Electronic Journal of Probability

Electron. J. Probab. **28** (2023), article no. 67, 1–59.
ISSN: 1083-6489 https://doi.org/10.1214/23-EJP952

A Markov process for a continuum infinite particle system with attraction*

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Abstract

An infinite system of point particles placed in $\mathbb{R}^d$ is studied. The particles are of two types; they perform random walks in the course of which those of distinct type repel each other. The interaction of this kind induces an effective multi-body attraction of the same type particles, which leads to the multiplicity of states of thermal equilibrium in such systems. The pure states of the system are locally finite counting measures on $\mathbb{R}^d$. The set of such states Γ^2 is equipped with the vague topology and the corresponding Borel σ-field. For a special class $\mathcal{P}_{\rm exp}$ of probability measures defined on Γ^2, we prove the existence of a family $\{P_{t,\mu} : t \geq 0, \ \mu \in \mathcal{P}_{\rm exp}\}$ of probability measures defined on the space of càdlàg paths with values in Γ^2, which is a unique solution of the restricted martingale problem for the mentioned stochastic dynamics. Thereby, the corresponding Markov process is specified.

Keywords: measure-valued Markov process; point process; martingale solution; Fokker-Planck equation; stochastic semigroup.
MSC2020 subject classifications: 60J25; 60J75; 60G55; 35Q84.
Submitted to EJP on July 17, 2022, final version accepted on May 1, 2023.

Contents

*Supported by the Deutsche Forschungsgemeinschaft (DFG) through the SFB 1238 “Taming uncertainty and profiting from randomness and low regularity in analysis, statistics and their applications”.
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1 Introduction

The stochastic dynamics of infinite systems of interacting particles placed in a continuous habitat is an actual and highly demanding subject of modern probability theory. In its comprehensive version, one deals with stochastic processes. Thus far, Markov processes have been constructed only for those particle systems where one cannot expect phase transitions, i.e., multiplicity of states of thermal equilibrium existing at the same values of the external parameters. In the present work, we deal with a system, for which such phase transitions are possible [6, 13, 15, 19], that ought to have an essential impact on its stochastic dynamics, cf. [14]. Namely, the object we study is an infinite collection of point particles of two types placed in $X = \mathbb{R}^d$, $d \geq 1$. The particles perform random walks (jumps) in the course of which those belonging to the same type (component) do not interact, whereas different type particles repel each other. This model can be viewed as a jump version of the Widom-Rowlinson model or the continuum two-state Potts model, see [6, 15] and [13, 19], respectively, as well as the literature quoted in these publications. By integrating out the coordinates of one of the components, one obtains a single-component particle system with a multi-body attraction (cf. subsect. 4.3.1 below), responsible for phase transitions – the multiplicity of states of thermal equilibrium, see [17] for more on this issue.

Similarly as in [16], we construct the process in question by solving a *restricted martingale problem*, cf. [9, page 79]. The basic aspects of this construction can be outlined as follows. The starting point is the evolution of states $\mu_0 \to \mu_t$ of the considered model obtained in [1, Theorem 3.5]; whereas the final outcome is the family of càdlàg path measures that solves the mentioned martingale problem. The one-dimensional marginals of these path measures coincide with the corresponding μ_t constructed in [1].

Let us now present the main ingredients of our theory. First, we set the state space – the collection Γ^2 of all possibly infinite configurations in X, by which we mean the following. Let γ be an integer valued Radon measure on X. For each ball $B_r(x) := \{y \in X : |x-y| \leq r\}$, $r > 0$, one thus has $\gamma(B_r(x)) \in \mathbb{N}_0$. For $x \in X$, we set $n_\gamma(x) = \inf_{r>0} \gamma(B_r(x))$, and also $p(\gamma) = \{x \in X : n_\gamma(x) > 0\}$. Each such γ can be associated with a locally finite system of point 'particles' such that each $x \in p(\gamma)$ is occupied by $n_\gamma(x)$ of them, cf. [18]. Keeping this in mind, we will call γ and $p(\gamma)$ *configuration* and *ground configuration*, correspondingly. The set of all such configurations γ is denoted by Γ. Since we are going to consider a two-component system, its state space is $\Gamma^2 = \Gamma \times \Gamma$, consisting of the pairs $\gamma = (\gamma_0, \gamma_1)$, $\gamma_i \in \Gamma$. In the sequel, γ without indices will always denote such a pair, whereas γ_i will stand for the configuration of particles of type $i = 0, 1$. Then the ground configuration of $\gamma = (\gamma_0, \gamma_1)$ is $p(\gamma) = p(\gamma_0) \cup p(\gamma_1)$. We also set $n_\gamma(x) = n_{\gamma_0}(x) + n_{\gamma_1}(x)$. If $n_\gamma(x) = 1$ for each $x \in p(\gamma)$, then γ is called a *simple* configuration. The set of all simple configurations is then

$$\breve{\Gamma}^2 = \{\gamma \in \Gamma^2 : \forall x \in p(\gamma) \ n_\gamma(x) = 1\}. \tag{1.1}$$

As mentioned above, the configurations are assumed to be locally finite, i.e., each γ_i takes finite values on every compact $\Lambda \subset X$. Let $x_1, x_2, \ldots,$ be an enumeration of a given γ_i. Then by $\sum_{x \in \gamma_i} g(x)$ we mean $\sum_j g(x_j)$, where $g : X \to \mathbb{R}$ is a suitable function. Obviously, this interpretation of $\sum_{x \in \gamma_i} g(x)$ is independent of the enumeration used in the second sum. Note also that $\sum_{x \in \gamma_i} g(x) = \int_X g(x)\gamma_i(dx)$, see (3.2) below as well as [18] for more on this subject. Then Γ is equipped with the vague topology, which is the weakest topology that makes continuous the maps $\gamma_i \mapsto \sum_{x \in \gamma_i} g(x)$, $g \in C_{\rm cs}(X)$, where the latter is the collection of all continuous compactly supported numerical functions. Correspondingly, the set $\Gamma^2 = \Gamma \times \Gamma$ is equipped with the product topology, and thereby with the Borel σ-field $\mathcal{B}(\Gamma^2)$. This allows us to employ probability measures defined thereon, the set of which is denoted by $\mathcal{P}(\Gamma^2)$. Their evolution is described by the Fokker-Planck equation

$$\mu_{t_2}(F) = \mu_{t_1}(F) + \int_{t_1}^{t_2} \mu_s(LF) ds, \qquad t_2 > t_1 \geq 0, \tag{1.2}$$

see [2] for a general theory of such and similar objects. In (1.2), we use the notation $\mu(F) = \int F d\mu$ and L is the Kolmogorov operator, which in the considered case has the following form

$$\begin{aligned} (LF)(\gamma) &= \sum_{x \in \gamma_0} \int_X a_0(x-y) \exp\left(- \sum_{z \in \gamma_1} \phi_0(z-y) \right) \left[F(\gamma \setminus x \cup_0 y) - F(\gamma) \right] dy \qquad (1.3) \\ &+ \sum_{x \in \gamma_1} \int_X a_1(x-y) \exp\left(- \sum_{z \in \gamma_0} \phi_1(z-y) \right) \left[F(\gamma \setminus x \cup_1 y) - F(\gamma) \right] dy. \end{aligned}$$

Here and in the sequel, by writing $\gamma \cup_i y$ we mean the element of Γ^2 obtained from γ by adding $y \in X$ to its component γ_i, $i = 0, 1$. Likewise, by writing $\gamma \setminus x$ we mean the configuration obtained from γ by subtracting x from the corresponding γ_i if it is clear which i is meant. Otherwise, we indicate it explicitly, see the next section for more detail.

The first summand in (1.3) describes the following elementary act: a particle located at $x \in \gamma_0$ instantly changes its position (jumps) to $y \in X$ with rate

$$c_0(x, y; \gamma) = a_0(x - y) \exp\left(- \sum_{z \in \gamma_1} \phi_0(z - y) \right). \tag{1.4}$$

It depends on γ_1 through the multiplier $\exp\left(- \sum_{z \in \gamma_1} \phi_0(z - y) \right)$, $\phi_0 \geq 0$, the role of which is diminishing the free jump rate $a_0(x - y)$ if the target point is 'close' to γ_1. In view of this, we shall call a_i and ϕ_i, $i = 0, 1$, jump and repulsion kernels, respectively.

As is typical for infinite particle systems, among the states $\mathcal{P}(\Gamma^2)$ one distinguishes a proper subset to which the evolution described by (1.2) is restricted. In [1], there was introduced a subset $\mathcal{P}_{\rm exp} \subset \mathcal{P}(\Gamma^2)$, cf. Definition 3.1 below, consisting of *sub-Poissonian measures*, and then constructed a map $t \mapsto \mu_t \in \mathcal{P}_{\rm exp}$ corresponding to (1.2) in the following sense. For a certain class of (unbounded) functions $F : \Gamma^2 \to \mathbb{R}$, it was shown that: (a) LF belongs to this class; (b) each such F is μ-integrable for all $\mu \in \mathcal{P}_{\rm exp}$; (c) the mentioned map satisfies (1.2). Our present results are essentially based on this construction. In a sense, we 'superpose' the mentioned map $t \mapsto \mu_t \in \mathcal{P}_{\rm exp}$ and obtain a family of càdlàg path measures $\{P_{s,\mu} : s \geq 0, \mu \in \mathcal{P}_{\rm exp}\}$, which is the unique solution of the *restricted initial value martingale problem* corresponding to (1.3), see [9, page 79], and is such that the one dimensional marginal of $P_{s,\mu}$ corresponding to $t > s$ is μ_t if $\mu_s = \mu$. This construction consists of the following steps:

(a) We pick a subset $\Gamma^2_* \subset \Gamma^2$ and equip it with a topology that makes this set a Polish space, continuously embedded in Γ^2, and such that $\mu(\Gamma^2_*) = 1$ for all $\mu \in \mathcal{P}_{\rm exp}$. This enlarges the set of continuous functions $F : \Gamma^2 \to \mathbb{R}$ and allows us to redefine the members of $\mathcal{P}_{\rm exp}$ as measures on Γ^2_*. Then we construct a sufficiently massive set $\mathcal{D}(L)$ of bounded continuous functions $F : \Gamma^2_* \to \mathbb{R}$, which will serve as the domain of the Kolmogorov operator. Its crucial property is that LF remains bounded for all $F \in \mathcal{D}(L)$.

(b) We prove that any solution $t \mapsto \mu_t \in \mathcal{P}(\Gamma^2_*)$ of the Fokker-Planck equation (1.2) with $F \in \mathcal{D}(L)$ and $\mu_0 \in \mathcal{P}_{\rm exp}$ is such that $\mu_t \in \mathcal{P}_{\rm exp}$ for all $t > 0$. Thereby, we prove that there is only one such solution given by the map $t \mapsto \mu_t \in \mathcal{P}_{\rm exp}$ constructed in [1].

(c) Then we introduce auxiliary models described by L^σ, $\sigma \in [0, 1]$ obtained by replacing $a_i(x - y) \to a_i^\sigma(x, y)$, in such a way that $L^0 = L$, whereas L^σ with $\sigma \in (0, 1]$ admits constructing transition functions p_t^σ, by means of which we obtain Markov processes $\mathcal{X}^\sigma$ with values in Γ^2_*.

(d) Thereafter, we prove that the finite-dimensional distributions of $\mathcal{X}^\sigma$ satisfy Chentsov-like estimates, uniformly in $\sigma \in (0, 1]$. By this we get that: (i) each $\mathcal{X}^\sigma$ has a càdlàg modification, which corresponds to the existence of families $\{P^\sigma_{s,\mu} : s \geq 0, \mu \in \mathcal{P}_{\rm exp}\}$, $\sigma \in (0, 1]$, of càdlàg path measures; (ii) as $\sigma \to 0$, the measures $P^\sigma_{s,\mu}$ have accumulation points which solve the restricted initial value martingale problem for $(L, \mathcal{D}(L), \mathcal{P}_{\rm exp})$. Then we prove that all these accumulation points coincide as their one-dimensional marginals solve (1.2), which has a unique solution, that was proved in (b). Thereby, we obtain the unique solution of the mentioned martingale problem $\{P_{s,\mu} : s \geq 0,\ \mu \in \mathcal{P}_{\rm exp}\}$.

(e) Finally, we prove that the constructed Markov process with probability one takes values in $\Gamma^2_* \cap \breve{\Gamma}^2$, see (1.1).

The structure of this paper is as follows. In Sect. 2, we collect the notations used herein. In Sect. 3, we introduce the main ingredients of our construction, among which are the spaces of tempered configurations Γ^2_*, $\breve{\Gamma}^2_*$, the basic classes of bounded continuous functions $F : \Gamma^2_* \to \mathbb{R}$, and the set of *sub-Poissonian measures* $\mathcal{P}_{\rm exp} \subset \mathcal{P}(\Gamma^2)$, see Definition 3.1. In Sect. 4, we formulate our assumptions concerning the properties of the parameters a_i and ϕ_i that appear in (1.3), (1.4). Next, we introduce the corresponding spaces of càdlàg paths and the very notion of a solution of the restricted initial value martingale problem for our model, see Definition 4.3. Then the result of this work is formulated in Theorem 4.5, followed by a number of comments. The remaining sections are dedicated to the proof of Theorem 4.5. In Sect. 5, we reformulate the corresponding results of [1] in the form adapted to the present context, as well as develop a number of additional technicalities. The basic result of Sect. 6 is Lemma 6.1 which states that every solution of the Fokker-Planck equation (1.2) for our model lies in $\mathcal{P}_{\rm exp}$ whenever μ_0 is in $\mathcal{P}_{\rm exp}$. Its proof is mostly based on combinatorial estimates obtained in [16] and those derived here in subsect. 5.2. Then we prove that (1.2) has a unique solution $t \mapsto \mu_t$, constructed in fact in [1], see Lemma 6.3. By means of Lemmas 6.1 and 6.3 we then prove that the martingale problem can have at most one solution. In Sect. 7, we introduce L^σ and show that the solution $t \mapsto \mu^\sigma_t$ of the Fokker-Planck equation for L^σ, $\sigma \in (0,1]$, has the property $\mu^\sigma_t \Rightarrow \mu_t$ as $\sigma \to 0$, where μ_t is the solution corresponding to the main model and $\Rightarrow$ denotes weak convergence. In Sect. 8, we obtain the evolution of states $t \mapsto \hat{\mu}^\sigma_t$, $\sigma \in (0,1]$, by constructing stochastic semigroups $S^\sigma = \{S^\sigma(t)\}_{t\geq 0}$ acting in the Banach space of signed measures on Γ^2_*, see Lemma 8.3. This construction becomes possible due to the modification $a_i(x-y) \to a^\sigma_i(x,y)$ and is based on a perturbation technique developed in [23]. By construction, $t \mapsto \hat{\mu}^\sigma_t$ solves the Fokker-Planck equation for L^σ, which by Lemma 6.3 yields $\hat{\mu}^\sigma_t = \mu^\sigma_t$. At the same time, by means of the semigroups S^σ we get the corresponding transition functions p^σ_t, and thus Markov processes $\mathcal{X}^\sigma$ with values in Γ^2_*. Thereafter, in Lemma 8.6 we show that these processes satisfy Chentsov-like estimates, uniform in $\sigma \in (0,1]$. By means of this result, in Sect. 9 we complete the proof of Theorem 4.5, including the property mentioned in item (e) above.

2 Notations

In view of the size of this work, for the reader convenience we collect here essential notations and notions used throughout the paper.

2.1 Sets and spaces

- The considered particle system dwells in $X = \mathbb{R}^d$, $d \geq 1$. By Λ we always denote a compact subset of X, its Euclidean volume is denoted $|\Lambda|$; $\mathbb{R}_+ = [0, +\infty)$; $\mathbb{N}$ – the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$; $B_r(y) = \{x \in \mathbb{R}^d : |x-y| \leq r\}$, $B_r = B_r(0)$, $r > 0$ and $y \in \mathbb{R}^d$. For a finite set Δ, by $|\Delta|$ we mean its cardinality.
- A Polish space is a separable topological space, the topology of which is consistent with a complete metric, see, e.g., [8, Chapt. 8]. Subsets of such spaces are usually denoted by $\mathbb{A}, \mathbb{B}$, whereas A, B (with indices) are reserved for denoting linear operators. For a Polish space E, by $C_{\rm b}(E)$ and $B_{\rm b}(E)$ we denote the sets of bounded continuous and bounded measurable functions $g : E \to \mathbb{R}$, respectively; $\mathcal{B}(E)$ denotes the Borel σ-field of subsets of E. By $\mathcal{P}(E)$ we denote the set of all probability measures defined on $(E, \mathcal{B}(E))$. For a suitable set Δ, by $\mathbb{1}_\Delta$ we denote the indicator of Δ.
- Γ stands for the set of all locally finite counting measures on X, interpreted also as configurations of point particles with possible multiple locations, see [18] and (3.1) below. By Γ_0 we mean the subset of Γ consisting of finite configurations, i.e., such that $\gamma(X) < \infty$; by Γ^2 we denote the set of configurations of the two-component particle system which we consider. That is, $\Gamma^2 = \Gamma \times \Gamma$ consists of

$\gamma = (\gamma_0, \gamma_1)$, $\gamma_i \in \Gamma$ with $i = 0, 1$ always indicating the particle type. The set of simple configurations $\breve{\Gamma}^2$ is defied in (1.1). The set of tempered configurations Γ_* is defined in (3.32) by means of $\psi(x) = (1 + |x|^{d+1})^{-1}$; then $\Gamma^2_* = \Gamma_* \times \Gamma_*$, see also (3.31). The metric properties of Γ^2_* are described in Lemma 3.4. Finally, $\breve{\Gamma}^2_*$ stands for $\Gamma^2_* \cap \breve{\Gamma}^2$, see (3.31). The relationships between these sets (Polish spaces) are described in Corollary 3.5.

- By $\mathcal{P}_{\rm exp}$ we denote the set of sub-Poissonian measures, see Definition 3.1, which is one of the basic notions of this research. Their essential properties are given in Proposition 3.2 and (3.34). By $\mathcal{M}$ (with indices) we denote the Banach spaces of signed measures on Γ^2_*, see also (8.1), (8.2), (8.4).
- By $\mathfrak{D}_{[s,+\infty)}(\Gamma^2_*)$ and $\mathfrak{D}_{[s,+\infty)}(\breve{\Gamma}^2_*)$, $s \geq 0$, we denote the spaces of càdlàg maps $\gamma : [s, +\infty) \to \Gamma^2_*$ and $\gamma : [s, +\infty) \to \breve{\Gamma}^2_*$, respectively. Equipped with Skorohod's metric they become Polish metric spaces, see subsect. 3.2.

2.2 Functions, measures, operators

- By x, y, z we always denote elements (points) of $X = \mathbb{R}^d$; for $k \in \mathbb{N}$, we write $\mathbf{x}^k = (x_1, \dots, x_k) \in X^k$. By small letters $f, g, u, v, \theta, \psi, \phi$ we denote numerical functions defined on X or X^k. Important classes of such functions Θ_ψ, Θ^+_ψ are defined in (3.38). By means of the function $\psi_\sigma(x) = (1 + \sigma|x|^{d+1})^{-1}$ we modify the model (1.3). For $v_{i,1}, \dots v_{i,k} \in C_{\rm b}(X)$, $i = 0, 1$ and $k \in \mathbb{N}$, we write $\mathbf{v}_i(\mathbf{x}) = v_{i,1}(x_1) \cdots v_{i,k}(x_k)$, see (3.54) and also (3.4). For an integrable $\theta : X \to \mathbb{R}_+$, we write $\langle \theta \rangle = \int_X \theta(x) dx$. Numerical functions defined of Γ^2 or Γ^2_* are denoted by capital letters F, H, etc. By F with indices we usually denote functions on Γ^2, Γ^2_*, whereas G (with indices) are defined on finite configurations. Significant examples of such functions are F^θ, see (3.16), Ψ (3.30), $\widetilde{F}^\theta_\tau$, see (3.49) and Proposition 3.8, $\widehat{F}^m_\tau$ (3.55). The class of functions $\mathcal{D}(L)$ is introduced in Definition 4.1, its properties are given in Proposition 4.2. By F, G, K we denote numerical functions defined on the path spaces.
- Measures on Γ^2 and Γ^2_* are usually denoted by μ with indices. Their correlation measures $\chi^{(m)}_\mu$ are defined in (3.7). By π_κ we denote the Poisson measure, see (3.31). Probability measures on the path spaces $\mathfrak{D}_{[s,+\infty)}(\Gamma^2_*)$ are denoted by P with indices. For suitable measure and function, we write $\mu(f) = \int f d\mu$.
- By L and L^σ we denote the Kolmogorov operator (1.3) and its modifications. By $L^{\dagger,\sigma}$ we denote the operators dual to L^σ, see (8.13), (8.14). Their domains are set in (8.16). By $\widehat{L}$ and L^Δ with indices we denote the counterparts of L acting on functions G and correlation functions, respectively, see (5.23) and (5.9).

3 Preliminaries

3.1 Configurations spaces and correlation measures

By Γ we denote the standard set of Radon counting measures on X, i.e., $\gamma(\Lambda) \in \mathbb{N}_0$ for each $\gamma \in \Gamma$ and a compact $\Lambda \subset X$. Then we also define $n_\gamma(x) = \inf_{r>0} \gamma(B_r(x))$ and $p(\gamma) = \{x \in X : n_\gamma(x) > 0\}$. Thus, $p(\gamma)$ is a locally finite subset of X, see (1.1). For $x \in p(\gamma)$, by $\gamma \setminus x$ we denote the element of Γ such that $n_{\gamma\setminus x}(x) = n_\gamma(x) - 1$ and $n_{\gamma\setminus x}(y) = n_\gamma(y)$ whenever $y \neq x$. Similarly, $\gamma \cup x$, $x \in X$, denotes the measure such that $n_{\gamma\cup x}(x) = n_\gamma(x) + 1$ and $n_{\gamma\cup x}(y) = n_\gamma(y)$ for $y \neq x$. For simplicity, with a certain abuse of notations we write

$$\sum_{x\in\gamma} g(x) = \int_X g(x)\gamma(dx) = \sum_{x\in p(\gamma)} n_\gamma(x) g(x), \tag{3.1}$$

where g is a positive numerical function. Note that the left-hand side of (3.1) can also be interpreted as $\sum_j g(x_j)$ for a certain enumeration $\mathbb{N} \ni j \mapsto x_j$ of the elements of $p(\gamma)$, in which each $x \in p(\gamma)$ is repeated $n_\gamma(x)$ times, see [18] for more detail. In the same way, we will understand sums

$$\sum_{x\in\gamma}\sum_{y\in\gamma\setminus x} g(x,y) = \int_X\int_X g(x,y)\gamma(dx)\gamma(dy) - \int_X g(x,x)\gamma(dx),$$

that can also be generalized to all $m \in \mathbb{N}$

$$\begin{aligned}
&\sum_{x_1\in\gamma}\sum_{x_2\in\gamma\setminus x_1}\cdots\sum_{x_m\in\gamma\setminus\{x_1,\dots,x_{m-1}\}} g(x_1,\dots x_m) \qquad (3.2)\\
&= \sum_{G\in\mathbb{K}_m}(-1)^{l_G}\int_{X^{n_G}} g_G(y_1,\dots,y_{n_G})\gamma(dy_1)\cdots\gamma(dy_{n_G})\\
&= \sum_{G\in\mathbb{K}_m}(-1)^{l_G}\sum_{y_1\in\gamma}\cdots\sum_{y_{n_G}\in\gamma} g_G(y_1,\dots,y_{n_G}),
\end{aligned}$$

where $\mathbb{K}_m$ is the collection of all graphs with vertices $\{1,2,\dots,m\}$, l_G and n_G are the number of edges and the connected components of G, respectively, whereas $g_G(y_1,\dots,y_{n_G})$ is obtained from $g(x_1,\dots,x_m)$ by setting the arguments $x_{l_1},\dots x_{l_{s_j}}$ of the latter equal y_j where $l_1,\dots l_{s_j}$ are the vertices of the j-th connected component of G.

Since the particles which we consider are of two types, their pure states are set to be pairs $\gamma = (\gamma_0,\gamma_1)$ such that $\gamma_i \in \Gamma$, $i = 0,1$. Thus, $\Gamma^2 = \Gamma\times\Gamma$ is the set of all pure states of the system. Correspondingly, we set $n_\gamma(x) = n_{\gamma_0}(x) + n_{\gamma_1}(x)$ and $p(\gamma) = p(\gamma_0)\cup p(\gamma_1)$. We will call $p(\gamma)$ the *ground* configuration for γ.

For $\gamma \in \Gamma^2$ and $m = (m_0,m_1) \in \mathbb{N}_0^2$, the counting measure $Q_\gamma^{(m)}$ on $X^{m_0}\times X^{m_1}$ is defined by its values on compact subsets $\Delta \subset X^{m_0}\times X^{m_1}$ in the following way. For $m_0 = m_1 = 0$, we set $Q_\gamma^{(m)} \equiv 1$ for each γ. For $m_0 > 0$, $m_1 = 0$ and $\Delta \subset X^{m_0}$, $Q_\gamma^{(m)}(\Delta)$ is equal to the number of different ordered strings $(i_1,\dots,i_{m_0})$ such that $\mathbf{x} := (x_{i_1},\dots,x_{i_{m_0}}) \in \Delta$. Likewise one defined $Q_\gamma^{(m)}$ for $m_0 = 0$ and $m_1 > 0$. For $m \in \mathbb{N}^2$, $Q_\gamma^{(m)}(\Delta)$ is equal to the number of different ordered strings $(i_1,\dots,i_{m_0})$ and $(j_1,\dots,j_{m_1})$ such that $(\mathbf{x},\mathbf{y}) \in \Delta$, where $\mathbf{x} = (x_{i_1},\dots,x_{m_0})$, $x_l \in \gamma_0$, and $\mathbf{y} = (y_{j_1},\dots,y_{m_1})$, $y_l \in \gamma_1$. It is obvious that this definition is independent of the enumerations of both γ_i. Then we get, cf. (3.2),

$$\begin{aligned}
&Q_\gamma^{(m)}(\Delta) \qquad (3.3)\\
&= \sum_{x_1\in\gamma_0}\sum_{x_2\in\gamma_0\setminus x_1}\cdots\sum_{x_{m_0}\in\gamma_0\setminus\{x_1,\dots,x_{m_0-1}\}}\sum_{y_1\in\gamma_1}\sum_{y_2\in\gamma_1\setminus y_1}\cdots\sum_{y_{m_1}\in\gamma_1\setminus\{y_1,\dots,y_{m_1-1}\}} \mathbb{1}_\Delta(\mathbf{x},\mathbf{y}).
\end{aligned}$$

To simplify notations, for suitable φ_0, φ, $k \in \mathbb{N}$ and $m \in \mathbb{N}_0^2$, we write

$$\begin{aligned}
&\sum_{\mathbf{x}^k\in\gamma_0}\varphi_0(\mathbf{x}^k) = \sum_{x_1\in\gamma_0}\sum_{x_2\in\gamma_0\setminus x_1}\cdots\sum_{x_k\in\gamma_0\setminus\{x_1,\dots,x_{k-1}\}}\varphi_0(x_1,\dots,x_k), \qquad (3.4)\\
&\sum_{(\mathbf{x}^{m_0},\mathbf{y}^{m_1})\in\gamma}\varphi(\mathbf{x}^{m_0},\mathbf{y}^{m_1}) = \sum_{\mathbf{x}^{m_0}\in\gamma_0}\sum_{\mathbf{y}^{m_1}\in\gamma_1}\varphi(x_1,\dots,x_{m_0},y_1,\dots,y_{m_1}).
\end{aligned}$$

As above, we will write $\mathbf{x}$ instead of $\mathbf{x}^k$ if the dimension k is clear from the context. For a compact $\Lambda \subset X$ and $\gamma_i \in \Gamma$, $i = 0,1$, we let $N_\Lambda(\gamma_i)$ be the number of the elements of γ_i contained in Λ. Then

$$N_\Lambda(\gamma_i) = \sum_{x\in\gamma_i}\mathbb{1}_\Lambda(x) = \gamma_i(\Lambda), \qquad (3.5)$$

that is, $N_\Lambda(\gamma_i) = Q^{(m)}_\gamma(\Lambda)$ for the corresponding m, see (3.3). For $p \in \mathbb{N}$, we have, cf. [16, page 8],

$$N^p_\Lambda(\gamma_i) = \left[\sum_{x\in\gamma_i} \mathbb{1}_\Lambda(x)\right]^p = \sum_{l=1}^p S(p,l) \sum_{\mathbf{x}^l\in\gamma_i} \mathbb{1}_\Lambda(x_1)\cdots \mathbb{1}_\Lambda(x_l), \tag{3.6}$$

where $S(p,l)$ is Stirling's number of second kind = the number of ways to divide p labeled items into l unlabeled groups. Below, expressions like that on the right-hand side of (3.6) with $p = 0$ are set to be identically equal to one.

It can be shown, cf. [18, Theorem 1], that the map $\gamma \mapsto Q^{(m)}_\gamma(\Delta)$ is measurable for all compact Δ and $m \in \mathbb{N}^2_0$. However, it may be unbounded. Let $\mathcal{P}(\Gamma^2)$ be the set of all probability measures defined on the Polish space Γ^2. For a given $\mu \in \mathcal{P}(\Gamma^2)$ and $m \in \mathbb{N}^2_0$, set

$$\chi^{(m)}_\mu = \int_{\Gamma^2} Q^{(m)}_\gamma \mu(d\gamma), \tag{3.7}$$

which exists for at least $m = (0,0)$. If it does for a given positive m, we call it *correlation measure* corresponding to these μ and m. If $\chi^{(m)}_\mu(\Delta) < \infty$ for all $m \in \mathbb{N}^2_0$ and compact Δ, we say that μ has *finite correlations*. In this case, each $\chi^{(m)}_\mu$ is a Radon measure on $X^{m_0} \times X^{m_1}$.

3.2 Sub-Poissonian measures

We begin by recalling that $C_{\rm cs}(X)$ is dense in $L^1(X) := L^1(X,dx)$, see e.g., [5, Theorem 4.12, page 97].

For $k \in \mathbb{N}$ and $\theta \in C_{\rm cs}(X)$, by $\theta^{\otimes k}$ we denote the function such that $\theta^{\otimes k}(x_1,\dots,x_k) = \theta(x_1)\cdots\theta(x_k)$, which we extend to $k = 0$ by setting $\theta^{\otimes 0} \equiv 1$. Likewise, for $\theta_0, \theta_1 \in C_{\rm cs}(X)$ and $m \in \mathbb{N}^2_0$, we set

$$\theta^{\otimes m}(\mathbf{x},\mathbf{y}) = \theta_0(x_1)\cdots\theta_0(x_{m_0})\theta_1(y_1)\cdots\theta_1(y_{m_1}). \tag{3.8}$$

Definition 3.1. *The set of sub-Poissonian measures $\mathcal{P}_{\rm exp}$ consists of all those $\mu \in \mathcal{P}(\Gamma^2)$ that have finite correlations such that, for each $m \in \mathbb{N}^2_0$ and $\theta_0, \theta_1 \in C_{\rm cs}(X)$, the following holds*

$$\left|\chi^{(m)}_\mu(\theta^{\otimes m})\right| \le \varkappa^{|m|} \|\theta_0\|^{m_0}_{L^1(X)} \|\theta_1\|^{m_1}_{L^1(X)}, \qquad |m| := m_0 + m_1, \tag{3.9}$$

with some μ-dependent $\varkappa > 0$.

The aforementioned density and the estimate in (3.9) imply that the map $(\theta_0,\theta_1) \mapsto \chi^{(m)}_\mu(\theta^{\otimes m})$ can be extended to a continuous homogeneous polynomial on $L^1(X) \times L^1(X)$. In this case, there exists a unique positive and symmetric $k^{(m)}_\mu \in L^\infty(X^{m_0} \times X^{m_1})$ such that, see (3.8),

$$\begin{aligned} \chi^{(m)}_\mu(\theta^{\otimes m}) &= \int_{X^{m_0}\times X^{m_1}} k^{(m)}_\mu(x_1,\dots,x_{m_0}; y_1,\dots,y_{m_1}) \\ &\times \theta_0(x_1)\cdots\theta_0(x_{m_0})\theta_1(y_1)\cdots\theta_1(y_{m_1}) dx_1\cdots dx_{m_0} dy_1\cdots dy_{m_1} \\ &=: \int_{X^{m_0}\times X^{m_1}} k^{(m)}_\mu(\mathbf{x},\mathbf{y})\theta^{\otimes m}(\mathbf{x},\mathbf{y}) d^{m_0}\mathbf{x} d^{m_1}\mathbf{y} =: \langle\!\langle k^{(m)}_\mu, \theta^{\otimes m}\rangle\!\rangle. \end{aligned} \tag{3.10}$$

The mentioned symmetricity means that

$$k^{(m)}_\mu(x_1,\dots,x_{m_0}; y_1,\dots,y_{m_1}) = k^{(m)}_\mu(x_{\sigma_0(1)},\dots,x_{\sigma_0(m_0)}; y_{\sigma_1(1)},\dots,y_{\sigma_1(m_1)}), \tag{3.11}$$

holding for all corresponding permutations σ_0, σ_1, whereas the positivity and the bound in (3.9) yield

$$0 \le k^{(m)}_\mu(\mathbf{x},\mathbf{y}) \le \varkappa^{|m|}, \tag{3.12}$$

holding for Lebesgue-almost all $(\mathbf{x},\mathbf{y}) \in X^{m_0} \times X^{m_1}$. The upper estimate in (3.12) is known as Ruelle's bound [20]. Noteworthy, for each μ,

$$k_\mu^{(0,0)} = 1, \tag{3.13}$$

which readily follows by the very definition of the counting measure Q_γ and (3.7).

