OFF-DIAGONAL LOWER ESTIMATES AND HÖLDER REGULARITY OF THE HEAT KERNEL

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Abstract. We study the heat kernel of a regular symmetric Dirichlet form on a metric space with doubling measure, in particular, a connection between the properties of the jump measure and the long time behaviour of the heat kernel. Under appropriate optimal hypotheses, we obtain the Hölder regularity and lower estimates of the heat kernel.

Dedicated to the memory of Professor Ka-Sing Lau.

Contents

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1. Introduction

In this paper, we are concerned with the heat kernel lower estimates for regular symmetric Dirichlet forms on metric spaces with doubling measures.

Let (M, d) be a locally compact separable metric space and let μ be a Radon measure on M with full support. A triple (M, d, μ) is called a *metric measure space*. Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on $L^2 := L^2(M, \mu)$. Let $\{P_t\}_{t>0}$ be the heat semigroup in L^2 associated with $(\mathcal{E}, \mathcal{F})$, that is, $P_t = e^{t\mathcal{L}}$, $t > 0$, where $\mathcal{L}$ is the generator of $(\mathcal{E}, \mathcal{F})$.

Note that P_t is a bounded self-adjoint operator in L^2. If, for any $t > 0$, the operator P_t has an integral kernel then the latter will be denoted by $p_t(x, y)$ and will be referred to as the *heat kernel* of $(\mathcal{E}, \mathcal{F})$. The heat kernel coincides with the transition density of the Hunt process associated with $(\mathcal{E}, \mathcal{F})$.

For the sake of Introduction, assume that $(\mathcal{E}, \mathcal{F})$ is of jump type and that it is determined by a jump kernel $J(x, y)$, that is,

$$\mathcal{E}(u, v) = \iint_{M\times M} (u(x) - u(y))(v(x) - v(y))J(x, y)d\mu(x)d\mu(y). \tag{1.1}$$

Our main goal is investigation of the influence of the jump kernel on the heat kernel long time behaviour.

For example, consider in $\mathbb{R}^n$ the following jump kernel

$$J(x, y) = \frac{1}{|x - y|^{n+\beta}}, \tag{1.2}$$

where $0 < \beta < 2$. In this case $\mathcal{E}(u, v)$ is a regular Dirichlet form with the generator const $(-\Delta)^{\beta/2}$, and its heat kernel admits the following two-sided estimate:

$$p_t(x, y) \simeq \frac{1}{t^{n/\beta}} \left(1 + \frac{|x - y|}{t^{1/\beta}}\right)^{-(n+\beta)} \simeq \frac{1}{t^{n/\beta}} \wedge \frac{t}{|x - y|^{n+\beta}}, \tag{1.3}$$

for all $t > 0$ and $x, y \in \mathbb{R}^n$. Here $\wedge$ means *minimum* and $\simeq$ means *comparable*, that is, the ratio of the both sides is bounded from above and below by positive constants (in this case, for all $t > 0$ and $x, y \in \mathbb{R}^n$).

For simplicity of presentation, we assume throughout Introduction that the metric measure space (M, d, μ) is α-regular for some $\alpha > 0$, that is, for any metric ball $B(x, r)$ in M,

$$\mu(B(x, r)) \simeq r^\alpha, \tag{1.4}$$

although the main results of this paper are stated and proved under a weaker hypothesis of the volume doubling.

Assume further that the jump kernel satisfies for some $\beta > 0$ the estimate

$$J(x, y) \simeq \frac{1}{d(x, y)^{\alpha+\beta}}, \tag{J}$$

for all $x, y \in M$. A natural question arises whether the heat kernel of $(\mathcal{E}, \mathcal{F})$ exists and satisfies an estimate similar to (1.3), that is, whether the following estimate holds:

$$p_t(x,y) \simeq \frac{C}{t^{\alpha/\beta}}\left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)} \simeq \frac{1}{t^{\alpha/\beta}} \wedge \frac{t}{d(x,y)^{\alpha+\beta}}, \tag{1.5}$$

for all $t > 0$ and $x, y \in M$. If $\beta < 2$ then the answer is affirmative; moreover, by a result of [4], the following equivalence holds:

$$(J) \Leftrightarrow (1.5).$$

However, if $\beta \geq 2$ then one more hypothesis is needed: a so called *generalized capacity condition* that will be denoted by (Gcap). This condition ensures the existence of cutoff functions with controlled energy and will be rigorously formulated in the next section. It was proved independently in [6] and [10] that, for any $\beta > 0$,

$$(J) + (\mathrm{Gcap}) \Leftrightarrow (1.5). \tag{1.6}$$

It is natural to ask then what kind of heat kernel bounds can be ensured if the jump kernel satisfies instead of (J) some weaker hypotheses. This problem has been addressed in a series of papers of the authors [12], [13], [14] which is concluded with the present work.

