ASYMPTOTIC BEHAVIOR OF THE HEAT SEMIGROUP ON CERTAIN RIEMANNIAN MANIFOLDS

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Abstract. We show that, on a complete, connected and non-compact Riemannian manifold of non-negative Ricci curvature, the solution to the heat equation with L^1 initial data behaves asymptotically as the mass times the heat kernel. In contrast to the previously known results in negatively curved contexts, the radial assumption is not required. Moreover, we provide a counterexample such that this asymptotic phenomenon fails in sup norm on manifolds with two Euclidean ends.

1. Introduction

Let $\mathcal{M}$ be a Riemannian manifold of dimension $n \geq 2$ and Δ be the Laplace-Beltrami operator on $\mathcal{M}$. It is well understood that the long-time behavior of solutions to the heat equation

$$\partial_t u(t,x) = \Delta_x u(t,x), \qquad u(0,x) = u_0(x), \qquad t>0,\ x \in \mathcal{M} \tag{1.1}$$

is very much related to the global geometry of $\mathcal{M}$. This applies also to the heat kernel $h_t(x,y)$ that is the minimal positive fundamental solution of the heat equation or, equivalently, the integral kernel of the heat semigroup $\exp(t\Delta)$ (see for instance [6, 15, 11]).

If the initial function u_0 belong to the space $L^p(M,\mu)$ with $p \in [1,\infty)$ (where μ is the Riemannian measure on $\mathcal{M}$) then the Cauchy problem (1.1) has a unique solution u such that $u(t,\cdot) \in L^p$ for any $t>0$, and this solution is given by

$$u(t,x) = \int_M h_t(x,y)\, u_0(y)\, \mathrm{d}\mu(y). \tag{1.2}$$

The same is true for the case $p=\infty$ provided $\mathcal{M}$ is stochastically complete. Hence, by a solution of (1.1) we always mean the function (1.2).

The aim of this paper is to investigate the connection between the long-time behavior of the solution $u(t,x)$ of (1.1) and that of the heat kernel $h_t(x,y)$. Let the initial function u_0 belong to $L^1(\mathcal{M})$, and denote by $M = \int_{\mathcal{M}} \mathrm{d}x\, u_0(x)$ its mass. In the case when $\mathcal{M} = \mathbb{R}^n$ with the Euclidean metric, the heat kernel is given by

$$h_t(x,y) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4t}}$$

and the solution to (1.1) satisfies as $t \to \infty$

$$\|u(t,\,.\,) - M h_t(\,.\,,x_0)\|_{L^1(\mathbb{R}^n)} \longrightarrow 0 \tag{1.3}$$

and

$$t^{\frac{n}{2}}\, \|u(t,\,.\,) - M h_t(\,.\,,x_0)\|_{L^\infty(\mathbb{R}^n)} \longrightarrow 0 \tag{1.4}$$

By interpolation, a similar convergence holds with respect to any L^p norm when $1<p<\infty$:

$$t^{\frac{n}{2p'}}\, \|u(t,\,.\,) - M h_t(\,.\,,0)\|_{L^p(\mathbb{R}^n)} \longrightarrow 0$$

where p' is the Hölder conjugate of p.

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Note that (1.3) and (1.4) hold for *any* choice of x_0, which means that in long run the solution $u(t,x)$ and the heat kernel $h_t(x,x_0)$ "forget" about the initial function u_0 resp. initial point x_0. We refer to a recent survey [18] for more details about this property in the Euclidean setting.

The convergence properties (1.3) and (1.4) have an interesting probabilistic meaning. Let $\{X_t\}$ be Brownian motion on $\mathcal{M}$ whose transition density is $h_t(x,y)$. Then (1.4) means, in particular, that X_t eventually "forgets" about its starting point x_0, which corresponds to the fact that X_t escapes to ∞ rotating chaotically in angular direction.

The situation is drastically different in hyperbolic spaces. It was shown by Vázquez [19] that (1.3) fails for a general initial function $u_0 \in L^1(\mathbb{H}^n)$ but is still true if u_0 is spherically symmetric around x_0. Similar results were obtained in [2] in a more general setting of symmetric spaces of non-compact type by using tools of harmonic analysis. Note that these spaces have nonpositive sectional curvature. Recall that in hyperbolic spaces Brownian motion X_t tends to escape to ∞ along geodesics, which means that it "remembers" at least the direction of the starting point x_0.