For $m = (m_0, m_1) \in \mathbb{N}_0^2$, $\theta = (\theta_0, \theta_1)$, $\theta_i \in C_{\rm cs}(X)$, $i = 0, 1$, we set, cf. (3.3),

$$\begin{aligned} H_\theta^{(m)}(\gamma) &= H_{\theta_0}^{(m_0)}(\gamma_0) H_{\theta_1}^{(m_1)}(\gamma_1), \qquad (3.14)\\ H_{\theta_i}^{(m_i)}(\gamma_i) &= \sum_{x_1\in\gamma_i} \sum_{x_2 \in \gamma_i \setminus x_1} \cdots \sum_{x_{m_i} \in \gamma_i \setminus \{x_1, \dots, x_{m_i-1}\}} \theta_i(x_1)\cdots\theta_i(x_{m_i}),\\ &= \sum_{\mathbf{x}\in\gamma_i} \theta_i^{\otimes m_i}(\mathbf{x}), \qquad m_i \geq 1, \qquad i = 0,1, \end{aligned}$$

and $H_{\theta_i}^{(0)}(\gamma_i) \equiv 1$. Then by means of (3.7) we rewrite (3.10) as follows

$$\chi_\mu^{(m)}(\theta^{\otimes m}) = \mu(H_\theta^{(m)}).$$

Now for $n = (n_0, n_1) \in \mathbb{N}_0^2$, let us consider

$$\bar{H}_\theta^{(n)}(\gamma) = \sum_{m_0=0}^{n_0} \sum_{m_1=0}^{n_1} \frac{1}{m_0! m_1!} H_\theta^{(m)}(\gamma),$$

which is obviously finite for all $\gamma \in \Gamma$. For every $\mu \in \mathcal{P}_{\rm exp}$, by (3.9) we have that

$$\mu(\bar{H}_\theta^{(n)}) \leq \exp\left[\varkappa(\|\theta_0\|_{L^1(X)} + \|\theta_1\|_{L^1(X)})\right], \tag{3.15}$$

where $\varkappa$ is as in (3.9). By (3.15), for $\theta = (\theta_0, \theta_1) \in L^1(X) \times L^1(X)$, the sequence $\{\bar{H}_\theta^{(n)}(\gamma)\}_{n\in\mathbb{N}_0^2}$ is μ-almost everywhere convergent to

$$\begin{aligned} F^\theta(\gamma) &= F^{\theta_0}(\gamma_0) F^{\theta_1}(\gamma_1), \qquad (3.16)\\ F^{\theta_i}(\gamma_i) &:= \prod_{x\in\gamma_i} (1 + \theta_i(x)) = \exp\left(\sum_{x\in\gamma_i} \log(1+\theta_i(x))\right). \end{aligned}$$

Moreover, by (3.15) it follows that each F^θ, $\theta \in L^1(X) \times L^1(X)$ is μ-integrable and

$$\mu(F^\theta) \leq \exp\left[\varkappa(\|\theta_0\|_{L^1(X)} + \|\theta_1\|_{L^1(X)})\right]. \tag{3.17}$$

This means that the map $L^1(X) \times L^1(X) \ni \theta \mapsto \mu(F^\theta) \in \mathbb{R}$ is an exponential type real entire function, which is reflected in the notation $\mathcal{P}_{\rm exp}$. Then borrowing terminology from the theory of entire functions, we will call the *type of* μ the least $\varkappa$ that verifies (3.9), (3.12). For the homogeneous Poisson measure π_κ, $\kappa = (\kappa_0, \kappa_1)$, $\kappa_0, \kappa_1 > 0$, we have

$$k_{\pi_\kappa}^{(m)}(\mathbf{x},\mathbf{y}) = \kappa_0^{m_0} \kappa_1^{m_1}, \qquad (\mathbf{x},\mathbf{y}) \in X^{m_0} \times X^{m_1}, \tag{3.18}$$

which yields, see (3.10) and (3.17),

$$\pi_\kappa(F^\theta) = \exp\left(\kappa_0 \int_X \theta_0(x)dx + \kappa_1 \int_X \theta_1(x)dx\right). \tag{3.19}$$

Hence, the type of $\pi_\kappa \in \mathcal{P}_{\rm exp}$ is $\varkappa = \max\{\kappa_0; \kappa_1\}$. In general, a Poisson measure, π_χ, is completely characterized by the pair $\chi = (\chi_0, \chi_1)$ of its intensity measures in such a way that, see (3.10),

$$\chi_{\pi_\chi}^{(m)}(\theta^{\otimes m}) = \left(\int_X \theta_0(x)\chi_0(dx)\right)^{m_0} \left(\int_X \theta_1(x)\chi_1(dx)\right)^{m_1}.$$

Note that π_χ is sub-Poissonian in the sense of Definition 3.1 if and only if $\chi_i(dx) = \varrho_i(x)dx$ with $\varrho_i \in L^\infty(X)$, $i = 0, 1$.

For a symmetric $G^{(m)} \in C_{\rm cs}(X^{m_0} \times X^{m_1})$, see (3.11), by (3.3) we have, cf. (3.4),

$$Q^{(m)}_\gamma (G^{(m)}) = \sum_{(\mathbf{x},\mathbf{y})\in\gamma} G^{(m)}(\mathbf{x},\mathbf{y}) =: m_0! m_1! (KG^{(m)})(\gamma), \tag{3.20}$$

which by (3.7) yields

$$\begin{aligned} \chi^{(m)}_\mu (G^{(m)}) &= \int_{X^{m_0}\times X^{m_1}} k^{(m)}_\mu (\mathbf{x},\mathbf{y}) G^{(m)}(\mathbf{x},\mathbf{y}) d^{m_0}\mathbf{x} d^{m_1}\mathbf{y} \\ &= m_0! m_1! \mu(KG^{(m)}) =: m_0! m_1! \langle\!\langle k^{(m)}_\mu, G^{(m)} \rangle\!\rangle. \end{aligned} \tag{3.21}$$

In view of (3.12), this can be continued to all $G^{(m)} \in L^1(X^{m_0} \times X^{m_1})$. For positive $G^{(m)}$, by (3.12) one also gets

$$\mu(KG^{(m)}) \leq \pi_\kappa(KG^{(m)}), \qquad \kappa_0 = \kappa_1 = \varkappa, \tag{3.22}$$

which, in particular, justifies the name *sub-Poissonian*. Let us now consider the following important version of (3.22). For a compact $\Lambda \subset X$, we let $N_\Lambda(\gamma) = N_\Lambda(\gamma_0) + N_\Lambda(\gamma_1)$, see (3.5). Then for $n \in \mathbb{N}$, by (3.6) we have

$$N^n_\Lambda(\gamma) = \sum_{p=0}^n \binom{n}{p} \sum_{l_0=1}^p \sum_{l_1=1}^{n-p} S(p,l_0) S(n-p,l_1) \sum_{(\mathbf{x}^{l_0},\mathbf{y}^{l_1})\in\gamma} \mathbb{1}_\Lambda(\mathbf{x}^{l_0},\mathbf{y}^{l_1}),$$

which for $\mu \in \mathcal{P}_{\rm exp}$ yields

$$\begin{aligned} \mu(N^n_\Lambda) &= \sum_{p=0}^n \binom{n}{p} \sum_{l_0=1}^p \sum_{l_1=1}^{n-p} S(p,l_0) S(n-p,l_1) \\ &\times \int_{X^{l_0}\times X^{l_1}} k^{(l_0,l_1)}_\mu (\mathbf{x}^{l_0},\mathbf{y}^{l_1}) \mathbb{1}_\Lambda(\mathbf{x}^{l_0},\mathbf{y}^{l_1}) d\mathbf{x}^{l_0} d\mathbf{y}^{l_1} \\ &\leq \sum_{p=0}^n \binom{n}{p} \sum_{l_0=1}^p \sum_{l_1=1}^{n-p} S(p,l_0) S(n-p,l_1) (\varkappa|\Lambda|)^{l_0+l_1} \\ &= \sum_{p=0}^n \binom{n}{p} T_p(\varkappa|\Lambda|) T_{n-p}(\varkappa|\Lambda|) = T_n(2\varkappa|\Lambda|), \end{aligned} \tag{3.23}$$

where $|\Lambda|$ is the Euclidean volume (Lebesgue measure) of Λ and T_n, $n \in \mathbb{N}$, are Touchard's polynomials, attributed also to J. A. Grunert, S. Ramanujan, and others, see [3, page 6]. Along with the already mentioned ones, sub-Poissonian measures have the following significant property. Recall that the set of simple configurations $\breve{\Gamma}^2$ is defined in (1.1).

Proposition 3.2. *For each $\mu \in \mathcal{P}_{\rm exp}$, it follows that $\mu(\breve{\Gamma}^2) = 1$.*

Proof. For a compact $\Lambda \subset X$, $N \in \mathbb{N}$ and $\epsilon \in (0,1)$, we set

$$h_{\Lambda,N}(x,y) = \mathbb{1}_\Lambda(x)\mathbb{1}_\Lambda(y) \min\{N; |x-y|^{-d\epsilon}\}, \quad x,y \in X, \tag{3.24}$$

$$H_{\Lambda,N}(\gamma) = \sum_{x\in\gamma_0} \sum_{y\in\gamma_0\setminus x} h_{\Lambda,N}(x,y) + \sum_{x\in\gamma_1} \sum_{y\in\gamma_1\setminus x} h_{\Lambda,N}(x,y) + \sum_{x\in\gamma_0} \sum_{y\in\gamma_1} h_{\Lambda,N}(x,y).$$

According to (3.21) and (3.12) we have

$$\mu(H_{\Lambda,N}) = \int_{\Lambda^2} \left(k_\mu^{(2,0)}(x,y) + k^{(1,1)}(x,y) + k_\mu^{(0,2)}(x,y)\right) h_{\Lambda,N}(x,y)dxdy \leq 3\varkappa^2 \mathcal{I}_{\Lambda,N}, \tag{3.25}$$

$$\mathcal{I}_{\Lambda,N} := \int_{\Lambda^2} h_{\Lambda,N}(x,y)dxdy =: \mathcal{I}^{(1)}_{\Lambda,N}(r) + \mathcal{I}^{(2)}_{\Lambda,N}(r),$$

where, for a certain $r > 0$, we set and then get

$$\mathcal{I}^{(1)}_{\Lambda,N}(r) = \int_\Lambda \left(\int_{\Lambda\cap B_r(x)} h_{\Lambda,N}(x,y)dy\right) dx \leq \int_\Lambda \left(\int_{B_r} \frac{dz}{|z|^{d\epsilon}}\right) dx = \frac{c_d r^{d(1-\epsilon)}}{d(1-\epsilon)}|\Lambda|, \tag{3.26}$$

$$\mathcal{I}^{(2)}_{\Lambda,N}(r) = \int_\Lambda \left(\int_{\Lambda\cap B^c_r(x)} h_{\Lambda,N}(x,y)dy\right) dx \leq \frac{1}{r^{d\epsilon}}|\Lambda|^2.$$

Here $B^c_r(x) = X \setminus B_r(x)$, and $|\Lambda|$ and c_d/d denote the Euclidean volume of Λ and the unit ball in X, respectively. We apply these estimates in (3.25) and obtain that

$$\mu(H_{\Lambda,N}) \leq C_{\mu,\Lambda},$$

for a suitable $C_{\mu,\Lambda}$ that is independent of N. Clearly, $0 \leq \mu(H_{\Lambda,N}) \leq \mu(H_{\Lambda,N+1})$, which by the monotone convergence theorem yields that the pointwise limit

$$\lim_{N\to+\infty} H_{\Lambda,N}(\gamma) =: H_\Lambda(\gamma) \tag{3.27}$$

is finite for μ-almost all γ, i.e., for all γ belonging to some $\breve{\Gamma}^2_{\mu,\Lambda}$ such that $\mu(\breve{\Gamma}^2_{\mu,\Lambda}) = 1$. For $C > 0$, we set $\breve{\Gamma}^2_C = \{\gamma : H_\Lambda(\gamma) \leq C\}$. Then $|x-y| \geq C^{-1/d\epsilon}$ for each $x, y \in (\gamma_0\cap\Lambda)\cup(\gamma_1\cap\Lambda)$ and each $\gamma \in \breve{\Gamma}^2_C$. That is, $\gamma_\Lambda := \gamma \cap \Lambda = (\gamma_0 \cap \Lambda, \gamma_1 \cap \Lambda)$ is simple whenever $\gamma \in \breve{\Gamma}^2_C$. This yields that $\cup_{k\in\mathbb{N}_0}\breve{\Gamma}^2_{C+k} \subset \breve{\Gamma}^2_{s,\Lambda}$, where the latter is the set of all those $\gamma \in \Gamma^2$ for which γ_Λ is simple. At the same time, $\cup_{k\in\mathbb{N}_0}\breve{\Gamma}^2_{C+k} \supset \breve{\Gamma}^2_{\mu,\Lambda}$; hence, $\mu(\breve{\Gamma}^2_{s,\Lambda}) = 1$. Note that $\breve{\Gamma}^2_{s,\Lambda}$ is an open subset of Γ^2, cf. the proof of Lemma 3.4 below. Now we take an ascending sequence $\{\Lambda_k\}$ that exhausts X, and obtain

$$\breve{\Gamma}^2 = \bigcap_k \breve{\Gamma}^2_{s,\Lambda_k}, \tag{3.28}$$

which completes the proof. □

3.3 Tempered configurations

Since we are going to essentially exploit the sub-Poissonian measures, it might be reasonable to restrict our consideration to subsets of Γ^2 the complements of which are null-sets for each $\mu \in \mathcal{P}_{\rm exp}$. To this end, we introduce the following function of $x \in X := \mathbb{R}^d$

$$\psi(x) = \frac{1}{1+|x|^{d+1}}, \qquad \langle\psi\rangle := \int_X \psi(x)dx, \tag{3.29}$$

and also

$$\Psi(\gamma) = \Psi(\gamma_0) + \Psi(\gamma_1) = \sum_{x\in\gamma_0} \psi(x) + \sum_{y\in\gamma_1} \psi(y). \tag{3.30}$$

Then we define

$$\Gamma^{(n)}_* = \{\gamma \in \Gamma^2 : \Psi(\gamma) \leq n\}, \quad \Gamma^2_* = \bigcup_{n\in\mathbb{N}} \Gamma^{(n)}_*, \quad \breve{\Gamma}^2_* = \Gamma^2_* \cap \breve{\Gamma}^2. \tag{3.31}$$

Elements of Γ^2_* (resp. $\breve{\Gamma}^2_*$) are called *tempered configurations* (resp. *tempered simple configurations*). Clearly, $\Gamma^2_* \in \mathcal{B}(\Gamma^2)$ as $\Gamma^{(n)}_* \in \mathcal{B}(\Gamma^2)$ for all $n \in \mathbb{N}$. According to (3.31), we can also write

$$\Gamma^2_* = \Gamma_* \times \Gamma_*, \qquad \Gamma_* := \{\gamma_i \in \Gamma : \Psi(\gamma_i) < \infty\}, \ \ i = 0, 1. \tag{3.32}$$

By (3.21), for $\mu \in \mathcal{P}_{\rm exp}$, we then have

$$\mu(\Psi) = \int_X \left(k_\mu^{(1,0)}(x) + k_\mu^{(0,1)}(x) \right) \psi(x) dx \leq 2\varkappa \langle \psi \rangle, \tag{3.33}$$

which by Proposition 3.2 yields

$$\forall \mu \in \mathcal{P}_{\rm exp} \qquad \mu(\breve{\Gamma}^2_*) = \mu(\Gamma^2_*) = 1. \tag{3.34}$$

This crucial property of the elements of $\mathcal{P}_{\rm exp}$ will allow us to consider mostly configurations belonging to Γ^2_*. In particular, this means that we will use the following sub-fields of $\mathcal{B}(\Gamma^2)$:

$$\breve{\mathcal{A}}_* = \{\mathbb{A} \in \mathcal{B}(\Gamma^2) : \mathbb{A} \subset \breve{\Gamma}^2_*\}, \qquad \mathcal{A}_* = \{\mathbb{A} \in \mathcal{B}(\Gamma^2) : \mathbb{A} \subset \Gamma^2_*\}. \tag{3.35}$$

Performing the same calculations as in obtaining (3.23) one readily gets

$$\mu(\Psi^n) \leq T_n(\varkappa \langle \psi \rangle), \qquad n \in \mathbb{N}, \tag{3.36}$$

which can be used to get the following estimate

$$\int_{\Gamma^2} \exp\left(\beta \Psi(\gamma)\right) \mu(d\gamma) \leq \exp\left(2\varkappa \langle \psi \rangle (e^\beta - 1)\right), \qquad \beta > 0, \tag{3.37}$$

holding for $\mu \in \mathcal{P}_{\rm exp}$ of type $\leq \varkappa$.

Now we recall that $C_{\rm b}(X)$ (resp. $B_{\rm b}(X)$) stands for the set of all bounded continuous (resp. bounded measurable) functions $g : X \to \mathbb{R}$. For ψ defined in (3.29), we then set

$$\begin{aligned} \Theta_\psi &= \{\theta(x) = g(x)\psi(x) : g \in C_{\rm b}(X), \ \ g(x) \geq 0\}, \qquad (3.38) \\ \Theta^+_\psi &= \{\theta \in \Theta_\psi : \theta(x) > 0\}. \end{aligned}$$

Clearly, each $\theta \in \Theta_\psi$ is integrable. For such θ, we also define

$$c_\theta = \sup_{x \in X} \frac{1}{\psi(x)} \log\left(1 + \theta(x)\right), \qquad \bar{c}_\theta := e^{c_\theta} - 1. \tag{3.39}$$

Then

$$0 \leq \theta(x) \leq \bar{c}_\theta \psi(x), \qquad \theta \in \Theta_\psi. \tag{3.40}$$

Next we define the following measures on X

$$(\psi\gamma_i)(dx) = \psi(x)\gamma_i(dx),, \qquad \gamma_i \in \Gamma_*, \ \ i = 0, 1. \tag{3.41}$$

Then

$$\Psi(\gamma) = (\psi\gamma_0)(X) + (\psi\gamma_1)(X), \qquad (\psi\gamma_i)(X) = \sum_{x \in \gamma_i} \psi(x) = \Psi(\gamma_i), \ \ i = 0, 1. \tag{3.42}$$

Let $\mathcal{N}$ be the set of all positive finite Borel measures on X. In view of (3.42) and (3.32), we have that $\psi\gamma_i \in \mathcal{N}$ for each $\gamma_i \in \Gamma_*$, $i = 0, 1$. Consider

$$C^L_{\rm b}(X) = \{g \in C_{\rm b}(X) : \|g\|_L < \infty\}, \quad \|g\|_L := \sup_{x,y \in X, \ x \neq y} \frac{|g(x) - g(y)|}{|x - y|},$$

and then define

$$\|g\|_{BL} = \|g\|_L + \sup_{x\in X} |g(x)|, \qquad g \in C_{\rm b}^L(X),$$

and also

$$\upsilon(\nu,\nu') = \max\left\{1; \sup_{g:\|g\|_{BL}\le 1} |\nu(g) - \nu'(g)|\right\}, \qquad \nu,\nu' \in \mathcal{N}. \tag{3.43}$$

Proposition 3.3. *[10, Theorem 18] The following three types of the convergence of a sequence $\{\nu_n\} \subset \mathcal{N}$ to a certain $\nu \in \mathcal{N}$ are equivalent:*

(i) $\nu_n(g) \to \nu(g)$ *for all* $g \in C_{\rm b}(X)$*;*
(ii) $\nu_n(g) \to \nu(g)$ *for all* $g \in C_{\rm b}^L(X)$*;*
(iii) $\upsilon(\nu_n,\nu) \to 0$.

That is, υ metrizes the weak convergence of the elements of $\mathcal{N}$. Our aim now is to metrize Γ^2_*. In view of (3.32), to this end it is enough to metrize Γ_*. Set

$$\varTheta_\psi^{BL} = \{\theta(x) = g(x)\psi(x) : \|g\|_{BL} \le 1\}, \tag{3.44}$$

and then define

$$\upsilon^*(\gamma,\gamma') = \upsilon(\psi\gamma_0,\psi\gamma_0') + \upsilon(\psi\gamma_1,\psi\gamma_1'). \tag{3.45}$$

Note that

$$\upsilon(\psi\gamma_i,\psi\gamma_i') = \max\left\{1; \sup_{\theta\in\varTheta_\psi^{BL}} \left|\sum_{x\in\gamma_i}\theta(x) - \sum_{x\in\gamma_i'}\theta(x)\right|\right\}, \qquad \gamma_i,\gamma_i' \in \Gamma_*, \quad i=0,1. \tag{3.46}$$

Before going further, we recall that the set of simple configurations is defined in (1.1), see also (3.28) and (3.31).

Lemma 3.4. *The metric space (Γ^2_*,υ^*) is complete and separable. Its metric topology is the weakest topology that makes continuous all the maps $\Gamma^2_* \ni \gamma \mapsto \sum_{i=0,1}\sum_{x\in\gamma_i}\theta_i(x)$, $\theta_0,\theta_1 \in \varTheta_\psi$. The set $\breve{\Gamma}^2_*$ defined in (3.31) is a G_δ subset of the Polish space Γ^2_*, and hence is also a Polish space. Its completion in the metric defined in (3.45) is Γ^2_*.*

Proof. The completeness of (Γ^2_*,υ^*) follows from the completeness of (Γ_*,υ_*), which was obtained in [16, Lemma 2.7]. The second part of the statement follows by the corresponding property of $\breve{\Gamma}_*$ obtained *ibid.* □

The following formula summarizes the relationship between the configuration sets

$$\breve{\Gamma}^2_* \subset \Gamma^2_* \subset \Gamma^2. \tag{3.47}$$

Recall that each of them is a Polish space with the topology as discussed above. Let $\mathcal{B}(\breve{\Gamma}^2_*)$ and $\mathcal{B}(\Gamma^2_*)$ be the corresponding Borel σ-fields, that can be compared with the σ-fields introduced in (3.35).

Corollary 3.5. *The embeddings in (3.47) are continuous. Therefore, $\mathcal{B}(\breve{\Gamma}^2_*) = \breve{\mathcal{A}}_* = \{\mathbb{A} \in \mathcal{B}(\Gamma^2_*) : \mathbb{A} \subset \breve{\Gamma}^2_*\}$ and $\mathcal{B}(\Gamma^2_*) = \mathcal{A}_*$.*

Proof. The continuity of $\Gamma^2_* \subset \Gamma^2$ follows by the fact that $C_{\rm cs}(X)$ is a proper subset of $\varTheta_\psi$. The other one follows by Lemma 3.4. The stated equality of the σ-fields follows by Kuratowski's theorem [22, Theorem 3.9, page 21]. □

Remark 3.6. The aforementioned equality of the σ-fields allows one to redefine each $\mu \in \mathcal{P}(\Gamma^2)$ possessing the property $\mu(\breve{\Gamma}^2_*) = 1$ as a probability measure on $(\Gamma^2_*,\mathcal{B}(\Gamma^2_*))$ or $(\breve{\Gamma}^2_*,\mathcal{B}(\breve{\Gamma}^2_*))$. By (3.34) this relates to all $\mu \in \mathcal{P}_{\rm exp}$.

3.4 Families of test functions

For $\theta \in \Theta_\psi$, see (3.38), we set

$$\varsigma^\theta_\tau(x) = \tau - \frac{1}{\psi(x)} \log(1+\theta(x)), \qquad \Sigma = \{\varsigma^\theta_\tau : \theta \in \Theta_\psi,\ \tau > c_\theta\}, \tag{3.48}$$

where c_θ is as in (3.39). Then $\Sigma \subset C_{\rm b}(X)$ and its elements are separated away from zero. It is closed with respect to pointwise addition since $\theta + \theta' + \theta\theta'$ belongs to Θ_ψ whenever $\theta, \theta' \in \Theta_\psi$. For $\tau_i > c_{\theta_i}$ and $\gamma_i \in \Gamma_*$, $i = 0, 1$, we define, see (3.41),

$$\widetilde{F}^{\theta_i}_{\tau_i}(\gamma_i) = \prod_{x\in\gamma_i} (1+\theta_i(x)) e^{-\tau_i \psi(x)} = \exp\left(-(\psi\gamma_i)(\varsigma^{\theta_i}_{\tau_i})\right), \tag{3.49}$$

$$\widetilde{F}^{\theta}_{\tau}(\gamma) = \widetilde{F}^{\theta_0}_{\tau_0}(\gamma_0)\widetilde{F}^{\theta_1}_{\tau_1}(\gamma_1),$$

and also

$$\widetilde{\mathcal{F}} = \{\widetilde{F}^\theta_\tau : \tau = (\tau_0, \tau_1),\ \tau_i > c_{\theta_i},\ \theta_i \in \Theta_\psi,\ i = 0,1\}, \tag{3.50}$$

which includes the case $\widetilde{F}^\theta_\tau \equiv 1$ corresponding to the zero τ and θ. Note that in expressions like those in (3.49), (3.50), by θ we understand (θ_0, θ_1), $\theta_i \in \Theta_\psi$.

Definition 3.7. *[11, page 111] A sequence $\{\Phi_n\}_{n\in\mathbb{N}} \subset B_{\rm b}(\Gamma^2_*)$ is said to boundedly and pointwise (bp-) converge to a given $\Phi \in B_{\rm b}(\Gamma^2_*)$ if it converges pointwise and*

$$\sup_{n\in\mathbb{N}} \sup_{\gamma\in\Gamma^2_*} |\Phi_n(\gamma)| < \infty.$$

The bp-closure of a set $M \subset B_{\rm b}(\Gamma^2_*)$ is the smallest subset of $B_{\rm b}(\Gamma^2_*)$ that contains M and is closed under the bp-convergence. In a similar way, one understands also the bp-convergence of a sequence of functions $\phi_n : X \to \mathbb{R}$.

It is quite standard, see [11, Proposition 4.2, page 111] or [9, Lemma 3.2.1, page 41], that $C_{\rm b}(X)$ contains a countable family of nonnegative functions, $\{g_j\}_{j\in\mathbb{N}}$, which is *convergence determining* and such that its linear span is bp-dense in $B_{\rm b}(X)$. This means that a sequence $\{\nu_n\} \subset \mathcal{N}$ weakly converges to a certain ν if and only if $\nu_n(g_j) \to \nu(g_j)$, $n \to +\infty$ for all $j \in \mathbb{N}$. One may take such a family containing the constant function $g(x) \equiv 1$ and closed with respect to pointwise addition. Moreover, one may assume that

$$\forall j \in \mathbb{N} \qquad \inf_{x\in X} g_j(x) =: \zeta_j > 0. \tag{3.51}$$

If this is not the case for a given g_j, in place of it one may take $\tilde{g}_j(x) = g_j(x) + \zeta_j$ with some $\zeta_j > 0$. The new set $\{\tilde{g}_j\}$ has both mentioned properties and also satisfies (3.51). Then assuming the latter we conclude that

$$\Sigma_0 := \{g_j\}_{j\in\mathbb{N}} \subset \Sigma, \tag{3.52}$$

where the latter is defined in (3.48). To see this, for a given g_j, take $\tau_j \geq \sup_x g_j(x)$ and then set

$$\theta_j(x) = \exp\left([\tau_j - g_j(x)]\psi(x)\right) - 1. \tag{3.53}$$

Clearly, $\theta_j(x) \geq 0$. Since $\psi^n(x) \leq \psi(x)$, $n \in \mathbb{N}$, we have that $\theta_j(x) \leq e^{\tau_j}\psi(x)$, and hence $\{\theta_j\}_{j\in\mathbb{N}} \subset \Theta_\psi$, see (3.38). At the same time, $\varsigma^{\theta_j}_{\tau_j} = g_j$ and $c_{\theta_j} = \sup_x(\tau_j - g_j(x)) < \tau_j$ in view of (3.51). By (3.53), (3.52) and [11, Theorem 3.4.5, page 113], see also [9, page 43], one readily proves the following statement.

Proposition 3.8. *The set $\widetilde{\mathcal{F}}$ defined in (3.50) is closed with respect to pointwise multiplication. Additionally:*

(i) It is separating: $\mu_1(F) = \mu_2(F)$, holding for all $F \in \widetilde{\mathcal{F}}$, implies $\mu_1 = \mu_2$ for all $\mu_1, \mu_2 \in \mathcal{P}(\Gamma^2_*)$.

(ii) It is convergence determining: if a sequence $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathcal{P}(\Gamma^2_*)$ is such that $\mu_n(F) \to \mu(F)$, $n \to +\infty$ for all $F \in \widetilde{\mathcal{F}}$ and some $\mu \in \mathcal{P}(\Gamma^2_*)$, then $\mu_n(F) \to \mu(F)$ for all $F \in C_{\rm b}(\Gamma^2_*)$.

(iii) The set $B_{\rm b}(\Gamma^2_*)$ is the bp-closure of the linear span of $\widetilde{\mathcal{F}}$.

Now we introduce another class of functions $F : \Gamma^2_* \to \mathbb{R}$ which we then use to define the domain of L. For $k \in \mathbb{N}$, $v_{i,1}, \dots, v_{i,k} \in C_{\rm b}(X)$ and $\gamma_i \in \Gamma_*$, $i = 0, 1$, we write

$$\mathbf{v}_i(\mathbf{x}^k) = v_{i,1}(x_1)\cdots v_{i,k}(x_k), \qquad \gamma_i \setminus \mathbf{x}^k = \gamma_i \setminus \{x_1, \dots, x_k\}, \tag{3.54}$$

see subsect. 3.1. As is (3.3), we will omit k if the dimension of $\mathbf{x}$ is known from the context. Then for $m = (m_0, m_1) \in \mathbb{N}^2_0$, $v_{i,j} \in \Theta^+_\psi$ (see (3.38)) and $\tau = (\tau_0, \tau_1)$, $\tau_i > 0$, we set, see also (3.4) and (3.18),

$$\widehat{F}^{m_i}_{\tau_i}(\mathbf{v}_i|\gamma_i) = \sum_{\mathbf{x}^{m_i}\in\gamma_i} \mathbf{v}_i(\mathbf{x}^{m_i}) \exp\left(-\tau_i \Psi(\gamma_i \setminus \mathbf{x}^{m_i})\right), \qquad i = 0, 1, \tag{3.55}$$

$$\widehat{F}^{m}_{\tau}(\mathbf{v}|\gamma) = \widehat{F}^{m_0}_{\tau_0}(\mathbf{v}_0|\gamma_0)\widehat{F}^{m_1}_{\tau_1}(\mathbf{v}_1|\gamma_1), \qquad \gamma \in \Gamma^2_*,$$

$$\widehat{\mathcal{F}} = \{\widehat{F}^m_\tau(\mathbf{v}|\cdot) : m \in \mathbb{N}^2_0,\ v_{i,j} \in \Theta^+_\psi,\ \tau_i > 0,\ i = 0, 1\}.$$

Here $\widehat{F}^{(0,0)}_\tau \equiv 0$, which is also an element of $\widehat{\mathcal{F}}$.

Proposition 3.9. *For each $m = (m_0, m_1) \in \mathbb{N}^2_0$, $\tau = (\tau_0, \tau_1) > 0$ and $v_{i,j} \in \Theta^+_\psi$, $j = 1, \dots, m_j$, $i = 0, 1$, it follows that $\widehat{F}^m_\tau(\mathbf{v}|\cdot) \in C_{\rm b}(\Gamma^2_*)$.*

Proof. Clearly, it suffices to show that $\widehat{F}^{m_i}_{\tau_i}(\mathbf{v}_i|\cdot) \in C_{\rm b}(\Gamma_*)$. To prove the continuity in question we rewrite the first line of (3.55) in the form

$$\widehat{F}^{m_i}_{\tau_i}(\mathbf{v}_i|\gamma_i) = \exp\left(-\tau_i\Psi(\gamma_i)\right) \sum_{x_1\in\gamma_i} \sum_{x_2\in\gamma_i\setminus x_1} \cdots \sum_{x_{m_i}\in\gamma_i\setminus\{x_1,\dots,x_{m_i-1}\}} u_1(x_1)\cdots u_{m_i}(x_{m_i}), \tag{3.56}$$

$$u_j(x) := v_{i,j}(x)e^{\tau_i\psi(x)}, \qquad j = 1, \dots, m_i.$$

Obviously, every u_j belongs to Θ^+_ψ. For each m_i, by (3.2) the sum in (3.56) can be written as the sum of the products of the functions

$$\gamma_i \mapsto U_{l_1,\dots l_s}(\gamma_i) = \sum_{y\in\gamma_i} u_{l_1}(y)\cdots u_{l_s}(y).$$

Each such $U_{l_1,\dots l_s}$ is continuous as Θ^+_ψ is closed under pointwise multiplication. Obviously, $\gamma_i \mapsto \Psi(\gamma_i)$ is also continuous, which yields the continuity of the function defined in (3.56). To prove its boundedness, we use the estimate $u_j(x) \leq c\psi(x)e^{\tau_i\psi(x)} \leq ce^\tau\psi(x)$. Then, see (3.4),

$$\widehat{F}^{m_i}_{\tau_i}(\mathbf{v}_i|\gamma_i) \leq \exp\left(-\tau_i\Psi(\gamma_i)\right) \sum_{\mathbf{x}\in\gamma_i} u^{\otimes m_i}(\mathbf{x}) \leq c^{m_i}e^{m_i\tau_i} \exp\left(-\tau_i\Psi(\gamma_i)\right) \left(\sum_{x\in\gamma_i}\psi(x)\right)^{m_i} \tag{3.57}$$

$$= c^{m_i}\Psi^{m_i}(\gamma_i) \exp\left(- \tau_i[\Psi(\gamma_i) - m_i]\right) \leq \left(\frac{cm_i}{\tau_i}\right)^{m_i} e^{m_i(\tau_i-1)},$$

which yields the proof. □

4 The statement

4.1 The Kolmogorov operator

Regarding the parameters of the Kolmogorov operator introduced in (1.3), we will assume that a_i, ϕ_i, $i = 0, 1$, are measurable functions taking nonnegative values. We also assume that the following holds

$$\max_{i=0,1} \sup_x a_i(x) =: \|a\| < \infty, \tag{4.1}$$

$$\max_{i=0,1} \int_X \left(1 - e^{-\phi_i(x)}\right) dx =: \varphi < \infty, \tag{4.2}$$

and

$$\int_X |x|^l a_i(x) dx =: \bar{a}_i^{(l)} < \infty, \qquad \text{for } \ i = 0, 1 \text{ and } \ l = 0, 1, \dots, d+1. \tag{4.3}$$

Definition 4.1. *By $\mathcal{D}(L)$ we denote the linear span of the set $\widetilde{\mathcal{F}} \cup \widehat{\mathcal{F}}$, where $\widetilde{\mathcal{F}}$ and $\widehat{\mathcal{F}}$ are defined in (3.50) and (3.55), respectively.*

Our aim now is to show that $L\widehat{F}_\tau^m(\mathbf{v}|\cdot) \in B_{\rm b}(\Gamma_*^2)$ holding for each $\widehat{F}_\tau^m(\mathbf{v}|\cdot) \in \widehat{\mathcal{F}}$. To this end, for $x \in \gamma_i$, $y \in X$ and a suitable F, we define

$$(\nabla_i^{y,x} F)(\gamma_i) = F(\gamma_i \setminus x \cup y) - F(\gamma_i). \tag{4.4}$$

By (3.55) we can write

$$\widehat{F}_{\tau_i}^{m_i}(\mathbf{v}_i|\gamma_i) = \sum_{z \in \gamma_i} v_{i,1}(z) \widehat{F}_{\tau_i}^{m_i - 1}(\mathbf{v}_i^1|\gamma_i \setminus z), \tag{4.5}$$

where $\mathbf{v}_i^1$ is obtained by setting $j = 1$ in the formula

$$\mathbf{v}_i^j(\mathbf{x}^{m_i - 1}) = v_{i,1}(x_1) \cdots v_{i,j-1}(x_{j-1}) v_{i,j+1}(x_{j+1}) \cdots v_{i,m_i}(x_{m_i}), \tag{4.6}$$

see (3.54), and

$$\widehat{F}_{\tau_i}^{m_i - 1}(\mathbf{v}_i^1|\gamma_i \setminus z) = \sum_{\mathbf{x}^{m_i - 1} \in \gamma_i \setminus z} \mathbf{v}_i^1(\mathbf{x}^{m_i - 1}) \exp\left(-\tau_i \Psi((\gamma_i \setminus z) \setminus \mathbf{x}^{m_i - 1})\right).$$