We replace (J) by some integral estimates of the jump kernel as follows. The pointwise upper bound of J is replaced by the hypothesis about the *tail of the jump kernel*:

$$\int_{B(x,r)^c} J(x,y)d\mu(y) \leq \frac{C}{r^\beta}, \tag{TJ}$$

for all $x \in M$ and $r > 0$, while the pointwise lower bound of J is replaced by an appropriate *Poincaré inequality* that is denoted by (PI). A detailed definition of the latter will be given in the next section. It is easy to verify that (J) implies (TJ) and (PI) but not vice versa.

The first main result of the present paper – Theorem 2.9, says that the hypotheses (TJ), (PI) and (Gcap) imply the following *near-diagonal lower estimate* of the heat kernel

$$p_t(x,y) \geq ct^{-\alpha/\beta} \ \text{ if } d(x,y) \leq \delta t^{1/\beta}, \tag{NLE}$$

for some $c, \delta > 0$. Moreover, under the standing hypothesis (TJ), we have the equivalence

$$(\mathrm{PI}) + (\mathrm{Gcap}) \Leftrightarrow (\mathrm{LLE}), \tag{1.7}$$

where (LLE) denotes a similar near-diagonal lower bound of the *Dirichlet* heat kernels in balls, which is a somewhat stronger condition than (NLE) (see the next section for a detailed definition).

Note that, under the hypotheses of Theorem 2.9, one cannot ensure an off-diagonal lower estimate of the form

$$p_t(x,y) \geq \frac{c}{t^{\alpha/\beta}}\left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-N},$$

for all $t > 0$ and $x, y \in M$, whatever $N > 0$ is, as it was shown by a counterexample in [1].

However, if we replace in (1.7) the Poincaré inequality (PI) by a stronger hypothesis – the pointwise lower estimate of the jump kernel

$$J(x,y) \geq \frac{c}{d(x,y)^{\alpha+\beta}}, \tag{$J_\geq$}$$

then we do obtain a full off-diagonal lower estimate

$$p_t(x,y) \geq \frac{c}{t^{\alpha/\beta}}\left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}. \tag{LE}$$

This follows from our second main result in this paper – Theorem 2.12, that says the following: under the standing hypothesis (TJ), the following equivalence holds:

$$(J_\geq) + (\mathrm{Gcap}) \Leftrightarrow (\mathrm{LLE}) + (\mathrm{LE})$$

(see also Corollary 2.13).

As far as *upper* bounds of the heat kernel are concerned, this problem under weaker assumptions on J has been addressed in our companion paper [14]. For any $q \geq 1$, let us introduce the following hypothesis about the *L^q-tail of the jump kernel*:

$$\left(\int_{B(x,r)^c} J(x,y)^q d\mu(y)\right)^{1/q} \leq \frac{C}{r^{\alpha/q'+\beta}}, \tag{TJ_q}$$

for all $x \in M$ and $r > 0$, where q is the Hölder conjugate to q. For example, if $q = 1$ then (TJ_q) coincides with (TJ). It is easy to see that (TJ_q) gets stronger when q increases.

By a result of [14], if $q \geq 2$ then (TJ_q), (Gcap), and a certain Faber-Krahn inequality imply the following upper bound of the heat kernel:

$$p_t(x,y) \leq \frac{C}{t^{\alpha/\beta}}\left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha/q'+\beta)} \simeq C\left(\frac{1}{t^{\alpha/(\beta q')}} \wedge \frac{t}{d(x,y)^{\alpha/q'+\beta}}\right)\frac{1}{t^{\alpha/(\beta q)}}\ , \tag{UE_q}$$

for all $t > 0$ and $x, y \in M$. Combining the results of the present paper with those of [14] yields the following implication for any $2 \leq q \leq \infty$:

$$(\mathrm{J}_{\geq}) + (\mathrm{Gcap}) + (\mathrm{TJ}_q) \Rightarrow (\mathrm{UE}_q) + (\mathrm{LE})$$

(cf. Theorem 2.19 and Corollary 2.20).

