{B}_M(\mathbb{R}^d)$, a $\theta \in (0,1)$ such that (1.3) holds and $r(\cdot)$ belongs to $\mathcal{B}_M(\mathbb{R}^d)$. This would allow one to simplify formulations of several results in the literature, where it was supposed that $p(\cdot)$ is of the form (1.3) with $r(\cdot) \in \mathcal{B}_M(\mathbb{R}^d)$ and some sufficiently small $\theta \in (0,1)$ (see, e.g., [**7**, Corollary 2.1, Theorem 3.2], [**8**, Theorem 1.2], [**9**, Theorem 2.1]). The second author asked in [**10**, Section 4.4] whether for a given $p(\cdot) \in \mathcal{B}_M(\mathbb{R}^d)$ satisfying $1 < p_- \le p_+ < \infty$, one can find a number $\tau_{p(\cdot)} \in (0,1]$ such that the variable exponent $r(\cdot)$ defined by (1.3) belongs to $\mathcal{B}_M(\mathbb{R}^d)$ *for every* $\theta \in (0, \tau_{p(\cdot)}]$.

Our main result is the following refinement of Theorem 1.2, which gives positive answers to the above questions.

THEOREM 1.3 (Main result). *Let* $p(\cdot) : \mathbb{R}^d \to [1, \infty]$ *be a measurable function satisfying* $1 < p_- \le p_+ < \infty$. *Then* $p(\cdot) \in \mathcal{B}_M(\mathbb{R}^d)$ *if and only if for every* $q \in (1, \infty)$, *there exists a number* $\Theta_{p(\cdot),q} \in (0,1)$ *such that for every* $\theta \in (0, \Theta_{p(\cdot),q}]$ *the variable exponent* $r(\cdot)$ *defined by* (1.4) *belongs to* $\mathcal{B}_M(\mathbb{R}^d)$.

Note that representation (1.4) implies that $0 < \theta/q \le 1/p(x) \le \theta/q + 1 - \theta < 1$ for $\theta > 0$, whence $1 < p_- \le p_+ < \infty$.

The paper is organized as follows. In Section 2, we formulate an interpolation lemma due to Cruz-Uribe [**1**], which immediately implies the proof of the sufficiency portion of Theorem 1.3. In Section 3, we show that if $p(\cdot) \in \mathcal{B}_M(\mathbb{R}^d)$ satisfies $1 < p_- \le p_+ < \infty$, then the variable exponents $\left(\frac{1}{t}\left(\frac{p(\cdot)}{s}\right)'\right)'$ belong to $\mathcal{B}_M(\mathbb{R}^d)$ for all $s, t \ge 1$ sufficiently close to 1. Based on this result, we complete the proof of the necessity portion of Theorem 1.3 in Section 4.

## 2. Proof of the sufficiency portion of Theorem 1.3

The sufficiency portion is an immedte corollary of the following result obtained by Cruz-Uribe [**1**, Corollary 3] (see also [**6**, Corollary 2.5] for the case $1 < (p_j)_- \le (p_j)_+ < \infty$, $j = 0, 1$) and the boundedness of the Hardy-Littlewood maximal operator $M$ on the standard Lebesgue space $L^q(\mathbb{R}^d)$ with $q \in (1, \infty)$.

LEMMA 2.1. *If* $p_i(\cdot) \in \mathcal{B}_M(\mathbb{R}^d)$ *for* $i = 0, 1$, *then for every* $\theta \in (0,1)$, *the variable exponent* $p_\theta(\cdot)$ *defined by*

$$\frac{1}{p_\theta(x)} = \frac{\theta}{p_0(x)} + \frac{1-\theta}{p_1(x)}, \quad x \in \mathbb{R}^d, \tag{2.1}$$

*belongs to* $\mathcal{B}_M(\mathbb{R}^d)$ *and*

$$\|M\|_{L^{p_\theta(\cdot)}\to L^{p_\theta(\cdot)}} \le 96\|M\|^{\theta}_{L^{p_0(\cdot)}\to L^{p_0(\cdot)}}\|M\|^{1-\theta}_{L^{p_1(\cdot)}\to L^{p_1(\cdot)}}. \tag{2.2}$$

