



On Nonlinear Markov Processes in the Sense of McKean

Marco Rehmeier¹ · Michael Röckner^{1,2}

Received: 18 October 2024 / Revised: 19 May 2025 / Accepted: 1 June 2025
© The Author(s) 2025

Abstract

We study nonlinear time-inhomogeneous Markov processes in the sense of McKean's (Proc Natl Acad Sci USA 56(6):1907–1911, 1966) seminal work. These are given as families of laws $\mathbb{P}_{s,\zeta}$, $s \geq 0$, on path space, where ζ runs through a set of admissible initial probability measures on \mathbb{R}^d . In this paper, we concentrate on the case where every $\mathbb{P}_{s,\zeta}$ is given as the path law of a solution to a McKean–Vlasov stochastic differential equation (SDE), where the latter is allowed to have merely measurable coefficients, which in particular are not necessarily weakly continuous in the measure variable. Our main result is the identification of general and checkable conditions on such general McKean–Vlasov SDEs, which imply that the path laws of their solutions form a nonlinear Markov process. Our notion of nonlinear Markov property is in McKean's spirit, but more general in order to include processes whose one-dimensional time marginal densities solve a nonlinear parabolic partial differential equation, more precisely, a nonlinear Fokker–Planck–Kolmogorov equation, such as Burgers' equation, the porous media equation and variants thereof with transport-type drift, and also the very recently studied two-dimensional vorticity Navier–Stokes equation and the p -Laplace equation. In all these cases, the associated McKean–Vlasov SDEs are such that both their diffusion and drift coefficients singularly depend (i.e., Nemytskii type) on the one-dimensional time marginals of their solutions. We stress that for our main result the nonlinear Fokker–Planck–Kolmogorov equations do not have to be well posed. Thus, we establish a one-to-one correspondence between solution flows of a large class of nonlinear parabolic PDEs and nonlinear Markov processes.

Keywords Nonlinear Fokker–Planck–Kolmogorov equation · McKean–Vlasov stochastic differential equation · Nonlinear Markov process · Porous media equation · Flow selection · Extremality

✉ Marco Rehmeier
mrehmeier@math.uni-bielefeld.de
Michael Röckner
roeckner@math.uni-bielefeld.de

¹ Faculty of Mathematics, Bielefeld University, 33615 Bielefeld, Germany

² Academy of Mathematics and System Sciences, CAS, Beijing, China

Mathematics Subject Classification (2020) Primary 60H30 · 35Q84 · 35K55 · 60J25; Secondary 60J45 · 60J60

1 Introduction

In this work, we study nonlinear Markov processes in the sense of McKean’s seminal work [32]. We suggest a modified definition of such processes (see Definition 2.1), derive sufficient criteria for their construction, and present and analyze several classes of examples. These include nonlinear Markov processes associated with the classical porous media equation in all spatial dimensions $d \geq 1$, with their Barenblatt solutions as one-dimensional time marginal densities, and variants thereof with more general diffusivity functions and additional drift of transport type.

1.1 Linear Situation

Let us briefly recall how the theory of classical “linear” Markov processes is linked to stochastic differential equations (SDEs) and linear parabolic Fokker–Planck–Kolmogorov equations (FPKEs). If

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad t \geq s, \quad X_s = x,$$

where B is a standard d -dimensional Brownian motion, has a unique probabilistic weak solution $X^{s,x}$ for each $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, then the corresponding path laws $\mathbb{P}_{s,x}$ form a classical Markov process (see, e.g., [19] or [43]), i.e., denoting the natural projections on the space of continuous paths $C([s, \infty), \mathbb{R}^d)$ by $\pi_t^s, \pi_t^s(w) := w(t)$, the Markov property

$$\mathbb{P}_{s,\zeta}(\pi_t^s \in A | \sigma(\pi_\tau^s, s \leq \tau \leq r)) = \mathbb{P}_{r,\pi_r^s}(\pi_t^r \in A) \quad \mathbb{P}_{s,\zeta} - \text{a.s. for all } A \in \mathcal{B}(\mathbb{R}^d) \quad (\ell\text{MP})$$

holds for all $\zeta \in \mathcal{P}$ (the space of Borel probability measures on \mathbb{R}^d) and $0 \leq s \leq r \leq t$, where $\mathbb{P}_{s,\zeta} := \int_{\mathbb{R}^d} \mathbb{P}_{s,x} \zeta(dx)$. By Itô’s formula, the one-dimensional time marginals, $\mu_t^{s,\zeta}, t \geq s$, of the latter solve the linear FPKE

$$\partial_t \mu_t = \partial_{ij}^2 (a_{ij}(t, x) \mu_t) - \partial_i (b_i(t, x) \mu_t), \quad t \geq s, \quad \mu_s = \zeta \quad (1)$$

(in the Schwartz distributional sense), where $a = (a_{ij})_{i,j \leq d} = \frac{1}{2} \sigma \sigma^T$ and we use Einstein summation convention. In particular, $(\mu_t^{s,\zeta})_{t \geq s}$ is a continuous curve in \mathcal{P} with the weak topology. Vice versa, if this linear FPKE has a unique weakly continuous probability measure-valued solution for all probability initial conditions and if the coefficients satisfy a mild integrability condition with respect to these solutions, then by the Ambrosio–Figalli–Trevisan superposition principle [1, 20, 46] (see also [17]) and [44, Thm.6.2.3], the above SDE is weakly well posed and the family of path laws

$\mathbb{P}_{s,x}$ of its unique weak solutions with Dirac initial conditions is a classical Markov process. For a comparison of this linear and our nonlinear case, see Sect. 2.1.

1.2 McKean's Vision

McKean seems to have been the first who realized that the core Markovian feature, i.e., that the path law $\mathbb{P}_{s,\zeta}$ restricted to $\sigma(\pi_\tau^s, r \leq \tau)$ and conditioned on $\sigma(\pi_\tau^s, s \leq \tau \leq r)$ is a function of s, r and the current position π_r^s only, is more general than the specific formula (\mathcal{LMP}) (see also the formulation in [44, p.145]). In [32], he suggested a generalization by replacing \mathbb{P}_{r,π_r^s} by (a regular conditional probability of) $\mathbb{P}_{r,\mu_r^{s,\zeta}}[\cdot | \pi_r^r = \pi_r^s]$, where $\mu_r^{s,\zeta}$ denotes the one-dimensional time marginal of $\mathbb{P}_{s,\zeta}$ at r . McKean's envisioned program was to connect this more general Markov property to nonlinear parabolic partial differential equations (PDEs), more precisely to nonlinear FPKEs of *Nemytskii type*, i.e., to equations of the form

$$\partial_t u = \partial_{ij}^2(a_{ij}(t, u, \cdot)u) - \partial_i(b_i(t, u, \cdot)u), \quad t \geq s, \quad u(t, x)dx \xrightarrow{t \rightarrow s} \zeta \quad (2)$$

by constructing a family of path measures $\mathbb{P}_{s,\zeta}$ satisfying this more general property, whose one-dimensional time marginals have densities which solve such a PDE and, thereby, to provide a natural and rich probabilistic representation for solutions to (2). In a later paper, McKean indicated how to construct such families as path laws of solutions to distribution-dependent SDEs (DDSDEs, also called *McKean–Vlasov equations of Nemytskii type*) of the form

$$\begin{aligned} dX_t &= b(t, u(t, X_t), X_t)dt + \sigma(t, u(t, X_t), X_t)dB_t, \\ u(t, x)dx &= \mathcal{L}_{X_t}, \quad u(t, x)dx \xrightarrow{t \rightarrow s} \zeta \end{aligned} \quad (3)$$

(\mathcal{L}_{X_t} = distribution of X_t), see [33]. In addition to special examples, such as discrete state space models from kinetic gas theory and DDSDEs with unit diffusion and drift $b(t, u(t, \cdot), x) = \int_{\mathbb{R}} (y - x)u(t, y)dy$, he proposed to study very interesting models of type (2), such as Burgers' equation and the porous media equation in dimension $d = 1$. Though, to the best of our knowledge, he did not go beyond such suggestions in [32] or subsequent works, he was clearly envisioning a much larger theory.

1.3 Goals of Our Paper

The aim of this paper is to first present a theory of nonlinear Markov processes based on McKean's suggestions, including a precise definition of such processes, and discuss some of their basic properties. Second, we shall prove a result which specifies sufficient and checkable conditions to construct from solutions to a general *nonlinear* FPKE

$$\partial_t \mu_t = \partial_{ij}^2(a_{ij}(t, \mu_t, \cdot)\mu_t) - \partial_i(b_i(t, \mu_t, \cdot)\mu_t) \quad (4)$$

(of which (2) is a special case) nonlinear Markov processes consisting of solution path laws to the associated DDSDE

$$dX_t = b(t, \mathcal{L}_{X_t}, X_t)dt + \sigma(t, \mathcal{L}_{X_t}, X_t)dB_t \quad (5)$$

(of which (3) is a special case) with one-dimensional time marginals given by these FPKE solutions. Third, we shall apply this result to a large class of examples. Our long-term goal is to develop an as rich theory as in the linear case, where the correspondence between linear (i.e., classical) Markov processes and linear PDE theory has become a very versatile and important tool to solve problems about the linear PDE through its corresponding linear Markov process and vice versa.

Our definition of a nonlinear Markov process is in the spirit of [32], but more general since we allow a possibly restricted class $\mathcal{P}_0 \subseteq \mathcal{P}$ of initial data, see Definition 2.1. This is of high relevance since then our theory also applies to examples in which (2) can only be solved for initial measures from a restricted class \mathcal{P}_0 , which is quite common for such nonlinear equations of Nemytskii type.

Let us now describe our main result, Theorem 3.8. The starting point is a family of weakly continuous probability measure-valued solutions $\{\mu^{s,\zeta}\}_{(s,\zeta) \in \mathbb{R}_+ \times \mathcal{P}_0}$, $\mu^{s,\zeta} = (\mu_t^{s,\zeta})_{t \geq s}$, $\mu_s^{s,\zeta} = \zeta$, to (4) and corresponding solution path laws $\{\mathbb{P}_{s,\zeta}\}_{(s,\zeta) \in \mathbb{R}_+ \times \mathcal{P}_0}$ to the associated DDSDE (5) such that

$$\mu_t^{s,\zeta} = \mathbb{P}_{s,\zeta} \circ (\pi_t^s)^{-1} = \mathcal{L}_{X_{s,t}^\zeta} \in \mathcal{P}_0, \quad t \geq s. \quad (6)$$

(Such solution path laws exist by the aforementioned superposition principle and its nonlinear extension [10, 11].) Here $\mathcal{P}_0 \subseteq \mathcal{P}$ is an arbitrary prescribed set of admissible initial conditions. Our main result identifies a pair of sufficient conditions implying:

- 1) each $\mathbb{P}_{s,\zeta}$ is the unique (!) solution path law with (6),

and

- 2) $\{\mathbb{P}_{s,\zeta}\}_{(s,\zeta) \in \mathbb{R}_+ \times \mathcal{P}_0}$ is a nonlinear Markov process in the sense of Definition 2.1.

Hence, we establish a one-to-one correspondence between solution flows for (4) and nonlinear Markov processes associated with (5).

Contrary to what one might expect, we do not need the nonlinear FPKE to be well posed. Instead, our first condition is that the prescribed solutions satisfy the *flow property*, i.e.,

$$\mu_t^{s,\zeta} \in \mathcal{P}_0, \quad \mu_t^{s,\zeta} = \mu_t^{r,\mu_r^{s,\zeta}}, \quad \forall 0 \leq s \leq r \leq t, \quad \zeta \in \mathcal{P}_0. \quad (7)$$

Such solution flows may, for instance, be obtained by selection ([37]) or by their construction via nonlinear semigroup methods (see, e.g., [2]) (and, of course, (7) is always valid in well-posed cases). The map $\mathcal{P}_0 \ni \zeta \mapsto \mu_t^{s,\zeta}$ is in general not linear on its domain, even when \mathcal{P}_0 is a convex set, e.g., $\mathcal{P}_0 = \mathcal{P}$. Therefore, we call Markov

processes as in Definition 2.1 *nonlinear Markov processes*. This nonlinearity is one main difference from the theory of linear Markov processes, where $\mathcal{P}_0 = \mathcal{P}$, and where one has the *Chapman–Kolmogorov equations*, i.e.,

$$\mu_t^{s,\delta_x} = \int_{\mathbb{R}^d} \mu_t^{r,\delta_y} \mu_r^{s,\delta_x}(\mathrm{d}y), \quad \forall 0 \leq s \leq r \leq t, x \in \mathbb{R}^d,$$

which imply (7).

Our second condition is a geometric one: Each $\mu^{s,\zeta}$ should be an extreme point in the convex set of all weakly continuous probability solutions with initial datum (s, ζ) to the linearized FPKE obtained by fixing a priori the nonlinear solution $\mu^{s,\zeta}$ in the measure variable of the coefficients in (4). For the connection between extremality and Markov property for martingale problems in the linear case, see the fundamental paper [45].

Our main result, Theorem 3.8, states that under these two conditions, assertions (1) and (2) mentioned above hold. Two main ingredients of the proof are Theorem 3.4 and Lemma 3.5. Theorem 3.4 ensures the uniqueness of $\mathbb{P}_{s,\zeta}$, and Lemma 3.5 gives a useful equivalent formulation of the extremality condition in terms of a restricted uniqueness condition for the corresponding linearized FPKEs. This restricted uniqueness condition is checkable in many applications, see Sect. 4. We point out that Theorem 3.4 and Lemma 3.5 are new results themselves.

An important consequence of our proof is Corollary 3.10, where we prove that it is sufficient to prove the extremality condition only for solutions $\mu^{s,\zeta}$ with ζ from a strict subset \mathfrak{P}_0 of all admissible initial conditions \mathcal{P}_0 , if $\mu_t^{s,\zeta}$ for $t > s$ belongs to \mathfrak{P}_0 for all $\zeta \in \mathcal{P}_0$. Then, we still prove (2), and (1) is still valid for all $\zeta \in \mathfrak{P}_0$. This extension is crucial to treat cases (as, for instance, the porous media equation) where \mathcal{P}_0 contains Dirac measures, since in many cases it turns out that our extremality condition can only be verified for ζ which are absolutely continuous w.r.t. Lebesgue measure, compare Sect. 4.

Finally, we mention that even for the linear special case, i.e., when the underlying FPKE is of type (1), our results yield new outcomes as well, see Sect. 3.3.

1.4 Applications

Our two sufficient conditions mentioned above are, of course, in particular satisfied when the nonlinear and the corresponding linearized FPKEs are well posed in the class of weakly continuous probability solutions, and we briefly present this case at the beginning of Sect. 4. Usually, well-posedness for the nonlinear equation is obtained by assuming (Lipschitz) continuity of the coefficients in their measure variable w.r.t. a Wasserstein distance.

All our main examples, i.e., Sect. 4.2 (i–iv), are not of this type, but of Nemytskii type, namely generalized porous media equations with drift in all spatial dimensions $d \geq 1$ (in particular including *nonlinear distorted Brownian motion*, see [3, 12, 38]), Burgers' equation, equations of porous media type where the Laplacian is replaced by a fractional Laplacian so that the corresponding DDSDEs have Lévy noise (see

also [6, 13, 40]), and the classical porous media equation with its Barenblatt solutions. This is the first work in which the nonlinear Markov property for solutions to the DDSDEs associated with each of these nonlinear PDEs is proved. In these cases, the coefficients are only measurable with respect to the Borel σ -algebra of the weak topology on \mathcal{P} (see the beginning of Sect. 4.2). As we show in Sect. 4.2, we are able to apply the theory developed in this work to all these examples. As already mentioned, in particular the Barenblatt solutions to the porous media equation, $d \geq 1$, uniquely determine a nonlinear Markov process with these solutions as one-dimensional time marginals, cf. Sect. 4.2 (iv). We consider this as a central new application of our main result.

While, in the spirit of McKean's vision, our focus is on nonlinear FPKEs of Nemytskii type (2) and the associated DDSDEs (3), as already indicated above, our main result and our examples also cover nonlinear Markov processes with general probability measures as one-dimensional time marginals, arising as solution flows to (4). In this case, the coefficients of the associated DDSDE (5) do not depend pointwise on the one-dimensional time marginal density (which in general need not exist), but on the one-dimensional time marginal as a measure.

Finally, we mention the very recent papers [13, 47], in which an earlier arXiv-version of our work was applied to the $2D$ vorticity Navier–Stokes equation and the p -Laplace equation, respectively (see (v) and (vi) in Sect. 4.2). This earlier version was already applied in [7].

1.5 Related Literature

Concerning the related literature, we would like to mention the very elaborate book by Kolokoltsov [29], in which nonlinear FPKEs with coefficients with Lipschitz continuous dependence in their measure variable (e.g., with respect to a Wasserstein distance) and their relation to DDSDEs are studied. The book contains well-posedness results for nonlinear FPKEs and their associated linearized equations, possibly including an additional jump operator. For such cases, the author associates with these well-posed equations a family of linear Markov processes in the following way: If for each initial datum $\zeta \in \mathcal{P}$ the linearized FPKE, obtained by fixing the unique solution curve μ^ζ to the nonlinear FPKE with initial datum ζ in the measure component of the coefficients, is well posed, then there is a unique linear Markov process related to each such linearized equation. More precisely, it is given by the path laws of the unique weak solutions to the stochastic equation associated with this linearized FPKE. The author defines this family of linear Markov processes as a nonlinear Markov process. As mentioned before, all our main examples in the present work are of Nemytskii type (2) and hence fail to satisfy the assumptions in [29]. Our notion of nonlinear Markov processes, however, is sufficiently general to also cover these cases.