According to (4.4) and (4.5) we then get

$$\nabla_i^{y,x} \widehat{F}_{\tau_i}^{m_i}(\mathbf{v}_i|\gamma_i) = [v_{i,1}(y) - v_{i,1}(x)] \widehat{F}_{\tau_i}^{m_i - 1}(\mathbf{v}_i^1|\gamma_i \setminus x) + \sum_{z \in \gamma_i \setminus x} v_{i,1}(z) \nabla_i^{y,x} \widehat{F}_{\tau_i}^{m_i - 1}(\mathbf{v}_i^1|\gamma_i \setminus z). \tag{4.7}$$

By iterating the latter we arrive at, see also (4.6),

$$\begin{aligned} \nabla_i^{y,x} \widehat{F}_{\tau_i}^{m_i}(\mathbf{v}_i|\gamma_i) = & \sum_{j=1}^{m_i} [v_{i,j}(y) - v_{i,j}(x)] \widehat{F}_{\tau_i}^{m_i - 1}(\mathbf{v}_i^j|\gamma_i \setminus x) \\ & + \left(e^{-\tau_i \psi(y)} - e^{-\tau_i \psi(x)}\right) \widehat{F}_{\tau_i}^{m_i}(\mathbf{v}_i|\gamma_i \setminus x). \end{aligned} \tag{4.8}$$

For $\theta \in \Theta_\psi$ and a_i as in (4.1), (4.3), we set

$$(a_i * \theta)(x) = \int_X a_i(x - y)\theta(y) dy = \int_X a_i(y)\theta(x - y) dy, \qquad i = 0, 1. \tag{4.9}$$

Then $a_i * \theta \in C_{\rm b}(X)$. Moreover, by (3.40) and (4.3) we obtain

$$\begin{aligned}(a_i * \theta)(x) &\leq \bar{c}_\theta \psi(x) \int_X \left(1+|x|^{d+1}\right) a_i(x-y)\psi(y)dy \\ &\leq \bar{c}_\theta \psi(x)\left[\bar{a}_i^{(0)} + \int_X \left(|x-y|+|y|\right)^{d+1} a_i(x-y)\psi(y)dy\right] \\ &= \bar{c}_\theta \psi(x)\left[\bar{a}_i^{(0)} + \sum_{l=0}^{d+1}\binom{d+1}{l}\int_X |x-y|^{d+1-l}|y|^l\psi(y)a_i(x-y)dy\right] \\ &\leq \bar{c}_\theta \psi(x)\left[\bar{a}_i^{(0)} + \sum_{l=0}^{d+1}\binom{d+1}{l}\bar{a}_i^{(l)}\right] =: \bar{c}_\theta\bar{\alpha}_i\psi(x),\end{aligned} \tag{4.10}$$

where we have used also the fact that $|y|^l\psi(y)\leq 1$ for all $l=0,\dots,d+1$. Therefore, for each $\theta\in\Theta_\psi$, it follows that

$$\begin{aligned}(a_i\theta)(x) &:= (a_i*\theta)(x)+\theta(x) \leq \bar{c}_\theta(\bar{\alpha}_i+1)\psi(x)\leq \bar{c}_\theta c_a\psi(x), \\ c_a &:= \max\{\bar{\alpha}_0;\bar{\alpha}_1\}+1.\end{aligned} \tag{4.11}$$

At the same time,

$$e^{-\tau_i\psi(y)} - e^{-\tau_i\psi(x)} \leq \tau_i\psi(y)\psi(x)\big||x|^{d+1}-|y|^{d+1}\big|, \qquad i=0,1, \tag{4.12}$$

which after calculations similar to those in (4.10) yields

$$\int_X a_i(x-y)\left|e^{-\tau_i\psi(y)} - e^{-\tau_i\psi(x)}\right|dy \leq \tau_i c_a\psi(x), \tag{4.13}$$

where c_a is as in (4.11). For $\mathbf{v}_i$ as in (3.54) and $a_i\theta$ as in (4.11), we set, cf. (4.6),

$$(a_i^j\mathbf{v}_i)(\mathbf{x}^{m_i}) = v_{i,1}(x_1)\cdots v_{i,j-1}(x_{j-1})(a_iv_{i,j})(x_j)v_{i,j+1}(x_{j+1})\cdots v_{i,m_i}(x_{m_i}). \tag{4.14}$$

Then by (3.55), (4.7), (4.8) and also by (4.11), (4.13), (4.14) we arrive at

$$\begin{aligned}\left|L\widehat{F}^m_\tau(\mathbf{v}|\gamma)\right| &\leq \left(\int_X\sum_{x\in\gamma_0}a_0(x-y)\left|\nabla_0^{y,x}\widehat{F}^{m_0}_{\tau_0}(\mathbf{v}_0|\gamma_0)\right|dy\right)\widehat{F}^{m_1}_{\tau_1}(\mathbf{v}_1|\gamma_1) \\ &+ \left(\int_X\sum_{x\in\gamma_1}a_1(x-y)\left|\nabla_1^{y,x}\widehat{F}^{m_1}_{\tau_1}(\mathbf{v}_1|\gamma_1)\right|dy\right)\widehat{F}^{m_0}_{\tau_0}(\mathbf{v}_0\gamma_0) \\ &\leq \left(\sum_{j=1}^{m_0}\widehat{F}^{m_0}_{\tau_0}(a_0^j\mathbf{v}_0|\gamma_0)+\tau_0c_a\bar{c}(\mathbf{v}_0)\widehat{F}^{m_0+1}_{\tau_0}(\gamma_0)\right)\widehat{F}^{m_1}_{\tau_1}(\mathbf{v}_1|\gamma_1) \\ &+ \left(\sum_{j=1}^{m_1}\widehat{F}^{m_1}_{\tau_1}(a_1^j\mathbf{v}_1|\gamma_1)+\tau_1c_a\bar{c}(\mathbf{v}_1)\widehat{F}^{m_1+1}_{\tau_1}(\gamma_1)\right)\widehat{F}^{m_0}_{\tau_0}(\mathbf{v}_0|\gamma_0),\end{aligned} \tag{4.15}$$

where $\bar{c}(\mathbf{v}_i)=\max_j\bar{c}_{v_{i,j}}$, see (3.39), and

$$\widehat{F}^{m_i}_{\tau_i}(\gamma_i) = \widehat{F}^{m_i}_{\tau_i}(\mathbf{v}_i|\gamma_i), \quad \text{with} \quad \mathbf{v}_i(\mathbf{x})=\psi(x_1)\cdots\psi(x_{m_i}), \tag{4.16}$$

see the first line of (3.55). Then the boundedness of $L\widehat{F}^m_\tau(\mathbf{v}|\cdot)$ follows by Proposition 3.9. Let us prove now that $L\widetilde{F}^\theta_\tau \in B_{\rm b}(\Gamma_*)$, holding for all $\theta_i \in \Theta_\psi$ and $\tau_i > c_{\theta_i}$, $i=0,1$, see (3.49). According to (4.4) we have

$$\nabla^{y,x}_i \widetilde{F}^{\theta_i}_{\tau_i}(\gamma_i) = \left(\left[e^{-\tau_i\psi(y)} - e^{-\tau_i\psi(x)}\right] + \left[\theta_i(y)e^{-\tau_i\psi(y)} - \theta_i(x)e^{-\tau_i\psi(x)}\right]\right)\widetilde{F}^{\theta_i}_{\tau_i}(\gamma_i \setminus x).$$

Then by means of (4.12) and (4.10) we obtain

$$\left|L\widetilde{F}^\theta_\tau(\gamma)\right| \le \widetilde{Q}(\gamma)\widetilde{F}^\theta_{\tau-\tau^0}(\gamma),$$

$$\begin{aligned} \widetilde{Q}(\gamma) &:= (\tau_0+\bar{c}_{\theta_0})c_a e^{\tau_0}\Psi(\gamma_0)e^{-\tau^0_0\Psi(\gamma_0)} + (\tau_1+\bar{c}_{\theta_1})c_a e^{\tau_1}\Psi(\gamma_1)e^{-\tau^0_1\Psi(\gamma_1)} \\ &\le e^{\tau_0}c_a\frac{\tau_0+\bar{c}_{\theta_0}}{e\tau^0_0} + e^{\tau_1}c_a\frac{\tau_1+\bar{c}_{\theta_1}}{e\tau^0_1}, \end{aligned}$$

where $\tau^0 = (\tau^0_0, \tau^0_1)$, $\tau^0_i > 0$, is chosen in such a way that $\tau_i - \tau^0_i > c_{\theta_i}$, which is possible since $\tau_i > c_{\theta_i}$. Then the boundedness in question follows by (3.49). The next statement summarizes the properties of $\mathcal{D}(L)$.

Proposition 4.2. *The set $\mathcal{D}(L)$ introduced in Definition 4.1 has the following properties:*

(a) $\mathcal{D}(L) \subset C_{\rm b}(\Gamma^2_*)$; $L : \mathcal{D}(L) \to B_{\rm b}(\Gamma^2_*)$.
(b) The set $B_{\rm b}(\Gamma^2_*)$ is the bp-closure of $\mathcal{D}(L)$.
(c) $\mathcal{D}(L)$ is separating for $\mathcal{P}_{\rm exp}$.
(d) For each $F \in \widetilde{\mathcal{F}}$, the measure $F\mu/\mu(F)$ belongs to $\mathcal{P}_{\rm exp}$.

Proof. Claim (a) has been just proved. Claims (b) and (c) follow by Proposition 3.8. Claim (d) holds true since $F \in \widetilde{\mathcal{F}}$ is positive and bounded, and thus multiplication does not affect the property defined in (3.7), (3.9). □

4.2 Formulating the result

Following [9, Chapter 5] – and similarly as in [16] – we will obtain the Markov process in question by solving a *restricted initial value martingale problem* for $(L, \mathcal{D}(L), \mathcal{P}_{\rm exp})$. Here we explicitly employ the complete metric υ^* of Γ^2_*, defined in (3.46). Since the elements of $\mathcal{P}_{\rm exp}$ "do not distinguish" between multiple and single configurations, see Proposition 3.2, one may expect that the constructed Markov process has the corresponding property. We will show that it does. Note that the direct construction of the process with values in $\breve{\Gamma}^2_*$ is rather impossible in this way as the latter space is not complete in υ^*.

To proceed further, we introduce the corresponding spaces of càdlàg paths. By $\mathfrak{D}_{\mathbb{R}_+}(\breve{\Gamma}^2_*)$ and $\mathfrak{D}_{\mathbb{R}_+}(\Gamma^2_*)$ we denote the spaces of càdlàg maps $[0,+\infty) =: \mathbb{R}_+ \ni t \mapsto \gamma_t \in \breve{\Gamma}^2_*$ and $\mathbb{R}_+ \ni t \mapsto \gamma_t \in \Gamma^2_*$, respectively. Then the evaluation maps are $\varpi_t(\gamma) = \gamma_t$, $\gamma \in \mathfrak{D}_{\mathbb{R}_+}(\Gamma^2_*)$, $t \in \mathbb{R}_+$; hence,

$$\varpi_t^{-1}(\mathbb{A}) = \{\gamma \in \mathfrak{D}_{\mathbb{R}_+}(\Gamma^2_*) : \varpi_t(\gamma) = \gamma_t \in \mathbb{A}\}, \qquad \mathbb{A} \in \mathcal{B}(\Gamma^2_*).$$

Analogously one defines $\mathfrak{D}_{[s,+\infty)}(\breve{\Gamma}^2_*)$, $\mathfrak{D}_{[s,+\infty)}(\Gamma^2_*)$, $s > 0$. For $s,t \ge 0$, $s < t$, by $\mathfrak{F}^0_{s,t}$ we denote the σ-field of subsets of $\mathfrak{D}_{[s,+\infty)}(\Gamma^2_*)$ generated by the family $\{\varpi_u : u \in [s,t]\}$. Then we set

$$\mathfrak{F}_{s,t} = \bigcap_{\varepsilon>0}\mathfrak{F}^0_{s,t+\varepsilon}, \qquad \mathfrak{F}_{s,+\infty} = \bigvee_{n\in\mathbb{N}}\mathfrak{F}_{s,s+n}.$$

In the next definition – which is an adaptation of the corresponding definition in [9, Section 5.1, pages 78, 79] – we deal with families of probability measures $\{P_{s,\mu} : s \ge$

$0, \mu \in \mathcal{P}_{\rm exp}\}$ defined on $(\mathfrak{D}_{[s,+\infty)}(\Gamma^2_*), \mathfrak{F}_{s,+\infty})$. Depending on the context, each $\mu \in \mathcal{P}_{\rm exp}$ is considered as a measure either on $\breve{\Gamma}^2_*$ or Γ^2_*, see Remark 3.6. Since $\breve{\Gamma}^2_*$ and Γ^2_* are Polish spaces, both $\mathfrak{D}_{[s,+\infty)}(\breve{\Gamma}^2_*)$ and $\mathfrak{D}_{[s,+\infty)}(\Gamma^2_*)$ are also Polish. The latter one is complete in Skorohod's metric, see [11, Theorem 5.6, page 121]. Then a probability measure P on $(\mathfrak{D}_{[s,+\infty)}(\Gamma^2_*), \mathfrak{F}_{s,+\infty})$ with the property $P(\mathfrak{D}_{[s,+\infty)}(\breve{\Gamma}^2_*)) = 1$ can be redefined as a measure on $\mathfrak{D}_{[s,+\infty)}(\breve{\Gamma}^2_*)$, that holds for all $s \geq 0$.

Definition 4.3. *A family of probability measures $\{P_{s,\mu} : s \geq 0,\ \mu \in \mathcal{P}_{\rm exp}\}$ is said to be a solution of the restricted initial value martingale problem for $(L, \mathcal{D}(L), \mathcal{P}_{\rm exp})$ if for all $s \geq 0$ and $\mu \in \mathcal{P}_{\rm exp}$, the following holds: (a) $P_{s,\mu} \circ \varpi_s^{-1} = \mu$; (b) $P_{s,\mu} \circ \varpi_t^{-1} \in \mathcal{P}_{\rm exp}$ for all $t > s$; (c) for each $F \in \mathcal{D}(L)$, $t_2 \geq t_1 \geq s$ and any bounded function $\mathsf{G} : \mathfrak{D}_{[s,+\infty)}(\Gamma^2_*) \to \mathbb{R}$ which is $\mathfrak{F}_{s,t_1}$-measurable, the function*

$$\mathsf{H}(\gamma) \quad := \quad \left[F(\varpi_{t_2}(\gamma)) - F(\varpi_{t_1}(\gamma)) - \int_{t_1}^{t_2} (LF)(\varpi_u(\gamma))du\right] \mathsf{G}(\gamma) \tag{4.17}$$

is such that

$$\int_{\mathfrak{D}_{[s,+\infty)}(\Gamma^2_*)} \mathsf{H}(\gamma) P_{s,\mu}(d\gamma) = 0.$$

The restricted initial value martingale problem is well-posed if for each $s \geq 0$ and $\mu \in \mathcal{P}_{\rm exp}$, there exists a unique path measure $P_{s,\mu}$ satisfying conditions (a), (b) and (c) mentioned above.

Remark 4.4. Instead of taking all G as in claim (c) of Definition 4.3, it is enough to take in the form

$$\mathsf{G}(\gamma) = F_1(\varpi_{s_1}(\gamma)) \cdots F_m(\varpi_{s_m}(\gamma)), \tag{4.18}$$

with all possible choices of $m \in \mathbb{N}$, $F_1, \ldots, F_m \in \widetilde{\mathcal{F}}$ (see Proposition 3.8), and $s \leq s_1 < s_2 < \cdots < s_m \leq t_1$, see [11, eq. (3.4), page 174].

Now we can formulate our result.

Theorem 4.5. *For the model defined in (1.3) and satisfying (4.1) – (4.3), the following is true:*

- *(i) The restricted initial value martingale problem for $(L, \mathcal{D}(L), \mathcal{P}_{\rm exp})$ is well-posed in the sense of Definition 4.3.*
- *(ii) Its solution has the property $P_{s,\mu}(\mathfrak{D}_{[s,+\infty)}(\breve{\Gamma}^2_*)) = 1$, holding for all $s \geq 0$ and $\mu \in \mathcal{P}_{\rm exp}$.*
- *(iii) The stochastic process related to the family*

$$(\mathfrak{D}_{[s,+\infty)}(\Gamma^2_*), \mathfrak{F}_{s,+\infty}, \{\mathfrak{F}_{s,t} : t \geq s\}, \{P_{s,\mu} : \mu \in \mathcal{P}_{\rm exp}\})_{s \geq 0}$$

is Markov. This means that, for all $t > s$ and $\mathbb{B} \in \mathfrak{F}_{t,+\infty}$, the following holds

$$P_{s,\mu}(\mathbb{B}|\mathfrak{F}_{s,t}) = P_{s,\mu}(\mathbb{B}|\mathfrak{F}_t), \qquad P_{s,\mu} \ - \ \text{almost surely}.$$

Here $\mathfrak{F}_t$ is the smallest σ-field of subsets of $\mathfrak{D}_{[s,+\infty)}(\Gamma^2_*)$ that contains all $\varpi_t^{-1}(\mathbb{A})$, $\mathbb{A} \in \mathcal{B}(\Gamma^2_*)$.

In the proof of Theorem 4.5, we crucially use the Fokker-Planck equation (1.2).

Definition 4.6. *For a given $s \geq 0$, a map $[s, +\infty) \ni t \mapsto \mu_t \in \mathcal{P}(\Gamma^2_*)$ is said to be measurable if the maps $[s, +\infty) \ni t \mapsto \mu_t(\mathbb{A}) \in \mathbb{R}$ are measurable for all $\mathbb{A} \in \mathcal{B}(\Gamma^2_*)$. Such a map is said to be a solution of the Fokker-Planck equation for $(L, \mathcal{D}(L))$ if for each $F \in \mathcal{D}(L)$ and any $t_2 > t_1 \geq s$, the equality in (1.2) holds true.*

Note that $LF \in B_{\rm b}(\Gamma_*)$; hence, the integral in the right-hand side (1.2) is well defined for measurable $t \mapsto \mu_t$.

Remark 4.7. By taking $\mathsf{G} \equiv 1$ in (4.17) one comes to the following conclusion. Let $\{P_{s,\mu} : s \geq 0, \mu \in \mathcal{P}_{\rm exp}\}$ be a solution as in Definition 4.3. Then for each s and $\mu \in \mathcal{P}_{\rm exp}$, the map $[s, +\infty) \ni t \mapsto P_{s,\mu} \circ \varpi_t^{-1}$ solves (1.2) for all $t_2 > t_1 \geq s$.

4.3 Comments

4.3.1 Concerning the model

In statistical physics, the first model where attraction is induced by an inter-component repulsion was proposed by Widom and Rowlinson in [24]. A mathematically rigorous proof that the Gibbs states in this model can be multiple was done by Ruelle in [21]. In both these works, the repulsion is of the hard-core type, which in our case corresponds to $\phi_0(x) = \phi_1(x) = \ell_r(|x|)$, $r > 0$, with $\ell_r(\rho) = 0$ for $\rho > r$, and $\ell_r(\rho) = +\infty$ for $\rho \leq r$. In the single-component version of the Widom-Rowlinson model, the energy of the multiparticle attraction induced by the hard core repulsion in a finite configuration $\eta_0 \subset \Gamma$ is given by the formula, see [6, eq. (1.1)],

$$U(\eta_0) = V(\eta_0) - |\eta_0||B_r|, \tag{4.19}$$

where $|B_r|$ is the volume of B_r and $V(\eta_0)$ is the volume of $\cup_{x \in \eta_0} B_r(x)$. The relationship between the single- and the two-component versions was analyzed in detain in [6], see also [15] where the interaction of the Curie-Weiss type (in place of the hard-core repulsion) was studied. A significant feature of (4.19) is that this interaction is super-stable in the sense of [20], see [6, eq. (1.2)]. For such interactions, the states of thermal equilibrium (Gibbs states) have correlation functions that satisfy (3.12), see [20], which means that the Gibbs states are sub-Poissonian. This is one more argument in favor of using such states. Note that our assumption (4.2) covers the case of hard core repulsion mentioned above. In [1], the results of which we will use in the remaining part of this work, the repulsion kernels ϕ_i were assumed bounded and integrable, which is a stronger version of (4.2) that does not cover the hard core repulsion. However, the boundedness was used there only in the part where the mesoscopic limit of the model was studied. That is, the part of [1] the results of which we will use here remains valid if one assumes only (4.2).

4.3.2 Concerning the method

In this work, we mostly follow the scheme elaborated by us in [16]. It has two basic ingredients: (a) proving existence and uniqueness for the Fokker-Planck equation, where existence is obtained by means of the corresponding results of [1]; (b) approximating the initial model by some models for which the process can be constructed directly by means of the corresponding transition functions. Of course, here we faced some additional technical problems related to a more complex nature of the model.

Another type of stochastic dynamics in infinite particle systems which is even more popular in the literature than the systems with ‘conservation of the number of particles’ is the so called ‘birth-and-death’ dynamics. Here the particles appear and disappear, also under the influence of the existing members of the population. Up to the best of our knowledge, by now uniqueness for the corresponding Markov processes was obtained only for infinite particle systems with independent disappearance, see [12]. We plan to modify our methods to cover also the case of systems with a logistic-type disappearance repulsion.

5 The evolution of sub-Poissonian states

As mentioned above, in [1] there was constructed a map $t \mapsto \mu_t \in \mathcal{P}_{\rm exp}$ which describes the evolution of states of the model (1.3). Here we show that this map is the unique solution of the Fokker-Planck equation (1.2), which is then used in the proof of Theorem 4.5. In this section, we outline the construction realized in [1] in the form adapted to the present context, which includes also passing to states on the space of multiple configurations Γ^2_*. This is possible since $\mu(\breve{\Gamma}^2_*) = \mu(\Gamma^2_*) = 1$, that holds for all $\mu \in \mathcal{P}_{\rm exp}$, see Remark 3.6.

The key idea of [1] may be described as follows. Since each $\mu \in \mathcal{P}_{\rm exp}$ is fully characterized by its correlation functions $k_\mu^{(m)}$, $m \in \mathbb{N}_0^2$, see Definition 3.1 and (3.10), instead of solving (1.2) directly one can pass to the evolution equation for the corresponding correlation functions defined in appropriate Banach spaces. An addition task, however, will be to prove that its solutions are correlation functions – an analog of the classical moment problem in this setting.

5.1 The evolution of correlation functions

For $m \in \mathbb{N}_0^2$, let a symmetric $G^{(m)}$ be in $C_{\rm cs}(X^{m_0} \times X^{m_1})$, see (3.20). As above, $m = (0,0)$ corresponds to constant functions. Let $G := \{G^{(m)}\}_{m\in\mathbb{N}_0^2}$ be a collection of such functions. We equip the set of all such collections with the usual (member-wise) linear operations and then write $G(\eta) = G^{(m)}(\mathbf{x}, \mathbf{y})$ for $\eta = (\eta_0, \eta_1)$, $\eta_0 = \{x_1, \dots, x_{m_0}\}$, $\eta_1 = \{y_1, \dots, y_{m_1}\}$, and $(\mathbf{x}, \mathbf{y}) = (x_1, \dots, x_{m_0}; y_1, \dots, y_{m_1})$, cf. (3.8), (3.10). Each $\eta = (\eta_0, \eta_1)$ is a pair of finite configurations, and thus $\eta \in \Gamma^2$. That is, η_i is a finite configuration of particles of type $i = 0, 1$; by Γ_0 we denote the subset of Γ consisting of all finite (possibly multiple) configurations. Let $\mathcal{G}_{\rm fin}$ denote the set of all aforementioned collections G verifying $G^{(m)} \equiv 0$ for all $m_0 + m_1 =: |m| > N_G$ for some $N_G \in \mathbb{N}$. Then the map K as in (3.20) can be defined on $\mathcal{G}_{\rm fin}$ by the formula

$$(KG)(\gamma) = \sum_{\eta\subset\gamma} G(\eta) = \sum_{\eta_0\subset\gamma_0} \sum_{\eta_1\subset\gamma_1} G(\eta_0, \eta_1) \tag{5.1}$$

$$= \sum_{m_0=0}^{\infty} \sum_{m_1=0}^{\infty} \frac{1}{m_0! m_1!} \sum_{(\mathbf{x},\mathbf{y})\in\gamma} G^{(m)}(\mathbf{x}, \mathbf{y}).$$

For $\mu \in \mathcal{P}_{\rm exp}$ and $G \in \mathcal{G}_{\rm fin}$, by (3.22) KG is μ-integrable and the following holds, cf. (3.10),

$$\begin{aligned} \mu(KG) &= \sum_{m\in\mathbb{N}_0^2} \frac{1}{m_0! m_1!} \int_{X^{m_0}\times X^{m_1}} k_\mu^{(m)}(\mathbf{x}, \mathbf{y}) G^{(m)}(\mathbf{x}, \mathbf{y}) d^{m_0}\mathbf{x} d^{m_1}\mathbf{y} \\ &=: \sum_{m\in\mathbb{N}_0^2} \frac{1}{m_0! m_1!} \langle\!\langle k_\mu^{(m)}, G^{(m)} \rangle\!\rangle =: \langle\!\langle k_\mu, G \rangle\!\rangle \\ &=: \int_{\Gamma_0} \int_{\Gamma_0} k_\mu(\eta_0, \eta_1) G(\eta_0, \eta_1) \lambda(d\eta_0) \lambda(d\eta_1). \end{aligned} \tag{5.2}$$

Here k_μ is the collection of the correlation functions $k_\mu^{(m)}$, $m \in \mathbb{N}_0^2$, that can also be considered as a function $k_\mu : \Gamma_0^2 \to \mathbb{R}$ such that

$$k_\mu(\eta) = k_\mu(\eta_0, \eta_1) = k^{(m)}(\mathbf{x}, \mathbf{y}), \quad \eta_0 = \{x_1, \dots, x_{m_0}\}, \;\; \eta_1 = \{y_1, \dots, y_{m_1}\}. \tag{5.3}$$

The integrals in (5.2) are understood in the following way, cf. (3.21),

$$\int_{\Gamma_0} \int_{\Gamma_0} k_\mu(\eta_0, \eta_1) G(\eta_0, \eta_1) \lambda(d\eta_0) \lambda(d\eta_1) \tag{5.4}$$

$$= \sum_{m\in\mathbb{N}_0^2} \frac{1}{m_0!m_1!} \int_{X^{m_0}\times X^{m_0}} k_\mu^{(m)}(\mathbf{x},\mathbf{y}) G^{(m)}(\mathbf{x},\mathbf{y}) d^{m_0}\mathbf{x} d^{m_1}\mathbf{y}.$$

In (5.3) and (5.4), k_μ – similarly as G in (5.2) – is the collection of symmetric $k_\mu^{(m)} \in L^\infty(X^{m_0}\times X^{m_1})$, considered as an element of the corresponding real linear space, which we denote by $\mathcal{K}$. Keeping in mind that we deal with $\mu(LF) = \mu(LKG)$, see (1.2), assume that we are given L^Δ such that

$$\mu(LKG) = \langle\!\langle L^\Delta k_\mu, G\rangle\!\rangle. \tag{5.5}$$

This L^Δ can be calculated explicitly, see [1, eq. (2.23)]. To present it here, we define

$$\tau_x^i(y) = e^{-\phi_i(x-y)}, \qquad t_x^i(y) = \tau_x^i(y) - 1, \qquad x,y\in X, \quad i=0,1, \tag{5.6}$$

and

$$\begin{aligned}(\Upsilon_y^0 k)(\eta_0,\eta_1) &= \int_{\Gamma_0} k(\eta_0,\eta_1\cup\xi) e(t_y^0;\xi)\lambda(d\xi), \\ (\Upsilon_y^1 k)(\eta_0,\eta_1) &= \int_{\Gamma_0} k(\eta_0\cup\xi,\eta_1) e(t_y^1;\xi)\lambda(d\xi),\end{aligned} \tag{5.7}$$

where, for an appropriate $\theta : X\to\mathbb{R}$ and $\xi\in\Gamma_0$, we write

$$e(\theta;\xi) = \prod_{x\in\xi}\theta(x).$$

The expressions in (5.7) are to be understood in the following way. For a given $m\in\mathbb{N}_0^2$, one sets

$$\begin{aligned}&(\Upsilon_y^0 k)^{(m)}(\mathbf{x},\mathbf{y}) = k^{(m)}(\mathbf{x},\mathbf{y}) \\ &+\sum_{n=1}^\infty \frac{1}{n!}\int_{X^n} k^{(m_0,m_1+n)}(x_1,\dots,x_{m_0};y_1,\dots,y_{m_1},z_1,\dots,z_n)\prod_{j=1}^n t_y^0(z_j)dz_1\cdots dz_n.\end{aligned} \tag{5.8}$$

The convergence of the series and the integrals will be shown below. In the same way, one defines also the second line of (5.7). Now the operator satisfying (5.5) presents in the following form

$$\begin{aligned}(L^\Delta k)(\eta_0,\eta_1) &= \sum_{y\in\eta_0}\int_X a_0(x-y)e(\tau_y^0;\eta_1)(\Upsilon_y^0 k)(\eta_0\setminus y\cup x,\eta_1)dx \\ &- \sum_{x\in\eta_0}\int_X a_0(x-y)e(\tau_y^0;\eta_1)(\Upsilon_y^0 k)(\eta_0,\eta_1)dy \\ &+ \sum_{y\in\eta_1}\int_X a_1(x-y)e(\tau_y^1;\eta_0)(\Upsilon_y^1 k)(\eta_0,\eta_1\setminus y\cup x)dx \\ &- \sum_{x\in\eta_1}\int_X a_1(x-y)e(\tau_y^1;\eta_0)(\Upsilon_y^1 k)(\eta_0,\eta_1)dy.\end{aligned} \tag{5.9}$$

For $\vartheta\in\mathbb{R}$ and $k\in\mathcal{K}$, see (5.3), we set

$$\|k\|_\vartheta = \operatorname*{ess\,sup}_{\xi_0,\xi_1\in\Gamma_0} |k(\xi_0,\xi_1)| \exp\Big(-\vartheta(|\xi_0|+|\xi_1|)\Big) \tag{5.10}$$

$$= \sup_{m\in\mathbb{N}_0^2} e^{-\vartheta(m_0+m_1)}\left(\operatorname*{ess\,sup}_{(\mathbf{x},\mathbf{y})\in X^{m_0}\times X^{m_1}} |k^{(m)}(\mathbf{x},\mathbf{y})|\right),$$

and then introduce

$$\mathcal{K}_\vartheta = \{k\in\mathcal{K} : \|k\|_\vartheta < \infty\}, \qquad \vartheta\in\mathbb{R}, \tag{5.11}$$

which is a real Banach space of weighted L^∞-type. By (5.10) one readily gets that $\|k\|_{\vartheta'} \le \|k\|_\vartheta$ whenever $\vartheta' > \vartheta$, which yields

$$\mathcal{K}_\vartheta \hookrightarrow \mathcal{K}_{\vartheta'}, \qquad \vartheta' > \vartheta, \tag{5.12}$$

where $\hookrightarrow$ denotes continuous embedding.

Let us turn now to the following issue. Given $k\in\mathcal{K}$, under which conditions is this k the correlation function for some $\mu\in\mathcal{P}(\Gamma^2)$? By (5.10) and Definition 3.1 one concludes, that $k_\mu\in\mathcal{K}_\vartheta$ with $\vartheta=\log\varkappa$ for $\mu\in\mathcal{P}_{\rm exp}$, where $\varkappa$ is the type of μ. At the same time, if $G\in\mathcal{G}_{\rm fin}$ is such that $(KG)(\gamma)\ge 0$, by (3.12) and (5.2) it follows that $\langle\!\langle k_\mu, G\rangle\!\rangle \ge 0$. Set $\mathcal{G}^\star_{\rm fin} = \{G\in\mathcal{G}_{\rm fin} : (KG)(\gamma)\ge 0,\ \gamma\in\Gamma^2\}$, and also

$$\mathcal{K}^\star = \{k\in\mathcal{K} : k^{(0,0)} = 1 \text{ and } \langle\!\langle k, G\rangle\!\rangle \ge 0\ \forall G\in\mathcal{G}^\star_{\rm fin}\}, \qquad \mathcal{K}^\star_\vartheta = \mathcal{K}^\star\cap\mathcal{K}_\vartheta,\ \ \vartheta\in\mathbb{R}. \tag{5.13}$$

It is known [1, Proposition 2.2], see also [18] for a more comprehensive discussion, that each $k\in\mathcal{K}^\star_\vartheta$ is the correlation function of a unique $\mu\in\mathcal{P}_{\rm exp}$ the type of which does not exceed e^ϑ. That is, $k\in\mathcal{K}_\vartheta$ is the correlation function of a unique sub-Poissonian state μ if and only if $k\in\mathcal{K}^\star_\vartheta$.

Proposition 5.1. *Let $k\in\mathcal{K}^\star_\vartheta$. Then $\|k\|_{\vartheta'} = 1$ for each $\vartheta' > \vartheta$.*

Proof. Firstly, we note that $\|k\|_\vartheta \ge 1$ for each $\vartheta\in\mathbb{R}$ since $k(\varnothing,\varnothing) = k^{(0,0)} = 1$, see (3.13). By (3.12) and the fact that $k\in\mathcal{K}_\vartheta$, it follows that $\|k^{(m_1,m_2)}\|_{L^\infty} \le e^{\vartheta(m_1+m_2)}$, which yields the proof. □

By (5.9) and (5.10) for $\vartheta\in\mathbb{R}$ we then have, see [1, eq. (3.10)] for more detail,

$$|(L^\Delta k)(\eta_0,\eta_1)| \le 4\alpha\|k\|_\vartheta e^{\vartheta(|\eta_0|+|\eta_1|)}\left(|\eta_0|+|\eta_1|\right)\exp\left(\varphi e^\vartheta\right), \tag{5.14}$$

where φ is as in (4.2) and

$$\alpha := \max_{i=0,1} \bar{a}_i^{(0)}, \tag{5.15}$$

see (4.3). This estimate settles the convergence issue in (5.8). It also implies

$$\|L^\Delta k\|_{\vartheta'} \le \frac{4\alpha\|k\|_\vartheta}{e(\vartheta'-\vartheta)}\exp\left(\varphi e^\vartheta\right), \quad \vartheta' > \vartheta, \tag{5.16}$$

which allows one to define the corresponding bounded linear operators acting from $\mathcal{K}_\vartheta$ to $\mathcal{K}_{\vartheta'}$. Along with them, we define an unbounded linear operator $L^\Delta_{\vartheta'}$, $\vartheta'\in\mathbb{R}$, which acts in $\mathcal{K}_{\vartheta'}$ according to (5.9) with domain

$$\mathcal{D}(L^\Delta_{\vartheta'}) = \{k\in\mathcal{K} : L^\Delta k\in\mathcal{L}_{\vartheta'}\}. \tag{5.17}$$

By (5.16) one concludes that

$$\mathcal{K}_\vartheta \subset \mathcal{D}(L^\Delta_{\vartheta'}), \qquad \vartheta < \vartheta'. \tag{5.18}$$

Now we fix $\vartheta\in\mathbb{R}$ and consider the following Cauchy problem in the Banach space $\mathcal{K}_\vartheta$

$$\frac{d}{dt}k_t = L^\Delta_\vartheta k_t, \qquad k_t|_{t=0} = k_0. \tag{5.19}$$

Definition 5.2. *By a solution of (5.19) on the time interval,* $[0,T)$, $T>0$, *we mean a continuous map* $[0,T)\ni t\mapsto k_t\in\mathcal{D}(L^\Delta_\vartheta)\subset\mathcal{K}_\vartheta$ *such that the map* $[0,T)\ni t\mapsto dk_t/dt\in\mathcal{K}_\vartheta$ *is also continuous and both equalities in (5.19) are verified.*

In view of the complex structure of (5.9), as well as of the fact that $\mathcal{K}_\vartheta$ is a weighted L^∞-type Banach space, it is barely possible to solve (5.19) with all $k_0\in\mathcal{D}(L^\Delta_\vartheta)$, e.g., by employing C_0-semigroup techniques. In [1], the solution was constructed for k_0 taken from $\mathcal{K}_{\vartheta_0}$ with $\vartheta_0<\vartheta$, see (5.18). Its characteristic feature is that k_t lies in some t-dependent $\mathcal{K}_{\vartheta'}$ such that, cf. (5.12),

$$\mathcal{K}_{\vartheta_0}\hookrightarrow\mathcal{K}_{\vartheta'}\hookrightarrow\mathcal{K}_\vartheta. \tag{5.20}$$

More precisely, the main result of [1] can be formulated as follows, see Theorem 3.5 *ibid*.

