# CONSUMPTION DECISION, PORTFOLIO CHOICE AND HEALTHCARE IRREVERSIBLE INVESTMENT

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Abstract. We propose a tractable dynamic framework for the joint determination of optimal consumption, portfolio choice, and healthcare irreversible investment. Our model is based on a Merton's portfolio and consumption problem, where, in addition, the agent can choose the time at which undertaking a costly lump sum health investment decision. Health depreciates with age and directly affects the agent's mortality force, so that investment into healthcare reduces the agent's mortality risk. The resulting optimization problem is formulated as a stochastic control-stopping problem with a random time-horizon and state-variables given by the agent's wealth and health capital. We transform this problem into its dual version, which is now a two-dimensional optimal stopping problem with interconnected dynamics and finite time-horizon. Regularity of the optimal stopping value function is derived and the related free boundary surface is proved to be Lipschitz continuous and it is characterized as the unique solution to a nonlinear integral equation. In the original coordinates, the agent thus invests into healthcare whenever her wealth exceeds an age- and health-dependent transformed version of the optimal stopping boundary.

**Keywords:** Optimal timing of health investment; Optimal consumption; Optimal portfolio choice; Duality; Optimal stopping; Free boundary; Stochastic control.

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**JEL Classification:** G11, E21, I13.

## 1. Introduction

It has been recognized that expenditures on medical services, annual physical exams, and exercise can be viewed as investments in health capital and analyzed using the tools of capital theory. This approach has enabled economists to derive propositions about the pattern of healthcare spending over an individual's lifetime and to describe the behavior of health capital over the life cycle. For example, a demand-for-health model developed by [Grossman, 1972] extended the human capital theory by explicitly incorporating health and recognizing that there are both consumption and investment motives for investing in health. The basic features of the model are (1) that health can be viewed as a durable capital stock that produces an output of healthy time, (2) that individuals inherit an initial stock of health that depreciates with age, (3) that the stock of health can be increased by investment, and (4) that the individual demands health (a) for its utility enhancing effects (the consumption motive), and (b) for its effect on the amount of healthy time (the investments motive).

Based on aforementioned standard model assumptions, various health economists have enhanced the Grossman's dynamic health investment model. These enhancements address, for example, the introduction of uncertainty into the theoretical model (see, e.g., [Cropper, 1977], [Ehrlich, 2000] and [Bolin and Caputo, 2020]) or the distribution of health within the family (see [Jacobson, 2000] and [Bolin et al., 2001], among many others).

Empirical evidence suggests that health crucially influences an agent's financial decisions (see, e.g., [Rosen and Wu, 2004]; [Smith, 2009]; [Atella et al., 2012]). In particular, literature reveals that health status is positively correlated with income, consumption asset holdings, and negatively correlated with health expenditures. To account for this fact, [Hugonnier et al., 2013] proposed a dynamic framework for the joint determination of optimal consumption, portfolio holdings, and

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health investment. They solve for the optimal rules in closed form and provide estimates of the parameters that confirm the relevance of all the main characteristics of the model. More recently, [Guasoni and Huang, 2019] focuses on a representative household that makes consumption, investment and healthcare spending decisions in order to maximise welfare under time-separable utilities. In [Guasoni and Huang, 2019], the Gompertz law of mortality is taken as state variable, in addition to wealth, and the resulting optimal stochastic control problem is reduced to the study of a nonlinear ordinary differential equation. This is shown to have a unique solution, which has an explicit expression in the old-age limit. Further, [Aurand and Huang, 2021] studies optimal consumption, investment, and healthcare spending under Epstein–Zin preferences.

As in [Hugonnier et al., 2013], also in this paper, we combine two well-accepted frameworks from the Financial and Health Economics literature within a unified setup. However, differently to [Hugonnier et al., 2013], health-related decisions are approached from a different viewpoint. Specifically, we start from a [Merton, 1971]'s portfolio and consumption choice problem and append to this model the determination of the time of health investment (e.g., buying a preventive health insurance). Meanwhile, the above essential features of [Grossman, 1972]'s canonical model are retained.

In what follows, a distinction is made and maintained between curative and precautionary health investments. This distinction is an important one since the two types of investment may behave quite differently over the life cycle. Specifically, curative health investments are defined to have direct effects on the stock of health or the rate of depreciation of health, or both, and are produced using medical-care goods and services. As a result, curative health investments are not among an individual's choice variables. In contrast, precautionary health investments are defined as those that are under the control of an agent and indirectly affect the rate of depreciation of a stock of health by directly affecting the stock of health itself. That is, the current stock of health, which is directly influenced by precautionary health investments, determines the rate of depreciation of health. The view that precautionary health investments indirectly affect the rate of depreciation of health in the aforementioned manner is consistent with ample medical evidence. Therefore, we focus on precautionary health investments in our model (see (2.4) in Section 2).

The introduction of the option to choose the time for a health precautionary investment raises several questions. First, if an individual is faced with the choice of when buying preventive health insurance, how should she optimally behave? In particular, a non-trivial trade-off arises: If the agent invests into healthcare too early, then she reduces her wealth, thus affecting consumption and portfolio choice; if she invests in health too late, then this will negatively impact on the utility and the survival probability. Second, how do optimal consumption and investment strategies react to the introduction of health factors?

The answers to these questions, which are collected in Section 5, are somewhat intuitively convincing. We show that it is optimal to invest in health when the agent's wealth first reaches an endogenously determined boundary surface, which depends on the agent's age and health status. Intuitively, if the agent is sufficiently rich (her wealth exceeds the corresponding boundary), then health investments should be performed immediately; otherwise, it is optimal to wait for an increase of the wealth.

1.1. **Overview of the mathematical analysis.** From a mathematical point of view, our model leads to a *random time-horizon, two-dimensional stochastic control problem with discretionary stopping*. The time-horizon is given as the minimum between the agent's random time of death, $\eta$, and the maximal expected biological longevity $T < \infty$[1]. We assume that $\eta$ does not need to depend on the financial market. In other words, we do not assume that $\eta$ is a stopping time of the filtration $\mathbb{F}$ generated by the asset prices. The conditional distribution function of $\eta$ is assumed to depend on the agent's health status and by means of health investment the agent slows the rate of mortality, which in turn changes the distribution of $\eta$.

[1]This can be also thought of as the maximal age at which insurance companies enable to enter a preventive healthcare program.

The two coordinates of the state process are the wealth process $X$ and the health capital process $H$. The agent chooses the consumption rate $c$ and the portfolio $\pi$, as well as the time $\tau$ at which undertaking a lump sum investment into health. At time $\tau$ the dynamics of $H$ and $X$ change, since the health capital is increased through the investment (so that the mortality rate is decreased), while the agent's wealth reduces. The aim is then to maximize the intertemporal utility from consumption and health status, up to the random time $\eta \wedge T$.

Problems with a similar structure arise, for instance, in retirement time choice models, where the agent has to consume and invest in risky assets, and to decide when to retire (see, e.g., [Jin Choi and Shim, 2006], [Yang and Koo, 2018]). Combined stochastic control/optimal stopping problems also arise in Mathematical Finance, namely, in the context of pricing American contingent claims under constraints and utility maximization problem with discretionary stopping; see, e.g., [Karatzas and Kou, 1998] and [Karatzas and Wang, 2000]. In order to tame the intricate mathematical structure of our problem, where the consumption and portfolio choice nontrivially interact with the investment decision, we combine a duality and a free-boundary approach, and proceed in our analysis as it follows.

**Step 1.** First, we conduct successive transformations (see Section 3) and formulate the original stochastic control-stopping problem (with value function $V$) in terms of its dual problem by martingale and duality methods (similar to [Karatzas and Wang, 2000] or [Yang and Koo, 2018]).

**Step 2.** We study the dual problem (with value function $J$), which is a *finite time-horizon, two-dimensional optimal stopping problem with interconnected dynamics.* The dual variable $Z$ (Lagrange multiplier) evolves as a geometric Brownian motion, whose drift depends on the health capital process $H$. Moreover, $H$ affects the mortality rate and thus the exponential discount factor appearing in the stopping functional. The coupling between the two components of the state process makes the study of the optimal stopping problem quite intricate.

It is also worth pointing out that the health capital process $H$ does not possess any diffusive term, which leads to a novel analysis of the regularity of $J$. As a matter of fact, the process $(Z, H)$ is a degenerate diffusion process (in the sense that the differential operator of $(Z, H)$ is a degenerate parabolic operator) so that the study of the regularity of $J$ in the interior of its continuation region cannot hinge on classical analytic existence results for parabolic PDEs (notice that the differential operator in our case does not even satisfy the Hörmander condition required in [Peskir, 2022]).

Additional technical difficulties arise when trying to infer properties of the optimal stopping boundary $b$. In fact, due to the generic time and health dependence of the mortality force, we were unable to establish any monotonicity for the mapping $(t, h) \mapsto b(t, h)$. It is well known in optimal stopping and free-boundary theory that monotonicity of $b$ is the key to a rigorous study of the regularity of the boundary (e.g. continuity) and of the value function (e.g. continuous differentiability). The interested reader may consult the introduction in [De Angelis and Stabile, 2019a] for a deeper discussion.

We overcome these major technical hurdles by proving that the optimal boundary is in fact a locally Lipschitz-continuous function of time $t$ and health capital $h$, without employing neither monotonicity of the boundary nor classical results on interior regularity for parabolic PDEs. In order to achieve this goal, we rely only upon probabilistic methods which are specifically designed to tackle our problem.

As a matter of fact, we first prove that $\widehat{J}$ (given by the difference of $J$ and the smooth payoff of immediate stopping; see (4.1)) is locally Lipschitz continuous and obtain probabilistic representations of its weak-derivatives (cf. [De Angelis and Stabile, 2019a]). Then, through a suitable application of the method developed in [De Angelis and Stabile, 2019b], by means of a version of the implicit function theorem for Lipschitz mappings (cf. [Papi, 2005]), we can show that the free boundary surface $(t, h) \to b(t, h)$ is locally-Lipschitz continuous. This enables us to prove that the optimal stopping time $(t, z, h) \mapsto \tau^*(t, z, h)$ is continuous, which in turn gives that $\widehat{J}$ is a continuously differentiable functions of its three variables. Being that the dual process $Z$ is the only diffusive one, the $C^1$-property of $\widehat{J}$ implies that $\widehat{J}_{zz}$ admits a continuous extension to the closure of the continuation region. Notice that it is in fact this regularity that could had not been derived from standard results on PDEs nor from [Peskir, 2022], and it is in fact this regularity that allows (via an application of a weak version of Dynkin's formula) to derive an integral equation which is uniquely solved by the free boundary.

**Step 3.** After proving the strict convexity of $J$ through techniques that employ stability results for viscosity solutions (cf. Proposition 5.1), we can come back to the original coordinates' system and via the duality relations obtain the optimal consumption and portfolio policies, as well as the optimal investment time, in terms of the optimal stopping boundary and value function (cf. Section 5).

In summary, our contribution is at least twofold. On the one hand, we contribute to the literature concerning health investment problems in the consumption-portfolio framework. To the best of our knowledge, our paper is the first that integrates timing decisions for irreversible investment into health within the portfolio-consumption literature. From a mathematical point of view, even though the literature on stochastic control with discretionary stopping problems is extensive (in different contexts), our study on a finite time-horizon two-dimensional optimal stopping problem with interconnected dynamics and non monotone boundary constitutes a novelty. Very recently, [Cai et al., 2022] study the pricing of American put options in the Black-Scholes market with a stochastic interest rate and finite-time maturity, which results into a finite time-horizon two-dimensional optimal stopping problem. However, in [Cai et al., 2022] monotonicity of the free boundary can be obtained due to the problem's mathematical structure.

1.2. **Plan of the paper.** The rest of the paper is organized as follows. In Section 2, we introduce the model. We transform the original stochastic control-stopping problem into a pure stopping problem in Section 3, while in Section 4 we study the dual optimal stopping problem. In Section 5, we provide the optimal health investment boundary, optimal consumption plan and optimal portfolio in primal variables, and in Section 6 we conclude. Appendix A contains technical estimates, Appendix B collects the proofs of some results of Section 4, whereas Appendix C provides some auxiliary results needed in the paper.

## 2. Setting and problem formulation

2.1. **Setting.** Let $T < \infty$ be a fixed time-horizon, representing either the maximal biological longevity from the initial time $t \in [0, T]$ or the maximal age at which a preventive health-care insurance program can be stipulated. Also, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, endowed with a filtration $\mathbb{F} := \{\mathcal{F}_s, t \leq s \leq T\}$ satisfying the usual conditions. We assume that there exists a random variable $\Theta$, constructed on $(\Omega, \mathcal{F})$, independent of $\mathcal{F}_T$ and such that

$$\mathbb{P}[\Theta > v] = e^{-v}, \; v \geq 0.$$

Consider an agent whose lifespan is determined through health capital. In the spirit of [Hugonnier et al., 2013], we model the mortality rate process $M^H := \{M_s^H, t \leq s \leq T\}$ as

$$M_s^H = m^0 + m^1 H_s^{-\kappa}, \tag{2.1}$$

for some non-negative constants $m^0, m^1$, and $\kappa > 0$. Here, $H := \{H_s, t \leq s \leq T\}$ is the $\mathbb{F}$-adapted health capital process. Notice that in (2.1) the endogenous part of the mortality process is a function of the agent's current health status rather than of her current health investment (see, e.g., [Guasoni and Huang, 2019]). This assumption excludes the possibility of freely altering the mortality rate by health investment.

We define the random death time of the agent $\eta$ as[2]

$$\eta := \inf\left\{s \geq t : \int_t^s M_u^H du \geq \Theta\right\},$$

which is such that $\{\eta \geq s\} = \left\{\int_t^s M_u^H du \leq \Theta\right\}$, where we have assumed that $\int_0^T M_u^H du < \infty$ a.s.

[2]Notice that $\eta$ is not an $\mathbb{F}$-stopping time. It is instead a $\mathbb{G}$-stopping time, where $\mathcal{G}_s := \mathcal{F}_s \vee \sigma(\mathbb{1}_{\{\eta \leq u\}}; t \leq u \leq s)$, for $s \geq 0$. This is the enlarged filtration generated by the underlying filtration $\mathbb{F}$ and the process $\{\mathbb{1}_{\{\eta \leq u\}}, t \leq u \leq s\}$. The filtration $\mathbb{G}$ is the smallest one which contains $\mathbb{F}$ and such that $\eta$ is a $\mathbb{G}$-stopping time (see, e.g., Chapter 7 in [Jeanblanc et al., 2009]).

The conditional distribution function of $\eta$ is such that (see, e.g., Lemma 7.3.2.1 in [Jeanblanc et al., 2009]),

$$\mathbb{P}[\eta > s|\mathcal{F}_{s'}] = \exp\bigg\{ - \int_t^s M_u^H du \bigg\}, \quad s' \geq s \geq t. \tag{2.2}$$

In particular, $\mathbb{P}[\eta > s|\mathcal{F}_s] = \exp\big(-\int_t^s M_u^H du\big)$.

Let $\tau$ be an $\mathbb{F}$-stopping time representing the time at which the agent invests in health. Before investing in health, the agent's health status $H^1 := \{H_s^1, t \leq s \leq \tau\}$ evolves as

$$dH_s^1 = -\delta H_s^1 ds, \text{ for all } s \in (t, \tau], \ H_t^1 = h > 0, \tag{2.3}$$

where $\delta > 0$ represents the decay rate of the health. After investing in healthcare a positive amount $I$, the agent's health status $H^2 := \{H_s^2, s \geq \tau\}$ increases by the deterministic positive amount $f(I)$, so that

$$dH_s^2 = (-\delta H_s^2 + f(I))ds, \text{ for all } s > \tau, \ H_\tau^2 = he^{-\delta(\tau - t)}. \tag{2.4}$$

From (2.3) and (2.4) one then has that the overall health capital $H$ evolves as

$$dH_s = (-\delta H_s + f(I)\mathbb{1}_{\{s>\tau\}})ds, \text{ for all } s \in (t, T], \ H_t = h > 0.$$

**Remark 2.1.** *(1) The fact that health investment is positive is a standard requirement in Health Economics. Health investment is irreversible in the sense that the agent cannot reduce her health through negative expenditure. Irreversibility of investment is a key economic feature that makes health fundamentally different from financial assets or housing (see, e.g.,* [Yogo, 2016]*).*

*(2) The state equation (2.4) is similar to* [Bolin and Caputo, 2020]. *In (2.4), $f(\cdot)$ is a health production function, mapping precautionary health investment into the gross rate of change of the health stock. In* [Bolin and Caputo, 2020] *it is assumed that $f(\cdot) \in C^2$ and $f'(I) > 0$; that is, the marginal product of health investment is positive. The health production function therefore captures the direct influence of precautionary health investment on the health stock, a defining feature of such investment as it is discussed in the Introduction.*

*(3) A classical model for force of mortality is the so-called Gompertz-Makeham law (see for instance* [Makeham, 1860]*), which corresponds to*

$$M_s = Ae^{B(s-t)} + C, \ s \geq t, \ M_t = C.$$

*Here, $A$ is known as the baseline mortality, the term $B$ can be thought of as the 'actuarial aging rate', in that its magnitude determines how fast the rate of dying will increase with the addition of extra years, while $C$ is a constant representing age-independent mortality. As a matter of fact, before health investment, our choice of the mortality rate reads as $M_s^H = m^0 + m^1 h^{-\kappa} e^{\delta\kappa(s-t)}$ (cf. (2.1) and (2.3)), which has a structure compatible to the Gompertz-Makeham law.*

We assume that the agent also invests in a financial market with two assets. One of them is a risk-free bond, whose price $S^0 := \{S_s^0, t \leq s \leq T\}$ evolves as

$$dS_s^0 = rS_s^0 ds, \quad S_t^0 = s^0 > 0,$$

where $r > 0$ is a constant risk-free rate. The second one is a stock, whose price is denoted by $S := \{S_s, t \leq s \leq T\}$ and it satisfies the stochastic differential equation

$$dS_s = \mu S_s dt + \sigma S_s dB_s, \quad S_t = s > 0,$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$ are given constants. Here, $B := \{B_s, t \leq s \leq T\}$ is an $\mathbb{F}$-adapted standard Brownian motion under $\mathbb{P}$.

The agent also consumes from her wealth, while investing in the financial market. Denoting by $\pi_s$ the amount of wealth invested in the stock at time $s$, the agent then chooses $\pi_s$ as well as the rate of spending in consumption $c_s$ at time $s$. Therefore, the agent's wealth $X := \{X_s^{c,\pi,\tau}, s \in [t, T]\}$ evolves as

$$dX_s^{c,\pi,\tau} = [\pi_s(\mu - r) + rX_s^{c,\pi,\tau} - c_s - I\mathbb{1}_{\{s\geq\tau\}}]ds + \pi_s\sigma dB_s, \quad X_t^{c,\pi,\tau} = x > 0. \tag{2.5}$$

In the following, we shall simply write $X$ to denote $X^{c,\pi,\tau}$, where needed.

2.2. **The optimization problem.** Here and in the sequel, we set $\mathcal{O} := [0,T] \times \mathbb{R}^2_+$ with $\mathbb{R}_+ := (0,\infty)$, we denote by $\mathcal{S}_{t,s}$ the class of $\mathbb{F}$-stopping times $\tau : \Omega \to [t,s]$ for $t \leq s \leq T$, and let $\mathcal{S} := \mathcal{S}_{t,T}$. Then we introduce the class of admissible strategies as it follows.

**Definition 2.1.** *Let $(t,x,h) \in \mathcal{O}$ be given and fixed. The triplet of choices $(c,\pi,\tau)$ is called an* ***admissible strategy*** *for $(t,x,h)$, and we write $(c,\pi,\tau) \in \mathcal{A}(t,x,h)$, if it satisfies the following conditions:*

*(i) $c$ and $\pi$ are progressively measurable with respect to $\mathbb{F}$, $\tau \in \mathcal{S}$;*
*(ii) $c_s \geq 0$ for all $s \in [t,T]$ and $\int_t^T (c_s + |\pi_s|^2) ds < \infty$ $\mathbb{P}$-a.s.;*
*(iii) $X_s^{c,\pi,\tau} > g(s)\mathbb{1}_{\{s \geq \tau\}}$ for all $s \in [t,T]$, where $g(s) = \frac{I}{r}(1 - e^{-r(T-s)})$.*

The function $g$ in Condition (iii) is the present value of the future health payment of the agent. Due to (iii) above the agent is able to consume and invest as long as her wealth level is above $g(s)$ at time $s \geq \tau$. Before health investment, she should keep her wealth positive for further consumption or financial investment.