Our main result is the following theorem that deals with manifolds of non-negative Ricci curvature. Denote by $B(x,r)$ the geodesic ball on $\mathcal{M}$ of radius r centered at $x \in \mathcal{M}$ and set $V(x,r) = \mu(B(x,r))$.

Theorem 1.1. *Let $\mathcal{M}$ be a complete, connected and non-compact Riemannian manifold of non-negative Ricci curvature. Fix a base point $x_0 \in \mathcal{M}$ and suppose that $u_0 \in L^1(\mathcal{M})$. Then the solution to the heat equation (1.1) satisfies as $t \to \infty$*

$$\|u(t,\,.\,) - Mh_t(\,.\,,x_0)\|_{L^1(\mathcal{M})} \to 0 \tag{1.5}$$

and

$$\|\,|u(t,\,.\,) - Mh_t(\,.\,,x_0)|\,V(\cdot,\sqrt{t})\|_{L^\infty(\mathcal{M})} \to 0. \tag{1.6}$$

Remark 1.2. By interpolation between (1.5) and (1.6), we obtain for any $p \in (1,\infty)$

$$\|\,|u(t,\,.\,) - Mh_t(\,.\,,x_0)|\,V(\cdot,\sqrt{t})^{1/p'}\|_{L^p(\mathcal{M})} \longrightarrow 0. \tag{1.7}$$

In Section 2 we will give a short review of the estimates of the heat kernel and its derivatives that are the main ingredients in our proof. In Section 3 we prove Theorem 1.1. An essential idea of the proof is to describe the critical region where the heat kernel concentrates.

2. Heat kernel estimates

From now on, $\mathcal{M}$ denotes a complete, connected, non-compact Riemannian manifold of dimension $n \geq 2$. Let μ be the Riemannian measure on $\mathcal{M}$. Let $d(x,y)$ be the geodesic distance between two points $x,y \in \mathcal{M}$, and $V(x,r) = \mu(B(x,r))$ be the Riemannian volume of the geodesic ball $B(x,r)$ of radius r centered at $x \in \mathcal{M}$.

Throughout the paper we follow the convention that $C, C_1, ...$ denote large positive constants whereas $c, c_1, ...$ are small positive constants. These constants may depend on $\mathcal{M}$ but do not depend on the variables x,y,t. Moreover, the notation $A \lesssim B$ between two positive expressions means that $A \leq CB$, and $A \asymp B$ means $cB \leq A \leq CB$.

We say that $\mathcal{M}$ satisfies the volume doubling property if, for all $x \in \mathcal{M}$ and $r > 0$, we have

$$V(x,2r) \leq C\,V(x,r). \tag{2.1}$$

It follows from (2.1) that there exist some positive constants $\nu, \nu' > 0$ such that

$$c\left(\frac{R}{r}\right)^{\nu'} \leq \frac{V(x,R)}{V(x,r)} \leq C\left(\frac{R}{r}\right)^{\nu} \tag{2.2}$$

for all $x \in \mathcal{M}$ and $0 < r \leq R$ (see for instance [11, Section 15.6]). Moreover, (2.2) implies that, for all $x,y \in \mathcal{M}$ and $r > 0$,

$$\frac{V(x,r)}{V(y,r)} \leq C\left(1 + \frac{d(x,y)}{r}\right)^{\nu}. \tag{2.3}$$

The integral kernel $h_t(x,y)$ of the heat semigroup $\exp(t\Delta)$ is the smallest positive fundamental solution to the heat equation (1.1). It is known that $h_t(x,y)$ is positive and smooth in (t,x,y), symmetric in x,y, and satisfied the semigroup identity (see for instance [16], [11]). Besides, for all $y\in M$ and $t>0$

$$\int_{\mathcal{M}} h_t(x,y)\,\mathrm{d}\mu(x)\le 1.$$

The manifold $\mathcal{M}$ is called stochastically complete if for all $y\in M$ and $t>0$

$$\int_{\mathcal{M}} h_t(x,y)\mathrm{d}\mu(x)=1.$$

It is known that if $\mathcal{M}$ is geodesically complete and, for some $x_0\in\mathcal{M}$ and all large enough r,

$$V(x_0,r)\le e^{Cr^2}$$

then $\mathcal{M}$ is stochastically complete. In particular, the volume doubling property (2.1) implies that $\mathcal{M}$ is stochastically complete.