This paper was strongly motivated by the results in [38], where (among other results) it was shown that uniqueness (in a restricted class) of distributional solutions to nonlinear FPKEs and their linearizations implies weak uniqueness (in a restricted class) of the associated DDSDEs, and as a consequence, the nonlinear Markov property of the path laws of the weak solutions to the latter was shown (see [38, Cor.4.6]). In compar-

ison with this, our Theorem 3.8, does not require such a uniqueness property, but only the existence of a solution flow satisfying an extremality condition. This enhances the realm of applications substantially (see Chapter 4) and, in particular, covers the so-called *nonlinear distorted Brownian motion* (NLDBM), which was already studied as a main example in [38], thus identifying NLDBM as a nonlinear Markov process in the sense of our Definition 2.1 (see Sect. 4.2 (i)).

We are not aware of any other paper on nonlinear Markov processes realizing McKean's program after [32, 33] in general, or specifically in the Nemytskii case. However, if one restricts oneself to finding a probabilistic representation for the solutions of (2) as one-dimensional time marginal laws of a stochastic process without proving that it is a nonlinear Markov process, there are several papers on this in the literature, e.g., [8, 9, 14, 16, 18, 27, 31]. Here we would like to draw special attention to the pioneering work [15], which treats the classical porous media equation on \mathbb{R}^1 on the basis of very nice original ideas and without using the much later discovered technique from [10, 11] (see also [4]), which in turn are based on the application of the already mentioned superposition principle to the associated linearized porous media equation. As mentioned before, our results in the present paper apply to the classical porous media equation on \mathbb{R}^d for all dimensions $d \geq 1$, see Sect. 4.2 (iv).

Finally, we mention the recent book [39], which contains results on DDSDEs with reflected or killed boundary, a topic to which the results of this paper may be extended in future work.

1.6 Structure of the Paper

Section 2 contains our definition of nonlinear Markov processes, as well as Proposition 2.3, which shows that also in the nonlinear case Markovian path laws are uniquely determined by suitably chosen one-dimensional time marginals. A comparison to classical (linear) Markov processes is contained in Sect. 2.1. After introducing all relevant equations and definitions of solutions and well-posedness in Sect. 3.1, we formulate and prove our main results in Sect. 3.2. In Sect. 3.3, we present new results for the classical linear case, which are obtained as special cases of Theorems 3.4 and 3.8. Section 4 contains applications of our main result, i.e., we associate nonlinear Markov processes to a large class of nonlinear FPKEs and McKean–Vlasov SDEs.

1.6.1 Notation

The Euclidean norm and inner product on \mathbb{R}^d are denoted by $|\cdot|$ and $\langle \cdot, \cdot \rangle$, and we set $\mathbb{R}_+ := [0, \infty)$. For topological spaces X and Y , $C(X, Y)$ denotes the space of continuous functions $f : X \rightarrow Y$, $\mathcal{B}(X)$ the Borel σ -algebra on X and $\mathcal{P}(X)$ the space of all probability measures on $\mathcal{B}(X)$, equipped with the weak topology, i.e., the initial topology of the maps $\mathcal{P}(X) \ni \mu \mapsto \int_X f(x) \mu(dx)$, for all bounded $f \in C(X, \mathbb{R})$. We let δ_x denote the Dirac measure in $x \in X$, and for $\mu \in \mathcal{P}(X)$ and $h : X \rightarrow \mathbb{R}$, we sometimes write $\mu(h) := \int_X h(x) \mu(dx)$, provided the integral exists. For $X = \mathbb{R}^d$, we write \mathcal{P} instead of $\mathcal{P}(\mathbb{R}^d)$ and \mathcal{P}_a for the space of measures $\mu \in \mathcal{P}$, which are

absolutely continuous with respect to Lebesgue measure dx , i.e., $\mu \ll dx$. If $\mu \ll \nu$ and $\nu \ll \mu$, we write $\mu \sim \nu$.

Let $\mathcal{B}_b^{(+)}(\mathbb{R}^d)$ denote the space of bounded (nonnegative) Borel functions $f : \mathbb{R}^d \rightarrow \mathbb{R}$ and $C_b(\mathbb{R}^d)$ its subspace of continuous functions. For $m \in \mathbb{N}_0$, $C_{(c)}^m(\mathbb{R}^d)$ are the spaces of (compactly supported) continuous functions with continuous partial derivatives up to order m , and we write $C_{(c)}(\mathbb{R}^d)$ instead of $C_{(c)}^0(\mathbb{R}^d)$. We use the standard notation $L^p(\mathbb{R}^d; \mathbb{R}^m) = \{g : \mathbb{R}^d \rightarrow \mathbb{R}^m \text{ Borel}, |g|_{L^p} < \infty\}$ with the usual L^p -norms $|\cdot|_p$ for $p \in [1, \infty]$. The corresponding local spaces are denoted $L_{\text{loc}}^p(\mathbb{R}^d; \mathbb{R}^m)$. For $m = 1$, we write $L^p(\mathbb{R}^d)$ instead of $L^p(\mathbb{R}^d; \mathbb{R})$.

The path space $C([s, \infty), \mathbb{R}^d)$ with the topology of locally uniform convergence is denoted by Ω_s with Borel σ -algebra $\mathcal{B}(\Omega_s) = \sigma(\pi_t^s, t \geq s)$, where $\pi_t^s, t \geq s$, are the usual projections on Ω_s . We also use the notation $\mathcal{F}_{s,r} := \sigma(\pi_\tau^s, s \leq \tau \leq r)$ and $\Pi_r^s : \Omega_s \rightarrow \Omega_r, \Pi_r^s : w \mapsto w|_{[r, \infty)}$ for $s \leq r$.

2 Nonlinear Markov Processes: Definition and Comparison to Linear Case

Definition 2.1 Let $\mathcal{P}_0 \subseteq \mathcal{P}$. A *nonlinear Markov process* is a family $(\mathbb{P}_{s,\zeta})_{(s,\zeta) \in \mathbb{R}_+ \times \mathcal{P}_0}$ of probability measures $\mathbb{P}_{s,\zeta}$ on $\mathcal{B}(\Omega_s)$ such that

- (i) The marginals $\mathbb{P}_{s,\zeta} \circ (\pi_t^s)^{-1} =: \mu_t^{s,\zeta}$ belong to \mathcal{P}_0 for all $0 \leq s \leq t$ and $\zeta \in \mathcal{P}_0$, and $\mu_s^{s,\zeta} = \zeta$.
- (ii) The *nonlinear Markov property* holds, i.e., for all $0 \leq s \leq r \leq t, \zeta \in \mathcal{P}_0$

$$\mathbb{P}_{s,\zeta}(\pi_t^s \in A | \mathcal{F}_{s,r})(\cdot) = p_{(s,\zeta),(r,\pi_r^s(\cdot))}(\pi_t^r \in A) \quad \mathbb{P}_{s,\zeta} - \text{a.s. for all } A \in \mathcal{B}(\mathbb{R}^d), \quad (\text{MP})$$

where $p_{(s,\zeta),(r,y)}, y \in \mathbb{R}^d$, is a regular conditional probability kernel from \mathbb{R}^d to $\mathcal{B}(\Omega_r)$ of $\mathbb{P}_{r,\mu_r^{s,\zeta}}[\cdot | \pi_r^r = y], y \in \mathbb{R}^d$ (i.e., in particular $p_{(s,\zeta),(r,y)} \in \mathcal{P}(\Omega_r)$ and $p_{(s,\zeta),(r,y)}(\pi_r^r = y) = 1$).

The term *nonlinear Markov property* originates from the fact that in the situation of Definition 2.1 the map $\mathcal{P}_0 \ni \zeta \mapsto \mu_t^{s,\zeta}$ is, in general, not linear on its domain, even if \mathcal{P}_0 is a convex set (which we do not assume). For the special case of a classical linear Markov process with $\mathcal{P}_0 = \mathcal{P}$, this map is linear on its domain, see Remark 2.5.

Remark 2.2 (i) Consider the situation of Definition 2.1, and let $(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}_0$ and $s \leq r$. By definition of $p_{(s,\zeta),(r,y)}, y \in \mathbb{R}^d$, one has

$$\mathbb{P}_{r,\mu_r^{s,\zeta}}(\cdot) = \int_{\mathbb{R}^d} p_{(s,\zeta),(r,y)}(\cdot) \mu_r^{s,\zeta}(dy) = \int_{\Omega_s} p_{(s,\zeta),(r,\pi_r^s(w))}(\cdot) \mathbb{P}_{s,\zeta}(dw). \quad (8)$$

- (ii) Let $0 \leq s \leq r \leq t$ and $\zeta \in \mathcal{P}_0$. By a monotone class argument, (MP) is equivalent to

$$\mathbb{E}_{s,\zeta}[h(\pi_t^s)|\mathcal{F}_{s,r}](\cdot) = p_{(s,\zeta),(r,\pi_r^s(\cdot))}(h(\pi_t^r)) \quad \mathbb{P}_{s,\zeta} - \text{a.s. for all } h \in \mathcal{B}_b(\mathbb{R}^d).$$

- (iii) The one-dimensional time marginals $\mu_t^{s,\zeta} = \mathbb{P}_{s,\zeta} \circ (\pi_t^s)^{-1}$ of a nonlinear Markov process satisfy the flow property, i.e.,

$$\mu_t^{s,\zeta} = \mu_t^{r,\mu_r^{s,\zeta}}, \quad \forall 0 \leq s \leq r \leq t, \zeta \in \mathcal{P}_0.$$

Indeed, by (8), for all $A \in \mathcal{B}(\mathbb{R}^d)$:

$$\begin{aligned} \mu_t^{s,\zeta}(A) &= \mathbb{E}_{s,\zeta}[\mathbb{P}_{s,\zeta}(\pi_t^s \in A | \mathcal{F}_{s,r})] \\ &= \mathbb{E}_{s,\zeta}[p_{(s,\zeta),(r,\pi_r^s)}(\pi_t^r \in A)] = \mathbb{P}_{r,\mu_r^{s,\zeta}}(\pi_t^r \in A) = \mu_t^{r,\mu_r^{s,\zeta}}(A). \end{aligned}$$

The following proposition shows that the finite-dimensional distributions of the path measures $\mathbb{P}_{s,\zeta}$ of a nonlinear Markov process (and hence the path measures itself) are uniquely determined by the family of one-dimensional time marginals $p_{r,t}^{s,\zeta}(x, dz)$, $s \leq r$, $x \in \mathbb{R}^d$, defined in (9).

Proposition 2.3 *Let $(\mathbb{P}_{s,\zeta})_{(s,\zeta) \in \mathbb{R}_+ \times \mathcal{P}_0}$ be a nonlinear Markov process. For $\zeta \in \mathcal{P}_0$, $0 \leq s \leq r \leq t$ and $x \in \mathbb{R}^d$, define $p_{r,t}^{s,\zeta}(x, dz) \in \mathcal{P}$ by*

$$p_{r,t}^{s,\zeta}(x, dz) := p_{(s,\zeta),(r,x)} \circ (\pi_t^r)^{-1}(dz) (= \mathbb{P}_{r,\mu_r^{s,\zeta}}[\cdot | \pi_r^r = x] \circ (\pi_t^r)^{-1}), \quad (9)$$

which is uniquely determined for $\mu_r^{s,\zeta}$ -a.e. $x \in \mathbb{R}^d$. Then for $n \in \mathbb{N}_0$, $f \in \mathcal{B}_b((\mathbb{R}^d)^{n+1})$ and $s \leq t_0 < \dots < t_n$:

$$\begin{aligned} &\mathbb{E}_{s,\zeta}[f(\pi_{t_0}^s, \dots, \pi_{t_n}^s)] < \\ &= \int_{\mathbb{R}^d} \left(\dots \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} f(x_0, \dots, x_n) p_{t_{n-1},t_n}^{s,\zeta}(x_{n-1}, dx_n) \right) p_{t_{n-2},t_{n-1}}^{s,\zeta}(x_{n-2}, dx_{n-1}) \dots \right) \\ &\mu_{t_0}^{s,\zeta}(dx_0). \end{aligned} \quad (10)$$

Proof Let $(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}_0$. By a monotone class argument, it suffices to consider $f = f_0 \otimes \dots \otimes f_n$ with $f_i \in \mathcal{B}_b(\mathbb{R}^d)$, $n \in \mathbb{N}_0$, and to proceed by induction over n . For $n = 0$,

$$\mathbb{E}_{s,\zeta}[f_0(\pi_{t_0}^s)] = \int_{\mathbb{R}^d} f_0(x_0) \mu_{t_0}^{s,\zeta}(dx_0)$$

holds by definition of $\mu_{t_0}^{s,\zeta}$. For $f_i \in \mathcal{B}_b(\mathbb{R}^d)$, $0 \leq i \leq n+1$, and $s \leq t_0 < \dots < t_{n+1}$, we have

$$\begin{aligned} \mathbb{E}_{s,\zeta}[\Pi_{i=0}^{n+1} f_i(\pi_{t_i}^s)] &= \mathbb{E}_{s,\zeta}[\Pi_{i=0}^n f_i(\pi_{t_i}^s) \mathbb{E}_{s,\zeta}[f_{n+1}(\pi_{t_{n+1}}^s) | \mathcal{F}_{s,t_n}]] = \mathbb{E}_{s,\zeta}[\Pi_{i=0}^n f_i(\pi_{t_i}^s) p_{(s,\zeta), (t_n, \pi_{t_n}^s)}(f_{n+1}(\pi_{t_{n+1}}^s))] \\ &= \int \left(\dots \int \Pi_{i=0}^n f_i(x_i) \left(\int_{\mathbb{R}^d} f_{n+1}(x_{n+1}) p_{t_n, t_{n+1}}^{s,\zeta}(x_n, dx_{n+1}) \right) p_{t_{n-1}, t_n}^{s,\zeta}(x_{n-1}, dx_n) \dots \right) \\ &\quad \mu_{t_0}^{s,\zeta}(dx_0), \end{aligned}$$

where we used the nonlinear Markov property and Remark 2.2 (ii) for the second equality and the induction assumption for the third equality. \square

Remark 2.4 (i) Hence, similarly as in the linear case, the path measure $\mathbb{P}_{s,\zeta}$ can be reconstructed from one-dimensional time marginals $p_{r,t}^{s,\zeta}(x, \cdot)$, $t \geq r \geq s \geq 0$, $x \in \mathbb{R}^d$. However, the nonlinear nature becomes evident via the fact that even in the case $\mathcal{P}_0 = \mathcal{P}$ it is usually not true that $p_{r,t}^{s,\zeta}(x, \cdot) = \mathbb{P}_{r,\delta_x} \circ (\pi_t^r)^{-1}$, see Remark 4.1.

(ii) By Proposition 2.3, the nonlinear Markov property extends to

$$\mathbb{E}_{s,\zeta}[h(\pi_{t_1}^s, \dots, \pi_{t_n}^s) | \mathcal{F}_{s,r}](\cdot) = p_{(s,\zeta), (r, \pi_r^s(\cdot))}(h(\pi_{t_1}^r, \dots, \pi_{t_n}^r)) \quad \mathbb{P}_{s,\zeta} - \text{a.s.}$$

for all $0 \leq s \leq r \leq t_1 < \dots < t_n$, $n \in \mathbb{N}$, $\zeta \in \mathcal{P}_0$ and $h \in \mathcal{B}_b((\mathbb{R}^d)^n)$. By another application of the monotone class theorem, this generalizes to

$$\mathbb{E}_{s,\zeta}[G \circ \Pi_r^s | \mathcal{F}_{s,r}](\cdot) = p_{(s,\zeta), (r, \pi_r^s(\cdot))}(G) \quad \mathbb{P}_{s,\zeta} - \text{a.s.}$$

for all $0 \leq s \leq r$, $\zeta \in \mathcal{P}_0$ and $G \in \mathcal{B}_b(\Omega_r)$. Note that a function $\tilde{G} : \Omega_s \rightarrow \mathbb{R}$ is bounded and $\sigma(\pi_\tau^s, \tau \geq r)$ -measurable if and only if $\tilde{G} = G \circ \Pi_r^s$ with $G \in \mathcal{B}_b(\Omega_r)$.

2.1 Comparison with Linear Markov Processes

Here we comment on the connection to classical linear Markov processes. More precisely, we show that our Definition 2.1 is a proper extension of the linear definition. Afterward we briefly recall how classical Markov processes typically arise from linear Kolmogorov operators.