Proposition 5.3. *For each* $\mu\in\mathcal{P}_{\rm exp}$ *and* $T>0$, *the Cauchy problem in (5.19) with* $\vartheta=\vartheta(T):=\log\varkappa+\alpha T$ *has a unique solution* $k_t\in\mathcal{K}^\star_{\vartheta(T)}$, *where* $\varkappa$ *is the type of* μ *and* α *is as in (5.15).*

Remark 5.4. The proof of Proposition 5.3 is performed in the following three steps. First one shows that the Cauchy problem (5.19) with $k_0\in\mathcal{K}_{\vartheta_0}$, $\vartheta_0<\vartheta$, has a unique local solution $k_t\in\mathcal{K}_\vartheta$, see (5.20), i.e., existing for $t\in[0,T(\vartheta,\vartheta_0))$ with

$$T(\vartheta,\vartheta_0)=\frac{\vartheta-\vartheta_0}{4\alpha}\exp\left(-\varphi e^{\vartheta}\right). \tag{5.21}$$

The next (and the hardest) step is showing that, given $k_0\in\mathcal{K}^\star$, the solution k_t lies in $\mathcal{K}^\star$ and hence is the correlation function of a unique $\mu_t\in\mathcal{P}_{\rm exp}$. Finally, by means of the positivity as in (5.13) one makes continuation of the local solution k_t to all $t>0$ in such a way that $k_t\in\mathcal{K}_{\vartheta(t)}$ with $\vartheta(t)=\vartheta_0+\alpha t$, cf. (5.18).

5.2 The predual evolution

Along with the evolution $t\mapsto k_t$ described in Proposition 5.3 we will need the following one. Assume that we are given $\widehat{L}$ such that, cf. (5.5),

$$\langle\!\langle L^\Delta k,G\rangle\!\rangle=\langle\!\langle k,\widehat{L}G\rangle\!\rangle, \tag{5.22}$$

holding for all appropriate $k\in\mathcal{K}$ and $G\in\mathcal{G}_{\rm fin}$. This operator can be derived similarly as L^Δ given in (5.9). It has the following form

$$\begin{aligned}&(\widehat{L}G)(\eta_0,\eta_1) \qquad\qquad (5.23)\\ &=\sum_{x\in\eta_0}\int_X\sum_{\xi\subset\eta_1}e(\tau^0_y;\eta_1\setminus\xi)e(t^0_y;\xi)\left[G(\eta_0\setminus x\cup y,\eta_1\setminus\xi)-G(\eta_0,\eta_1\setminus\xi)\right]dy\\ &+\sum_{x\in\eta_1}\int_X\sum_{\xi\subset\eta_0}e(\tau^1_y;\eta_0\setminus\xi)e(t^1_y;\xi)\left[G(\eta_0\setminus\xi,\eta_1\setminus x\cup y)-G(\eta_0\setminus\xi,\eta_1)\right]dy,\end{aligned}$$

with t^i_y and τ^i_y given in (5.6). Obviously, $\widehat{L}$ is defined for each $G\in\mathcal{G}_{\rm fin}$. Our aim now is to extend it to G taken from the spaces predual to those defined in (5.11). To this end, we introduce the norm

$$|G|_\vartheta=\int_{\Gamma_0}\int_{\Gamma_0}|G(\xi_0,\xi_1)|\exp\left(\vartheta(|\xi_0|+|\xi_1|)\right)\lambda(d\xi_0)\lambda(d\xi_1), \tag{5.24}$$

and then define, cf. (5.11),

$$\mathcal{G}_\vartheta=\{G:|G|_\vartheta<\infty\}.$$

Thus, each $\mathcal{G}_\vartheta$ is a weighted L^1-type Banach space. Noteworthy, cf. (5.12),

$$\mathcal{G}_{\vartheta'} \hookrightarrow \mathcal{G}_\vartheta, \qquad \vartheta < \vartheta'. \tag{5.25}$$

By employing (5.23), similarly as in (5.16) we get

$$|\widehat{L}G|_\vartheta \leq \frac{4\alpha |G|_{\vartheta'}}{e(\vartheta' - \vartheta)} \exp\left(\varphi e^{\vartheta}\right), \quad \vartheta' > \vartheta. \tag{5.26}$$

The latter formula allows one to define (by induction in n) the iterations of $\widehat{L}$, cf. (5.25),

$$(\widehat{L})^n_{\vartheta\vartheta'} : \mathcal{G}_{\vartheta'} \to \mathcal{G}_\vartheta, \qquad n \in \mathbb{N},$$

the operator norms of which obey

$$\|(\widehat{L})^n_{\vartheta\vartheta'}\| \leq n^n \left(\frac{4\alpha}{\vartheta' - \vartheta}\right)^n \exp\Big(n(\varphi e^{\vartheta'} - 1)\Big). \tag{5.27}$$

Then we introduce the operators

$$\varSigma_{\vartheta\vartheta'}(t) = 1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} (\widehat{L})^n_{\vartheta\vartheta'}, \tag{5.28}$$

where the series converges in the norm of the Banach space $\mathcal{L}(\mathcal{G}_{\vartheta'}, \mathcal{G}_\vartheta)$ of bounded linear operators acting from $\mathcal{G}_{\vartheta'}$ to $\mathcal{G}_\vartheta$ – uniformly on compact subsets of $[0, T(\vartheta', \vartheta))$, with $T(\vartheta', \vartheta)$ defined in (5.21). The latter fact readily follows by (5.27). Then, for $t < T(\vartheta', \vartheta)$, we can set

$$G_t = \varSigma_{\vartheta\vartheta'}(t) G, \qquad G \in \mathcal{G}_{\vartheta'}. \tag{5.29}$$

For a given ϑ, let $k \in \mathcal{K}_\vartheta$ be the correlation function of a certain $\mu \in \mathcal{P}_{\rm exp}$. According to Proposition 5.3, see also Remark 5.4, there exists the map $t \mapsto k_t$, $k_0 = k$, that solves (5.19) and is such that $k_t \in \mathcal{K}^\star_{\vartheta(t)}$ with $\vartheta(t) = \vartheta + \alpha t$. Let ϑ and ϑ' be as in (5.29). For $t < T(\vartheta', \vartheta)$, by (5.21) it follows that

$$\vartheta(t) < \vartheta + \alpha T(\vartheta', \vartheta) < \vartheta',$$

which means that $\mathcal{K}_{\vartheta(t)} \subset \mathcal{K}_{\vartheta'}$. The continuation mentioned in Remark 5.4 was done in [1] by showing that the solution – a priori lying in $\mathcal{K}_{\vartheta'}$ – is in fact in $\mathcal{K}_{\vartheta(t)}$. For $t < T(\vartheta', \vartheta)$, it can be obtained similarly as in (5.29). By induction in n one defines bounded operators $(L^\Delta)^n_{\vartheta'\vartheta} : \mathcal{K}_\vartheta \to \mathcal{K}_{\vartheta'}$, $n \geq 2$, the norms of which are estimated as in (5.27). Then one sets

$$k_t = \varXi_{\vartheta'\vartheta}(t) k_0, \qquad \varXi_{\vartheta'\vartheta}(t) = 1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} (L^\Delta)^n_{\vartheta'\vartheta}, \qquad t < T(\vartheta', \vartheta). \tag{5.30}$$

For each $t < T(\vartheta', \vartheta)$, one finds $\vartheta'' \in (\vartheta, \vartheta')$ such that $t < T(\vartheta'', \vartheta)$, see (5.21), which means that $\varXi_{\vartheta'\vartheta}(t)$ maps $\mathcal{K}_\vartheta$ to $\mathcal{D}(L^\Delta_{\vartheta'})$, see (5.12), (5.17). Furthermore, by the absolute convergence of the series in (5.30) – in the norm of the Banach space $\mathcal{L}(\mathcal{K}_\vartheta, \mathcal{K}_{\vartheta'})$ – it follows that the map $t \mapsto \varXi_{\vartheta'\vartheta}(t)$ is continuously differentiable in this norm and the following holds

$$\frac{d}{dt} \varXi_{\vartheta'\vartheta}(t) = \varXi_{\vartheta'\vartheta''}(t) L^\Delta_{\vartheta''\vartheta} = L^\Delta_{\vartheta'\vartheta''} \varXi_{\vartheta''\vartheta}(t) = L^\Delta_{\vartheta'} \varXi_{\vartheta'\vartheta}(t), \qquad t < T(\vartheta', \vartheta), \tag{5.31}$$

which yields that k_t as in (5.30) solves (5.19). Then the steps mentioned in Remark 5.4 amount to the following. For fixed ϑ, ϑ', one constructs $\varXi_{\vartheta'\vartheta}(t)$, $t < T(\vartheta', \vartheta)$, and shows by (5.31) that k_t as in (5.30) solves the corresponding Cauchy problem. Then one takes $k_0 = k_\mu \in \mathcal{K}^\star_{\vartheta_0}$ and shows that k_t, $t < T(\vartheta, \vartheta_0)$ for some $\vartheta > \vartheta_0$, obtained as just mentioned, lies in $\mathcal{K}^\star$. Finally, by the positivity as in (5.13) one proves that this k_t lies in $\mathcal{K}^\star_{\vartheta(t)}$, $\vartheta(t) = \vartheta_0 + \alpha t < \vartheta$ for $t < T(\vartheta, \vartheta_0)$. The continuation to $s > t$ is then performed by applying $\Xi_{\vartheta\vartheta(t)}(s)$ to k_t, see [1, Lemma 5.5] fore more detail.

Complementary information concerning the operator norms of the maps $t \mapsto \Sigma_{\vartheta\vartheta'}(t)$ and $t \mapsto \Xi_{\vartheta'\vartheta}(t)$ is given by the following estimates

$$\|\Sigma_{\vartheta\vartheta'}(t)\| \leq \frac{T(\vartheta', \vartheta)}{T(\vartheta', \vartheta) - t}, \qquad |\Xi_{\vartheta'\vartheta}(t)| \leq \frac{T(\vartheta', \vartheta)}{T(\vartheta', \vartheta) - t}, \qquad t < T(\vartheta', \vartheta), \tag{5.32}$$

which readily follow by (5.27) and the corresponding estimate of $(L^\Delta)^n_{\vartheta'\vartheta}$, respectively. By means of (5.31) and (5.30) we also obtain the following.

Proposition 5.5. *Given ϑ and $\vartheta' > \vartheta$, let $k_0 \in \mathcal{K}_\vartheta$ be the correlation function of a certain $\mu \in \mathcal{P}_{\rm exp}$ and then k_t be the solution as in Proposition 5.3. Let also G be in $\mathcal{G}_{\vartheta'}$. Then for each $t < T(\vartheta', \vartheta)$ the following holds*

$$\langle\!\langle k_t, G \rangle\!\rangle = \langle\!\langle k_0, G_t \rangle\!\rangle, \tag{5.33}$$

where G_t is as in (5.29).

We end up this section by producing appropriate extensions of the map $G \mapsto KG$ defined in (5.1) for $G \in \mathcal{G}_{\rm fin}$. Set $\mathcal{G}_\infty = \cap_{\vartheta \in \mathbb{R}} \mathcal{G}_\vartheta$. As is usual, we do not distinguish between the elements of $\mathcal{G}_\infty$ and the measurable functions $G : \Gamma^2_0 \to \mathbb{R}$ for which the integrals in the right-hand side of (5.24) are finite for all $\vartheta \in \mathbb{R}$. Then $\mathcal{G}_{\rm fin} \subset \mathcal{G}_\infty$. We recall that each measurable $G : \Gamma^2_0 \to \mathbb{R}$ is a collection $\{G^{(m)}\}_{m \in \mathbb{N}^2_0}$ of symmetric (cf. (3.11)) Borel functions. Similarly as in [18, Theorem 1], one can show that, for each such G and $m \in \mathbb{N}^2_0$, the map

$$\Gamma^2_* \ni \gamma \mapsto \sum_{(\mathbf{x}, \mathbf{y}) \in \gamma} G^{(m)}(\mathbf{x}, \mathbf{y})$$

is $\mathcal{B}(\Gamma^2_*)$-measurable. Then also the functions (possibly taking infinite values)

$$F_G(\gamma) := \sum_{m \in \mathbb{N}^2_0} \frac{1}{m_0! m_1!} \sum_{(\mathbf{x}, \mathbf{y}) \in \gamma} |G^{(m)}(\mathbf{x}, \mathbf{y})|, \qquad G \in \mathcal{G}_\infty, \quad \gamma \in \Gamma^2_*,$$

enjoy this property; hence, the sets

$$\Gamma^2_G = \bigcup_{n \in \mathbb{N}} \{\gamma \in \Gamma^2_* : F_G(\gamma) \leq n\}, \quad G \in \mathcal{G}_\infty,$$

are $\mathcal{B}(\Gamma^2_*)$-measurable. Moreover, Γ^2_* itself is $\Gamma^2_{G_\psi}$ for G_ψ such that $G^{(1,0)}_\psi(x) = G^{(0,1)}_\psi(x) = \psi(x)$ and $G^{(m)}_\psi = 0$ whenever $|m| \neq 1$, see (3.31). Let $\mu \in \mathcal{P}_{\rm exp}$ be of type e^ϑ for some $\vartheta \in \mathbb{R}$. By (5.2) we thus have

$$\mu(F_G) \leq |G|_\vartheta.$$

Similarly as in (3.33) and (3.34) we then get that $\mu(\Gamma^2_G) = 1$. Therefore, for each $\mu \in \mathcal{P}_{\rm exp}$ and $G \in \mathcal{G}_\infty$, the series in

$$(KG)(\gamma) := \sum_{m \in \mathbb{N}^2_0} \frac{1}{m_0! m_1!} \sum_{(\mathbf{x}, \mathbf{y}) \in \gamma} G^{(m)}(\mathbf{x}, \mathbf{y}) \tag{5.34}$$

absolutely converges, μ-almost everywhere on Γ^2_*. This includes also the Poisson measures π_κ with all $\kappa > 0$, see (3.19). By means of these argument we obtain the following conclusion.

Proposition 5.6. *Let $\mu \in \mathcal{P}_{\rm exp}$ be of type e^{ϑ} for some $\vartheta \in \mathbb{R}$. Then the map $G \mapsto KG$ as in (5.34) gives rise to the bounded linear operator K acting from the Banach space $\mathcal{G}_\vartheta$ to the Banach space $L^1(\Gamma^2_*, \mu)$, such that*

$$\mu(KG) = \langle\!\langle k_\mu, G\rangle\!\rangle.$$

Moreover, if G belongs to $\mathcal{G}_{\vartheta'}$ for some $\vartheta' > \vartheta$, then

$$\mu(LKG) = \mu(K\widehat{L}G) = \langle\!\langle k_\mu, \widehat{L}G\rangle\!\rangle,$$

where $\widehat{L} : \mathcal{G}_{\vartheta'} \to \mathcal{G}_\vartheta$ is the linear operator defined in (5.26).

6 Uniqueness

6.1 Solving the Fokker-Planck equation

We begin by recalling Definition 4.6, in which we mention maps $t \mapsto \mu_t \in \mathcal{P}(\Gamma^2_*)$.

Lemma 6.1. *Let $\mu_0 \in \mathcal{P}_{\rm exp}$ be of type $\varkappa_0 = e^{\vartheta_0}$ and consider the Fokker-Planck equation (1.2) with the initial condition $\mu_t|_{t=0} = \mu_0$ and all choices of $F \in \widehat{\mathcal{F}}$, see (3.55) and Definition 4.1. Assume that $t \mapsto \mu_t$ is a solution of (1.2) with such μ_0 and F. Then $\mu_t \in \mathcal{P}_{\rm exp}$; moreover, for each $T > 0$, there exists $\vartheta_T > \vartheta_0$ such that the type of μ_t does not exceed e^{ϑ_T} for all $t \leq T$.*

Note that in this lemma we assume that only μ_0 is sub-Poissonian, and that $t \mapsto \mu_t$ solves (1.2) only with a part of $\mathcal{D}(L)$. Before proceeding further, we recall that the families of functions $\widetilde{\mathcal{F}}$ and $\widehat{\mathcal{F}}$ were introduced in (3.50) and (3.55), respectively.

Proposition 6.2. *Set $\mathcal{F}_\infty = \{F = KG : G \in \mathcal{G}_\infty\}$, see (5.34). Then both $\widetilde{\mathcal{F}}$ and $\widehat{\mathcal{F}}$ are subsets of $\mathcal{F}_\infty$.*

Proof. By (3.49) and then by (5.1) one readily gets that

$$\begin{aligned}\widetilde{F}^{\theta}_{\tau}(\gamma) &= \left(\sum_{m_0=0}^{\infty} \sum_{\{x_1,\dots,x_{m_0}\}\subset\gamma_0} \widetilde{G}^{(m_0)}_{\tau_0,\theta_0}(x_1,\dots,x_{m_0})\right)\left(\sum_{m_1=0}^{\infty} \sum_{\{y_1,\dots,y_{m_1}\}\subset\gamma_1} \widetilde{G}^{(m_1)}_{\tau_1,\theta_1}(y_1,\dots,y_{m_1})\right) \\ &= (K\widetilde{G}_{\tau,\theta})(\gamma), \qquad \widetilde{G}_{\tau,\theta}(\eta_0,\eta_1) = \widetilde{G}_{\tau_0,\theta_0}(\eta_0)\widetilde{G}_{\tau_1,\theta_1}(\eta_1), \end{aligned} \tag{6.1}$$

where

$$\widetilde{G}^{(m_i)}_{\tau_i,\theta_i}(x_1,\dots,x_{m_i}) = \prod_{j=1}^{m_i} \theta_i^{\tau_i}(x_j), \quad \theta_i^{\tau_i}(x) := \theta_i(x)e^{-\tau_i\psi(x)} + e^{-\tau_i\psi(x)} - 1, \;\; i = 0,1. \tag{6.2}$$

Clearly, $\theta_i^\tau \in L^1(X)$ for each $\tau \geq 0$, $\theta_i \in \Theta_\psi$, $i = 0,1$. Hence, $\widetilde{G}_{\tau,\theta} \in \mathcal{G}_\vartheta$ for any $\vartheta \in \mathbb{R}$, which yields $\widetilde{\mathcal{F}} \subset \mathcal{F}_\infty$.

Now by the first line in (3.55) we have, see (3.4),

$$\begin{aligned}\widehat{F}^{m_i}_{\tau_i}(\mathbf{v}_i|\gamma_i) &= \sum_{\mathbf{x}^{m_i}\in\gamma_i} \mathbf{v}_i(\mathbf{x}^{m_i}) \prod_{x\in\gamma_i\setminus\mathbf{x}^{m_i}} (1+\varsigma_i(x)) \\ &= \sum_{\eta_i\subset\gamma_i} R^{m_i}(\mathbf{v}_i|\eta_i) \prod_{x\in\gamma_i\setminus\mathbf{x}^{m_i}} (1+\varsigma_i(x)),\end{aligned} \tag{6.3}$$

where $\varsigma_i(x) = e^{-\tau_i\psi(x)} - 1$, and, see (3.54),

$$R^{m_i}(\mathbf{v}_i|\eta_i) = \begin{cases} \sum_{\sigma\in S_{m_i}} v_{i,1}(x_{\sigma(1)})\cdots v_{i,m_i}(x_{\sigma(m_i)}), & \text{if} \quad \eta_i = \{x_1,\dots,x_{m_i}\}, \\ 0, & \text{otherwise}. \end{cases} \tag{6.4}$$

Now we open the brackets in the product in (6.3) and get

$$\begin{aligned}\widehat{F}^{m_i}_{\tau_i}(\mathbf{v}_i|\gamma_i) &= \sum_{\eta_i\subset\gamma_i}\widehat{G}^{(m_i)}_{\tau_i}(\mathbf{v}_i|\eta_i), \\ \widehat{G}^{(m_i)}_{\tau_i}(\mathbf{v}_i|\eta_i) &:= \sum_{\xi_i\subset\eta_i} R^{m_i}(\mathbf{v}_i|\xi_i)\prod_{x\in\eta_i\setminus\xi_i}\varsigma_i(x).\end{aligned} \tag{6.5}$$

To complete the proof we have to show the corresponding integrability of $\widehat{G}^{(m_i)}_{\tau_i}(\mathbf{v}_i|\cdot)$. Since $v_{i,j}\in\Theta^+_\psi$ and $\tau_i>0$, we have

$$\left|\widehat{G}^{(m_i)}_{\tau_i}(\mathbf{v}_i|\eta_i)\right| \le \sum_{\xi_i\subset\eta_i} R^{m_i}(\mathbf{v}_i|\xi_i)\prod_{x\in\eta_i\setminus\xi_i}[\tau_i\psi(x)],$$

and hence

$$\begin{aligned}\int_{\Gamma_0}\left|\widehat{G}^{(m_i)}_{\tau_i}(\mathbf{v}_i|\eta)\right| e^{\vartheta|\eta|}\lambda(d\eta) &\le \int_{\Gamma_0}\int_{\Gamma_0} e^{\vartheta|\xi|}R^{m_i}(\mathbf{v}_i|\xi)e^{\vartheta|\eta|}\prod_{x\in\eta}[\tau_i\psi(x)]\,\lambda(d\xi)\lambda(d\eta) \\ &= e^{m_i\vartheta}\langle v_{1,i}\rangle\cdots\langle v_{m_i,i}\rangle\exp\left(\tau_i e^{\vartheta}\langle\psi\rangle\right),\end{aligned} \tag{6.6}$$

where $\langle v_{j,i}\rangle$, $j=1,\dots,m_i$, and $\langle\psi\rangle$ are the $L^1(X)$-norms of these functions. Similarly as in (6.1) we then have

$$\widehat{F}^m_\tau(\mathbf{v}|\gamma) = (K\widehat{G}^m_\tau(\mathbf{v}|\cdot))(\gamma), \quad \widehat{G}^m_\tau(\mathbf{v}|\eta) = \widehat{G}^{m_0}_{\tau_0}(\mathbf{v}_0|\eta_0)\widehat{G}^{m_1}_{\tau_1}(\mathbf{v}_1|\eta_1), \tag{6.7}$$

which completes the proof. □

Lemma 6.3. *For each $\mu\in\mathcal{P}_{\rm exp}$, the Fokker-Planck equation (1.2) with $\mu_0=\mu$ has exactly one solution.*

Proof. Existence: Let $t\mapsto k_t$ be as in Proposition 5.3 with $k_0=k_\mu$. Since k_t solves (5.19), it follows that

$$k_{t_2}-k_{t_1} = \int_{t_1}^{t_2} L^\Delta_{\vartheta(T)}k_s ds, \tag{6.8}$$

holding for all $t_2>t_1\ge 0$ and $T>t_2$. Let $\mu_t\in\mathcal{P}_{\rm exp}$ be the unique measure for which k_t is the correlation function, see Remark 5.4. Then for each $G\in\mathcal{G}_\infty$, we have

$$\mu_{t_j}(KG) = \langle\!\langle k_{t_j}, G\rangle\!\rangle, \qquad j=1,2.$$

For each $\vartheta\in\mathbb{R}$, the map $\mathcal{K}_\vartheta\ni k\mapsto\langle\!\langle k,G\rangle\!\rangle$ is linear and bounded – hence continuous. Then by (6.8) and Proposition 5.6 for $F=KG$ we get

$$\begin{aligned}\mu_{t_2}(F)-\mu_{t_1}(F) &= \langle\!\langle \int_{t_1}^{t_2} L^\Delta_{\vartheta(T)}k_s ds, G\rangle\!\rangle = \int_{t_1}^{t_2}\langle\!\langle L^\Delta_{\vartheta(T)}k_s, G\rangle\!\rangle ds \\ &= \int_{t_1}^{t_2}\langle\!\langle k_s, \widehat{L}G\rangle\!\rangle ds = \int_{t_1}^{t_2}\mu_s(LF)ds.\end{aligned} \tag{6.9}$$

Now we can take $G=\widehat{G}^m_\tau(\mathbf{v}|\cdot)$, see (6.7), or $G=\widetilde{G}_{\tau,\theta}$, see (6.1), (6.2), and conclude that the map $t\mapsto\mu_t$ is a solution of (1.2) according to Definition 4.6.

Uniqueness: Let $t\mapsto\tilde{\mu}_t$ be another solution satisfying $\tilde{\mu}_t|_{t=0}=\mu$. Let also ϑ_0 be such that $k_0=k_\mu\in\mathcal{K}_{\vartheta_0}$. For a fixed $T>0$ and each $t\le T$, by Lemma 6.1 it follows that $\tilde{\mu}_t\in\mathcal{P}_{\rm exp}$ and its type does not exceed e^{ϑ_T}. That is, the correlation function $\tilde{k}_t$ of this measure $\tilde{\mu}_t$ lies in $\mathcal{K}_{\vartheta_T}$. Without any harm we may take ϑ_T big enough so that

$$\sup_{s\in[0,T]}\|\tilde{k}_s\|_{\vartheta_T} = 1, \tag{6.10}$$

see Proposition 5.1, and also $\vartheta_T \geq \vartheta(T) = \vartheta_0 + \alpha T$, see Proposition 5.3.

It is known, see [1, eqs. (4.6) – (4.8)], that the map $[\vartheta_T, +\infty) \ni \vartheta \mapsto T(\vartheta, \vartheta_T)$, see (5.21), attains maximum $T_*(\vartheta_T)$ at $\tilde{\vartheta}_T = \vartheta_T + \delta(\vartheta_T)$, where

$$T_*(\vartheta_T) = \frac{\delta(\vartheta_T)}{4\alpha} \exp\left(-\frac{1}{\delta(\vartheta_T)}\right), \tag{6.11}$$

and $\delta(\vartheta_T)$ is the unique solution of the equation

$$\delta e^{\delta} = \exp\left(-\vartheta_T - \log \varphi\right).$$

According to our assumption $\tilde{k}_t \in \mathcal{K}_{\vartheta_T} \subset \mathcal{D}(L^{\Delta}_{\tilde{\vartheta}_T})$, see (5.18), and

$$\tilde{\mu}_t(KG) - \mu(KG) = \langle\!\langle \tilde{k}_t - k_0, G \rangle\!\rangle = \int_0^t \langle\!\langle L^{\Delta}_{\tilde{\vartheta}_T} \tilde{k}_s, G \rangle\!\rangle ds, \tag{6.12}$$

holding for all $t \leq T$ and G such that $KG \in \widetilde{\mathcal{F}} \cup \widehat{\mathcal{F}}$. That is, G is either $\widetilde{G}_{\tau,\theta}$ (6.1) or $\widehat{G}^m_\tau(\mathbf{v}|\cdot)$ (6.7). The integrations in (6.12) were interchanges for the same reasons as in (6.9). Let us prove that (6.12) holds for all $G \in \mathcal{G}_\infty$. By (5.16) and (6.10) we have

$$\|L^{\Delta}_{\tilde{\vartheta}_T} \tilde{k}_s\|_{\tilde{\vartheta}_T} \leq 1/eT_*(\vartheta_T), \tag{6.13}$$

holding for all $s \leq T$. Now we fix

$$t < \min\{T; T_*(\vartheta_T)\}, \tag{6.14}$$

and set

$$q = \tilde{k}_t - k_0 - \int_0^t L^{\Delta}_{\tilde{\vartheta}_T} \tilde{k}_s ds.$$

By Proposition 5.1, and then by (6.10) and (6.13), we get

$$\|q\|_{\tilde{\vartheta}_T} \leq 2 + t/eT_*(\vartheta_T). \tag{6.15}$$

Then for $G = \widehat{G}^m_\tau(\mathbf{v}|\cdot)$ with $\tau_i \leq 1$, $i = 0, 1$, by (6.12) it follows that $\langle\!\langle q, G \rangle\!\rangle = 0$. At the same time, by (6.7), (6.6), (6.5) and (6.15), we have

$$\begin{aligned} |\langle\!\langle q, G \rangle\!\rangle| &\leq \|q\|_{\tilde{\vartheta}_T} |G|_{\tilde{\vartheta}_T} \\ &\leq (2 + t/eT_*(\vartheta_T)) \exp\left((m_0 + m_1)\tilde{\vartheta}_T + 2\langle\psi\rangle e^{\tilde{\vartheta}_T}\right) \prod_{j_0=1}^{m_0} \prod_{j_1=1}^{m_1} \langle v_{j,i} \rangle. \end{aligned} \tag{6.16}$$

Let G_ε denote $\widehat{G}^m_\tau(\mathbf{v}|\cdot)$ with $\tau_0 = \tau_1 = \varepsilon \leq 1$. Then by the dominated convergence theorem and (6.16) we get

$$\langle\!\langle q, G_0 \rangle\!\rangle = \lim_{\varepsilon \to 0} \langle\!\langle q, G_\varepsilon \rangle\!\rangle = 0, \tag{6.17}$$

where G_0 is the pointwise limit of G_ε as $\varepsilon \to 0$. That is, see (6.5) and (6.4),

$$G_0(\eta) = \widehat{G}^{m_0}_0(\mathbf{v}_0|\eta_0) \widehat{G}^{m_1}_0(\mathbf{v}_1|\eta_1) = R^{m_0}(\mathbf{v}_0|\eta_0) R^{m_1}(\mathbf{v}_1|\eta_1). \tag{6.18}$$

Now we use this G_0 in (6.17) and obtain, see (3.54),

$$\begin{aligned} &\int_{X^{m_0} \times X^{m_1}} q^{(m)}(\mathbf{x}, \mathbf{y}) \mathbf{v}_0(\mathbf{x}) \mathbf{v}_1(\mathbf{y}) d^{m_0}\mathbf{x} d^{m_1}\mathbf{y} \\ &= \int_{X^{m_0} \times X^{m_1}} q^{(m_0,m_1)}(x_1, \ldots, x_{m_0}; y_1, \ldots, y_{m_1}) v_{0,1}(x_1) \cdots v_{0,m_0}(x_{m_0}) \end{aligned} \tag{6.19}$$

$$\times v_{1,1}(y_1)\cdots v_{1,m_1}(x_{m_1})dx_1\cdots dx_{m_0}dy_1\cdots dy_{m_1}=0,$$

holding for all $m=(m_0,m_1)\in\mathbb{N}_0^2$ and $v_{i,j}\in\Theta_\psi^+$. For each $m\in\mathbb{N}_0^2$, the set of functions $(\mathbf{x},\mathbf{y})\mapsto\mathbf{v}_0(\mathbf{x})\mathbf{v}_1(\mathbf{y})$, $v_{i,j}\in\Theta_\psi^+$, is closed with respect to the pointwise multiplication and separates the points of $X^{m_0}\times X^{m_1}$. Such functions vanish at infinity and are everywhere positive, see (3.38). Then by the corresponding version of the of the Stone-Weierstrass theorem [4], the linear span of this set is dense (in the supremum norm) in the algebra $C_0(X^{m_0}\times X^{m_1})$ of continuous functions that vanish at infinity (recall that $X=\mathbb{R}^d$, hence $X^{m_0}\times X^{m_1}$ is locally compact). At the same time, $C_0(X^{m_0}\times X^{m_1})\cap L^1(X^{m_0}\times X^{m_1})$ is dense in $L^1(X^{m_0}\times X^{m_1})$ as its subset $C_{\rm cs}(X^{m_0}\times X^{m_1})$ has this property. Thus,

$$\langle\!\langle q^{(m)},G^{(m)}\rangle\!\rangle=0,$$

holding for all $G^{(m)}\in L^1(X^{m_0}\times X^{m_1})$. The extension of the latter to

$$\langle\!\langle q,G\rangle\!\rangle=0,\qquad\text{for}\qquad G\in\mathcal{G}_\infty,$$

is standard, which yields the validity of (6.12) for all such G. By (5.22), (6.9) and (6.12) we have

$$\langle\!\langle\tilde{k}_t,G\rangle\!\rangle=\langle\!\langle k_0,G\rangle\!\rangle+\int_0^t\langle\!\langle L^{\Delta}_{\tilde\vartheta_T\vartheta_T}\tilde{k}_s,G\rangle\!\rangle ds \tag{6.20}$$

$$=\langle\!\langle k_0,G\rangle\!\rangle+\int_0^t\langle\!\langle\tilde{k}_s,\widehat{L}_{\vartheta_T\tilde\vartheta_T}G\rangle\!\rangle ds\qquad G\in\mathcal{G}_\infty.$$

Note that, for $G\in\mathcal{G}_\infty$, $\widehat{L}_{\vartheta_T\tilde\vartheta_T}G\in\mathcal{G}_{\vartheta_T}$, where the latter space is predual to $\mathcal{K}_{\vartheta_T}$, and $\tilde{k}_s\in\mathcal{K}_{\vartheta_T}$ for all $s\le t\le T$. For $G\in\mathcal{G}_\infty$, the action of $\widehat{L}_{\vartheta_T\tilde\vartheta_T}$ on G is the same as in (5.23), that by (5.26) yields $G_1:=\widehat{L}_{\vartheta_T\tilde\vartheta_T}G\in\mathcal{G}_\infty$. Therefore, one can write (6.20) also for G_1. Repeating this procedure n times we arrive at the following

$$\langle\!\langle\tilde{k}_t,G\rangle\!\rangle=\langle\!\langle k_0,G\rangle\!\rangle+t\langle\!\langle k_0,\widehat{L}_{\vartheta_T\tilde\vartheta_T}G\rangle\!\rangle+\frac{t^2}{2}\langle\!\langle k_0,(\widehat{L}_{\vartheta_T\tilde\vartheta_T})^2G\rangle\!\rangle \tag{6.21}$$

$$+\cdots+\frac{t^{n-1}}{(n-1)!}\langle\!\langle k_0,(\widehat{L}_{\vartheta_T\tilde\vartheta_T})^{n-1}G\rangle\!\rangle+\int_0^t\int_0^{t_1}\cdots\int_0^{t_{n-1}}\langle\!\langle\tilde{k}_{t_n},(\widehat{L}_{\vartheta_T\tilde\vartheta_T})^nG\rangle\!\rangle dt_1\cdots dt_n.$$

Let k_t be the solution as in (6.8). Our choice of ϑ_T is such that $k_t\in\mathcal{K}_{\vartheta_T}$, hence (6.21) can also be written for this k_t, which yields

$$\langle\!\langle\tilde{k}_t-k_t,G\rangle\!\rangle=\int_0^t\int_0^{t_1}\cdots\int_0^{t_{n-1}}\langle\!\langle\tilde{k}_{t_n}-k_{t_n},(\widehat{L}_{\vartheta_T\tilde\vartheta_T})^nG\rangle\!\rangle dt_1\cdots dt_n.$$

Now by (5.27) we obtain from the latter, see (6.14),

$$\left|\langle\!\langle\tilde{k}_t-k_t,G\rangle\!\rangle\right|\le 2\frac{n^n}{n!e^n}\left(\frac{t}{T_*(\vartheta_T)}\right)^n|G|_{\vartheta_0}\to 0,\qquad\text{as } n\to+\infty.$$

Thus, $\tilde{k}_t=k_t$ for t satisfying (6.14). The continuation of this equality to all t can be made by repeating this construction, similarly as in [1, the proof of Theorem 3.5, pages 659, 660]. Now the equality $\tilde\mu_t=\mu_t$ follows by the fact that each $\mu\in\mathcal{P}_{\rm exp}$ is identified by its correlation function, see Remark 5.4. □

6.2 Useful estimates

The aim of this subsection is to prepare the proof of Lemma 6.1. A priori a solution μ_t need not be a sub-Poissonian state, so one can speak of $\mu_t(F)$ only for bounded F, in particular of $\mu_t(\widehat{F}_\tau(\mathbf{v}|\cdot))$. At the same time, $\widehat{F}_\tau(\mathbf{v}|\cdot)$ is bounded for positive τ_i only, see the proof of Proposition 3.9. Assume that we have obtained an estimate of $\mu_t(\widehat{F}_\tau(\mathbf{v}|\cdots))$ that is uniform in τ, which might allow for passing to the limit $\max_i \tau_i \to 0$. Assume further that this limit satisfies an estimate similar to (3.8) with a certain t-dependent $\varkappa$. Then the proof will follow with the help of Definition 3.1. Let us then turn to obtaining such estimates. Here we will mostly follow the way elaborated in [16].