From the perspective of time $t$, the agent's aim is then to maximize the expected utility

$$\mathbb{E}\left[\int_t^{\eta \wedge T} e^{-\rho(s-t)} u(c_s, H_s) ds \middle| \mathcal{F}_t\right] \tag{2.6}$$

over all $(c,\pi,\tau) \in \mathcal{A}(t,x,h)$. In (2.6), $\rho$ is a positive discount rate and $u(c,h) = c^\alpha h^{1-\alpha}$, where $0 < \alpha < 1$. Thanks to Fubini's Theorem and the tower property, we can disentangle the market risk and the mortality risk and write

$$\begin{aligned}
\mathbb{E}\left[\int_t^{\eta \wedge T} e^{-\rho(s-t)} u(c_s, H_s) ds \middle| \mathcal{F}_t\right] &= \mathbb{E}\left[\int_t^T e^{-\rho(s-t)} u(c_s, H_s) \mathbb{1}_{\{s<\eta\}} ds \middle| \mathcal{F}_t\right] \\
&= \int_t^T \mathbb{E}\left[e^{-\rho(s-t)} u(c_s, H_s) \mathbb{1}_{\{s<\eta\}} \middle| \mathcal{F}_t\right] ds \\
&= \int_t^T \mathbb{E}\left[\mathbb{E}\left[e^{-\rho(s-t)} u(c_s, H_s) \mathbb{1}_{\{s<\eta\}} \middle| \mathcal{F}_s\right] \middle| \mathcal{F}_t\right] ds \\
&= \int_t^T \mathbb{E}\left[e^{-\rho(s-t)} u(c_s, H_s) \mathbb{E}\left[\mathbb{1}_{\{s<\eta\}} \middle| \mathcal{F}_s\right] \middle| \mathcal{F}_t\right] ds \\
&= \mathbb{E}\left[\int_t^T e^{-\int_t^s (\rho + M_u^H) du} u(c_s, H_s) ds \middle| \mathcal{F}_t\right],
\end{aligned}$$

where (2.2) has been also employed. Hence, given the Markovian setting, the agent aims at determining

$$V(t,x,h) := \sup_{(c,\pi,\tau) \in \mathcal{A}(t,x,h)} \mathbb{E}_{t,x,h}\left[\int_t^T e^{-\int_t^s (\rho + M_u^H) du} u(c_s, H_s) ds\right], \tag{2.7}$$

where $\mathbb{E}_{t,x,h}$ denote the expectation under $\mathbb{P}_{t,x,h}(\cdot) := \mathbb{P}(\cdot | X_t = x, H_t = h)$. In the rest of the paper, we shall focus on (2.7).

## 3. From control-stopping to pure stopping

3.1. **The static budget constraint.** We define the market price of risk $\theta := \frac{\mu - r}{\sigma}$. For $\tau \in \mathcal{S}$, an application of Itô's formula to the process $\{e^{-ru - \theta B_u - \frac{1}{2}\theta^2 u}(X_u - g(u)\mathbb{1}_{\{u \geq \tau\}}), u \in [t,s]\}$, leads on

$\{s \geq \tau\} \cap \{\tau = t\}$ to

$$e^{-rs-\theta B_s-\frac{1}{2}\theta^2 s}(X_s - g(s)) + \int_t^s e^{-ru-\theta B_u-\frac{1}{2}\theta^2 u} c_u du$$

$$= e^{-rt-\theta B_t-\frac{1}{2}\theta^2 t}(x - g(t)) + \int_t^s e^{-ru-\theta B_u-\frac{1}{2}\theta^2 u}\Big(\pi_u \sigma - \theta(X_u - g(u))\Big) dB_u, \tag{3.1}$$

and on $\{s < \tau\}$ to

$$e^{-rs-\theta B_s-\frac{1}{2}\theta^2 s} X_s + \int_t^s e^{-ru-\theta B_u-\frac{1}{2}\theta^2 u} c_u du$$

$$= e^{-rt-\theta B_t-\frac{1}{2}\theta^2 t} x + \int_t^s e^{-ru-\theta B_u-\frac{1}{2}\theta^2 u}(\pi_u \sigma - \theta X_u) dB_u. \tag{3.2}$$

Since $X_s - g(s)\mathbb{1}_{\{s \geq \tau\}} > 0$ for any $s \in [t, T]$, we can deduce that $X_\tau > g(\tau) \geq 0$. For an admissible plan $(c, \pi, \tau) \in \mathcal{A}(t, x, h)$, the left-hand side of (3.2) is nonnegative for $s \leq \tau$, and so the Itô's integral on the right-hand side is not only a continuous $\mathbb{P}$-local martingale, but also a supermartingale by Fatou's Lemma. Thus, letting $\gamma_{s,t} := e^{-r(s-t)-\theta(B_s-B_t)-\frac{1}{2}\theta^2(s-t)}$, the optional sampling theorem implies the so-called budget constraint:

$$\mathbb{E}_{t,x,h}\big[\gamma_{s,t} X_s\big] + \mathbb{E}_{t,x,h}\left[\int_t^s \gamma_{u,t} c_u du\right] \leq x, \text{ if } 0 \leq t \leq s \leq \tau. \tag{3.3}$$

By similar arguments on (3.1) we also have

$$\mathbb{E}_{t,x,h}\big[\gamma_{s,t}(X_s - g(s))\big] + \mathbb{E}_{t,x,h}\left[\int_t^s \gamma_{u,t} c_u du\right] \leq x - g(t), \text{ if } 0 \leq t = \tau \leq s \leq T. \tag{3.4}$$

3.2. **The agent's optimization problem after health investment.** In this subsection we will consider the agent's optimization problem after health investment, and over this time period only consumption and portfolio choice have to be determined. Formally, the model in the previous section accommodates to this case if we let $\tau = t$, where $t$ is the fixed starting time, and the mortality rate is set to be $M_u^{H^2}, u \geq t$. Then, letting $\mathcal{A}_t(t, x, h) := \{(c, \pi) : (c, \pi, t) \in \mathcal{A}(t, x, h)\}$, where the subscript $t$ indicates that the investment time into health $\tau$ is equal to $t$, the agent's value function after health investment is

$$\widehat{V}(t, x, h) := \sup_{(c,\pi) \in \mathcal{A}_t(t,x,h)} \mathbb{E}_{t,x,h}\left[\int_t^T e^{-\int_t^s (\rho + M_u^{H^2}) du} u(c_s, H_s^2) ds\right], \tag{3.5}$$

with $H^2$ as defined in (2.4).

From the budget constraint (3.4), recalling that $\gamma_{s,t} = e^{-r(s-t)-\theta(B_s-B_t)-\frac{1}{2}\theta^2(s-t)}$ and for any pair $(c, \pi) \in \mathcal{A}_t(t, x, h)$ with a Lagrange multiplier $z > 0$, we have

$$\begin{aligned}
&\mathbb{E}_{t,x,h}\left[\int_t^T e^{-\int_t^s (\rho + M_u^{H^2}) du} u(c_s, H_s^2) ds\right] \\
&\leq \mathbb{E}_{t,x,h}\left[\int_t^T e^{-\int_t^s (\rho + M_u^{H^2}) du} u(c_s, H_s^2) ds\right] - z\mathbb{E}_{t,x,h}\left[\int_t^T \gamma_{s,t} c_s ds\right] + z(x - g(t)) \\
&= \mathbb{E}_{t,x,h}\left[\int_t^T e^{-\int_t^s (\rho + M_u^{H^2}) du} u(c_s, H_s^2) ds\right] - \mathbb{E}_{t,x,h}\left[\int_t^T e^{-\int_t^s (\rho + M_u^{H^2}) du} z P_s^2(h) c_s ds\right] + z(x - g(t)) \\
&= \mathbb{E}_{t,x,h}\left[\int_t^T e^{-\int_t^s (\rho + M_u^{H^2}) du}\Big(u(c_s, H_s^2) - z P_s^2(h) c_s\Big) ds\right] + z(x - g(t)) \\
&\leq \mathbb{E}_{t,x,h}\left[\int_t^T e^{-\int_t^s (\rho + M_u^{H^2}) du} \widehat{u}(z P_s^2(h), H_s^2) ds\right] + z(x - g(t)),
\end{aligned} \tag{3.6}$$

where

$$P_s^2(h) := \gamma_{s,t} e^{\int_t^s (\rho + M_u^{H^2}) du} \quad \text{and} \quad \widehat{u}(z,h) := \sup_{c \geq 0}[u(c,h) - cz]. \tag{3.7}$$

Let then $Z_s^2 := zP_s^2(h)$. By Itô's formula, we obtain that the dual variable $Z^2$ satisfies

$$dZ_s^2 = (\rho - r + M_s^{H^2}) Z_s^2 ds - \theta Z_s^2 dB_s, \quad Z_t^2 = z,$$

and we set

$$W(t,z,h) := \mathbb{E}_{t,z,h}\left[\int_t^T e^{-\int_t^s (\rho + M_u^{H^2}) du} \widehat{u}(Z_s^2, H_s^2) ds\right],$$

with $\mathbb{E}_{t,z,h}$ being the expectation under $\mathbb{P}$ conditioned on $Z_t^2 = z$ and $H_t^2 = h$. Hence,

$$\mathbb{E}_{t,x,h}\left[\int_t^T e^{-\int_t^s (\rho + M_u^{H^2}) du} u(c_s, H_s^2) ds\right] \leq W(t,z,h) + z(x - g(t)),$$

for $z > 0$ and $(t,x,h) \in \mathcal{O}$.

**Proposition 3.1.** *One has $W \in C^{1,2,1}(\mathcal{O})$. Moreover, $W$ satisfies*

$$-\widehat{\mathcal{L}} W = \widehat{u}, \ \text{on } [0,T) \times \mathbb{R}_+^2, \ \text{and} \quad W(T,z,h) = 0, \tag{3.8}$$

*where*

$$\widehat{\mathcal{L}} W := W_t + \frac{1}{2}\theta^2 z^2 W_{zz} + (\rho - r + m^0 + m^1 h^{-\kappa}) z W_z + (-\delta h + f(I)) W_h - (\rho + m^0 + m^1 h^{-\kappa}) W. \tag{3.9}$$

*Proof.* First, we compute the convex dual of $u(c,h) = c^\alpha h^{1-\alpha}$ in (3.7); that is,

$$\widehat{u}(z,h) = (1-\alpha)(\frac{z}{\alpha})^{\frac{\alpha}{\alpha-1}} h. \tag{3.10}$$

From (2.4) and the boundary condition $H_\tau^2 = H_t^2 = h$ (recall that $\tau = t$ this subsection), we have

$$H_s^2 = h e^{-\delta(s-t)} + \frac{f(I)}{\delta}(1 - e^{-\delta(s-t)}), \ \forall s \geq t. \tag{3.11}$$

Therefore, by (3.10) and (3.11) we rewrite $W(t,z,h)$ as follows

$$\begin{aligned} W(t,z,h) &= \mathbb{E}_{t,z,h}\left[\int_t^T e^{-\int_t^s (\rho + M_u^{H^2}) du} \widehat{u}(Z_s^2, H_s^2) ds\right] \\ &= \mathbb{E}_{t,z,h}\left[\int_t^T e^{-\int_t^s (\rho + M_u^{H^2}) du} (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} (Z_s^2)^{\frac{\alpha}{\alpha-1}} H_s^2 ds\right] \\ &= (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} z^{\frac{\alpha}{\alpha-1}} \int_t^T e^{-\int_t^s (\rho + M_u^{H^2}) du} \mathbb{E}[(P_s^2(h))^{\frac{\alpha}{\alpha-1}}]\Big(h e^{-\delta(s-t)} + \frac{f(I)}{\delta}(1 - e^{-\delta(s-t)})\Big) ds \\ &= (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} z^{\frac{\alpha}{\alpha-1}} \int_t^T e^{\frac{\int_t^s (\rho + M_u^{H^2}) du}{\alpha - 1}} e^{\int_t^s \left(\frac{\alpha(-r - \frac{1}{2}\theta^2)}{\alpha-1} + \frac{1}{2}\frac{\theta^2\alpha^2}{(\alpha-1)^2}\right) du}\Big(h e^{-\delta(s-t)} + \frac{f(I)}{\delta}(1 - e^{-\delta(s-t)})\Big) ds, \end{aligned} \tag{3.12}$$

where we have used the definition of $P_s^2(h)$ as in (3.7) and the fact that

$$\begin{aligned} \mathbb{E}[(P_s^2(h))^{\frac{\alpha}{\alpha-1}}] &= \mathbb{E}[(\gamma_{s,t} e^{\int_t^s (\rho + M_u^{H^2}) du})^{\frac{\alpha}{\alpha-1}}] = e^{\frac{\alpha}{\alpha-1}\int_t^s (\rho + M_u^{H^2}) du} \mathbb{E}[\gamma_{s,t}^{\frac{\alpha}{\alpha-1}}] \\ &= e^{\frac{\alpha}{\alpha-1}\int_t^s (\rho + M_u^{H^2}) du} \mathbb{E}[(e^{-r(s-t) - \theta(B_s - B_t) - \frac{1}{2}\theta^2(s-t)})^{\frac{\alpha}{\alpha-1}}] \\ &= e^{\int_t^s \left(\frac{\alpha}{\alpha-1}(\rho + M_u^{H^2} - r - \frac{1}{2}\theta^2) + \frac{1}{2}\frac{\theta^2\alpha^2}{(\alpha-1)^2}\right) du}. \end{aligned}$$

Thus, it is easy to see that $W \in C^{1,2,1}(\mathcal{O})$. Hence, it satisfies (3.8) by the well-known Feynman-Kac formula (see, e.g., Chapter 4 in [Karatzas and Shreve, 1998a]). $\square$

It is possible to relate $\widehat{V}$ to $W$ through the following duality relation.

**Theorem 3.1.** *The following dual relations hold:*

$$\widehat{V}(t,x,h) = \inf_{z>0}[W(t,z,h) + z(x-g(t))], \quad W(t,z,h) = \sup_{x>g(t)}[\widehat{V}(t,x,h) - z(x-g(t))].$$

*Proof.* Since $(c,\pi) \in \mathcal{A}_t(t,x,h)$ is arbitrary, taking the supremum over $(c,\pi) \in \mathcal{A}_t(t,x,h)$ on the left-hand side of (3.6) and recalling (3.5), we get, for any $z > 0$,

$$\widehat{V}(t,x,h) \le \mathbb{E}_{t,z,h}\left[\int_t^T e^{-\int_t^s(\rho+M_u^{H^2})du}\widehat{u}(Z_s^2, H_s^2)ds\right] + z(x-g(t)), \tag{3.13}$$

and thus

$$W(t,z,h) \ge \sup_{x>g(t)}[\widehat{V}(t,x,h) - z(x-g(t))].$$

Further, from (3.13) we have

$$\widehat{V}(t,x,h) \le \inf_{z>0}[W(t,z,h) + z(x-g(t))].$$

For the reverse inequalities, observe that the equality in (3.6) holds if and only if

$$c_s = \mathcal{I}^u(Z_s^2, H_s^2), \tag{3.14}$$

and

$$\mathbb{E}_{t,x,h}\left[\int_t^T \gamma_{s,t}c_s ds\right] = x - g(t), \tag{3.15}$$

where we denote by $\mathcal{I}^u$ the inverse of the marginal utility function $u_c(\cdot,h)$.

Then, assuming (3.15) (we will prove its validity later), we define

$$\mathcal{X}(t,z,h) := \mathbb{E}_{t,z,h}\left[\int_t^T \gamma_{s,t}\mathcal{I}^u(Z_s^2,H_s^2)ds\right], \quad \mathcal{Y}(t,x,h) := \mathbb{E}_{t,x,h}\left[\int_t^T e^{-\int_t^s(\rho+M_u^{H^2})du}u(c_s,H_s^2)ds\right],$$

and notice that (3.6), (3.14) and (3.15) yield

$$\begin{aligned}\mathcal{Y}(t,g(t)+\mathcal{X}(t,z,h),h) &= W(t,z,h) + z(g(t)+\mathcal{X}(t,z,h)-g(t)) \\ &= W(t,z,h) + z\mathcal{X}(t,z,h) \le \widehat{V}(t,x,h),\end{aligned}$$

where the last inequality is due to $\mathcal{Y}(t,g(t)+\mathcal{X}(t,z,h),h) \le \widehat{V}(t,x,h)$. The last display inequality thus provides

$$W(t,z,h) \le \sup_{x>g(t)}[\widehat{V}(t,x,h) - z(x-g(t))] \quad \text{and} \quad \widehat{V}(t,x,h) \ge \inf_{z>0}[W(t,z,h)+z(x-g(t))].$$

It thus remains only to show that equality (3.15) indeed holds. As a matter of fact, Lemma C.1 guarantees the existence of a candidate optimal portfolio process $\pi^*$ such that $(c^*,\pi^*) \in \mathcal{A}_t(t,x,h)$ and (3.15) holds, where $c_s^* = \mathcal{I}^u(Z_s^2, H_s^2)$ is candidate optimal consumption process. By Theorem 3.6.3 in [Karatzas and Shreve, 1998b] or Lemma 6.2 in [Karatzas and Wang, 2000], one can then show that $(c^*,\pi^*)$ is optimal for the optimization problem $\widehat{V}$.

□

3.3. **The dual optimal stopping problem.** From the agent's problem in (2.7), by the dynamic programming principle we can deduce that for any $(t,x,h) \in \mathcal{O}$,

(3.16)
$$V(t,x,h) = \sup_{(c,\pi,\tau)\in\mathcal{A}(t,x,h)} \mathbb{E}_{t,x,h}\left[\int_t^\tau e^{-\int_t^s(\rho+M_u^{H^1})du}u(c_s,H_s^1)ds + e^{-\int_t^\tau(\rho+M_u^{H^1})du}\widehat{V}(\tau,X_\tau,H_\tau^1)\right].$$

In the sequel, whenever necessary, we write $X^x$ to show the dependency on the initial datum (similarly, $H^{1,h}$ denotes the process $H^1$ with initial state $h$). Now, for any $(t,x,h) \in \mathcal{O}$ and Lagrange multiplier $z>0$, from the budget constraint (3.3) and (3.16), letting $P^1_s(h) := \gamma_{s,t} e^{\int_t^s (\rho + M_u^{H^{1,h}})du}$ and $M_s^{H^{1,h}} = m^0 + m^1 (H^{1,h}_s)^{-\kappa}$ with $H^1_s$ as in (2.3), we have

$$
\begin{aligned}
&\mathbb{E}_{t,x,h}\bigg[\int_t^T e^{-\int_t^s(\rho+M_u^H)du} u(c_s,H_s)ds\bigg] \\
&\leq \sup_{(c,\pi,\tau)\in\mathcal{A}(t,x,h)} \mathbb{E}_{t,x,h}\bigg[\int_t^\tau e^{-\int_t^s(\rho+M_u^{H^1})du} u(c_s,H^1_s)ds + e^{-\int_t^\tau(\rho+M_u^{H^1})du}\widehat{V}(\tau,X_\tau,H^1_\tau)\bigg] \\
&- z\mathbb{E}_{t,x,h}\bigg[\gamma_{\tau,t}X_\tau + \int_t^\tau \gamma_{s,t}c_s ds\bigg] + zx \\
&= \sup_{(c,\pi,\tau)\in\mathcal{A}(t,x,h)} \mathbb{E}_{t,x,h}\bigg[\int_t^\tau e^{-\int_t^s(\rho+M_u^{H^1})du}\Big(u(c_s,H^1_s) - zP^1_s(h)c_s\Big)ds \\
&+ e^{-\int_t^\tau(\rho+M_u^{H^1})du}\widehat{V}(\tau,X_\tau,H^1_\tau) - e^{-\int_t^\tau(\rho+M_u^{H^1})du} zP^1_\tau(h)X_\tau\bigg] + zx \\
&= \sup_{(c,\pi,\tau)\in\mathcal{A}(t,x,h)} \mathbb{E}_{t,x,h}\bigg[\int_t^\tau e^{-\int_t^s(\rho+M_u^{H^1})du}\Big(u(c_s,H^1_s) - zP^1_s(h)c_s\Big)ds \\
&+ e^{-\int_t^\tau(\rho+M_u^{H^1})du}\Big(\widehat{V}(\tau,X_\tau,H^1_\tau) - zP^1_\tau(h)X_\tau + zP^1_\tau(h)g(\tau) - zP^1_\tau(h)g(\tau)\Big)\bigg] + zx
\end{aligned}
$$

(3.17)

$$
\leq \sup_{\tau\in\mathcal{S}} \mathbb{E}_{t,z,h}\bigg[\int_t^\tau e^{-\int_t^s(\rho+M_u^{H^1})du}\widehat{u}(Z^1_s,H^1_s)ds + e^{-\int_t^\tau(\rho+M_u^{H^1})du}\Big(W(\tau,Z^1_\tau,H^1_\tau) - Z^1_\tau g(\tau)\Big)\bigg] + zx,
$$

where we recall that $\widehat{u}(z,h) = \sup_{c\geq 0}[u(c,h) - cz]$, and we have defined $Z^1_s := zP^1_s(h)$ such that

$$
dZ^1_s = (\rho - r + M_s^{H^1})Z^1_s ds - \theta Z^1_s dB_s,\ s\in(t,\tau],\quad Z^1_t = z > 0. \tag{3.18}
$$

With a slight abuse in the notation, and when no confusion arise, we also write $\mathbb{E}_{t,z,h}$ to indicate the expectation under $\mathbb{P}_{t,z,h}(\cdot) := \mathbb{P}(\cdot | Z^1_t = z, H^1_t = h)$. Hence, defining the value function $J(t,z,h)$ as

(3.19)

$$
J(t,z,h) := \sup_{\tau\in\mathcal{S}} \mathbb{E}_{t,z,h}\bigg[\int_t^\tau e^{-\int_t^s(\rho+M_u^{H^1})du}\widehat{u}(Z^1_s,H^1_s)ds + e^{-\int_t^\tau(\rho+M_u^{H^1})du}\Big(W(\tau,Z^1_\tau,H^1_\tau) - Z^1_\tau g(\tau)\Big)\bigg],
$$

we have a finite-horizon, two-dimensional optimal stopping problem, with interconnected dynamics $(Z^1,H^1)$ as in (3.18) and (2.3).

