LOCAL LAWS FOR SPARSE SAMPLE COVARIANCE MATRICES WITHOUT THE TRUNCATION CONDITION

F. GÖTZE*, A. TIKHOMIROV, AND D. TIMUSHEV

Abstract. We consider sparse sample covariance matrices $\frac{1}{np_n}\mathbf{X}\mathbf{X}^*$, where $\mathbf{X}$ is a sparse matrix of order $n \times m$ with the sparse probability p_n. We prove the local Marchenko–Pastur law in some complex domain assuming that $np_n > \log^\beta n$, $\beta > 0$ and some $(4+\delta)$-moment condition is fulfilled, $\delta > 0$.

1. Introduction

Sample covariance matrices are of great practical importance for problems of multivariate statistical analysis and such rapidly developing areas as the theory of wireless communication and deep learning. Another significant area of application of sample covariance matrices is graph theory. The adjacency matrix of an undirected graph is asymmetric, so the study of its singular values leads to the sample covariance matrix. If we assume that the probability p_n of having graph edges tends to zero as the number of vertices n increases to infinity, we get to the concept of sparse random matrices.

Sparse Wigner random matrices have been considered in a number of papers (see [1, 2, 3, 4]) where many results have been obtained. With the symmetrization of sample covariance matrices it is possible to apply this results in the case when the observation matrix is square. However, when the sample size is greater than observation dimension, the spectral limit distribution has the singularity in zero, which requires different approaches.

The limit spectral distribution of sparse sample covariance matrices with sparsity $np_n \sim n^\varepsilon$, ($\varepsilon > 0$ is arbitrary small) was studied in [5, 6]. In particular, a local law was proved under the assumption that the matrix elements satisfy the moments condition $\mathbb{E}\,|X_{jk}|^q \le (Cq)^{cq}$. In the paper [7] the case of the sparsity $np_n \sim \log^\alpha n$, for some $\alpha > 1$ was considered, assuming that the moments of the matrix elements satisfy the conditions $\mathbb{E}\,|X_{jk}|^{4+\delta} \le C < \infty$, $|X_{jk}| \le c_1(np_n)^{\frac{1}{2}-\varkappa}$, for some $\varkappa > 0$. Under this assumptions the local Marchenko–Pastur law was proved in some complex domain $z \in \mathcal{D}$ with $\operatorname{Im} z > v_0 > 0$, where v_0 is of order $\log^4 n/n$ and the domain bound not depend on p_n while $np_n > \log^\beta n$.

This work is devoted to the case, when the elements X_{jk} are not truncated, and only the conditions $\mathbb{E}\,|X_{jk}|^{4+\delta} \le C < \infty$, $np_n \sim \log^\alpha n$, for some $\alpha > 1$ are fulfilled. We prove the local Marchenko–Pastur law in some complex domain $u + iv \in \mathcal{D}_\mu$ with the

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real part contained in the support of the Marchenko–Pastur distribution and separated from the support ends.

2. Main results

Let $m = m(n)$, $m \geq n$. Consider independent identically distributed zero mean random variables X_{jk}, $1 \leq j \leq n$, $1 \leq k \leq m$ with $\mathbb{E}\, X_{jk}^2 = 1$ and independent of that set independent Bernoulli random variables ξ_{jk}, $1 \leq j \leq n$, $1 \leq k \leq m$ with $\mathbb{E}\, \xi_{jk} = p_n$. In addition suppose that $np_n \to \infty$ as $n \to \infty$.

Observe the sequence of sparse sample covariance random matrices

$$\mathbf{X} = \frac{1}{\sqrt{mp_n}} (\xi_{jk} X_{jk})_{1 \leq j \leq n, 1 \leq k \leq m}.$$

Denote by $s_1 \geq \cdots \geq s_n$ the singular values of $\mathbf{X}$ and define the symmetrized empirical spectral distribution function (ESD) of the sample covariance matrix $\mathbf{W} = \mathbf{X}\mathbf{X}^*$:

$$F_n(x) = \frac{1}{2n} \sum_{j=1}^{n} \Big(\mathbb{I}\{s_j \leq x\} + \mathbb{I}\{-s_j \leq x\} \Big),$$

where $\mathbb{I}\{A\}$ stands for the event A indicator.

Note that $F_n(x)$ is the ESD of the block matrix

$$\mathbf{V} = \begin{bmatrix} \mathbf{O}_n & \mathbf{X} \\ \mathbf{X}^* & \mathbf{O}_m \end{bmatrix},$$

where $\mathbf{O}_k$ is $k \times k$ matrix with zero elements.

Denote $\mathbf{R} = \mathbf{R}(z)$ the resolvent matrix of $\mathbf{V}$:

$$\mathbf{R} = (\mathbf{V} - z\mathbf{I})^{-1}.$$

Let $y = y(n) = \frac{n}{m}$ and $G_y(x)$ — the symmetrized Marchenko–Pastur distribution function with the density

$$g_y(x) = \frac{1}{2\pi y |x|} \sqrt{(x^2 - a^2)(b^2 - x^2)}\, \mathbb{I}\{a^2 \leq x^2 \leq b^2\},$$

where $a = 1 - \sqrt{y}$, $\quad b = 1 + \sqrt{y}$. We shall assume that $y \leq y_0 < 1$ for $n, m \geq 1$. Denote by $S_y(z)$ the Stieltjes transform of the distribution function $G_y(x)$ and $s_n(z)$ the Stieltjes transform of the distribution function $F_n(x)$. We have

$$\begin{aligned} S_y(z) =& \frac{-z + \frac{1-y}{z} + \sqrt{(z - \frac{1-y}{z})^2 - 4y}}{2y}, \\ s_n(z) =& \frac{1}{2n} \Big[\sum_{j=1}^{n} \frac{1}{s_j - z} + \sum_{j=1}^{n} \frac{1}{-s_j - z} \Big] = \frac{1}{n} \sum_{j=1}^{n} \frac{z}{s_j^2 - z^2} = \frac{1}{n} \sum_{j=1}^{n} R_{jj}. \end{aligned}$$

The last equality follows from Schur complement (see [7, Section 3]). Put

$$b(z) = z - \frac{1-y}{z} + 2yS_y(z) = -\frac{1}{S_y(z)} + yS_y(z). \tag{2.1}$$

In this paper we prove so called Marchenko–Pastur law for sparse sample covariance matrices. Let

$$\Lambda_n := \Lambda_n(z) = s_n(z) - S_y(z).$$

For constant $\delta > 0$ define the value $\varkappa = \varkappa(\delta) := \frac{\delta}{2(4+\delta)}$ and consider the following conditions:

- the condition $(C0)$: for some $c_0 > 0$ and all $n \ge 1$ we have $np_n \ge c_0 \log^{\frac{2}{\varkappa}} n$;
- the condition $(C1)$: for some $\delta > 0$ we have $\mu_{4+\delta} := \mathbb{E}\,|X_{11}|^{4+\delta} < \infty$;
- the condition $(C2)$: there exists a constant $c_1 > 0$ such that for all $1 \le j \le n$, $1 \le k \le m$ we have $|X_{jk}| \le c_1 (np_n)^{\frac{1}{2}-\varkappa}$ almost surely.