Let $(\mathbb{P}_{s,x})_{(s,x) \in \mathbb{R}_+ \times \mathbb{R}^d}$ be a linear normal time-inhomogeneous *Markov process*, i.e., for each (s, x) , $\mathbb{P}_{s,x}$ is a probability measure on $\mathcal{B}(\Omega_s)$ with $\mathbb{P}_{s,x} \circ (\pi_s^s)^{-1} = \delta_x$, such that

- (i) $x \mapsto \mathbb{P}_{s,x}(B)$ is measurable for all $s \geq 0$ and $B \in \mathcal{B}(\Omega_s)$,
- (ii) the linear Markov property holds, i.e., for all $s \leq r \leq t$

$$\mathbb{P}_{s,x}(\pi_t^s \in A | \mathcal{F}_{s,r})(\cdot) = \mathbb{P}_{r, \pi_r^s(\cdot)}(\pi_t^r \in A) \quad \mathbb{P}_{s,x} - \text{a.s. for all } A \in \mathcal{B}(\mathbb{R}^d).$$

Remark 2.5 It is natural to set $\mathbb{P}_{s,\zeta} := \int_{\mathbb{R}^d} \mathbb{P}_{s,y} \zeta(dy)$ for any $s \geq 0$ and non-Dirac $\zeta \in \mathcal{P}$. Then $\mathbb{P}_{s,\zeta} \circ (\pi_s^s)^{-1} = \zeta$, and we again define $\mu_t^{s,\zeta}$, $t \geq s$, as in Definition 2.1 (i). Then clearly $\mu_t^{s,\zeta} = \int_{\mathbb{R}^d} \mu_t^{s,y} \zeta(dy)$, and for any $0 \leq s \leq t$, the map $\mathcal{P} \ni \zeta \mapsto \mu_t^{s,\zeta}$ is linear on its domain.

Proposition 2.6 $(\mathbb{P}_{s,\zeta})_{(s,\zeta) \in \mathbb{R}_+ \times \mathcal{P}}$ is a nonlinear Markov process with $\mathcal{P}_0 = \mathcal{P}$ in the sense of Definition 2.1.

Proof We have $\mathbb{P}_{r,\mu_r^{s,\zeta}} = \int_{\mathbb{R}^d} \mathbb{P}_{r,y} \mu_r^{s,\zeta}(dy)$ and $\mathbb{P}_{r,y}$ is concentrated on $\{\pi_r^r = y\}$. Hence, $\mathbb{P}_{r,y}$, $y \in \mathbb{R}^d$, is a regular conditional probability kernel of $\mathbb{P}_{r,\mu_r^{s,\zeta}}[\cdot | \pi_r^r = y]$, $y \in \mathbb{R}^d$, and thus, (MP) holds with $p_{(s,\zeta),(r,\pi_r^s(\cdot))} = \mathbb{P}_{r,\pi_r^s(\cdot)}$. \square

Remark 2.7 The transition probabilities $\mu_t^{s,x} := \mu_t^{s,\delta_x} = \mathbb{P}_{s,x} \circ (\pi_t^s)^{-1}$ of a linear Markov process satisfy the Chapman–Kolmogorov equations

$$\mu_t^{s,x}(A) = \int_{\mathbb{R}^d} \mu_t^{r,y}(A) \mu_r^{s,x}(dy), \quad \forall 0 \leq s \leq r \leq t, x \in \mathbb{R}^d \text{ and } A \in \mathcal{B}(\mathbb{R}^d). \quad (\text{CK})$$

They also satisfy the flow property, since

$$\begin{aligned} \mu_t^{s,\zeta}(A) &= \mathbb{E}_{s,\zeta}[\mathbb{P}_{s,\zeta}(\pi_t^s \in A | \mathcal{F}_{s,r})] \\ &= \mathbb{E}_{s,\zeta}[\mathbb{P}_{r,\pi_r^s}(\pi_t^r \in A)] = \int_{\mathbb{R}^d} \mathbb{P}_{r,y}(\pi_t^r \in A) \mu_r^{s,\zeta}(dy) \\ &= \mathbb{P}_{r,\mu_r^{s,\zeta}}(\pi_t^r \in A) = \mu_t^{r,\mu_r^{s,\zeta}}(A). \end{aligned}$$

(Of course, this also follows directly from Proposition 2.6 and Remark 2.2 (iii).) To us, it seems that the Chapman–Kolmogorov equations do not have a natural counterpart in the nonlinear case, since even in the rather special case $\mathcal{P}_0 = \mathcal{P}$, (CK) will usually not be satisfied. Indeed, (CK) is not satisfied when the curves $(\mu_t^{s,x})_{t \geq s}$ are solutions to a nonlinear FPKE, which is the case for all our examples in Sect. 4. From our point of view, with regard to nonlinear Markov processes, the flow property is the key property on the level of the one-dimensional time marginals, see also our main result, Theorem 3.8.

2.1.1 Linear Markov Processes from Linear Kolmogorov Operators

Before we continue with the nonlinear case, let us recall how linear Markov processes typically arise from linear Kolmogorov operators L of type

$$L_t \varphi(x) = a_{ij}(t, x) \partial_{ij}^2 \varphi(x) + b_i(t, x) \partial_i \varphi(x), \quad \varphi \in C^2(\mathbb{R}^d),$$

where a_{ij} and b_i , $1 \leq i, j \leq d$, are Borel coefficients on $\mathbb{R}_+ \times \mathbb{R}^d$ such that $a := (a_{ij})_{i,j \leq d}$ is symmetric and nonnegative definite. The stochastic differential equation (SDE) associated with L is

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad t \geq s,$$

where $\sigma = (\sigma_{ij})_{1 \leq i, j \leq d}$ is a square root of a so that $1/2\sigma\sigma^T = a$, and B is a Brownian motion in \mathbb{R}^d . Probabilistic weak solutions to this SDE are equivalent to solutions to the martingale problem for L (see [28, Prop.4.11] and also [44]). By Itô's formula and the Ambrosio–Figalli–Trevisan superposition principle (see, for instance, [46, Thm.2.5]), $X^{s,x}$ is a weak solution to the SDE with $X_s^{s,x} = x$, if and only if its one-dimensional time marginal curve $t \mapsto \mu_t^{s,x}$ solves the Cauchy problem of the FPKE

$$\partial_t \mu_t = L_t^* \mu_t, \quad t \geq s, \quad \mu_s = \delta_x, \quad (11)$$

in the distributional sense (L^* denotes the formal dual operator of L) and satisfies a mild integrability condition with respect to a and b . By the aforementioned superposition principle and by [28, Prop.4.11], if either the Cauchy problem for the martingale problem, the SDE or the FPKE is well posed for every initial datum (s, x) , then all three problems are well posed. In this case, the path laws $\mathbb{P}_{s,x}$ of the unique weak solutions to the SDE (which are exactly the martingale problem solutions) constitute a linear Markov process.

The question whether a Markov process exists in ill-posed regimes is more delicate, but positive answers are known: If no uniqueness is known at all, at least for continuous bounded coefficients one can select one particular martingale solution $\mathbb{P}_{s,x}$ for each initial datum such that the selected family constitutes a Markov process, cf. [44, Ch.12]. If partial well-posedness holds in the sense that for suitably rich subclasses of initial conditions $\mathcal{R}_s \subseteq \mathcal{P}$, the Cauchy problem (11) from initial time s has a unique solution in a suitable subclass $\mathcal{R}_{[s,\infty)}$, then for all $\zeta \in \mathcal{R}_s$ the corresponding martingale problem has a unique solution $\mathbb{P}_{s,\zeta}$ with one-dimensional time marginals in $\mathcal{R}_{[s,\infty)}$, and the path measures $\mathbb{P}_{s,\zeta}$ induce a Markov process via disintegration, see [46, Prop.2.8].

3 Construction of Nonlinear Markov Processes: Main Result

In this section, we present our main result on the construction of nonlinear Markov processes from prescribed one-dimensional time marginals given as solution curves to nonlinear FPKEs. First, let us introduce the analytic and stochastic equations associated with a nonlinear Kolmogorov operator.

3.1 Equations and Solutions

Let $a_{ij}, b_i : \mathbb{R}_+ \times \mathcal{P} \times \mathbb{R}^d \rightarrow \mathbb{R}$, $1 \leq i, j \leq d$, be Borel maps (with respect to the weak topology on \mathcal{P}), set $a := (a_{ij})_{i,j \leq d}$, $b := (b_i)_{i \leq d}$, and consider for $(t, \mu) \in \mathbb{R}_+ \times \mathcal{P}$

the nonlinear Kolmogorov operator

$$L_{t,\mu}\varphi(x) = a_{ij}(t, \mu, x)\partial_{ij}^2\varphi(x) + b_i(t, \mu, x)\partial_i\varphi(x), \quad \varphi \in C^2(\mathbb{R}^d). \quad (12)$$

We assume $a = \frac{1}{2}\sigma\sigma^T$ for some Borel map $\sigma : \mathbb{R}_+ \times \mathcal{P} \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$, with coefficients σ_{ij} , and we denote by B an \mathbb{R}^d -standard Brownian motion. For a random variable Z , let \mathcal{L}_Z denote its distribution. In particular, for a process X with paths in Ω_s , \mathcal{L}_X denotes its path law on Ω_s , and \mathcal{L}_{X_t} , $t \geq s$, its one-dimensional time marginals on \mathbb{R}^d .

As in the linear case, three nonlinear problems are associated with L and an initial datum $(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}$: On the level of path measures the distribution-dependent stochastic differential equation (also *McKean–Vlasov equation*)

$$dX_t = b(t, \mathcal{L}_{X_t}, X_t)dt + \sigma(t, \mathcal{L}_{X_t}, X_t)dB_t, \quad t \geq s, \quad \mathcal{L}_{X_s} = \zeta, \quad (\text{DDSDE})$$

(the same equation as in (5)) and the nonlinear martingale problem, which consists of finding $P \in \mathcal{P}(\Omega_s)$ such that

$$P \circ (\pi_s^s)^{-1} = \zeta \text{ and } \varphi(\pi_t^s) - \int_s^t L_{r, \mathcal{L}_{\pi_r^s}} \varphi(\pi_r^s) dr, \quad t \geq s, \text{ is a martingale } \forall \varphi \in C_c^2(\mathbb{R}^d) \quad (\text{MGP})$$

with respect to $(\mathcal{F}_{s,t})_{t \geq s}$. On state space level, one has the nonlinear FPKE

$$\partial_t \mu_t = L_{t, \mu_t}^* \mu_t, \quad t \geq s, \quad \mu_s = \zeta, \quad (\text{FPE})$$

where $L_{t, \mu}^*$ denotes the formal dual operator of $L_{t, \mu}$. (This is the same equation as (4).) For an initial datum $(s, \eta) \in \mathbb{R}_+ \times \mathcal{P}$, from (FPE), one obtains a *linear* FPKE by fixing a weakly continuous curve $[s, \infty) \ni t \mapsto \mu_t \in \mathcal{P}$ in a and b :

$$\partial_t \nu_t = L_{t, \mu_t}^* \nu_t, \quad t \geq s, \quad \nu_s = \eta. \quad (\ell\text{FPE})$$

Analogously, one obtains a (non-distribution-dependent) SDE from (DDSDE):

$$dX_t = b(t, \mu_t, X_t)dt + \sigma(t, \mu_t, X_t)dB_t, \quad t \geq s, \quad \mathcal{L}_{X_s} = \eta, \quad (\ell\text{DDSDE})$$

without any a priori relation between X_t and μ_t , and a linear martingale problem, namely to find $P \in \mathcal{P}(\Omega_s)$ such that

$$P \circ (\pi_s^s)^{-1} = \eta \text{ and } \varphi(\pi_t^s) - \int_s^t L_{r, \mu_r} \varphi(\pi_r^s) dr, \quad t \geq s, \text{ is a martingale } \forall \varphi \in C_c^2(\mathbb{R}^d) \quad (\ell\text{MGP})$$

with respect to $(\mathcal{F}_{s,t})_{t \geq s}$.

Definition 3.1 Let $a_{ij}, b_i, \sigma_{ij} : \mathbb{R}_+ \times \mathcal{P} \times \mathbb{R}^d \rightarrow \mathbb{R}$ be as above, $s \geq 0$ and $\zeta, \eta \in \mathcal{P}$.

(a) **Solutions to (DDSDE), (MGP), (FPE) and linear counterparts.**

- (i) A stochastic process $(X_t)_{t \geq s}$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq s}, \mathbb{P}, B)$ with an \mathbb{R}^d -valued (\mathcal{F}_t) -adapted Brownian motion B is a *(probabilistic weak) solution to (DDSDE)* from (s, ζ) , if it is (\mathcal{F}_t) -adapted, $\mathcal{L}_{X_s} = \zeta$,

$$\mathbb{E} \left[\int_s^T |b(t, \mathcal{L}_{X_t}, X_t)| + |\sigma(t, \mathcal{L}_{X_t}, X_t)|^2 dt \right] < \infty, \quad \forall T > s, \quad (13)$$

and \mathbb{P} -a.s.

$$X_t - X_s = \int_s^t b(r, \mathcal{L}_{X_r}, X_r) dr + \int_s^t \sigma(r, \mathcal{L}_{X_r}, X_r) dB_r, \quad \forall t \geq s. \quad (14)$$

- (ii) A process X as in (i) is a solution to (ℓDDSDE) from (s, η) , if $\mathcal{L}_{X_s} = \eta$ and (13) and (14) hold with μ_t instead of \mathcal{L}_{X_t} , where $[s, \infty) \ni t \mapsto \mu_t \in \mathcal{P}$ is weakly continuous.
- (iii) $P \in \mathcal{P}(\Omega_s)$ is a solution to (MGP) from (s, ζ) , if

$$\mathbb{E}_P \left[\int_s^T |b(t, \mathcal{L}_{\pi_t^s}, \pi_t^s)| + |a(t, \mathcal{L}_{\pi_t^s}, \pi_t^s)| dt \right] < \infty, \quad \forall T > s, \quad (15)$$

and P solves (MGP) .

- (iv) Similarly, P is a solution to (ℓMGP) from (s, η) , if (15) holds, with μ_t replacing $\mathcal{L}_{\pi_t^s}$, and P solves (ℓMGP) .
- (v) A weakly continuous curve $\mu : [s, \infty) \ni t \mapsto \mu_t$ in \mathcal{P} is a solution to (FPE) from (s, ζ) , if

$$\int_s^T \int_{\mathbb{R}^d} |a(t, \mu_t, x)| + |b(t, \mu_t, x)| \mu_t(dx) dt < \infty, \quad \forall T > s,$$

and for all $\varphi \in C_c^2$

$$\int_{\mathbb{R}^d} \varphi(x) \mu_t(dx) - \int_{\mathbb{R}^d} \varphi(x) \zeta(dx) = \int_s^t \int_{\mathbb{R}^d} L_{\tau, \mu_\tau} \varphi(x) \mu_\tau(dx) d\tau, \quad \forall t \geq s.$$

- (vi) For $\mathcal{P}_0 \subseteq \mathcal{P}$, a family $\{\mu^{s, \zeta}\}_{(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}_0}$, $\mu^{s, \zeta} = (\mu_t^{s, \zeta})_{t \geq s}$, of solutions to (FPE) with initial datum $\mu_s^{s, \zeta} = \zeta$ is a *solution flow*, if it satisfies the flow

property, i.e., if $\mu_t^{s,\zeta} \in \mathcal{P}_0$ for all $s \leq t$, $\zeta \in \mathcal{P}_0$, and

$$\mu_t^{s,\zeta} = \mu_t^{r,\mu_r^{s,\zeta}}, \quad \forall 0 \leq s \leq r \leq t, \zeta \in \mathcal{P}_0.$$

- (vii) A weakly continuous curve $v : [s, \infty) \ni t \mapsto v_t$ in \mathcal{P} is a *solution to (ℓFPE)* from (s, η) , if

$$\int_s^T \int_{\mathbb{R}^d} |a(t, \mu_t, x)| + |b(t, \mu_t, x)| v_t(dx) dt < \infty, \quad \forall T > s,$$

and for all $\varphi \in C_c^2(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \varphi(x) v_t(dx) - \int_{\mathbb{R}^d} \varphi \eta(dx) = \int_s^t \int_{\mathbb{R}^d} L_{\tau, \mu_\tau} \varphi(x) v_\tau(dx) d\tau, \quad \forall t \geq s.$$

(b) Uniqueness of solutions.

- (i) Solutions to (DDSDE) from (s, ζ) are weakly unique, if for any two solutions X^1, X^2 , $\mathcal{L}_{X_s^i} = \zeta$, $i \in \{1, 2\}$, implies $\mathcal{L}_{X^1} = \mathcal{L}_{X^2}$.
- (ii) Solutions to (MGP) from (s, ζ) are unique, if for any two solutions P^1, P^2 , $P^i \circ (\pi_s^s)^{-1} = \zeta$, $i \in \{1, 2\}$, implies $P^1 = P^2$ on $\mathcal{B}(\Omega_s)$.
- (iii) Solutions to (FPE) from (s, ζ) are unique, if for any two solutions μ^1, μ^2 , $\mu_s^i = \zeta$, $i \in \{1, 2\}$, implies $\mu_t^1 = \mu_t^2$ for all $t \geq s$.
- (iv) For a fixed weakly continuous curve $[s, \infty) \ni t \mapsto \mu_t \in \mathcal{P}$, uniqueness of solutions to (ℓDDSDE), (ℓMGP) and (ℓFPE) from an initial condition $(s, \eta) \in \mathbb{R}_+ \times \mathcal{P}$ is defined analogously.