In the following sections, we perform a detailed probabilistic study of (3.19). Before doing that, we have the following theorem that establishes a dual relation between the original problem (2.7) and the optimal stopping problem (3.19).

**Theorem 3.2.** *The following duality relations hold:*

$$
V(t,x,h) = \inf_{z>0}[J(t,z,h) + zx], \quad J(t,z,h) = \sup_{x>0}[V(t,x,h) - zx].
$$

*Proof.* Since $(c,\pi,\tau)\in\mathcal{A}(t,x,h)$ is arbitrary, taking the supremum over $(c,\pi,\tau)\in\mathcal{A}(t,x,h)$ on the left-hand side of (3.17), we get, for any $z>0, x>0$,

$$
V(t,x,h) \leq J(t,z,h) + zx,
$$

so that $V(t,x,h) \leq \inf_{z>0}[J(t,z,h)+zx]$ and $J(t,z,h) \geq \sup_{x>0}[V(t,x,h) - zx]$.

For the inverse inequality, observe that equality holds in (3.17) if and only if

$$
c_s = \mathcal{I}^u(Z^1_s, H^1_s), \quad W(t,z,h) = \sup_{x>g(t)}[\widehat{V}(t,x,h) - z(x - g(t))],
$$

and

$$\mathbb{E}_{t,x,h}\left[\gamma_{\tau,t}X_\tau + \int_t^\tau \gamma_{s,t}c_s ds\right] = x, \tag{3.20}$$

where we recall that $\mathcal{I}^u$ denotes the inverse of the marginal utility function $u_c(\cdot, h)$. From Lemma C.2, we know that there exists a portfolio process $\pi^*$ such that (3.20) holds. From Theorem 3.1, we also know that $W(t,z,h) = \sup_{x>g(t)}[\widehat{V}(t,x,h) - z(x-g(t))]$.

Next we define

$$\bar{\mathcal{X}}(t,z,h) := \mathbb{E}_{t,z,h}\left[\int_t^T \gamma_{s,t}\mathcal{I}^u(Z_s^1, H_s^1)ds\right], \quad \bar{\mathcal{Z}}(x) := \mathbb{E}_{t,x,h}[\gamma_{\tau,t}X_\tau],$$

and

$$\bar{\mathcal{Y}}(t,x,h) := \mathbb{E}_{t,x,h}\left[\int_t^\tau e^{-\int_t^s(\rho+M_u^{H^1})du}u(c_s, H_s^1)ds + e^{-\int_t^\tau(\rho+M_u^{H^1})du}\widehat{V}(\tau, X_\tau, H_\tau^1)\right].$$

Then by (3.17) and (3.20) we have

$$\bar{\mathcal{Y}}(t, \bar{\mathcal{X}}(t,z,h) + \bar{\mathcal{Z}}(x), h) = J(t,z,h) + z(\bar{\mathcal{X}}(t,z,h) + \bar{\mathcal{Z}}(x)) \leq V(t,x,h),$$

where the last inequality is due to $\bar{\mathcal{Y}}(t, \bar{\mathcal{X}}(t,z,h) + \bar{\mathcal{Z}}(x), h) \leq V(t,x,h)$. This in turn gives

$$V(t,x,h) \geq \inf_{z>0}[J(t,z,h) + zx],$$

which completes the proof. $\square$

## 4. Study of the dual optimal stopping problem

4.1. **Preliminary properties of the value function.** To study the optimal stopping problem (3.19), we find it convenient to introduce the function

$$\widehat{J}(t,z,h) := J(t,z,h) - \widehat{W}(t,z,h) \tag{4.1}$$

with

$$\widehat{W}(t,z,h) := W(t,z,h) - zg(t). \tag{4.2}$$

Applying Itô's formula to $\{e^{-\int_t^s(\rho+M_u^{H^1})du}[W(s,Z_s^1,H_s^1) - Z_s^1 g(s)], s \in [t,\tau]\}$, and taking conditional expectations we have

$$\begin{aligned}&\mathbb{E}_{t,z,h}\left[e^{-\int_t^\tau(\rho+M_s^{H^1})ds}\Big(W(\tau, Z_\tau^1, H_\tau^1) - Z_\tau^1 g(\tau)\Big)\right] = W(t,z,h) - zg(t) + \\ &\mathbb{E}_{t,z,h}\left[\int_t^\tau e^{-\int_t^s(\rho+M_u^{H^1})du}\mathcal{L}\Big(W(s,Z_s^1,H_s^1) - Z_s^1 g(s)\Big)ds\right],\end{aligned}$$

where, for any $F \in C^{1,2,1}(\mathcal{O})$, the second order differential operator $\mathcal{L}$ is such that

$$\mathcal{L}F := F_t + \frac{1}{2}\theta^2 z^2 F_{zz} + (\rho - r + m^0 + m^1 h^{-\kappa})zF_z - \delta h F_h - (\rho + m^0 + m^1 h^{-\kappa})F. \tag{4.3}$$

Combining (3.19), (4.1) and (4.2), we have

$$\begin{aligned}\widehat{J}(t,z,h) &= \sup_{t\leq\tau\leq T}\mathbb{E}_{t,z,h}\left[\int_t^\tau e^{-\int_t^s(\rho+M_u^{H^1})du}\widehat{u}(Z_s^1, H_s^1)ds\right. \\ &\quad \left.+ \int_t^\tau e^{-\int_t^s(\rho+M_u^{H^1})du}\Big(\mathcal{L}(W(s,Z_s^1,H_s^1) - Z_s^1 g(s))\Big)ds\right] \\ &= \sup_{t\leq\tau\leq T}\mathbb{E}_{t,z,h}\left[\int_t^\tau e^{-\int_t^s(\rho+M_u^{H^1})du}\Big(Z_s^1 I - f(I)W_h(s,Z_s^1,H_s^1)\Big)ds\right] \\ &= \sup_{t\leq\tau\leq T}\mathbb{E}_{t,z,h}\left[\int_t^\tau z\gamma_{s,t} I ds - \int_t^\tau e^{-\int_t^s(\rho+M_u^{H^1})du}f(I)W_h(s,Z_s^1,H_s^1)ds\right],\end{aligned} \tag{4.4}$$

where we have used the fact that (cf. (3.8))

$$\begin{aligned}\mathcal{L}(W(s,z,h)-zg(s)) &= \mathcal{L}W(s,z,h)-\mathcal{L}(zg(s)) \\ &= \widehat{\mathcal{L}}W(s,z,h)-f(I)W_h(s,z,h)-\mathcal{L}(zg(s)) \\ &= -\widehat{u}(z,h)-f(I)W_h(s,z,h)+Iz,\end{aligned}$$

with $\mathcal{L}F=\widehat{\mathcal{L}}F-f(I)W_h$ (cf. (3.9)), for any $F\in C^{1,2,1}(\mathcal{O})$.

Notice now that the process $(Z^1,H^1)$ is time-homogeneous, so that

$$\mathrm{Law}[(u,Z^1_u,H^1_u)_{u\geq t}|Z^1_t=z,H^1_t=h]=\mathrm{Law}[(t+s,Z^1_s,H^1_s)_{s\geq 0}|Z^1_0=z,H^1_0=h].$$

Let $\mathbb{E}_{z,h}$ be the expectation under $\mathbb{P}_{z,h}(\cdot):=\mathbb{P}(\cdot|Z^1_0=z,H^1_0=h)$. Hence, from (4.4),

$$\begin{aligned}\widehat{J}(t,z,h) &= \sup_{0\leq\tau\leq T-t}\mathbb{E}_{z,h}\bigg[\int_0^\tau e^{-\int_0^s(\rho+M_u^{H^1})du}\Big(Z^1_sI-f(I)W_h(t+s,Z^1_s,H^1_s)\Big)ds\bigg] \\ &= \sup_{0\leq\tau\leq T-t}\mathbb{E}_{z,h}\bigg[\int_0^\tau ze^{-rs-\theta B_s-\frac{1}{2}\theta^2 s}Ids-\int_0^\tau e^{-\int_0^s(\rho+M_u^{H^1})du}f(I)W_h(t+s,Z^1_s,H^1_s)ds\bigg],\end{aligned} \tag{4.5}$$

with (cf. also (3.18))

$$Z^1_s=z\gamma_{s,0}e^{\int_0^s(\rho+M_u^{H^1})du}\quad\text{and}\quad H^1_s=he^{-\delta s}. \tag{4.6}$$

In the following, when needed, we shall write $Z^{1,z}$ as the solution to (3.18) such that $Z^1_0=z>0$. As a matter of fact, the state process in (4.5) is the time-space Markov process $(Y_s)_{s\in[0,T-t]}$ defined by $Y_0=(t,z,h)$ and $Y_s:=(t+s,Z^1_s,H^1_s)$.

As usual in optimal stopping theory, we let

$$\mathcal{W}:=\{(t,z,h)\in\mathcal{O}:\widehat{J}(t,z,h)>0\},\quad \mathcal{I}:=\{(t,z,h)\in\mathcal{O}:\widehat{J}(t,z,h)=0\} \tag{4.7}$$

be the so-called continuation (waiting) and stopping (investing) regions, respectively. We denote by $\partial\mathcal{W}$ the boundary of the set $\mathcal{W}$.

Since, for any stopping time $\tau$, the mapping $(z,h)\to\mathbb{E}_{z,h}\Big[\int_0^\tau e^{-\int_0^s(\rho+M_u^{H^1})du}\Big(Z^1_sI-f(I)W_h(t+s,Z^1_s,H^1_s)\Big)ds\Big]$ is continuous, then $\widehat{J}$ is lower semicontinuous on $\mathcal{O}$. Hence, $\mathcal{W}$ is open, $\mathcal{I}$ is closed, and introducing the stopping time

$$\tau^*(t,z,h):=\inf\{s\geq 0:(t+s,Z^1_s,H^1_s)\in\mathcal{I}\}\wedge(T-t),\quad \mathbb{P}_{z,h}-a.s.,$$

with $\inf\emptyset=+\infty$, one has that $\tau^*(t,z,h)$ is optimal for $\widehat{J}(t,z,h)$ (see, e.g., Corollary I.2.9 in [Peskir and Shiryaev, 2006]).

**Proposition 4.1.** *The function $\widehat{J}$ is such that $0\leq\widehat{J}(t,z,h)\leq\frac{Iz}{r}(1-e^{-r(T-t)})$ for all $(t,z,h)\in\mathcal{O}$.*

*Proof.* From (4.5) it is clear that $\widehat{J}$ is nonnegative. Moreover, again from (4.5), and since $W_h(t,z,h)\geq 0$ (cf. Lemma A.1), we find that

$$\begin{aligned}&\sup_{0\leq\tau\leq T-t}\mathbb{E}_{z,h}\bigg[\int_0^\tau ze^{-rs-\theta B_s-\frac{1}{2}\theta^2 s}Ids-\int_0^\tau e^{-\int_0^s(\rho+M_u^{H^1})du}f(I)W_h(t+s,Z^1_s,H^1_s)ds\bigg] \\ &\leq\mathbb{E}_{z,h}\bigg[\int_0^{T-t}ze^{-rs-\theta B_s-\frac{1}{2}\theta^2 s}Ids\bigg]=\frac{Iz}{r}(1-e^{-r(T-t)}),\end{aligned}$$

which implies the claim. □

The next lemma shows that $\mathcal{I}$ as in (4.7) is nonempty.

**Lemma 4.1.** *One has $\mathcal{I}\neq\emptyset$.*

*Proof.* Suppose that $\mathcal{I} = \emptyset$, then for all $(t, z, h) \in \mathcal{O}$ we have

$$0 \le \widehat{J}(t, z, h) = \mathbb{E}\Big[\int_0^{T-t} z e^{-rs-\theta B_s - \frac{1}{2}\theta^2 s} I ds - \int_0^{T-t} e^{-\int_0^s (\rho + M_u^{H^{1,h}}) du} f(I) W_h(t+s, Z_s^{1,z}, H_s^{1,h}) ds\Big].$$

However, taking $z \downarrow 0$, the right-hand side above converges to $-\infty$ due to $\lim_{z\to 0} W_h(t, z, h) = \infty$ (as shown in Lemma A.1), which is a contradiction. □

Since $W_h(t, \cdot, h)$ is strictly decreasing (cf. (A.1) in Lemma A.1 in the Appendix A), the next monotonicity of $\widehat{J}$ follows.

**Proposition 4.2.** *$z \mapsto \widehat{J}(t, z, h)$ is non-decreasing for all $(t, h) \in [0, T) \times \mathbb{R}_+$.*

On the other hand, we notice that it is hard to determine whether $h \to \widehat{J}(t, z, h)$ is monotonic or not, since it is not clear if $h \to e^{-\int_0^s (\rho + M_u^{H^1}) du} f(I) W_h(t+s, Z_s^1, H_s^1)$ in (4.5) is monotonic. Similarly, we cannot conclude that $t \to \widehat{J}(t, z, h)$ is monotonic.

The next technical result states properties of $\widehat{J}$ that will be useful in the study of the regularity of the boundary $\partial\mathcal{W}$.

**Proposition 4.3.** *The function $\widehat{J}$ is locally Lipschitz continuous on $\mathcal{O}$ and for a.e. $(t, z, h)$ we have the following probabilistic representation formulas*

$$\begin{aligned}\widehat{J}_h(t, z, h) = -f(I)\mathbb{E}_{z,h}\Big[\int_0^{\tau^*} & e^{-\int_0^s (\rho + M_u^{H^1}) du}\Big[W_{hh}(t+s, Z_s^1, H_s^1)e^{-\delta s} \\ & + \frac{1}{1-\alpha} W_h(t+s, Z_s^1, H_s^1)\frac{m^1}{\delta} h^{-\kappa-1}(e^{\delta\kappa s} - 1)\Big] ds\Big],\end{aligned} \tag{4.8}$$

$$\widehat{J}_z(t, z, h) = \mathbb{E}_{z,h}\Big[\int_0^{\tau^*} e^{-rs-\theta B_s - \frac{1}{2}\theta^2 s}\Big(I - f(I) W_{hz}(t+s, Z_s^1, H_s^1)\Big) ds\Big], \tag{4.9}$$

*and*

$$\widehat{J}_t(t, z, h) = -f(I)\mathbb{E}_{z,h}\Big[\int_0^{\tau^*} e^{-\int_0^s (\rho + M_u^{H^1}) du} W_{ht}(t+s, Z_s^1, H_s^1) ds\Big], \tag{4.10}$$

*where $\tau^* := \tau^*(t, z, h)$ is the optimal stopping time for the problem with initial data $(t, z, h)$.*

*Proof.* The proof is given in Appendix B.1. □

We conclude with asymptotic limits of $\widehat{J}$.

**Proposition 4.4.** $\lim_{z\to 0} \widehat{J}(t, z, h) = 0$*,* $\lim_{z\to\infty} \widehat{J}(t, z, h) = \infty$ *for all $(t, h) \in [0, T) \times \mathbb{R}_+$.*

*Proof.* The proof is given in Appendix B.2. □

4.2. **Properties of the free boundary.** In this section, we show that the boundary $\partial\mathcal{W}$ can be represented by a function $b(t, h)$. We establish connectedness of the sets $\mathcal{W}$ and $\mathcal{I}$ with respect to the $z$-variable and finally prove (local) Lipshitz-continuity of $b$ with respect to both its variables.

First, we provide the shape of the continuation and stopping regions. Defining $\Gamma(t, h)$ as

$$\begin{aligned}\Gamma(t, h) := (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\Big[\int_0^{T-t} & e^{\frac{\alpha}{1-\alpha}(rs + \frac{1}{2}\theta^2 s) + \frac{\theta^2\alpha^2 s}{2(\alpha-1)^2}} e^{\int_0^s \left(\frac{\rho + M_u^{H^{2,h}}(h)}{\alpha - 1}\right) du} \times \\ & \times \Big[\frac{m^1\kappa}{1-\alpha}\Big(\int_0^s \big(he^{-\delta u} + \frac{f(I)}{\delta}(1 - e^{-\delta u})\big)^{-\kappa-1} e^{-\delta u} du\Big)\Big(he^{-\delta s} + \frac{f(I)}{\delta}(1 - e^{-\delta s})\Big) + e^{-\delta s}\Big] ds\Big],\end{aligned} \tag{4.11}$$

for all $(t, h) \in [0, T] \times \mathbb{R}_+$, the following result holds.

**Lemma 4.2.** *There exists a free boundary*

$$b : [0, T] \times \mathbb{R}_+ \to [0, \infty)$$

*such that*

$$\mathcal{I} = \{(t, z, h) \in \mathcal{O} : 0 < z \leq b(t, h)\}.$$

*Moreover, setting* $g(t, h) := (\frac{I}{f(I)})^{\alpha-1}\Gamma(t, h)^{1-\alpha}$*, one has* $b(t, h) \leq g(t, h)$ *for all* $(t, h) \in [0, T] \times \mathbb{R}_+$*, where* $\Gamma(t, h)$ *is defined in (4.11).*

*Proof.* Since $z \mapsto \widehat{J}(t, z, h)$ is nondecreasing by Proposition 4.2, we can define $b(t, h) := \sup\{z > 0 : \widehat{J}(t, z, h) \leq 0\}$ (with the convention $\sup \emptyset = 0$), so that $\mathcal{I} = \{(t, z, h) \in \mathcal{O} : 0 < z \leq b(t, h)\}$. Notice that $b > 0$ on $[0, T) \times \mathbb{R}_+$ since $\mathcal{I} \neq \emptyset$ by Lemma 4.1.

Next we show $b(t, h) \leq g(t, h)$. Noticing that, due to (4.5),

$$\mathcal{R} := \{(t, z, h) \in \mathcal{O} : zI - f(I)W_h(t, z, h) > 0\} \subseteq \mathcal{W},$$

we have

$$\mathcal{R}^C := \{(t, z, h) \in \mathcal{O} : zI - f(I)W_h(t, z, h) \leq 0\} \supseteq \mathcal{I}. \tag{4.12}$$

Recalling $W_h(t, z, h)$ as in (A.1), and using (4.11), we then write $W_h(t, z, h) = z^{\frac{\alpha}{\alpha-1}}\Gamma(t, h)$, so that $zI - f(I)W_h(t, z, h) \leq 0 \Leftrightarrow z^{\frac{1}{\alpha-1}} f(I)\Gamma(t, h) \geq I$ for all $(t, h) \in [0, T] \times \mathbb{R}_+$. But then, since $0 < \alpha < 1$, we have

$$(t, h, z) \in \mathcal{R}^C \iff z \leq \left(\frac{I}{f(I)\Gamma(t, h)}\right)^{\alpha-1} = \left(\frac{I}{f(I)}\right)^{\alpha-1}\Gamma(t, h)^{1-\alpha}.$$

Because $\mathcal{I} = \{(t, z, h) \in \mathcal{O} : 0 < z \leq b(t, h)\}$, by (4.12) we find

$$b(t, h) \leq g(t, h) = \left(\frac{I}{f(I)}\right)^{\alpha-1}\Gamma(t, h)^{1-\alpha}. \tag{4.13}$$

□

The first main result of this paper shows that the optimal boundary is locally Lipschitz-continuous on $[0, T] \times \mathbb{R}_+$. The local Lipschitz-continuity of the boundary has important consequences regarding the regularity of the value function $\widehat{J}$, as we will see in Proposition 4.6 below.

**Theorem 4.1.** *The free boundary* $b$ *is locally Lipschitz-continuous on* $[0, T] \times \mathbb{R}_+$*.*

*Proof.* The proof is given in Appendix B.3.

□

4.3. **Characterization of the free boundary and of the value function.** Given that $b$ is locally Lipschitz, the law of the iterated logarithm allows to prove the following result.