Introduce the quantity $v_0 = v_0(a_0) := a_0 n^{-1} \log^4 n$ with some positive constant a_0, and define the region

$$\mathcal{D}(a_0) := \{z = u + iv : (1 - \sqrt{y} - v)_+ \le |u| \le 1 + \sqrt{y} + v, V \ge v \ge v_0\}.$$

Let

$$\Gamma_n = 2C_0 \log n \Big(\frac{1}{nv} + \min\Big\{ \frac{1}{np|b(z)|}, \frac{1}{\sqrt{np}} \Big\} \Big),$$

$$d(z) = \frac{\operatorname{Im} b(z)}{|b(z)|},$$

and

$$d_n(z) := \frac{1}{nv} \left(d(z) + \frac{\log n}{nv|b(z)|} \right) + \frac{1}{np|b(z)|}.$$

Put

$$\mathcal{T}_n := \mathbb{I}\{|b(z)| \ge \Gamma_n\} \left(d_n(z) + d_n^{\frac{3}{4}}(z) \frac{1}{(nv)^{\frac{1}{4}}} + d_n^{\frac{1}{2}}(z) \frac{1}{(nv)^{\frac{1}{2}}} \right) + \mathbb{I}\{|b(z)| \le \Gamma_n\} \left(\left(\frac{\Gamma_n}{nv} \right)^{\frac{1}{2}} + \Gamma_n^{\frac{1}{2}} \left(\frac{\Gamma_n^{\frac{1}{2}}}{\sqrt{nv}} + \frac{1}{\sqrt{np}} \right) \right).$$

In the paper [7], assuming that the conditions $(C0)$–$(C2)$ are satisfied, the next theorem was proved:

Theorem 2.1. *Assume that the conditions $(C0)$–$(C2)$ are satisfied. Then for any $Q \ge 1$ there exist positive constants $C = C(Q, \delta, \mu_{4+\delta}, c_0, c_1)$, $K = K(Q, \delta, \mu_{4+\delta}, c_0, c_1)$, $a_0 = a_0(Q, \delta, \mu_{4+\delta}, c_0, c_1)$ such that for $z \in \mathcal{D}(a_0)$*

$$\Pr\Big\{ |\Lambda_n| \ge K\mathcal{T}_n \Big\} \le Cn^{-Q}.$$

This work is devoted to the case, when the elements X_{jk} are not truncated, and only the conditions $(C0)$–$(C1)$ are fulfilled. Let

$$\mathcal{D}_\mu = \{z = u + iv :\ 1 - \sqrt{y} + \mu \le |u| \le 1 + \sqrt{y} - \mu,\ V \ge v \ge v_0\},$$

for some $\mu > 0$. Note that $|b(z)|$ are bounded in domain $\mathcal{D}_\mu$, therefore

$$\Gamma_n = C_0 \log n \Big(\frac{1}{nv} + \frac{1}{np} \Big). \tag{2.2}$$

Without assumption $(C1)$ we get the following result.

Theorem 2.2. *Assume that the conditions* $(C0)$–$(C1)$ *are satisfied. Then for any* $\mu > 0$ *and* $Q \geq 1$ *there exist constants* $K = K(Q, \delta, \mu_{4+\delta}, \mu)$, $a_0 = a_0(Q, \delta, \mu_{4+\delta}, \mu)$ *depending on* Q, δ, $\mu_{4+\delta}$ *and* μ *such that*

$$\Pr\{|\Lambda_n| \leq K\Gamma_n\} \geq 1 - n^{-Q},$$

for all $z \in \mathcal{D}_\mu$ *and* Γ_n *defined in* (2.2).

Organization. The proof of the theorem is based on papers [8] and [7]. In **Section 3** we follow [7]. In our case the domain $\mathcal{D}_\mu$ is separated from the ends of the spectrum. This makes it possible to significantly simplify the estimates obtained there and so to prove Theorem 2.2. In **Section 4**, we show that the elements R_{jk} of the resolvent are bounded. For this, following [8], we introduce the so-called admissible and inadmissible configurations. Assuming that the configuration is admissible, we obtain conditional estimates for R_{jk}. Further, taking into account the small probability of inadmissible configurations, we obtain the estimate for the resolvent elements. In the **Section 5** we state and prove some auxiliary results.

Notation. We use C for large universal constants which maybe different from line by line. $S_y(z)$ and $s_n(z)$ denote the Stieltjes transforms of the symmetrized Marchenko–Pastur distribution and the spectral distribution function correspondingly. $R(z)$ denotes the resolvent matrix. Let $\mathbb{T} = \{1, \dots, n\}$, $\mathbb{J} \subset \mathbb{T}$ and $\mathbb{T}^{(1)} = \{1, \dots, m\}$, $\mathbb{K} \subset \mathbb{T}^{(1)}$. Consider σ-algebras $\mathfrak{M}^{(\mathbb{J},\mathbb{K})}$, generated by the elements of $\mathbf{X}$ with the exception of the rows with number from $\mathbb{J}$ and the columns with number from $\mathbb{K}$. We will write for brevity $\mathfrak{M}_j^{(\mathbb{J},\mathbb{K})}$ instead of $\mathfrak{M}^{(\mathbb{J}\cup\{j\},\mathbb{K})}$ and $\mathfrak{M}_{l+n}^{(\mathbb{J},\mathbb{K})}$ instead of $\mathfrak{M}^{(\mathbb{J},\mathbb{K}\cup\{l\})}$. By symbol $\mathbf{X}^{(\mathbb{J},\mathbb{K})}$ we denote the matrix $\mathbf{X}$ which rows with numbers in $\mathbb{J}$ are deleted, and which columns with numbers in $\mathbb{K}$ are deleted too. In a similar way, we will denote all objects defined via $\mathbf{X}^{(\mathbb{J},\mathbb{K})}$, such that the resolvent matrix $\mathbf{R}^{(\mathbb{J},\mathbb{K})}$, the ESD Stieltjes transform $s_n^{(\mathbb{J},\mathbb{K})}$, $\Lambda_n^{(\mathbb{J},\mathbb{K})}$ and so on. The symbol $\mathbb{E}_j$ denotes the conditional expectation with respect to the σ-algebra $\mathfrak{M}_j$, and $\mathbb{E}_{l+n}$ — with respect to σ-algebra $\mathfrak{M}_{l+n}$. Let $\mathbb{J}^c = \mathbb{T} \setminus \mathbb{J}$, $\mathbb{K}^c = \mathbb{T}^{(1)} \setminus \mathbb{K}$.

3. Proof of Theorem 2.2

For the diagonal elements of $\mathbf{R}$ we can write

$$R_{jj}^{(\mathbb{J},\mathbb{K})} = S_y(z)\big(1 - \varepsilon_j^{(\mathbb{J},\mathbb{K})} R_{jj}^{(\mathbb{J},\mathbb{K})} + y\Lambda_n^{(\mathbb{J},\mathbb{K})} R_{jj}^{(\mathbb{J},\mathbb{K})}\big), \tag{3.1}$$

for $j \in \mathbb{J}^c$, and

$$R_{l+n,l+n}^{(\mathbb{J},\mathbb{K})} = -\frac{1}{z + yS_y(z)}\big(1 - \varepsilon_{l+n}^{(\mathbb{J},\mathbb{K})} R_{l+n,l+n}^{(\mathbb{J},\mathbb{K})} + y\Lambda_n^{(\mathbb{J},\mathbb{K})} R_{l+n,l+n}^{(\mathbb{J},\mathbb{K})}\big), \tag{3.2}$$

for $l \in \mathbb{K}^c$. Correction terms $\varepsilon_j^{(\mathbb{J},\mathbb{K})}$ for $j \in \mathbb{J}^c$ and $\varepsilon_{l+n}^{(\mathbb{J},\mathbb{K})}$ for $l \in \mathbb{K}^c$ are defined as

$$\begin{aligned}
\varepsilon_j^{(\mathbb{J},\mathbb{K})} &= \varepsilon_{j1}^{(\mathbb{J},\mathbb{K})} + \cdots + \varepsilon_{j3}^{(\mathbb{J},\mathbb{K})},\\
\varepsilon_{j1}^{(\mathbb{J},\mathbb{K})} &= \frac{1}{m}\sum_{l=1}^{m} R_{l+n,l+n}^{(\mathbb{J},\mathbb{K})} - \frac{1}{m}\sum_{l=1}^{m} R_{l+n,l+n}^{(\mathbb{J}\cup\{j\},\mathbb{K})},\\
\varepsilon_{j2}^{(\mathbb{J},\mathbb{K})} &= \frac{1}{mp}\sum_{l=1}^{m}(X_{jl}^2\xi_{jl} - p)R_{l+n,l+n}^{(\mathbb{J}\cup\{j\},\mathbb{K})},\\
\varepsilon_{j3}^{(\mathbb{J},\mathbb{K})} &= \frac{1}{mp}\sum_{1\le l\ne k\le m} X_{jl}X_{jk}\xi_{jl}\xi_{jk}R_{l+n,k+n}^{(\mathbb{J}\cup\{j\},\mathbb{K})};
\end{aligned}$$

and

$$\begin{aligned}
\varepsilon_{l+n}^{(\mathbb{J},\mathbb{K})} &= \varepsilon_{l+n,1}^{(\mathbb{J},\mathbb{K})} + \cdots + \varepsilon_{l+n,3}^{(\mathbb{J},\mathbb{K})},\\
\varepsilon_{l+n,1}^{(\mathbb{J},\mathbb{K})} &= \frac{1}{m}\sum_{j=1}^{n} R_{jj}^{(\mathbb{J},\mathbb{K})} - \frac{1}{m}\sum_{j=1}^{n} R_{jj}^{(\mathbb{J},\mathbb{K}\cup\{l+n\})},\\
\varepsilon_{l+n,2}^{(\mathbb{J},\mathbb{K})} &= \frac{1}{mp}\sum_{j=1}^{n}(X_{jl}^2\xi_{jl} - p)R_{jj}^{(\mathbb{J},\mathbb{K}\cup\{l+n\})},\\
\varepsilon_{l+n,3}^{(\mathbb{J},\mathbb{K})} &= \frac{1}{mp}\sum_{1\le j\ne k\le n} X_{jl}X_{kl}\xi_{jl}\xi_{kl}R_{jk}^{(\mathbb{J},\mathbb{K}\cup\{l+n\})}.
\end{aligned}$$