Our starting point is the estimate obtained in (4.15) the right-hand side of which is an element of $\mathcal{D}(L)$. Significantly, it is independent of the interaction terms ϕ_i, $i = 0, 1$, where both components appear in a multiplicative form, similarly as in $\widehat{F}_\tau(\mathbf{v}|\cdot)$ in (3.55). Another observation is that in the latter function all $v_{i,j}$ with the same $i = 0, 1$ can be different, whereas (3.9) is based on just two functions θ_0, θ_1. Keeping this fact in mind, we introduce the following functions. Fix $\theta_0, \theta_1 \in \Theta_\psi^+$ and set, cf. (3.55),

$$\varPhi_{\tau_i}^{m_i}(\theta_i|\gamma_i) = \widehat{F}_{\tau_i}^{m_i}(\mathbf{v}_i|\gamma_i)|_{v_{i,j}=\theta_i} = \sum_{\mathbf{x}^{m_i}\in\gamma_i} \theta_i^{\otimes m_i}(\mathbf{x}^{m_i}) \exp\left(-\tau_i \Psi(\gamma_i \setminus \mathbf{x}^{m_i}\right), \quad i = 0, 1. \tag{6.22}$$

Along with this, we also introduce

$$\begin{aligned} \varPhi_{\tau_i}^{m_i,\theta_i^1}(\theta_i|\gamma_i) &= \widehat{F}_{\tau_i}^{m_i}(\mathbf{v}_i|\gamma_i)|_{v_{i,1}=a_i\theta_i,\ v_{i,j}=\theta_i, j\geq 2}, \\ \varPhi_{\tau_i,1}^{m_i}(\theta_i|\gamma_i) &= m_i \varPhi_{\tau_i}^{m_i,\theta_i^1}(\theta_i|\gamma_i) + \tau_i c_a \bar{c}_{\theta_i} \widehat{F}_{\tau_i}^{m_i+1}(\gamma_i), \end{aligned} \tag{6.23}$$

where $\theta_i^1 := a_i\theta_i$ and $\bar{c}_{\theta_i}$ are as in (4.11) and in (3.39), respectively; $\widehat{F}_{\tau_i}^{m_i+1}(\gamma_i)$ is as in (4.16). Now we set

$$\varPhi_\tau^m(\theta|\gamma) = \varPhi_{\tau_0}^{m_0}(\theta_0|\gamma_0)\varPhi_{\tau_1}^{m_1}(\theta_1|\gamma_1), \tag{6.24}$$

for which by the estimate in (4.15) we then get

$$|L\varPhi_\tau^m(\theta|\gamma)| \leq \varPhi_{\tau_0,1}^{m_0}(\theta_0|\gamma_0)\varPhi_{\tau_1}^{m_1}(\theta_1|\gamma_1) + \varPhi_{\tau_0}^{m_0}(\theta_0|\gamma_0)\varPhi_{\tau_1,1}^{m_1}(\theta_1|\gamma_1) =: \varPhi_{\tau,1}^m(\theta|\gamma). \tag{6.25}$$

Each of the summands above is a linear combination of the corresponding functions $\widehat{F}_{\tau_i}^{\bar{m}_i}(\mathbf{v}_i|\cdot)$. Hence $\varPhi_{\tau,1}^m(\theta|\cdot) \in \mathcal{D}(L)$, and one can estimate $L\varPhi_{\tau,1}^m(\theta|\cdot)$ by repeating the above procedure based on (4.15). This yields

$$\begin{aligned} \left|L\varPhi_{\tau,1}^m(\theta|\cdot)(\gamma)\right| &\leq \left(\int_X \sum_{x\in\gamma_0} a_0(x-y)\left|\nabla_0^{y,x}\varPhi_{\tau_0,1}^{m_0}(\theta_0|\gamma_0)\right| dy\right)\varPhi_{\tau_1}^{m_1}(\theta_1|\gamma_1) \\ &+ \left(\int_X \sum_{x\in\gamma_1} a_1(x-y)\left|\nabla_1^{y,x}\varPhi_{\tau_1}^{m_1}(\theta_1|\gamma_1)\right| dy\right)\varPhi_{\tau_0,1}^{m_0}(\theta_0|\gamma_0) \\ &+ \left(\int_X \sum_{x\in\gamma_0} a_0(x-y)\left|\nabla_0^{y,x}\varPhi_{\tau_0}^{m_0}(\theta_0|\gamma_0)\right| dy\right)\varPhi_{\tau_1,1}^{m_1}(\theta_1|\gamma_1) \\ &+ \left(\int_X \sum_{x\in\gamma_1} a_1(x-y)\left|\nabla_1^{y,x}\varPhi_{\tau_1,1}^{m_1}(\theta_1|\gamma_1)\right| dy\right)\varPhi_{\tau_0}^{m_0}(\theta_0|\gamma_0). \end{aligned} \tag{6.26}$$

Each of the summands of the right-hand side of (6.26) can be estimated in the same way as in the last two lines of (4.15). This procedure was systematically elaborated in [16], which we are going to use now. To describe it, we introduce the following notions. First, for $l \in \mathbb{N}$ and θ_i, $i = 0, 1$, we define, see (4.11),

$$\theta_i^l = a_i \theta_i^{l-1}, \qquad \theta_i^0 := \theta_i. \tag{6.27}$$

Then as in [16, page 28], for $p \in \mathbb{N}$ and $q \in \mathbb{N}_0$, by $\mathcal{C}_{p,q}$ we denote the set of all integer-valued sequences $c = \{c_l\}_{l \in \mathbb{N}_0} \subset \mathbb{N}_0$ such that

$$c_0 + c_1 + \cdots + c_l + \cdots = p, \qquad c_1 + 2c_2 + \cdots + lc_l + \cdots = q. \tag{6.28}$$

For instance, $\mathcal{C}_{p,0}$ is a singleton consisting of $c = \{p, 0, \ldots, 0, \ldots\}$, $\mathcal{C}_{p,2}$ consists of $c = \{p-1, 0, 1, 0, \ldots\}$ and $c = \{p-2, 2, 0, \ldots\}$ for $p \geq 2$. Thereafter, we set

$$\begin{aligned} C_{p,q}(c) &= \frac{p!q!}{c_0!c_1!c_2!\cdots(0!)^{c_0}(1!)^{c_1}(2!)^{c_2}\cdots}, \qquad c \in \mathcal{C}_{p,q}, \\ w_k(p,q) &= \varDelta^k p^q = \frac{1}{k!}\sum_{l=0}^{k}(-1)^{k-l}\binom{k}{l}(p+l)^q, \qquad k \in \mathbb{N}_0. \end{aligned} \tag{6.29}$$

Note that $\varDelta$ is the step-one forward difference operator for which

$$\varDelta^k p^q = 0, \qquad \text{for } k > q. \tag{6.30}$$

Next, for $c \in \mathcal{C}_{m_i,q}$, we write $\mathbf{v}_i^c(\mathbf{x}^{m_i}) = \prod_{j=1}^{m_i} v_{i,j}(x_j)$, see (3.54), where the number of $v_{i,j}$ equal to θ_i is c_0, the number of $v_{i,j}$ equal to θ_i^1 is c_1, see (6.27), the number of $v_{i,j}$ equal to θ_i^2 is c_2, etc, cf. (6.23). Thereafter, for $\theta_i \in \varTheta_\psi^+$, $i = 0, 1$, such that

$$\bar{c}_{\theta_i} = 1, \tag{6.31}$$

see (3.39), we set

$$\varPhi_{\tau_i,q}^{m_i}(\theta_i|\gamma_i) = \sum_{c \in \mathcal{C}_{m_i,q}} C_{m_i,q}(c)\widehat{F}_{\tau_i}^{m_i}(\mathbf{v}_i^c|\gamma_i) + c_a^q \sum_{k=1}^{q} \tau_i^k w_k(m_i,q)\widehat{F}_{\tau_i}^{m_i+k}(\gamma_i), \tag{6.32}$$

see (4.16). For $q = 0$ (resp. $q = 1$), this function coincides with that given in the first (resp. second) line of (6.23). Let us now denote, cf. (4.8), (4.15),

$$\mathcal{L}_i \varPhi_{\tau_i,q}^{m_i}(\theta_i|\gamma_i) = \int_X \sum_{x \in \gamma_i} a_i(x-y)\left|\nabla_i^{y,x}\varPhi_{\tau_i,q}^{m_i}(\theta_i|\gamma_i)\right| dy, \quad i = 0, 1.$$

In [16, Appendix], the following was proved, see also (5.24) *ibid*.

$$\mathcal{L}_i \varPhi_{\tau_i,q}^{m_i}(\theta_i|\gamma_i) \leq \varPhi_{\tau_i,q+1}^{m_i}(\theta_i|\gamma_i), \qquad i = 0, 1, \tag{6.33}$$

holding for all $\theta_i \in \varTheta_\psi^+$ satisfying (6.31), $m_i \in \mathbb{N}$, $q \in \mathbb{N}_0$ and $\tau_i \in (0, 1]$. By means of (6.33) we then get from (6.26) the following estimate

$$\begin{aligned} \left|L\varPhi_{\tau,1}^m(\theta|\gamma)\right| &\leq \varPhi_{\tau_0,2}^{m_0}(\theta_0|\gamma_0)\varPhi_{\tau_1}^{m_1}(\theta_1|\gamma_1) + 2\varPhi_{\tau_0,1}^{m_0}(\theta_0|\gamma_0)\varPhi_{\tau_1,1}^{m_1}(\theta_1|\gamma_1) \\ &+ \varPhi_{\tau_0}^{m_0}(\theta_0|\gamma_0)\varPhi_{\tau_1,2}^{m_1}(\theta_1|\gamma_1). \end{aligned}$$

The estimates obtained in (6.25), (6.26) can be summarized as follows. Set

$$\varPhi_{\tau,q}^m(\theta|\gamma) = \sum_{l=0}^{q}\binom{q}{l}\varPhi_{\tau_0,q-l}^{m_0}(\theta_0|\gamma_0)\varPhi_{\tau_1,l}^{m_1}(\theta_1|\gamma_1). \tag{6.34}$$

Then the main result of this subsection is the following estimate

$$\left|L\varPhi_{\tau,q}^m(\theta|\gamma)\right| \leq \varPhi_{\tau,q+1}^m(\theta|\gamma), \tag{6.35}$$

holding for all $q \in \mathbb{N}_0$, $m \in \mathbb{N}_0^2$, $\tau = (\tau_0, \tau_1)$, $\tau_i \in (0, 1]$, and $\theta = (\theta_0, \theta_1)$, $\theta_i \in \varTheta_\psi^+$ satisfying (6.31). The first step in the proof of (6.35) is made as in (4.15), first estimate. Next, one applies (6.33).

6.3 Proving Lemma 6.1

By (6.34) and (6.32), and then by Proposition 3.9, $\Phi^m_{\tau,q}(\theta|\cdot)$ is a bounded continuous function of $\gamma \in \Gamma^2_*$. However, the upper bound of it may depend on q. Our aim is to estimate this dependence.

Proposition 6.4. *For each $\varepsilon \in (0,1)$, $\tau = (\tau_0, \tau_1)$, $\tau_0, \tau_1 \in (0,1]$, and $m = (m_0, m_1) \in \mathbb{N}_0^2$, there exists $\bar{C} > 0$, dependent on ε, τ and m, such that the following holds*

$$\forall q \in \mathbb{N}_0 \qquad \Phi^m_{\tau,q}(\theta|\gamma) \leq \frac{q!}{\rho_\varepsilon^q}\bar{C}, \qquad \rho_\varepsilon := \frac{1}{c_a}\log(1+\varepsilon), \tag{6.36}$$

uniformly in $\gamma \in \Gamma^2_*$ and $\theta = (\theta_0, \theta_1)$, $\theta_0, \theta_1 \in \Theta^+_\psi$ satisfying (6.31).

Proof. Introduce

$$V^m_\tau(\rho|\gamma) = \sum_{q=0}^{\infty} \frac{\rho^q}{q!}\Phi^m_{\tau,q}(\theta|\gamma), \qquad \rho \geq 0, \tag{6.37}$$

where m, τ and θ are as assumed. Let us estimate the growth of this function. By (6.27) and (4.11), (6.31), we have

$$\theta^l_i(x) \leq c^l_a \psi(x),$$

which we use to get the following

$$\widehat{F}^{m_i}_{\tau_i}(\mathbf{v}^c_i|\gamma_i) \leq c_a^{c_1+2c_2+\cdots}\widehat{F}^{m_i}_{\tau_i}(\gamma_i) = c^q_a \widehat{F}^{m_i}_{\tau_i}(\gamma_i), \quad c \in \mathcal{C}_{m_i,q},$$

where we used the second equality in (6.28). Now we employ the fact, see (6.29), that

$$\sum_{c\in\mathcal{C}_{p,q}} C_{p,q}(c) = p^q = \Delta^0 p^q = w_0(p,q), \tag{6.38}$$

which was proved in [16, Appendix], and obtain from (6.32) the following estimate

$$\Phi^{m_i}_{\tau_i,q}(\theta_i|\gamma_i) \leq c^q_a \sum_{k=0}^{q} \tau^k_i w_k(m_i,q)\widehat{F}^{m_i+k}_{\tau_i}(\gamma_i).$$

We use the latter in (6.34) and then in (6.37) to get the following

$$\begin{aligned} V^m_\tau(\rho|\gamma) \;&\leq\; \sum_{q=0}^{\infty}\frac{(c_a\rho)^q}{q!}\sum_{l=0}^{q}\frac{q!}{l!(q-l)!} \\ &\times\; \sum_{k_0=0}^{q-l}\sum_{k_1=0}^{l}\tau_0^{k_0}\tau_1^{k_1}w_{k_0}(m_0,q-l)w_{k_1}(m_1,l)\widehat{F}^{m_0+k_0}_{\tau_0}(\gamma_0)\widehat{F}^{m_1+k_1}_{\tau_1}(\gamma_1) \\ &=\; \sum_{k_0=0}^{\infty}\sum_{k_1=0}^{\infty}\frac{\tau_0^{k_0}\tau_1^{k_1}}{k_0!k_1!}W^{m_0}_{\tau_0,k_0}(\rho|\gamma_0)W^{m_1}_{\tau_1,k_1}(\rho|\gamma_1), \end{aligned} \tag{6.39}$$

where we also used the fact that $w_k(p,q) = 0$ whenever $k > q$, see (6.30). The functions that appear in the last line of (6.39) are

$$\begin{aligned} W^{m_i}_{\tau_i,k_i}(\rho|\gamma_i) \;&:=\; \left(\sum_{l=0}^{\infty}\frac{(c_a\rho)^l}{l!}w_{k_i}(m_i,l)\right)\widehat{F}^{m_i+k_i}_{\tau_i}(\gamma_i) \\ &=\; \left(e^{c_a\rho}-1\right)^{k_i} e^{m_i c_a \rho}\widehat{F}^{m_i+k_i}_{\tau_i}(\gamma_i), \end{aligned} \tag{6.40}$$

where the second line was derived by means of the second line of (6.29). Then we have

$$V^m_\tau(\rho|\gamma) \;\leq\; e^{(m_0+m_1)c_a\rho}Y^{m_0}_{\tau_0}(\rho|\gamma_0)Y^{m_1}_{\tau_1}(\rho|\gamma_1), \tag{6.41}$$

$$Y_{\tau_i}^{m_i}(\rho|\gamma_i) \quad := \quad \sum_{k=0}^{\infty} \frac{(\tau_i \ell(\rho))^k}{k!} \widehat{F}_{\tau_i}^{m_i+k}(\gamma_i), \quad i=0,1,$$

where $\ell(\rho) = e^{c_a \rho} - 1$. By means of the estimate obtained in (3.56), (3.57) with $u_j(x) = \psi(x) e^{\tau_i \psi(x)} \leq e^{\tau_i} \psi(x)$ and $\tau_i \leq 1$, we then obtain

$$\widehat{F}_{\tau_i}^{m_i+k}(\gamma_i) \leq \left(\frac{m_i + k}{\tau_i} \right)^{m_i+k},$$

which yields that both series in (6.41) converge whenever $e^{c_a \rho} - 1 < 1$. Take $\varepsilon \in (0,1)$ and ρ_ε as in (6.36), then set

$$\bar{Y}_i = \frac{1}{\tau_i^{m_i}} \sum_{k=0}^{\infty} \frac{\varepsilon^k}{k!} (m_i + k)^{m_i+k}, \qquad i=0,1.$$

Now (6.36) follows by

$$V_\tau^m(\theta|\gamma) \leq (1+\varepsilon)^{m_0+m_1} \bar{Y}_0 \bar{Y}_1 =: \bar{C},$$

see (6.37) and (6.41). □

Proof of Lemma 6.1. By (6.34), each $\Phi_\tau^m(\theta|\cdot)$ is a linear combination of the elements of $\widehat{\mathcal{F}}$, and hence $\Phi_{\tau,q}^m(\theta|\cdot) \in \mathcal{D}(L)$, see (3.55) and Definition 4.1. If $t \mapsto \mu_t$ solves (1.2), see Definition 4.6, then

$$\begin{aligned} \mu_t(\Phi_{\tau,q}^m(\theta|\cdot)) &= \mu_0(\Phi_{\tau,q}^m(\theta|\cdot)) + \int_0^t \mu_s(L\Phi_{\tau,q}^m(\theta|\cdot)) ds \\ &\leq \mu_0(\Phi_{\tau,q}^m(\theta|\cdot)) + \int_0^t \mu_s(\Phi_{\tau,q+1}^m(\theta|\cdot)) ds, \end{aligned}$$

see (6.35). Now we repeat this estimate due times and arrive at the following one

$$\begin{aligned} \mu_t(\Phi_\tau^m(\theta|\cdot)) \quad &\leq \quad \sum_{q=0}^{n-1} \frac{t^q}{q!} \mu_0(\Phi_{\tau,q}^m(\theta|\cdot)) + \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \mu_{t_n}(\Phi_{\tau,n}^m(\theta|\cdot)) dt_n dt_{n-1} \cdots dt_1 \\ &\leq \quad \sum_{q=0}^{n-1} \frac{t^q}{q!} \mu_0(\Phi_{\tau,q}^m(\theta|\cdot)) + \left(\frac{t}{\rho_\varepsilon} \right)^n \bar{C}, \end{aligned}$$

where we also used (6.36) and the fact that μ_t is a probability measure. For $t \in (0, \rho_\varepsilon)$, the second summand in the last line vanishes as $n \to +\infty$, which yields

$$\mu_t(\Phi_\tau^m(\theta|\cdot)) \leq \sum_{q=0}^{\infty} \frac{t^q}{q!} \mu_0(\Phi_{\tau,q}^m(\theta|\cdot)), \qquad t < \log(1+\varepsilon)/c_a. \tag{6.42}$$

Now we recall that $\widehat{F}_{\tau_i}^{m_i}(\mathbf{v}_i|\cdot)$ can be written as the $K\widehat{G}_{\tau_i}^{m_i}(\mathbf{v}_i|\cdot)$, see (6.5). Since $\widehat{F}_{\tau_i}^{m_i}(\cdot)$ is a particular case of $\widehat{F}_{\tau_i}^{m_i}(\mathbf{v}_i|\cdot)$, see (4.16), we can also write it as $K\widehat{G}_{\tau_i}^{m_i}(\cdot)$, where $\widehat{G}_{\tau_i}^{m_i}(\cdot)$ is obtained by the corresponding choice of $\mathbf{v}_i$ in (6.4), (6.5). This allows us to write

$$\begin{aligned} \Phi_{\tau_i,q}^{m_i}(\theta_i|\gamma_i) \quad &= \quad \sum_{\eta_i \subset \gamma_i} \Pi_{\tau_i,q}^{m_i}(\theta_i|\eta_i), \qquad i=0,1, \\ \Pi_{\tau_i,q}^{m_i}(\theta_i|\eta_i) \quad &= \quad \sum_{c \in \mathcal{C}_{m_i,q}} C_{m_i,q}(c) \widehat{G}_{\tau_i}^{m_i}(\mathbf{v}_i^c|\eta_i) + c_a^q \sum_{k=1}^{q} \tau_i^k w_k(m_i,q) \widehat{G}_{\tau_i}^{m_i+k}(\eta_i), \end{aligned}$$

where $\mathbf{v}_i^c$ is as in (6.32). Then by (6.34) we obtain

$$\Phi^m_{\tau,q}(\theta|\gamma) = \sum_{\eta_0\subset\gamma_0}\sum_{\eta_1\subset\gamma_1}\sum_{l=0}^{q}\binom{q}{l}\Pi^{m_0}_{\tau_0,q-l}(\theta_0|\eta_0)\Pi^{m_1}_{\tau_1,l}(\theta_1|\eta_1).$$

Now we may use the fact that μ_0 is sub-Poissonian and write, see (5.1), (5.2),

$$\mu_0(\Phi^m_{\tau,q}(\theta|\cdot)) = \sum_{l=0}^{q}\binom{q}{l}\int_{\Gamma_0}\int_{\Gamma_0}k_{\mu_0}(\eta_0,\eta_1)\Pi^{m_0}_{\tau_0,q-l}(\theta_0|\eta_0)\Pi^{m_1}_{\tau_1,l}(\theta_1|\eta_1)\lambda(d\eta_0)\lambda(d\eta_1).$$

Let e^{ϑ_0} be the type of μ_0. Then by (3.22) we have

$$\mu_0(\Phi^m_{\tau,q}(\theta|\cdot)) \le \sum_{l=0}^{q}\binom{q}{l}\Omega^{m_0}_{\tau_0,q-l}(\theta_0)\Omega^{m_1}_{\tau_1,l}(\theta_1),$$

which yields in (6.42),

$$\mu_t(\Phi^m_{\tau}(\theta|\cdot)) \le \widehat{\Omega}^{m_0}_{\tau_0}(\theta_0|t)\widehat{\Omega}^{m_1}_{\tau_1}(\theta_1|t), \qquad t < \log(1+\varepsilon)/c_a, \tag{6.43}$$

where

$$\widehat{\Omega}^{m_i}_{\tau_i}(\theta_i|t) = \sum_{q=0}^{\infty}\frac{t^q}{q!}\Omega^{m_i}_{\tau_i,q}(\theta_i), \qquad i=0,1, \tag{6.44}$$

and, see (6.6),

$$\begin{aligned}\Omega^{m_i}_{\tau_i,n}(\theta_i) &:= \int_{\Gamma_0} e^{\vartheta_0|\eta_i|}\left|\Pi^{m_i}_{\tau_1,n}(\theta_i|\eta_i)\right|\lambda(d\eta_i) \qquad (6.45)\\ &\le \sum_{c\in\mathcal{C}_{m_i,n}} C_{m_i,n}(c)\int_{\Gamma_0} e^{\vartheta_0|\eta_i|}\left|\widehat{G}^{m_i}_{\tau_i}(\mathbf{v}_i^c|\eta_i)\right|\lambda(d\eta_i)\\ &+ c_a^n\sum_{k=1}^{n}\tau_i^k w_k(m_i,n)\int_{\Gamma_0} e^{\vartheta_0|\eta_i|}\left|\widehat{G}^{m_i+k}_{\tau_i}(\eta_i)\right|\lambda(d\eta_i)\\ &\le \exp\left(m_i\vartheta_0+\tau_i\langle\psi\rangle e^{\vartheta_0}\right)\sum_{c\in\mathcal{C}_{m_i,n}} C_{m_i,n}(c)\langle\theta_i\rangle^{c_0}\langle\theta_i^1\rangle^{c_1}\cdots\langle\theta_i^l\rangle^{c_l}\cdots\\ &+ c_a^n\sum_{k=1}^{n}\tau_i^k w_k(m_i,n)\langle\psi\rangle^{m_i+k}\exp\left((m_i+k)\vartheta_0+\tau_i\langle\psi\rangle e^{\vartheta_0}\right).\end{aligned}$$

By (6.27) and (4.9), (4.11), for $\theta_i\in\Theta^+_\psi$, we have, see (5.15),

$$\langle\theta_i^l\rangle \le (\alpha+1)^l\langle\theta_i\rangle$$

since

$$\langle\theta_i^l\rangle = \int_X\int_X a_i(x-y)\theta_i^{l-1}(y)dxdy + \langle\theta_i^{l-1}\rangle \le (\alpha+1)\langle\theta_i^{l-1}\rangle.$$

Then, see (6.28),

$$\langle\theta_i\rangle^{c_0}\langle\theta_i^1\rangle^{c_1}\cdots\langle\theta_i^l\rangle^{c_l}\cdots \le \langle\theta_i\rangle^{c_0+c_1+\cdots}(\alpha+1)^{c_1+2c_2+\cdots} = \langle\theta_i^l\rangle^{m_i}(\alpha+1)^n.$$

We use this in (6.45) and obtain, see also (6.38),

$$\Omega^{m_i}_{\tau_i,n}(\theta_i) \le \exp\left(m_i\vartheta_0+\tau_i\langle\psi\rangle e^{\vartheta_0}\right)\Big[\langle\theta_i\rangle^{m_i}[m_i(\alpha+1)]^n$$

$$+c_a^n \langle\psi\rangle^{m_i} \sum_{k=1}^{n} w_k(m_i,n) \left(\tau_i \langle\psi\rangle e^{\vartheta_0}\right)^k \Bigg].$$

Now we use this in (6.44), and finally arrive at the following estimate

$$\begin{aligned} \widehat{\Omega}^{m_i}_{\tau_i}(\theta_i|t) \quad &\le \quad \exp\left(m_i\vartheta_0 + \tau_i\langle\psi\rangle e^{\vartheta_0}\right)\left[e^{(\alpha+1)m_i t}\langle\theta_i\rangle^{m_i}\right. \\ &+ \quad \left.\langle\psi\rangle^{m_i}\sum_{q=0}^{\infty}\frac{(c_a t)^q}{q!}\sum_{k=1}^{\infty} w_k(m_i,q)\left(\tau_i\langle\psi\rangle e^{\vartheta_0}\right)^k\right], \end{aligned} \tag{6.46}$$

where we took into account that $w_k(m_i,q) = 0$ for $k > q$, see (6.30). By (6.29) we have

$$\begin{aligned} &\sum_{q=0}^{\infty}\frac{(c_a t)^q}{q!}\sum_{k=1}^{\infty} w_k(m_i,q)\left(\tau_i\langle\psi\rangle e^{\vartheta_0}\right)^k \\ &= \sum_{k=1}^{\infty}\frac{1}{k!}\left(\tau_i\langle\psi\rangle e^{\vartheta_0}\right)^k \sum_{l=0}^{k}(-1)^{k-l}\binom{k}{l}\sum_{q=0}^{\infty}\frac{(c_a t)^q}{q!}(m_i+l)^q \\ &= e^{m_i c_a t}\sum_{k=1}^{\infty}\frac{1}{k!}\left(\tau_i\langle\psi\rangle e^{\vartheta_0}\right)^k \sum_{l=0}^{k}(-1)^{k-l}\binom{k}{l} e^{l c_a t} \\ &= e^{m_i c_a t}\left[\exp\left((e^{c_a t}-1)\tau_i\langle\psi\rangle e^{\vartheta_0}\right) - 1\right] \\ &\le \tau_i\varepsilon(1+\varepsilon)^{m_i}\exp\left(\langle\psi\rangle e^{\vartheta_0}\right), \end{aligned} \tag{6.47}$$

where t is as in (6.43) and $\tau_i \le 1$, $\varepsilon < 1$. We use now (6.47) in (6.46) and then turn (6.43) into the following estimate

$$\begin{aligned} \mu_t(\Phi^m_\tau(\theta|\cdot)) \quad &\le \quad \exp\left((m_0+m_1)\vartheta_0 + (\tau_0+\tau_1)\langle\psi\rangle e^{\vartheta_0}\right) \\ &\times \quad \left[e^{(\alpha+1)m_0 t}\langle\theta_0\rangle^{m_0} + \tau_0\varepsilon(1+\varepsilon)^{m_0}\exp\left(\langle\psi\rangle e^{\vartheta_0}\right)\right] \\ &\times \quad \left[e^{(\alpha+1)m_1 t}\langle\theta_1\rangle^{m_1} + \tau_1\varepsilon(1+\varepsilon)^{m_1}\exp\left(\langle\psi\rangle e^{\vartheta_0}\right)\right], \quad t < \log(1+\varepsilon)/c_a. \end{aligned} \tag{6.48}$$

For each $\gamma \in \Gamma^2_*$ and a decreasing sequence of positive $\tau_k \to 0$, the sequence $\Phi^m_\tau(\theta|\gamma)$, $\tau_i = \tau^k$, $i = 0,1$, is nondecreasing, see (6.32) and (3.55). By (6.48) and the Beppo Levi monotone convergence lemma we conclude that the pointwise limit, see (6.24), (6.22) and (3.8), (3.14),

$$\Phi^m_0(\theta|\gamma) = \lim_{k\to+\infty}\Phi^m_{\tau^k}(\theta|\gamma) = H^m_\theta(\gamma) = \sum_{(\mathbf{x}^{m_0},\mathbf{y}^{m_1})\in\gamma}\theta^{\otimes m}(\mathbf{x}^{m_0},\mathbf{y}^{m_1}), \tag{6.49}$$

is μ_t-integrable. Moreover, by the same lemma and (6.48) it follows that

$$\mu_t(\Phi^m_0(\theta|\cdot)) = \chi^{(m)}_{\mu_t}(\theta^{\otimes m}) \le \varkappa_t^{|m|}\|\theta_0\|^{m_0}_{L^1(X)}\|\theta_1\|^{m_1}_{L^1(X)}, \tag{6.50}$$

with

$$\varkappa_t = e^{\vartheta_0 + (\alpha+1)t}, \tag{6.51}$$

which by (3.9) yields the property in question for $t < \log(1+\varepsilon)/c_a$. Since this length of the validity interval is independent of μ_0, the further continuation can be done by the literal repetition of the procedure above. □

Remark 6.5. According to Proposition 5.3, see also [1, Theorem 3.5], the type of the solution μ_t obtained in Lemma 6.3 does not exceed $\exp(\vartheta_0 + \alpha t)$, which is a more precise estimate than that given in (6.50), (6.51).

7 Existence: approximating models

The aim of this section is introducing approximating models, for which the corresponding processes can be constructed by employing explicitly derived Markov transition functions. By this result the process in question will be obtained as the limit of such approximating processes. Similarly as in [16], the Kolmogorov operators for the approximating models are obtained from that in (1.3).