**Lemma 4.3.** *Let* $(t, z, h) \in \mathcal{O}$ *and set*

$$\hat{\tau}(t, z, h) := \inf\{s \geq 0 : Z_s^{1,z} < b(t + s, H_s^{1,h})\} \wedge (T - t).$$

*Then* $\hat{\tau}(t, z, h) = \tau^*(t, z, h)$ *a.s., where* $\tau^*(t, z, h) = \inf\{s \geq 0 : Z_s^{1,z} \leq b(t + s, H_s^{1,h})\} \wedge (T - t)$*.*

*Proof.* The proof is given in Appendix B.4.

□

The previous lemma in turn yields the following continuity property of $\tau^*$, which will then be fundamental in the proof of Proposition 4.6 below.

**Proposition 4.5.** *One has that* $\mathcal{O} \ni (t, z, h) \mapsto \tau^*(t, z, h) \in [0, T - t]$ *is continuous.*

*Proof.* The proof exploits arguments as in the proof of Proposition 5.2 in [De Angelis and Ekström, 2017]. □

**Proposition 4.6.** *The value function* $\widehat{J} \in C^{1,1,1}(\mathcal{O}) \cap C^{1,2,1}(\mathcal{W})$ *and solves the boundary value problem*

$$\begin{cases} \mathcal{L}\widehat{J}(t,z,h) = -Iz + f(I)W_h(t,z,h), \ (t,z,h) \in \mathcal{W}, \\ \widehat{J}(t,z,h) = 0, \ (t,z,h) \in \mathcal{I} \cap \{t < T\}, \\ \widehat{J}(T,z,h) = 0, \ (z,h) \in \mathbb{R}^2_+, \\ \widehat{J}_t(t,z,h) = \widehat{J}_z(t,z,h) = \widehat{J}_h(t,z,h) = 0 \ \text{on} \ \partial\mathcal{W} \cap \{t < T\}. \end{cases}$$

*Moreover, for all* $\epsilon > 0$, $\widehat{J}_{zz}$ *admits a continuous extension to the closure of* $\mathcal{W} \cap \{t < T - \epsilon\}$.

*Proof.* First we show that the function $\widehat{J}$ is continuously differentiable over $\mathcal{O}$. From the representations of $\widehat{J}_h, \widehat{J}_t$ and $\widehat{J}_z$ in Proposition 4.3, and the continuity of $(t,z,h) \mapsto \tau^*(t,z,h)$ (cf. Proposition 4.5), we conclude that those weak derivatives are in fact continuous and therefore that $\widehat{J} \in C^{1,1,1}(\mathcal{W}) \cap C^{1,1,1}(\mathring{\mathcal{I}})$, where $\mathring{\mathcal{I}}$ denotes the interior of $\mathcal{I}$. In particular, $\widehat{J}_t = \widehat{J}_h = \widehat{J}_z = 0$ on $\mathring{\mathcal{I}}$. It thus remains to analyze the regularity of $\widehat{J}$ across $\partial\mathcal{W}$.

Fix a point $(t_0, z_0, h_0) \in \partial\mathcal{W} \cap \{t < T\}$ and take a sequence $(t_n, z_n, h_n)_{n\geq 1} \subseteq \mathcal{W}$ with $(t_n, z_n, h_n) \to (t_0, z_0, h_0)$ as $n \to \infty$. Continuity of $(t,z,h) \mapsto \tau^*(t,z,h)$ implies that $\tau^*(t_n, z_n, h_n) \to \tau^*(t_0, z_0, h_0) = 0, \mathbb{P}$-a.s. as $n \to \infty$. Again, from Proposition 4.3, dominated convergence yields that $\widehat{J}_h(t_n, z_n, h_n) \to 0, \widehat{J}_z(t_n, z_n, h_n) \to 0$ and $\widehat{J}_t(t_n, z_n, h_n) \to 0$. Since $(t_0, z_0, h_0)$ and the sequence $(t_n, z_n, h_n)$ were arbitrary, we get $\widehat{J} \in C^{1,1,1}(\mathcal{O})$.

On the other hand, by Corollary 6 in [Peskir, 2022], $\widehat{J}$ solves in the sense of distributions

$$\mathcal{L}\widehat{J}(t,z,h) = -Iz + f(I)W_h(t,z,h), \ (t,z,h) \in \mathcal{W}, \tag{4.14}$$

and it is such that

$$\begin{aligned} \widehat{J}(t,z,h) &= 0, \ (t,z,h) \in \mathcal{I} \cap \{t < T\}, \\ \widehat{J}(T,z,h) &= 0, \ (z,h) \in \mathbb{R}^2_+. \end{aligned}$$

However, $\widehat{J} \in C^{1,1,1}(\mathcal{O})$, so that $\widehat{J}_{zz} \in C^0(\mathcal{W})$, upon recalling the definition of $\mathcal{L}$ as in (4.3) and that $W_h \in C^0(\mathcal{O})$. Then, taking any $(t_0, z_0, h_0) \in \partial\mathcal{W} \cap \{t < T\}$ and making limits in (4.14) as $(t,z,h) \to (t_0, z_0, h_0)$ with $(t,z,h) \in \mathcal{W}$, we also find

$$\lim_{(t,z,h)\to(t_0,z_0,h_0)} \frac{\theta^2 z^2}{2} \widehat{J}_{zz}(t,z,h) = -Iz_0 + f(I)W_h(t_0, z_0, h_0),$$

upon using $\widehat{J}(t_0, z_0, h_0) = \widehat{J}_h(t_0, z_0, h_0) = \widehat{J}_z(t_0, z_0, h_0) = \widehat{J}_t(t_0, z_0, h_0) = 0$. This shows, as claimed, that $\widehat{J}_{zz}$ admits a continuous extension to the closure of $\mathcal{W} \cap \{t < T - \epsilon\}$, for all $\epsilon > 0$. ☐

**Corollary 4.1.** *Recall (4.1). The function* $J \in C^{1,2,1}(\mathcal{W}) \cap C^{1,1,1}(\mathcal{O})$ *and solves the boundary value problem*

$$\begin{aligned} &\mathcal{L}J(t,z,h) = -\widehat{u}(z,h), \ (t,z,h) \in \mathcal{W}, \\ &J(t,z,h) = \widehat{W}(t,z,h), \ (t,z,h) \in \mathcal{I} \cap \{t < T\}, \\ &J(T,z,h) = \widehat{W}(T,z,h) = 0, \ (z,h) \in \mathbb{R}^2_+, \\ &J_t(t,z,h) = \widehat{W}_t(t,z,h), \ J_z(t,z,h) = \widehat{W}_z(t,z,h), \ J_h(t,z,h) = \widehat{W}_h(t,z,h) \ \text{on} \ \partial\mathcal{W} \cap \{t < T\}. \end{aligned}$$

**Remark 4.1.** *It is worth noting that standard PDE arguments could not be directly applied in the proof of Proposition 4.6 due to the fully degenerate diffusion process* $(Z^1, H^1)$. *Therefore, we had to hinge on a novel series of intermediate results. First, we find the locally Lipschitz continuity of* $\widehat{J}$ *(cf. Proposition 4.3) and then establish the locally Lipschitz continuity of free boundary without relying upon continuity of* $\widehat{J}_z, \widehat{J}_h$ *and* $\widehat{J}_t$ *(cf. Theorem 4.1). Finally, we upgrade the regularity of* $\widehat{J}$ *using the continuity of the optimal stopping time (cf. Propositions 4.5 and 4.6).*

We are now in the conditions of determining a nonlinear integral equation that characterizes uniquely the free boundary. As a byproduct, such a characterization will result also into an integral representation for the value function $\widehat{J}$. This is accomplished by the next theorem, which exploits the regularity properties of $\widehat{J}$ proved so far.

**Theorem 4.2.** *For all $(t,z,h) \in \mathcal{O}$, $\widehat{J}$ from (4.5) has the representation*

(4.15)
$$\widehat{J}(t,z,h) = \mathbb{E}\bigg[\int_0^{T-t} e^{-\int_0^s (\rho + M_u^{H^{1,h}})du}\Big(IZ_s^{1,z} - f(I)W_h(t+s, Z_s^{1,z}, H_s^{1,h})\Big)\mathbb{1}_{\{Z_s^{1,z} \geq b(t+s, H_s^{1,h})\}}ds\bigg].$$

*Moreover, the optimal boundary $b$ is the unique continuous solution to the following nonlinear integral equation: For all $(t,h) \in [0,T] \times \mathbb{R}_+$,*

(4.16)
$$0 = \mathbb{E}\bigg[\int_0^{T-t} e^{-\int_0^s (\rho + M_u^{H^{1,h}})du}\Big(IZ_s^{1,b(t,h)} - f(I)W_h(t+s, Z_s^{1,b(t,h)}, H_s^{1,h})\Big)\mathbb{1}_{\{Z_s^{1,b(t,h)} \geq b(t+s, H_s^{1,h})\}}ds\bigg],$$

*with $\lim_{t\uparrow T} b(t,h) = 0$ and such that $0 \leq b(t,h) \leq g(t,h)$ (cf. Lemma 4.2).*

*Proof.* **Step 1**. We start by deriving (4.15). Let $(t,z,h) \in \mathcal{O}$ be given and fixed, let $(K_m)_{m\geq 0}$ be a sequence of compact sets increasing to $[0,T] \times \mathbb{R}_+^2$ and define

$$\tau_m := \inf\{s \geq 0 : (t+s, Z_s^{1,z}, H_s^{1,h}) \notin K_m\} \wedge (T-t).$$

Since $\widehat{J} \in C^{1,1,1}(\mathcal{O})$, $\widehat{J}_{zz} \in L^\infty_{loc}(\mathcal{O})$, and $\mathbb{P}[(t+s, Z_s^{1,z}, H_s^{1,h}) \in \partial\mathcal{W}] = 0$ for all $s \in [0, T-t)$, we can apply a weak version of Dynkin's formula (see, e.g., [Bensoussan and Lions, 1982], Lemma 8.1 and Th. 8.5, pp. 183-186) so to obtain

$$\widehat{J}(t,z,h) = \mathbb{E}\bigg[e^{-\int_0^{\tau_m}(\rho + M_u^{H^{1,h}})du}\widehat{J}(t+\tau_m, Z_{\tau_m}^{1,z}, H_{\tau_m}^{1,h}) - \int_0^{\tau_m} e^{-\int_0^s(\rho + M_u^{H^{1,h}})du}\mathcal{L}\widehat{J}(t+s, Z_s^{1,z}, H_s^{1,h})ds\bigg].$$

Therefore, using (4.14), we also find

$$\begin{aligned}\widehat{J}(t,z,h) = \mathbb{E}\bigg[&e^{-\int_0^{\tau_m}(\rho + M_u^{H^{1,h}})du}\widehat{J}(t+\tau_m, Z_{\tau_m}^{1,z}, H_{\tau_m}^{1,h}) \\ &+ \int_0^{\tau_m} e^{-\int_0^s(\rho + M_u^{H^{1,h}})du}\Big(IZ_s^{1,z} - f(I)W_h(t+s, Z_s^{1,z}, H_s^{1,h})\Big)\mathbb{1}_{\{Z_s^{1,z} \geq b(t+s, H_s^{1,h})\}}ds\bigg],\end{aligned}$$

where we have used again that $\mathbb{P}[(t+s, Z_s^{1,z}, H_s^{1,h}) \in \partial\mathcal{W}] = 0$.

Finally, we take $m \uparrow \infty$, apply the dominated convergence theorem, and use that $\tau_m \uparrow (T-t)$ and $\widehat{J}(T,z,h) = 0$ (cf. Proposition 4.6) to obtain (4.15).

**Step 2.** Next, we find the limit value of $b(t,h)$ when $t \to T$. Firstly, the limit $b(T-,h) := \lim_{t\to T} b(t,h)$ exists, since $b$ is locally Lipschitz on $[0,T] \times \mathbb{R}_+$. Noticing that $b(t,h) \leq g(t,h)$ for all $(t,h) \in [0,T] \times \mathbb{R}_+$ by Lemma 4.2, we find $0 \leq b(T-,h) \leq g(T,h) = 0$ due to (4.13), which proves the claim.

**Step 3.** Given that (4.15) holds for any $(t,z,h) \in \mathcal{O}$, we can take $z = b(t,h)$ in (4.15), which leads to (4.16), upon using that $\widehat{J}(t, b(t,h), h) = 0$ (cf. Proposition 4.6). The fact that $b$ is the unique continuous solution to (4.16) can be proved by following the four-step procedure from the proof of uniqueness provided in Theorem 3.1 of [Peskir, 2005]. Since the present setting does not create additional difficulties we omit further details.

□

## 5. Optimal solution in terms of the primal variables

In the previous section, we studied the properties of the dual value function $J(t,z,h)$ and used $(t,z,h)$, where $t$ denotes time, $z$ denotes marginal utility and $h$ denotes health capital, as the coordinate system for the study. In this section, we will come back to study of the value function $V(t,x,h)$ in the original coordinate system $(t,x,h)$, where $x$ denotes the wealth of the agent.

**Proposition 5.1.** *The function $J$ in (3.19) is strictly convex with respect to $z$.*

*Proof.* By the duality relation in Theorem 3.2, we see that $z \mapsto J(t,z,h)$ is convex on $\mathbb{R}_+$. From Corollary 4.1, we know that $J(t,z,h) = \widehat{W}(t,z,h)$ for all $(t,z,h) \in \mathcal{I}$. Since $W$ is strictly convex with respect to $z$ (cf. (3.12)), and recalling that $\widehat{W} = W - zg(t)$, we conclude that $J$ is strictly convex with respect to $z$ for $(t,z,h) \in \mathring{\mathcal{I}}$.

To prove strict convexity of $J$ on $\mathcal{W}$, our argument is inspired by Lemma A.7 in [Federico et al., 2017]. By Corollary 4.1, $J$ satisfies in the classical sense on $\mathcal{W}$ the linear PDE

$$J_t + \frac{1}{2}\theta^2 z^2 J_{zz} + (\rho - r + m^0 + m^1 h^{-\kappa}) z J_z - \delta h J_h - (\rho + m^0 + m^1 h^{-\kappa}) J = -\widehat{u}(z,h). \tag{5.1}$$

Fix now $(t_0,z_0,h_0) \in \mathcal{W}$ and let $B_R(t_0,z_0,h_0)$ be the open ball of radius $R > 0$ centered in $(t_0,z_0,h_0)$, such that $\overline{B_R(t_0,z_0,h_0)} \subset \mathcal{W}$. Then, for $(t,z,h) \in B_R(t_0,z_0,h_0)$,

$$J_t + \frac{1}{2}\theta^2 z^2 J_{zz} + (\rho - r + m^0 + m^1 h^{-\kappa}) z J_z - \delta h J_h - (\rho + m^0 + m^1 h^{-\kappa}) J + \widehat{u}(z,h) = 0, \tag{5.2}$$

and, for $\epsilon > 0$ small enough,

$$\begin{aligned} &J_t(t,z+\epsilon,h) + \frac{1}{2}\theta^2 z^2 J_{zz}(t,z+\epsilon,h) + (\rho - r + m^0 + m^1 h^{-\kappa}) z J_z(t,z+\epsilon,h) \\ &- \delta h J_h(t,z+\epsilon,h) - (\rho + m^0 + m^1 h^{-\kappa}) J(t,z+\epsilon,h) + \widehat{u}(z+\epsilon,h) \\ &+ \frac{1}{2}\theta^2 \epsilon^2 J_{zz}(t,z+\epsilon,h) + \theta^2 z\epsilon J_{zz}(t,z+\epsilon,h) + (\rho - r + m^0 + m^1 h^{-\kappa}) \epsilon J_z(t,z+\epsilon,h) = 0. \end{aligned} \tag{5.3}$$

Hence, setting

$$J^\epsilon(t,z,h) := \frac{J(t,z+\epsilon,h) - J(t,z,h)}{\epsilon},$$

we find from (5.2) and (5.3)

$$\begin{aligned} &J^\epsilon_t + \frac{1}{2}\theta^2 z^2 J^\epsilon_{zz} + (\rho - r + m^0 + m^1 h^{-\kappa}) z J^\epsilon_z - \delta h J^\epsilon_h - (\rho + m^0 + m^1 h^{-\kappa}) J^\epsilon + \frac{\widehat{u}(z+\epsilon,h) - \widehat{u}(z,h)}{\epsilon} \\ &+ \frac{1}{2}\theta^2 \epsilon J_{zz}(t,z+\epsilon,h) + \theta^2 z J_{zz}(t,z+\epsilon,h) + (\rho - r + m^0 + m^1 h^{-\kappa}) J_z(t,z+\epsilon,h) = 0. \end{aligned}$$

Since $J$ is continuously differentiable over $[0,T) \times \mathbb{R}^2_+$ (cf. Corollary 4.1), then $J^\epsilon \to J_z$ locally uniformly over $[0,T) \times \mathbb{R}^2_+$. On the other hand, by continuity of $J_z$ and $J_{zz}$, we have that $J_z(t,z+\epsilon,h) \to J_z(t,z,h)$ and $J_{zz}(t,z+\epsilon,h) \to J_{zz}(t,z,h)$. Hence, by Proposition 5.9 in Chapter 4 of [Yong and Zhou, 1999] we have that $v := J_z$ is a viscosity solution to

$$v_t + \frac{1}{2}\theta^2 z^2 v_{zz} + (\rho - r + m^0 + m^1 h^{-\kappa} + \theta^2) z v_z - \delta h v_h - r v + \widehat{u}_z = 0, \text{ on } B_R(t_0,z_0,h_0)$$

with boundary condition $v(t,z,h) = J_z(t,z,h)$ on $\partial B_R(t_0,z_0,h_0)$ and $v(T,z,h) = 0$.

Let us now repeat the argument, and define

$$v^\epsilon(t,z,h) := \frac{v(t,z+\epsilon,h) - v(t,z,h)}{\epsilon},$$

which, due again to Corollary 4.1, converges uniformly over compacts of $\mathcal{W}$ to $v_z$, i.e. $J_{zz}$. Also, $v^\epsilon$ is a viscosity solution to

$$\begin{aligned} &v^\epsilon_t + \frac{1}{2}\theta^2 z^2 v^\epsilon_{zz} + (\rho - r + m^0 + m^1 h^{-\kappa} + \theta^2) z v^\epsilon_z - \delta h v^\epsilon_h - r v^\epsilon + \frac{\widehat{u}_z(z+\epsilon,h) - \widehat{u}_z(z,h)}{\epsilon} \\ &+ \frac{1}{2}\theta^2 \epsilon v_{zz}(t,z+\epsilon,h) + \theta^2 z v_{zz}(t,z+\epsilon,h) + (\rho - r + m^0 + m^1 h^{-\kappa} + \theta^2) v_z(t,z+\epsilon,h) = 0. \end{aligned}$$

Applying once more Proposition 5.9 in Chapter 4 of [Yong and Zhou, 1999] we have that $w := v_z$ solves in the viscosity sense on $B_R(t_0, z_0, h_0)$

(5.4)

$$w_t + \frac{1}{2}\theta^2 z^2 w_{zz} + (\rho - r + m^0 + m^1 h^{-\kappa} + 2\theta^2) z w_z - \delta h w_h + (\rho - 2r + m^0 + m^1 h^{-\kappa} + \theta^2) w + \widehat{u}_{zz} = 0,$$

with the boundary condition $w(t,z,h) = J_{zz}(t,z,h)$ on $\partial B_R(t_0,z_0,h_0)$ and $w(T,z,h) = 0$. Actually, since $J_{zz}$ is continuous on $\mathcal{W}$, the boundary problem associated to Equation (5.4) admits a unique viscosity solution (cf. Corollary 8.1 in Chapter V of [Fleming and Soner, 2006]).

Define now the second-order differential operator $\widetilde{\mathcal{L}}F := F_t + \frac{1}{2}\theta^2 z^2 F_{zz} + (2\theta^2 + \rho - r + m^0 + m^1 h^{-\kappa}) z F_z - \delta h F_h + (\rho + m^0 + m^1 h^{-\kappa} + \theta^2 - 2r) F$ and let $\tau_R := \inf\{s \geq 0 : (t+s, \widetilde{Z}^{1,z}_s, H^{1,h}_s)\} \notin B_R(t_0,z_0,h_0)\} \wedge (T-t)$, where $\widetilde{Z}$ is the process such that $(t+s, \widetilde{Z}_s, H^1_s)$ has infinitesimal generator $\widetilde{\mathcal{L}}$. Then, introducing

$$m(t,z,h) := \mathbb{E}_{z,h}\left[\int_0^{\tau_R} e^{-\int_0^s (\rho + M^{H^1}_u) du} \widehat{u}_{zz}(\widetilde{Z}^1_s, H^1_s) ds + e^{-\int_0^{\tau_R} (\rho + M^{H^1}_u) du} J_{zz}(t+\tau_R, \widetilde{Z}^1_{\tau_R}, H^1_{\tau_R})\right],$$

we see that $m(t,z,h) > 0$ since $J_{zz} \geq 0$ on $\mathcal{O}$ by convexity and $\widehat{u}_{zz} > 0$ (cf. (3.10)). However, $m$ is a viscosity solution to (5.4) and therefore, by uniqueness, $m \equiv J_{zz}$. It thus follows that $J_{zz} > 0$ on $B_R(t_0,z_0,h_0)$ and we then conclude by arbitrariness of $(t_0,z_0,h_0) \in \mathcal{W}$ and of $R > 0$.