Summing the equation (3.1) ($\mathbb{J} = \emptyset$, $\mathbb{K} = \emptyset$), we get the self-consistent equation

$$s_n(z) = S_y(z)(1 + T_n - y\Lambda_n s_n(z)),$$

with the error term

$$T_n = \frac{1}{n}\sum_{j=1}^{n}\varepsilon_j R_{jj}.$$

The proof of Theorem 2.2 is based on the following theorem.

Theorem 3.1. *Under the conditions of the Theorem 2.2, for any $\mu > 0$, there exist constants $C = C(\delta, \mu_{4+\delta}, c_0)$, $a_0 = a_0(\delta, \mu_{4+\delta}, c_0)$, such that*

$$\mathbb{E}\,|T_n|^q\mathbb{I}\{\mathcal{Q}\} \le C^q\Big(\frac{1}{nv} + \frac{1}{np}\Big)^q \log^q n,$$

for all $z \in \mathcal{D}_\mu$.

Proof. The proof repeats [7][Theorem 3], taking into account that $0 < \varepsilon < \operatorname{Im} b(z)$ for some $\varepsilon > 0$ and $\operatorname{Im} b(z)$, $|b(z)|$ are bounded in domain $\mathcal{D}_\mu$. The arguments of [7][Theorem 3] also require that the condition $\Pr\{\mathcal{B}\} \le Cn^{-Q}$ be satisfied (see [7][p. 17]). But Lemma 4.1 implies $\Pr\{\mathcal{B}; \mathcal{Q}\} \le Cn^{-Q}$. □

Proof of Theorem 2.2. First of all, we note that [7, Lemma 8] gives the bound

$$|\Lambda_n| \le C|T_n|$$

in domain $\mathcal{D}_\mu$. We have

$$\Pr\{|\Lambda_n| \ge K\Gamma_n\} \le \Pr\{|\Lambda_n| \ge K\Gamma_n; \mathcal{Q}\} + \Pr\{\mathcal{Q}^c\}.$$

[7, Corollary 3] implies

$$\Pr\{\mathcal{Q}\} \ge 1 - Cn^{-Q}.$$

Applying Markov inequality and combining the last inequality and Theorem 3.1, we get

$$\Pr\{|\Lambda_n| \ge K\Gamma_n\} \le \frac{\mathbb{E}\,|T_n|^q\mathbb{I}\{\mathcal{Q}\}}{K^q\Gamma_n^q} + Cn^{-Q} \le \Big(\frac{C}{K}\Big)^q.$$

By choosing a sufficiently large K value and $q \sim \log n$, we obtained the proof. □

4. Estimate of R_{jk}

We shall use the notations of [7].

Let $s_0 > 1$ be some positive constant depending on δ, V. For any $0 < v \le V$ we define k_v as

$$k_v = k_v(V) := \min\{l \ge 0 : s_0^l v \ge V\}.$$

For given $\gamma > 0$ consider the event

$$\mathcal{Q}_\gamma(v) := \big\{|\Lambda_n(u + iv)| \le \gamma, \text{ for all } u\big\}$$

and the event

$$\mathcal{Q} := \widehat{\mathcal{Q}}_\gamma(v) = \bigcap_{l=0}^{k_v} \mathcal{Q}_\gamma(s_0^l v).$$

For the proof of main result it is enough to estimate the entries of the resolvent matrix. We prove the next Lemma.

Lemma 4.1. *Under conditions of Theorem 2.2 there exists a constant H such that for $z \in \mathcal{D}_\mu$*

$$\Pr\{\max_{1\le j,k\le n+m} |R_{jk}| > H; \mathcal{Q}\} \le Cn^{-c\log n \log n}.$$

Following the work of Aggarwal (see [8]), we introduce the configuration matrix $\mathbf{L} = (L_{jk})$. Set events

$$A_{jk} = \{|X_{jk}| \ge C(np)^{\frac{1}{2}-\varkappa}\}.$$

Define the matrix $\mathbf{L}$ with elements

$$L_{jk} = \xi_{jk}\mathbb{I}\{A_{jk}\}.$$

Note that

$$\mathbb{E}\, L_{jk} \le \frac{\mu_{4+\delta}}{n^2p}.$$

Introduce the configuration matrix $\mathbf{L_V}$:

$$\mathbf{L_V} = \begin{bmatrix} \mathbf{O} & \mathbf{L} \\ \mathbf{L}^T & \mathbf{O} \end{bmatrix}.$$

Definition. We call j and k *linked* (with respect to $\mathbf{L_V}$), if $L_{jk} = 1$. Otherwise we call them *unlinked.*

Definition. If there exists a sequence $j = j_1, j_2, \dots, j_r = k$ such that j_ν is linked to $j_{\nu+1}$ for each $\nu \in [1, r-1]$, then j and k are called *connected.*

Definition. We call an index j *deviant* if there exists some index k such that j and k are linked. Otherwise we call j *typical.*

Let

$$\mathcal{D}_{\mathbf{L}} = \{j \in [1, n+m] : j \text{ is deviant}\}, \quad \mathcal{T}_{\mathbf{L}} = \{j \in [1, n+m] : j \text{ is typical}\}.$$

Definition. We call $\mathbf{L_V}$ *deviant-inadmissible* if there exist at least $\sqrt{\frac{n}{p}}$, deviant indices. We call $\mathbf{L_V}$ *connected-inadmissible* if there exist distinct indices $j_1, j_2, \dots, j_r$, $r = [\log n]$, that are pairwise connected. We call the configuration $\mathbf{L_V}$ *inadmissible*, if it is either deviant-inadmissible or connected-inadmissible. Otherwise, the configuration is called admissible.

Define $\mathcal{A}$ as the set of all admissible configurations of size $n+m$. Let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ be the event that the configuration $\mathbf{L_V}$ is inadmissible, $\mathcal{C}_1$ be the event that the configuration $\mathbf{L_V}$ is deviant-inadmissible, and $\mathcal{C}_2$ be the event that the configuration $\mathbf{L_V}$ is connected-inadmissible.

Lemma 4.2. *Under the conditions of Theorem 2.2 the bound*

$$\Pr\{\mathcal{C}\} \le C n^{-c \log\log n}$$

is valid.

Proof. First, we estimate $\Pr\{\mathcal{C}_1\}$. The event $\mathcal{C}_1$ implies that there are at least $\sqrt{\frac{n}{p}}$ deviant indices, which in turn gives that there is at least $\sqrt{\frac{n}{p}}$ pairs $\{j, k\}$ such that $j \in [1, n]$, $k \in [1, m]$ and $L_{jk} = 1$. Hence

$$\Pr\{\mathcal{C}_1\} \le \sum_{j=\sqrt{\frac{n}{p}}}^{n} \binom{nm}{j} \left(\frac{C}{n^2 p}\right)^j.$$

By Stirling's formula, we have

$$\binom{nm}{j} \left(\frac{C}{n^2 p}\right)^j \le C \left(\frac{C}{\sqrt{np}}\right)^j$$

for $\sqrt{\frac{n}{p}} \le j \le n$. This yields

$$\Pr\{\mathcal{C}_1\} \le C \left(\frac{C}{\sqrt{np}}\right)^{\sqrt{\frac{n}{p}}}.$$