Note that inequality (2.2) is stated in [**1**] with the constant 48, which seems to be a typo. This result was obtained as a consequence of the pointwise inequality $|T_f f| \le Mf \le 2T_f|f|$, where each $T_f$ is a linear integral operator with a positive kernel. On the other hand, it was shown in [**1**, Theorem 1] that if $T$ is a linear integral operator with a positive kernel that satisfies $\|Tf\|_{p_i(\cdot)} \le B_i\|f\|_{p_i(\cdot)}$ for $i = 0, 1$ and all $f \in L^{p_i(\cdot)}(\mathbb{R}^d)$ with $B_i$ independent of $f$, then

$$\|Tf\|_{p_\theta(\cdot)} \le 48 B_0^\theta B_1^{1-\theta}\|f\|_{p_\theta(\cdot)}.$$

## 3. Doubly iterated "left-openness and then duality" trick

Our construction is similar to that of the proof of [**11**, Theorem 4.1]. It is based on the consecutive application of the "left-openness" of the class $\mathcal{B}_M(\mathbb{R}^d)$ (see Theorem 1.1(c)) and then the "duality" of the class $\mathcal{B}_M(\mathbb{R}^d)$ (see Theorem 1.1(b)). In order to succeed, we repeat this procedure two times. The main novelty is that we can guarantee that the constructed exponents belong to $\mathcal{B}_M(\mathbb{R}^d)$ in certain ranges of parameters.

Lemma 3.1. *If* $p(\cdot) \in \mathcal{B}_M(\mathbb{R}^d)$ *satisfies* $1 < p_- \le p_+ < \infty$*, then there exist* $s_0, t_0 \in (1, \infty)$ *such that*

$$\left(\frac{1}{t}\left(\frac{p(\cdot)}{s}\right)'\right)' \in \mathcal{B}_M(\mathbb{R}^d) \quad \textit{for all} \quad s \in [1, s_0], \quad t \in [1, t_0]. \tag{3.1}$$

Proof. By Theorem 1.1(c), there exists a number $s_0 \in (1, \infty)$ such that $p(\cdot)/s_0 \in \mathcal{B}_M(\mathbb{R}^d)$. Then it follows from Theorem 1.1(b) that $p'(\cdot)$ and $(p(\cdot)/s_0)'$ belong to $\mathcal{B}_M(\mathbb{R}^d)$. Applying Theorem 1.1(c) once again, we see that there exist $t_1, t_2 \in (1, \infty)$ such that

$$\frac{p'(\cdot)}{t_1} \in \mathcal{B}_M(\mathbb{R}^d), \quad \frac{1}{t_2}\left(\frac{p(\cdot)}{s_0}\right)' \in \mathcal{B}_M(\mathbb{R}^d).$$

It follows from Lemma 2.1 (one can employ also a more elementary argument using Jensen's inequality as in [**6**, p. 43]) that

$$\frac{p'(\cdot)}{t} \in \mathcal{B}_M(\mathbb{R}^d) \text{ for all } t \in [1, t_1], \quad \frac{1}{t}\left(\frac{p(\cdot)}{s_0}\right)' \in \mathcal{B}_M(\mathbb{R}^d) \text{ for all } t \in [1, t_2].$$

Put

$$t_0 := \min\{t_1, t_2\}.$$

Then it is clear that $t_0 \in (1, \infty)$ and

$$\frac{p'(\cdot)}{t} \in \mathcal{B}_M(\mathbb{R}^d), \quad \frac{1}{t}\left(\frac{p(\cdot)}{s_0}\right)' \in \mathcal{B}_M(\mathbb{R}^d) \quad \text{for all} \quad t \in [1, t_0]. \tag{3.2}$$

Take any $s \in [1, s_0]$ and set $\theta := \frac{s_0 - s}{s_0 - 1} \in [0, 1]$. Then $s = \theta + (1-\theta)s_0$. Further, for $x \in \mathbb{R}^d$, we have

$$\left(\frac{1}{t}\left(\frac{p(x)}{s}\right)'\right)^{-1} = t\left(\frac{p(x)/s}{p(x)/s - 1}\right)^{-1} = t\frac{p(x) - s}{p(x)}$$

$$
\begin{aligned}
&= t\frac{p(x)(\theta+(1-\theta))-(\theta+(1-\theta)s_0)}{p(x)}\\
&= t\theta\frac{p(x)-1}{p(x)}+t(1-\theta)\frac{p(x)-s_0}{p(x)}\\
&= \theta\left(\frac{p'(x)}{t}\right)^{-1}+(1-\theta)\left(\frac{1}{t}\left(\frac{p(x)}{s_0}\right)'\right)^{-1}. \qquad (3.3)
\end{aligned}
$$

It follows from (3.2)–(3.3) and Lemma 2.1 that

$$\frac{1}{t}\left(\frac{p(\cdot)}{s}\right)' \in \mathcal{B}_M(\mathbb{R}^d) \quad \text{for all} \quad s\in[1,s_0], \quad t\in[1,t_0].$$