Assume now that the Ricci curvature of $\mathcal{M}$ is non-negative. Then the following two-sided estimates of the heat kernel were proved by Li and Yau [13]:

$$\frac{c_1}{V(x,\sqrt{t})}\exp\Big(-C_1\frac{d^2(x,y)}{t}\Big)\le h_t(x,y)\le\frac{C_2}{V(x,\sqrt{t})}\exp\Big(-c_2\frac{d^2(x,y)}{t}\Big). \tag{2.4}$$

Besides, Li and Yau proved in [13] also the following gradient estimate for any positive solution $u(t,x)$ of the heat equation $\partial_t u=\Delta u$ on $\mathbb{R}_+\times\mathcal{M}$:

$$\frac{|\nabla u|^2}{u^2}-\frac{\partial_t u}{u}\le\frac{C}{t} \tag{2.5}$$

with $C=\frac{n}{2}$. By a result of [10], the upper bound of the heat kernel in (2.4), that is,

$$h_t(x,y)\le\frac{C}{V(x,\sqrt{t})}\exp\Big(-c\,\frac{d^2(x,y)}{t}\Big), \tag{2.6}$$

implies the following estimate of the time derivative:

$$\Big|\frac{\partial h_t}{\partial t}(x,y)\Big|\le\frac{C}{t\,V(x,\sqrt{t})}\exp\Big(-c\frac{d^2(x,y)}{t}\Big). \tag{2.7}$$

It follows from (2.5) that

$$|\nabla u|^2\le u\partial_t u+\frac{C}{t}u^2$$

and, hence,

$$|\nabla u|\le\sqrt{u\,|\partial_t u|}+\frac{C}{\sqrt{t}}u.$$

Applying this for the function $u(t,y)=h_t(x,y)$ and combining with (2.6) and (2.7), we obtain that

$$|\nabla_y h_t(x,y)|\le\frac{C}{\sqrt{t}\,V(x,\sqrt{t})}\exp\Big(-c\,\frac{d^2(x,y)}{t}\Big). \tag{2.8}$$

This estimate of ∇h_t plays essential role in our work. It is known that the upper bound (2.6) of h_t (and, hence, its consequence (2.7)) hold on a larger class of Riemannian manifolds satisfying a so called *relative Faber-Krahn inequality* (see, for example, [11]). However, the estimate (2.8) of the gradient ∇h_t is much more subtle and requires more serious hypothesis, for example, non-negative Ricci curvature as we assume here.

Note also that the lower bound in (2.4) will not be used in this work. Equivalent conditions for the two-sided estimate (2.4) of the heat kernel are also known: these are the volume doubling (2.1) and a certain *Poincaré inequality* (see [9, 14, 15]).

Finally, let us observe that on spaces of essentially negative curvature the above estimates of the heat kernel typically fail: for example, these are hyperbolic spaces [7], non-compact symmetric spaces [1], asymptotically hyperbolic manifolds [5], fractal like manifolds [3].

Now let us reformulate Theorem 1.1 in a bit more general way.

Theorem 2.1. *Let $\mathcal{M}$ be a geodesically complete non-compact manifold that satisfies the following conditions:*

- *the volume doubling condition (2.1);*
- *the upper bound (2.6) of the heat kernel;*
- *the upper bound (2.8) of the gradient of the heat kernel.*

Then the conclusions of Theorem 1.1 are satisfied.

Remark 2.2. Apart from the manifolds with nonnegative Ricci curvature, the above-described manifolds cover the co-compact covering manifolds whose deck transformation has polynomial growth (see [8]), as well as the Lie groups with polynomial growth if one considers the heat kernel associated with a suitable sub-Laplacian (see [17, Ch.VIII, §2]). In fact, we do not need any differential operator like Laplacian (or sub-Laplacian). All we need is a function $h_t(x,y)$ satisfying certain estimates, and the result can be formulated as a property of integral operators.

We prove Theorem 2.1 in the next section. In the next two lemmas we describe some consequences of the hypotheses of Theorem 2.1, in particular, the critical annulus where the heat kernel concentrates.