The estimate $\Pr\{\mathcal{C}_2\}$ almost repeats the proof of the bound for $\Pr\{\Delta_2\}$ in Lemma 3.11 of [8]. The event $\mathcal{C}_2$ implies that there exists a sequence of indices $\mathcal{S} = \{i_1, i_2, \dots, i_r\}$ such that at least $r-1$ pair (i_j, i_k) are linked. We have

$$\Pr\{\mathcal{C}_2\} \le \binom{n+m}{r} \binom{r^2}{r-1} \left(\frac{C}{n^2 p}\right)^{r-1}.$$

Applying Stirling's formula, get

$$\Pr\{\mathcal{C}_2\} \le n^{-C\log\log n}.$$

□

Now we fix the admissible configuration $\mathbf{L}_\mathbf{V}$. Let $R < \sqrt{\frac{n}{p}}$ denotes the number of the deviant indices. Consider the matrix $\mathbf{V}_\mathbf{L} = (V_\mathbf{L}(j,k))$ with entries

$$V_\mathbf{L}(j,k) = \begin{cases} 0, \text{ if } 1 \le j,k \le n \text{ or } n+1 \le j,k \le n+m, \\ \xi_{jk}a_{jk}, \text{ if } 1 \le j \le n, n+1 \le k \le n+m \text{ and } L_{jk} = 0, \\ \xi_{jk}b_{jk}, \text{ if } 1 \le j \le n, n+1 \le k \le n+m \text{ and } L_{jk} = 1, \\ \overline{V}_{kj}, \text{ if } n+1 \le j \le n+m,\ 1 \le k \le n. \end{cases}$$

Here a_{jk} (resp. b_{jk}) are independent random variables with the distributions

$$\Pr\{a_{jk} \in G\} = \Pr\{X_{jk} \in G \big| \mathcal{A}_{jk}^c\}$$

and

$$\Pr\{b_{jk} \in G\} = \Pr\{X_{jk} \in G \big| \mathcal{A}_{jk}\}.$$

The permutation of rows and columns gives the matrix

$$\mathbf{V} = \begin{bmatrix} \mathbf{V}_{11} & \mathbf{V}_{12} \\ \mathbf{V}_{12}^* & \mathbf{V}_{22} \end{bmatrix}.$$

The Hermitian matrix $\mathbf{V}_{11}$ of size $R \times R$ consists of type b elements and has the form

$$\mathbf{V}_{11} = \begin{bmatrix} \mathbf{B}_1 & 0\ldots & 0\ldots \\ 0\ldots & \mathbf{B}_2 & \ldots \\ \ldots & \ldots & \ldots \\ 0\ldots & 0\ldots & \mathbf{B}_L \end{bmatrix},$$

where $\mathbf{B}_\nu$ are Hermitian matrices of order $r_\nu \le r$, $\nu = 1,\ldots,L$. The matrix $\mathbf{V}_{12}$ of size $R \times (m+n-R)$ consists of type a elements and has the form

$$\mathbf{V}_{12} = \begin{bmatrix} \mathbf{O}_1 & \mathbf{A}_1 \end{bmatrix},$$

where $\mathbf{O}_1$ is a matrix of size $R \times m$ with zero elements, the matrix $\mathbf{A}_1$ is $R \times (n-R)$ with elements distributed by type a. The Hermitian matrix $\mathbf{V}_{22}$ of size $(n+m-R) \times (n+m-R)$ has the form

$$\mathbf{V}_{22} = \begin{bmatrix} \mathbf{O}_{11} & \mathbf{A}_2 \\ \mathbf{A}_2^* & \mathbf{O}_{22} \end{bmatrix}.$$

Here the square matrices $\mathbf{O}_{11}$ and $\mathbf{O}_{22}$ have zero elements and the orders m and $n-R$ respectively, and the matrix $\mathbf{A}_2$ is $m \times (n-R)$ with elements distributed by type a. The resolvent $\mathbf{R}(z) = (\mathbf{V} - z\mathbf{I})^{-1}$ can be represented as

$$\mathbf{R} = \begin{bmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} \\ \mathbf{R}_{12}^T & \mathbf{R}_{22} \end{bmatrix},$$

where

$$\begin{aligned}\mathbf{R}_{11} =&(\mathbf{V}_{11}-z\mathbf{I}-\mathbf{V}_{12}(\mathbf{V}_{22}-z\mathbf{I})^{-1}\mathbf{V}_{12}^*)^{-1},\\ \mathbf{R}_{12} =&(\mathbf{V}_{12}(\mathbf{V}_{22}-z\mathbf{I})^{-1}\mathbf{V}_{12}^*-\mathbf{V}_{11}+z\mathbf{I})^{-1}\mathbf{V}_{12}(\mathbf{V}_{22}-z\mathbf{I})^{-1},\\ \mathbf{R}_{22} =&(\mathbf{V}_{22}-z\mathbf{I})^{-1}+(\mathbf{V}_{22}-z\mathbf{I})^{-1}\mathbf{V}_{12}^*\\ &\times(\mathbf{V}_{11}-z\mathbf{I}-\mathbf{V}_{12}(\mathbf{V}_{22}-z\mathbf{I})^{-1}\mathbf{V}_{12}^*)^{-1}\mathbf{V}_{12}(\mathbf{V}_{22}-z\mathbf{I})^{-1}.\end{aligned}$$

We will be primarily interested in estimating the spectral norm of the matrix $\mathbf{R}_{11}$ since it majorizes all elements of the matrix $\mathbf{R}_{11}$. Note that the dimension of the matrix $\mathbf{R}_{11}$ is equal to $R\times R$, where $R<\sqrt{\frac{n}{p}}$. Introduce a random matrix

$$\mathbf{Y}=\mathbf{V}_{12}(\mathbf{V}_{22}-z\mathbf{I})^{-1}\mathbf{V}_{12}^*.$$

Note that

$$\mathbf{R}^{(\mathbb{J})}=(\mathbf{V}_{22}-z\mathbf{I})^{-1}=\begin{bmatrix}\mathbf{R}_{11}^{(\mathbb{J})} & \mathbf{R}_{12}^{(\mathbb{J})}\\ \mathbf{R}_{12}^{(\mathbb{J})T} & \mathbf{R}_{22}^{(\mathbb{J})}\end{bmatrix}.$$

Given the form of the matrices $\mathbf{V}_{12}$ and $\mathbf{V}_{22}$, we find that

$$\mathbf{Y}=\mathbf{A}_1\mathbf{R}_{22}^{(\mathbb{J})}\mathbf{A}_1^*.$$

In these notation

$$\mathbf{R}_{11}=(\mathbf{V}_{11}-z\mathbf{I}-\mathbf{Y})^{-1}.$$

In what follows we shall assume that $\mathbf{L}_{\mathbf{V}}$ is admissible. We prove that for the resolvent matrix $\mathbf{R}$ all entries are bounded conditioning by admissible $\mathbf{L}_{\mathbf{V}}$.

Lemma 4.3. *Let $\mathbf{L}_{\mathbf{V}}$ be admissible. Under conditions of Theorem 2.2 there exists a constant H such that for $z\in\mathcal{D}_\mu$*

$$\Pr\{\max_{1\le j,k\le n+m}|R_{jk}|>H;\mathcal{Q}\}\le Cn^{-c\log\log n}.$$

Note that $\mathcal{T}_{\mathbf{L}}\cup\mathcal{D}_{\mathbf{L}}=[1,n+m]$, $\mathbb{J}\subset[1,n+m]$. We introduce the events

$$\mathcal{C}_1(v,k)=\bigcap_{|\mathbb{J}|\le k}\left\{\max_{j,l\in\mathcal{T}_{\mathbf{L}}}|R_{jl}^{(\mathbb{J})}(u+iv)|\le H_1\right\}$$

and

$$\mathcal{C}_2(v,k)=\bigcap_{|\mathbb{J}|\le k}\left\{\max_{j\in\mathcal{D}_{\mathbf{L}},1\le l\le n+m}|R_{jl}^{(\mathbb{J})}(u+iv)|\le H_2\right\}.$$

The following lemma holds.

Lemma 4.4. *Under the conditions of the Theorem 2.2, the inequalities*

$$\Pr\left\{\mathcal{C}_1(v,k-1);\mathcal{C}_1(sv,k)\cap\mathcal{C}_2(sv,k)\cap\mathcal{Q}\right\}\ge1-Cn^{-c\log\log n}\tag{4.1}$$

and

$$\Pr\left\{\mathcal{C}_2(v,k-1);\mathcal{C}_1(sv,k)\cap\mathcal{C}_2(sv,k)\cap\mathcal{Q}\right\}\ge1-Cn^{-c\log\log n}\tag{4.2}$$

are valid.