Note that there is a mismatch in the exponents of the off-diagonal terms in (LE) and (UE_q) that are $\alpha + \beta$ and $\alpha/q' + \beta$, respectively. The gap between these exponents is in general unavoidable as an example in [1] shows. However, if $q = \infty$ then $q' = 1$, and the two exponents coincide. In this case we recover the equivalence (1.6).

In the main body of the paper our results are stated and proved in a more general form as follows.

(1) Instead of volume regularity, we assume the volume doubling condition, so that the volume function $V(x,r) = \mu(B(x,r))$ explicitly enters the heat kernel estimates.
(2) Instead of the scaling function r^β that appears in (TJ) as well as in (Gcap) and (PI), we use a more general scaling function $W(x,r)$ depending also on the space variable $x \in M$, which causes additional difficulties in the proof.
(3) The Dirichlet form $(\mathcal{E},\mathcal{F})$ may contain a local part, that is, $(\mathcal{E},\mathcal{F})$ may be an arbitrary regular Dirichlet form without killing part.
(4) The hypotheses (Gcap) and (PI) are assumed in a localized form, that is, for a bounded range of radii of balls involved, which, in particular, allows to include bounded metric spaces. In this case, the heat kernel estimates are valid for a bounded range of time.
(5) Together with heat kernel lower estimates, we obtain also the Hölder regularity of the heat kernel.

In Section 2 we give all necessary definitions and formulate our main results in full generality. In a short Section 3, we recall some general properties of the energy measure of $(\mathcal{E},\mathcal{F})$. In Section 4 we change the metric d so that in the new metric the scaling function does not depend on the space variable. In Section 5 we prove an oscillation inequality that is a central technical result. It is used, in particular, in Section 6 to prove the Hölder continuity of the heat kernel. In Section 7 we prove Theorem 2.9. In Section 8 we prove Theorems 2.12 and 2.19. Appendix contains some auxiliary results.

Notation. Letters c, C, C', C_1, C_2, etc. are used to denote positive numbers, depending on the constants in the hypotheses, whose values may change at each occurrence. For a function u on M, we denote by $\mathrm{supp}(u)$ the support of u that is, the minimal closed subset of M so that $u = 0$ a.e. outside it. For an open set U, the notation $A \Subset U$ means that A is a precompact subset of U with $\overline{A} \subset U$.

2. Statement of the main results

Now we give precise statements of our results. For any $x \in M$ and $r > 0$, consider an open metric ball

$$B(x,r) := \{y \in M : d(y,x) < r\}$$

and its volume

$$V(x,r) := \mu(B(x,r)).$$

For any ball $B = B(x,r)$ and any $\lambda > 0$, set

$$\lambda B := B(x, \lambda r).$$

Definition 2.1 (Volume doubling condition)**.** We say that a measure μ on (M,d) satisfies the condition (VD) if there exists a constant $C \geq 1$ such that, for all $x \in M$ and all $r > 0$,

$$V(x,2r) \leq CV(x,r). \tag{2.1}$$

Condition (VD) implies that $0 < V(x,r) < \infty$ for all $r > 0$.

It is known that condition (VD) is equivalent to the following: there exists a positive number α such that, for all $x,y \in M$ and all $0 < r \leq R < \infty$,

$$\frac{V(x,R)}{V(y,r)} \leq C\left(\frac{d(x,y)+R}{r}\right)^{\alpha}, \tag{2.2}$$

where constant C can be taken the same as in (VD). In particular, for all $x \in M$ and all $0 < r \leq R < \infty$,

$$\frac{V(x,R)}{V(x,r)} \leq C\left(\frac{R}{r}\right)^{\alpha}. \tag{2.3}$$

Let us fix throughout the paper a parameter $\overline{R} \in (0, \operatorname{diam} M]$, where $\operatorname{diam} M$ is the diameter of M.