Lemma 2.3. *Under the hypothesis (2.1) we have, for any $x_0 \in M$ and $t>0$,*

$$\int_M \frac{\mathrm{d}\mu(x)}{V(x_0,\sqrt{t})}\,\exp\left(-\,c\,\frac{d\,(x,x_0)^2}{t}\right) \lesssim 1 \tag{2.9}$$

and, for any $N\in\mathbb{N}$ and all $r\geq\sqrt{t}$,

$$\int_{B(x_0,r)^c} \frac{\mathrm{d}\mu(x)}{V(x_0,\sqrt{t})}\,\exp\left(-\,c\,\frac{d\,(x,x_0)^2}{t}\right) \lesssim \left(\frac{r}{\sqrt{t}}\right)^{-N}. \tag{2.10}$$

Proof. Let us prove first (2.10). Using and (2.6), we obtain

$$\begin{aligned}\int_{d(x,x_0)\geq r} h_t(x,x_0)\,\mathrm{d}\mu(x) &= \sum_{j=0}^{\infty}\int_{2^j r\leq d(x,x_0)<2^{j+1}r} h_t(x,x_0)\,\mathrm{d}\mu(x)\\ &\lesssim \sum_{j=0}^{\infty}\frac{V(x_0,2^{j+1}r)}{V(x_0,\sqrt{t})}\,\exp\left(-\,c\,\frac{(2^j r)^2}{t}\right)\\ &\lesssim \sum_{j=0}^{\infty}\left(\frac{2^{j+1}r}{\sqrt{t}}\right)^{\nu}\exp\left(-\,c\,\frac{2^{2j}r^2}{t}\right).\end{aligned}$$

Since $\exp(-s)\leq(N!)\,s^{-N}$ for any $s>0$ and $N\in\mathbb{N}$, we obtain that

$$\int_{d(x,x_0)\geq r} h_t(x,x_0)\,\mathrm{d}\mu(x) \lesssim \sum_{j=0}^{\infty}\left(\frac{r}{\sqrt{t}}\right)^{\nu}2^{\nu j-2Nj}\left(\frac{t}{r^2}\right)^{N} \lesssim \left(\frac{r}{\sqrt{t}}\right)^{\nu-2N},$$

which proves (2.10).

In order to prove (2.9) we apply (2.10) with $r=\sqrt{t}$ and $N=1$ and obtain

$$\begin{aligned}\int_M \frac{\mathrm{d}\mu(x)}{V(x_0,\sqrt{t})}\,\exp\left(-\,c\,\frac{d\,(x,x_0)^2}{t}\right) &= \left(\int_{B(x_0,r)^c}+\int_{B(x_0,r)}\right)\frac{\mathrm{d}\mu(x)}{V(x_0,\sqrt{t})}\,\exp\left(-\,c\,\frac{d\,(x,x_0)^2}{t}\right)\\ &\lesssim 1+\int_{B(x_0,\sqrt{t})}\frac{\mathrm{d}\mu(x)}{V(x_0,\sqrt{t})}=2.\end{aligned}$$

□

The next lemma describes the annulus where the heat kernel concentrates. Let us fix a point $x_0 \in \mathcal{M}$ and a positive function $\varphi(t)$ such that $\varphi(t) \to 0$ as $t \to \infty$. For any $t > 0$, define the following annulus in $\mathcal{M}$:

$$\Omega_t = \left\{ x \in \mathcal{M} \;\middle|\; \varphi(t)\sqrt{t} \le d(x, x_0) \le \frac{\sqrt{t}}{\varphi(t)} \right\}. \tag{2.11}$$

Lemma 2.4. *Under the hypotheses (2.1) and (2.6), we have for all large enough t*

$$\int_{\mathcal{M}\setminus\Omega_t} h_t(x, x_0)\,\mathrm{d}\mu(x) \lesssim \varphi(t)^{\nu'} \tag{2.12}$$

where ν' is the exponent from (2.2). Consequently,

$$\int_{\Omega_t} h_t(x, x_0)\,\mathrm{d}\mu(x) \longrightarrow 1 \quad \text{as } t \longrightarrow \infty. \tag{2.13}$$