Proof. For simplicity, we assume that $k = 1$. We begin by proving the inequality (4.1). Since both indices are typical, the corresponding matrix elements in the rows (and columns) with numbers j, k are of type a. Consider the diagonal elements. For $j \in \mathcal{T}_{\mathbf{L}} \cap [1, n]$ the equality

$$R_{jj} = yS_y(z)\Big(1 + \varepsilon_j R_{jj} + \Lambda_n R_{jj}\Big)$$

holds. For $\omega \in \mathcal{Q}$ we have

$$|\Lambda_n| \le \frac{1}{2}.$$

Hence,

$$|R_{jj}|\mathbb{I}\{\mathcal{Q}\} \le 2\sqrt{y}(1 + |\varepsilon_j||R_{jj}|)\mathbb{I}\{\mathcal{Q}\}.$$

Let

$$\varepsilon_j = \varepsilon_{j1} + \varepsilon_{j2} + \varepsilon_{j3}$$

with

$$\begin{aligned}
\varepsilon_{j1} =& \frac{1}{m}\sum_{l=1}^{m} R^{(j)}_{l+n,l+n} - \frac{1}{m}\sum_{l=1}^{m} R_{l+n,l+n}, \\
\varepsilon_{j2} =& \frac{1}{mp}\sum_{l=1}^{m} (a_{jl}^2\xi_{jl} - p)R^{(j)}_{l+n,l+n}, \\
\varepsilon_{j3} =& \frac{1}{mp}\sum_{l,t=1}^{m} a_{jl}a_{jt}\xi_{jl}\xi_{jt}R^{(j)}_{l+n,t+n}.
\end{aligned}$$

Note that for admissible configurations

$$|\mathcal{D}_{\mathbf{L}}| \le \sqrt{\frac{n}{p}}.$$

By [7, Lemma 1],

$$|\varepsilon_{j1}| \le \frac{C}{nv}.$$

Next, note that

$$\frac{1}{n}\sum_{l=1}^{m} |R^{(j)}_{l+n,l+n}|^2 \mathbb{I}\{\mathcal{C}_1(sv, 1)\}\mathbb{I}\{\mathcal{C}_2(sv, 1)\}\mathbb{I}\{\mathcal{Q}\} \le H_2^2 s^2 + H_1^2 s^2$$

and

$$\frac{1}{n}\sum_{l=1}^{m} |R^{(j)}_{l+n,l+n}|^q \mathbb{I}\{\mathcal{C}_1(sv, 1)\}\mathbb{I}\{\mathcal{C}_2(sv, 1)\}\mathbb{I}\{\mathcal{Q}\} \le H_2^q s^q + H_1^q s^q.$$

We used here the so-called multiplicative inequality: for any $s \ge 1$

$$|R_{jj}(u + iv)| \le s|R_{jj}(u + isv)|.$$

Given the above, get

$$\mathbb{E}\,|\varepsilon_{j2}|^q\mathbb{I}\{\mathcal{C}_1(sv,1)\}\mathbb{I}\{\mathcal{C}_2(sv,1)\}\mathbb{I}\{\mathcal{Q}\} \le \frac{C^q q^{\frac{q}{2}} s^q H_2^q}{(np)^{\frac{q}{2}}} + \frac{C^q s^q q^{\frac{q}{2}} H_1^q}{(np)^{\frac{q}{2}}} + \frac{C^q q^q H_2^q s^q}{(np)^{2\varkappa q+1}} + \frac{C^q q^q H_1^q s^q}{(np)^{2\varkappa q+1}}.$$

Similarly,

$$\mathbb{E}\,|\varepsilon_{j3}|^q|R_{jj}|^q\mathbb{I}\{\mathcal{C}_1(sv,1)\}\mathbb{I}\{\mathcal{C}_2(sv,1)\}\mathbb{I}\{\mathcal{Q}\} \le \frac{C^q q^q}{(nv)^q} a_n^{\frac{q}{2}}(z) + \frac{C^q q^{\frac{3q}{2}} s^{\frac{q}{2}} H_2^{\frac{q}{2}}}{(nv)^{\frac{q}{2}}(np)^{\varkappa q+1}} + \frac{C^q q^{\frac{3q}{2}} s^{\frac{3q}{2}} H_1^{\frac{q}{2}}}{(nv)^{\frac{q}{2}}(np)^{\varkappa q+1}} + \frac{C^q q^{2q} s^{2q} H_2^{2q}}{(np)^{2\varkappa q+2}} + \frac{C^q q^{2q} s^{2q} H_1^{2q} p}{(np)^{2\varkappa q+2}}.$$

Here we used the fact that

$$|R_{jk}(u+iv)| \le |R_{jk}(u+iv)| + (s-1)v|[\mathbf{R}(u+iv)\mathbf{R}(u+sv)]_{jk}| \le |R_{jk}(u+isv)| + (s-1)\sqrt{\operatorname{Im} R_{jj}\operatorname{Im} R_{kk}} \le sH_1$$

for $j,k \in \mathcal{T}_{\mathbf{L}}$, and

$$|R_{jk}(u+iv)| \le sH_2$$

in the case $j \in \mathcal{D}_{\mathbf{L}}$ or $k \in \mathbf{D_L}$.

If $(np)^{2\varkappa}|b(z)| \ge Cq^2 sH_1$ and $(np)^{\varkappa} > CqsH_1$, then H_1 and H_2 can be chosen so that

$$\mathbb{E}\,|R_{jj}|^q\mathbb{I}\{\mathcal{C}_1(sv,1)\}\mathbb{I}\{\mathcal{C}_2(sv,1)\}\mathbb{I}\{\mathcal{Q}\} \le H_1^p.$$

Now consider the case of deviant indices. Let $j \in \mathcal{D}_{\mathbf{L}}$ and k be arbitrary. Consider the matrix

$$\mathbf{Y} = \mathbf{V}_{12}(\mathbf{V}_{22} - z\mathbf{I})^{-1}\mathbf{V}_{12}^* = \mathbf{A}_1\mathbf{R}_{22}^{(\mathbb{J})}\mathbf{A}_1^*.$$

We estimate the matrix $\mathbf{Y}$ elementwise. We start with off-diagonal elements. Consider Y_{12}. The equality

$$Y_{12} = \frac{1}{mp}\sum_{l,t} a_{1l}\overline{a}_{2t}[\mathbf{R}_{22}^{(\mathbb{J})}]_{lt}$$

holds. Note that $\{a_{1l}\}$ and $\{a_{2t}\}$ are independent. We can apply the lemma 5.1 with $\mathbf{A} = \mathbf{R}_{22}^{(\mathbb{J})}$. By the assumption $\mathcal{C}_1 \cap \mathcal{C}_2 \cap \mathcal{Q}$ we get

$$\|\mathbf{A}\|^2 \le \gamma\frac{na_n(z)}{v} + \frac{r}{v},$$

and

$$\sum_{j=1}^n \mathcal{L}_j^q \le \frac{H_2^{\frac{q}{2}} s^{\frac{q}{2}}}{pv^{\frac{q}{2}}|b(z)|^{\frac{q}{2}}} + \frac{H_1^{\frac{q}{2}} s^{\frac{q}{2}} n}{v^{\frac{q}{2}}}.$$

Finally,

$$\sum_{i,j\in\mathbb{T}\setminus\mathbb{J}} |[\mathbf{R}_{22}^{(\mathbb{J})}]_{ij}|^q \le \frac{H_2^{q-2}s^{q-2}n}{|b(z)|^{q-2}v}(\gamma a_n(z) + \frac{r}{nv}).$$

Further, we have

$$\mu_\xi^{(q)}, \mu_\eta^{(q)} \le p(np)^{-2-\varkappa q}$$

for $q \ge 4+\delta$, and

$$\mu_\xi^{(q)}, \mu_\eta^{(q)} \le p\mu_{4+\delta}^{\frac{q}{4+\delta}}/(np)^{\frac{q}{2}}$$

for $q \le 4$. Combining all the estimates, we obtain

$$\begin{aligned}
\mathcal{A}_1 \le & \frac{C^q q^{\frac{q}{2}}}{n^q}\Big(\frac{1}{q^{\frac{q}{2}}} + \frac{q^{\frac{q}{2}}}{(np)^{\frac{q}{2}\varkappa}} + \frac{1}{(np)^2}\frac{q^q}{(np)^{q\varkappa}}\Big),\\
\mathcal{A}_2 \le & \frac{C^q q^{\frac{3q}{2}}}{n^{\frac{q}{2}+1}(np)^{q\varkappa}},\\
\mathcal{A}_3 \le & \frac{C^q q^{2q}}{n^2(np)^{2\varkappa q+2}}.
\end{aligned}$$