Definition 2.2 (Reverse volume doubling condition)**.** We say that μ satisfies the condition (RVD) if there exist positive numbers C, α' such that, for all x in M and for all $0 < r \leq R < \overline{R}$,

$$\frac{V(x,R)}{V(x,r)} \geq C^{-1}\left(\frac{R}{r}\right)^{\alpha'}. \tag{2.4}$$

Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form on L^2. Recall that any regular symmetric Dirichlet form $(\mathcal{E},\mathcal{F})$ in L^2 admits the following unique *Beurling-Deny decomposition* (cf. [8, Theorem 3.2.1 and Theorem 4.5.2]):

$$\mathcal{E}(u,v) = \mathcal{E}^{(L)}(u,v) + \mathcal{E}^{(J)}(u,v) + \mathcal{E}^{(K)}(u,v), \tag{2.5}$$

where $\mathcal{E}^{(L)}$ is the *local part* (or *diffusion part*) associated with a unique Radon measure $d\Gamma^{(L)}$ (the notions $\mathcal{E}^{(L)}(u,v), d\Gamma^{(L)}(u,v)$ are instead denoted by $\mathcal{E}^{(c)}(u,v), \frac{1}{2}d\mu^c_{\langle u,v\rangle}$ respectively in [8, see formula (3.2.22) on p.126]):

$$\mathcal{E}^{(L)}(u,v) = \int_M d\Gamma^{(L)}(u,v),$$

$\mathcal{E}^{(J)}$ is the jump part associated with a unique Radon measure j defined on $M \times M \setminus \text{diag}$:

$$\mathcal{E}^{(J)}(u,v) = \iint_{M\times M\setminus \text{diag}} (u(x)-u(y))(v(x)-v(y))dj(x,y), \tag{2.6}$$

and finally, $\mathcal{E}^{(K)}$ is the *killing part*. We assume throughout the paper that $\mathcal{E}^{(K)} \equiv 0$ and, thus,

$$\mathcal{E}(u,v) = \mathcal{E}^{(L)}(u,v) + \mathcal{E}^{(J)}(u,v). \tag{2.7}$$

For simplicity, we set $j = 0$ on the diagonal of $M \times M$ so that the integral in (2.6) can be extended to the entire space $M \times M$.

Let us fix a *scaling function* $W : M \times [0,\infty] \to [0,\infty]$ such that, for each $x \in M$, the function $W(x,\cdot)$ is strictly increasing, and $W(x,0) = 0$, $W(x,\infty) = \infty$. Assume also that there exist three positive numbers C, β_1, β_2 ($\beta_1 \le \beta_2$) such that, for all $0 < r \le R < \infty$ and for all $x, y \in M$ with $d(x,y) \le R$,

$$C^{-1}\left(\frac{R}{r}\right)^{\beta_1} \le \frac{W(x,R)}{W(y,r)} \le C\left(\frac{R}{r}\right)^{\beta_2}. \tag{2.8}$$

Clearly, we have by (2.8) that, for all $x \in M$ and all $0 < r \le R < \infty$

$$C^{-1}\left(\frac{R}{r}\right)^{1/\beta_2} \le \frac{W^{-1}(x,R)}{W^{-1}(x,r)} \le C\left(\frac{R}{r}\right)^{1/\beta_1}, \tag{2.9}$$

where $W^{-1}(x,\cdot)$ is the inverse function of $W(x,\cdot)$ for every $x \in M$.

The function W will determine the space/time scaling of the Hunt process of the Dirichlet form $(\mathcal{E},\mathcal{F})$. A typical example of a scaling function is $W(x,r) = r^\beta$ as was considered in Introduction. For example, if $M = \mathbb{R}^n$ and $(\mathcal{E},\mathcal{F})$ is the classical Dirichlet integral

$$\mathcal{E}(u,u) = \int_{\mathbb{R}^n} |\nabla u|^2 \, d\mu$$

then $\beta = 2$. For the jump type Dirichlet form in $\mathbb{R}^n$ with the jump kernel (1.2), β can be any number from $(0,2)$. If M is a fractal space and $(\mathcal{E},\mathcal{F})$ is a self-similar strongly local Dirichlet form then typically $\beta > 2$, for example, for the Sierpinski gasket in $\mathbb{R}^2$ we have $\beta = \frac{\log 5}{\log 2}$. This value of β is called the *walk dimension* of the fractal.

For any metric ball $B := B(x,r)$, set

$$W(B) := W(x,r).$$

Despite of notation, $W(B)$ is *not* a function of a ball as a subset of M, but is a function of a pair (x,r) as it may happen that $B(x_1,r_1) = B(x_2,r_2)$ whereas $W(x_1,r_1) \ne W(x_2,r_2)$.

Let $U \subset M$ be an open set, A be a Borel subset of U and $\overline{\kappa} \ge 1$ be a number. A *$\overline{\kappa}$-cutoff function* of the pair (A,U) is any function ϕ in $\mathcal{F}$ such that

- $0 \le \phi \le \overline{\kappa}$ μ-a.e. in M;
- $\phi \ge 1$ μ-a.e. in A;
- $\phi = 0$ μ-a.e. in U^c.