□

From Theorem 3.2, for any $(t,z,h) \in \mathcal{O}$, we know that $V(t,x,h) = \inf_{z>0}[J(t,z,h) + zx]$. Since $z \mapsto J(t,z,h) + zx$ is strictly convex (cf. Proposition 5.1), then there exists an unique solution $z^*(t,x,h) > 0$ such that

$$V(t,x,h) = J(t, z^*(t,x,h), h) + x z^*(t,x,h), \tag{5.5}$$

where $z^*(t,x,h) := \mathcal{I}^J(t,-x,h)$ and $\mathcal{I}^J$ is the inverse function of $J_z$. Moreover, $z^* \in C(\mathcal{O})$, and for any $(t,h) \in [0,T] \times \mathbb{R}_+$, $z^*(t,x,h)$ is strictly decreasing with respect to $x$, which is a bijection form. Hence, for any $(t,h) \in [0,T] \times \mathbb{R}_+$, $z^*(t,\cdot,h)$ has an inverse function $x^*(t,\cdot,h)$, which is continuous, strictly decreasing, and maps $\mathbb{R}_+$ to $\mathbb{R}_+$.

**Proposition 5.2.** *One has $V \in C^{1,1,1}(\mathcal{O})$ and $V_{xx} \in L^\infty_{loc}(\mathcal{O})$.*

*Proof.* From (5.5), using that $J_z(t, z^*(t,x,h), h) = -x$, one has

$$\begin{aligned}
V_t &= J_t(t, z^*(t,x,h), h) + J_z(t, z^*(t,x,h), h) z^*_t(t,x,h) + x z^*_t(t,x,h) = J_t(t, z^*(t,x,h), h),\\
V_h &= J_h(t, z^*(t,x,h), h) + J_z(t, z^*(t,x,h), h) z^*_h(t,x,h) + x z^*_h(t,x,h) = J_h(t, z^*(t,x,h), h),\\
V_x &= J_z(t, z^*(t,x,h), h) z^*_x(t,x,h) + z^*(t,x,h) + x z^*_x(t,x,h) = z^*(t,x,h),
\end{aligned}$$

$$V_{xx} = z^*_x(t,x,h) = -\frac{1}{J_{zz}(t, z^*(t,x,h), h)}, \text{ in the a.e. sense.} \tag{5.6}$$

The proof is then completed due to Theorem 4.6. □

Let us now define

$$\begin{cases} \widehat{b}(t,h) := x^*(t, b(t,h), h), \\ \mathcal{W}_x := \{(t,x,h) \in \mathcal{O} : (t, z^*(t,x,h), h) \in \mathcal{W}\}, \\ \mathcal{I}_x := \{(t,x,h) \in \mathcal{O} : (t, z^*(t,x,h), h) \in \mathcal{I}\}. \end{cases} \tag{5.7}$$

Then, by Lemma 4.2 we have

$$\mathcal{W}_x := \{(t,x,h) \in \mathcal{O} : 0 < x < \widehat{b}(t,h)\}, \quad \mathcal{I}_x := \{(t,x,h) \in \mathcal{O} : x \geq \widehat{b}(t,h)\}, \tag{5.8}$$

so that we can express the optimal investment time into health in terms of the initial coordinates as: $\tau^*(t,x,h) = \inf\{s \geq 0 : X^x_s \geq \widehat{b}(t+s, H^{1,h}_s)\} \wedge (T-t)$.

Recalling that $J \geq \widehat{W}$ on $\mathcal{O}$ by (3.19), we notice that if $J(t, z^*(t,x,h), h) = \widehat{W}(t, z^*(t,x,h), h)$, then the function $z \mapsto (J - \widehat{W})(t,z,h)$ attains its minimum value 0 at $(t, z^*(t,x,h), h)$. Hence,

$$J_z(t, z^*(t,x,h), h) = \widehat{W}_z(t, z^*(t,x,h), h) = -x.$$

This means that $z^*(t,x,h)$ is a stationary point of the convex function $z \mapsto \widehat{W}(t,z,h) + zx$, so that

$$\widehat{W}(t, z^*(t,x,h), h) + xz^*(t,x,h) = \min_z(\widehat{W}(t,z,h) + zx) = \min_z[W(t,z,h) - z(x - g(t))] = \widehat{V}(t,x,h),$$

by (4.2) and Theorem 3.1. Together with (5.5), we obtain $V(t,x,h) = \widehat{V}(t,x,h)$.

On the other hand, if $V(t,x,h) = \widehat{V}(t,x,h)$, then by (5.5), Theorem 3.1 and (4.2)

$$J(t, z^*(t,x,h), h) + xz^*(t,x,h) = \inf_z(\widehat{W}(t,z,h) + zx) \leq \widehat{W}(t, z^*(t,x,h), h) + xz^*(t,x,h). \tag{5.9}$$

Hence, since $J \geq \widehat{W}$ on $\mathcal{O}$, $J(t, z^*(t,x,h), h) = \widehat{W}(t, z^*(t,x,h), h)$.

Combining these two arguments we have that

$$\{(t,x,h) \in \mathcal{O} : V(t,x,h) = \widehat{V}(t,x,h)\} = \{(t,x,h) \in \mathcal{O} : J(t, z^*(t,x,h), h) = \widehat{W}(t, z^*(t,x,h), h)\}.$$

This, together with (5.8), leads to express the optimal investment time in the original coordinates as

$$\begin{cases} \tau^*(t,x,h) = \inf\{s \geq 0 : X_s^x \geq \widehat{b}(t+s, H_s^{1,h})\} \wedge (T-t) \\ \qquad\qquad\;\; = \inf\{s \geq 0 : V(t+s, X_s^x, H_s^{1,h}) = \widehat{V}(t+s, X_s^x, H_s^{1,h})\} \wedge (T-t). \end{cases} \tag{5.10}$$

Due to the regularity of $V$ and the dual relations between $V$ and $J$ (cf. Proposition 5.2), from Corollary 4.1 we can deduce that $V$ is a solution in the a.e. sense to the HJB equation

(5.11)

$$0 = \max\Big\{\widehat{V} - V, \sup_{c,\pi}\Big[V_t + \frac{1}{2}\sigma^2\pi^2 V_{xx} + (\pi(\mu - r) + rx - c)V_x + u(c,h) - \delta h V_h - (\rho + m^0 + m^1 h^{-\kappa})V\Big]\Big\}.$$

Then a standard verification argument leads to the following result.

**Theorem 5.1.** *Let $(t,x,h) \in \mathcal{O}$ and recall that $\mathcal{I}^u(\cdot,h)$ denotes the inverse of $u_c(\cdot,h)$. Then $c^*(t,x,h) := \mathcal{I}^u(V_x(t,x,h),h)$ and $\pi^*(t,x,h) := -\frac{\theta V_x(t,x,h)}{\sigma V_{xx}(t,x,h)}$ (a.e. on $\mathcal{O}$) define the optimal feedback maps, while $\tau^* = \inf\{s \geq 0 : V(t+s, X_s^x, H_s^{1,h}) \leq \widehat{V}(t+s, X_s^x, H_s^{1,h})\} \wedge (T-t)$ is the optimal investment time. Hence, $c_s^* = c^*(s, X_s, H_s), \pi_s^* = \pi^*(s, X_s, H_s)$ and $\tau^*, \mathbb{P}_{t,x,h}$-a.s., provide an optimal control triple.*

Thanks to (5.7) and Proposition 5.2 we can finally express the optimal health investment threshold $\widehat{b}$ and the optimal portfolio $\pi$ in terms of $b$ and $z^*$, respectively.

**Proposition 5.3.** *One has that $\widehat{b}(t,h) = -W_z(t, b(t,h), h) + g(t)$, for any $(t,h) \in [0,T] \times \mathbb{R}_+$, and $\pi^*(t,x,h) = \frac{\theta}{\sigma} z^*(t,x,h) J_{zz}(t, z^*(t,x,h), h)$ for almost all $(t,x,h) \in \mathcal{O}$.*

*Proof.* We know that $\widehat{b}(t,h) = x^*(t, b(t,h), h)$, where $x^*(t,\cdot,h)$ is the inverse function of $z^*(t,\cdot,h)$. Since $J_z(t, z^*(t,x,h), h) = -x$, by taking $x = x^*(t,z,h)$, computations show that

$$J_z(t,z,h) = J_z(t, z^*(t, x^*(t,z,h), h), h) = -x^*(t,z,h).$$

Hence, from Corollary 4.1 and (4.2) we have

$$\begin{aligned} \widehat{b}(t,h) &= x^*(t, b(t,h), h) = -J_z(t, b(t,h), h) = -\widehat{W}_z(t, b(t,h), h) \\ &= -W_z(t, b(t,h), h) + g(t). \end{aligned}$$

To prove the second statement, we notice that $V_x(t,x,h) = z^*(t,x,h), V_{xx}(t,x,h) = z_x^*(t,x,h) = -\frac{1}{J_{zz}(t,z^*,h)}$ (cf. (5.6)), which then yield

$$\pi^*(t,x,h) = -\frac{\theta V_x(t,x,h)}{\sigma V_{xx}(t,x,h)} = \frac{\theta}{\sigma} z^*(t,x,h) J_{zz}(t, z^*(t,x,h), h), \text{ a.e. on } \mathcal{O}.$$

□

## 6. Conclusions

In this paper, we study a consumption/portfolio problem in which the agent can also choose the time at which making an irreversible precautionary investment into health, thus facing a trade-off between a costly health investment and the reduction of her mortality rate. The optimization problem is formulated as a stochastic control-stopping problem over a random time horizon, which contains two state variables: wealth and health capital.

We first transform by martingale and duality methods the original problem into its dual problem, which is a finite time-horizon two-dimensional optimal stopping problem. We then study the optimal stopping problem by probabilistic arguments. Due to the lack of monotonicity of the optimal stopping boundary, we prove the boundary's Lipschitz-continuity in order to deduce regularity properties of the optimal stopping problem's value function. Furthermore, we provide an integral equation uniquely characterizing of optimal boundary. Finally, we obtain the optimal strategies in terms of the primal variables and show that the agent invests in health optimally whenever her wealth reaches a boundary surface, which depends on the agent's age and health capital.

There are many directions towards this work can be generalized and further investigated. By performing a thorough probabilistic analysis on the regularity of the free boundary, we are able to provide a complete characterization of the optimal timing of health investment through a non-linear integral equation. A survey of numerical methods for equations of this kind may be found in [Atkinson, 1992] (see also classical textbook like [Delves and Mohamed, 1988]). These methods could be employed (and further generalized) to solve our Equation (4.16). However, since they are certainly nontrivial, we believe that such numerical computation falls outside the scopes of our work. The numerical study will also shed light on the sensitivity of the free boundary with respect to the model's parameters, as well as on the economic and actuarial insights of the optimal solution. This analysis is also left for future research. A further immediate question regards the possibility of studying not only when it is optimal to invest into health, but also how much. Like in consumption choices (see [Hindy and Huang, 1993] and [Bank and Riedel, 2001]), the agent can invest in health at "gulps" at any moment, as well as at finite rates over intervals. Therefore, we can model the health investment $I_t$—representing the cumulative amount of health investment paid from time zero up to $t$—as a singular control and study the corresponding optimal health investment strategy under a stochastic regular-singular control framework. We leave this fascinating and challenging research question for future research.

## Appendix A. Technical Estimates

**Lemma A.1.** *Let* $C_0(h) := \frac{m^1 h^{-\kappa-1}(h+\frac{f(I)}{\delta})}{(1-\alpha)\delta} e^{\delta\kappa T} + 1$ *and* $c_1 := \frac{1}{2}(\frac{\alpha\theta}{1-\alpha})^2 + \frac{\alpha}{1-\alpha}(r + \frac{1}{2}\theta^2) > 0$. *Then*

$$0 \leq W_h(t,z,h) \leq z^{\frac{\alpha}{\alpha-1}}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} C_0(h)\frac{e^{c_1 T}}{c_1}, \quad \forall (t,z,h) \in \mathcal{O}.$$

*Moreover,*

$$\lim_{z\to\infty} W_h(t,z,h) = 0 \quad \textit{and} \quad \lim_{z\to 0} W_h(t,z,h) = \infty \quad \textit{for } (t,h) \in [0,T) \times \mathbb{R}_+.$$

*Proof.* From (3.12) we compute the partial derivative with respect to $h$,

$$W_h(t,z,h) = z^{\frac{\alpha}{\alpha-1}}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} \Big[ \int_0^{T-t} e^{\frac{\alpha}{1-\alpha}(rs+\frac{1}{2}\theta^2 s)+\frac{\theta^2\alpha^2 s}{2(\alpha-1)^2}} e^{\int_0^s \frac{(\rho+M_u^{H^{2,h}}(h))}{\alpha-1}du} \times$$

$$\times \Big[\frac{m^1\kappa}{1-\alpha}\Big(\int_0^s (he^{-\delta u} + \frac{f(I)}{\delta}(1-e^{-\delta u}))^{-\kappa-1} e^{-\delta u} du\Big)\Big(he^{-\delta s} + \frac{f(I)}{\delta}(1-e^{-\delta s})\Big) + e^{-\delta s}\Big] ds\Big] \tag{A.1}$$

with

$$M_u^{H^{2,h}}(h) := m^0 + m^1(H_u^{2,h})^{-\kappa} = m^0 + m^1(he^{-\delta u} + \frac{f(I)}{\delta}(1-e^{-\delta u}))^{-\kappa}, \ u \geq 0. \tag{A.2}$$

Since $\alpha < 1$, then $W_h(t,z,h) \geq 0$ for any $(t,z,h) \in \mathcal{O}$. On the other hand, since $he^{-\delta s} \leq he^{-\delta s} + \frac{f(I)}{\delta}(1-e^{-\delta s}) \leq h + \frac{f(I)}{\delta}$, then $(he^{-\delta u} + \frac{f(I)}{\delta}(1-e^{-\delta u}))^{-\kappa-1} \leq h^{-\kappa-1}e^{(\kappa+1)\delta s}$. Therefore, from (A.1) we have

$$\begin{aligned}
&\frac{m^1\kappa}{1-\alpha}\bigg(\int_0^s (he^{-\delta u} + \frac{f(I)}{\delta}(1-e^{-\delta u}))^{-\kappa-1}e^{-\delta u}du)(he^{-\delta s} + \frac{f(I)}{\delta}(1-e^{-\delta s})\bigg) + e^{-\delta s} \\
&\leq \frac{m^1\kappa}{1-\alpha}\bigg(\int_0^s h^{-\kappa-1}e^{(\kappa+1)\delta u}e^{-\delta u}du\bigg)\Big(h + \frac{f(I)}{\delta}\Big) + e^{-\delta s} \leq C_0(h).
\end{aligned}$$

Combining the above inequality with (A.1), we have

$$W_h(t,z,h) \leq z^{\frac{\alpha}{\alpha-1}}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}C_0(h)\frac{e^{c_1T}}{c_1},$$

where we have used the fact that

$$\int_0^{T-t} e^{\frac{\alpha}{1-\alpha}(rs+\frac{1}{2}\theta^2 s)+\frac{\theta^2\alpha^2 s}{2(\alpha-1)^2}}e^{\int_0^s \frac{(\rho+M_u^{H^{2,h}}(h))}{\alpha-1}du}ds \leq \int_0^{T-t} e^{c_1 s}ds \leq \frac{e^{c_1T}}{c_1}, \tag{A.3}$$

upon using that $e^{\int_0^s\left(\frac{\rho+M_u^{H^{2,h}}(h)}{\alpha-1}\right)du} \leq 1$, since $\alpha < 1$.

Finally, it is easy to see that $\lim_{z\to\infty} W_h(t,z,h) = 0$ and $\lim_{z\to 0} W_h(t,z,h) = \infty$ from (A.1). $\square$

## Appendix B. Proofs from Section 4

### B.1. **Poof of Proposition 4.3.**

*Proof.* From (4.5) one has

$$\widehat{J}(t,z,h) = \mathbb{E}_{z,h}\bigg[\int_0^{\tau^*} ze^{-rs-\theta B_s-\frac{1}{2}\theta^2 s}Ids - \int_0^{\tau^*} e^{-\int_0^s(\rho+M_u^{H^1})du}f(I)W_h(t+s,Z_s^1,H_s^1)ds\bigg].$$

Here we show that $\widehat{J}(t,z,\cdot)$ is locally Lipschitz and (4.8) holds for a.e. $h \in \mathbb{R}_+$ and each given $(t,z) \in [0,T]\times\mathbb{R}_+$ (with the null set where $\widehat{J}(t,z,\cdot)$ is not differentiable being a priori dependent on $(t,z)$). Similar arguments, that we omit for brevity, also show that $\widehat{J}$ is locally Lipschitz in $t$ and $z$.

First we obtain bounds for the left and right derivatives of $\widehat{J}(t,z,\cdot)$. Fix $(t,z,h) \in \mathcal{O}$, pick $\epsilon > 0$, and notice that $\tau^*$ is suboptimal in $\widehat{J}(t,z,h+\epsilon)$ (and independent of $\epsilon$). Then, denoting by $Z^{1,z,\epsilon}$ the solution to (3.18), where $H^1$ is such that $H_0^1 = h+\epsilon$, we obtain

$$\begin{aligned}
&\widehat{J}(t,z,h+\epsilon) - \widehat{J}(t,z,h) \\
&\geq -f(I)\mathbb{E}\bigg[\int_0^{\tau^*} e^{-\int_0^s(\rho+M_u^{H^{1,h+\epsilon}})du}W_h(t+s,Z_s^{1,z,\epsilon},H_s^{1,h+\epsilon}) \\
&- e^{-\int_0^s(\rho+M_u^{H^{1,h}})du}W_h(t+s,Z_s^{1,z},H_s^{1,h})ds\bigg] \\
&= -f(I)\mathbb{E}\bigg[\int_0^{\tau^*}\Big(e^{-\int_0^s(\rho+M_u^{H^{1,h+\epsilon}})du} - e^{-\int_0^s(\rho+M_u^{H^{1,h}})du}\Big)W_h(t+s,Z_s^{1,z,\epsilon},H_s^{1,h+\epsilon}) \\
&+ e^{-\int_0^s(\rho+M_u^{H^{1,h}})du}\Big(W_h(t+s,Z_s^{1,z,\epsilon},H_s^{1,h+\epsilon}) - W_h(t+s,Z_s^{1,z},H_s^{1,h})\Big)ds\bigg] \\
&= -f(I)\epsilon\mathbb{E}\bigg[\int_0^{\tau^*}\frac{\Big(e^{-\int_0^s(\rho+M_u^{H^{1,h+\epsilon}})du} - e^{-\int_0^s(\rho+M_u^{H^{1,h}})du}\Big)}{\epsilon}W_h(t+s,Z_s^{1,z,\epsilon},H_s^{1,h+\epsilon}) \\
&+ e^{-\int_0^s(\rho+M_u^{H^{1,h}})du}\frac{\Big(W_h(t+s,Z_s^{1,z,\epsilon},H_s^{1,h+\epsilon}) - W_h(t+s,Z_s^{1,z},H_s^{1,h})\Big)}{\epsilon}ds\bigg]
\end{aligned}$$

$$= -f(I)\epsilon\mathbb{E}\bigg[\int_0^{\tau^*} e^{-\int_0^s(\rho+M_u^{H^{1,h_\epsilon}})du}\bigg(\int_0^s m^1\kappa(H_u^{1,h_\epsilon})^{-\kappa-1}\frac{\partial H_u^{1,h}}{\partial h}\Big|_{h=h_\epsilon}du\bigg)W_h(t+s,Z_s^{1,z,\epsilon},H_s^{1,h+\epsilon})$$