Finally we get, for $np \ge C\log n^{\frac{1}{\varkappa}}$,

$$\begin{aligned}
\mathbb{E}\,|Y_{12}|^q \mathbb{I}\{\mathcal{C}_1\}\mathbb{I}\{\mathcal{C}_2\}\mathbb{I}\{\mathcal{Q}\} \le & \frac{C^q q^{\frac{q}{2}}}{(nv)^{\frac{q}{2}}}\Big(a_n^{\frac{q}{2}}(z) + \frac{r^{\frac{q}{2}}}{(nv)^{\frac{q}{2}}}\Big) + \frac{C^q H_2^{\frac{q}{2}} s^{\frac{q}{2}} q^{\frac{3q}{2}}}{(nv)^{\frac{q}{2}}(np)^{q\varkappa+1}|b(z)|^{\frac{q}{2}}}\\
& + \frac{C^q q^{\frac{3q}{2}} H_1^{\frac{q}{2}} s^{\frac{q}{2}}}{(nv)^{\frac{q}{2}}(np)^{q\varkappa}} + \frac{C^q q^{2q} H_1^q s^q}{(np)^{2\varkappa q+2}} + \frac{C^q H_2^q s^q}{|b(z)|^q (np)^{2\varkappa q+3}}.
\end{aligned}$$

Applying Chebyshev's inequality with $q \sim \log n$, we conclude that

$$\Pr\Big\{|Y_{12}| \ge C\log n\Big(\frac{a_n(z)}{\sqrt{nv}} + \frac{\log^{\frac{3}{2}} n}{\sqrt{nv}(np)^{\varkappa}|b(z)|^{\frac{1}{2}}} + \frac{\log^2 n}{(np)^{2\varkappa}|b(z)|}\Big)\Big\} \le Cn^{-c\log\log n}. \quad (4.3)$$

Now consider the diagonal elements.

$$Y_{11} = \sum_{l,t} a_{1l}a_{1t}[\mathbf{R}_{22}^{(\mathbb{J})}]_{lt}.$$

Represent Y_{11} as

$$Y_{11} = \sum_l a_{1l}^2[\mathbf{R}_{22}^{(\mathbb{J})}]_{ll} + \sum_{l\ne t} a_{1l}a_{1t}[\mathbf{R}_{22}^{(\mathbb{J})}]_{lt} =: \widehat{Y}_{11} + \widetilde{Y}_{11}.$$

Applying the inequality for quadratic forms, obtain

$$\begin{aligned}
\mathbb{E}\,|\widetilde{Y}_{11}|^q \mathbb{I}\{\mathcal{C}_1(sv,1)\}\mathbb{I}\{\mathcal{C}_2(sv,1)\}\mathbb{I}\{\mathcal{Q}\} \le & C^q\Big(q^q(\mathbb{E}\,|a_{11}|^2)^q\,\mathbb{E}\,\|\mathbf{R}^{(\mathbb{J})}\|^q \mathbb{I}\{\mathcal{C}_2(sv,1)\}\mathbb{I}\{\mathcal{Q}\}\\
& + q^{\frac{3q}{2}}\mu^{(q)}(\mathbb{E}\,|a_{11}|^2)^{\frac{q}{2}}\sum_l \mathbb{E}\,\Big(\sum_t |[\mathbf{R}_{22}^{(\mathbb{J})}]_{lt}|^2\Big)^{\frac{q}{2}}\\
& + q^{2q}(\mu^{(q)})^2\sum_{l,t}|[\mathbf{R}_{22}^{(\mathbb{J})}]_{lt}|^q\Big)\mathbb{I}\{\mathcal{C}_1(sv,1)\}\mathbb{I}\{\mathcal{C}_2(sv,1)\}\mathbb{I}\{\mathcal{Q}\}.
\end{aligned}$$

From here it is easy to get

$$\begin{aligned}
\mathbb{E}\,|\widetilde{Y}_{11}|^q \mathbb{I}\{\mathcal{C}_1(sv,1)\}\mathbb{I}\{\mathcal{C}_2(sv,1)\}\mathbb{I}\{\mathcal{Q}\} \le & C^q\Big(\frac{q^q r^{\frac{q}{2}} a_n^{\frac{q}{2}}(z)}{(nv)^{\frac{q}{2}}} + \frac{q^{\frac{3q}{2}}C^q}{(nv)^{\frac{q}{2}}(np)^{\varkappa q+2}|b(z)|^{\frac{q}{2}}}\\
& + \frac{C^q q^{2q}}{(np)^{2\varkappa q+3}|b(z)|^q} + \frac{C^q}{(np)^{2\varkappa q+2}}\Big).
\end{aligned}$$

This yields

$$\Pr\Big\{|\widetilde{Y}_{11}| \ge C\Big(\frac{\log^2 n \log\log n a_n^{\frac12}(z)}{\sqrt{nv}} + \frac{\log^{\frac52} n}{\sqrt{nv}(np)^{\varkappa}\sqrt{|b(z)|}} + \frac{\log^3 n}{(np)^{2\varkappa}|b(z)|}\Big); \mathcal{C}_1(sv,1)\cap\mathcal{C}_2(sv,1)\cap\mathcal{Q}\Big\} \le Cn^{-\log\log n}.$$

Now consider $\widehat{Y}_{11}$. We have

$$\begin{aligned}\widehat{Y}_{11} =& \frac{y}{n}\sum_l R_{ll}^{(\mathbb{J})} + \sum_l (a_{1l}^2 - \mathbb{E}\, a_{1l}^2)R_{ll}^{(\mathbb{J})}\\ =& yS_y(z) - \frac{1-y}{z} + y\Lambda_n(z) + \frac{r}{nv} + \sum_l (a_{1l}^2 - \mathbb{E}\, a_{1l}^2)R_{ll}^{(\mathbb{J})}.\end{aligned}$$

By Rosenthal's inequality,

$$\mathbb{E}\Big|\sum_l (a_{1l}^2 - \mathbb{E}\, a_{1l}^2)R_{ll}^{(\mathbb{J})}\Big|^q \mathbb{I}\{\mathcal{C}_1(sv,1)\}\mathbb{I}\{\mathcal{C}_2(sv,1)\}\mathbb{I}\{\mathcal{Q}\} \le C^q\Big(\frac{q^{\frac q2}s^q}{(np)^{\frac q2}} + \frac{q^{\frac q2}s^q}{(np)^q|b(z)|^q} + \frac{s^q q^q}{(np)^{2\varkappa q+2}|b(z)|^q}\Big).$$

The obtained bounds give

$$\Pr\Big\{\Big|\widehat{Y}_{11} - \Big(-\frac{1-y}{z} + yS_y(z)\Big)\Big| \ge C\Big(\gamma a_n(z) + \frac{r}{nv} + \frac{\log^{\frac32} n}{(np)^{\frac12}} + \frac{\log^{\frac32}}{(np)|b(z)|} + \frac{\log^2 n}{(np)^{2\varkappa}|b(z)|}\Big); \mathcal{C}_1\cap\mathcal{C}_2\cap\mathcal{Q}\Big\} \le Cn^{-\log\log n}.$$

Summing up the estimates for $\widehat{Y}_{11}$ and $\widetilde{Y}_{11}$, we conclude that

$$\Pr\Big\{\Big|Y_{11} - \Big(yS_y(z) - \frac{1-y}{z}\Big)\Big| \ge \mathcal{G}_1 + \mathcal{G}_2; \mathcal{C}_1\cap\mathcal{C}_2\cap\mathcal{Q}\Big\} \le Cn^{-c\log n},$$

where

$$\begin{aligned}\mathcal{G}_1 =& \gamma a_n(z),\\ \mathcal{G}_2 =& C\Big(\frac{r}{nv} + \frac{\log^{\frac32} n}{(np)^{\frac12}} + \frac{\log^{\frac32} n}{(np)|b(z)|} + \frac{\log^2 n}{(np)^{2\varkappa}|b(z)|}\\ &+ \frac{\log^2 n \log\log n a_n^{\frac12}(z)}{\sqrt{nv}} + \frac{\log^{\frac52} n}{\sqrt{nv}(np)^{\varkappa}\sqrt{|b(z)|}} + \frac{\log^3 n}{(np)^{2\varkappa}|b(z)|}\Big).\end{aligned}$$