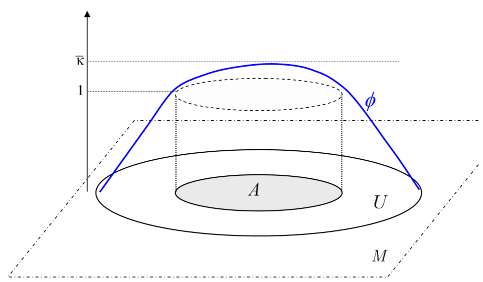

FIGURE 1. A function $\phi \in \overline{\kappa}$-cutoff$(A,U)$

We denote by $\overline{\kappa}$-cutoff(A,U) the collection of all $\overline{\kappa}$-cutoff functions of the pair (A,U). Any 1-cutoff function for $\overline{\kappa} = 1$ will be simply referred to as a *cutoff function*. Clearly, $\phi \in \mathcal{F}$ is a cutoff function of (A,U) if and only if $0 \le \phi \le 1$, $\phi|_A = 1$ and $\phi|_{U^c} = 0$. Denote by

$$\text{cutoff}(A,U) := 1\text{-}\,\text{cutoff}(A,U).$$

Note that for every $\overline{\kappa} \ge 1$,

$$\text{cutoff}(A,U) \subset \overline{\kappa}\text{-}\,\text{cutoff}(A,U),$$

and that, if $\phi \in \overline{\kappa}$-cutoff$(A, U)$, then $1 \wedge \phi \in$ cutoff(A, U). It is known that if $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form in L^2, then cutoff(A, U) is not empty for any nonempty precompact A with $\overline{A} \subset U$.

Define a function space $\mathcal{F}'$ by

$$\mathcal{F}' := \{v + a : v \in \mathcal{F},\ a \in \mathbb{R}\},$$

that is, $\mathcal{F}'$ is a vector space that contains $\mathcal{F}$ and constants.

Definition 2.3 (Generalized capacity condition)**.** We say that condition (Gcap) is satisfied if there exist two numbers $\overline{\kappa} \geq 1, C > 0$ such that, for any $u \in \mathcal{F}' \cap L^\infty$ and for any pair of concentric balls $B_0 := B(x_0, R)$, $B := B(x_0, R + r)$ with $x_0 \in M$ and $0 < R < R + r < \overline{R}$, there exists $\phi \in \overline{\kappa}$-cutoff$(B_0, B)$ such that

$$\mathcal{E}(u^2\phi, \phi) \leq \sup_{x \in B} \frac{C}{W(x, r)} \int_B u^2 d\mu. \tag{2.10}$$

We remark that the function ϕ in (Gcap) may depend on u, but the constants $\overline{\kappa}, C$ are independent of u, B_0, B.

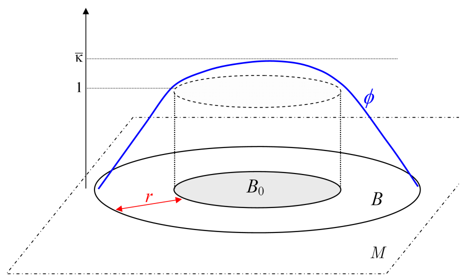

FIGURE 2. Function ϕ in (2.10)

For any open set $U \subset M$ and a Borel set $A \subset U$, define the capacity of the pair (A, U) by

$$\text{cap}(A, U) := \inf\{\mathcal{E}(\varphi, \varphi) : \text{ for any } \varphi \in \text{cutoff}(A, U)\}.$$

Definition 2.4 (Capacity upper bound)**.** We say that the condition ($\text{Cap}_\leq$) is satisfied if there exists a constant $C > 0$ such that for all balls B of radius R less than $\overline{R}$

$$\text{cap}(\frac{1}{2}B, B) \leq C \frac{\mu(B)}{W(B)}. \tag{2.11}$$

Note that the following implication is obvious:

$$(\text{Gcap}) \Rightarrow (\text{Cap}_\leq). \tag{2.12}$$

Indeed, using (Gcap) with $u \equiv 1$, we see that there exists a function

$$\phi \in \overline{\kappa}\text{- cutoff}(\frac{1}{2}B, B)$$

such that

$$\mathcal{E}(\phi, \phi) \leq \frac{C}{W(B)} \int_B u^2 d\mu = C \frac{\mu(B)}{W(B)}.$$

Replacing ϕ by $\widetilde{\phi} := 1 \wedge \phi \in \text{cutoff}(\frac{1}{2}B, B)$ and then using the Markov property

$$\mathcal{E}(\widetilde{\phi}, \widetilde{\phi}) \leq \mathcal{E}(\phi, \phi),$$

we obtain that $\mathcal{E}(\widetilde{\phi}, \widetilde{\phi})$ satisfies the same estimate, which implies the condition ($\text{Cap}_\leq$).