For each of these problems, *uniqueness in a subclass of solutions* means any two solutions from a specified subclass with identical initial datum coincide.

Remark 3.2 The following relations between solutions to the above problems are well known.

- (i) The path law P of a weak solution X to (DDSDE), resp. (ℓDDSDE), solves (MGP), resp. (ℓMGP), and vice versa, for a solution P to (MGP), resp. (ℓMGP), there exists a weak solution X to (DDSDE), resp. (ℓDDSDE) such that $P = \mathcal{L}_X$. These equivalences are in one-to-one correspondence, i.e., existence and uniqueness of solutions for the [linear] stochastic equation and the [linear] martingale problem are equivalent, see, for instance, [28, Prop.4.11].
- (ii) By Itô's formula, a solution P to (MGP), resp. (ℓMGP), induces a solution to the nonlinear, resp. linear FPKE via its curve of one-dimensional time marginals. Conversely, by the Ambrosio–Figalli–Trevisan superposition principle, for any solution $(v_t)_{t \geq s}$ to (ℓFPE) there is a solution to (ℓMGP) with marginals $(v_t)_{t \geq s}$, cf. [1, 17, 20, 46]. An analogous version of this result for nonlinear equations is due to Barbu and the second-named author of this paper, see [10, 11]. Hence,

uniqueness for the [linear] martingale problem yields uniqueness for the [linear] FPKE. Conversely, uniqueness of (ℓFPE) for sufficiently many initial data implies uniqueness for (ℓMGP) . To obtain uniqueness of solutions to (MGP) , one needs uniqueness not only for (FPE) , but also for the associated linearized FPKE, see [5, 38].

Remark 3.3 If $\{\mathbb{P}_{s,\zeta}\}_{(s,\zeta)\in\mathbb{R}_+\times\mathcal{P}_0}$ is a nonlinear Markov process, consisting of solution laws to an equation of type (DDSDE) , its one-dimensional time marginal curves $(\mu_t^{s,\zeta})_{t\geq s}$, $\mu_t^{s,\zeta} := \mathbb{P}_{s,\zeta} \circ (\pi_t^s)^{-1}$, solve (FPE) , and $(p_{r,t}^{s,\zeta}(x, dz))_{t\geq r}$ as in (9) solves (ℓFPE) with $\mu_t^{s,\zeta}$ replacing μ_t , and with initial datum (r, δ_x) for $\mu_r^{s,\zeta}$ -a.e. x . This follows from [46, Prop.2.8] and part (ii) of the previous remark. Hence, if solutions to (ℓFPE) with $\eta = \delta_x$ and $\mu_t^{s,\zeta}$ replacing μ_t are unique for every $(s, \zeta, x) \in \mathbb{R}_+ \times \mathcal{P}_0 \times \mathbb{R}^d$, then the kernels from (9) are the unique solutions to these linear equations. Then $p_{r,t}^{s,\zeta}$, $s \leq r \leq t$, are the transition kernels of a linear time-inhomogeneous Markov process $\{P_{r,x}^{s,\zeta}\}_{(r,x)\in[s,\infty)\times\mathbb{R}^d}$. Their relation to the nonlinear Markov process is given by

$$\mathbb{P}_{r,\mu_r^{s,\zeta}} = \int_{\mathbb{R}^d} P_{r,x}^{s,\zeta} d\mu_r^{s,\zeta}(x), \quad \forall 0 \leq s \leq r, \zeta \in \mathcal{P}_0$$

(i.e., the RHS is the convex mixture of the path laws of the corresponding linear Markov process). In this case, (10) shows that the finite-dimensional marginals of $\mathbb{P}_{s,\zeta}$ (and hence its path law) are uniquely determined by the transition kernels of a linear Markov process, which depends, however, on (s, ζ) .

3.2 Main Result

We want to construct nonlinear Markov processes with prescribed one-dimensional time marginals, which are given as solutions to nonlinear FPKEs, see Theorem 3.8, which is the main result of this paper. Before its formulation and proof, we present another main result of this section, Theorem 3.4, which gives a new condition for uniqueness of solutions to (DDSDE) with prescribed one-dimensional time marginals $(\mu_t^{s,\zeta})_{t\geq s\geq 0, \zeta\in\mathcal{P}_0}$ in terms of the extremality of each $(\mu_t^{s,\zeta})_{t\geq s}$ in the set of solutions to (ℓFPE) , with $\mu_t^{s,\zeta}$ replacing μ_t .

To this end, we introduce the following notation. For $(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}$, we denote the space of all solutions to (FPE) from (s, ζ) by $M^{s,\zeta}$ and, for a weakly continuous curve $\eta : [s, \infty) \ni t \mapsto \eta_t \in \mathcal{P}$, the space of all solutions to (ℓFPE) from (s, ζ) , with η_t replacing μ_t , by $M_\eta^{s,\zeta}$. Here, as throughout the paper, solutions to (FPE) , resp. (ℓFPE) , are understood as in Definition 3.1 (v), resp. (vii), i.e., in particular all elements of $M^{s,\zeta}$ and $M_\eta^{s,\zeta}$ are weakly continuous curves of probability measures.

Clearly $M_\eta^{s,\zeta}$ is a convex set for all $(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}$ and η as above, and we recall that μ is an *extremal point* of $M_\eta^{s,\zeta}$, if $\mu \in M_\eta^{s,\zeta}$ and $\mu = \alpha\mu^1 + (1 - \alpha)\mu^2$ for

$\alpha \in (0, 1)$ and $\mu^1, \mu^2 \in M_{\eta}^{s,\zeta}$ implies $\mu^1 = \mu^2$. The set of all extremal points of $M_{\eta}^{s,\zeta}$ is denoted by $M_{\eta,\text{ex}}^{s,\zeta}$.

Theorem 3.4 Let $\mathcal{P}_0 \subseteq \mathcal{P}$ and assume that $\{\mu^{s,\zeta}\}_{(s,\zeta) \in \mathbb{R}_+ \times \mathcal{P}_0}$ is a solution flow to (FPE) such that $\mu^{s,\zeta} \in M_{\mu^{s,\zeta},\text{ex}}^{s,\zeta}$ for each $(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}_0$. Then for every $(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}_0$, there is a unique weak solution $X^{s,\zeta}$ to the corresponding equation (DDSDE) with initial condition (s, ζ) and one-dimensional time marginals equal to $(\mu_t^{s,\zeta})_{t \geq s}$.

Regarding the proof of Theorem 3.4, we first provide a convenient characterization of the extremality assumption in terms of an as mild as possible restricted uniqueness property of the linearized equations (ℓ FPE) associated with (FPE). For a \mathcal{P} -valued curve $\mu = (\mu_t)_{t \geq s}$ and $C > 0$, set

$$\begin{aligned} \mathcal{A}_{s,\leq}(\mu, C) &:= \{(\eta_t)_{t \geq s} \in C([s, \infty), \mathcal{P}) : \eta_t \leq C\mu_t, \forall t \geq s\}, \\ \mathcal{A}_{s,\leq}(\mu) &:= \bigcup_{C>0} \mathcal{A}_{s,\leq}(\mu, C). \end{aligned}$$

In the following lemma, we consider a linear FPKE and denote the space of its solutions from $(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}$ by $M^{s,\zeta}$, which is no abuse of notation, since linear FPKEs are special cases of nonlinear FPKEs of type (FPE) with coefficients independent of their measure variable. Moreover, in this linear case we denote the set of extremal points of $M^{s,\zeta}$ by $M_{\text{ex}}^{s,\zeta}$. For a set A , $\#A$ denotes its cardinality.

Lemma 3.5 Consider a linear FPKE, i.e., (FPE) with coefficients independent of their measure variable. Let $(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}$ and $\mu = (\mu_t)_{t \geq s} \in M^{s,\zeta}$. Then

$$\#(M^{s,\zeta} \cap \mathcal{A}_{s,\leq}(\mu)) = 1 \iff \mu \in M_{\text{ex}}^{s,\zeta}.$$

Proof Fix $(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}$. Clearly, $\mu \in M^{s,\zeta} \cap \mathcal{A}_{s,\leq}(\mu)$. First, suppose $\mu \notin M_{\text{ex}}^{s,\zeta}$, i.e., there are $\mu^i, i \in \{1, 2\}$, in $M^{s,\zeta}$ and $\alpha \in (0, 1)$ such that

$$\mu_t = \alpha \mu_t^1 + (1 - \alpha) \mu_t^2, \quad t \geq s, \quad (16)$$

and $\mu^1 \neq \mu^2$. Then (16) implies $\mu^i \in M^{s,\zeta} \cap \mathcal{A}_{s,\leq}(\mu)$, $i \in \{1, 2\}$, and hence, $\#(M^{s,\zeta} \cap \mathcal{A}_{s,\leq}(\mu)) \geq 2$.

Now assume $\mu \in M_{\text{ex}}^{s,\zeta}$ and let $\nu \in M^{s,\zeta} \cap \mathcal{A}_{s,\leq}(\mu)$. Then for every $t \geq s$ there is $\varrho_t : \mathbb{R}^d \rightarrow \mathbb{R}_+, \mathcal{B}(\mathbb{R}^d)$ -measurable, such that $\nu_t = \varrho_t \mu_t$, and $\varrho_t \leq C$ for all $t \geq s$ for some $C \in (1, \infty)$. Furthermore, for $t \geq s$,

$$\mu_t = \frac{1}{C} \varrho_t \mu_t + (1 - \frac{1}{C} \varrho_t) \mu_t = \frac{1}{C} \nu_t + (1 - \frac{1}{C}) \lambda_t,$$

where $\lambda_t := \frac{1 - \frac{1}{C} \varrho_t}{1 - \frac{1}{C}} \mu_t$. Clearly, $\lambda_t(\mathbb{R}^d) = 1$, and thus, $(\lambda_t)_{t \geq s} \in M^{s,\zeta}$. Since $\mu^{s,\zeta} \in M_{\text{ex}}^{s,\zeta}$, it follows that $\mu_t = \nu_t$. \square

Remark 3.6 By Lemma 3.5, one obtains an equivalent formulation of Theorem 3.4, which states that the restricted uniqueness in the subclasses $\mathcal{A}_{s,\leq}(\mu^{s,\zeta})$ of solutions to (ℓFPE) , with $\mu_t^{s,\zeta}$ replacing μ_t , for all $(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}_0$ implies the restricted uniqueness in the subclasses of solutions with one-dimensional time marginals $(\mu_t^{s,\zeta})_{t \geq s}$ of solutions to (DDSDE) for all $(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}_0$.

As a further preparation for the proofs of Theorem 3.4 and Theorem 3.8, we need Part (ii) of Lemma 3.7. Its Part (i) is not used in this paper, but is of independent interest. We postpone the proof of Lemma 3.7 until the end of this section.

Lemma 3.7 Consider a linear FPKE, i.e., (FPE) with coefficients independent of their measure variable, and let $(s_0, \zeta_0) \in \mathbb{R}_+ \times \mathcal{P}$. Then the following holds.

- (i) If solutions (in the sense of Definition 3.1 (v) with a, b independent of $\mu \in \mathcal{P}$) are unique from (s_0, ζ_0) , then solutions are also unique from any (s_0, η) such that $\eta \in \mathcal{P}$ and $\eta \sim \zeta_0$.
- (ii) If $(v_t^{s_0, \zeta_0})_{t \geq s}$ is the unique solution in $\mathcal{A}_{s_0, \leq}(v^{s_0, \zeta_0})$ from (s_0, ζ_0) , then in this class solutions are also unique from any $(s_0, g \zeta_0)$ with $g \in \mathcal{B}_b^+(\mathbb{R}^d)$, $\int_{\mathbb{R}^d} g(x) \zeta_0(dx) = 1$, and $\delta \leq g$ for some $\delta > 0$.

We proceed to the proof of Theorem 3.4.

Proof of Theorem 3.4 The existence of a weak solution $X^{s,\zeta}$ to (DDSDE) follows from [11, Sect.2]. Concerning uniqueness, note that by the assumption and Lemma 3.5, for each $0 \leq s \leq r$, $\zeta \in \mathcal{P}_0$, (ℓFPE) , with $\mu_t^{s,\zeta} = \mu_t^{r, \mu_r^{s,\zeta}}$ replacing μ_t , has a unique solution from $(r, \mu_r^{s,\zeta})$ in $\mathcal{A}_{r, \leq}(\mu^{r, \mu_r^{s,\zeta}})$.

Claim: Together with Lemma 3.7 (ii), this implies that for any $(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}_0$ and $r \geq s$, even solutions to the linearized martingale problem (ℓMGP) , with $\mu_t^{r, \mu_r^{s,\zeta}}$ replacing μ_t , with one-dimensional time marginals in $\mathcal{A}_{r, \leq}(\mu^{r, \mu_r^{s,\zeta}})$ are unique from $(r, \mu_r^{s,\zeta})$. (By the superposition principle, this is indeed stronger than uniqueness for the linearized FPKEs.)

Proof of Claim: In fact, this uniqueness for (ℓMGP) can be obtained by Lemma 3.7 (ii) and a refinement of the proof of [46, Lem.2.12] as follows: Fix $(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}_0$, $r \geq s$, and let P^1 and P^2 be two solutions to (ℓMGP) , with $\mu_t^{s,\zeta} = \mu_t^{r, \mu_r^{s,\zeta}}$ replacing μ_t , with one-dimensional time marginals in $\mathcal{A}_{r, \leq}(\mu^{r, \mu_r^{s,\zeta}})$ from $(r, \mu_r^{s,\zeta})$. Then the one-dimensional time marginal curves $(P_t^i)_{t \geq r}$, $P_t^i := P^i \circ (\pi_t^r)^{-1}$, solve (ℓFPE) , with $\mu_t^{r, \mu_r^{s,\zeta}} = \mu_t^{s,\zeta}$ replacing μ_t , from $(r, \mu_r^{s,\zeta})$, and hence,

$$P_t^i = \mu_t^{r, \mu_r^{s,\zeta}} = \mu_t^{s,\zeta}, \quad \forall t \geq r, \quad i \in \{1, 2\}. \quad (17)$$

For $n \in \mathbb{N}$, let

$$\mathcal{H}_r^{(n)} := \{\Pi_{i=1}^n h_i(\pi_{t_i}^r) \mid h_i \in \mathcal{B}_b^+(\mathbb{R}^d), h_i \geq c_i \text{ for some } c_i > 0, r \leq t_1 < \dots < t_n\},$$

$$\mathcal{H}_r := \bigcup_{n \in \mathbb{N}} \mathcal{H}_r^{(n)}$$

and note that \mathcal{H}_r is closed under pointwise multiplication and $\sigma(\mathcal{H}_r) = \mathcal{B}(\Omega_r)$. Hence, by induction in $n \in \mathbb{N}$ and a monotone class argument, it suffices to prove

$$\mathbb{E}_{P^1}[H] = \mathbb{E}_{P^2}[H] \text{ for all } H \in \mathcal{H}_r^{(n)} \quad (18)$$

for each $n \in \mathbb{N}$. For $n = 1$, (18) holds by (17). For the induction step from n to $n + 1$, fix $r \leq t_1 < \dots < t_n < t_{n+1}$ and functions $h_i, i \in \{1, \dots, n + 1\}$, as specified in the definition of $\mathcal{H}_r^{(n+1)}$ above, and set

$$\varrho := \frac{\prod_{i=1}^n h_i(\pi_{t_i}^r)}{\mathbb{E}_{P^1}[\prod_{i=1}^n h_i(\pi_{t_i}^r)]},$$

where the denominator is greater or equal to $\prod_{i=1}^n c_i > 0$. Note that

$$\frac{1}{c} \leq \varrho \leq c \text{ pointwise for some } c > 1, \quad (19)$$

and $\mathbb{E}_{P^i}[\varrho] = 1$ for $i \in \{1, 2\}$ by the induction hypothesis. Since for every $f \in \mathcal{B}_b^+(\mathbb{R}^d)$ we have

$$\begin{aligned} & \int_{\Omega_r} f(\pi_{t_n}^r(w)) (\varrho P^i)(dw) \\ &= \left[\int (\prod_{i=1}^n h_i(\pi_{t_i}^r(w))) f(\pi_{t_n}^r(w)) P^i(dw) \right] (\mathbb{E}_{P^1}[\prod_{i=1}^n h_i(\pi_{t_i}^r)])^{-1}, \end{aligned}$$

and since the induction hypothesis implies that these terms are equal for $i \in \{1, 2\}$, it follows that

$$(\varrho P^1) \circ (\pi_{t_n}^r)^{-1} = (\varrho P^2) \circ (\pi_{t_n}^r)^{-1}. \quad (20)$$