Applying Theorem 1.1(b) one more time, one arrives at (3.1). □

## 4. Proof of the necessity portion of Theorem 1.3

Suppose that $q\in(1,\infty)$ and $p(\cdot)$ satisfies $1<p_-\le p_+<\infty$ and belongs to $\mathcal{B}_M(\mathbb{R}^d)$. We need to prove that $r(\cdot)$ defined by (1.4) belongs to $\mathcal{B}_M(\mathbb{R}^d)$ for all sufficiently small positive values of $\theta$. We will show that one can choose $s$ and $t$ in such a way that $r(\cdot)=(\frac{1}{t}(\frac{p(\cdot)}{s})')'$, and use Lemma 3.1 to conclude that $r(\cdot)\in\mathcal{B}_M(\mathbb{R}^d)$.

Now, (1.4) is equivalent to

$$\frac{1}{r(x)}=\frac{1}{1-\theta}\frac{1}{p(x)}-\frac{\theta}{1-\theta}\frac{1}{q}, \quad x\in\mathbb{R}^d,$$

while

$$\frac{1}{(\frac{1}{t}(\frac{p(x)}{s})')'}=1-t\left(1-\frac{s}{p(x)}\right)=st\frac{1}{p(x)}-(t-1), \quad x\in\mathbb{R}^d.$$

So, we need to take $s$ and $t$ such that

$$st=\frac{1}{1-\theta} \qquad \text{and} \qquad t-1=\frac{\theta}{1-\theta}\frac{1}{q}.$$

An easy calculation shows that these equations are equivalent to

$$\theta=1-\frac{1}{st} \qquad \text{and} \qquad t=\frac{q-1}{q-s}.$$

Let $s_0\in(1,\infty)$ and $t_0\in(1,\infty)$ be such that (3.1) holds. Put

$$t(s):=\frac{q-1}{q-s}, \quad 1<s<q. \qquad (4.1)$$

Since $1<t(s)\to1$ as $s\to1$, there exists $s_1\in(1,s_0]$ such that

$$1<t(s)\le t_0 \quad \text{for all} \quad s\in(1,s_1],$$

Let

$$\theta(s):=1-\frac{1}{st(s)}=1-\frac{q-s}{s(q-1)}=\frac{q(s-1)}{s(q-1)}=\frac{q}{q-1}\left(1-\frac{1}{s}\right). \qquad (4.2)$$

Then $0<\theta(s)\to0$ as $s\to1$. So, there exists $s_2\in(1,s_1]$ such that

$$\Theta_{p(\cdot),q}:=\theta(s_2)\in(0,1).$$

It is clear from (4.2) that $\theta(\cdot)$ is an increasing continuous function. Then

$$\theta((1,s_2])=(0,\Theta_{p(\cdot),q}].$$

Take any $\theta \in (0, \Theta_{p(\cdot),q}]$. It follows from the above that there exists a unique $s \in (1, s_2]$ such that $\theta(s) = \theta$. For this $s$ and $t := t(s)$,

$$r_\theta(\cdot) := \left( \frac{1}{t} \left( \frac{p(\cdot)}{s} \right)' \right)' \in \mathcal{B}_M(\mathbb{R}^d) \tag{4.3}$$

according to (3.1).

It follows from (4.1) that

$$q = \frac{st-1}{t-1}. \tag{4.4}$$

Combining (4.2), (4.3), (4.4), we get for $x \in \mathbb{R}^d$,

$$\begin{aligned} \frac{\theta}{q} + \frac{1-\theta}{r_\theta(x)} &= \frac{\theta}{q} + (1-\theta) \left( 1 - \frac{1}{\frac{1}{t}\left(\frac{p(x)}{s}\right)'} \right) \\ &= \left(1 - \frac{1}{st}\right) \frac{t-1}{st-1} + \frac{1}{st} \left( 1 - t \left( 1 - \frac{s}{p(x)} \right) \right) \\ &= \frac{t-1}{st} + \frac{1}{st}(1-t) + \frac{1}{st} \cdot \frac{ts}{p(x)} \\ &= \frac{1}{p(x)}. \end{aligned}$$

Thus $r_\theta(\cdot)$ satisfies (1.4) for every $\theta \in (0, \Theta_{p(\cdot),q}]$. □

**Acknowledgments.** This work was supported by national funds through the FCT – Fundação para a Ciência e a Tecnologia, I.P. (Portuguese Foundation for Science and Technology) within the scope of the project UIDB/00297/2020 (Centro de Matemática e Aplicações). Lars Diening was also funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – SFB 1283/2 2021 – 317210226.