Proof. Since $\mathcal{M}$ is stochastically complete, (2.13) follows from (2.12) and $\varphi(t) \to 0$. Since

$$\mathcal{M} \setminus \Omega_t = B\left(x_0, \varphi(t)\sqrt{t}\right) \cup B^c\left(x_0, \varphi(t)/\sqrt{t}\right)$$

we estimate the integrals over $B\left(x_0, \varphi(t)\sqrt{t}\right)$ and $B^c\left(x_0, \sqrt{t}/\sqrt{t}\right)$ separately. Assume that t is large enough so that $\varphi(t) < 1$. Using (2.6) and (2.2) we obtain

$$\int_{d(x,x_0)<\varphi(t)\sqrt{t}} h_t(x, x_0)\,\mathrm{d}\mu(x) \lesssim \frac{V(x_0, \varphi(t)\sqrt{t})}{V(x_0, \sqrt{t})} \lesssim \varphi(t)^{\nu'}.$$

Using (2.6) and (2.10) with $r = \frac{\sqrt{t}}{\varphi(t)}$ and any $N \in \mathbb{N}$, we obtain

$$\begin{aligned}
\int_{d(x,x_0)>\frac{\sqrt{t}}{\varphi(t)}} h_t(x, x_0)\,\mathrm{d}\mu(x) &\lesssim \int_{B(x_0,r)^c} \frac{\mathrm{d}\mu(x)}{V(x_0, \sqrt{t})} \exp\left(-c\,\frac{d(x, x_0)^2}{t}\right) \\
&\lesssim \left(\frac{r}{\sqrt{t}}\right)^{-N} = \varphi(t)^N,
\end{aligned}$$

whence the claim follows. □

3. Proof of the main theorem

We start the proof of Theorem 2.1 with continuous compactly supported initial data and prove the asymptotic properties of the solution in the L^1 norm, working separately outside and inside the critical region Ω_t. Then, we show that these properties remain valid for all L^1 initial data by using a density argument. The L^∞ convergence is proved in the same spirit.

Proposition 3.1. *Let $x_0 \in \mathcal{M}$ and $u_0 \in \mathcal{C}_c(B(x_0, a))$ for some $a > 0$. Assume that $\varphi(t)$ is a positive function such that $\varphi(t) \to 0$ and $\varphi(t)\sqrt{t} \to \infty$ as $t \to \infty$. Then the solution (1.2) satisfies*

$$\|u(t,\,.\,) - Mh_t(\,.\,, x_0)\|_{L^1(\mathcal{M}\setminus\Omega_t)} \lesssim \varphi(t)^{\nu'} \tag{3.1}$$

and

$$\|u(t,\,.\,) - Mh_t(\,.\,, x_0)\|_{L^1(\Omega_t)} \lesssim t^{-\frac{1}{2}}, \tag{3.2}$$

where Ω_t is defined by (2.11), ν' is the exponent from (2.2) and

$$M = \int_{\mathcal{M}} u_0(x)\,\mathrm{d}\mu(x).$$

Consequently,

$$\|u(t,\,.\,) - Mh_t(\,.\,, x_0)\|_{L^1(\mathcal{M})} \lesssim t^{-\eta} \tag{3.3}$$

for any $\eta < \min(\nu', 1)/2$.

Proof. By Lemma 2.4 we have

$$\|h_t(\,.\,,x_0)\|_{L^1(\mathcal{M}\smallsetminus\Omega_t)} \lesssim \varphi(t)^{\nu'}$$

so that (3.1) will follow if we prove that

$$\|u(t,\,.\,)\|_{L^1(\mathcal{M}\smallsetminus\Omega_t)} \lesssim \varphi(t)^{\nu'}. \tag{3.4}$$

We write

$$\begin{aligned}\|u(t,\,.\,)\|_{L^1(\mathcal{M}\smallsetminus\Omega_t)} &= \int_{\mathcal{M}\smallsetminus\Omega_t}\left|\int_{B(x_0,a)} h_t(x,y)\,u_0(y)\,\mathrm{d}\mu(x)\right|\mathrm{d}\mu(y)\\ &\le \int_{B(x_0,a)}|u_0(y)|\left\{\int_{\mathcal{M}\smallsetminus\Omega_t} h_t(x,y)\,\mathrm{d}\mu(x)\right\}\mathrm{d}\mu(y).\end{aligned}$$