7.1 The models

We begin by introducing the basic function, cf. (3.29)

$$\psi_\sigma(x) = \frac{1}{1+\sigma|x|^{d+1}}, \qquad \sigma \in (0,1], \ \ x \in X = \mathbb{R}^d, \tag{7.1}$$

and then define

$$a_i^\sigma(x,y) = a_i(x-y)\psi_\sigma(x), \qquad i = 0,1, \tag{7.2}$$

where a_i are the jump kernels that appear in (1.3). It is clear that these a_i^σ satisfy, cf. (4.1), (4.3),

$$\max_{i=0,1} \sup_{(x,y)\in X^2} a_i^\sigma(x,y) \leq \|a\|, \tag{7.3}$$

$$\max\left\{ \sup_{y\in X} \int_X |x|^l a_i^\sigma(x,y)dx; \ \sup_{y\in X} \int_X |x|^l a_i^\sigma(y,x)dx \right\} \leq \bar{a}_i^{(l)}, \quad l = 0,\dots,d+1, \quad i = 0,1.$$

The Kolmogorov operator corresponding to the approximation model is obtained by replacing in (1.3) $a_i(x-y)$ with $a_i^\sigma(x,y)$, $i=0,1$; that is, it has the form

$$\begin{aligned} (L^\sigma F)(\gamma) \ &= \ \sum_{x\in\gamma_0} \int_X a_0^\sigma(x,y) \exp\left(- \sum_{z\in\gamma_1} \phi_0(z-y) \right) [F(\gamma\setminus x \cup_0 y) - F(\gamma)]\, dy \\ &+ \ \sum_{x\in\gamma_1} \int_X a_1^\sigma(x,y) \exp\left(- \sum_{z\in\gamma_0} \phi_1(z-y) \right) [F(\gamma\setminus x \cup_1 y) - F(\gamma)]\, dy. \end{aligned} \tag{7.4}$$

Noteworthy, in the approximating model the kernels corresponding to the jumps from x to y get smaller if x goes away from the origin. Now we introduce $L^{\Delta,\sigma}$ by replacing in (5.9) $a_i(x-y)$ with $a_i^\sigma(x,y)$, $i=0,1$, where one should take into account that $a_i^\sigma(x,y) \neq a_i^\sigma(y,x)$. Then, cf. (5.5),

$$\mu(L^\sigma KG) = \langle\!\langle L^{\Delta,\sigma} k_\mu, G \rangle\!\rangle, \qquad \sigma \in [0,1], \tag{7.5}$$

holding for each $\mu \in \mathcal{P}_{\rm exp}$ and $G \in \mathcal{G}_\infty$. For $\sigma = 0$, $L^{\Delta,\sigma}$ coincides with the operator given in (5.9). Clearly, for all $\sigma \in [0,1]$, $L^{\Delta,\sigma}$ satisfies (5.16) and similar estimates, which allows one to define bounded operators $(L^{\Delta,\sigma})^n_{\vartheta'\vartheta}$, $n \in \mathbb{N}$, and thus construct, cf. (5.30),

$$\varXi^\sigma_{\vartheta'\vartheta}(t) = 1 + \sum_{n=1}^\infty \frac{t^n}{n!} (L^{\Delta,\sigma})^n_{\vartheta'\vartheta}, \qquad t < T(\vartheta',\vartheta), \tag{7.6}$$

where the latter is the same as in (5.21). Similarly, one obtains $\widehat{L}^\sigma$ by making the aforementioned replacements in (5.23), and then defines, cf. (5.28),

$$\varSigma^\sigma_{\vartheta\vartheta'}(t) = 1 + \sum_{n=1}^\infty \frac{t^n}{n!} (\widehat{L}^\sigma)^n_{\vartheta\vartheta'}, \qquad t < T(\vartheta',\vartheta). \tag{7.7}$$

Thereafter, one sets

$$k_t^\sigma = \varXi^\sigma_{\vartheta'\vartheta}(t) k_0, \quad G_t^\sigma = \varSigma^\sigma_{\vartheta\vartheta'}(t) G_0, \qquad t < T(\vartheta',\vartheta), \tag{7.8}$$

holding for each $k_0 \in \mathcal{K}_\vartheta$, $G_0 \in \mathcal{G}_{\vartheta'}$, $\vartheta' > \vartheta$. For $\sigma = 0$, both k_t^σ and G_t^σ coincide with those that appear in (5.29), (5.32), etc. Let us now consider the Fokker-Planck equation

$$\mu_{t_2}(F) = \mu_{t_1}(F) + \int_{t_1}^{t_2} \mu_s(L^\sigma F)ds, \qquad F \in \mathcal{D}(L), \tag{7.9}$$

where the latter is as in Definition 4.1. Since L^σ satisfies all the estimates used in the proof of Lemma 6.3, see, e.g., (7.3), we have the following.

Proposition 7.1. *For each $\mu \in \mathcal{P}_{\rm exp}$ and $\sigma \in [0,1]$, the Fokker-Planck equation (7.9) with $\mu_0 = \mu$ has exactly one solution $t \mapsto \mu_t^\sigma \in \mathcal{P}_{\rm exp}$ defined by the map $t \mapsto k_t^\sigma \in \mathcal{K}^\star$ constructed with the help of (7.6), (7.8), similarly as in the case $\sigma = 0$, see Remark 5.4. Let also ϑ be such that $k_0 \in \mathcal{K}_\vartheta$. Then for each $\sigma \in [0,1]$, $\vartheta' > \vartheta$ and $G \in \mathcal{G}_{\vartheta'}$, the following holds, see (5.33),*

$$\langle\!\langle k_t^\sigma, G\rangle\!\rangle = \langle\!\langle k_0, G_t^\sigma\rangle\!\rangle, \qquad t < T(\vartheta', \vartheta), \tag{7.10}$$

where $G_t^\sigma = \Sigma^\sigma_{\vartheta\vartheta'}(t)G$, see (7.7), (7.8).

7.2 The weak convergence

Our aim is to prove that $\mu_t^\sigma \Rightarrow \mu_t$ as $\sigma \to 0$. We begin by proving the following statement.

Proposition 7.2. *Let $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathcal{P}_{\rm exp}$ be such that the type of each μ_n does not exceed e^ϑ, $\vartheta \in \mathbb{R}$, and $\mu_n \Rightarrow \mu$ for some $\mu \in \mathcal{P}(\Gamma_*^2)$. Then $\mu \in \mathcal{P}_{\rm exp}$ and its type $\leq e^\vartheta$. Moreover, for each $G \in \mathcal{G}_\vartheta$, it follows that*

$$\langle\!\langle k_{\mu_n}, G\rangle\!\rangle \to \langle\!\langle k_\mu, G\rangle\!\rangle, \qquad n \to +\infty. \tag{7.11}$$

Proof. Since $\widehat{F}^m_\tau(\mathbf{v}|\cdot) \in C_{\rm b}(\Gamma_*^2)$, see Proposition 3.9 and (3.55), the assumed convergence yields $\mu_n(F) \to \mu(F)$, holding for all $F \in \widehat{\mathcal{F}}$, including $F = \Phi^m_\tau$, see (6.24), (6.22). Therefore, by (6.49), (6.50) we have

$$\mu(\Phi^m_\tau) \leq \sup_n \mu_n(\Phi^m_\tau) \leq e^{|m|\vartheta}\|\theta_0\|^{m_0}_{L^1(X)}\|\theta_1\|^{m_1}_{L^1(X)},$$

holding for all $\theta_0, \theta_1 \in \Theta^+_\psi$. As in the proof of Lemma 6.1, this yields $\mu \in \mathcal{P}_{\rm exp}$ and its type does not exceed e^ϑ. The validity of (7.11) follows by the fact just mentioned. □

Now we prove that the solutions of the Fokker-Planck equations (1.2) and (7.9) have the property $\mu_t^\sigma \Rightarrow \mu_t$ as $\sigma \to 0$, holding for each $t > 0$. We obtain this result by proving a bit more general statement, which will be used in the subsequent part of this paper.

Lemma 7.3. *Let $\{\mu^\sigma\}_{\sigma\in(0,1]} \subset \mathcal{P}_{\rm exp}$ be such that the type of each μ^σ does not exceed e^{ϑ_0} for some $\vartheta_0 \in \mathbb{R}$, and $\mu^\sigma \Rightarrow \mu$ as $\sigma \to 0$. Let also $t \mapsto \mu_t^\sigma$, $\sigma \in (0,1]$, $\mu_t^\sigma|_{t=0} = \mu^\sigma$, be the solution of the Fokker-Planck equation (7.9) mentioned in Proposition 7.1. Then for each $t > 0$, it follows that $\mu_t^\sigma \Rightarrow \mu_t$ as $\sigma \to 0$, where μ_t is the solution of (1.2) with $\mu_t|_{t=0} = \mu$.*

Noteworthy, by Proposition 7.2 it follows that the limiting measure μ in Lemma 7.3 is sub-Poissonian and its type does not exceed e^{ϑ_0}. The proof of Lemma 7.3 is based on the following statement.

Lemma 7.4. *For a given $t > 0$, let k_t^σ and k_t be the correlation functions of the measures μ_t^σ and μ_t mentioned in Lemma 7.3. Then there exists $\tilde{\vartheta}(t) \in \mathbb{R}$ such that*

$$\forall G \in \mathcal{G}_{\tilde{\vartheta}(t)} \qquad \langle\!\langle k_t^\sigma, G\rangle\!\rangle \to \langle\!\langle k_t, G\rangle\!\rangle, \qquad \text{as } \ \sigma \to 0. \tag{7.12}$$

Proof. As the type of each μ^σ does not exceed e^{ϑ_0}, both k_t^σ and k_t lie in $\mathcal{K}_{\vartheta(t)}$ with $\vartheta(t) = \vartheta_0 + \alpha t$, see Remark 5.4 and the proof of Proposition 7.1. Moreover, k_t^σ and

G satisfy (7.10) with appropriate ϑ, ϑ'. Recall that the map $\vartheta' \mapsto T(\vartheta', \vartheta)$ attains its maximum $T_*(\vartheta)$ given in (6.11).

Let now the convergence stated in (7.12) hold for a given $t \geq 0$. By the assumed convergence $\mu^\sigma \Rightarrow \mu$ and Proposition 7.2 this certainly holds for $t = 0$. Our aim is to prove that it holds also for all $t + s$, $s \leq s_0$, with a possibly t-dependent $s_0 > 0$. Set $\bar{\vartheta}_t = \vartheta(t) + \delta(\vartheta(t))$, see (6.11). For $s < T_*(\vartheta(t))$, the norm of $\varXi_{\bar{\vartheta}_t \vartheta(t)}(s)$ satisfies

$$\|\varXi_{\bar{\vartheta}_t \vartheta(t)}(s)\| \leq \frac{T_*(\vartheta(t))}{T_*(\vartheta(t)) - s},$$

see (5.32). In the same way, one estimates also the norm of $\varXi^\sigma_{\bar{\vartheta}_t \vartheta(t)}(s)$, $\sigma \in (0,1]$ since the norms of the corresponding $(L^{\Delta,\sigma})^n_{\bar{\vartheta}_t \vartheta(t)}$ have the same bounds as for $\sigma = 0$. For $\sigma \in (0,1]$, we write

$$q^\sigma_s = k_{t+s} - k^\sigma_{t+s} = \varXi_{\bar{\vartheta}_t \vartheta(t)}(s) k_t - \varXi^\sigma_{\bar{\vartheta}_t \vartheta(t)}(s) k^\sigma_t, \qquad s < T_*(\vartheta(t)). \tag{7.13}$$

Note that

$$\forall G \in \mathcal{G}_{\vartheta(t)} \qquad \langle\!\langle q^\sigma_0, G \rangle\!\rangle \to 0 \qquad \text{as} \ \ \sigma \to 0. \tag{7.14}$$

At the same time, (7.13) can be written in the form

$$\begin{aligned} q^\sigma_s &= \varXi_{\bar{\vartheta}_t \vartheta(t)}(s) q^\sigma_0 - \varPi^\sigma_{\bar{\vartheta}_t \vartheta(t)}(s) k^\sigma_t, \qquad (7.15)\\ \varPi^\sigma_{\bar{\vartheta}_t \vartheta(t)}(s) &:= \int_0^s \frac{d}{du} \left[\varXi_{\bar{\vartheta}_t \vartheta}(s-u) \varXi^\sigma_{\vartheta \vartheta(t)}(u) \right] du, \end{aligned}$$

where s and $\vartheta \in (\vartheta(t), \bar{\vartheta}_t)$ are chosen in such a way that

$$s < \min\{T(\bar{\vartheta}_t, \vartheta); T(\vartheta, \vartheta(t))\}, \tag{7.16}$$

and hence the Bochner integral in the second line of (7.15) makes sense, see (7.6). Since the map $(\vartheta, \vartheta') \mapsto T(\vartheta', \vartheta)$ is continuous, see (5.21), one can pick $\vartheta_1 < \vartheta$ and $\vartheta_2 > \vartheta$, ϑ being as in (7.16), such that

$$s < \min\{T(\bar{\vartheta}_t, \vartheta_2); T(\vartheta_1, \vartheta(t))\}. \tag{7.17}$$

Keeping this in mind, we use an evident identical extension of (5.31) to all $\sigma \leq 1$ and obtain

$$\begin{aligned} \varPi^\sigma_{\bar{\vartheta}_t \vartheta(t)}(s) &= - \int_0^s \varXi_{\bar{\vartheta}_t \vartheta_2}(s-u) \widetilde{L}^{\Delta,\sigma}_{\vartheta_2 \vartheta_1} \varXi^\sigma_{\vartheta_1 \vartheta(t)}(u) du, \qquad (7.18)\\ \widetilde{L}^{\Delta,\sigma}_{\vartheta_2 \vartheta_1} &:= L^\Delta_{\vartheta_2 \vartheta_1} - L^{\Delta,\sigma}_{\vartheta_2 \vartheta_1}. \end{aligned}$$

We apply this in (7.15) and get

$$q^\sigma_s = \varXi_{\bar{\vartheta}_t \vartheta(t)}(s) q^\sigma_0 + \int_0^s \varXi_{\bar{\vartheta}_t \vartheta_2}(s-u) \widetilde{L}^{\Delta,\sigma}_{\vartheta_2 \vartheta_1} k^\sigma_{t+u} du.$$

Note that $\widetilde{L}^{\Delta,\sigma}$ can be written in the same form as L^Δ, see (5.9), in which $a_i(x-y)$, $i = 0,1$, ought to be replaced by $\tilde{a}^\sigma_i(x,y) := a_i(x-y)(1-\psi_\sigma(x))$. Now let us turn to picking s_0 and ϑ_j, $j = 1,2$, such that (7.17) holds for $s \leq s_0$. First we set $\vartheta_1 = \vartheta(t) + \delta(\vartheta(t))/2$, see (6.11). By (5.21) and (6.11) we then get

$$T(\bar{\vartheta}_t, \vartheta_1) = T_*(\vartheta(t))/2 < T(\vartheta_1, \vartheta(t)).$$

Now we fix some $\epsilon \in (0,1)$ and set

$$s_0 = \epsilon T_*(\vartheta(t))/2 = \epsilon T(\bar{\vartheta}_t, \vartheta_1). \tag{7.19}$$

Since the map $\vartheta \mapsto T(\vartheta', \vartheta)$ is continuous, one can pick $\vartheta_2 \in (\vartheta_1, \bar{\vartheta}_t)$ such that $s_0 < T(\bar{\vartheta}_t, \vartheta_2)$, see (7.19). Then (7.17) holds for these ϑ_j, $j = 1,2$, and $s \leq s_0$. Now we take $G \in \mathcal{G}_{\bar{\vartheta}_t}$ and set

$$G_s = \Sigma_{\vartheta_2 \bar{\vartheta}_t}(s) G. \tag{7.20}$$

Note that $G_s \in \mathcal{G}_{\vartheta_2} \subset \mathcal{G}_{\vartheta(t)}$; that is, G_s can be considered as an element of $\mathcal{G}_{\vartheta(t)}$ since

$$G_s = I_{\vartheta(t)\vartheta_2} \Sigma_{\vartheta_2 \bar{\vartheta}_t}(s) G,$$

where $I_{\vartheta(t)\vartheta_2} = \Sigma_{\vartheta(t)\vartheta_2}(0)$ is the embedding operator. For these G and G_s, by (7.17) and (7.10) we then have

$$\langle\!\langle q_s^\sigma, G \rangle\!\rangle = \langle\!\langle q_0^\sigma, G_s \rangle\!\rangle + R^\sigma(s), \tag{7.21}$$

$$R^\sigma(s) := \int_0^s \langle\!\langle \widetilde{L}^{\Delta,\sigma}_{\vartheta_2\vartheta_1} k^\sigma_{t+u}, G_{s-u} \rangle\!\rangle du.$$

In view of (7.14), it remains to prove that $R^\sigma(s) \to 0$ as $\sigma \to 0$. To this end, we split $R^\sigma(s)$ into four terms in accord with the structure of $\widetilde{L}^{\Delta,\sigma}$, see (5.9). Thus, we write

$$R^\sigma(s) = \sum_{j=1}^4 R_j^\sigma(s), \tag{7.22}$$

with

$$\begin{aligned}
R_1^\sigma(s) &= \int_0^s \Bigg(\int_{\Gamma_0}\int_{\Gamma_0} \Bigg(\sum_{y\in\eta_0} \int_X \tilde{a}_0^\sigma(x,y) e(\tau_y^0;\eta_1)(\Upsilon_y^0 k_{t+u}^\sigma)(\eta_0\setminus y \cup x, \eta_1) dx \Bigg) \\
&\times G_{s-u}(\eta_0,\eta_1)\lambda(d\eta_0)\lambda(d\eta_1)\Bigg) du, \qquad (7.23)\\
&= \int_0^s \Bigg(\int_{\Gamma_0}\int_{\Gamma_0} \Bigg(\int_X\int_X \tilde{a}_0^\sigma(x,y) e(\tau_y^0;\eta_1)(\Upsilon_y^0 k_{t+u}^\sigma)(\eta_0 \cup x, \eta_1) \\
&\times G_{s-u}(\eta_0\cup y,\eta_1) dx dy\Bigg)\lambda(d\eta_0)\lambda(d\eta_1)\Bigg) du \\
R_2^\sigma(s) &= -\int_0^s \Bigg(\int_{\Gamma_0}\int_{\Gamma_0} \Bigg(\int_X\int_X \tilde{a}_0^\sigma(x,y) e(\tau_y^0;\eta_1)(\Upsilon_y^0 k_{t+u}^\sigma)(\eta_0 \cup x, \eta_1) \\
&\times G_{s-u}(\eta_0\cup x,\eta_1) dx dy\Bigg)\lambda(d\eta_0)\lambda(d\eta_1)\Bigg) du, \\
R_3^\sigma(s) &= \int_0^s \Bigg(\int_{\Gamma_0}\int_{\Gamma_0} \Bigg(\int_X\int_X \tilde{a}_1^\sigma(x,y) e(\tau_y^1;\eta_0)(\Upsilon_y^1 k_{t+u}^\sigma)(\eta_0, \eta_1\cup x) \\
&\times G_{s-u}(\eta_0,\eta_1\cup y) dx dy\Bigg)\lambda(d\eta_0)\lambda(d\eta_1)\Bigg) du, \\
R_4^\sigma(s) &= -\int_0^s \Bigg(\int_{\Gamma_0}\int_{\Gamma_0} \Bigg(\int_X\int_X \tilde{a}_1^\sigma(x,y) e(\tau_y^1;\eta_1)(\Upsilon_y^1 k_{t+u}^\sigma)(\eta_0, \eta_1) \\
&\times G_{s-u}(\eta_0,\eta_1\cup x) dx dy\Bigg)\lambda(d\eta_0)\lambda(d\eta_1)\Bigg) du.
\end{aligned}$$

By (5.10), (5.6), (5.7) and (4.2) for each $\vartheta \in \mathbb{R}$, $i = 0, 1$, $s \ge 0$ and $(\eta_0, \eta_1) \in \Gamma_0^2$, we have

$$\begin{aligned}\left|(\Upsilon_y^i k_s^\sigma)(\eta_0,\eta_1)\right| &\le \|k_s^\sigma\|_\vartheta \exp\Big(\vartheta(|\eta_0|+|\eta_1|)\Big)\int_{\Gamma_0} e^{\vartheta|\xi|} e\left(|t_y^i|;\xi\right)\lambda(d\xi) \qquad (7.24)\\ &= \|k_s^\sigma\|_\vartheta \exp\Big(\vartheta(|\eta_0|+|\eta_1|)\Big)\sum_{n=0}^\infty \frac{1}{n!}e^{n\vartheta}\left(\int_X \left[1-e^{-\phi_i(x-y)}\right]dx\right)^n\\ &\le \|k_s^\sigma\|_\vartheta \exp\Big(\vartheta(|\eta_0|+|\eta_1|)+\varphi e^\vartheta\Big).\end{aligned}$$

By (7.18) we know that $k_{t+s}^\sigma \in \mathcal{K}^\star_{\vartheta(t+u)} \subset \mathcal{K}^\star_{\vartheta_1}$, which by (3.12) implies $\|k_{t+s}^\sigma\|_{\vartheta_1} \le 1$. We take this into account in (7.23), and also that $\tau_y^i(x) \le 1$, see (5.6), and then estimate the summands in (7.22) as follows

$$\left|R_j^\sigma(s)\right| \le \int_X r_j^\sigma(y) g_j(y) dy, \qquad (7.25)$$

with

$$r_1^\sigma(y) = \int_X (1-\psi_\sigma(x))a_0(x-y)dx, \quad r_3^\sigma(y) = \int_X (1-\psi_\sigma(x))a_1(x-y)dx, \qquad (7.26)$$
$$r_2^\sigma(y) = (1-\psi_\sigma(y))\bar{a}_0^0, \qquad r_4^\sigma(y) = (1-\psi_\sigma(y))\bar{a}_1^0,$$

see (4.3). It is clear that $r_j^\sigma(y) \le r_j^1(y)$, $j = 1,\dots,4$, and

$$\forall y \in X \qquad r_j^\sigma(y) \to 0 \quad \sigma \to 0, \ \ = 1,\dots,4. \qquad (7.27)$$

Furthermore,

$$g_1(y) = g_2(y) = c(\vartheta_1)\int_0^s \int_{\Gamma_0}\int_{\Gamma_0} |G_u(\eta_0\cup y,\eta_1)|\, e^{\vartheta_1(|\eta_0|+|\eta_1|)}\lambda(d\eta_0)\lambda(d\eta_1)du, \qquad (7.28)$$
$$g_3(y) = g_4(y) = c(\vartheta_1)\int_0^s \int_{\Gamma_0}\int_{\Gamma_0} |G_u(\eta_0,\eta_1\cup y)|\, e^{\vartheta_1(|\eta_0|+|\eta_1|)}\lambda(d\eta_0)\lambda(d\eta_1)du,$$

where $c(\vartheta_1) = \exp(\vartheta_1 + \varphi e^{\vartheta_1})$. Let us show that each g_j, $j = 1,\dots,4$, is integrable for all $s \le s_0$. Since $G_u \in \mathcal{G}_{\vartheta_2}$, by (5.32) and (7.20) for $u \le s \le s_0$, see (7.19), we have

$$|G_u|_{\vartheta_2} \le \frac{T(\bar{\vartheta}_t,\vartheta_2)}{T(\bar{\vartheta}_t,\vartheta_2)-s_0}|G|_{\bar{\vartheta}_t} =: C_G. \qquad (7.29)$$

By (7.28) we then have

$$\begin{aligned}\int_X g_1(y)dy &\le c(\vartheta_1)\int_0^s \left(\int_{\Gamma_0}\int_{\Gamma_0}\int_X |G_u(\eta_0\cup y,\eta_1)|\, e^{\vartheta_1(|\eta_0|+|\eta_1|)}\lambda(d\eta_0)\lambda(d\eta_1)dy\right)du\\ &= c(\vartheta_1)e^{-\vartheta_1}\int_0^s \left(\int_{\Gamma_0}\int_{\Gamma_0} |\eta_0|\,|G_u(\eta_0,\eta_1)|\, e^{\vartheta_1(|\eta_0|+|\eta_1|)}\lambda(d\eta_0)\lambda(d\eta_1)\right)du\\ &\le \frac{sc(\vartheta_1)}{(\vartheta_2-\vartheta_1)e^{1+\vartheta_1}}C_G. \qquad (7.30)\end{aligned}$$

Clearly, the same estimate holds for the remaining g_j. Then by the dominated convergence theorem and (7.27), (7.25) it follows that $R^\sigma(s) \to 0$ as $\sigma \to 0$, holding for all $s \le s_0$, see (7.19). By (7.14) and (7.21) this yields $\langle\!\langle q_s^\sigma, G\rangle\!\rangle \to 0$ as $\sigma \to 0$.

To complete the proof of this statement, let us consider the following sequences, cf. (7.19),

$$t_l = t_{l-1} + s_{0l}, \quad s_{0l} = \epsilon T_*(\vartheta_{t_{l-1}})/2, \quad t_0 = 0, \quad l \in \mathbb{N}. \tag{7.31}$$

Now we may use the construction just made and the induction in l, which yields (7.12) holding for all $t \leq t_l$. Thus, the proof will follow if we show $t_l \to +\infty$ as $l \to +\infty$. Set $\sup_l t_l =: t_*$. By (7.31) we have $t_l = s_{01} + \cdots + s_{0l}$. Hence, $t_* < \infty$ yields $s_{0l} \to 0$, $l \to +\infty$. By passing to the limit in the second formula in (7.31) we then get $T_*(\vartheta_{t_*}) = 0$, which is impossible, see (6.11). □

Proof of Lemma 7.3. By Lemma 7.4 it follows that $\mu_t^\sigma(F) \to \mu_t(F)$, $\sigma \to 0$, holding for all $F \in \mathcal{F}_\infty$, which by Proposition 6.2 yields that $\mu_t^\sigma(F) \to \mu_t(F)$, $\sigma \to 0$, for all $F \in \widetilde{\mathcal{F}}$. Then the property in question follows by claim (ii) of Proposition 3.8. □

We end up this subsection with the following complement to Lemma 7.3. For $F \in \widetilde{\mathcal{F}}$, see (3.49), and a sequence $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathcal{P}_{\rm exp}$ as in Proposition 7.2, consider

$$\tilde{\mu}_n(d\gamma) = C_n^{-1} F(\gamma)\mu_n(d\gamma), \qquad n \in \mathbb{N}. \tag{7.32}$$

where

$$C_n = \mu_n(F) > 0, \tag{7.33}$$

since each $F \in \widetilde{\mathcal{F}}$ is strictly positive.

Proposition 7.5. *Let $\tilde{\mu}_n$ and μ_n be as in (7.32) and assume that $\mu_n \Rightarrow \mu$ as $n \to +\infty$. Then $\tilde{\mu}_n \Rightarrow \tilde{\mu}$, where*

$$\tilde{\mu}(d\gamma) = C^{-1} F(\gamma)\mu(d\gamma), \qquad C = \mu(F).$$

Proof. By assumption, $C_n \to C$. Take any $F' \in \widetilde{\mathcal{F}}$ and set $F'' = F'F$, which is an element of $\widetilde{\mathcal{F}}$ since the latter is closed under multiplication, see Proposition 3.8. Then $\tilde{\mu}_n(F') = C_n^{-1}\mu_n(F'') \to C^{-1}\mu(F'')$ as $n \to +\infty$. Since $\widetilde{\mathcal{F}}$ is convergence determining, see claim (ii) of Proposition 3.8, the sequence $\{\tilde{\mu}_n\}_{n\in\mathbb{N}}$ converges to some $\tilde{\mu} \in \mathcal{P}_{\rm exp}$ (by Proposition 7.2), such that $\tilde{\mu}(F') = C^{-1}\mu(F'')$. This implies that $\tilde{\mu}$ is as stated since $\widetilde{\mathcal{F}}$ is separating. □

8 Existence: approximating processes

The aim of this section is proving Theorem 4.5 by constructing path measures for the model described by L^σ introduced in the preceding section. This will be done in a direct way by means of the corresponding Markov transition functions.

8.1 The transition function

We start by introducing the real linear space of signed measures on Γ_*^2, see [8, Chapter 4], which we denote by $\mathcal{M}$. That is, each $\mu \in \mathcal{M}$ is a σ-additive map $\mu : \mathcal{B}(\Gamma_*^2) \to \mathbb{R}$ taking finite values only. Let $\mathcal{M}^+$ be the set of $\mu \in \mathcal{M}$ such that $\mu(\mathbb{A}) \geq 0$ for all $\mathbb{A} \in \mathcal{B}(\Gamma_*^2)$. Then the Jordan decomposition of a given $\mu \in \mathcal{M}$ is the unique representation $\mu = \mu^+ - \mu^-$, $\mu^\pm \in \mathcal{M}^+$, in view of which the cone $\mathcal{M}^+$ is generating. Set $|\mu| = \mu^+ + \mu^-$. Then

$$\|\mu\| = |\mu|(\Gamma_*^2) \tag{8.1}$$

is a norm, additive on the cone $\mathcal{M}^+$. According to [8, Proposition 4.1.8, page 119], $\mathcal{M}$ is a Banach space with this norm. Set $\Psi_1 = 1 + \Psi$, where the latter was defined in (3.30), and then define

$$\mathcal{M}_n = \{\mu \in \mathcal{M} : \|\mu\|_n := |\mu|(\Psi_1^n) < \infty\}, \qquad n \in \mathbb{N}. \tag{8.2}$$

By the same [8, Proposition 4.1.8, page 119] $\mathcal{M}_n$ with the norm $\|\mu\|_n$ is also a real Banach space. In the sequel, we extend (8.2) to $n = 0$ by setting $\mathcal{M}_0 = \mathcal{M}$ and $\|\mu\|_0 = \|\mu\|$. Additionally, for $n \in \mathbb{N}_0$, we set

$$\varphi_n(\mu) = \mu(\Psi_1^n). \tag{8.3}$$

Now for $\beta > 0$, define

$$\|\mu\|_\beta = \int_{\Gamma_*^2} \exp\left(\beta\Psi(\gamma)\right)|\mu|(d\gamma), \quad \mathcal{M}_\beta = \{\mu \in \mathcal{M} : \|\mu\|_\beta < \infty\}, \tag{8.4}$$

and also

$$\varphi_\beta(\mu) = \int_{\Gamma_*^2} \exp\left(\beta\Psi(\gamma)\right)\mu(d\gamma). \tag{8.5}$$

It is clear that

$$\forall \mu \in \mathcal{M}_+ \qquad \|\mu\|_n = \varphi_n(\mu), \qquad \|\mu\|_\beta = \varphi_\beta(\mu), \tag{8.6}$$

holding for all $n \in \mathbb{N}_0$ and $\beta > 0$. In our construction, we essentially use the cones of positive elements

$$\mathcal{M}_n^+ = \mathcal{M}_n \cap \mathcal{M}^+, \qquad \mathcal{M}_\beta^+ = \mathcal{M}_\beta \cap \mathcal{M}^+, \qquad \beta > 0, \; n \in \mathbb{N}. \tag{8.7}$$

For a given $\mathcal{N} \subset \mathcal{M}$, by $\overline{\mathcal{N}}$ we denote the closure of $\mathcal{N}$ in $\|\cdot\|$ defined in (8.1). The proof of the next statement is completely analogous to that of [16, Lemma 7.4 and Corollary 7.5 pages 39, 40], and thus is omitted here.

Proposition 8.1. *For each $n \in \mathbb{N}$ and $\beta > 0$, it follows that $\overline{\mathcal{M}_\beta} = \overline{\mathcal{M}_n} = \mathcal{M}$ and also $\overline{\mathcal{M}_\beta^+} = \overline{\mathcal{M}_n^+} = \mathcal{M}^+$.*

Finally, we denote $\mathcal{M}^{+,1} = \mathcal{P}(\Gamma_*^2)$ and also

$$\mathcal{M}_\beta^{+,1} = \mathcal{M}^{+,1} \cap \mathcal{M}_\beta, \qquad \mathcal{M}_n^{+,1} = \mathcal{M}^{+,1} \cap \mathcal{M}_n. \tag{8.8}$$

By (3.37) it follows that

$$\forall \beta > 0 \;\; \forall n \in \mathbb{N} \qquad \mathcal{P}_{\exp} \subset \mathcal{M}_\beta^{+,1} \subset \mathcal{M}_n^{+,1}. \tag{8.9}$$

Now for $\sigma \in (0,1]$, we set, cf. (7.4),

$$\begin{aligned} \Psi^\sigma(\gamma) &= \sum_{x\in\gamma_0} \int_X a_0^\sigma(x,y) \exp\left(-\sum_{z\in\gamma_1} \phi_0(z-y)\right) dy \\ &+ \sum_{x\in\gamma_1} \int_X a_1^\sigma(x,y) \exp\left(-\sum_{z\in\gamma_0} \phi_1(z-y)\right) dy, \qquad \gamma \in \Gamma_*^2. \end{aligned}$$

By (3.29) and (7.1) it follows that $\psi(x) \le \psi_\sigma(x) \le \psi(x)/\sigma$, $\sigma \in (0,1]$. For these values of σ, by (3.30) and (4.3), (5.15) we then have

$$\Psi^\sigma(\gamma) \le (\alpha/\sigma)\Psi(\gamma), \qquad \gamma \in \Gamma_*^2. \tag{8.10}$$

As mentioned above, the transition function in question will be constructed directly, i.e., by the formula

$$p_t^\sigma(\gamma, \cdot) = S^\sigma(t)\delta_\gamma(\cdot), \qquad t > 0, \;\; \gamma \in \Gamma_*^2, \tag{8.11}$$

where δ_γ is the Dirac measure centered at γ and $S^\sigma = \{S^\sigma(t)\}_{t\ge 0}$ is the stochastic semigroup of bounded linear operators acting in $\mathcal{M}$, generated by the dual $L^{\dagger,\sigma}$ of L^σ defined in (7.4). The mentioned duality means that

$$\mu(L^\sigma F) = (L^{\dagger,\sigma}\mu)(F), \qquad F \in \mathcal{D}(L). \tag{8.12}$$

Recall that the domains of all L^σ, $\sigma \in [0,1]$ are the same, i.e., are as in Definition 4.1. By (7.4) and (7.2) we then get

$$\begin{aligned}(L^{\dagger,\sigma}\mu)(\mathbb{A}) &= -\int_{\Gamma_*^2} \mathbb{1}_{\mathbb{A}}(\gamma)\Psi^\sigma(\gamma)\mu(d\gamma) + \int_{\Gamma_*^2} \Omega^\sigma(\mathbb{A}|\gamma)\mu(d\gamma) \\ &=: (A\mu)(\mathbb{A}) + (B\mu)(\mathbb{A}), \quad \mathbb{A} \in \mathcal{B}(\Gamma_*^2).\end{aligned} \tag{8.13}$$

Here

$$\begin{aligned}\Omega^\sigma(\mathbb{A}|\gamma) &= \sum_{x\in\gamma_0}\int_X a_0^\sigma(x,y)\exp\left(-\sum_{z\in\gamma_1}\phi_0(z-y)\right)\mathbb{1}_{\mathbb{A}}(\gamma_0\setminus x\cup_0 y,\gamma_1)dy \\ &+ \sum_{x\in\gamma_1}\int_X a_1^\sigma(x,y)\exp\left(-\sum_{z\in\gamma_0}\phi_1(z-y)\right)\mathbb{1}_{\mathbb{A}}(\gamma_0,\gamma_1\setminus x\cup_1 y)dy.\end{aligned} \tag{8.14}$$

Note that A in (8.13) is just the multiplication operator by Ψ^σ, and the following holds

$$\Omega^\sigma(\Gamma_*^2|\gamma) = \Psi^\sigma(\gamma). \tag{8.15}$$

Now we set

$$\mathcal{D}(L^{\dagger,\sigma}) = \{\mu \in \mathcal{M} : |\mu|(\Psi^\sigma) < \infty\}, \tag{8.16}$$

which might have sense if we show that B can act on $\mu \in \mathcal{D}(L^{\dagger,\sigma})$. By writing $\mu = \mu^+ - \mu^-$ we conclude that it is enough to show $B\mu \in \mathcal{M}$ for positive $\mu \in \mathcal{D}(L^{\dagger,\sigma})$ only. Since B itself is positive, by (8.6) and (8.15) we have that

$$\|B\mu\| = (B\mu)(\Gamma_*^2) = \int_{\Gamma_*^2}\Psi^\sigma(\gamma)\mu(d\gamma) = \|A\mu\|, \tag{8.17}$$

which yields $L^{\dagger,\sigma} : \mathcal{D}(L^{\dagger,\sigma}) \to \mathcal{M}$. Clearly, $(A, \mathcal{D}(L^{\dagger,\sigma}))$ is closed and the following holds

$$\mathcal{M}_1 \subset \mathcal{D}(L^{\dagger,\sigma}), \tag{8.18}$$

see (8.10) and (8.2).