(B.1)

$$+ e^{-\int_0^s(\rho+M_u^{H^{1,h}})du}\bigg(\frac{\partial H_s^{1,h}}{\partial h}\Big|_{h=h_\epsilon}W_{hh}(t+s,Z_s^{1,z,h_\epsilon},H_s^{1,h_\epsilon}) + W_{hz}(t+s,Z_s^{1,z,h_\epsilon},H_s^{1,h_\epsilon})\frac{\partial Z_s^{1,z,h}}{\partial h}\Big|_{h=h_\epsilon}\bigg)ds\bigg],$$

for some $h_\epsilon \in (h, h+\epsilon)$, where the last step has used the mean value theorem. Dividing (B.1) by $\epsilon$ and taking limits as $\epsilon \downarrow 0$ gives

$$\liminf_{\epsilon\to 0}\frac{\widehat{J}(t,z,h+\epsilon)-\widehat{J}(t,z,h)}{\epsilon}$$

$$\geq -f(I)\mathbb{E}\bigg[\int_0^{\tau^*} e^{-\int_0^s(\rho+M_u^{H^{1,h}})du}\bigg[\Big(\int_0^s m^1\kappa(H_u^{1,h})^{-\kappa-1}\frac{\partial H_u^{1,h}}{\partial h}du\Big)W_h(t+s,Z_s^{1,z},H_s^{1,h})$$

$$\text{(B.2)}\qquad + \frac{\partial H_s^{1,h}}{\partial h}W_{hh}(t+s,Z_s^{1,z},H_s^{1,h}) + W_{hz}(t+s,Z_s^{1,z},H_s^{1,h})\frac{\partial Z_s^{1,z}}{\partial h}\bigg]ds\bigg].$$

Since symmetric arguments applied to $\widehat{J}(t,z,h)-\widehat{J}(t,z,h-\epsilon)$ lead to the reverse inequality, we obtain

$$\limsup_{\epsilon\to 0}\frac{\widehat{J}(t,z,h)-\widehat{J}(t,z,h-\epsilon)}{\epsilon}$$

$$\leq -f(I)\mathbb{E}\bigg[\int_0^{\tau^*} e^{-\int_0^s(\rho+M_u^{H^{1,h}})du}\bigg[\Big(\int_0^s m^1\kappa(H_u^{1,h})^{-\kappa-1}\frac{\partial H_u^{1,h}}{\partial h}du\Big)W_h(t+s,Z_s^{1,z},H_s^{1,h})$$

$$+ \frac{\partial H_s^{1,h}}{\partial h}W_{hh}(t+s,Z_s^{1,z},H_s^{1,h}) + W_{hz}(t+s,Z_s^{1,z},H_s^{1,h})\frac{\partial Z_s^{1,z}}{\partial h}\bigg]ds\bigg].$$

It now remains to show that $\widehat{J}(t,z,\cdot)$ is locally Lipschitz, so that a.e. $h \in \mathbb{R}_+$ is a point of differentiability. With the same notation as above, let $\tau^*_\epsilon := \tau^*(t,z,h+\epsilon)$ be optimal for the problem with initial data $(t,z,h+\epsilon)$. By arguments analogous to those used previously we find

$$\widehat{J}(t,z,h+\epsilon)-\widehat{J}(t,z,h)$$

$$\leq -f(I)\mathbb{E}\bigg[\int_0^{\tau^*_\epsilon} e^{-\int_0^s(\rho+M_u^{H^{1,h+\epsilon}})du}W_h(t+s,Z_s^{1,z,\epsilon},H_s^{1,h+\epsilon}) - e^{-\int_0^s(\rho+M_u^{H^{1,h}})du}W_h(t+s,Z_s^{1,z},H_s^{1,h})ds\bigg]$$

(B.3)

$$\leq -f(I)\mathbb{E}\bigg[\int_0^{\tau^*_\epsilon} e^{-\int_0^s(\rho+M_u^{H^{1,h}})du}\Big(W_h(t+s,Z_s^{1,z,\epsilon},H_s^{1,h+\epsilon}) - W_h(t+s,Z_s^{1,z},H_s^{1,h})\Big)ds\bigg],$$

due to $e^{-\int_0^s(\rho+M_u^{H^{1,h}})du} \leq e^{-\int_0^s(\rho+M_u^{H^{1,h+\epsilon}})du}$ and $W_h \geq 0$ (cf. Lemma A.1). Then, by the Hölder inequality, we can write from (B.3)

$$\widehat{J}(t,z,h+\epsilon)-\widehat{J}(t,z,h)$$

(B.4)

$$\leq \mathbb{E}\bigg[\int_0^T f(I)^2 e^{-\int_0^s 2(\rho+M_u^{H^{1,h}})du}ds\bigg]^{\frac{1}{2}}\mathbb{E}\bigg[\int_0^T \Big|W_h(t+s,Z_s^{1,z,\epsilon},H_s^{1,h+\epsilon}) - W_h(t+s,Z_s^{1,z},H_s^{1,h})\Big|^2 ds\bigg]^{\frac{1}{2}}.$$

Clearly, since $\rho + M_u^{H^{1,h}} > 0$, $\mathbb{E}\Big[\int_0^T e^{-2\int_0^s(\rho+M_u^{H^{1,h}})du}ds\Big] \leq T$, and because $W_h(t,\cdot,\cdot)$ is continuously differentiable (cf. (A.1)), there exists a positive function $c(t,z,h)$ such that, for all $(z_1,h_1)$ and $(z_2,h_2)$, $|W_h(t,z,h_1)-W_h(t,z,h_2)| + |W_h(t,z_2,h)-W_h(t,z_1,h)| \leq c(t,z,h)(|z_2-z_1|+|h_1-h_2|)$.

Therefore, from (B.4) we have

$$\widehat{J}(t,z,h+\epsilon)-\widehat{J}(t,z,h) \leq c(t,z,h)\epsilon.$$

The estimate in (B.3) and (B.2) imply $|\widehat{J}(t,z,h+\epsilon)-\widehat{J}(t,z,h)|\le \widehat{c}(t,z,h)\epsilon$, for some other constant $\widehat{c}(t,z,h)>0$ which can be taken uniform over compact sets. Symmetric arguments allow to prove also that $|\widehat{J}(t,z,h)-\widehat{J}(t,z,h-\epsilon)|\le \widehat{c}(t,z,h)\epsilon$. Therefore, $\widehat{J}(t,z,\cdot)$ is locally Lipschitz and for almost all $(t,z,h)\in\mathcal{O}$

$$\widehat{J}_h(t,z,h) = -f(I)\mathbb{E}\Big[\int_0^{\tau^*} e^{-\int_0^s(\rho+M_u^{H^{1,h}})du}\Big[\Big(\int_0^s m^1\kappa(H_u^{1,h})^{-\kappa-1}\frac{\partial H_u^{1,h}}{\partial h}du\Big)W_h(t+s,Z_s^{1,z},H_s^{1,h})$$

$$+\frac{\partial H_s^{1,h}}{\partial h}W_{hh}(t+s,Z_s^{1,z},H_s^{1,h})+W_{hz}(t+s,Z_s^{1,z},H_s^{1,h})\frac{\partial Z_s^{1,z}}{\partial h}\Big]ds\Big]. \tag{B.5}$$

To simplify (B.5), we need to compute $W_h, W_{hh}$ and $W_{hz}$ firstly. From (A.1) we have

$$W_h(t+s,Z_s^{1,z},H_s^{1,h}) = Z_s^{1,z\,\frac{\alpha}{\alpha-1}}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\Big[\int_0^{T-(t+s)} e^{\frac{\alpha}{1-\alpha}(rs+\frac{1}{2}\theta^2 s)+\frac{\theta^2\alpha^2 s}{2(\alpha-1)^2}}e^{\int_0^s\frac{(\rho+M_u^{H^{2,h}}(H^1))}{\alpha-1}du}\times$$

$$\times\Big[\frac{m^1\kappa}{1-\alpha}\Big(\int_0^s(H_u^{1,h}e^{-\delta u}+\frac{f(I)}{\delta}(1-e^{-\delta u}))^{-\kappa-1}e^{-\delta u}du\Big)\Big(H_s^{1,h}e^{-\delta s}+\frac{f(I)}{\delta}(1-e^{-\delta s})\Big)+e^{-\delta s}\Big]ds\Big]$$

$$= Z_s^{1,z\,\frac{\alpha}{\alpha-1}}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\Big[\int_0^{T-(t+s)} e^{\frac{\alpha}{1-\alpha}(rs+\frac{1}{2}\theta^2 s)+\frac{\theta^2\alpha^2 s}{2(\alpha-1)^2}}e^{\int_0^s\frac{(\rho+M_u^{H^{2,h}}(H^1))}{\alpha-1}du}\times$$

$$\times\Big[\frac{m^1\kappa}{1-\alpha}\Big(\int_0^s(he^{-2\delta u}+\frac{f(I)}{\delta}(1-e^{-\delta u}))^{-\kappa-1}e^{-\delta u}du\Big)\Big(he^{-2\delta s}+\frac{f(I)}{\delta}(1-e^{-\delta s})\Big)+e^{-\delta s}\Big]ds\Big], \tag{B.6}$$

where we have used the fact that $H_s^1=he^{-\delta s}$ in (4.6) and from (A.2)

$$M_u^{H^{2,h}}(H^1) = m^0 + m^1(he^{-2\delta s}+\frac{f(I)}{\delta}(1-e^{-\delta s}))^{-\kappa}. \tag{B.7}$$

Furthermore, by (A.1) we have

$$W_{hz}(t+s,Z_s^{1,z},H_s^{1,h}) = -Z_s^{1,z\,\frac{1}{\alpha-1}}\alpha^{\frac{1}{1-\alpha}}\Big[\int_0^{T-(t+s)} e^{\frac{\alpha}{1-\alpha}(rs+\frac{1}{2}\theta^2 s)+\frac{\theta^2\alpha^2 s}{2(\alpha-1)^2}}e^{\int_0^s\frac{(\rho+M_u^{H^{2,h}}(H^1))}{\alpha-1}du}\times$$

$$\times\Big[\frac{m^1\kappa}{1-\alpha}\Big(\int_0^s(he^{-2\delta u}+\frac{f(I)}{\delta}(1-e^{-\delta u}))^{-\kappa-1}e^{-\delta u}du\Big)\Big(he^{-2\delta s}+\frac{f(I)}{\delta}(1-e^{-\delta s})\Big)+e^{-\delta s}\Big]ds\Big]. \tag{B.8}$$

From (4.6) we observe that

$$\frac{\partial Z_s^{1,z}}{\partial h} = -Z_s^{1,z}\int_0^s m^1\kappa(H_u^{1,h})^{-\kappa-1}e^{-\delta u}du = -Z_s^{1,z}\int_0^s m^1\kappa h^{-\kappa-1}e^{\delta\kappa u}du,$$

which, combined with (B.8), gives

$$W_{hz}(t+s,Z_s^{1,z},H_s^{1,h})\frac{\partial Z_s^{1,z}}{\partial h} = W_h(t+s,Z_s^{1,z},H_s^{1,h})\frac{\alpha}{1-\alpha}\int_0^s m^1\kappa h^{-\kappa-1}e^{\delta\kappa u}du,$$

$$m^1\kappa\int_0^s h^{-\kappa-1}e^{\delta\kappa u}du W_h(t+s,Z_s^{1,z},H_s^{1,h}) + W_{hz}(t+s,Z_s^{1,z},H_s^{1,h})\frac{\partial Z_s^{1,z}}{\partial h}$$

$$= \frac{W_h(t+s,Z_s^{1,z},H_s^{1,h})}{1-\alpha}\frac{m^1h^{-\kappa-1}}{\delta}(e^{\delta\kappa s}-1). \tag{B.9}$$

Therefore, combining (B.2) and (B.9), we have

$$\widehat{J}_h(t,z,h) = -f(I)\mathbb{E}_{z,h}\Big[\int_0^{\tau^*} e^{-\int_0^s(\rho+M_u^{H^1})du}\Big[W_{hh}(t+s,Z_s^1,H_s^1)e^{-\delta s}$$

$$+\frac{1}{1-\alpha}W_h(t+s,Z_s^1,H_s^1)\frac{m^1}{\delta}h^{-\kappa-1}(e^{\delta\kappa s}-1)\Big]ds\Big],$$

which completes the proof. $\square$

B.2. **Proof of Proposition 4.4.**

*Proof.* Since $\widehat{J}(t,z,h) \geq \mathbb{E}\Big[\int_0^{T-t} ze^{-rs-\theta B_s - \frac{1}{2}\theta^2 s} I ds - \int_0^{T-t} e^{-\int_0^s (\rho + M_u^{H^{1,h}}) du} f(I) W_h(t+s, Z_s^{1,z}, H_s^{1,h}) ds\Big]$ (cf. (4.5)), and by Lemma A.1 we know that $\lim_{z\to\infty} W_h(t,z,h) = 0$, we can conclude that $\lim_{z\to\infty} \widehat{J}(t,z,h) = \infty$. The fact that $\lim_{z\to 0} \widehat{J}(t,z,h) = 0$ directly follows from the bounds of $\widehat{J}$ in Proposition 4.1. $\square$

B.3. **Proof of Theorem 4.1.**

*Proof.* The proof is organized in five steps.

**Step 1.** For $\epsilon > 0$, define the function

$$F^\epsilon(t,z,h) := \widehat{J}(t,z,h) - \epsilon.$$

Let now $(t,z,h) \in \mathcal{W}$, $\lambda^\epsilon, L_1^\epsilon, L_2^\epsilon \geq 0$ (possibly depending on $(t,z,h)$), and, for $u \in \mathbb{R}$, denote by $B_\delta(u) := \{u' \in \mathbb{R} : |u'-u| < \delta\}, \delta > 0$. Since $F^\epsilon$ is locally Lipschitz continuous in $\mathcal{O}$ (cf. Proposition 4.3), if the following conditions are satisfied

(i) $F^\epsilon(t,z,h) = 0$;
(ii) $||F_z^\epsilon(t,z,h)||_\infty^{-1} < \lambda^\epsilon$;
(iii) $||F_t^\epsilon(B_\delta(t) \times B_\delta(z) \times B_\delta(h))||_\infty \leq L_1^\epsilon$ and $||F_h^\epsilon(B_\delta(t) \times B_\delta(z) \times B_\delta(h))||_\infty \leq L_2^\epsilon$,

then a version of the implicit function theorem (see, e.g., the Corollary at p.256 in [Clarke, 1990] or Theorem 3.1 in [Papi, 2005]) implies that, for suitable $\delta' > 0$, there exists a unique continuous function $b_\epsilon(t,h) : (t-\delta', t+\delta') \times (h-\delta', h+\delta') \mapsto (z-\delta', z+\delta')$ such that

$$\widehat{J}(t, b_\epsilon(t,h), h) = \epsilon \quad \text{in } (t-\delta', t+\delta') \times (h-\delta', h+\delta').$$

Also, the following inequalities hold true:

$$\begin{aligned} &|b_\epsilon(t_1,h) - b_\epsilon(t_2,h)| \leq \lambda^\epsilon L_1^\epsilon |t_1 - t_2|, \ \forall\, t_1,t_2 \in (t-\delta', t+\delta'), \\ &|b_\epsilon(t,h_1) - b_\epsilon(t,h_2)| \leq \lambda^\epsilon L_2^\epsilon |h_1 - h_2|, \ \forall\, h_1,h_2 \in (h-\delta', h+\delta'). \end{aligned} \tag{B.10}$$

According to Proposition 4.3, we have $\widehat{J}_z(t,z,h) > 0$ for a.e. $z$ inside $\mathcal{W}$ due to $W_{hz}(t,z,h) \leq 0$ in (B.8). Then, by Propositions 4.2 and 4.4 it clearly follows that such a $b_\epsilon$ above indeed exists, and also $b_\epsilon(t,h) > b(t,h) > 0$.

Moreover, the family $(b_\epsilon)_{\epsilon>0}$ decreases as $\epsilon \to 0$, so that its limit $b_0$ exists. Such a limit is such that the mapping $(t,h) \mapsto b_0(t,h)$, is upper semicontinuous, as decreasing limit of continuous functions, and $b_0(t,h) \geq b(t,h)$. Since $\widehat{J}(t, b_\epsilon(t,h), h) = \epsilon$, it is clear that taking limits as $\epsilon \to 0$, we get $\widehat{J}(t, b_0(h,t), h) = 0$ by continuity of $\widehat{J}$ (cf. Proposition 4.3), and therefore $b_0(t,h) \leq b(t,h)$ due to the definition of the stopping region $\mathcal{I}$ in Lemma 4.2. Hence,

$$\lim_{\epsilon\to 0} b_\epsilon(t,h) = b(t,h), \quad \text{for all } (t,h) \in [0,T] \times \mathbb{R}_+. \tag{B.11}$$

**Step 2.** We here prove that $b_\epsilon(t,h)$ is bounded uniformly in $\epsilon$. Clearly, we can restrict the attention to $\epsilon \in (0,\epsilon_0)$ for some $\epsilon_0 > 0$. From Lemma 4.2 we know that $b(t,h) \leq g(t,h)$. Since now $\lim_{\epsilon\to 0} b_\epsilon(t,h) = b(t,h)$ (cf. (B.11)), we thus have that $0 \leq b_\epsilon(t,h) \leq 1 + g(t,h), \forall \epsilon \in (0,\epsilon_0)$, which provides the desired uniform bound.

**Step 3.** According to Step 1, we need to verify conditions (ii) and (iii).

Step 3-(a). We here determine an **upper bound for** $|\widehat{J}_h(t, b_\epsilon(t,h), h)|$. Recalling $\widehat{J}_h(t,z,h)$ as in Proposition 4.3, we have

$$\widehat{J}_h(t, b_\epsilon(t,h), h) = -f(I)\mathbb{E}_{b_\epsilon(t,h),h}\Big[\int_0^{\tau^*} e^{-\int_0^s(\rho+M_u^{H^1})du}\Big[W_{hh}(t+s, Z_s^1, H_s^1)e^{-\delta s}+ \frac{1}{1-\alpha}W_h(t+s, Z_s^1, H_s^1)\frac{m^1}{\delta}h^{-\kappa-1}(e^{\delta\kappa s}-1)\Big]ds\Big].$$

Since $W_h(t,z,h) > 0$ for all $(t,z,h) \in [0,T) \times \mathbb{R}_+^2$ (cf. (A.1)), we have

$$|\widehat{J}_h(t, b_\epsilon(t,h), h)| \leq f(I)\mathbb{E}_{b_\epsilon(t,h),h}\Big[\int_0^{\tau^*} e^{-\int_0^s(\rho+M_u^{H^1})du}\Big[\Big|W_{hh}(t+s, Z_s^1, H_s^1)\Big|e^{-\delta s} + \frac{1}{1-\alpha}W_h(t+s, Z_s^1, H_s^1)\frac{m^1}{\delta}h^{-\kappa-1}(e^{\delta\kappa s}-1)\Big]ds\Big]. \tag{B.12}$$

To proceed further, we determine $\mathbb{P}_{b_\epsilon(t,h),h}$-a.s. upper bounds for $W_h(t+s, Z_s^1, H_s^1)$ and $|W_{hh}(t+s, Z_s^1, H_s^1)|$. Firstly, we give the upper bound of $W_h(t+s, Z_s^1, H_s^1)$. From (B.6) we have

$$W_h(t+s, Z_s^1, H_s^1) = Z_s^{1\frac{\alpha}{\alpha-1}}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\Big[\int_0^{T-(t+s)} e^{\frac{\alpha}{1-\alpha}(rs+\frac{1}{2}\theta^2 s)+\frac{\theta^2\alpha^2 s}{2(\alpha-1)^2}} e^{\int_0^s \frac{(\rho+M_u^{H^{2,h}}(H^1))}{\alpha-1}du} \times \times \Big[\frac{m^1\kappa}{1-\alpha}\Big(\int_0^s (he^{-2\delta u}+\frac{f(I)}{\delta}(1-e^{-\delta u}))^{-\kappa-1}e^{-\delta u}du\Big)\Big(he^{-2\delta s}+\frac{f(I)}{\delta}(1-e^{-\delta s})\Big)+e^{-\delta s}\Big]ds\Big], \tag{B.13}$$

where $M_u^{H^{2,h}}(h)$ is defined in (A.2).