Notice that $x\in\mathcal{M}\smallsetminus\Omega_t$ and $y\in B(x_0,a)$ imply $x\in\mathcal{M}\smallsetminus\widetilde{\Omega}_{t,y}$, where

$$\widetilde{\Omega}_{t,y} = \left\{x\in\mathcal{M}\,\middle|\,2\,\varphi(t)\sqrt{t}\le d(x,y)\le \frac{1}{2}\,\frac{\sqrt{t}}{\varphi(t)}\right\}$$

provided t is large enough. Indeed, if $x\in\widetilde{\Omega}_{t,y}$ then

$$d(x,x_0)\le d(x,y)+d(y,x_0)\le \frac{1}{2}\,\frac{\sqrt{t}}{\varphi(t)}+a\le\frac{\sqrt{t}}{\varphi(t)}$$

and

$$d(x,x_0)\ge d(x,y)-d(y,x_0)\ge 2\,\varphi(t)\sqrt{t}-a\ge\ \varphi(t)\sqrt{t}$$

for t large enough, since $\varphi(t)\to 0$ and $\varphi(t)\sqrt{t}\to\infty$ as $t\to\infty$. It follows that $x\in\Omega_t$.

Applying (2.12) with $\widetilde{\Omega}_{t,y}$ instead of Ω_t we obtain

$$\int_{\mathcal{M}\smallsetminus\Omega_t} h_t(x,y)\,\mathrm{d}\mu(x)\le\int_{\mathcal{M}\smallsetminus\widetilde{\Omega}_{t,y}} h_t(x,y)\,\mathrm{d}\mu(x)\lesssim\varphi(t)^{\nu'},$$

whence (3.4) follows.

Now, let us turn to (3.2). Observe that

$$\begin{aligned}u(t,x)-Mh_t(x,x_0) &= \int_{\mathcal{M}} u_0(y)\,\big(h_t(x,y)-h_t(x,x_0)\big)\,\mathrm{d}\mu(y)\\ &= \int_{B(x_0,a)} u_0(y)\,\big(h_t(x,y)-h_t(x,x_0)\big)\,\mathrm{d}\mu(y).\end{aligned}\tag{3.5}$$

According to the mean value theorem, there exists $y^*\in\mathcal{M}$ lying on the minimal geodesic segment joining x_0 and y such that

$$|h_t(x,y)-h_t(x,x_0)|\le d(x_0,y)\,|\nabla h_t(x,y^*)|$$

where $\nabla=\nabla_y$. By the gradient estimate (2.8) we have

$$|\nabla h_t(x,y^*)|\le\frac{C}{\sqrt{t}\,V(x,\sqrt{t})}\exp\left(-c\,\frac{d^2(x,y_*)}{t}\right).$$

For $x\in\Omega_t$ we have $d\,(x\,,x_0)>\sqrt{t}\varphi\,(t)$ while $d\,(x,y^*)\le a$. Hence, for large enough t we obtain

$$d^2(x,y^*)\ge\big(d(x,x_0)-d(x_0,y^*)\big)^2\ge\frac{d^2(x,x_0)}{2} \tag{3.6}$$

It follows that

$$|u(t,x)-Mh_t(x,x_0)|\le\frac{a\,C}{\sqrt{t}\,V(x,\sqrt{t})}\exp\left(-c\,\frac{d^2(x,x_0)}{2t}\right)\int_{B(x_0,a)}|u_0(y)|\,\mathrm{d}\mu(y). \tag{3.7}$$

On the other hand, (2.3) implies that

$$\begin{aligned}\frac{1}{V(x,\sqrt{t})}\exp\left(-c\,\frac{d^2(x,x_0)}{2t}\right) &\lesssim \left(\frac{d(x,x_0)}{\sqrt{t}}+1\right)^{\nu}\frac{1}{V(x_0,\sqrt{t})}\exp\left(-c\,\frac{d^2(x,x_0)}{2t}\right)\\ &\lesssim \frac{1}{V(x_0,\sqrt{t})}\exp\left(-c\,\frac{d^2(x,x_0)}{4t}\right). \qquad (3.8)\end{aligned}$$