## References

[1] D. Cruz-Uribe. Interpolation of positive operators on variable Lebesgue spaces. *Math. Inequal. Appl.*, 15(3):639–644, 2012.

[2] D. Cruz-Uribe, A. Fiorenza, and C. J. Neugebauer. The maximal function on variable $L^p$ spaces. *Ann. Acad. Sci. Fenn. Math.*, 28(1):223–238, 2003.

[3] D. V. Cruz-Uribe and A. Fiorenza. *Variable Lebesgue spaces.* Applied and Numerical Harmonic Analysis. Birkhäuser/Springer, Heidelberg, 2013. Foundations and harmonic analysis.

[4] L. Diening. Maximal function on Musielak-Orlicz spaces and generalized Lebesgue spaces. *Bull. Sci. Math.*, 129(8):657–700, 2005.

[5] L. Diening, P. Harjulehto, P. Hästö, and M. Růžička. *Lebesgue and Sobolev spaces with variable exponents*, volume 2017 of *Lecture Notes in Mathematics.* Springer, Heidelberg, 2011.

[6] L. Diening, P. Hästö, and A. Nekvinda. Open problems in variable Lebesgue and Sobolev spaces. In *FSDONA04 Proceedings*, pages 38–58. Czech Acad. Sci., Milovy, Czech Republic, 2004.

[7] A. Fiorenza, A. Gogatishvili, and T. Kopaliani. Estimates for imaginary powers of the Laplace operator in variable Lebesgue spaces and applications. *Izv. Nats. Akad. Nauk Armenii Mat.*, 49(5):11–22, 2014.

[8] A. Gogatishvili and T. Kopaliani. On the Rubio de Francia's theorem in variable Lebesgue spaces. *Bull. TICMI*, 18(1):3–10, 2014.

[9] A. Gogatishvili and T. Kopaliani. Maximal multiplier operators in $L^{p(\cdot)}(\mathbb{R}^n)$ spaces. *Bull. Sci. Math.*, 140(4):86–97, 2016.

[10] A. Y. Karlovich. Algebras of continuous Fourier multipliers on variable Lebesgue spaces. *Mediterr. J. Math.*, 17(4):Paper No. 102, 19, 2020.

[11] A. Y. Karlovich and I. M. Spitkovsky. Pseudodifferential operators on variable Lebesgue spaces. In *Operator theory, pseudo-differential equations, and mathematical physics*, volume 228 of *Oper. Theory Adv. Appl.*, pages 173–183. Birkhäuser/Springer Basel AG, Basel, 2013.

[12] V. Kokilashvili, A. Meskhi, H. Rafeiro, and S. Samko. *Integral operators in non-standard function spaces. Vol. 1*, volume 248 of *Operator Theory: Advances and Applications*. Birkhäuser/Springer, [Cham], 2016. Variable exponent Lebesgue and amalgam spaces.

[13] A. K. Lerner. Some remarks on the Hardy-Littlewood maximal function on variable $L^p$ spaces. *Math. Z.*, 251(3):509–521, 2005.

[14] A. Nekvinda. Hardy-Littlewood maximal operator on $L^{p(x)}(\mathbb{R})$. *Math. Inequal. Appl.*, 7(2):255–265, 2004.

[15] A. Nekvinda. Maximal operator on variable Lebesgue spaces for almost monotone radial exponent. *J. Math. Anal. Appl.*, 337(2):1345–1365, 2008.

[16] L. Pick and M. Růžička. An example of a space $L^{p(x)}$ on which the Hardy-Littlewood maximal operator is not bounded. *Expo. Math.*, 19(4):369–371, 2001.

[17] V. Rabinovich and S. Samko. Boundedness and Fredholmness of pseudodifferential operators in variable exponent spaces. *Integral Equations Operator Theory*, 60(4):507–537, 2008.

Universität Bielefeld, Fakultät für Mathematik, Postfach 10 01 31, D-33501 Bielefeld, Germany

*Email address*: lars.diening@uni-bielefeld.de

Centro de Matemática e Aplicações, Departamento de Matemática, Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa, Quinta da Torre, 2829–516 Caparica, Portugal

*Email address*: oyk@fct.unl.pt

Department of Mathematics, King's College London, Strand, London WC2R 2LS, United Kingdom and Technische Universität Dresden, Fakultät Mathematik, 01062 Dresden, Germany

*Email address*: eugene.shargorodsky@kcl.ac.uk