It is easy to show that if

$$|b(z)| \ge C\log n^{\frac32} n\Big(\frac{1}{\sqrt{nv}} + \frac{1}{(np)^{\varkappa}}\Big),$$

then

$$\Pr\Big\{\Big|Y_{11} - \Big(yS_y(z) - \frac{1-y}{z}\Big)\Big| \le \gamma|b(z)|; \mathcal{C}_1\cap\mathcal{C}_2\cap\mathcal{Q}\Big\} \le Cn^{-c\log n}$$

with an arbitrarily small constant γ. From this and the inequality (4.3) it follows that

$$\Pr\Big\{\Big\|\mathbf{Y}-\Big(yS_y(z)-\frac{1-y}{z}\Big)\mathbf{I}\Big\|\ge\gamma|b(z)|;\mathcal{C}_1\cap\mathcal{C}_2\cap\mathcal{Q}\Big\}\ge 1-Cn^{-c\log n}.$$

Since the matrix $\mathbf{V}_{11}$ is Hermitian (the eigenvalues are real), and

$$\operatorname{Im}\Big(z-\frac{1-y}{z}+yS_y(z)\Big)\ge\frac{\sqrt{2}}{2}|b(z)|,$$

we find that

$$\Pr\Big\{\Big\|\Big(\mathbf{V}_{11}-\Big(z-\frac{1-y}{z}+yS_y(z)\Big)-\mathbf{Y}\Big)^{-1}\Big\|\le\frac{C}{|b(z)|};\mathcal{C}_1\cap\mathcal{C}_2\cap\mathcal{Q}\Big\}\ge 1-Cn^{-c\log\log n}.$$

This, in particular, implies that

$$\Pr\Big\{|R_{jk}|\le H_2;\mathcal{C}_1(sv,1)\cap\mathcal{C}_2(sv,1)\cap\mathcal{Q}\Big\}\ge 1-Cn^{-c\log\log n},$$

for $j\in\mathcal{D}$. The last statement completes the proof of the Lemma 4.4. □

Proof of Lemma 4.3. Let $k=|\mathbb{J}|$. Lemma 4.4 and inequality $\max_{j,l}|R^{(\mathbb{J})}_{jl}(V)|\le V^{-1}$ imply

$$\Pr\Big\{\mathcal{C}_1(v,k-1);\mathcal{Q}\Big\}\ge 1-Cn^{-c\log\log n},$$

$$\Pr\Big\{\mathcal{C}_2(v,k-1);\mathcal{Q}\Big\}\ge 1-Cn^{-c\log\log n},$$

for $V/s_0\le v\le V$. We may repeat this procedure $L(v_0,s_0)$ times and obtain

$$\Pr\{\max_{1\le j,k\le n+m}|R_{jk}(v)|>H;\mathcal{Q}\}\le Cn^{-c\log\log n},$$

for $v\ge V/s_0^L=v_0$. □

Proof of Lemma 4.1. We recall that Lemma 4.2 gives

$$\Pr\{\mathbf{L}_{\mathbf{V}}\notin\mathcal{A}\}\le Cn^{-c\log\log n}.$$

It implies Lemma 4.1. □

5. Appendix

Let $\xi_1,\dots,\xi_n$ and $\eta_1,\dots,\eta_n$ be mutually independent random variables, $A=(a_{ij})_{i,j=1}^n$. Define

$$\mathcal{L}_j^2=\sum_{i=1}^n|a_{ij}|^2.$$

Note that

$$\|A\|^2=\sum_{j=1}^n\mathcal{L}_j^2.$$

Lemma 5.1. *For any $q \geq 2$ the inequality*

$$\mathbb{E}\,|\sum_{i,j=1}^{n} a_{ij}\xi_i\eta_j|^q \leq C^q\big(\mathcal{A}_1\|A\|^q + \mathcal{A}_2(\sum_{j=1}^{n}\mathcal{L}_j^q) + \mathcal{A}_3(\sum_{i,j=1}^{n}|a_{ij}|^q)\big)$$

holds, where

$$\begin{aligned}\mathcal{A}_1 =& q^{\frac{3q}{2}}(\sigma_\xi^{2q} + \sigma_\eta^{2q}),\\ \mathcal{A}_2 =& q^{\frac{3q}{2}}(\sigma_\xi^{(2q)} + \sigma_\eta^{2q})^{\frac{q-6}{2(q-4}}(\mu_\xi^{(\frac{q}{2})})^{\frac{2(q-2)}{q-4}},\\ \mathcal{A}_3 =& q^{2q}\mu_\xi^{(q)}\mu_\eta^{(q)}.\end{aligned}$$

Proof. Let $A = \sum_{i,j=1}^{n} a_{ij}\xi_i\eta_j = \sum_{i=1}^{n}\xi_i(\sum_{j=1}^{n} a_{ij}\eta_j)$. Applying Rosenthal's inequality, we get

$$\begin{aligned}A \leq C^q(q^{\frac{q}{2}}\sigma_\xi^q\,\mathbb{E}\Big(\sum_{i=1}^{n}(\sum_{j=1}^{n}\eta_j a_{ij})^2\Big)^{\frac{q}{2}} + q^q\mu_\xi^{(q)}\sum_{i=1}^{n}\mathbb{E}\,|\sum_{j=1}^{n}a_{ij}\eta_j|^q\Big)\\ =: C^q q^{\frac{q}{2}}\sigma_\xi^q A_1 + C^q q^q\mu_\xi^{(q)}A_2.\end{aligned}$$

Using the triangle inequality, we obtain

$$\begin{aligned}A_1 \leq 2^{\frac{q}{2}}\,\mathbb{E}\Big(\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}^2\eta_j^2\Big)^{\frac{q}{2}} + 2^{\frac{q}{2}}\,\mathbb{E}\Big(\sum_{i\neq j}\eta_i\eta_j(\sum_{l=1}^{n}a_{il}a_{lj})\Big)^{\frac{q}{2}}\\ =: 2^{\frac{q}{2}}(A_{11} + A_{12}).\end{aligned}$$

Further,

$$A_{11} \leq 2^{\frac{q}{2}}\Big((\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}^2)^{\frac{q}{2}}\sigma_\eta^q + \mathbb{E}\Big(\sum_{j=1}^{n}(\eta_j^2 - \sigma_\eta^2)(\sum_{i=1}^{n}a_{ij}^2)\Big)^{\frac{q}{2}}\Big).$$

Applying Rosenthal's inequality again, we conclude that

$$\begin{aligned}\mathbb{E}\Big(\sum_{j=1}^{n}(\eta_j^2 - \sigma_\eta^2)(\sum_{i=1}^{n}a_{ij}^2)\Big)^{\frac{q}{2}} \leq C^q q^{\frac{q}{4}}\Big(\sum_{j=1}^{n}(\sum_{i=1}^{n}a_{ij}^2)^2\Big)^{\frac{q}{4}}(\mu_\eta^{(4)})^{\frac{q}{4}}\\ + C^q q^{\frac{q}{2}}\mu_\eta^{(q)}\sum_{j=1}^{n}(\sum_{i=1}^{n}a_{ij}^2)^{\frac{q}{2}}.\end{aligned}$$

To estimate A_{12}, we use the inequality for quadratic forms from [9]. We have

$$\begin{aligned}A_{12} \leq& C^q q^{\frac{q}{2}}\sigma_\eta^q\Big(\sum_{i\neq j}(\sum_{l=1}^{n}a_{il}a_{lj})^2\Big)^{\frac{q}{4}} + C^q q^{\frac{3q}{4}}\mu_\eta^{(\frac{q}{2})}\sigma_\eta^{\frac{q}{2}}\sum_{j=1}^{n}\Big(\sum_{i=1}^{n}(\sum_{l=1}^{n}a_{il}a_{lj})^2\Big)^{\frac{q}{4}}\\ &+ C^q\Big(q^q(\mu_\eta^{(\frac{q}{2})})^2\Big(\sum_{i\neq j}(\sum_{l=1}^{n}a_{il}a_{lj})^{\frac{q}{2}}\Big).\end{aligned}$$