By [46, Lem.2.6], the path measures $(\varrho P^i) \circ (\Pi_{t_n}^r)^{-1}$, $i \in \{1, 2\}$, on $\mathcal{B}(\Omega_{t_n})$ solve (ℓ MGP), with $\mu_t^{t_n, \mu_{t_n}^{s, \zeta}}$ replacing μ_t , from time t_n and, by (20), with identical initial condition. Consequently, their curves of one-dimensional time marginals $\eta^i = (\eta_t^i)_{t \geq t_n} := ((\varrho P^i) \circ (\pi_{t_n}^r)^{-1})_{t \geq t_n}$, $i \in \{1, 2\}$, solve (ℓ FPE), with $\mu_t^{t_n, \mu_{t_n}^{s, \zeta}} = \mu_t^{s, \zeta}$ replacing μ_t . For any $A \in \mathcal{B}(\mathbb{R}^d)$ and $t \geq t_n$, we have by (17)

$$\eta_t^i(A) = \int_{\Omega_r} \varrho(w) \mathbb{1}_A(\pi_t^r(w)) P^i(dw) \leq c P_t^i(A) = c \mu_t^{s, \zeta}, \quad i \in \{1, 2\},$$

for c as in (19), and consequently $\eta^i \in \mathcal{A}_{t_n, \leq}(\mu^{t_n, \mu_{t_n}^{s, \zeta}})$. Similarly, $\eta_t^i(A) \geq \frac{1}{c} \mu_t^{s, \zeta}(A)$ for all $A \in \mathcal{B}(\mathbb{R}^d)$ and $t \geq t_n$. In particular, for $t = t_n$, it follows that $\eta_{t_n}^i = g \mu_{t_n}^{s, \zeta}$ for some measurable $g : \mathbb{R}^d \rightarrow \mathbb{R}_+$ such that $\frac{1}{c_0} \leq g \leq c_0$ pointwise for some $c_0 > 1$,

and $\int_{\mathbb{R}^d} g(x) \mu_{t_n}^{s,\zeta} = 1$. By the assumption, Lemma 3.5 and Lemma 3.7 (ii), we obtain $(\eta_t^1)_{t \geq t_n} = (\eta_t^2)_{t \geq t_n}$, so in particular $\eta_{t_{n+1}}^1 = \eta_{t_{n+1}}^2$. Now we conclude

$$\frac{\mathbb{E}_{P^1}[\Pi_{i=1}^{n+1} h_i(\pi_{t_i}^r)]}{\mathbb{E}_{P^1}[\Pi_{i=1}^n h_i(\pi_{t_i}^r)]} = \int_{\Omega_r} \varrho(w) h_{n+1}(\pi_{t_{n+1}}^r(w)) P^i(dw) = \int_{\mathbb{R}^d} h_{n+1}(x) \eta_{t_{n+1}}^i(dx)$$

for $i \in \{1, 2\}$ and conclude

$$\mathbb{E}_{P^1}[\Pi_{i=1}^{n+1} h_i(\pi_{t_i}^r)] = \mathbb{E}_{P^2}[\Pi_{i=1}^{n+1} h_i(\pi_{t_i}^r)],$$

which gives (18) for $n + 1$, and hence completes the proof of the claim.

Consequently, for any such initial datum (s, ζ) , solutions to the corresponding nonlinear martingale problem (MGP) with one-dimensional time marginals $(\mu_t^{s,\zeta})_{t \geq s}$ are unique. Now the uniqueness part of the assertion follows from the equivalence of (MGP) and (DDSDE) mentioned in Remark 3.2 (i). \square

Now we state and prove our main result of this paper on the existence of nonlinear Markov processes with prescribed one-dimensional time marginals, which are given as a solution flow to a nonlinear FPKE of type (FPE).

Theorem 3.8 *Let $\mathcal{P}_0 \subseteq \mathcal{P}$ and assume that $\{\mu^{s,\zeta}\}_{(s,\zeta) \in \mathbb{R}_+ \times \mathcal{P}_0}$ is a solution flow to (FPE) such that $\mu^{s,\zeta} \in M_{\mu^{s,\zeta}, ex}^{s,\zeta}$ for each $(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}_0$. Then the path laws $\mathbb{P}_{s,\zeta} := \mathcal{L}_{X^{s,\zeta}}, (s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}_0$, of the unique weak solution to (DDSDE) with one-dimensional time marginals $(\mu_t^{s,\zeta})_{t \geq s}$ from Theorem 3.4 constitute a nonlinear Markov process.*

Combining Theorem 3.8 and Lemma 3.5, one obtains the following corollary, which provides a convenient formulation to be checked in applications, compare with Sect. 4.

Corollary 3.9 *Let $\mathcal{P}_0 \subseteq \mathcal{P}$ and assume that $\{\mu^{s,\zeta}\}_{(s,\zeta) \in \mathbb{R}_+ \times \mathcal{P}_0}$ is a solution flow to (FPE) such that for every $(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}_0$, $\mu^{s,\zeta}$ is the unique solution to (ℓ FPE), with $\mu_t^{s,\zeta}$ replacing μ_t , in $\mathcal{A}_{s,\leq}(\mu^{s,\zeta})$ from (s, ζ) . Then the path laws $\mathbb{P}_{s,\zeta} := \mathcal{L}_{X^{s,\zeta}}, (s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}_0$, of the unique weak solution to (DDSDE) with one-dimensional time marginals $(\mu_t^{s,\zeta})_{t \geq s}$ from Theorem 3.4 constitute a nonlinear Markov process.*

Proof of Theorem 3.8 By Theorem 3.4, the family $(\mathbb{P}_{s,\zeta})_{(s,\zeta) \in \mathbb{R}_+ \times \mathcal{P}_0}$ is such that for all $(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}_0$, the following holds:

- (i) $\mathbb{P}_{s,\zeta} \in \mathcal{P}(\Omega_s)$ and $\mathbb{P}_{s,\zeta} \circ (\pi_t^s)^{-1} = \mu_t^{s,\zeta}$ for all $t \geq s$,
- (ii) $\mathbb{P}_{s,\zeta}$ solves (MGP), as well as (ℓ MGP), with $\mu_t^{s,\zeta}$ replacing μ_t , from (s, ζ) ,
- (iii) $\mathbb{P}_{s,\zeta}$ is the path law of the unique weak solution to (DDSDE) with one-dimensional time marginals $(\mu_t^{s,\zeta})_{t \geq s}$.

To prove the nonlinear Markov property, let $0 \leq s \leq r \leq t$ and $\zeta \in \mathcal{P}_0$. Disintegrating $\mathbb{P}_{r,\mu_r^{s,\zeta}}$ with respect to $\mu_r^{s,\zeta}$ yields, as already mentioned in Remark 2.2 (i),

$$\mathbb{P}_{r,\mu_r^{s,\zeta}}(\cdot) = \int_{\mathbb{R}^d} p_{(s,\zeta),(r,y)}(\cdot) \mu_r^{s,\zeta}(dy) \quad (21)$$

as measures on $\mathcal{B}(\Omega_r)$, where the $\mu_r^{s,\zeta}$ -almost surely determined family $p_{(s,\zeta),(r,y)}$, $y \in \mathbb{R}^d$, of Borel probability measures on Ω_r is as in Definition 2.1. By [46, Prop.2.8], for $\mu_r^{s,\zeta}$ -a.e. $y \in \mathbb{R}^d$, $p_{(s,\zeta),(r,y)}$ solves (ℓ MGP), with $\mu_t^{r,\mu_r^{s,\zeta}}$ replacing μ_t , from (r, δ_y) . Hence, for any $\varrho \in \mathcal{B}_b^+(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \varrho(x) \mu_r^{s,\zeta}(dx) = 1$, the measure $\mathbb{P}_\varrho \in \mathcal{P}(\Omega_r)$,

$$\mathbb{P}_\varrho := \int_{\mathbb{R}^d} p_{(s,\zeta),(r,y)} \varrho(y) \mu_r^{s,\zeta}(dy), \quad (22)$$

solves the same linearized martingale problem with initial datum $(r, \varrho \mu_r^{s,\zeta})$. Let $n \in \mathbb{N}$, $s \leq t_1 < \dots < t_n \leq r$, $h \in \mathcal{B}_b^+(\mathbb{R}^d)^n$ such that $a^{-1} \leq h \leq a$ for some $a > 1$, and let $\tilde{g} : \mathbb{R}^d \rightarrow \mathbb{R}_+$ be the bounded, $\mu_r^{s,\zeta}$ -a.s. uniquely determined map such that

$$\mathbb{E}_{s,\zeta}[h(\pi_{t_1}^s, \dots, \pi_{t_n}^s) | \sigma(\pi_r^s)] = \tilde{g}(\pi_r^s) \quad \mathbb{P}_{s,\zeta} - a.s.$$

Let $g := c_0 \tilde{g}$, where $c_0 > 0$ is such that $\int_{\mathbb{R}^d} g(x) \mu_r^{s,\zeta}(dx) = 1$, and let \mathbb{P}_g be as in (22), with g replacing ϱ , with initial condition $(r, g \mu_r^{s,\zeta})$. Also, consider $\theta : \Omega_s \rightarrow \mathbb{R}$, $\theta := c_0 h(\pi_{t_1}^s, \dots, \pi_{t_n}^s)$, i.e., $\mathbb{E}_{s,\zeta}[\theta] = 1$. Set

$$\mathbb{P}^\theta := (\theta \mathbb{P}_{s,\zeta}) \circ (\Pi_r^s)^{-1}.$$

Note that $g(\mathbb{R}^d)$, $\theta(\Omega_s) \subseteq [a^{-1}c_0, ac_0]$ $\mu_r^{s,\zeta}$ -a.s., so in particular $g \mu_r^{s,\zeta} \sim \mu_r^{s,\zeta}$. By [46, Prop.2.6], also \mathbb{P}^θ solves (ℓ MGP), with $\mu_t^{s,\zeta}$ replacing μ_t , and its initial datum is $(r, g \mu_r^{s,\zeta})$, since for all $A \in \mathcal{B}(\mathbb{R}^d)$

$$\begin{aligned} \mathbb{P}^\theta \circ (\pi_r^r)^{-1}(A) &= \int_{\Omega_s} \mathbb{1}_A(\pi_r^s(w)) \theta(w) \mathbb{P}_{s,\zeta}(dw) \\ &= \int_{\Omega_s} \mathbb{1}_A(\pi_r^s(w)) g(\pi_r^s(w)) \mathbb{P}_{s,\zeta}(dw) = \int_A g(y) \mu_r^{s,\zeta}(dy). \end{aligned}$$

In particular, both one-dimensional time marginal curves $(\mathbb{P}_g \circ (\pi_t^r)^{-1})_{t \geq r}$ and $(\mathbb{P}^\theta \circ (\pi_t^r)^{-1})_{t \geq r}$ solve (ℓ FPE), with $\mu_t^{s,\zeta}$ replacing μ_t , from $(r, g \mu_r^{s,\zeta})$. Moreover,

$$\mathbb{P}^\theta \circ (\pi_t^r)^{-1}, \mathbb{P}_g \circ (\pi_t^r)^{-1} \leq ac_0 \mu_t^{r,\mu_r^{s,\zeta}}, \quad \forall t \geq r, \quad (23)$$

i.e., both these one-dimensional time marginal curves belong to $\mathcal{A}_{r,\leq}(\mu^{r,\mu_r^{s,\zeta}})$. Indeed, (23) can be seen as follows. For all $t \geq r$ and $A \in \mathcal{B}(\mathbb{R}^d)$,

$$\begin{aligned}\mathbb{P}^\theta \circ (\pi_t^r)^{-1}(A) &= \int_{\Omega_s} \theta(w) \mathbb{1}_A(\pi_t^s(w)) \mathbb{P}_{s,\zeta}(dw) \\ &\leq c_0 a \int_{\Omega_s} \mathbb{1}_A(\pi_t^s(w)) \mathbb{P}_{s,\zeta}(dw) = c_0 a \mu_t^{s,\zeta}(A) = c_0 a \mu_t^{r,\mu_r^{s,\zeta}}(A).\end{aligned}$$

Similarly, by (21),

$$\begin{aligned}\mathbb{P}_g \circ (\pi_t^r)^{-1}(A) &= \int_{\mathbb{R}^d} p_{(s,\zeta),(r,y)}(\pi_t^r \in A) g(y) \mu_r^{s,\zeta}(dy) \\ &\leq a c_0 \mathbb{P}_{r,\mu_r^{s,\zeta}}(\pi_t^r \in A) = a c_0 \mu_t^{r,\mu_r^{s,\zeta}}(A).\end{aligned}$$

Hence, by the assumption, Lemma 3.5 and Lemma 3.7 (ii)

$$\mathbb{P}_g \circ (\pi_t^r)^{-1} = \mathbb{P}^\theta \circ (\pi_t^r)^{-1}, \quad \forall t \geq r,$$

and therefore, for $t \geq r$ and $A \in \mathcal{B}(\mathbb{R}^d)$

$$\begin{aligned}\mathbb{E}_{s,\zeta} [h(\pi_{t_1}^s, \dots, \pi_{t_n}^s) \mathbb{1}_{\pi_t^s \in A}] \\ &= c_0^{-1} \mathbb{P}^\theta \circ (\pi_t^r)^{-1}(A) = c_0^{-1} \mathbb{P}_g \circ (\pi_t^r)^{-1}(A) \\ &= c_0^{-1} \int_{\Omega_s} p_{(s,\zeta),(r,\pi_r^s(\omega))}(\pi_t^r \in A) g(\pi_r^s(\omega)) \mathbb{P}_{s,\zeta}(d\omega) \\ &= \int_{\Omega_s} p_{(s,\zeta),(r,\pi_r^s(\omega))}(\pi_t^r \in A) h(\pi_{t_1}^s(\omega), \dots, \pi_{t_n}^s(\omega)) \mathbb{P}_{s,\zeta}(d\omega).\end{aligned}$$

Here we used the $\sigma(\pi_r^s)$ -measurability of $\Omega_s \ni \omega \mapsto p_{(s,\zeta),(r,\pi_r^s(\omega))}(\pi_t^r \in A)$ for the final equality. By a monotone class argument, (MP) follows. \square

Since the nonlinear Markov property is obviously always fulfilled for $s = r$, the following corollary follows from the proof of Theorem 3.8. We will make use of it in the examples in Sect. 4 to allow singular initial data (e.g., Dirac measures) for nonlinear Markov processes.

Corollary 3.10 *Let $\mathfrak{P}_0 \subseteq \mathcal{P}_0 \subseteq \mathcal{P}$ and let $\{\mu^{s,\zeta}\}_{(s,\zeta) \in \mathbb{R}_+ \times \mathcal{P}_0}$ be a solution flow to (FPE) such that $\mu_t^{s,\zeta} \in \mathfrak{P}_0$ for all $0 \leq s \leq t$, $\zeta \in \mathfrak{P}_0$ (then $\{\mu^{s,\zeta}\}_{(s,\zeta) \in \mathbb{R}_+ \times \mathfrak{P}_0}$ is a solution flow) and $\mu_t^{s,\zeta} \in \mathfrak{P}_0$ for all $0 \leq s < t$, $\zeta \in \mathcal{P}_0$. Also assume $\mu^{s,\zeta} \in M_{\mu^{s,\zeta},ex}^{s,\zeta}$ for all $(s, \zeta) \in \mathbb{R}_+ \times \mathfrak{P}_0$.*

Then there exists a nonlinear Markov process $(\mathbb{P}_{s,\zeta})_{(s,\zeta) \in \mathbb{R}_+ \times \mathcal{P}_0}$ such that $\mathbb{P}_{s,\zeta} \circ (\pi_t^s)^{-1} = \mu_t^{s,\zeta}$ for every $(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}_0$ and $t \geq s$, consisting of path laws of

weak solutions to the corresponding (DDSDE). Moreover, for $\zeta \in \mathfrak{P}_0$, $\mathbb{P}_{s,\zeta}$ is the path law of the unique weak solution to (DDSDE) with one-dimensional time marginals $(\mu_t^{s,\zeta})_{t \geq s}$.

A typical application of Corollary 3.10 is as follows: $\mathfrak{P}_0 = \mathcal{P}_a \cap L^\infty$, $\mathcal{P}_0 = \mathcal{P}$, one has a solution flow $\{\mu^{s,\zeta}\}_{(s,\zeta) \in \mathbb{R}_+ \times \mathcal{P}}$ to (FPE), every solution $(\mu_t)_{t \geq s}$ to (FPE) is absolutely continuous with respect to Lebesgue measure with bounded density for every $t > s$, even if the initial datum $\mu_s = \zeta$ is a singular probability measure (e.g., a Dirac measure) (also called $L^1 - L^\infty$ -regularization), and for every initial measure $\zeta \in \mathcal{P}_a \cap L^\infty$, solutions to (FPE) are unique in $\bigcap_{T>s} L^\infty((s, T) \times \mathbb{R}^d)$. See the second part of (i) in Sect. 4.2 for a concrete example.

Similarly to Corollary 3.9, by Lemma 3.5, one can alternatively formulate Corollary 3.10 in terms of the restricted uniqueness of the corresponding linearized FPKes.

Now we give the postponed proof of Lemma 3.7.