Remark 8.2. Note that $\delta_\gamma \in \mathcal{D}(L^{\dagger,\sigma})$, since $\delta_\gamma(\Psi^\sigma) = \Psi^\sigma(\gamma) < \infty$, holding for all $\gamma \in \Gamma_*^2$, see (8.10) and (3.32). At the same time, δ_γ is evidently not sub-Poissonian.

Along with constructing the semigroup S^σ, see (8.11), in Lemma 8.3 below we obtain a number of complementary results, which we then exploit for proving Theorem 4.5. To this end, for $n \in \mathbb{N}$ and a positive μ, let us consider, cf. (8.3), (8.13) and (8.14),

$$\begin{aligned}\varphi_n(B\mu) &= \int_{\Gamma_*^2}\Psi_1^n(\gamma)(B\mu)(d\gamma) = \int_{\Gamma_*^2}\int_{\Gamma_*^2}\Psi_1^n(\gamma)\Omega^\sigma(d\gamma|\gamma')\mu(d\gamma') \\ &= \int_{\Gamma_*^2}\left(\sum_{x\in\gamma_0}\int_X a_0^\sigma(x,y)\exp\left(-\sum_{z\in\gamma_1}\phi_0(z-y)\right)\Psi_1^n(\gamma\setminus x\cup_0 y)dy\right)\mu(d\gamma) \\ &+ \int_{\Gamma_*^2}\left(\sum_{x\in\gamma_1}\int_X a_1^\sigma(x,y)\exp\left(-\sum_{z\in\gamma_0}\phi_1(z-y)\right)\Psi_1^n(\gamma\setminus x\cup_1 y)dy\right)\mu(d\gamma).\end{aligned} \tag{8.19}$$

Since $\Psi_1(\gamma\setminus x\cup_i y) = 1 + \Psi(\gamma\setminus x\cup_i y) = \Psi_1(\gamma) + \psi(y) - \psi(x)$, $i = 0,1$, see (3.30), then

$$\Psi_1^n(\gamma\setminus x\cup_i y) \le 2^n\Psi_1^n(\gamma), \tag{8.20}$$

which by (8.10) yields in (8.19) the following estimate

$$\forall \mu \in \mathcal{M}^+_{n+1} \qquad \|B\mu\|_n = \varphi_n(B\mu) \le 2^n \alpha \sigma^{-1} \|\mu\|_{n+1}, \tag{8.21}$$

and hence

$$\forall n \in \mathbb{N}_0 \qquad B : \mathcal{M}^+_{n+1} \to \mathcal{M}^+_n. \tag{8.22}$$

In a similar way, one shows that $\|A\mu\|_n \le (\alpha/\sigma)\|\mu\|_{n+1}$ and

$$-A : \mathcal{M}^+_{n+1} \to \mathcal{M}^+_n, \tag{8.23}$$

which finally yields that $L^{\dagger,\sigma} : \mathcal{M}^+_{n+1} \to \mathcal{M}^+_n$, holding for all $n \in \mathbb{N}_0$. By means of (8.21) and the corresponding estimate for A we then define bounded linear operators

$$(L^{\dagger,\sigma})^l_{n,n+l} : \mathcal{M}_n \to \mathcal{M}_{n+l}, \qquad l \in \mathbb{N}, \tag{8.24}$$

the norms of which satisfy

$$\|(L^{\dagger,\sigma})^l_{n,n+l}\| \le \left(\frac{\alpha}{\sigma}\right)^l (2^n+1)(2^{n+1}+1)\cdots(2^{n+l-1}+1). \tag{8.25}$$

Next, similarly as in (5.27) we also define bounded operators $(L^{\dagger,\sigma}_{\beta',\beta})^l : \mathcal{M}_\beta \to \mathcal{M}_{\beta'}$, $\beta > \beta' > 0$, see (8.4). To estimate their norms, for a given $\mu = \mu^+ - \mu^- \in \mathcal{D}(L^{\dagger,\sigma})$, we write

$$L^{\dagger,\sigma}\mu = (A+B)(\mu^+ - \mu^-) = (B\mu^+ - A\mu^-) - (B\mu^- - A\mu^+) =: \mu_1^+ - \mu_1^-.$$

It is clear that $\mu_1^\pm \in \mathcal{M}^+$. Then

$$\|L^{\dagger,\sigma}\mu\|_{\beta'} \le \|\mu_1^+\|_{\beta'} + \|\mu_1^-\|_{\beta'} = \|A\mu^+\|_{\beta'} + \|A\mu^-\|_{\beta'} + \|B\mu^+\|_{\beta'} + \|B\mu^-\|_{\beta'}. \tag{8.26}$$

Here we used the additivity of the norms on the positive cone as well as the positivity of B and $-A$, see (8.22), (8.23). Now by (8.10) we have

$$\Psi^\sigma(\gamma)\exp\left(\beta'\Psi(\gamma)\right) \le \frac{\alpha}{e\sigma(\beta-\beta')}\exp\left(\beta\Psi(\gamma)\right),$$

which for $\mu \in \mathcal{M}^+_\beta$ yields

$$\|A\mu\|_{\beta'} \le \frac{\alpha}{e\sigma(\beta-\beta')}\|\mu\|_\beta. \tag{8.27}$$

Next, similarly as in (8.19), (8.21) by (8.10) we have

$$\begin{aligned}
&\int_{\Gamma^2_*} \exp\left(\beta'\Psi(\gamma)\right)(B\mu)(d\gamma) = \int_{\Gamma^2_*}\int_{\Gamma^2_*} \exp\left(\beta'\Psi(\gamma')\right)\Omega^\beta(d\gamma'|\gamma)\mu(d\gamma) \\
&= \int_{\Gamma^2_*} \exp\left(\beta'\Psi(\gamma)\right)\Bigg(\sum_{x\in\gamma_0}\int_X a_0^\sigma(x,y)\exp\Bigg(-\sum_{z\in\gamma_1}\phi_0(z-y) + \beta'(\psi(y)-\psi(x))\Bigg)dy \\
&+ \sum_{x\in\gamma_1}\int_X a_1^\sigma(x,y)\exp\Bigg(-\sum_{z\in\gamma_0}\phi_1(z-y) + \beta'(\psi(y)-\psi(x))\Bigg)dy\Bigg)\mu(d\gamma) \\
&\le e^{\beta'}\int_{\Gamma^2_*}\Psi^\sigma(\gamma)\exp\left(\beta'\Psi(\gamma)\right)\mu(d\gamma) \le \frac{e^{\beta'}\alpha}{\sigma e(\beta-\beta')}\|\mu\|_\beta, \qquad \mu \in \mathcal{M}^+_\beta.
\end{aligned}$$

We combine now this estimate with (8.27) and obtain in (8.26) the following, cf. (5.27)

$$\|(L^{\dagger,\sigma}_{\beta',\beta})^l\| \le \left(\frac{l}{eT_\sigma(\beta,\beta')}\right)^l, \qquad l \in \mathbb{N}, \tag{8.28}$$

with

$$T_\sigma(\beta,\beta') = \frac{\sigma(\beta-\beta')}{\alpha e^{\beta}}. \tag{8.29}$$

By (8.24)), for each $l \in \mathbb{N}$ and $\mu \in \mathcal{M}_\beta$, we have that $(L^{\dagger,\sigma})^l \mu \in \mathcal{M}_{\beta'}$, $\beta' < \beta$, and the following holds

$$(L^{\dagger,\sigma}_{\beta',\beta})^l \mu = (L^{\dagger,\sigma})^l \mu, \qquad l \in \mathbb{N}. \tag{8.30}$$

In the next statement, we employ a perturbation technique for constructing stochastic semigroups of bounded linear operators in ordered Banach spaces with norms additive on the cones of positive elements. Note that the spaces defined in (8.2), (8.4) have this property, see (8.6), (8.7). The details of this technique can be found in our previous work [16, subsect. 7.1.1]. Here we just recall that a semigroup $S = \{S(t)\}_{t\geq 0}$ of such operators is called stochastic (resp. substochastic) if each $S(t)$ is positive and $\|S(t)u\| = \|u\|$ (resp. $\|S(t)u\| \leq \|u\|$) for each positive u and $t > 0$. Also, for a given $n \in \mathbb{N}$ and

$$\mathcal{D}^\sigma_n := \{\mu \in \mathcal{M}_n : \ |\mu|(\Psi^\sigma \Psi_1^n) < \infty\}, \tag{8.31}$$

cf. (8.2), (8.16), by the trace of A in $\mathcal{M}_n$ we mean the operator $(A, \mathcal{D}^\sigma_n)$ acting therein.

Lemma 8.3. *For each $\sigma \in (0,1]$, the closure of $(L^{\dagger,\sigma}, \mathcal{D}(L^{\dagger,\sigma}))$, see (8.16), is the generator of a stochastic semigroup, $S^\sigma = \{S^\sigma(t)\}_{t\geq 0}$, in $\mathcal{M}$ such that $S^\sigma(t) : \mathcal{M}_n \to \mathcal{M}_n$, holding for each $t > 0$ and $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, the restrictions $S^\sigma(t)|_{\mathcal{M}_n}$ constitute a C_0-semigroup on $\mathcal{M}_n$. Additionally, for each $\beta > 0$ and $\beta' \in (0,\beta)$, $S^\sigma(t) : \mathcal{M}^+_\beta \to \mathcal{M}^+_{\beta'}$ for $t < T_\sigma(\beta,\beta')$, see (8.29).*

Proof. We basically follow the way of proving [16, Lemma 7.6], based on the Thieme-Voigt theorem [23] in the form adapted to the context of the present work, see [16, Assumption 7.1 and Proposition 7.2]. Thus, we begin by mentioning that all the items of Assumption 7.1 *ibid* are satisfied. That is: (i) each $\mathcal{M}_n$ is dense in $\mathcal{M}$, see Proposition 8.1; (ii) each $\mathcal{M}_n$ is a Banach space (by the aforementioned [8, Proposition 4.1.8, page 119]); (iii) each cone $\mathcal{M}^+_n$, $n \in \mathbb{N}$, is $\mathcal{M}^+ \cap \mathcal{M}_n$ and $\|\cdot\|_n$ is additive on this cone, see (8.6); (iv) each $\mathcal{M}^+_n$ is dense in $\mathcal{M}^+$, see Proposition 8.1. Now we can apply [16, Proposition 7.2], which amounts to checking that:

- (i) $-A$ and B map $\mathcal{D}(L^{\dagger,\sigma}) \cap \mathcal{M}^+$ to $\mathcal{M}^+$, which follows by the very definition of A and (8.17);
- (ii) $(A, \mathcal{D}(L^{\dagger,\sigma}))$ generates a substochastic semigroup, $S^\sigma_0 = \{S^\sigma_0(t)\}_{t\geq 0}$, such that (a) $S^\sigma_0(t) : \mathcal{M}_n \to \mathcal{M}_n$, (b) the restrictions $S^\sigma_0(t)|_{\mathcal{M}_n}$ constitute a C_0-semigroup on $\mathcal{M}_n$ generated by the trace of A in $\mathcal{M}_n$;
- (iii) $B : \mathcal{D}^\sigma_n \to \mathcal{M}_n$ and $\varphi((A+B)\mu) = 0$, holding for all $\mu \in \mathcal{D}(L^{\dagger,\sigma}) \cap \mathcal{M}^+$;
- (iv) there exist positive c_n and ε_n such that the following holds

$$\forall \mu \in \mathcal{D}^\sigma_n \cap \mathcal{M}^+ \qquad \varphi_n((A+B)\mu) \leq c_n \varphi_n(\mu) - \varepsilon_n \|A\mu\|. \tag{8.32}$$

The semigroup S^σ_0 mentioned in item (ii) consists of the multiplication operators

$$(S^\sigma_0(t)\mu)(d\gamma) = \exp\left(-t\Psi^\sigma(\gamma)\right)\mu(d\gamma), \tag{8.33}$$

which is certainly such that (a) holds for each $n \in \mathbb{N}$. To check the strong continuity of S^σ_0, we take $\mu \in \mathcal{M}^+$ and $\varepsilon > 0$, and then show that $\|\mu - S^\sigma_0(t)\mu\| < \varepsilon$ whenever t is smaller than an ε-specific $\delta > 0$. The validity of such estimates for an arbitrary $\mu \in \mathcal{M}$ then simply follows by the Jordan decomposition. Since $\mathcal{D}(L^{\dagger,\sigma})$ is dense in $\mathcal{M}$, see (8.18) and Proposition 8.1, one finds $\mu' \in \mathcal{D}(L^{\dagger,\sigma}) \cap \mathcal{M}^+$ such that $\|\mu - \mu'\| < \varepsilon/3$. By (8.33) and (8.10) we then have

$$\|\mu - S^\sigma_0(t)\mu\| \leq \|\mu - \mu'\| + \|S^\sigma_0(t)(\mu - \mu')\| + \|\mu' - S^\sigma_0(t)\mu'\| \tag{8.34}$$

$$\leq 2\|\mu - \mu'\| + t\int_{\Gamma^2_*} \Psi^\sigma(\gamma)\mu'(d\gamma) \leq \frac{2}{3}\varepsilon + \frac{t\alpha}{\sigma}\|\mu'\|_1 < \varepsilon,$$

for $t < \sigma\varepsilon/3\alpha\|\mu'\|_1$. Moreover, (8.33) can be considered as the definition of bounded linear operators acting in a given $\mathcal{M}_n$. These operators constitute a C_0 semigroup, which can be proved similarly as in (8.34). Its generator is then obviously the trace of A in $\mathcal{M}_n$, see (8.31). Thus, it remains to prove the validity of (8.32), that is, the validity of

$$\int_{\Gamma^2_*} \Psi^n_1(\gamma)(L^{\dagger,\sigma}\mu)(d\gamma) + \varepsilon_n \int_{\Gamma^2_*} \Psi^\sigma(\gamma)\mu(d\gamma) \leq c_n \int_{\Gamma^2 + *} \Psi^n_1(\gamma)\mu(d\gamma), \tag{8.35}$$

holding for all $\mu \in \mathcal{D}^\sigma_n \cap \mathcal{M}^+$ and certain positive c_n and ε_n. This clearly amounts to proving that each of the summands in the left-hand side of (8.35) is $\leq (c_n/2)\mu(\Psi^n_1)$ with a properly chosen c_n. We begin by proving this for the first summand. By (8.12) we have that

$$\int_{\Gamma^2_*} \Psi^n_1(\gamma)(L^{\dagger,\sigma}\mu)(d\gamma) = \int_{\Gamma^2_*} (L^\sigma \Psi^n_1)(\gamma)\mu(d\gamma) \tag{8.36}$$

Similarly as in obtaining (8.20) we have

$$|\Psi^n_1(\gamma \setminus x \cup_i y) - \Psi^n_1(\gamma)| \leq (2^n - 1)\,|\psi(y) - \psi(x)|\,\Psi^{n-1}_1(\gamma). \tag{8.37}$$

Set

$$b_i(x) = \int_X a_i(x-y)\,|\psi(y) - \psi(x)|\,dy, \qquad i = 0, 1. \tag{8.38}$$

Assume first that $|x| \geq |y|$. Then

$$\begin{aligned} |\psi(y) - \psi(x)| &= \psi(y) - \psi(x) = \left[|x|^{d+1} - |y|^{d+1}\right]\psi(x)\psi(y) \\ &\leq \psi(x)\left[(|x-y| + |y|)^{d+1} - |y|^{d+1}\right]\psi(y) \\ &= \psi(x)\sum_{l=1}^{d+1}\binom{d+1}{l}|x-y|^l|y|^{d+1-l}\psi(y) \\ &\leq \psi(x)\sum_{l=1}^{d+1}\binom{d+1}{l}|x-y|^l =: \psi(x)h(x-y), \end{aligned} \tag{8.39}$$

where we have used the fact that $|y|^{d+1-l}\psi(y) \leq 1$ for all $l \geq 1$ and $y \in X$. For $|x| < |y|$, we have

$$\begin{aligned} |\psi(y) - \psi(x)| &= (|y|^{d+1} - |x|^{d+1})\psi(x)\psi(y) \\ &\leq \psi(x)\left[(|y-x| + |x|)^{d+1} - |x|^{d+1}\right]\psi(y) \\ &\leq \psi(x)\sum_{l=1}^{d+1}\binom{d+1}{l}|x-y|^l|y|^{d+1-l}\psi(y) \\ &\leq \psi(x)h(x-y). \end{aligned}$$

Now we use these two estimates in (8.38) and obtain

$$b_i(x) \leq \psi(x)\bar{\alpha}, \qquad \bar{\alpha} := \max_{i=0,1}\int_X a_i(x)h(x)dx, \tag{8.40}$$

cf. (5.15) and (4.3). By (8.37) and the latter estimate we obtain

$$|(L^\sigma \Psi_1^n)(\gamma)| \leq \sum_{x\in\gamma_0} \int_X a_0(x-y) \, |\Psi_1^n(\gamma \setminus x \cup_0 y) - \Psi_1^n(\gamma)| \, dy \qquad (8.41)$$

$$+ \sum_{x\in\gamma_1} \int_X a_1(x-y) \, |\Psi_1^n(\gamma \setminus x \cup_1 y) - \Psi_1^n(\gamma)| \, dy$$

$$\leq (2^n - 1)\bar{\alpha}\Psi_1^{n-1}(\gamma) \left(\sum_{x\in\gamma_0} \psi(x) + \sum_{x\in\gamma_1} \psi(x) \right) \leq (2^n - 1)\bar{\alpha}\Psi_1^n(\gamma),$$

holding for all $\sigma \in [0,1]$, including $\sigma = 0$. By (8.41) and (8.36) we then have

$$\int_{\Gamma_*^2} \Psi_1^n(\gamma)(L^{\dagger,\sigma}\mu)(d\gamma) \leq (2^n-1)\bar{\alpha} \int_{\Gamma_*^2} \Psi_1^n(\gamma)\mu(d\gamma). \qquad (8.42)$$

By (8.10) we have $\Psi^\sigma(\gamma) \leq (\alpha/\sigma)\Psi_1^n(\gamma)$ holding for all $n \in \mathbb{N}$ and $\gamma \in \Gamma_*^2$, which then yields

$$\int_{\Gamma_*^2} \Psi^\sigma(\gamma)\mu(d\gamma) \leq \frac{\alpha}{\sigma} \int_{\Gamma_*^2} \Psi_1^n(\gamma)\mu(d\gamma).$$

The latter estimate together with (8.42) yields the validity of (8.35) with $\varepsilon_n = 1$ and $c_n = (2^n-1)\bar{\alpha} + \alpha/\sigma$.

It remains now to prove the concluding statement of the lemma. We proceed by defining the following bounded operators

$$S^\sigma_{\beta',\beta}(t) = I_{\beta',\beta} + \sum_{l=1}^\infty \frac{t^l}{l!} (L^{\dagger,\sigma}_{\beta',\beta})^l, \qquad t < T_\sigma(\beta,\beta'), \qquad (8.43)$$

acting from $\mathcal{M}_\beta$ to $\mathcal{M}_{\beta'}$, see (8.29). Here the powers $(L^{\dagger,\sigma}_{\beta',\beta})^l$ satisfy (8.28) and $I_{\beta',\beta}$ is the embedding operator. By (8.43) and (8.30), for each $\mu \in \mathcal{M}_\beta$, one has

$$S^\sigma_{\beta',\beta}(t)\mu = S^\sigma(t)\mu, \qquad (8.44)$$

where $S^\sigma(t)$ is the same as in the first part of the lemma. Then the positivity of $S^\sigma_{\beta',\beta}(t)$ follows by the positivity of the latter. This completes the whole proof. □

Thus, the lemma just proved yields the existence of the semigroup S^σ which we use in (8.11) to obtain the Markov transition function p_t^σ. The fact that $t \mapsto p_t^\sigma$ satisfies the corresponding conditions, see [11, eqs. (1.3)–(1.6), page 156], follows directly by (8.11). We will use this function to construct the finite dimensional marginals of the stochastic process corresponding to the approximating model described by L^σ. This will be done in the next subsection.

8.2 Constructing path measures

By means of the semigroup S^σ constructed in Lemma 8.3 we may have

$$\hat{\mu}_t^\sigma(\cdot) = (S^\sigma(t)\mu)(\cdot) = \int_{\Gamma_*^2} p_t^\sigma(\gamma,\cdot)\mu(d\gamma), \qquad \mu \in \mathcal{M}. \qquad (8.45)$$

Recall that here $\sigma \in (0,1]$ and S^σ is stochastic. The latter means that $\hat{\mu}_t^\sigma$ is in $\mathcal{P}(\Gamma_*^2)$ whenever μ has this property. Moreover, $\hat{\mu}_t^\sigma$ may be in $\mathcal{M}_n \cap \mathcal{P}(\Gamma_*^2)$ under the corresponding condition. However, so far we do not know whether S^σ preserves $\mathcal{P}_{\exp}$.

Lemma 8.4. *For given $\mu \in \mathcal{P}_{\rm exp}$, let $t \mapsto \mu_t^\sigma \in \mathcal{P}_{\rm exp}$, $t > 0$, be the solution of (7.9), see Proposition 7.1. Let also $\hat{\mu}_t^\sigma$ be as in (8.45) with the same μ. Then, for all $t > 0$, it follows that $\mu_t^\sigma = \hat{\mu}_t^\sigma$.*

Proof. By (8.9) and (8.18) it follows that $\mathcal{P}_{\rm exp} \subset \mathcal{D}(L^{\dagger,\sigma})$, which means that $t \mapsto \hat{\mu}_t^\sigma$ is the classical solution of the initial value problem

$$\frac{d}{dt}\hat{\mu}_t^\sigma = L^{\dagger,\sigma}\hat{\mu}_t^\sigma, \qquad \hat{\mu}_t^\sigma|_{t=0} = \mu, \tag{8.46}$$

which by (8.12) yields that $t \mapsto \hat{\mu}_t^\sigma$ solves (7.9). Then the proof follows by Proposition 7.1. □

It is a standard fact that the transition function p_t^σ determines the finite dimensional distributions of a Markov process, $\mathcal{X}^\sigma$, with values in Γ_*^2, see [11, Theorem 1.1, page 157]. Our aim now is to prove that it has càdlàg paths. To this end, we employ Chentsov-like arguments, cf. [7] and [16, Proposition 7.8], and thus the metric υ^*, see (3.45), (3.46). By Lemma 3.4 it is complete. Set

$$w_u^\sigma(\gamma) = \int_{\Gamma_*^2} \upsilon^*(\gamma,\gamma') p_u^\sigma(\gamma, d\gamma'), \tag{8.47}$$

$$W_{u,v}^\sigma(\gamma) = \int_{\Gamma_*^2} \upsilon^*(\gamma,\gamma') w_u^\sigma(\gamma') p_v^\sigma(\gamma, d\gamma'), \quad u, v \geq 0.$$

Thereafter, for a triple $t_3 > t_2 > t_1 \geq 0$, consider

$$\mathcal{W}^\sigma(t_1,t_2,t_3) = \int_{\Gamma_*^2} W_{t_3-t_2,t_2-t_1}^\sigma(\gamma)\hat{\mu}_{t_1}^\sigma(d\gamma) = \int_{\Gamma_*^2} W_{t_3-t_2,t_2-t_1}^\sigma(\gamma)\mu_{t_1}^\sigma(d\gamma). \tag{8.48}$$

Note that this $\mathcal{W}^\sigma(t_1,t_2,t_3)$ depends also on $\mu = \mu_t^\sigma|_{t=0}$, see Lemma 8.4. By combining [7, Theorem 1] and [11, Theorems 8.6–8.8, pages 137-139] we obtain the following statement.

Proposition 8.5. *Given $T > 0$, $\sigma \in (0,1]$, $s \geq 0$ and $\mu \in \mathcal{P}_{\rm exp}$, assume that there exist $C_\sigma > 0$ and $\delta > 0$ such that, for each triple that satisfies $t_1 \geq s$, $t_3 \leq T$ and $t_3 - t_1 < \delta$, the following holds*

$$\mathcal{W}^\sigma(t_1,t_2,t_3) \leq C_\sigma |t_3 - t_1|^2. \tag{8.49}$$

Then

(i) There exists a probability measure $P_{s,\mu}^\sigma$ on $\mathfrak{D}_{[0,+\infty)}(\Gamma_*^2)$ uniquely determined by its finite dimensional marginals, cf. [11, eq. (1.10), page 157], expressed by the formula

$$\begin{aligned} &P_{s,\mu}^\sigma(\{\gamma : \varpi_{t_n}(\gamma) \in \mathbb{A}_n, \varpi_{t_{n-1}}(\gamma) \in \mathbb{A}_{n-1}, \dots, \varpi_{t_1}(\gamma) \in \mathbb{A}_1, \varpi_0(\gamma) \in \mathbb{A}_0\}) \\ &= \int_{\mathbb{A}_{n-1}} \cdots \int_{\mathbb{A}_0} p_{t_n - t_{n-1}}^\sigma(\gamma_{n-1}, \mathbb{A}_n) p_{t_{n-1}-t_{n-2}}^\sigma(\gamma_{n-2}, d\gamma_{n-1}) \cdots p_{t_2-t_1}^\sigma(\gamma_1, d\gamma_2) \\ &\times p_{t_1}^\sigma(\gamma_0, d\gamma_1)\mu(d\gamma_0), \end{aligned} \tag{8.50}$$

holding for all $n \in \mathbb{N}$, $t_n > t_{n-1} \cdots t_1$ and $\mathbb{A}_j \in \mathcal{B}(\Gamma_*^2)$, $j = 0, \dots, n$.

(ii) If the estimate in (8.49) holds for all $\sigma \in (0,1]$ with one and the same $C > 0$,and the family $\{\hat{\mu}_t^\sigma\}_{\sigma\in(0,1]}$ is tight for all $t > 0$, then the family $\{P_{s,\mu}^\sigma\}_{\sigma\in(0,1]}$ of measures mentioned in (i) is also tight, and hence has accumulation points in the weak topology.

Note that the tightness mentioned in item (ii) of the latter statement follows by Prohorov's theorem and Lemmas 7.3 and 8.4.

Lemma 8.6. *For every $s \geq 0$ and $\mu \in \mathcal{P}_{\rm exp}$, the estimate as in (8.49) holds for all $\sigma \in (0,1]$ with one and the same $C > 0$, dependent on T only.*

Proof. For convenience, we take here $s = 0$ – the proof for $s > 0$ is completely analogous. Then we begin by recalling that δ_γ is in $\mathcal{D}(L^{\dagger,\sigma})$, see Remark 8.2. Thus, by (8.11) and the corresponding formulas, see e.g., [11, eq. (1.16), page 9], we have

$$p_t^\sigma(\gamma,\cdot) = \delta_\gamma(\cdot) + \int_0^t L^{\dagger,\sigma} p_s^\sigma(\gamma,\cdot) ds. \tag{8.51}$$

We use this in (8.47), which yields

$$\begin{aligned} w_u^\sigma(\gamma) &= w_0^\sigma(\gamma) + \int_0^u \left(\int_{\Gamma_*^2} \upsilon^*(\gamma,\gamma') L^{\dagger,\sigma} p_s^\sigma(\gamma, d\gamma') \right) ds \\ &= \int_0^u \left(\int_{\Gamma_*^2} \upsilon^*(\gamma,\gamma') L^{\dagger,\sigma} p_s^\sigma(\gamma, d\gamma') \right) ds \\ &= \int_0^u \left(\int_{\Gamma_*^2} L^\sigma \upsilon^*(\gamma,\gamma') p_s^\sigma(\gamma, d\gamma') \right) ds, \end{aligned} \tag{8.52}$$

see (8.12). The second equality in (8.52) follows by the fact that $w_0^\sigma(\gamma) = \upsilon^*(\gamma,\gamma) = 0$ as υ^* is a metric. The function $\gamma' \mapsto L^\sigma \upsilon^*(\gamma,\gamma') =: J_\gamma^\sigma(\gamma')$ has the following form, see (3.45), (3.46),

$$\begin{aligned} J_\gamma^\sigma(\gamma') &= \sum_{x\in\gamma_0'} \int_X a_0^\sigma(x,y) \exp\left(-\sum_{z\in\gamma_1} \phi_0(z-y) \right) [\upsilon^*(\gamma, \gamma'\setminus x \cup_0 y) - \upsilon^*(\gamma,\gamma')]\, dy \\ &+ \sum_{x\in\gamma_1'} \int_X a_1^\sigma(x,y) \exp\left(-\sum_{z\in\gamma_0} \phi_1(z-y) \right) [\upsilon^*(\gamma, \gamma'\setminus x \cup_1 y) - \upsilon^*(\gamma,\gamma')]\, dy. \end{aligned} \tag{8.53}$$

By the triangle inequality for the metric υ_* we then get

$$\begin{aligned} \left|J_\gamma^\sigma(\gamma')\right| &\leq \sum_{x\in\gamma_0'} \int_X a_0(x-y) \upsilon(\psi(\gamma_0'\setminus x\cup y), \psi\gamma_0') dy \\ &+ \sum_{x\in\gamma_1'} \int_X a_1(x-y) \upsilon(\psi(\gamma_1'\setminus x\cup y), \psi\gamma_1') dy. \end{aligned} \tag{8.54}$$

By (3.44) and (3.46) it follows that

$$\upsilon(\psi(\gamma_i'\setminus x\cup y), \psi\gamma_i') \leq \sup_{g:\|g\|_{BL}\leq 1} |g(x)\psi(x) - g(y)\psi(y)|. \tag{8.55}$$

Proceeding similarly as in (8.39) we obtain

$$\begin{aligned} &|g(x)\psi(x) - g(y)\psi(y)| = \psi(x)\psi(y) \left| \frac{g(x)}{\psi(y)} - \frac{g(y)}{\psi(x)} \right| \\ &\leq \psi(x)\psi(y)g(y) \left| \frac{1}{\psi(y)} - \frac{1}{\psi(x)} \right| + \psi(x)\,|g(x) - g(y)| \end{aligned}$$

$$\leq \psi(x) \left[|x-y| + \sum_{l=1}^{d+1} \binom{d+1}{l} |x-y|^l \right].$$

We apply the latter in (8.55), (8.54) and then arrive at the following, see (8.40) and (5.15),

$$\left| J^\sigma_\gamma(\gamma') \right| \leq (\alpha + \bar{\alpha}) \Psi(\gamma'), \tag{8.56}$$

which is uniform in σ. By (8.56) we then get from (8.52) the following estimate

$$w^\sigma_u(\gamma) \leq (\alpha + \bar{\alpha}) \int_0^u \kappa^\sigma_s(\gamma) ds, \qquad \kappa^\sigma_s(\gamma) := \int_{\Gamma^2_*} \Psi(\gamma') p^\sigma_s(\gamma, d\gamma'). \tag{8.57}$$

By (8.51), similarly as in (8.52) we have

$$\kappa^\sigma_s(\gamma) = \Psi(\gamma) + \int_0^s \left(\int_{\Gamma^2_*} L^\sigma \Psi(\gamma') p^\sigma_v(\gamma, d\gamma') \right) dv \tag{8.58}$$

Proceeding as in (8.41) we obtain

$$|L^\sigma \Psi(\gamma')| \leq \bar{\alpha} \Psi(\gamma'),$$

by which we obtain from (8.57), (8.58) the following

$$\kappa^\sigma_s(\gamma) \leq \Psi(\gamma) + \bar{\alpha} \int_0^s \kappa^\sigma_v(\gamma) dv, \tag{8.59}$$

which by the Grönwall inequality and (8.57) leads to

$$w^\sigma_u(\gamma) \leq (\alpha + \bar{\alpha}) u e^{\bar{\alpha} u} \Psi(\gamma). \tag{8.60}$$

Now we may pass to estimating $W^\sigma_{u,v}(\gamma)$. By the second line in (8.47) and (8.60) we have

$$W^\sigma_{u,v}(\gamma) \leq (\alpha + \bar{\alpha}) u e^{\bar{\alpha} u} V^\sigma_v(\gamma), \qquad V^\sigma_v(\gamma) := \int_{\Gamma^2_*} \upsilon^*(\gamma, \gamma') \Psi(\gamma') p^\sigma_v(\gamma, d\gamma'). \tag{8.61}$$

Here we again apply (8.51) and then get, cf. (8.52),

$$V^\sigma_v(\gamma) = \int_0^v \left(\int_{\Gamma^2_*} \left[L^\sigma \Psi(\gamma') \upsilon^*(\gamma, \gamma') \right] p^\sigma_s(\gamma, d\gamma') \right) ds \tag{8.62}$$

Proceeding as in (8.53) we get, see also (8.38),

$$\begin{aligned} |L^\sigma \Psi(\gamma') \upsilon^*(\gamma, \gamma')| \quad \leq & \quad \sum_{x \in \gamma'_0} \int_X a_0(x-y) \Big| \Psi(\gamma' \setminus x \cup_0 y) \upsilon^*(\gamma, \gamma' \setminus x \cup_0 y) \\ & \qquad\qquad - \Psi(\gamma') \upsilon^*(\gamma, \gamma') \Big| dy \\ & + \quad \sum_{x \in \gamma'_1} \int_X a_1(x-y) \Big| \Psi(\gamma' \setminus x \cup_1 y) \upsilon^*(\gamma, \gamma' \setminus x \cup_1 y) \\ & \qquad\qquad - \Psi(\gamma') \upsilon^*(\gamma, \gamma') \Big| dy \\ \leq & \quad 2 \Big(\sum_{x \in \gamma'_0} b_0(x) + \sum_{x \in \gamma'_1} b_1(x) \Big) \end{aligned}$$

$$
\begin{aligned}
&+ \quad \Psi(\gamma')\Big(\sum_{x\in\gamma_0} \int_X a_0(x-y) v^*(\gamma' \setminus x \cup_0 y, \gamma')dy \\
&\qquad + \sum_{x\in\gamma_0} \int_X a_1(x-y) v^*(\gamma' \setminus x \cup_1 y, \gamma')dy \Big) \\
&\leq \quad 2\bar{\alpha}\Psi(\gamma') + (\alpha+\bar{\alpha})\Psi^2(\gamma'),
\end{aligned}
$$

where we used also (8.40) and (8.54). Now we apply the latter estimate in (8.62) and obtain

$$
V_v^\sigma(\gamma) \leq 2\alpha \int_0^v \kappa_s^\sigma(\gamma)ds + (\alpha+\bar{\alpha}) \int_0^v K_s^\sigma(\gamma)ds, \tag{8.63}
$$

where $\kappa_s^\sigma(\gamma)$ is the same as in (8.57) and

$$
K_s^\sigma(\gamma) = \int_{\Gamma_*^2} \Psi^2(\gamma') p_s^\sigma(\gamma, d\gamma').
$$

By (8.51) we have, cf. (8.58), (8.59),

$$
K_s^\sigma(\gamma) = \Psi^2(\gamma) + \int_0^s \left(\int_{\Gamma_*^2} (L^\sigma \Psi^2(\gamma')) p_u^\sigma(\gamma, d\gamma') \right) du. \tag{8.64}
$$

Similarly as in (8.41) it follows that

$$
\left| (L^\sigma \Psi^2)(\gamma') \right| \leq 3\bar{\alpha}\Psi^2(\gamma'),
$$

by which and the Grönwall inequality we get from (8.64) the following estimate

$$
K_s^\sigma(\gamma) \leq \Psi^2(\gamma) e^{3\bar{\alpha}s}. \tag{8.65}
$$

Now we use (8.59) and (8.65) in (8.63) and arrive at

$$
V_v^\sigma(\gamma) \leq v\left(2\alpha e^{\bar{\alpha}v}\Psi(\gamma) + (\alpha+\bar{\alpha})e^{3\bar{\alpha}v}\Psi^2(\gamma)\right) \tag{8.66}
$$

Now we may turn to (8.48) where we use the estimate, see (3.36),

$$
\int_{\Gamma_*^2} \Psi^n(\gamma)\mu_{t_1}(d\gamma) \leq T_n(\varkappa\langle\psi\rangle), \qquad n = 1, 2,
$$

and the fact that $\mu_{t_1} \in \mathcal{P}_{\rm exp}$ of type not exceeding $\exp(\vartheta_0 + \alpha t_1) \leq \exp(\vartheta_0 + \alpha T) =: \varkappa$, see Proposition 5.3 and Remark 5.4. Here e^{ϑ_0} is the type of μ. By (8.61) and (8.66) we then conclude that $\mathcal{W}^\sigma(t_1, t_2, t_3)$ satisfies (8.49) with

$$
C = 2\alpha(\alpha+\bar{\alpha})e^{2\bar{\alpha}T} T_1(\varkappa\langle\psi\rangle) + (\alpha+\bar{\alpha})^2 e^{4\bar{\alpha}T} T_2(\varkappa\langle\psi\rangle),
$$

which ends the proof. □

9 Completing the proof

Here the hardest part is the proof of item (i), whereas the validity of (iii) is rather standard, see cf. [9, Theorem 5.1.2, claim (iv), page 80].