Summing up the above inequalities, we find that

$$
\begin{aligned}
A_1 \leq & C^q \sigma_\eta^q \Big(\sum_{i=1}^n \sum_{j=1}^n a_{ij}^2 \Big)^{\frac{q}{2}} + C^q (\mu_\eta^{(4)})^{\frac{q}{4}} q^{\frac{q}{4}} \Big(\sum_{j=1}^n \big(\sum_{i=1}^n a_{ij}^2 \big)^2 \Big)^{\frac{q}{4}} \\
& + C^q \Big(q^{\frac{q}{2}} \sigma_\eta^q \Big(\sum_{i \neq j} \big(\sum_{l=1}^n a_{il} a_{lj} \big)^2 \Big)^{\frac{q}{4}} + C^q q^{\frac{3q}{4}} \mu_\eta^{(\frac{q}{2})} \sigma_\eta^{\frac{q}{2}} \sum_{i=1}^n \Big(\sum_{j \neq i} \big(\sum_{l=1}^n a_{il} a_{lj} \big)^2 \Big)^{\frac{q}{4}} \\
& + C^q q^q (\mu_\eta^{(\frac{q}{2})})^2 \sum_{i \neq j} \big(\sum_{l=1}^n a_{il} a_{lj} \big)^{\frac{q}{2}} \Big) + C^q q^{\frac{q}{2}} \mu_\eta^{(q)} \sum_{j=1}^n \big(\sum_{i=1}^n a_{ij}^2 \big)^{\frac{q}{2}}.
\end{aligned}
$$

For A_2, by Rosenthal's inequality, we have

$$
A_2 \leq C^q \sigma_\eta^q q^{\frac{q}{2}} \sum_{i=1}^n \mathcal{L}_i^q + C^q q^q \mu_\eta^{(q)} \sum_{i=1}^n |a_{ij}|^q. \tag{5.1}
$$

Further note that

$$
\begin{aligned}
\Big(\sum_{j=1}^n \big(\sum_{i=1}^n a_{ij}^2 \big)^2 \Big)^{\frac{q}{4}} &\leq \Big(\sum_{i=1}^n \mathcal{L}_i^q \Big)^{\frac{q}{2(q-2)}} (\|A\|^q)^{\frac{q-4}{2(q-2)}}, \\
\Big(\sum_{i \neq j} \big(\sum_{l=1}^n a_{il} a_{lj} \big)^2 \Big)^{\frac{q}{4}} &\leq \|A\|^q, \\
\sum_{i=1}^n \Big(\sum_{j \neq i} \big(\sum_{l=1}^n a_{il} a_{lj} \big)^2 \Big)^{\frac{q}{4}} &\leq \Big(\sum_{j=1}^n \mathcal{L}_j^q \Big)^{\frac{(q-4)}{2(q-2)}} \big(\|A\|^q \big)^{\frac{q}{2(q-2)}}, \\
\Big(\sum_i \mathcal{L}_j^{\frac{q}{2}} \Big)^2 &\leq \Big(\sum_{j=1}^n \mathcal{L}_j^q \Big)^{\frac{(q-4)}{(q-2)}} \big(\|A\|^q \big)^{\frac{2}{(q-2)}}.
\end{aligned}
$$

For A we get the estimate

$$
A \leq C^q (B_1 + \ldots + B_8),
$$

where

$$\begin{aligned}
B_1 =& q^{\frac{q}{2}}\sigma_\xi^q\sigma_\eta^q\|A\|^q,\\
B_2 =& \sigma_\xi^q(\mu_\eta^{(4)})^{\frac{q}{4}}q^{\frac{3q}{4}}\Big(\sum_{j=1}^n \mathcal{L}_j^q\Big)^{\frac{q}{2(q-2)}}(\|A\|^q)^{\frac{q-4}{2(q-2)}},\\
B_3 =& q^q\sigma_\xi^q\sigma_\eta^q\Big(\sum_{j=1}^n \mathcal{L}_j^q\Big)^{\frac{q-4}{q-2}}\Big(\|A\|^q\Big)^{\frac{2}{q-2}},\\
B_4 =& q^{\frac{5q}{4}}\sigma_\xi^q\mu_\eta^{(\frac{q}{2})}\sigma_\eta^{\frac{q}{2}}\Big(\sum_{j=1}^n \mathcal{L}_j^q\Big)^{\frac{q-4}{2(q-2)}}\Big(\|A\|^q\Big)^{\frac{q}{2(q-2)}},\\
B_5 =& q^{\frac{3q}{2}}\sigma_\xi^q\big(\mu_\eta^{(\frac{q}{2})}\big)^2\Big(\sum_{j=1}^n \mathcal{L}_j^q\Big)^{\frac{q-4}{(q-2)}}\Big(\|A\|^q\Big)^{\frac{2}{(q-2)}},\\
B_6 =& q^q\sigma_\xi^q\mu_\eta^{(q)}\sum_{j=1}^n \mathcal{L}_j^q,\\
B_7 =& q^{\frac{3q}{2}}\sigma_\eta^q\mu_\xi^{(q)}\sum_{j=1}^n \mathcal{L}_j^q,\\
B_8 =& q^{2q}\mu_\xi^{(q)}\mu_\eta^{(q)}\sum_{i,j=1}^n |a_{ij}|^q.
\end{aligned}$$

Applying Young's inequality, we obtain the bounds

$$\begin{aligned}
B_2 \le& Cq^{\frac{3q}{4}}\Big(\sigma_\xi^4(\mu_\eta^{(4)})^{\frac{q-2}{2}}\sum_{j=1}^n \mathcal{L}_j^q + \sigma_\xi^{2q}\|A\|^q),\\
B_3 \le& C^q q^q\sigma_\xi^q\sigma_\eta^q\sum_{j=1}^n \mathcal{L}_j^q + C^q q^q\sigma_\xi^q\sigma_\eta^q\|A\|^q\Big),\\
B_4 \le& C^q q^{\frac{5q}{4}}\Big((\sigma_\xi^{2q}+\sigma_\eta^{2q})\|A\|^q + (\mu_\eta^{(\frac{q}{2})})^{\frac{2(q-2)}{q-4}}(\sigma_\xi^{q\frac{q-6}{q-4}} + \sigma_\eta^{q\frac{q-6}{q-4}})\sum_{j=1}^n \mathcal{L}_j^q\Big),\\
B_5 \le& C^q q^{\frac{3q}{2}}\Big(\sigma_\xi^{2q}\|A\|^q + \sigma_\xi^{\frac{q(q-6)}{q-4}}(\mu_\eta^{(\frac{q}{2})})^{\frac{2(q-2)}{q-4}}\sum_{j=1}^n \mathcal{L}_j^q\Big).
\end{aligned}$$

The last inequalities give

$$A \le C^q\big(\mathcal{A}_1\|A\|^q + \mathcal{A}_2(\sum_{j=1}^n \mathcal{L}_j^q) + \mathcal{A}_3(\sum_{i,j=1}^n |a_{ij}|^q)\big),$$

where

$$\begin{aligned}\mathcal{A}_1 =&q^{\frac{3q}{2}}(\sigma_\xi^{2q}+\sigma_\eta^{2q}),\\ \mathcal{A}_2 =&q^{\frac{3q}{2}}(\sigma_\xi^{(2q)}+\sigma_\eta^{2q})^{\frac{q-6}{2(q-4}}(\mu_\xi^{(\frac{q}{2})})^{\frac{2(q-2)}{q-4}},\\ \mathcal{A}_3 =&q^{2q}\mu_\xi^{(q)}\mu_\eta^{(q)}.\end{aligned}$$

Thus Lemma is proved. □

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Friedrich Götze, Faculty of Mathematics, Bielefeld University, Bielefeld, Germany

Email address: `goetze@math.uni-bielefeld.de`

Alexander N. Tikhomirov, Institute of Physics and Mathematics, Komi Science Center of Ural Division of RAS, Syktyvkar, Russia; and National Research University Higher School of Economics, Moscow, Russia

Email address: `tikhomirov@ipm.komisc.ru`

Dmitry A. Timushev, Institute of Physics and Mathematics, Komi Science Center of Ural Division of RAS, Syktyvkar, Russia

Email address: `timushev@ipm.komisc.ru`