Proof of Lemma 3.7 (i) Assume the claim is wrong, i.e., there is $\nu \sim \zeta_0$, $\nu \in \mathcal{P}$, such that there are two solutions $(\eta_t^i)_{t \geq s_0}$ with $\eta_{s_0}^i = \nu$, $i \in \{1, 2\}$ and $\eta_r^1 \neq \eta_r^2$ for some $r > s_0$. By the superposition principle, there are probability measures P^i on $\mathcal{B}(\Omega_{s_0})$ with one-dimensional time marginal curves $P_t^i = \eta_t^i$, $t \geq s_0$, which solve the corresponding linear martingale problem corresponding to the linear FPKE from the assumption. Disintegrating each P^i with respect to $\pi_{s_0}^{s_0}$ yields two probability kernels $\theta^i : \mathbb{R}^d \times \mathcal{B}(\Omega_{s_0}) \ni (y, D) \mapsto \theta_y^i(D)$ from \mathbb{R}^d to $\mathcal{B}(\Omega_{s_0})$ which are uniquely determined for ν -a.e. (hence also ζ_0 -a.e.) $y \in \mathbb{R}^d$, and for ν -a.e. (hence ζ_0 -a.e.) $y \in \mathbb{R}^d$, θ_y^i solves the linear martingale problem with initial datum (s_0, δ_y) , see [46, Prop.2.8]. Since

$$\int_{\mathbb{R}^d} \theta_y^1 \circ (\pi_r^{s_0})^{-1} \nu(dy) = P_r^1 = \eta_r^1 \neq \eta_r^2 = P_r^2 = \int_{\mathbb{R}^d} \theta_y^2 \circ (\pi_r^{s_0})^{-1} \nu(dy),$$

there is $A \in \mathcal{B}(\mathbb{R}^d)$ such that (w.l.o.g.) the Borel set $E := \{y \in \mathbb{R}^d : \theta_y^1(\pi_r^{s_0} \circ A) > \theta_y^2(\pi_r^{s_0} \circ A)\}$ has strictly positive ν - (hence strictly positive ζ_0 -) measure. For $i \in \{1, 2\}$, set

$$\tilde{\theta}_y^i := \begin{cases} \theta_y^i, & y \in E, \\ \theta_y^1, & y \in E^c. \end{cases}$$

Clearly, $\tilde{\theta}^i$ are again probability kernels from \mathbb{R}^d to $\mathcal{B}(\Omega_r)$, and the probability measures on $\mathcal{B}(\Omega_{s_0})$ given by $\int \tilde{\theta}_y^i \zeta_0(dy)$ both solve the linear martingale problem with initial datum (s_0, ζ_0) . Hence, the curves $(\nu_t^i)_{t \geq s_0}$,

$$\nu_t^i := \int_{\mathbb{R}^d} \tilde{\theta}_y^i \circ (\pi_t^{s_0})^{-1} \zeta_0(dy),$$

solve the linear FPKE from the assertion with initial datum (s_0, ζ_0) , and for A and r as above, we have

$$\begin{aligned} v_r^1(A) &= \int_E \theta_y^1 \circ (\pi_r^{s_0})^{-1}(A) \zeta_0(dy) + \int_{E^c} \theta_y^1 \circ (\pi_r^{s_0})^{-1}(A) \zeta_0(dy) \\ &> \int_E \theta_y^2 \circ (\pi_r^{s_0})^{-1}(A) \zeta_0(dy) + \int_{E^c} \theta_y^1 \circ (\pi_r^{s_0})^{-1}(A) \zeta_0(dy) = v_r^2(A), \end{aligned}$$

i.e., $v^1 \neq v^2$, which contradicts the assumption.

- (ii) We proceed as in (i), but assume in addition that $v = g \zeta_0$ with g as in the assertion and $\eta^i \in \mathcal{A}_{s_0, \leq}(\nu^{s_0, \zeta_0})$ for $i \in \{1, 2\}$. Then, denoting by $|\frac{1}{g}|_\infty$ the L^∞ -norm with respect to either of the equivalent measures ζ_0 and $g \zeta_0$ (note that we even assume $\delta \leq g$ pointwise), we have

$$\begin{aligned} v_r^i &\leq |1/g|_\infty \left(\int_{\mathbb{R}^d} \theta_y^i \circ (\pi_r^{s_0})^{-1} g(y) \zeta_0(dy) + \int_{\mathbb{R}^d} \theta_y^1 \circ (\pi_r^{s_0})^{-1} g(y) \zeta_0(dy) \right) \\ &= |1/g|_\infty (\eta_r^i + \eta_r^1), \end{aligned}$$

and since $\eta^i \in \mathcal{A}_{s, \leq}(\nu^{s_0, \zeta_0})$, also $v^i \in \mathcal{A}_{s_0, \leq}(\nu^{s, \zeta_0})$, which contradicts the assumption in (ii). \square

Remark 3.11 Due to Lemma 3.7 (ii) and Lemma 3.5, the assumption of Corollary 3.10 can slightly be further generalized as follows:

Let $\mathfrak{P}_0 \subseteq \mathcal{P}_0 \subseteq \mathcal{P}$ and let $\{\mu^{s, \zeta}\}_{(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}_0}$ be a solution flow to (FPE) such that $\mu_t^{s, \zeta} \in \mathfrak{P}_0$ for all $0 \leq s \leq t$, $\zeta \in \mathfrak{P}_0$ (then $\{\mu^{s, \zeta}\}_{(s, \zeta) \in \mathbb{R}_+ \times \mathfrak{P}_0}$ is a solution flow) and $\mu_t^{s, \zeta} \in \mathfrak{P}_0$ for all $0 \leq s < t$, $\zeta \in \mathcal{P}_0$. Also assume that for each $(s, \zeta) \in \mathbb{R}_+ \times \mathfrak{P}_0$, there is $\mathfrak{P}_0 \ni \eta = g \zeta$ with g bounded from above and from below by a strictly positive constant, such that solutions to (ℓ FPE), with $\mu^{s, \zeta}$ replacing μ_t , are unique in $\mathcal{A}_{s, \leq}(\mu^{s, \zeta})$ from $(s, g \zeta)$.

3.3 Implications for the Special Case of Linear FPKEs and Linear Markov Processes

Since linear FPKEs are special cases of nonlinear FPKEs, our results from the previous subsection particularly apply to the theory of linear FPKEs and linear Markov processes. It turns out that even in this classical situation, our general nonlinear theorems contain new linear results, which we explicitly state and discuss in this subsection. More precisely, we consider Theorem 3.4 and Theorem 3.8 for the case that (FPE) is linear, i.e., its coefficients are independent of their measure variable.

As before, we use the notation $M^{s, \zeta}$ and $M_{\text{ex}}^{s, \zeta}$ for the set of solutions to this equation from (s, ζ) and the set of extremal points in $M^{s, \zeta}$, respectively. In this case, the associated stochastic equation is an SDE whose coefficients are not distribution-dependent.

Proposition 3.12 (Linear version of Thm. 3.4) *Let $\mathcal{P}_0 \subseteq \mathcal{P}$ and assume that $\{\mu^{s,\zeta}\}_{(s,\zeta) \in \mathbb{R}_+ \times \mathcal{P}_0}$ is a solution flow to a linear FPKE such that $\mu^{s,\zeta} \in M_{\text{ex}}^{s,\zeta}$ for each $(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}_0$. Then for every $(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}_0$, there is a unique weak solution $X^{s,\zeta}$ to the corresponding SDE with initial condition (s, ζ) and one-dimensional time marginals equal to $(\mu_t^{s,\zeta})_{t \geq s}$.*

Similarly, as in the nonlinear case, replacing the assumption $\mu^{s,\zeta} \in M_{\text{ex}}^{s,\zeta}$ for all $(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}_0$ by the assumption that each $\mu^{s,\zeta}$ is the unique element in $M^{s,\zeta} \cap \mathcal{A}_{s,\leq}(\mu^{s,\zeta})$, gives an equivalent formulation of Proposition 3.12.

Remark 3.13 We would like to mention that a general method to construct solution flows to linear FPKEs in non-unique situations consisting of extremal points in $M^{s,\zeta}$, $(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}_0$, has been presented in [36, 37]. Indeed, although it was not pointed out in these papers, the extremality of the members of the constructed flows can easily be read off from the iterative selection presented in the proof of Theorem 3.1. in [37] (which contains the results of [36]). Hence, for any solution flow $\{\mu^{s,\zeta}\}_{(s,\zeta) \in \mathbb{R}_+ \times \mathcal{P}_0}$ selected from $(M^{s,\zeta})_{(s,\zeta) \in \mathbb{R}_+ \times \mathcal{P}_0}$ as in [37], there exists a unique family of weak solutions $(X^{s,\zeta})_{(s,\zeta) \in \mathbb{R}_+ \times \mathcal{P}_0}$ with $\mathcal{L}_{X_t^{s,\zeta}} = \mu_t^{s,\zeta}$ for all $0 \leq s \leq t$ and $\zeta \in \mathcal{P}_0$ to the corresponding SDE.

For the special case of a linear FPKE, the main result of our paper, Theorem 3.8, reads as follows.

Proposition 3.14 (Linear version of Theorem 3.8) *Let $\mathcal{P}_0 = \mathcal{P}$, and assume that $\{\mu^{s,\zeta}\}_{(s,\zeta) \in \mathbb{R}_+ \times \mathcal{P}}$ is a solution flow to a linear FPKE such that $\mu^{s,\zeta} \in M_{\text{ex}}^{s,\zeta}$ for each $(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}$. Then the family of the path laws $\mathbb{P}_{s,\zeta} = \mathcal{L}_{X^{s,\zeta}}$, $(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}$, of the unique weak solutions to the corresponding SDE with $\mathcal{L}_{X_t^{s,\zeta}} = \mu_t^{s,\zeta}$, given by Proposition 3.12, is a nonlinear Markov process. Moreover, if $x \mapsto \mu_t^{s,\delta_x}$ is measurable for all $0 \leq s \leq t$ and the flow also fulfills the Chapman–Kolmogorov equations (CK), then $(\mathbb{P}_{s,\zeta})_{(s,\zeta) \in \mathbb{R}_+ \times \mathcal{P}}$ is a linear Markov process, i.e., one has (LMP) and $\mathbb{P}_{s,\zeta} = \int_{\mathbb{R}^d} \mathbb{P}_{s,\delta_y} \zeta(dy)$ for all $(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}$.*

An equivalent formulation, i.e., the linear special case of Corollary 3.9, is obtained by replacing the assumption $\mu^{s,\zeta} \in M_{\text{ex}}^{s,\zeta}$ for all $(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}$ by the assumption that each $\mu^{s,\zeta}$ is the unique element in $M^{s,\zeta} \cap \mathcal{A}_{s,\leq}(\mu^{s,\zeta})$. As a consequence of Proposition 3.14 and Remark 3.13, the solution flows to linear FPKEs selected in [37] with $\mathcal{P}_0 = \mathcal{P}$ give rise to nonlinear Markov processes, consisting of the path laws $\mathbb{P}_{s,\zeta}$ of the unique weak solutions $X^{s,\zeta}$ to the corresponding SDE, given by Proposition 3.12. In [37, Sect.5], it was proved that at least for bounded and continuous coefficients, the solution flows constructed in [37] with $\mathcal{P}_0 = \mathcal{P}$ are measurable in the sense of Proposition 3.14 and satisfy (CK). Consequently, we have the following corollary. We stress that in general it is still an open question whether every solution flow with $\mathcal{P}_0 = \mathcal{P}$ to a linear FPKE satisfies the Chapman–Kolmogorov equations.

Corollary 3.15 *Let the coefficients of a linear FPKE be continuous in $x \in \mathbb{R}^d$ and bounded. Then, there exists a solution flow $\{\mu^{s,\zeta}\}_{(s,\zeta) \in \mathbb{R}_+ \times \mathcal{P}}$ to this equation such that*

$\mu^{s,\zeta} \in M_{ex}^{s,\zeta}$ for all $(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}$, and the path laws $\mathbb{P}_{s,\zeta} = \mathcal{L}_{X^{s,\zeta}}, (s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}$, of the unique weak solutions to the corresponding SDE with $\mathcal{L}_{X_t^{s,\zeta}} = \mu_t^{s,\zeta}$, given by Proposition 3.12, form a linear Markov process.

Remark 3.16 (i) Following the pioneering work [30], in [44, Ch.12], linear Markovian selections are constructed directly for the linear martingale problem associated with the linear FPKE under the same assumptions on the coefficients as in Corollary 3.15. However, to us it seems that a strength of selecting on the level of the one-dimensional time marginals instead of the path laws is that, as stated in Propositions 3.12 and 3.14 and Corollary 3.15, we obtain in addition that the Markovian path laws are the unique SDE solutions with the prescribed one-dimensional time marginals given by the solution flow for the linear FPKE. To the best of our knowledge, such a uniqueness result for the members of Markovian selections has not been obtained by selecting directly on the level of the path laws.

(ii) The connection between Markovian selections of path laws and their extremality in the convex space of all solutions to the corresponding martingale problem was studied in depth in [45]. In particular, in Theorem 2.10. the authors prove that for time-homogeneous, continuous and bounded coefficients, the Markovian selections from [44, Ch.12] consist of extremal points of the respective sets of solutions to the linear martingale problem.

We conclude this section by pointing out that Corollary 3.15 applies, for instance, to continuity equations (i.e., linear FPKEs with $a = 0$) with bounded and merely continuous (in $x \in \mathbb{R}^d$) vector fields b . It is well known that this includes plenty of interesting ill-posed examples.

4 Applications and Examples

4.1 Well-Posed Nonlinear FPKEs

Theorem 3.8 (more precisely, Corollary 3.9) particularly applies, if for any $(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}_0$ the nonlinear FPKE (FPE) has a unique solution $(\mu_t^{s,\zeta})_{t \geq s}$ from (s, ζ) such that $\mu_t^{s,\zeta} \in \mathcal{P}_0$ for all $t \geq s$, which is also the unique solution to (ℓ FPE), with $\mu^{s,\zeta}$ replacing μ_t , from (s, ζ) . The first part is particularly true when the associated McKean–Vlasov equation (DDSE) has a unique weak solution from (s, ζ) with one-dimensional time marginals in \mathcal{P}_0 . Such a uniqueness result was obtained in [21] for the case of Lipschitz continuous coefficients on $\mathbb{R}^d \times \mathcal{P}_p$, where for $p \geq 1$

$$\mathcal{P}_p := \left\{ \zeta \in \mathcal{P} : \int_{\mathbb{R}^d} |x|^p \zeta(dx) < \infty \right\}$$

is equipped with the p -Wasserstein distance \mathbb{W}_p . The following generalization to one-sided Lipschitz drift b was proved in [49]: For $p \geq 2$, (DDSE) has a unique weak solution $X^{s,\zeta}$ from $(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}_p$ with $\mu_t^{s,\zeta} := \mathcal{L}_{X_t^{s,\zeta}} \in \mathcal{P}_p$, if there is a strictly

positive, increasing continuous function K on \mathbb{R}_+ such that for all $t \geq 0, x, y \in \mathbb{R}^d$ and $\mu, \nu \in \mathcal{P}_p$:

- (A1) $|\sigma(t, \mu, x) - \sigma(t, \nu, y)|^2 \leq K(t)(|x - y|^2 + \mathbb{W}_p(\mu, \nu)^2)$,
 (A2) $2|b(t, \mu, x) - b(t, \nu, y), x - y| \leq K(t)(|x - y|^2 + \mathbb{W}_p(\mu, \nu))$,
 (A3) b is bounded on bounded sets in $\mathbb{R}_+ \times \mathcal{P}_p \times \mathbb{R}^d$, $b(t, \cdot, \cdot)$ is continuous on $\mathcal{P}_p \times \mathbb{R}^d$,
 and

$$|b(t, 0, \mu)|^p \leq K(t) \left(1 + \int_{\mathbb{R}^d} |z|^p \mu(dz) \right).$$

Under these assumptions, $\mu^{s, \zeta}$ is also the unique solution to (LFPE) from (s, ζ) , with $\mu_t^{s, \zeta}$ replacing μ_t . Hence, by Corollary 3.9, the path laws $\mathbb{P}_{s, \zeta} := \mathcal{L}_{X^{s, \zeta}}, (s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}_p$, form a nonlinear Markov process with one-dimensional time marginals $\mu_t^{s, \zeta}$, and the latter uniquely determine $\mathbb{P}_{s, \zeta}, (s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}_p$.

Further well-posedness results for equations of type (DDSDE) were obtained in [24–26, 34, 41]. While in these situations it has to be checked separately whether also the linear equations (LFPE) are well posed in the sense of Corollary 3.9, this is in general much easier than in the nonlinear case and thus often true in the above works. Then the path laws of these unique weak solutions to (DDSDE) form a nonlinear Markov process.

4.2 Restricted Well-Posed Nonlinear FPKEs of Nemytskii Type

We recall that we set $\mathcal{P}_a := \{\mu \in \mathcal{P} | \mu \ll dx\}$, and here, we also use the notation $\mathcal{P}_a^\infty := \{\mu \in \mathcal{P}_a | \frac{d\mu}{dx} \in L^\infty(\mathbb{R}^d)\}$.