Substituting (3.6) into the right-hand side of (3.7) and integrating in x over Ω_t, we obtain by (2.9)

$$\int_{\Omega_t}|u(t,x)\,-\,Mh_t(x,x_0)|\,\mathrm{d}\mu(x)\lesssim\frac{1}{\sqrt{t}}\int_{\Omega_t}\frac{\mathrm{d}\mu(x)}{V(x_0,\sqrt{t})}\exp\left(-c\,\frac{d^2(x,x_0)}{4t}\right)\lesssim\frac{1}{\sqrt{t}}.$$

Finally, (3.3) follows by adding up (3.1) and (3.2) with $\varphi(t)=t^{\varepsilon-\frac{1}{2}}$ with small enough ε. □

Next, we prove our main theorem.

Proof of Theorem 2.1. Given $u_0\in L^1(\mathcal{M})$, fix $\varepsilon>0$ and choose $\widetilde{u}_0\in\mathcal{C}_c(\mathcal{M})$ such that

$$\|u_0-\widetilde{u}_0\|_{L^1(\mathcal{M})}<\frac{\varepsilon}{3}.$$

Let us prove first (1.5). Setting $\widetilde{M}=\int_{\mathcal{M}}\widetilde{u}_0(y)\,\mathrm{d}\mu(y)$we have

$$|M\,-\,\widetilde{M}|\,\le\,\int_{\mathcal{M}}\mathrm{d}\mu(y)\,|u_0(y)\,-\,\widetilde{u}_0(y)|\,=\,\|u_0-\widetilde{u}_0\|_{L^1(\mathcal{M})}\,<\,\frac{\varepsilon}{3}. \qquad (3.9)$$

It follows that

$$\|Mh_t(\,.\,,x_0)\,-\,\widetilde{M}h_t(\,.\,,x_0)\|_{L^1(\mathcal{M})}\,\le\,|M\,-\,\widetilde{M}|\,\|h_t(\,.\,,x_0)\|_{L^1(\mathcal{M})}<\frac{\varepsilon}{3}. \qquad (3.10)$$

Let $\widetilde{u}(t,x)$ be the semigroup solution to the heat equation with the initial data $\widetilde{u}_0$. Then we have

$$\|u(t,\,.\,)\,-\,\widetilde{u}(t,\,.\,)\|_{L^1(\mathcal{M})}\,\le\,\int_{\mathcal{M}}|u_0(y)\,-\,\widetilde{u}_0(y)|\left\{\int_{\mathcal{M}}\mathrm{d}\mu(x)\,h_t(x,y)\right\}\mathrm{d}\mu(y)\,<\,\frac{\varepsilon}{3}. \qquad (3.11)$$

By the estimate (3.3) of Proposition 3.1, we have, for sufficiently large t,

$$\|\widetilde{u}(t,\,.\,)\,-\,\widetilde{M}h_t(\,.\,,x_0)\|_{L^1(\mathcal{M})}\,<\,\frac{\varepsilon}{3}. \qquad (3.12)$$

Combining (3.10), (3.11) and (3.12) together, we obtain

$$\|u(t,\,.\,)\,-\,Mh_t(\,.\,,x_0)\|_{L^1(\mathcal{M})}\,<\,\varepsilon$$

which finishes the proof of (1.5).

Let us turn to the sup norm convergence (1.6). Since $\widetilde{u}_0$ is compactly supported, we can apply (3.7) which gives for all x

$$|\widetilde{u}(t,x)\,-\,\widetilde{M}h_t(x,x_0)|\,\le\,\frac{C}{\sqrt{t}V(x,\sqrt{t})}\le\frac{\varepsilon}{V(x,\sqrt{t})} \qquad (3.13)$$

for sufficiently large t. Next, we have

$$|u(t,x)\,-\,\widetilde{u}(t,x)|\,\le\,\int_{\mathcal{M}}h_t(x,y)\,|u_0(y)-\widetilde{u}_0(y)|\,\mathrm{d}\mu(y)\,\le\frac{C\,\|u_0-\widetilde{u}_0\|_{L^1}}{V(x,\sqrt{t})}\lesssim\frac{\varepsilon}{V(x,\sqrt{t})} \qquad (3.14)$$

and

$$|Mh_t(x,x_0)\,-\,\widetilde{M}h_t(x,x_0)|\,=\,|M-\widetilde{M}|\,h_t\,(x,x_0)\lesssim\frac{\varepsilon}{V\,(x,\sqrt{t})}. \qquad (3.15)$$