Since $he^{-2\delta s} \leq he^{-2\delta s}+\frac{f(I)}{\delta}(1-e^{-\delta s}) \leq h+\frac{f(I)}{\delta}$, then $(he^{-2\delta u}+\frac{f(I)}{\delta}(1-e^{-\delta u}))^{-\kappa-1} \leq h^{-\kappa-1}e^{2(\kappa+1)\delta s}$. Therefore, from (B.13) we have

$$\begin{aligned}
&\frac{m^1\kappa}{1-\alpha}\Big(\int_0^s (he^{-2\delta u}+\frac{f(I)}{\delta}(1-e^{-\delta u}))^{-\kappa-1}e^{-\delta u}du)(he^{-2\delta s}+\frac{f(I)}{\delta}(1-e^{-\delta s})\Big)+e^{-\delta s}\\
&\leq \frac{m^1\kappa}{1-\alpha}\Big(\int_0^s h^{-\kappa-1}e^{2(\kappa+1)\delta u}e^{-\delta u}du\Big)\Big(h+\frac{f(I)}{\delta}\Big)+e^{-\delta s}\\
&\leq \frac{m^1 h^{-\kappa-1}\kappa(h+\frac{f(I)}{\delta})}{(1-\alpha)(2\kappa+1)\delta}e^{(2\kappa+1)\delta T}+1 =: C_1(h).
\end{aligned}$$

Combining the above inequality with (B.13), we have $\mathbb{P}_{b_\epsilon(t,h),h}$-a.s.

$$W_h(t+s, Z_s^1, H_s^1) \leq Z_s^{1\frac{\alpha}{\alpha-1}}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}C_1(h)N(t+s,h), \tag{B.14}$$

where (cf. (A.3))

$$N(t+s,h) := \Big[\int_0^{T-(t+s)} e^{\frac{\alpha}{1-\alpha}(rs+\frac{1}{2}\theta^2 s)+\frac{\theta^2\alpha^2 s}{2(\alpha-1)^2}} e^{\int_0^s \frac{(\rho+M_u^{H^{2,h}}(H^1))}{\alpha-1}du}ds\Big] \leq \frac{e^{c_1 T}}{c_1}. \tag{B.15}$$

We continue by obtaining an upper bound for $|W_{hh}(t+s, Z_s^1, H_s^1)|$. From (A.1) we compute

$$W_{hh}(t,z,h) = z^{\frac{\alpha}{\alpha-1}}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\Big[\int_0^{T-t} e^{c_1+\int_0^s \frac{(\rho+M_u^{H^{2,h}}(h))}{\alpha-1}du}\frac{m^1\kappa}{1-\alpha}\Big[\Big(\int_0^s (H_u^2)^{-\kappa-1}e^{-\delta u}du\Big)^2 \times \times\Big(\frac{m^1\kappa}{1-\alpha}H_s^2\Big)+2e^{-\delta s}\int_0^s (H_u^2)^{-\kappa-1}e^{-\delta u}du - H_s^2\int_0^s(\kappa+1)(H_u^2)^{-\kappa-2}e^{-2\delta u}du\Big]ds\Big],$$

so that we have $\mathbb{P}_{b_\epsilon(t,h),h}$-a.s.

$$
\begin{aligned}
&|W_{hh}(t+s, Z^1_s, H^1_s)| \\
&\le Z_s^{1\,\frac{\alpha}{\alpha-1}}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\Big[\int_0^{T-t-s} e^{c_1+\int_0^s \frac{(\rho+M_u^{H^{2,h}}(H^1))}{\alpha-1}du}\,\frac{m^1\kappa}{1-\alpha}\times \\
&\times\Big[\Big(\int_0^s (he^{-2\delta u}+\frac{f(I)}{\delta}(1-e^{-\delta u}))^{-\kappa-1}e^{-\delta u}du\Big)^2\Big(\frac{m^1\kappa}{1-\alpha}(he^{-2\delta s}+\frac{f(I)}{\delta}(1-e^{-\delta s}))\Big) \\
&+2e^{-\delta s}\int_0^s (he^{-2\delta u}+\frac{f(I)}{\delta}(1-e^{-\delta u}))^{-\kappa-1}e^{-\delta u}du \\
&+(he^{-2\delta s}+\frac{f(I)}{\delta}(1-e^{-\delta s}))\int_0^s(\kappa+1)(he^{-2\delta u}+\frac{f(I)}{\delta}(1-e^{-\delta u}))^{-\kappa-2}e^{-2\delta u}du\Big]ds\Big].
\end{aligned}
\tag{B.16}
$$

Since $he^{-2\delta s}\le he^{-2\delta s}+\frac{f(I)}{\delta}(1-e^{-\delta s})\le h+\frac{f(I)}{\delta}$, then $(he^{-2\delta u}+\frac{f(I)}{\delta}(1-e^{-\delta u}))^{-\kappa-1}\le h^{-\kappa-1}e^{2(\kappa+1)\delta u}$ and

$$
\begin{aligned}
&\Big(\int_0^s (he^{-2\delta u}+\frac{f(I)}{\delta}(1-e^{-\delta u}))^{-\kappa-1}e^{-\delta u}du\Big)^2\Big(\frac{m^1\kappa}{1-\alpha}(he^{-2\delta s}+\frac{f(I)}{\delta}(1-e^{-\delta s}))\Big) \\
&\le \frac{h^{-2\kappa-2}}{(2\kappa+1)^2\delta^2}e^{2(2\kappa+1)\delta T}\frac{m^1\kappa(h+\frac{f(I)}{\delta})}{1-\alpha}, \\
&2e^{-\delta s}\int_0^s (he^{-2\delta u}+\frac{f(I)}{\delta}(1-e^{-\delta u}))^{-\kappa-1}e^{-\delta u}du\le\frac{2h^{-\kappa-1}e^{(2\kappa+1)\delta T}}{(2\kappa+1)\delta}, \\
&\Big(he^{-2\delta s}+\frac{f(I)}{\delta}(1-e^{-\delta s})\Big)\int_0^s(\kappa+1)(he^{-2\delta u}+\frac{f(I)}{\delta}(1-e^{-\delta u}))^{-\kappa-2}e^{-2\delta u}du \\
&\le\frac{1}{2}h^{-\kappa-2}(h+\frac{f(I)}{\delta})e^{(2\kappa+2)\delta T}.
\end{aligned}
$$

By using the latter inequality in (B.16) we obtain $\mathbb{P}_{b_\epsilon(t,h),h}$-a.s.

$$
\begin{aligned}
|W_{hh}(t+s,Z^1_s,H^1_s)| &\le Z_s^{1\,\frac{\alpha}{\alpha-1}}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\Big[\int_0^{T-(t+s)} e^{c_1+\int_0^s \frac{(\rho+M_u^{H^{2,h}}(H^1))}{\alpha-1}du}C_2(h)ds\Big] \\
&= Z_s^{1\,\frac{\alpha}{\alpha-1}}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}C_2(h)N(s+t,h),
\end{aligned}
\tag{B.17}
$$

where $N(s+t,h)$ is defined in (B.15) and

$$
C_2(h):=\frac{h^{-2\kappa-2}}{(2\kappa+1)^2\delta^2}e^{2(2\kappa+1)\delta T}\frac{m^1\kappa(h+\frac{f(I)}{\delta})}{1-\alpha}+\frac{2h^{-\kappa-1}e^{(2\kappa+1)\delta T}}{(2\kappa+1)\delta}+\frac{1}{2}h^{-\kappa-2}(h+\frac{f(I)}{\delta})e^{(2\kappa+2)\delta T}.
$$

Then, (B.12), (B.14) and (B.17) yield

$$
\begin{aligned}
&|\widehat{J}_h(t,b_\epsilon(t,h),h)| \\
&\le f(I)\mathbb{E}_{b_\epsilon(t,h),h}\Big[\int_0^{\tau^*}e^{-\int_0^s(\rho+M_u^{H^1})du}\Big[Z_s^{1\,\frac{\alpha}{\alpha-1}}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}C_2(h)N(t+s,h) \\
&+Z_s^{1\,\frac{\alpha}{\alpha-1}}C_1(h)\alpha^{\frac{\alpha}{1-\alpha}}N(t+s,h)\frac{m^1}{\delta}h^{-\kappa-1}e^{\delta\kappa s}\Big]ds\Big] \\
&\le f(I)C_3(h)\mathbb{E}_{b_\epsilon(t,h),h}\Big[\int_0^{\tau^*}e^{-\int_0^s(\rho+M_u^{H^1})du}Z_s^{1\,\frac{\alpha}{\alpha-1}}N(t+s,h)(1+e^{\delta\kappa T})ds\Big] \\
&= f(I)C_3(h)(1+e^{\delta\kappa T})\mathbb{E}_{b_\epsilon(t,h),h}\Big[\int_0^{\tau^*}e^{-\int_0^s(\rho+M_u^{H^1})du}Z_s^{1\,\frac{\alpha}{\alpha-1}}N(t+s,h)ds\Big]
\end{aligned}
$$

(B.18)

$$\leq f(I)C_3(h)(1+e^{\delta\kappa T})(1+g(t,h))\mathbb{E}_{b_\epsilon(t,h),h}\Big[\int_0^{\tau^*} e^{-(rs+\theta B_s+\frac{1}{2}\theta^2 s)} Z_s^{1\,\frac{1}{\alpha-1}} N(t+s,h)ds\Big] := L_2^\epsilon(t,h),$$

where $C_3(h) := \max\{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}C_2(h), \frac{C_1(h)}{1-\alpha}\alpha^{\frac{\alpha}{1-\alpha}}\frac{m^1}{\delta}h^{-\kappa-1}\}$ and we have used the fact that $b_\epsilon(t,h) \leq g(t,h)+1$ in Step 2.

Step 3-(b). We here determine a **lower bound for** $\widehat{J}_z(t,b_\epsilon(t,h),h)$. From (4.9), we have

$$\widehat{J}_z(t,b_\epsilon(t,h),h) = \mathbb{E}_{b_\epsilon(t,h),h}\Big[\int_0^{\tau^*} e^{-rs-\theta B_s-\frac{1}{2}\theta^2 s}\Big(I - f(I)W_{hz}(t+s,Z_s^1,H_s^1)\Big)ds\Big], \tag{B.19}$$

and from (B.8) we have $\mathbb{P}_{b_\epsilon(t,h),h}$-a.s.

$$W_{hz}(t+s,Z_s^1,H_s^1) = -Z_s^{1\,\frac{1}{\alpha-1}}\alpha^{\frac{1}{1-\alpha}}\Big[\int_0^{T-(t+s)} e^{\frac{\alpha}{1-\alpha}(rs+\frac{1}{2}\theta^2 s)+\frac{\theta^2\alpha^2 s}{2(\alpha-1)^2}} e^{\int_0^s \frac{(\rho+M_u^{H^{2,h}}(H^1))}{\alpha-1}du}\times$$

(B.20)

$$\times\Big[\frac{m^1\kappa}{1-\alpha}\Big(\int_0^s (he^{-2\delta u}+\frac{f(I)}{\delta}(1-e^{-\delta u}))^{-\kappa-1}e^{-\delta u}du\Big)\Big(he^{-2\delta s}+\frac{f(I)}{\delta}(1-e^{-\delta s})\Big)+e^{-\delta s}\Big]ds\Big].$$

Since $he^{-2\delta s} \leq he^{-2\delta s}+\frac{f(I)}{\delta}(1-e^{-\delta s}) \leq h+\frac{f(I)}{\delta}$, then $(he^{-2\delta u}+\frac{f(I)}{\delta}(1-e^{-\delta u}))^{-\kappa-1} \geq (h+\frac{f(I)}{\delta})^{-\kappa-1}$, and therefore

$$\begin{aligned}&\Big[\frac{m^1\kappa}{1-\alpha}\Big(\int_0^s (he^{-2\delta u}+\frac{f(I)}{\delta}(1-e^{-\delta u}))^{-\kappa-1}e^{-\delta u}du\Big)\Big(he^{-2\delta s}+\frac{f(I)}{\delta}(1-e^{-\delta s})\Big)+e^{-\delta s}\Big]\\ &\geq \frac{m^1\kappa(h+\frac{f(I)}{\delta})^{-\kappa-1}}{1-\alpha}\frac{1-e^{-\delta s}}{\delta}he^{-2\delta s}+e^{-\delta s} \geq e^{-\delta s} \geq e^{-\delta T}.\end{aligned}$$

Hence, from (B.20) we can write $\mathbb{P}_{b_\epsilon(t,h),h}$-a.s.

$$\begin{aligned}W_{hz}(t+s,Z_s^1,H_s^1) &\leq -Z_s^{1\,\frac{1}{\alpha-1}}\alpha^{\frac{1}{1-\alpha}}\Big[\int_0^{T-(t+s)} e^{\frac{\alpha}{1-\alpha}(rs+\frac{1}{2}\theta^2 s)+\frac{\theta^2\alpha^2 s}{2(\alpha-1)^2}} e^{\int_0^s \frac{(\rho+M_u^{H^{2,h}}(H^1))}{\alpha-1}du} e^{-\delta T}ds\Big]\\ &= -Z_s^{1\,\frac{1}{\alpha-1}}\alpha^{\frac{1}{1-\alpha}}e^{-\delta T}N(t+s,h),\end{aligned}$$

where $N$ is defined in (B.15), so that from (B.19) we obtain

(B.21)

$$\widehat{J}_z(t,b_\epsilon(t,h),h) \geq f(I)e^{-\delta T}\alpha^{\frac{1}{1-\alpha}}\mathbb{E}_{b_\epsilon(t,h),h}\Big[\int_0^{\tau^*} e^{-(rs+\theta B_s+\frac{1}{2}\theta^2 s)} Z_s^{1\,\frac{1}{\alpha-1}} N(t+s,h)ds\Big] =: \frac{1}{\lambda_1^\epsilon(t,h)}.$$

From (B.10) (with $\lambda^\epsilon = \lambda_1^\epsilon$), (B.18) and (B.21) we conclude that the family of weak derivatives $(|\partial_h b_\epsilon(t,h)|)_{\epsilon\geq 0}$ is uniformly bounded; i.e.,

$$\begin{aligned}\sup_{\epsilon\geq 0}|\partial_h b_\epsilon(t,h)| &\leq \sup_{\epsilon\geq 0}(\lambda_1^\epsilon(t,h)L_2^\epsilon(t,h))\\ &= C_3(h)(1+e^{\delta\kappa T})\alpha^{\frac{1}{\alpha-1}}e^{\delta T}(1+g(t,h)).\end{aligned} \tag{B.22}$$

**Step 4.** We here show that $b_\epsilon$ is locally-Lipschitz continuous in $t$.

Step 4-(a). We here find an **upper bound for** $|\widehat{J}_t(t,b_\epsilon(t,h),h)|$**.** Recalling $W_h(t,z,h)$ as in (A.1), we have $\mathbb{P}_{b_\epsilon(t,h),h}$-a.s.

$$W_{ht}(t+s,Z_s^1,H_s^1) = -Z_s^{1\,\frac{\alpha}{\alpha-1}}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\Bigg[e^{c_1(T-t)+\int_0^{T-t}\frac{(\rho+M_u^{H^{2,h}}(H^1))}{\alpha-1}du}\times$$
$$\times\Big[\frac{m^1\kappa}{1-\alpha}\Big(\int_0^{T-(t+s)}\Big((he^{-2\delta u}+\frac{f(I)}{\delta}(1-e^{-\delta u}))^{-\kappa-1}e^{-\delta u}du\Big)\Big(he^{-\delta s}e^{-\delta(T-(t+s))}$$
$$+\frac{f(I)}{\delta}(1-e^{-\delta(T-t-s)})\Big)+e^{-\delta(T-t-s)}\Big]\Bigg]. \tag{B.23}$$

Also, since $\widehat{J}_t(t,z,h) = -f(I)\mathbb{E}_{z,h}\Big[\int_0^{\tau^*}e^{-\int_0^s(\rho+M_u^{H^1})du}W_{ht}(t+s,Z_s^1,H_s^1)ds\Big]$ by (4.10), (B.23) implies that

$$|\widehat{J}_t(t,b_\epsilon(t,h),h)| \le f(I)\mathbb{E}_{b_\epsilon(t,h),h}\Big[\int_0^{\tau^*}e^{-\int_0^s(\rho+M_u^{H^1})du}Z_s^{1\,\frac{\alpha}{\alpha-1}}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}O(t+s,h)ds\Big], \tag{B.24}$$

where

$$O(t+s,h) := e^{c_1(T-t)+\int_0^{T-t}\frac{(\rho+M_u^{H^{2,h}}(H^1))}{\alpha-1}du}\Big[\frac{m^1\kappa}{1-\alpha}\Big(\int_0^{T-(t+s)}\Big((he^{-2\delta u}+\frac{f(I)}{\delta}(1-e^{-\delta u}))^{-\kappa-1}e^{-\delta u}du\Big)\times$$
$$\times\Big(he^{-\delta s}e^{-\delta(T-(t+s))}+\frac{f(I)}{\delta}(1-e^{-\delta(T-t-s)})\Big)+e^{-\delta(T-t-s)}\Big]. \tag{B.25}$$

In order to obtain an upper bound for (B.25), we use $he^{-2\delta s} \le he^{-2\delta s}+\frac{f(I)}{\delta}(1-e^{-\delta s}) \le h+\frac{f(I)}{\delta}$, which gives $(he^{-2\delta u}+\frac{f(I)}{\delta}(1-e^{-\delta u}))^{-\kappa-1} \le h^{-\kappa-1}e^{2(\kappa+1)\delta u}$ and thus

$$\frac{m^1\kappa}{1-\alpha}\Big(\int_0^{T-(t+s)}(he^{-2\delta u}+\frac{f(I)}{\delta}(1-e^{-\delta u}))^{-\kappa-1}e^{-\delta u}du\Big)(he^{-\delta s}e^{-\delta(T-(t+s))}$$
$$+\frac{f(I)}{\delta}(1-e^{-\delta(T-t-s)}))+e^{-\delta(T-t-s)}$$
$$\le \frac{m^1\kappa}{1-\alpha}\int_0^{T-t-s}h^{-\kappa-1}e^{(2\kappa+2)\delta u}e^{-\delta u}(h+\frac{f(I)}{\delta})+1 \le \frac{m^1\kappa(h+\frac{f(I)}{\delta})^{-\kappa-1}}{(1-\alpha)(2\kappa+1)\delta}e^{(2\kappa+1)\delta T}+1 =: C_4(h).$$

Therefore, from (B.7) and (B.25) we have

$$O(t+s,h) \le C_4(h)e^{c_1(T-t)+\int_0^{T-t}\frac{(\rho+M_u^{H^{2,h}}(H^1))}{\alpha-1}du} \le C_4(h)e^{c_1T}, \tag{B.26}$$

where $c_1 = (r+\frac{1}{2}\theta^2)\frac{\alpha}{1-\alpha}+\frac{1}{2}(\frac{\alpha\theta}{\alpha-1})^2>0$. Then by (B.24) and (B.26) we know

$$|\widehat{J}_t(t,b_\epsilon(t,h),h)| \le f(I)\mathbb{E}_{b_\epsilon(t,h),h}\Big[\int_0^{\tau^*}e^{-\int_0^s(\rho+M_u^{H^1})du}Z_s^{1\,\frac{\alpha}{\alpha-1}}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}C_4(h)e^{c_1T}ds\Big]$$
$$= f(I)C_4(h)e^{c_1T}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\mathbb{E}_{b_\epsilon(t,h),h}\Big[\int_0^{\tau^*}e^{-\int_0^s(\rho+M_u^{H^1})du}Z_s^{1\,\frac{\alpha}{\alpha-1}}ds\Big]$$
$$= f(I)C_4(h)e^{c_1T}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}(b_\epsilon(t,h))^{\frac{\alpha}{\alpha-1}}\mathbb{E}_{b_\epsilon(t,h),h}\Big[\int_0^{\tau^*}e^{-\int_0^s(r+\theta B_u+\frac{1}{2}\theta^2)du}P_s^1(h)^{\frac{1}{\alpha-1}}ds\Big]$$
$$= f(I)C_4(h)e^{c_1T}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}(b_\epsilon(t,h))^{\frac{\alpha}{\alpha-1}}\mathbb{E}_{b_\epsilon(t,h),h}\Big[\int_0^{\tau^*}e^{(rs+\theta B_s+\frac{1}{2}\theta^2 s)\frac{\alpha}{1-\alpha}}(e^{\rho s+\int_0^s M_u^{H^1}du})^{\frac{1}{\alpha-1}}ds\Big]$$
$$\le f(I)C_4(h)e^{c_1T}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}(b_\epsilon(t,h))^{\frac{\alpha}{\alpha-1}}\mathbb{E}_{b_\epsilon(t,h),h}\Big[\int_0^{\tau^*}e^{(rs+\theta B_s+\frac{1}{2}\theta^2 s)\frac{\alpha}{1-\alpha}}ds\Big]$$

(B.27)

$$= f(I)C_4(h)e^{c_1T}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}(b_\epsilon(t,h))^{\frac{\alpha}{\alpha-1}}\mathbb{E}\bigg[\int_0^{\tau_\epsilon^*} e^{(rs+\theta B_s+\frac{1}{2}\theta^2 s)\frac{\alpha}{1-\alpha}}ds\bigg] := L_1^\epsilon(t,h),$$

where $\tau_\epsilon^* := \tau^*(t, b_\epsilon(t,h), h)$.

Step 4-(b). In this step we perform two changes of probability measures through a Girsanov argument in order to take care of the expectation on the very right-hand side of (B.27). We define the probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F}_T)$ as

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\Big\{-\theta B_T - \frac{1}{2}\theta^2 T\Big\}.$$

By Girsanov's Theorem, the process $B^{\mathbb{Q}} := \{B_s + \theta s, s \in [0,T]\}$ is a standard Brownian motion under the new measure $\mathbb{Q}$.