It would be ideal if in all our results (Gcap) could be replaced by the simpler condition ($\text{Cap}_\leq$), but so far there is no technique for that. Usually it is very difficult to verify (Gcap). However, there are some cases when (Gcap) is trivially satisfied (see [12, Section 4]).

Let $\mathcal{B}(M)$ be the sigma-algebra of Borel sets of M. Recall that a *transition kernel* is a map $J : M \times \mathcal{B}(M) \mapsto \mathbb{R}_+$ satisfying the following two properties:

- for every fixed x in M, the map $E \mapsto J(x, E)$ is a measure on $\mathcal{B}(M)$;
- for every fixed E in $\mathcal{B}(M)$, the map $x \mapsto J(x, E)$ is a non-negative measurable function on M.

Definition 2.5 (Tail estimate of jump measure)**.** We say that condition (TJ) is satisfied if there exists a transition kernel $J(x, E)$ on $M \times \mathcal{B}(M)$ such that

$$dj(x, y) = J(x, dy)d\mu(x) \text{ in } M \times M,$$

and, for any point x in M and any $R > 0$,

$$J(x, B(x, R)^c) = \int_{B(x,R)^c} J(x, dy) \le \frac{C}{W(x, R)}, \tag{2.13}$$

where $C \in [0, \infty)$ is a constant independent of x, R.

For example, if $W(x, R) = R^\beta$ for any $x \in M$ and $R > 0$ then the inequality (2.13) reads

$$J(x, B(x, R)^c) \le \frac{C}{R^\beta} \quad \text{for all } x \text{ in } M \text{ and } R > 0.$$

The latter condition was introduced in [1] in the setting of the ultra-metric spaces.

For a measurable function u and a measurable set A, let u_A denote the mean of the function u over A, that is,

$$u_A := \frac{1}{\mu(A)} \int_A u d\mu =: \fint_A u d\mu,$$

whenever the integral makes sense.

Definition 2.6 (Poincaré inequality)**.** We say that the *Poincaré inequality* (PI) holds if there exist constants $C > 0$ and $\kappa \in (0, 1]$ such that, for any ball $B := B(x_0, R)$ with $0 < R < \overline{R}$ and for any function $u \in \mathcal{F}' \cap L^\infty$,

$$\int_{\kappa B} |u - u_{\kappa B}|^2 d\mu \le CW(x_0, R) \int_B d\Gamma_B(u), \tag{2.14}$$

where

$$\int_B d\Gamma_B(u) = \int_B d\Gamma^{(L)}(u)(x) + \iint_{B \times B} (u(x) - u(y))^2 dj(x, y).$$

For example, if M is a complete manifold of non-negative Ricci curvature, d is the geodesic metric, μ is the Riemannian measure, and $\mathcal{E}$ is the Dirichlet integral, then (PI) holds with $W(x, R) = R^2$.

Definition 2.7 (Near-diagonal lower estimate)**.** We say that condition (NLE) holds if the heat kernel $p_t(x, y)$ exists and satisfies a *near-diagonal lower estimate*: for any $C_0 \ge 1$, there exist two constants $\delta, C > 0$ such that

$$p_t(x, y) \ge \frac{C^{-1}}{V(x, W^{-1}(x, t))} \tag{2.15}$$

for any $t < C_0 W(x, \overline{R})$ and $\mu \times \mu$-almost all $(x, y) \in M \times M$ such that

$$d(x, y) \le \delta W^{-1}(x, t).$$

We say that condition (sNLE) is satisfied if the function $p_t(x, y)$ has a version satisfying the semigroup identity

$$p_{t+s}(x, y) = \int_M p_t(x, z)p_s(z, y)dz$$

for any $t, s > 0$, $x, y \in M$, and satisfying (2.15) for any $t < C_0 W(x, \overline{R})$ and all $x, y \in M$ such that $d(x, y) \le \delta W^{-1}(x, t)$.