9.1 Proving item (i)

First we prove existence by employing the fact that, for a given $\mu \in \mathcal{P}_{\rm exp}$ and $s \geq 0$, the measure in question, $P_{s,\mu}$, is obtained as an accumulation point of the family $\{P^\sigma_{s,\mu}\}_{\sigma\in(0,1]}$. Our aim now is to prove that such accumulation points have properties (a), (b), (c) mentioned in Definition 4.3.

To check (a), we note that $P_{s,\mu}\circ \varpi_s^{-1} = \mu$ since $P^\sigma_{s,\mu}\circ\varpi_s^{-1} = \mu$ for all $\sigma\in(0,1]$. Furthermore, by Lemmas 8.4 and 7.3 it follows that $P^\sigma_{s,\mu}\circ\varpi_t^{-1}\Rightarrow \mu_t$ as $\sigma\to 0$, which yields $P_{s,\mu}\circ\varpi_t^{-1} = \mu_t$, holding for all accumulation points in view of Lemma 6.3. These facts yield the validity of (b) of Definition 4.3.

To check (c), we take G as in (4.18) with fixed $t_2 > t_1 > s$, $m\in\mathbb{N}$ and $s_1 < s_2 < \cdots < s_m$, $s_1 \geq s$, $s_m \leq t_1$. Then we recall that $\mu^\sigma_{s_1} = \hat{\mu}^\sigma_{s_1} = S^\sigma(s_1 - s)\mu$, the type of which does not exceed $e^{\vartheta(s_1-s)}$, $\vartheta(t) = \vartheta_0 + \alpha t$, see Lemma 8.4, and set $\chi^\sigma_{s_1} = C^{-1}_{1,\sigma} F_1 \mu^\sigma_{s_1}$, that is,

$$\chi^\sigma_{s_1}(d\gamma) = C^{-1}_{1,\sigma}F_1(\gamma)\mu^\sigma_{s_1}(d\gamma), \qquad C_{1,\sigma} := \mu^\sigma_{s_1}(F_1). \tag{9.1}$$

Note that $C_{1,\sigma} > 0$ since each $F\in\widetilde{\mathcal{F}}$ is strictly positive, see (3.49) and (7.33). By claim (d) of Proposition 4.2 it follows that $\chi^\sigma_{s_1}\in\mathcal{P}_{\rm exp}$, and its type does not exceed that of μ_{s_1}, and hence $\exp(\vartheta(s_1 - s))$. Then we define recursively

$$\tilde{\chi}^\sigma_{s_l}(d\gamma) = (S^\sigma(s_l - s_{l-1})\chi^\sigma_{s_{l-1}})(d\gamma) = \int_{\Gamma^2_*} p^\sigma_{s_l - s_{l-1}}(\gamma', d\gamma)\chi^\sigma_{s_{l-1}}(d\gamma'), \tag{9.2}$$

$$\chi^\sigma_{s_l}(d\gamma) = C^{-1}_{l,\sigma}F_l(\gamma)\tilde{\chi}^\sigma_{s_l}(d\gamma), \qquad C_{l,\sigma} := \tilde{\chi}^\sigma_{s_l}(F_l), \qquad l\leq m.$$

As above, for all $l\leq m$, $\chi^\sigma_{s_l}$ is sub-Poissonian of type $\leq \exp(\vartheta(s_l - s))$. Now we take $F\in\mathcal{D}(L)$, see Definition 4.1, $t\in[s_m, t_2]$, set

$$\mathsf{F}_t = F\circ\varpi_t, \qquad \mathsf{K}_t = (LF)\circ\varpi_t, \qquad \mathsf{K}^\sigma_t = (L^\sigma F)\circ\varpi_t, \quad \sigma\in(0.1], \tag{9.3}$$

and then consider $P^\sigma_{s,\mu}(\mathsf{F}_t\mathsf{G})$ with G as just discussed. By (8.50) it follows that

$$P^\sigma_{s,\mu}(\mathsf{F}_t\mathsf{G}) = C_\sigma P^\sigma_{s,\chi^\sigma_{s_m}}(\mathsf{F}_t) = C_\sigma P^\sigma_{s,\chi^\sigma_{s_m}}(F\circ\varpi_t) =: C_\sigma\mu^{\sigma,s_m}_t(F), \tag{9.4}$$

with $C_\sigma = P^\sigma_{s,\mu}(\mathsf{G}) > 0$. By (8.50)

$$\mu^{\sigma,s_m}_t = S^\sigma(t - s_m)\chi^\sigma_{s_m}, \tag{9.5}$$

and the type of μ^{σ,s_m}_t is $\leq e^{\vartheta(t-s)}$. By (9.5) it follows that

$$\mu^{\sigma,s_m}_{t_2}(F) - \mu^{\sigma,s_m}_{t_1}(F) = \int_{t_1}^{t_2}\mu^{\sigma,s_m}_t(L^\sigma F)dt = \int_{t_1}^{t_2} P^\sigma_{s,\mu}(\mathsf{F}_t\mathsf{G})dt,$$

see (9.4), which yields $P^\sigma_{s,\mu}(\mathsf{H}) = 0$, holding for all $\sigma\in(0,1]$.

Now let $P_{s,\mu}$ be an accumulation point of the family $\{P^\sigma_{s,\mu}\}_{\sigma\in(0,1]}$. By Lemmas 7.3 and 8.4, all such accumulation points have the same one dimensional marginals coinciding with μ_t. For this $P_{s,\mu}$, let $\{\sigma_n\}_{n\in\mathbb{N}}\subset(0,1]$, $\sigma_n\to 0$, be such that $P^{\sigma_n}_{s,\mu}\Rightarrow P_{s,\mu}$ as $n\to+\infty$. Then set, cf. (9.4),

$$C_n = C_{\sigma_n} = P^{\sigma_n}_{s,\mu}(\mathsf{G}), \qquad C_\infty = P_{s,\mu}(\mathsf{G}), \tag{9.6}$$

$$\mu^{s_m}_t(\mathbb{A}) = C^{-1}_\infty P_{s,\mu}(\mathsf{G}\mathbb{1}_{\mathbb{A}}\circ\varpi_t), \qquad \mathbb{A}\in\mathcal{B}(\Gamma^2_*), \qquad t\in[s_m, t_2].$$

Let us show that the assumed convergence $P^{\sigma_n}_{s,\mu}\Rightarrow P_{s,\mu}$ implies $\mu^{\sigma_n,s_m}_t\Rightarrow\mu^{s_m}_t$, as $n\to+\infty$. To this end, by $\tilde{\chi}_{s_l}$ we denote $\mu^{s_{l-1}}_{s_l}$, where $\mu^{s_{l-1}}_t$, $t\geq s_{l-1}$ is the solution of (1.2) with the initial condition $\chi_{s_l} := C^{-1}_{l-1,\infty}F_{l-1}\tilde{\chi}_{s_{l-1}}$, $l = 2,\dots m$, where $C_{l,\infty} = \tilde{\chi}_{s_l}(F_l)$, cf.

(9.2), and $\tilde{\chi}_{s_1} = \mu_{s_1} = P_{s,\mu} \circ \varpi_{s_1}^{-1}$, which solves (1.2) on $[s, s_1]$ with the initial condition μ. The assumed convergence of the path measures implies $\tilde{\chi}_{s_1}^{\sigma_n} \Rightarrow \tilde{\chi}_{s_1}$, see (9.1). By Lemma 7.3 this yields $\tilde{\chi}_{s_2}^{\sigma_n} \Rightarrow \tilde{\chi}_{s_2}$, and thus $\tilde{\chi}_{s_l}^{\sigma_n} \Rightarrow \tilde{\chi}_{s_l}$ for all $l \leq m$. Since $\mu_t^{s_m}$ defined in (9.6) is the solution of (1.2) on $[s_m, t]$ with the initial condition $\chi_{s_m} := C_{m,\infty}^{-1} F_m \tilde{\chi}_{s_m}$, this yields the convergence in question. By Proposition 7.2 this yields in turn that $\mu_t^{s_m} \in \mathcal{P}_{\exp}$ and the type of $\mu_t^{s_m}$ is $\leq \exp(\vartheta(t-s))$. Note that C_∞ defined in (9.6) is $C_{m,\infty}$.

Keeping the aforementioned facts in mind we write, see (4.17),

$$P_{s,\mu}(\mathsf{H}) = P_{s,\mu}(\mathsf{F}_{t_2}\mathsf{G}) - P_{s,\mu}(\mathsf{F}_{t_1}\mathsf{G}) - \int_{t_1}^{t_2} P_{s,\mu}(\mathsf{K}_t\mathsf{G})dt, \tag{9.7}$$

and also set

$$\begin{aligned} a_n(t) &= P_{s,\mu}(\mathsf{F}_t\mathsf{G}) - P_{s,\mu}^{\sigma_n}(\mathsf{F}_t\mathsf{G}), \\ b_n(t) &= P_{s,\mu}(\mathsf{K}_t\mathsf{G}) - P_{s,\mu}^{\sigma_n}(\mathsf{K}_t\mathsf{G}), \\ c_n(t) &= P_{s,\mu}^{\sigma_n}\left((\mathsf{K}_t - \mathsf{K}_t^{\sigma_n})\mathsf{G}\right). \end{aligned} \tag{9.8}$$

Since $P_{s,\mu}^{\sigma}(\mathsf{H}) = 0$, by (9.7) and (9.8) it follows that

$$\begin{aligned} P_{s,\mu}(\mathsf{H}) = P_{s,\mu}(\mathsf{H}) - P_{s,\mu}^{\sigma_n}(\mathsf{H}) = [a_n(t_2) - a_n(t_1)] \\ - \int_{t_1}^{t_2} b_n(t)dt - \int_{t_1}^{t_2} c_n(t)dt =: a_n - b_n - c_n. \end{aligned}$$

By $P_{s,\mu}^{\sigma_n} \Rightarrow P_{s,\mu}$ we have $a_n \to 0$ as $n \to +\infty$. However, the same conclusion for b_n and c_n does not follow in so simple way as LF and $L^\sigma F$ need not be continuous. To settle this case, by means of (9.6) we write

$$b_n = C_n \int_{t_1}^{t_2} \left(\mu_t^{s_m}(LF) - \mu_t^{\sigma_n, s_m}(LF)\right) dt + (C_\infty - C_n) \int_{t_1}^{t_2} \mu_t^{s_m}(LF)dt. \tag{9.9}$$

By item (a) of Proposition 4.2, LF is a bounded function; hence, the second summand in (9.9) vanishes as $n \to +\infty$ since $C_n \to C_\infty$ by the assumed weak convergence, see (9.6). To prove the same for the first summand – denote it $b_n^{(1)}$ – we employ the fact that $\mu_t^{\sigma_n, s_m}$ and $\mu_t^{s_m}$ are sub-Poissonian and each $F \in \mathcal{D}(L)$ can be written as KG with $G \in \widetilde{\mathcal{G}}_\infty$, see Proposition 6.2. Then

$$\mu_t^{s_m}(LF) - \mu_t^{\sigma_n, s_m}(LF) = \langle\!\langle k_{\mu_t^{s_m}} - k_{\mu_t^{\sigma_n, s_m}}, \widehat{L}G \rangle\!\rangle \to 0, \qquad n \to +\infty, \tag{9.10}$$

where we have taken into account that $\widehat{L}G \in \widetilde{\mathcal{G}}_\infty$ whenever $G \in \widetilde{\mathcal{G}}_\infty$, see (5.26), and also the fact that $\mu_t^{\sigma_n, s_m} \Rightarrow \mu_t^{s_m}$ implies the convergence of the integrals in (9.10), see Proposition 7.2. As mentioned above, LF is a bounded function (by claim (i) of Proposition 4.2), which means that both terms of the left-hand side of (9.10) are bounded by $\sup_{\gamma \in \Gamma_*^2} |(LF)(\gamma)|$. Together with the convergence $C_n \to C_\infty$ this yields $b_n^{(1)} \to 0$ as $n \to +\infty$.

Now we turn to c_n. By (9.3) and (9.4), and then by (7.5), we have

$$\begin{aligned} c_n(t) = C_n \left[\mu_t^{\sigma_n, s_m}(LKG) - \mu_t^{\sigma_n, s_m}(L^{\sigma_n}KG)\right] \\ = C_n \langle\!\langle (L^\Delta - L^{\Delta,\sigma_n}) k_{\mu_t^{\sigma_n, s_m}}, G \rangle\!\rangle = C_n \langle\!\langle \widetilde{L}^{\Delta,\sigma_n} k_{\mu_t^{\sigma_n, s_m}}, G \rangle\!\rangle, \end{aligned}$$

cf. (7.21). Here we have also taken into account that $\mu_t^{\sigma_n, s_m} \in \mathcal{P}_{\exp}$, that was established above, and the operator $\widetilde{L}^{\Delta,\sigma_n}$ is the same as in (7.21). To make precise in which spaces

$\mathcal{K}_\vartheta$ it acts, we will take into account that $G \in \mathcal{G}_\infty = \cap_{\vartheta\in\mathbb{R}}\mathcal{G}_\vartheta$, see Proposition 6.2, and that the type of each $\mu_t^{\sigma_n,s_m}$ does not exceed $\exp(\vartheta(t_2-s)) =: e^{\vartheta}$. Then we write, cf. (7.21) and (7.22),

$$\langle\!\langle \widetilde{L}^{\Delta,\sigma_n} k_{\mu_t^{\sigma_n,s_m}}, G\rangle\!\rangle =: R_n(t) = \sum_{j=1}^{4} R_{n,j}(t),$$

where

$$\begin{aligned}
R_{n,1}(t) &= \int_{\Gamma_0}\int_{\Gamma_0}\left(\sum_{y\in\eta_0}\int_X a_0^{\sigma_n}(x,y)e(\tau_y^0;\eta_1)(\Upsilon_y^0 k_{\mu_t^{\sigma_n,s_m}})(\eta_0\setminus y\cup x,\eta_1)dx\right)\\
&\times\ G(\eta_0,\eta_1)\lambda(d\eta_0)\lambda(d\eta_1)\\
&= \int_{\Gamma_0}\int_{\Gamma_0}\left(\int_X\int_X a_0^{\sigma_n}(x,y)e(\tau_y^0;\eta_1)(\Upsilon_y^0 k_{\mu_t^{\sigma_n,s_m}})(\eta_0\cup x,\eta_1)G(\eta_0\cup y,\eta_1)dxdy\right)\\
&\times\ \lambda(d\eta_0)\lambda(d\eta_1),
\end{aligned}$$

and likewise

$$\begin{aligned}
R_{n,2}(t) &= -\int_{\Gamma_0}\int_{\Gamma_0}\left(\int_X\int_X a_0^{\sigma_n}(x,y)e(\tau_y^0;\eta_1)(\Upsilon_y^0 k_{\mu_t^{\sigma_n,s_m}})(\eta_0\cup x,\eta_1)G(\eta_0\cup x,\eta_1)dxdy\right)\\
&\times\ \lambda(d\eta_0)\lambda(d\eta_1),\\
R_{n,3}(t) &= \int_{\Gamma_0}\int_{\Gamma_0}\left(\int_X\int_X a_1^{\sigma_n}(x,y)e(\tau_y^1;\eta_0)(\Upsilon_y^1 k_{\mu_t^{\sigma_n,s_m}})(\eta_0,\eta_1\cup x)G(\eta_0,\eta_1\cup y)dxdy\right)\\
&\times\ \lambda(d\eta_0)\lambda(d\eta_1),\\
R_{n,4}(t) &= -\int_{\Gamma_0}\int_{\Gamma_0}\left(\int_X\int_X a_1^{\sigma_n}(x,y)e(\tau_y^1;\eta_0)(\Upsilon_y^1 k_{\mu_t^{\sigma_n,s_m}})(\eta_0,\eta_1\cup x)G(\eta_0,\eta_1\cup x)dxdy\right)\\
&\times\ \lambda(d\eta_0)\lambda(d\eta_1).
\end{aligned}$$

Now we take into account that $k_{\mu_t^{\sigma_n,s_m}} \in \mathcal{K}_\vartheta$ and $G\in\mathcal{G}_\infty$, see above, employ (7.24), and then get, cf. (7.25),

$$|R_{n,j}(t)| \le \int_X r_j^{\sigma_n}(y) g_i(y)dy, \qquad j=1,\dots,4. \tag{9.11}$$

with $r_j^{\sigma_n}(y)$ given in (7.26) and

$$g_1(y) = g_2(y) = c(\vartheta)\int_{\Gamma_0}\int_{\Gamma_0}|G(\eta_0\cup y,\eta_1)|\exp\left(\vartheta|\eta_0|+\vartheta|\eta_1|\right)\lambda(d\eta_0)\lambda(d\eta_1), \tag{9.12}$$

$$g_3(y) = g_4(y) = c(\vartheta)\int_{\Gamma_0}\int_{\Gamma_0}|G(\eta_0,\eta_1\cup y)|\exp\left(\vartheta|\eta_0|+\vartheta|\eta_1|\right)\lambda(d\eta_0)\lambda(d\eta_1),$$

where $c(\vartheta)$ is the same as in (7.28). Note that the bound in (9.11) is uniform in $t\in[s_m,t_2]$, for which $k_{\mu_t^{\sigma_n,s_m}}\in\mathcal{K}_\vartheta$. Now, similarly as in (7.30), we get

$$\begin{aligned}
\int_X g_1(y)dy &\le c(\vartheta)e^{-\vartheta}\int_{\Gamma_0}\int_{\Gamma_0}|\eta_0|\,|G(\eta_0,\eta_1)|\exp\left(\vartheta|\eta_0|+\vartheta|\eta_1|\right)\lambda(d\eta_0)\lambda(d\eta_1)\\
&\le c(\vartheta)e^{-\vartheta-1-\log\varepsilon}\int_{\Gamma_0}\int_{\Gamma_0}|G(\eta_0,\eta_1)|\exp\Big((\vartheta+\varepsilon)\left(|\eta_0|+|\eta_1|\right)\Big)\lambda(d\eta_0)\lambda(d\eta_1)
\end{aligned}$$

$$= c(\vartheta)e^{-\vartheta-1-\log\varepsilon}|G|_{\vartheta+\varepsilon} < \infty$$

that holds for all $\varepsilon > 0$ as $G \in \mathcal{G}_\infty$. Similar estimates can be obtained for the remaining $|R_{n,j}(t)|$. By the dominated convergence theorem we then get that $R_n(t) \to 0$ as $n \to +\infty$, uniformly in $t \in [s_m, t_2]$, which together with the aforementioned convergence $C_n \to C_\infty$, yields $c_n \to 0$ as $n \to +\infty$. Therefore, for each limiting point $P_{s,\mu}$, it follows that $P_{s,\mu}(\mathsf{H}) = 0$, that yields the proof of item (c), and thus the existence in question.

Now we turn to uniqueness. To this end we employ the following fact.

Proposition 9.1. *Assume that two solutions* $\{P^{(j)}_{s,\mu} : s \geq 0,\ \mu \in \mathcal{P}_{\rm exp},\ j = 1, 2\}$*, see Definition 4.3, satisfy* $P^{(1)}_{s,\mu} \circ \varpi_t^{-1} = P^{(2)}_{s,\mu} \circ \varpi_t^{-1}$*, holding for all* $t \geq s$*,* $s \geq 0$*, and* $\mu \in \mathcal{P}_{\rm exp}$*. Then they coincide, i.e.,* $P^{(1)}_{s,\mu} = P^{(2)}_{s,\mu}$ *for all* $s \geq 0$ *and* $\mu \in \mathcal{P}_{\rm exp}$*.*

The proof of this statement – based on Lemma 6.3 – is completely analogous to that of [16, Lemma 5.4], and thus can be omitted here. Then the uniqueness in question is straightforward. This completes the proof of item (i) of the theorem.

9.2 Proving item (ii)

We begin by recalling that $X = \mathbb{R}^d$. Let $\{r_j\}_{j\in\mathbb{N}} \subset \mathbb{R}_+$ be a strictly increasing sequence such that $\lim_{j\to+\infty} r_j = +\infty$. Set $\Delta_j = \{x \in X : |x| < r_j\}$ and $\gamma_{i,j} = \gamma_i \cap \Delta_j$, $i = 0, 1$, $j \in \mathbb{N}$, $\gamma = (\gamma_0, \gamma_1) \in \Gamma^2_*$. We also will use the notation γ_j for $(\gamma_{0,j}, \gamma_{1,j})$. Then we define

$$\Gamma^2_{*,j} = \{\gamma \in \Gamma^2_* : \gamma_j \in \breve{\Gamma}^2\}.$$

That is, $\gamma \in \Gamma^2_*$ belongs to $\Gamma^2_{*,j}$ if and only if $n_0(x)+n_1(x) = 1$, holding for all $x \in p(\gamma)\cap\Delta_j$, see (1.1). By (3.31) we then have

$$\breve{\Gamma}^2_* = \bigcap_{j\in\mathbb{N}} \Gamma^2_{*,j}. \tag{9.13}$$

Similarly as in [16, Lemma 2.7], one proves that each $\Gamma^2_{*,j}$ is an open subset of Γ^2_*, see also Lemma 3.4. Define, cf. (3.24),

$$\begin{aligned} h_N(x,y) &= \psi(x)\psi(y)\min\{N; |x-y|^{-d\epsilon}\}, \qquad N \in \mathbb{N}, \\ H_N(\gamma) &= \sum_{x\in\gamma_0}\sum_{y\in\gamma_0\setminus x} h_N(x,y) + \sum_{x\in\gamma_1}\sum_{y\in\gamma_1\setminus x} h_N(x,y) + \sum_{x\in\gamma_0}\sum_{y\in\gamma_1} h_N(x,y), \end{aligned}$$

where $\epsilon \in (0,1)$ and $\psi(x)$ is as in (3.29). Now for $\mu \in \mathcal{P}_{\rm exp}$ of type $\varkappa$, similarly as in (3.25) we get

$$\mu(H_N) \leq 3\varkappa^2 \mathcal{I}_N, \tag{9.14}$$

where, cf. (3.26),

$$\begin{aligned} \mathcal{I}_N &= \int_{X^2} h_N(x,y)dxdy \leq \int_X \psi(x)\left(\int_X \psi(y)|x-y|^{-d\epsilon}dy\right)dx \\ &\leq \int_X \psi(x)\left(\int_{B_r}\frac{dz}{|z|^{d\epsilon}} + \frac{\langle\psi\rangle}{r^{d\epsilon}}\right)dx = \frac{c_d r^{d(1-\epsilon)}}{d(1-\epsilon)}\langle\psi\rangle + \frac{\langle\psi\rangle^2}{r^{d\epsilon}}. \end{aligned}$$

Then, similarly as in (3.27), we have that

$$H(\gamma) := \lim_{N\to+\infty} H_N(\gamma) < \infty$$

for μ-almost all $\gamma \in \breve{\Gamma}^2_*$. At the same time,

$$H_j(\gamma) := H(\gamma_j) \leq r_{\gamma_j}^{-d\epsilon}\left(|\gamma_0\cap\Delta_j| + |\gamma_1\cap\Delta_j|\right)^2,$$

holding for all $\gamma \in \Gamma^2_{*,j}$. Here r_{γ_j} is the minimal distance between two distinct $x, y \in (\gamma_0 \cap \Delta_j) \cup (\gamma_0 \cap \Delta_j)$, which is positive since the number of pairs of such points is finite and γ_j is simple.

Let $\{P_{s,\mu} : s \geq 0,\ \mu \in \mathcal{P}_{\rm exp}\}$ be the solution which exists and is unique according to item (i). Fix some $s \geq 0$ and $\mu \in \mathcal{P}_{\rm exp}$ of type $\varkappa$, and let $\mathcal{X}$ be the process corresponding to $P_{s,\mu}$. For $N \in \mathbb{N}$, we define the stopping time, cf. [11, page 180],

$$T_{N,j} = \inf\{t \geq s : H_j(\mathcal{X}(t)) > N\}.$$

Then for a fixed $j \in \mathbb{N}$ and $T_{N,j} \wedge t := \min\{t; T_{N,j}\}$, we set $\mathcal{Z}(t) = \lim_{N\to+\infty} \mathcal{X}(T_{N,j} \wedge t)$ and $T_j = \lim_{N\to+\infty} T_{N,j}$. Both limits exist as $T_{N,j} \leq T_{N+1,j}$. Let $\tilde{\mu}_t$ be the law of $\mathcal{Z}(t)$. For $\varPhi^m_\tau(\theta|\cdot) \in \mathcal{D}(L)$ defined in (6.24),

$$\varPhi^m_\tau(\theta|\mathcal{X}(t)) - \int_s^t L\varPhi^m_\tau(\theta|\mathcal{X}(u))du \tag{9.15}$$

is a right-continuous martingale. Then, similarly as in [11, page 180], by the optional sampling theorem, we can write

$$\mathbb{E}\left[\varPhi^m_\tau(\theta|\mathcal{X}(T_{N,j} \wedge t))\right] = \mathbb{E}\left[\varPhi^m_\tau(\theta|\mathcal{X}(s))\right] + \mathbb{E}\left[\int_0^{T_{N,j}\wedge t} L\varPhi^m_\tau(\theta|\mathcal{X}(u))du\right],$$

where we pass to the limit $N \to +\infty$ and get, see also (6.25),

$$\begin{aligned} \tilde{\mu}_t(\varPhi^m_\tau(\theta|\cdot)) &= \mu(\varPhi^m_\tau(\theta|\cdot)) + \lim_{N\to+\infty} \mathbb{E}\left[\int_0^{T_{N,j}\wedge t} L\varPhi^m_\tau(\theta|\mathcal{X}(u))du\right] \\ &\leq \mu(\varPhi^m_\tau(\theta|\cdot)) + \lim_{N\to+\infty} \mathbb{E}\left[\int_0^{T_{N,j}\wedge t} \left|L\varPhi^m_\tau(\theta|\mathcal{X}(u))\right| du\right] \\ &\leq \mu(\varPhi^m_\tau(\theta|\cdot)) + \mathbb{E}\left[\int_0^t \left|L\varPhi^m_\tau(\theta|\mathcal{X}(u))\right| du\right] \\ &\leq \mu(\varPhi^m_\tau(\theta|\cdot)) + \mathbb{E}\left[\int_0^t \varPhi^m_{\tau,1}(\theta|\mathcal{X}(u))du\right] \\ &= \mu(\varPhi^m_\tau(\theta|\cdot)) + \int_s^t \mu_u(\varPhi^m_{\tau,1}(\theta|\cdot))du. \end{aligned} \tag{9.16}$$

Here $\varPhi^m_{\tau,1}(\theta|\cdot) \in \mathcal{D}(L)$ is as in (6.25) and (6.34), and $\mu_u = P_{s,\mu} \circ \varpi_u^{-1}$ is the law of $\mathcal{X}(u)$. Similarly as in (6.42), by (9.16) we then obtain

$$\tilde{\mu}_t(\varPhi^m_\tau(\theta|\cdot)) \leq \sum_{q=0}^{\infty} \frac{(t-s)^q}{q!} \mu\left(\varPhi^m_{\tau,q}(\theta|\cdot)\right), \qquad t - s < \log(1+\varepsilon)/c_a.$$

Now we proceed here as in obtaining (6.48), which finally yields, see (6.50), (6.51,

$$\tilde{\mu}_t(\varPhi^m(\theta|\cdot)) := \lim_{\max\{\tau_0,\tau_1\}\to 0} \tilde{\mu}_t(\varPhi^m_\tau(\theta|\cdot)) \leq \varkappa_t^{|m|} \|\theta_0\|^{m_0}_{L^1(X)} \|\theta_1\|^{m_1}_{L^1(X)}, \tag{9.17}$$

where $t - s < \log(1+\varepsilon)/c_a$ and $\varkappa_t = \varkappa e^{(\alpha+1)(t-s)}$. By Definition 3.1 (9.17) yields $\tilde{\mu}_t \in \mathcal{P}_{\rm exp}$ and hence $\mathcal{Z}(t) \in \breve{\Gamma}^2_*$ (almost surely) for this t, see (3.34). Thus, $T_j > t$. Now we take $\delta \in (0,1)$ and then $s_1 = s + \delta\log(1+\varepsilon)/c_a$, take into account that $\mu_{s_1} = P_{s,\mu} \circ \varpi_{s_1}^{-1} \in \mathcal{P}_{\rm exp}$, and repeat the above procedure with s replaced by s_1. Since the type of μ_t is $\varkappa e^{\alpha(t-s)}$ – and hence is finite for all t – the construction can be repeated ad infinitum to cover the whole $[s, +\infty)$. This implies that the paths of $\mathcal{X}$ remain in $\mathfrak{D}_{[0,+\infty)}(\Gamma^2_{*,j})$ for all $j \in \mathbb{N}$, which by (9.13) yields the proof of item (ii).

References

[1] Barańska, J. and Kozitsky, Yu.: A Widom-Rowlinson jump dynamics in the continuum, *J. Dynam. Differential Equations* **30**, (2018), 637–665. MR3815580

[2] Bogachev, V. I., Krylov, N. V., Röckner, M. and Shaposhnikov, S. V.: Fokker-Planck-Kolmogorov equations. Mathematical Surveys and Monographs, 207. American Mathematical Society, Providence, RI, 2015. MR3443169

[3] Boyadzhiev, K. N.: Exponetial pnolynomials, Stirling numbers, and evaluation of some Gamma integrals, *Abstr. Appl. Analysis* Volume 2009, Article ID 168672, 18 pp. MR2545183

[4] de Branges, L.: The Stone-Weierstrass theorem, *Proc. Amer. Math. Soc.* **10**, (1959), 822–824. MR0113131

[5] Brezis, H.: Functional analysis, Sobolev spaces and partial differential equations. Universitext. Springer, New York, 2011. MR2759829

[6] Chayes, J. T., Chayes, L. and Kotecký, R.: The analysis of the Widim-Rowlinson model by stochastic geometric method, *Comm. Math. Phys.* **172**, (1995), 551–569. MR1354260

[7] Chentsov, N. N.: Weak convergence of stochastic processes whose trajectories have no discontinuities of the second kind and the “heuristic” approach to the Kolmogorov-Smirnov test, *Theory of Probability & Its Applications* **1**, (1956), 140–144.

[8] Cohn, D. L.: Measure theory. Second edition. Birkhäuser Advanced Texts: Basler Lehrbücher. Birkhäuser/Springer, New York, 2013. MR3098996

[9] Dawson, D. A.: Measure-valued Markov processes. École d’Été de Probabilités de Saint-Flour XXI–1991, 1–260, Lecture Notes in Math., 1541, Springer, Berlin, 1993. MR1242575

[10] Dudley, R. M.: Converfgence of Baire measures, *Studia Math.* **27**, (1966), 251–268. MR0200710

[11] Ethier, S. N. and Kurtz, T. G.:; Markov processes: characterization and convergence, Wiley, New York, 1986. MR0838085

[12] Garcia, N. L, and Kurtz, T. G.: Spatial birth and death processes as solutions of stochastic equations, *ALEA Lat. Am. J. Probab. Math. Stat.* **1**, (2006), 281–303. MR2249658

[13] Georgii, H.-O. and Häggström, O.: Phase transition in continuum Potts models, *Comm. Math. Phys.* **181**, (1996), 507–528. MR1414841

[14] Kissel, S. and Külske, Ch.: Dynamical Gibbs–non-Gibbs transitions in Curie-Weiss Widom-Rowlinson models, *Markov Process. Related Fields* **25**, (2019), 379–413. MR3966346

[15] Kozitsky, Yu. and Kozlovskii, M.: A phase transition in a Widom-Rowlinson model with Curie-Weiss interaction, *J. Stat. Mech. Theory Exp.* 2018, no. 7, 073202, 28 pp. MR3845477

[16] Kozitsky, Yu. and Röckner, M.: A Markov process for an infinite interacting particle system in the continuum, *Electron. J. Probab.* **26**, (2021), article no. 72, 1–53. MR4269202

[17] Lebowitz, J. L.: Statistical mechanics: A selective review of two central issues, *Rev. Mod. Phys.* **71**, (1999), S346.

[18] Lenard, A.: Correletion functions and the uniqueness of the state in classical statistical mechanics, *Comm. Math. Phys.* **30**, (1973), 35–44. MR0323270

[19] Mazel, A., Suhov, Yu. and Stuhl, I.: A classical WR model with q particle types, *J. Stat. Phys.* **159**, (2015), 1040–1086. MR3345410

[20] Ruelle, D.: Superstable interactions in classical statistical mechanics, *Comm. Math. Phys.* **18**, (1970), 127–159. MR0266565

[21] Ruelle, D.: Existence of a phase transition in a continuous classical system, *Phys. Rev. Lett.* **27**, (1971), 1040–1041.

[22] Parthasarathy, K. R.: Probability measures on metric spaces, Probability and Mathematical Statistics, No. 3 Academic Press, Inc., New York-London 1967. MR0226684

[23] Thieme, H. R. and Voigt, J.: Stochastic semigroups: their construction by perturbation and approximation, in M. R. Weber and J. Voigt (Eds.), *Positivity IV – Theory and Applications,* Tech. Univ. Dresden, Dresden, 2006, pp. 135–146. MR2239544

[24] Widom, B. and Rowlinson, J. S.: The new model for the study of liquid-vapor phase transitions, *J. Chem. Phys.* **52**, (1970), 1670–1684.

Acknowledgments. The author are grateful to both referees for careful reading of the manuscript and making valuable remarks.