From the expectation on the very right-hand side of (B.27) we then have

$$\begin{aligned}\mathbb{E}\bigg[\int_0^{\tau_\epsilon^*} e^{\frac{\alpha}{1-\alpha}(rs+\theta B_s+\frac{1}{2}\theta^2 s)}ds\bigg] &= \mathbb{E}_{\mathbb{Q}}\bigg[\int_0^{\tau_\epsilon^*} e^{\frac{\alpha}{1-\alpha}(rs+\frac{1}{2}\theta^2 s)}e^{\frac{\alpha}{1-\alpha}\theta B_s}e^{\theta B_s+\frac{1}{2}\theta^2 s}ds\bigg]\\ &= \mathbb{E}_{\mathbb{Q}}\bigg[\int_0^{\tau_\epsilon^*} e^{\frac{r\alpha}{1-\alpha}s}e^{\frac{-\theta^2}{2(1-\alpha)}s}e^{\frac{\theta}{1-\alpha}B_s^{\mathbb{Q}}}ds\bigg]. \qquad \text{(B.28)}\end{aligned}$$

Next, we define another auxiliary probability measure $\widetilde{\mathbb{Q}}$ on $(\Omega, \mathcal{F}_T)$ such that

$$\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}} = \exp\Big\{\frac{\theta}{1-\alpha}B_T^{\mathbb{Q}} - \frac{1}{2}\frac{\theta^2}{(1-\alpha)^2}T\Big\}.$$

By Girsanov's Theorem, we obtain the process $B^{\widetilde{\mathbb{Q}}} := \{B_s^{\mathbb{Q}} - \frac{\theta}{1-\alpha}s, s \in [0,T]\}$, which is a standard Brownian motion under the new measure $\widetilde{\mathbb{Q}}$.

From (B.28) finally find

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}}\bigg[\int_0^{\tau_\epsilon^*} e^{\frac{r\alpha}{1-\alpha}s}e^{\frac{-\theta^2}{2(1-\alpha)}s}e^{\frac{\theta}{1-\alpha}B_s^{\mathbb{Q}}}ds\bigg] = \mathbb{E}_{\mathbb{Q}}\bigg[\int_0^{\tau_\epsilon^*} e^{\frac{r\alpha}{1-\alpha}s}e^{\frac{-\theta^2}{2(1-\alpha)}s}e^{\frac{\theta}{1-\alpha}B_s^{\mathbb{Q}}}e^{-\frac{1}{2}\frac{\theta^2}{(1-\alpha)^2}s}e^{\frac{1}{2}\frac{\theta^2}{(1-\alpha)^2}s}ds\bigg]\\ &= \mathbb{E}_{\widetilde{\mathbb{Q}}}\bigg[\int_0^{\tau_\epsilon^*} e^{\frac{r\alpha}{1-\alpha}s}e^{\frac{\theta^2\alpha s}{2(1-\alpha)^2}}ds\bigg] \leq c_2\mathbb{E}_{\widetilde{\mathbb{Q}}}\bigg[\int_0^{\tau_\epsilon^*}ds\bigg] = c_2\mathbb{E}_{\widetilde{\mathbb{Q}}}[\tau_\epsilon^*], \qquad \text{(B.29)}\end{aligned}$$

where $c_2 := e^{\frac{r\alpha}{1-\alpha}T}e^{\frac{\theta^2\alpha T}{2(1-\alpha)^2}}$.

Step 4-(c). We here determine another **lower bound** for $\widehat{J}_z$. From Proposition 4.3, we have

$$\widehat{J}_z(t, b_\epsilon(t,h), h) = \mathbb{E}_{b_\epsilon(t,h),h}\bigg[\int_0^{\tau^*} e^{-rs-\theta B_s-\frac{1}{2}\theta^2 s}\Big(I - f(I)W_{hz}(t+s, Z_s^1, H_s^1)\Big)ds\bigg],$$

which, due to $W_{hz}(t,z,h) < 0$ for all $(t,z,h) \in [0,T) \times \mathbb{R}_+^2$ (cf. (B.8)), yields

$$\widehat{J}_z(t, b_\epsilon(t,h), h) \geq I\mathbb{E}\bigg[\int_0^{\tau_\epsilon^*} e^{-rs-\theta B_s-\frac{1}{2}\theta^2 s}ds\bigg] := \frac{1}{\lambda_2^\epsilon(t,h)}.$$

Recalling the measure $\mathbb{Q}$ from Step 4-(b) above, we change measure from $\mathbb{P}$ to $\mathbb{Q}$ in the right-hand side of the inequality above and obtain

$$\begin{aligned}\widehat{J}_z(t, b_\epsilon(t,h), h) &\geq I\mathbb{E}\bigg[\int_0^{\tau_\epsilon^*} e^{-rs-\theta B_s-\frac{1}{2}\theta^2 s}ds\bigg] = I\mathbb{E}_{\mathbb{Q}}\bigg[\int_0^{\tau_\epsilon^*} e^{-rs}ds\bigg]\\ &\geq Ie^{-rT}\mathbb{E}_{\mathbb{Q}}\bigg[\int_0^{\tau_\epsilon^*}ds\bigg] = Ie^{-rT}\mathbb{E}_{\mathbb{Q}}[\tau_\epsilon^*]. \qquad \text{(B.30)}\end{aligned}$$

Step 4-(d). Our aim here is to bound of $\frac{\mathbb{E}_{\widetilde{\mathbb{Q}}}[\tau^*_\epsilon]}{\mathbb{E}_{\mathbb{Q}}[\tau^*_\epsilon]}$, uniformly with respect to $\epsilon$. This term arises from the ratio of the two expectations in (B.29) and (B.30).

Since the dynamics of $Z^\epsilon_s$ under $\mathbb{P}$ are

$$dZ^\epsilon_s = (\rho - r + M^{H^1}_s)Z^\epsilon_s ds - \theta Z^\epsilon_s dB_s, \quad Z^\epsilon_0 = b_\epsilon(h,t),$$

those become under $\mathbb{Q}$ (remember that $B^{\mathbb{Q}} := \{B_s + \theta s, s \in [0,T]\}$)

$$dZ^\epsilon_s = (\rho - r + M^{H^1}_s + \theta^2)Z^\epsilon_s ds - \theta Z^\epsilon_s dB^{\mathbb{Q}}_s, \quad Z^\epsilon_0 = b_\epsilon(h,t),$$

while they are

$$dZ^\epsilon_s = (\rho - r + M^{H^1}_s + \theta^2 - \frac{\theta^2}{1-\alpha})Z^\epsilon_s ds - \theta Z^\epsilon_s dB^{\widetilde{\mathbb{Q}}}_s, \quad Z^\epsilon_0 = b_\epsilon(h,t),$$

under $\widetilde{\mathbb{Q}}$ (here recall that $B^{\widetilde{\mathbb{Q}}} := \{B^{\mathbb{Q}}_s - \frac{\theta}{1-\alpha}s, s \in [0,T]\}$).

Now, if on $(\Omega, \mathcal{F}, \mathbb{Q})$ we define

$$d\widetilde{Z}^\epsilon_s = (\rho - r + M^{H^1}_s + \theta^2 - \frac{\theta^2}{1-\alpha})\widetilde{Z}^\epsilon_s ds - \theta \widetilde{Z}^\epsilon_s dB^{\mathbb{Q}}_s, \quad \widetilde{Z}^\epsilon_0 = b_\epsilon(h,t).$$

and $\widetilde{\tau}^*_\epsilon := \inf\{s \in [0, T-t] : (t+s, \widetilde{Z}^\epsilon_s, H^1_s) \in \mathcal{I}\}$, then we see that

$$\mathrm{Law}(Z^\epsilon_s|\widetilde{\mathbb{Q}}) = \mathrm{Law}(\widetilde{Z}^\epsilon_s|\mathbb{Q}), \quad \mathrm{Law}(\tau^*_\epsilon|\widetilde{\mathbb{Q}}) = \mathrm{Law}(\widetilde{\tau}^*_\epsilon|\mathbb{Q}),$$

where $\tau^*_\epsilon := \inf\{s \in [0, T-t] : (t+s, Z^\epsilon_s, H^1_s) \in \mathcal{I}\}$. Moreover by the comparison principles for SDEs we have that $Z^\epsilon_s \geq \widetilde{Z}^\epsilon_s$, $\mathbb{Q}$-a.s., for all $s \in [0, T-t]$ since $\alpha < 1$, and therefore, we have $\tau^*_\epsilon \geq \widetilde{\tau}^*_\epsilon$, $\mathbb{Q}$-a.s., and

$$\mathbb{E}_{\widetilde{\mathbb{Q}}}[\tau^*_\epsilon] = \mathbb{E}_{\mathbb{Q}}[\widetilde{\tau}^*_\epsilon] \leq \mathbb{E}_{\mathbb{Q}}[\tau^*_\epsilon]. \tag{B.31}$$

Step 4-(e). Combining (B.10) (with $\lambda^\epsilon = \lambda^\epsilon_2$), (B.27), (B.28), (B.29), (B.30) and (B.31) we have

$$\begin{aligned}
\sup_{\epsilon \geq 0} |\partial_t b_\epsilon(t,h)| &\leq \sup_{\epsilon \geq 0} (\lambda^\epsilon_2(t,h) L^\epsilon_1(t,h)) \\
&\leq \sup_{\epsilon \geq 0} \frac{f(I) C_4(h) e^{c_1 T}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}(b_\epsilon(t,h))^{\frac{\alpha}{\alpha-1}}\mathbb{E}\left[\int_0^{\tau^*_\epsilon} e^{(rs + \theta B_s + \frac{1}{2}\theta^2 s)\frac{\alpha}{1-\alpha}} ds\right]}{I e^{-rT}\mathbb{E}_{\mathbb{Q}}[\tau^*_\epsilon]} \\
&\leq \sup_{\epsilon \geq 0} \frac{f(I) C_4(h) e^{c_1 T}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}(b_\epsilon(t,h))^{\frac{\alpha}{\alpha-1}} c_2 \mathbb{E}_{\widetilde{\mathbb{Q}}}[\tau^*_\epsilon]}{I e^{-rT}\mathbb{E}_{\mathbb{Q}}[\tau^*_\epsilon]} \\
&\leq \frac{f(I) C_4(h) e^{c_1 T}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}(b(t,h))^{\frac{\alpha}{\alpha-1}} c_2}{I e^{-rT}}.
\end{aligned} \tag{B.32}$$

**Step 5.** Combining the findings of the previous steps, by (B.10) we have that $b_\epsilon$ is locally-Lipschitz continuous, with Lipschitz constants that are independent of $\epsilon$ (see (B.18), (B.21) and (B.32)). Furthermore, the family $(b_\epsilon)_\epsilon$ is also uniformly bounded (cf. Step 2).

Hence, by Ascoli-Arzelà theorem we can extract a subsequence $(\epsilon_j)_{j \in \mathbb{N}}$ such that $b_{\epsilon_j} \to g$ uniformly, with $g$ being Lipschitz continuous with the same Lipschitz constant of $b_\epsilon$. However, $b_{\epsilon_j}$ converges to $b$ (cf. Step 1), which, by uniqueness of the limit, is then locally Lipschitz continuous.

□

B.4. **Proof of Lemma 4.3.**

*Proof.* The claim is trivial for $(t,z,h)$ such that $z < b(t,h)$, hence we fix $(t,z,h) \in \mathcal{O}$ with $z \geq b(t,h)$ in the subsequent proof. It is easy to check that $\hat{\tau}(t,z,h) \geq \tau^*(t,z,h)$ by their definitions. In order to show the reverse inequality, the rest of the proof is organized in two steps.

**Step 1.** We claim that

$$\hat{\tau}(t, b(t,h), h) = 0, \quad \mathbb{P}-a.s.$$

due to the Lipschitz continuity of $b(t,h)$ and the law of the iterated logarithm of Brownian motion. As a matter of fact, we fix a point $(t_0, z_0, h_0) \in \partial\mathcal{W} \cap \{t < T\}$ and take a sequence $(t_n, z_n, h_n)_{n\in\mathbb{N}} \subseteq \mathcal{W}$ with $(t_n, z_n, h_n) \to (t_0, z_0, h_0)$ as $n \to \infty$. We also fix $\omega \in \Omega_0$, with $\mathbb{P}(\Omega_0) > 0$, and assume that $\limsup_{n\to\infty} \hat{\tau}(t_n, z_n, h_n)(\omega) =: \lambda > 0$. Hence,

$$Z_s^{1,z_n}(\omega) \geq b(t_n + s, H_s^{1,h_n}), \quad \forall n \in \mathbb{N},\ \forall s \in [0, \frac{\lambda}{2}].$$

Upon using that $(t,h) \mapsto b(t_n + s, H_s^{1,h_n})$ is Lipschitz continuous (cf. Theorem 4.1), we let $n \to \infty$ and obtain

$$\begin{aligned} Z_s^{1,z_0}(\omega) &\geq b(t_0, h_0) + b(t_0 + s, H_s^{1,h_0}) - b(t_0, H_s^{1,h_0}) + b(t_0, H_s^{1,h_0}) - b(t_0, h_0) \\ &= b(t_0, h_0) + \int_0^s \partial_t b(t_0 + u, H_s^{1,h_0})du + \int_{h_0}^{h_0 e^{-\delta s}} \partial_h b(t_0, u)du. \end{aligned} \tag{B.33}$$

However, from (B.22) and (B.32) we have

$$\begin{aligned} \partial_h b(t_0, u) &\geq -\bar{\kappa}_1 C_3(u)(1 + g(t_0, u)), \\ \partial_t b(t_0 + u, H_s^{1,h_0}) &\geq -\bar{\kappa}_2 C_4(h_0 e^{-\delta s}) b(t_0 + u, h_0 e^{-\delta s})^{\frac{\alpha}{\alpha-1}}, \end{aligned}$$

where $\bar{\kappa}_1 := (1 + e^{\delta\kappa T})e^{\delta T}\alpha^{\frac{1}{\alpha-1}}$ and $\bar{\kappa}_2 := \frac{f(I)e^{c_1 T}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}c_2}{Ie^{-rT}}$, which used in (B.33) give

$$\begin{aligned} Z_s^{1,z_0}(\omega) &\geq b(t_0, h_0) - \int_0^s \bar{\kappa}_2 C_4(h_0 e^{-\delta s}) b(t_0 + u, h_0 e^{-\delta s})^{\frac{\alpha}{\alpha-1}} du - \int_{h_0}^{h_0 e^{-\delta s}} \bar{\kappa}_1 C_3(u)(1 + g(t_0, u))du \\ &= b(t_0, h_0) - \int_0^s \bar{\kappa}_2 C_4(h_0 e^{-\delta s}) b(t_0 + u, h_0 e^{-\delta s})^{\frac{\alpha}{\alpha-1}} du + \int_{h_0 e^{-\delta s}}^{h_0} \bar{\kappa}_1 C_3(u)(1 + g(t_0, u))du \\ &\geq b(t_0, h_0) - \int_0^s \bar{\kappa}_2 C_4(h_0 e^{-\delta s}) b(t_0 + u, h_0 e^{-\delta s})^{\frac{\alpha}{\alpha-1}} du \\ &\geq b(t_0, h_0) - \bar{k}_3(s)s, \end{aligned}$$

where $\bar{k}_3(s) := \bar{k}_2 C_4(h_0) \max_{u\in[0,\frac{\lambda}{2}]}[b(t_0 + u, h_0 e^{-\delta s})^{\frac{\alpha}{\alpha-1}}]$. Since $b(t_0, h_0) = z_0$, then we have

$$z_0 e^{(\rho - r - \frac{1}{2}\theta^2)s + \int_0^s M_u^H du} e^{-\theta B_s(\omega)} \geq z_0 - \bar{k}_3(s) \cdot s. \tag{B.34}$$

By the law of the iterated logarithm (cf. Theorem 9.23 in [Karatzas and Shreve, 1998a]), for all $\epsilon > 0$ we have (along a sequence of times converging to zero)

$$B_s(\omega) \geq (1 - \epsilon)\sqrt{2s \log(\log(\frac{1}{s}))}, \tag{B.35}$$

which combined with (B.34) yields

$$z_0 e^{(\rho - r - \frac{1}{2}\theta^2)s + \int_0^s M_u^H du} e^{-\theta(1-\epsilon)\sqrt{2s\log(\log(\frac{1}{s}))}} \geq z_0 - \bar{k}_3(s) \cdot s.$$

On the other hand, since $e^x = 1 + x + O(x^2)$ when $x \approx 0$, the last display equation implies (for $s$ small enough) that

$$z_0 \Big[1 - \theta(1 - \epsilon)\sqrt{2s \log(\log(\frac{1}{s}))} + (\rho - r - \frac{1}{2}\theta^2)s + \int_0^s M_u^H du\Big] \geq z_0 - \bar{k}_3(s) \cdot s,$$

which simplified gives

$$z_0\theta(1-\epsilon)\sqrt{2s\log(\log(\frac{1}{s}))}-z_0(\rho-r-\frac{1}{2}\theta^2)s-z_0\int_0^s M_u^H du\le \bar{k}_3(s)\cdot s. \tag{B.36}$$

Then dividing by $s$ and letting $s\downarrow 0$, we obtain that the left hand side of the inequality in (B.36) is $\infty$ (since $\sqrt{2s\log(\log(\frac{1}{s}))}/s\to\infty$ for $s\downarrow 0$), but the right hand side of the inequality in (B.36) is the constant $\bar{k}_3(0)$. Thus, we reach a contradiction and $\hat{\tau}(t,b(t,h),h)=0$, $\mathbb{P}$-a.s.

**Step 2.** In order to prove that $\hat{\tau}(t,z,h)\le\tau^*(t,z,h)$, one can use arguments as in the proof of Lemma 5.1 in [De Angelis and Ekström, 2017].

□

## Appendix C. Some auxiliary results

**Lemma C.1.** *Let $x>g(t)$ be given, let $c\ge 0$ be a consumption process satisfying*

$$\mathbb{E}_{t,x,h}\left[\int_t^T \gamma_{s,t}c_s ds\right]=x-g(t).$$

*Then, there exists a portfolio process $\pi$ such that the pair $(c,\pi)$ is admissible and*

$$X_s^{c,\pi,\tau}>g(s),\ \text{for } s\ge\tau.$$

*Proof.* Let us define $L_s:=\int_t^s\gamma_{u,t}c_u du$ and consider the nonnegative martingale

$$M_s:=\mathbb{E}[L_T|\mathcal{F}_s],\ t\le s\le T.$$

According to the martingale representation theorem, there is an $\mathbb{F}$-adapted process $\phi$ satisfying $\int_t^T||\phi_u||^2du<\infty$ almost surely and

$$M_s=M_t+\int_t^s\phi_u dB_u=x-g(t)+\int_t^s\phi_u dB_u,\ t\le s\le T.$$

Define then the nonnegative process $X$ by

$$X_s:=\frac{1}{\gamma_{s,t}}\mathbb{E}\left[\int_s^T\gamma_{u,t}c_u du\middle|\mathcal{F}_s\right]+g(s)=\frac{1}{\gamma_{s,t}}[M_s-L_s]+g(s),$$

so that $X_t=x, M_t=x-g(t)$. Itô's rule implies

$$d(e^{-rs}X_s)=-c_se^{-rs}ds-Ie^{-rs}ds+e^{-rs}\pi_s\sigma dB_s,$$

where $\pi_s:=\frac{1}{\gamma_{s,t}\sigma}[\phi_s+(M_s-L_s)\theta]$. It is easy to check that $\pi$ satisfies $\int_t^T|\pi_s|^2ds<\infty$ a.s. (see, e.g., Theorem 3.3.5 in [Karatzas and Shreve, 1998b]). We thus conclude that $X_s=X_s^{c,\pi,\tau}$ when $s\ge\tau$, by comparison with (2.5). Finally, since $X_s>g(s)$ for $s\ge\tau$, the pair $(c,\pi)$ is admissible, and $X_s^{c,\pi,\tau}>g(s)$, for $s\ge\tau$.

□

**Lemma C.2.** *For any $\tau\in\mathcal{S}$, let $x>0$ be given, let $c\ge 0$ be a consumption process. For any $\mathcal{F}_\tau$-measurable random variable $\phi$ with $\mathbb{P}[\phi>0]=1$ such that*

$$\mathbb{E}_{t,x,h}\left[\gamma_{\tau,t}\phi+\int_t^\tau\gamma_{s,t}c_s ds\right]=x,$$

*there exists a portfolio process $\pi$ such that the pair $(c,\pi)$ is admissible and*

$$X_s^{c,\pi,\tau}>0,\ \text{for } s\le\tau,\ \phi=X_\tau^{c,\pi,\tau}.$$

*Proof.* The proof is similar to Lemma 6.3 in [Karatzas and Wang, 2000], and we thus omit details.

□

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