Putting together (3.13), (3.14) and (3.15), we conclude that

$$|\,u(t,\,x\,)\,-\,Mh_t(\,x\,,x_0)|\lesssim\frac{\varepsilon}{V\,(x,\sqrt{t})}$$

whence (1.6) follows. □

4. A counterexample

Let $n \geq 3$. Consider a manifold $\mathcal{M} = \mathbb{R}^n \# \mathbb{R}^n$ that is a connected sum of two copies of $\mathbb{R}^n$. This means that $\mathcal{M}$ is a union of a compact part K and two Euclidean ends $E^{\pm} = \mathbb{R}^n \backslash B(0, R)$ (see Fig. 1).

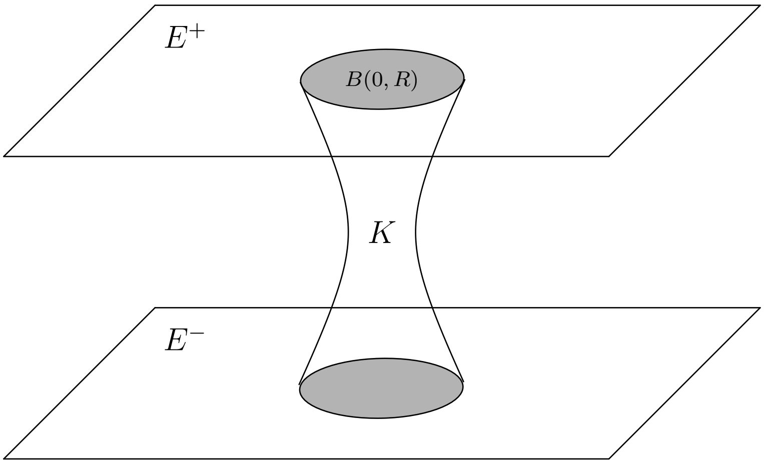

Figure 1. Manifold $\mathbb{R}^n \# \mathbb{R}^n$

Note that $V(x, R) \asymp R^n$ for all $x \in \mathcal{M}$, and the heat kernel on $\mathcal{M}$ satisfies the upper bound (2.6) (see, for instance, [12, Corollary 4.6, p.1946]). However, another essential hypothesis of Theorem 2.1, the gradient estimate (2.8), fails in the present setting. Indeed, it was shown in [4, Proposition 6.1] that, for large t,

$$\sup_{x,y \in \mathcal{M}} |\nabla_y h_t(x, y)| \geq c\, t^{-n/2},$$

which is incompatible with (2.8).

Let us verify that also the conclusion (1.6) of Theorem 2.1 fails in this setting. For that consider on E^+ for any $t > 1$ a point x_t with $|x_t| = \sqrt{t}$ and with a fixed direction $\omega = x_t/\sqrt{t}$. It was proved in [4, Proposition 6.1] that, for any $x \in K$,

$$\lim_{t \to \infty} t^{\frac{n}{2}} h_t(x_t, x) = \Phi(x) \tag{4.1}$$

where $\Phi(x)$ is a positive harmonic function on $\mathcal{M}$ that tends to a positive constant as $x \to \infty$ at the end E^+ and tends to 0 as $x \to \infty$ at the end E^-. Let us fix two points x_1 and x_2 in K such that $\Phi(x_1) \neq \Phi(x_2)$. Then (4.1) implies that

$$t^{\frac{n}{2}} \Big(h_t(x_t, x_1) - h_t(x_t, x_2) \Big) \longrightarrow \Big(\Phi(x_1) - \Phi(x_2) \Big) \qquad as \quad t \longrightarrow \infty \tag{4.2}$$

while (1.6) would imply that

$$t^{\frac{n}{2}} \| h_t(\,\cdot\,, x_1) - h_t(\,\cdot\,, x_2) \|_{L^\infty(\mathcal{M})} \longrightarrow 0 \qquad as \quad t \longrightarrow \infty\,,$$

hence,

$$t^{\frac{n}{2}} \left(h_t(x_t, x_1) - h_t(x_t, x_2) \right) \longrightarrow 0,$$

which contradicts to (4.2).

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