Without sufficient regularity in $(x, \mu) \in \mathbb{R}^d \times \mathcal{P}$, e.g., Lipschitz continuity with respect to a Wasserstein distance, or non-degeneracy for the diffusion coefficient, well-posedness for nonlinear (in fact, even for linear) FPKEs usually fails, but one might still be able to prove uniqueness in a subclass of solutions. Here we present such examples, giving rise to nonlinear Markov processes in the case of singular Nemytskii-type coefficients, i.e., for $\mu \in \mathcal{P}_a$ with $\mu = \frac{d\mu}{dx} dx$

$$b(t, \mu, x) := \tilde{b}\left(t, \frac{d\mu}{dx}(x), x\right), \quad a(t, \mu, x) := \tilde{a}\left(t, \frac{d\mu}{dx}(x), x\right) \quad (24)$$

for Borel coefficients \tilde{b}, \tilde{a} on $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^d$ with values in \mathbb{R}^d and $\mathbb{R}^{d \times d}$, respectively. Without further mentioning, we always consider the version of $\frac{d\mu}{dx}$ which is 0 on those $x \in \mathbb{R}^d$ for which $\lim_{r \rightarrow 0} dx(B_r(0))^{-1} \mu(B_r(x))$ does not exist in \mathbb{R} . (Here $B_r(0)$ denotes the Euclidean ball of radius $r > 0$ centered at x .) By Lebesgue's differentiation theorem, the set of such x is a dx -zero set. Since $(\mu, y) \mapsto \frac{d\mu}{dx}(y)$ is $\mathcal{B}(\mathcal{P}_a) \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable by [23, Sect.4.2.], it follows that b and a are product Borel measurable on $\mathbb{R}_+ \times \mathcal{P}_a \times \mathbb{R}^d$ for the weak topology on \mathcal{P}_a .

(i) **Generalized porous media equation.** Let

$$\tilde{b}(t, z, x) := b_0(z)D(x), \quad \tilde{a}(t, z, x) := \frac{\beta(z)}{z} \text{Id} \quad (25)$$

for Borel maps $b_0, \beta : \mathbb{R} \rightarrow \mathbb{R}$ and $D : \mathbb{R}^d \rightarrow \mathbb{R}$, where Id denotes the $d \times d$ -identity matrix. In this case, (FPE) is a generalized porous media equation for the densities $u_t(\cdot) = \frac{d\mu_t}{dx}(\cdot)$, namely

$$\partial_t u = \Delta \beta(u) - \text{div}(Db_0(u)u). \quad (\text{GPME})$$

A (distributional probability) solution to (GPME) from $(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}$ is a map $u : (s, \infty) \rightarrow \mathcal{P}_a$ such that $t \mapsto u_t dx =: \mu_t$ is a weakly continuous solution to (FPE) in the sense of Definition 3.1 (v) with $\lim_{t \rightarrow s} u_t dx = \zeta$ in the weak topology on \mathcal{P} . Suppose that the following assumptions are satisfied.

- (B1) $\beta(0) = 0, \beta \in C^2(\mathbb{R}), \beta' \geq 0$.
- (B2) $b_0 \in C^1(\mathbb{R}) \cap C_b(\mathbb{R}), b \geq 0$.
- (B3) $D \in L^\infty(\mathbb{R}^d; \mathbb{R}^d), \text{div } D \in L^2_{\text{loc}}(\mathbb{R}^d), (\text{div } D)^- \in L^\infty(\mathbb{R}^d)$.
- (B4) $\forall K \subset \mathbb{R}$ compact: $\exists \alpha_K > 0$ with $|b_0(r)r - b_0(s)s| \leq \alpha_K |\beta(r) - \beta(s)|$ $\forall r, s \in K$.

Then by [2, Thm.2.2] for $\zeta \in \mathcal{P}_0 := \mathcal{P}_a^\infty$, there is a (distributional weakly continuous probability) solution $u^{s,\zeta}$ to (GPME) from (s, ζ) such that $u^{s,\zeta} \in \bigcap_{T>s} L^\infty((s, T) \times \mathbb{R}^d)$, $u_t^{s,\zeta} \in \mathcal{P}_0$ for all $t \geq s$, and $\{u^{s,\zeta}\}_{(s,\zeta) \in \mathbb{R}_+ \times \mathcal{P}_0}$ is a solution flow. (We identify $u^{s,\zeta}$ with $(\mu_t^{s,\zeta})_{t \geq s}$, $\mu_t^{s,\zeta} := u^{s,\zeta}(t, x)dx$.) Moreover, by [5, Cor.4.2], the linearized equation (ℓ FPE) associated with (GPME) with initial datum (s, ζ) , with $\mu_t^{s,\zeta}$ replacing μ_t , has $\mu^{s,\zeta}$ as its unique (weakly continuous probability) solution in $\bigcap_{T>s} L^\infty((s, T) \times \mathbb{R}^d)$.

Therefore, Corollary 3.9 applies and gives the existence of a nonlinear Markov process $(\mathbb{P}_{s,\zeta})_{(s,\zeta) \in \mathbb{R}_+ \times \mathcal{P}_a^\infty}$ with one-dimensional time marginals $\mathbb{P}_{s,\zeta} \circ (\pi_t^s)^{-1} = u_t^{s,\zeta} dx$, which is uniquely determined by the solution flow. More precisely, since by [5, Cor.3.4], $u^{s,\zeta}$ is the unique (distributional weakly continuous probability) solution to (GPME) from (s, ζ) in $\bigcap_{T>s} L^\infty((s, T) \times \mathbb{R}^d)$, it follows from Theorem 3.4 and Lemma 3.5 that $\mathbb{P}_{s,\zeta}$ is the path law of the unique weak solution $X^{s,\zeta}$ to the corresponding McKean–Vlasov equation

$$\begin{aligned} dX_t &= b_0(u_t(X_t))D(X_t)dt + \sqrt{\frac{2\beta(u_t(X_t))}{u_t(X_t)}}dB_t, \\ \mathcal{L}_{X_t}(dx) &= u_t dx, \quad t \geq s, \quad \mathcal{L}_{X_s} = \zeta, \end{aligned} \quad (26)$$

under the constraint $u \in \bigcap_{T>s} L^\infty((s, T) \times \mathbb{R}^d)$. We stress that well-posedness for (GPME) or its linearized equations in a larger class than $\bigcap_{T>s} L^\infty((s, T) \times \mathbb{R}^d)$ is not known.

The special case $\gamma_1 \leq \beta' \leq \gamma_2$ for $\gamma_i > 0$ (which already implies (B4)) and $D = -\nabla\Phi$ for

$$\Phi \in C^1(\mathbb{R}^d), \Phi \geq 1, \lim_{|x| \rightarrow \infty} \Phi(x) = \infty, \Phi^{-m} \in L^1(\mathbb{R}^d) \text{ for some } m \geq 2,$$

called *nonlinear distorted Brownian motion (NLDBM)*, was already studied in [38].

Degenerate initial data. We continue this example with cases where the set of admissible initial data \mathcal{P}_0 includes singular measures, e.g., the Dirac measures. For such initial data, (restricted) uniqueness of solutions to (FPE) and (ℓ FPE) is difficult to obtain, in particular in the singular Nemytskii case (although there are positive results for pure diffusion equations, cf. [35]). However, if solutions started from singular measures are more regular after any strictly positive time, one can still obtain the nonlinear Markov property without any uniqueness for singular initial data. Let b, a be as in (24), (25), and assume the following:

(C1) Assumptions (B1)–(B4) hold. In addition assume:

(C2) $\beta'(r) \geq a|r|^{\alpha-1}$ and $|\beta(r)| \leq C|r|^\alpha$ for some $a, C > 0$ and $\alpha \geq 1$.

(C3) $D \in L^\infty \cap L^2(\mathbb{R}^d; \mathbb{R}^d)$, $\operatorname{div} D \in L^2(\mathbb{R}^d)$, $\operatorname{div} D \geq 0$.

Then by [2, Thm.5.2], for each $(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}$, there is a (weakly continuous probability) solution $u^{s, \zeta}$ to (GPME) from (s, ζ) such that $u^{s, \zeta} \in \bigcap_{r>s} L^\infty((r, \infty) \times \mathbb{R}^d)$. If $\zeta \in \mathcal{P}_a^\infty$, even $u^{s, \zeta} \in L^\infty((s, \infty) \times \mathbb{R}^d)$ and in this case, since (B1)–(B4) hold, it follows as in the first part of this example that $u^{s, \zeta}$ is the unique weakly continuous distributional probability solution to (GPME) from (s, ζ) in $\bigcap_{T>s} L^\infty((s, T) \times \mathbb{R}^d)$. Consequently, since for any $s < r$ and $\zeta \in \mathcal{P}$, both $(u_t^{s, \zeta})_{t \geq r}$ and $(u_t^{r, u_r^{s, \zeta} dx})_{t \geq r}$ solve (GPME) from $(r, u_r^{s, \zeta} dx)$ and belong to $\bigcap_{T>r} L^\infty((r, T) \times \mathbb{R}^d)$, it follows that $\{\mu^{s, \zeta}\}_{(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}}$, $\mu_t^{s, \zeta} := u_t^{s, \zeta} dx$, is a solution flow (in contrast to the first part of this example, this does not follow directly from the construction of $\mu^{s, \zeta}$ in [2]). Since for $s < r$ and $\zeta \in \mathcal{P}$ we have $(u_t^{s, \zeta})_{t \geq r} \in L^\infty((r, \infty) \times \mathbb{R}^d)$, it follows as in the first part of this example that the corresponding linearized equation with initial datum $(r, \mu_r^{s, \zeta})$ and with $\mu_t^{s, \zeta}$ replacing μ_t has $\mu^{s, \zeta} = \mu^{r, \mu_r^{s, \zeta}}$ as its unique solution in $L^\infty((r, \infty) \times \mathbb{R}^d)$. Hence, by Lemma 3.5 and Corollary 3.10, with $\mathfrak{P}_0 = \mathcal{P}_a^\infty$ and $\mathcal{P}_0 = \mathcal{P}$, there is a nonlinear Markov process $(\mathbb{P}_{s, \zeta})_{(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}}$ with one-dimensional time marginals $\mathbb{P}_{s, \zeta} \circ (\pi_t^s)^{-1} = \mu_t^{s, \zeta}$, consisting of path laws of weak solutions to (26). For $\zeta \in \mathcal{P}_a^\infty$, $\mathbb{P}_{s, \zeta}$ is the path law of the unique weak solution to (26) with one-dimensional time marginals in $\bigcap_{T>s} L^\infty((s, T) \times \mathbb{R}^d)$ (as in the first part of this example).

(ii) **Burgers' equation.** Consider Burgers' equation in \mathbb{R}^1 , i.e.,

$$\partial_t u = \frac{\partial^2 u}{\partial^2 x} - u \frac{\partial u}{\partial x},$$

which is one of the key examples McKean hinted at in his seminal work [32]. Here we put forward his program by realizing classical solutions to Burgers' equation as one-dimensional time marginals of a nonlinear Markov process. Since we restrict attention to smooth pointwise solutions, we consider the equivalent equation

$$\partial_t u = \frac{\partial^2 u}{\partial^2 x} - \frac{1}{2} \frac{\partial u^2}{\partial x} \quad (\text{BE})$$

instead, which is a nonlinear FPKE of Nemytskii type with coefficients

$$b(t, \mu, x) := \tilde{b}\left(\frac{d\mu}{dx}(x)\right) := \frac{1}{2} \frac{d\mu}{dx}(x) \text{ and } a(t, \mu, x) := 1. \quad (27)$$

For $(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}_a^\infty$, (BE) has a unique smooth pointwise (i.e., strong) solution $u^{s, \zeta}$ on (s, ∞) such that $u_t^{s, \zeta} \rightarrow \zeta$ in $L^1(dx)$ as $t \rightarrow s$, see, for instance, [22]. Furthermore, $0 \leq u_t^{s, \zeta} \leq |\zeta|_\infty$ and

$$\int_{\mathbb{R}} u_t^{s, \zeta}(x) dx = \int_{\mathbb{R}} \zeta(x) dx$$

for all $t \geq s$. Hence, $u^{s, \zeta}$ is in particular a distributional probability solution with initial datum (s, ζ) (as introduced in the first part of (i) above), and its uniqueness in the class of pointwise solutions implies that $\{u^{s, \zeta}\}_{(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}_a^\infty}$ is a solution flow for (BE). Since $u^{s, \zeta} \in L^\infty((s, \infty) \times \mathbb{R}) \cap L^1((s, T) \times \mathbb{R})$ for all $T > s$, the superposition principle [46, Thm.2.5] applies, i.e., there is a probabilistic weak solution $X^{s, \zeta}$ on (s, ∞) to

$$dX_t = \frac{1}{2} u_t(X_t) dt + dB_t, \quad \mathcal{L}_{X_t}(dx) = u_t(x) dx, \quad \mathcal{L}_{X_s} = \zeta, \quad (28)$$

with $u = u^{s, \zeta}$. Since $u^{s, \zeta} \in L^\infty((s, \infty) \times \mathbb{R})$, by [5, Cor.4.2] the linearized equation

$$\partial_t v = \frac{\partial^2 v}{\partial^2 x} - \frac{1}{2} \frac{\partial(u^{s, \zeta} v)}{\partial x}, \quad t \geq s, \quad v_s = \zeta$$

has $v = u^{s, \zeta}$ as its unique weakly continuous probability solution in $\bigcap_{T > s} L^\infty((s, T) \times \mathbb{R})$ for each $(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}_a^\infty$. More precisely, to see this one replaces \tilde{b} from (27) by a bounded C^1 map which coincides with \tilde{b} on $[0, \sup_{t \geq s, x \in \mathbb{R}^d} u_t^{s, \zeta}(x)]$. Consequently, by Corollary 3.9, the path laws $\mathbb{P}_{s, \zeta}$ of the unique solutions $X^{s, \zeta}$, $(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}_a^\infty$, to (28) from (s, ζ) with time marginal densities in $\bigcap_{T > s} L^\infty((s, T) \times \mathbb{R})$ form a uniquely determined nonlinear Markov process.

- (iii) **Distribution-dependent stochastic equations with Lévy noise.** For $\alpha \in (\frac{1}{2}, 1)$ and $\zeta \in \mathcal{P}_a^\infty$, consider the Cauchy problem for the fractional generalized porous

media equation

$$\partial_t u = (-\Delta)^\alpha \beta(u) - \operatorname{div} (Db(u)u), \quad t \geq s, \quad u_s(x)dx = \zeta(dx). \quad (29)$$

Since the operator $(-\Delta)^\alpha$ is non-local, such equations are related to non-local Kolmogorov operators, see [6]. Associated with this non-local equation is the distribution-dependent SDE

$$\begin{aligned} dX_t &= b_0(u_t(X_t))D(X_t)dt + \left(\frac{\beta(u_t(X_{t-}))}{u_t(X_{t-})} \right)^{\frac{1}{2\alpha}} dL_t, \\ \mathcal{L}_{X_t}(dx) &= u_t(x)dx, \quad t \geq s, \end{aligned} \quad (30)$$

where L is a d -dimensional isotropic $2s$ -stable process with Lévy measure $dz/|z|^{d+2\alpha}$. A probabilistic weak solution X to (30) is defined as in Def.(3.1) (i) with L instead of B , see [6] for details. Suppose the following assumptions hold.

- (D1) $\beta \in C^\infty(\mathbb{R})$, $\beta' > 0$, $\beta(0) = 0$.
- (D2) $D \in L^\infty(\mathbb{R}^d; \mathbb{R}^d)$, $\operatorname{div} D \in L^2_{\operatorname{loc}}(\mathbb{R}^d)$, $(\operatorname{div} D)^- \in L^\infty(\mathbb{R}^d)$.
- (D3) $b_0 \in C_b(\mathbb{R}) \cap C^1(\mathbb{R})$, $b \geq 0$.

Then, by [6, Thm.2.4, Thm.3.1], for $(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}_a^\infty$ there is a weakly continuous distributional solution $t \mapsto u_t^{s, \zeta} \in \mathcal{P}_a^\infty$ to (29) from (s, ζ) with the following properties. $\{u^{s, \zeta}\}_{(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}_a^\infty}$ is a solution flow in $\mathcal{P}_0 = \mathcal{P}_a^\infty$, and $u^{s, \zeta}$ is the unique weakly continuous probability solution to (29) in $\bigcap_{T>s} L^\infty((s, T) \times \mathbb{R}^d)$. Moreover, by [6, Thm.3.2], it is also the unique solution to the linearized equation

$$\partial_t v = (-\Delta)^\alpha \left(\frac{\beta(u^{s, \zeta})}{u^{s, \zeta}} v \right) - \operatorname{div} (Db(u^{s, \zeta})v), \quad t \geq s, \quad v(s, \cdot) = \zeta,$$

in $\bigcap_{T>s} L^\infty((s, T) \times \mathbb{R}^d)$. Consequently, by a non-local version of the superposition principle from [40], Corollary 3.9 also applies in this non-local case and implies the existence of a nonlinear Markov process $(\mathbb{P}_{s, \zeta})_{(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}_a^\infty}$, where $\mathbb{P}_{s, \zeta}$ is the path law of a càdlàg weak solution $X^{s, \zeta}$ to (30) with one-dimensional time marginals $u_t^{s, \zeta}(x)dx$. By [6, Thm.4.2], $X^{s, \zeta}$ is unique under the constraint $u \in \bigcap_{T>s} L^\infty((s, T) \times \mathbb{R}^d)$. The nonlinear Markov property of the solutions to (30) was already hinted at in [6, Remarks 4.3, 4.4] and is now finally indeed proved by the above.

- (iv) **Barenblatt solutions to the classical PME.** For the case of the classical porous media equation

$$\partial_t u = \Delta(|u|^{m-1}u), \quad m \geq 1,$$

it was shown in [35] that for any initial datum $(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}$, there is a unique weakly continuous distributional probability solution $u^{s, \zeta}$ in $\bigcap_{\tau>s, T>\tau} L^\infty((\tau, T) \times \mathbb{R}^d)$.

\mathbb{R}^d). In fact, it is shown that $u^{s,\zeta}$ is even L^1 continuous on (s, ∞) . Clearly, the uniqueness implies the flow property of the curves $t \mapsto u_t^{s,\zeta}(x)dx$. For $\zeta = \delta_{x_0}$, $u^{s,\zeta}$ is the *Barenblatt solution* (see [48]), given by

$$u_t^{s,\delta_{x_0}}(x) = (t-s)^{-\alpha} \left[(C - k|x-x_0|^2(t-s)^{-2\beta})^+ \right]^{\frac{1}{m-1}}, \quad t > s,$$

where $\alpha = \frac{d}{d(m-1)+2}$, $\beta = \frac{\alpha}{d}$, $k = \frac{\alpha(m-1)}{2md}$, $f^+ := f \vee 0$, and $C = C(m, d) > 0$ is chosen such that $\int_{\mathbb{R}^d} u_t^{s,\zeta}(x)dx = 1$ for all $t > s$. The corresponding McKean–Vlasov equation is

$$dX_t = \sqrt{2u_t(X_t)^{m-1}}dB_t, \quad \mathcal{L}_{X_t}(dx) = u_t(x)dx, \quad t \geq s, \quad \mathcal{L}_{X_s} = \zeta. \quad (31)$$

By Lemma 3.5, the results of the second part of (i) above and Corollary 3.10 with $\mathfrak{P}_0 = \mathcal{P}_a^\infty$ and $\mathcal{P}_0 = \mathcal{P}$, there is a nonlinear Markov process $(\mathbb{P}_{s,\zeta})_{(s,\zeta) \in \mathbb{R}_+ \times \mathcal{P}}$ consisting of path laws $\mathbb{P}_{s,\zeta}$ of weak solutions to (31) with one-dimensional time marginals given by $u_t^{s,\zeta}(x)dx$. For $\zeta \in \mathcal{P}_a^\infty$, $\mathbb{P}_{s,\zeta}$ is the path law of the unique weak solution to (31) with one-dimensional time marginals in $\bigcap_{\tau > s, T > \tau} L^\infty((\tau, T) \times \mathbb{R}^d)$. This way, the Barenblatt solutions have a probabilistic interpretation as time marginal densities of a nonlinear Markov process. In the special case $d = 1$, the identification of the Barenblatt solutions as one-dimensional time marginals of a stochastic process solving (31) was already obtained in [15], but no connection to any kind of nonlinear Markov property was drawn.

- (v) **2D vorticity Navier–Stokes equations.** Very recently, in [13], it was proved that Theorem 3.8 applies to the 2D vorticity Navier–Stokes equations, i.e., to

$$\partial_t u = \nu \Delta u - \operatorname{div}(vu), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^2, \quad (32)$$

which is of type (FPE), with $\mathcal{P}_0 = \mathcal{P}_a \cap L^4(\mathbb{R}^2)$. Here $\nu > 0$ and $u = \operatorname{curl} v$. For details, including the McKean–Vlasov equation associated with (32), we refer to [13].

- (vi) **Parabolic p -Laplace equation.** In the very recent paper [47], the p -Laplace equation

$$\partial_t u = \operatorname{div}(|\nabla u|^{p-2} \nabla u), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d, \quad (33)$$

was reinterpreted as a nonlinear FPKE of type (2) and it was proved there that our theory, more precisely Corollary 3.10, applies to the explicit fundamental solution (also called *Barenblatt solution*) of (33). This way, a uniquely determined nonlinear Markov process, consisting of solution path laws to the associated DDSDE and with one-dimensional time marginal densities given by this fundamental solution, was constructed. Due to its analogy to the case $p = 2$, which is the classical heat equation and Brownian motion, in [47] this nonlinear Markov process was called *p-Brownian motion*.

Remark 4.1 With regard to Remark 2.4, (i) we stress that in all the above examples, even in those where $\mathcal{P}_0 = \mathcal{P}$, it is not true that $p_{(s,\zeta),(r,y)} = \mathbb{P}_{r,\delta_y}$, since for the nonlinear equation (FPE) solutions are not stable under convex combinations in the initial datum. Hence, unlike for classical linear cases, for the nonlinear Markov processes from the previous examples one cannot expect to calculate the path laws $\mathbb{P}_{s,\zeta}$ via the one-dimensional time marginals $\mathbb{P}_{r,y} \circ (\pi_r^s)^{-1}$, $s \leq r$, $y \in \mathbb{R}^d$. Instead, one has the formula given in Proposition 2.3.

4.3 Ill-Posed Nonlinear FPKEs

Finally we point out again that the results from Sect. 3.2 apply when no uniqueness for the nonlinear FPKE is known at all, but one can select a solution flow $\{\mu^{s,\zeta}\}_{(s,\zeta) \in \mathbb{R}_+ \times \mathcal{P}_0}$ for (FPE), and if, additionally, one can establish the required restricted uniqueness for the linearized equations (ℓ FPE), with $\mu_t^{s,\zeta}$ replacing μ_t , as formulated in Corollary 3.9. Solution flows for ill-posed nonlinear FPKEs have been constructed in [37]. Since, in principle, (restricted) uniqueness for nonlinear equations is much harder to obtain than the mild restricted uniqueness condition for (ℓ FPE), it is often possible to construct nonlinear Markov processes in such nonlinear completely ill-posed cases, if one can select a solution flow.

Let us give the following particular example in dimension $d = 1$. In [42], it is shown that (DDSDE) with coefficients

$$\sigma = 0, \quad b(t, \mu, x) = \int_{\mathbb{R}^d} h(y) \mu(dy)$$

has more than one solution for a specifically constructed function $h \in C_c(\mathbb{R})$ and a suitable initial datum $\zeta \in \mathcal{P}$. Consequently, since the solutions presented in [42] have distinct one-dimensional time marginals, the corresponding nonlinear FPKE (FPE) with initial datum ζ is ill-posed as well. On the other hand, clearly Assumption B2 from [37] holds for σ and b as above, and, therefore, by [37, Prop.4.11, Rem.4.12] there is a solution flow $\{\mu^{s,\zeta}\}_{(s,\zeta) \in \mathbb{R}_+ \times \mathcal{P}}$ for (FPE) with $\mathcal{P}_0 = \mathcal{P}$. Moreover, for each $(s, \zeta) \in \mathbb{R}_+ \times \mathcal{P}$, the corresponding linearized equation (ℓ FPE), with $\mu_t^{s,\zeta}$ in place of μ_t , which is a continuity equation with bounded vector field only depending continuously on $t \geq s$, has $\mu^{s,\zeta}$ as its unique solution (in the sense of Definition 3.1). Consequently, by Corollary 3.9, there exists a nonlinear Markov process $(\mathbb{P}_{s,\zeta})_{(s,\zeta) \in \mathbb{R}_+ \times \mathcal{P}}$ such that the one-dimensional time marginals of $\mathbb{P}_{s,\zeta}$ are given by $(\mu_t^{s,\zeta})_{t \geq s}$, and each $\mathbb{P}_{s,\zeta}$ is the path law of the unique weak solution of the corresponding DDSDE with these one-dimensional time marginals. It is straightforward to construct further examples of this prototype, e.g., with nonzero diffusion term and space-dependent drift b .

Acknowledgements This study was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation)—Project-ID 317210226—SFB 1283.

Author Contributions Both authors contributed equally to each part of the paper.

Funding Open Access funding enabled and organized by Projekt DEAL.

Data Availability No datasets were generated or analyzed during the current study.

Declarations

Conflicts of interest The authors have no conflict of interest to declare that are relevant to the content of this article.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

1. Ambrosio, L.: Transport Equation and Cauchy Problem for Non-smooth Vector Fields, pp. 1–41. Springer, Berlin (2008)
2. Barbu, V., Röckner, M.: Solutions for nonlinear Fokker-Planck equations with measures as initial data and McKean-Vlasov equations. *J. Funct. Anal.* **280**(7), 108926 (2021)
3. Barbu, V., Röckner, M.: The evolution to equilibrium of solutions to nonlinear Fokker-Planck equation. *Indiana Univ. Math. J.* **72**(1), 89–131 (2023)
4. Barbu, V., Röckner, M.: Nonlinear Fokker-Planck equations with time-dependent coefficients. *SIAM J. Math. Anal.* **55**(1), 1–18 (2023)
5. Barbu, V., Röckner, M.: Uniqueness for nonlinear Fokker-Planck equations and for McKean-Vlasov SDEs: the degenerate case. *J. Funct. Anal.* **285**(4), 37 (2023)
6. Barbu, V., Röckner, M.: Nonlinear Fokker-Planck equations with fractional Laplacian and McKean-Vlasov SDEs with Lévy noise. *Probab. Theory Relat. Fields* **189**(3–4), 849–878 (2024)
7. Barbu, V., Röckner, M.: Nonlinear Fokker-Planck flows and their probabilistic counterparts. *Lecture Notes in Mathematics*, vol. 2353. Springer, Cham (2024)
8. Barbu, V., Röckner, M., Russo, F.: Probabilistic representation for solutions of an irregular porous media type equation: the degenerate case. *Probab. Theory Relat. Fields* **151**(1), 1–43 (2011)
9. Barbu, V., Röckner, M., Russo, F.: Doubly probabilistic representation for the stochastic porous media type equation. *Ann. de l'Institut Henri Poincaré Probab. et Stat.* **53**(4), 2043–2073 (2017)
10. Barbu, V., Röckner, M.: Probabilistic representation for solutions to nonlinear Fokker-Planck equations. *SIAM J. Math. Anal.* **50**(4), 4246–4260 (2018)
11. Barbu, V., Röckner, M.: From nonlinear Fokker-Planck equations to solutions of distribution dependent SDE. *Ann. Probab.* **48**(4), 1902–1920 (2020)
12. Barbu, V., Röckner, M.: The invariance principle for nonlinear Fokker-Planck equations. *J. Differ. Equ.* **315**, 200–221 (2022)
13. Barbu, V., Röckner, M., Zhang, D.: Uniqueness of distributional solutions to the 2D vorticity Navier–Stokes equation and its associated nonlinear Markov process. [arXiv:2309.13910](https://arxiv.org/abs/2309.13910), (2023)
14. Belaribi, N., Russo, F.: Uniqueness for Fokker-Planck equations with measurable coefficients and applications to the fast diffusion equation. *Electron. J. Probab.* **17**, 1–28 (2012)
15. Benachour, S., Chassaing, P., Roynette, B., Vallois, P.: Processus associés à l'équation des milieux poreux. *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze*, 4e série **23**(4), 793–832 (1996)
16. Blanchard, P., Röckner, M., Russo, F.: Probabilistic representation for solutions of an irregular porous media type equation. *Ann. Probab.* **38**(5), 1870–1900 (2010)
17. Bogachev, V.I., Röckner, M., Shaposhnikov, S.V.: On the Ambrosio-Figalli-Trevisan superposition principle for probability solutions to Fokker-Planck-Kolmogorov equations. *J. Dyn. Diff. Equat.* **33**(2), 715–739 (2021)

18. Ciotir, I., Russo, F.: Probabilistic representation for solutions of a porous media type equation with Neumann boundary condition: The case of the half-line. *Differ. Integral Equ.* **27**(1/2), 181–200 (2014)
19. Dynkin, E. B.: Markov processes. Vols. I, II, volume Band 121, 122 of Die Grundlehren der mathematischen Wissenschaften. Springer-Verlag, Berlin-Göttingen-Heidelberg; Academic Press, Inc., Publishers, New York, (1965)
20. Figalli, A.: Existence and uniqueness of martingale solutions for SDEs with rough or degenerate coefficients. *J. Funct. Anal.* **254**(1), 109–153 (2008)
21. Funaki, T.: A certain class of diffusion processes associated with nonlinear parabolic equations. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete* **67**(3), 331–348 (1984)
22. Godlewski, E., Raviart, P. A.: Hyperbolic Systems of Conservation Laws. Ellipses, (1991)
23. Grube, S.: Strong solutions to McKean-Vlasov Stochastic Differential Equations with Coefficients of Nemytskii-type. PhD thesis. Bielefeld University (2022)
24. Hammersley, W., Šiška, D., Szpruch, Ł.: McKean-Vlasov SDEs under measure dependent Lyapunov conditions. *Ann. Inst. Henri Poincaré Probab. Stat.* **57**(2), 1032–1057 (2021)
25. Huang, X., Wang, F.-Y.: Well-posedness for singular McKean-Vlasov stochastic differential equations. Available under <https://www.sfb1283.uni-bielefeld.de/preprints/sfb20110.pdf> (2020)
26. Huang, X., Wang, F.-Y.: Singular McKean-Vlasov (reflecting) SDEs with distribution dependent noise. *J. Math. Anal. Appl.* **514**(1), 126301 (2022)
27. Izydorczyk, L., Oudjane, N., Russo, F., Tessitore, G.: Fokker-Planck equations with terminal condition and related McKean probabilistic representation. *Nonlinear Differ. Equ. Appl.* **29**, 1–41 (2020)
28. Karatzas, I., Shreve, S.: Brownian motion and stochastic calculus, volume 113 of Graduate Texts in Mathematics. Springer-Verlag, New York, second edition (1991)
29. Kolokoltsov, V.N.: Nonlinear Markov Processes and Kinetic Equations. Cambridge Tracts in Mathematics. Cambridge University Press (2010)
30. Krylov, N.V.: On the selection of a Markov process from a system of processes and the construction of quasi-diffusion processes. *Math. USSR-Izvestiya* **7**(3), 691–709 (1973)
31. Le Cavit, A., Oudjane, N., Russo, F.: Probabilistic representation of a class of non-conservative nonlinear partial differential equations. *ALEA Lat. Am. J. Probab. Math. Stat.* **13**(2), 1189–1233 (2016)
32. McKean, H.P.: A class of Markov processes associated with nonlinear parabolic equations. *Proc. Natl. Acad. Sci. USA* **56**(6), 1907–1911 (1966)
33. McKean, H.P.: Propagation of chaos for a class of non-linear parabolic equations. 1967 stochastic differential equations. (Lecture Series in Differential Equations, Session 7, Catholic Univ., 1967) pp. 41–57 Air Force Office Sci. Res., Arlington, Va (1967)
34. Mishura, Y.S., Veretennikov, A.Y.: Existence and uniqueness theorems for solutions of McKean–Vlasov stochastic equations. *Theory of Probability and Mathematical Statistics* (2020)
35. Pierre, M.: Uniqueness of the solutions of $u_t - \Delta \phi(u) = 0$ with initial datum a measure. *Nonlinear Anal. Theory Methods Appl.* **6**(2), 175–187 (1982)
36. Rehmeier, M.: Existence of flows for linear Fokker-Planck-Kolmogorov equations and its connection to well-posedness. *J. Evol. Equ.* **21**(1), 17–31 (2021)
37. Rehmeier, M.: Flow selections for (nonlinear) Fokker-Planck-Kolmogorov equations. *J. Differ. Equ.* **328**, 105–132 (2022)
38. Ren, P., Rckner, M., Wang, F.-Y.: Linearization of nonlinear Fokker-Planck equations and applications. *J. Differ. Equ.* **322**, 1–37 (2022)
39. Ren, P., Wang, F.-Y.: Distribution Dependent Stochastic Differential Equations. WORLD SCIENTIFIC (2024)
40. Rckner, M., Xie, L., Zhang, X.: Superposition principle for non-local Fokker-Planck-Kolmogorov operators. *Probab. Theory Relat. Fields* **178**(3), 699–733 (2020)
41. Rckner, M., Zhang, X.: Well-posedness of distribution dependent SDEs with singular drifts. *Bernoulli* **27**(2), 1131–1158 (2021)
42. Scheutzow, M.: Uniqueness and non-uniqueness of solutions of Vlasov-McKean equations. *J. Aust. Math. Soc. Ser. Pure Math. Stat.* **43**(2), 246256 (1987)
43. Sharpe, M.: General Theory of Markov Processes. Academic Press (1988)
44. Stroock, D.W., Varadhan, S.: Multidimensional Diffusion Processes. Classics in Mathematics. Springer, Berlin, Heidelberg (2007)
45. Stroock, D.W., Yor, M.: On extremal solutions of martingale problems. *Ann. Sci. de l'École Normale Supérieure, Ser. 4* **13**(1), 95–164 (1980)

46. Trevisan, D.: Well-posedness of multidimensional diffusion processes with weakly differentiable coefficients. *Electron. J. Probab.* **21**, 1–41 (2016)
47. Rckner, M., Barbu, V., Rehmeier, M.: p -brownian motion and the p -laplacian. [arXiv:2409.18744](https://arxiv.org/abs/2409.18744) (2024)
48. Vázquez, J.L.: The porous medium equation. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, Oxford (2007)
49. Wang, F.-Y.: Distribution dependent SDEs for Landau type equations. *Stoch. Process. Appl.* **128**(2), 595–621 (2018)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.