For a non-empty open subset U of M, let $C_0(U)$ denote the space of all continuous functions with compact supports contained in U. Let $\mathcal{F}(U)$ be a vector space defined by

$$\mathcal{F}(U) = \text{ the closure of } \mathcal{F} \cap C_0(U) \text{ in the norm of } \sqrt{\mathcal{E}(\cdot) + \|\cdot\|_2^2}, \tag{2.16}$$

where $\mathcal{E}(u) := \mathcal{E}(u,u)$. By the theory of Dirichlet forms, $(\mathcal{E}, \mathcal{F}(U))$ is a regular Dirichlet form on $L^2(U)$ if $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(M,\mu)$ (see, for example, [8, Theorem 4.4.3]). In this case, denote the heat semigroup of $(\mathcal{E}, \mathcal{F}(U))$ by $\{P_t^U\}_{t>0}$. The integral kernel of $\{P_t^U\}_{t>0}$ (should it exist) is denoted by $p_t^U(x,y)$ and is referred to as the *heat kernel* of $(\mathcal{E}, \mathcal{F}(U))$ or the *Dirichlet heat kernel* of $(\mathcal{E}, \mathcal{F})$ in U.

Definition 2.8 (Localized lower estimate)**.** We say that condition (LLE) holds if the following two properties are satisfied:

1. for any bounded open set $\Omega \subset M$, the Dirichlet heat kernel $p_t^\Omega(x,y)$ exists;
2. there exist $C > 0$ and $\delta \in (0,1)$ such that, for any ball $B := B(x_0, R)$ with $R \in (0, \overline{R})$, for any $t \leq W(x_0, \delta R)$ and for μ-almost all $x, y \in B(x_0, \delta W^{-1}(x_0,t))$,

$$p_t^B(x,y) \geq \frac{C^{-1}}{V(x_0, W^{-1}(x_0,t))}. \tag{2.17}$$

We say that condition (sLLE) (*strong localized lower estimate*) holds if (LLE) holds and, in addition, the Dirichlet heat kernel $p_t^\Omega(x,y)$ is locally Hölder continuous in

$$(x,y,t) \in \Omega \times \Omega \times (0,\infty)$$

for any non-empty bounded open set $\Omega \subset M$.

In other words, the inequality (2.17) says that the *Dirichlet heat kernel* $p_t^B(x,y)$ satisfies the *near-diagonal* lower bound for x, y close to the center of B.

Under condition (sLLE), we can rephrase the inequality (2.17) in a simpler way: there exist some constants $C > 0, \delta \in (0,1)$ such that, for all $x \in M$, $0 < R < \overline{R}$ and all $t \leq W(x, \delta R)$,

$$p_t^B(x,y) \geq \frac{C^{-1}}{V(x, W^{-1}(x,t))} \quad \text{for all} \quad y \in B(x, \delta W^{-1}(x,t)).$$

The following theorem is our first main result that gives a lower estimate of the heat kernel.

Theorem 2.9. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2 without killing part. If conditions* (VD), (RVD) *and* (TJ) *hold, then*

$$\text{(PI)} + \text{(Gcap)} \;\Leftrightarrow\; \text{(sLLE)} \;\Leftrightarrow\; \text{(LLE)} \;\Rightarrow\; \text{(sNLE)}.$$

We will prove Theorem 2.9 in Section 7.5. The most difficult part is to show the implication

$$\text{(VD)} + \text{(RVD)} + \text{(TJ)} + \text{(PI)} + \text{(Gcap)} \;\Rightarrow\; \text{(sLLE)}, \tag{2.18}$$

which will be done in Section 7.

Let us turn to off-diagonal lower estimates of the heat kernel. For that we need two more conditions ($\text{J}_\geq$) and (LE). For all $x, y \in M$, denote

$$V(x,y) := V(x, d(x,y)) \;\text{ and }\; W(x,y) := W(x, d(x,y)).$$

Note that $V(x,y)$ and $W(x,y)$ are not symmetric in x, y in general.

Definition 2.10 (Lower bound of jump kernel)**.** We say that condition ($\text{J}_\geq$) is satisfied if there exists a non-negative function J (called the *jump kernel*) such that

$$dj(x,y) = J(x,y)d\mu(y)d\mu(x)$$

in $M \times M$, and, for $\mu \times \mu$-almost all $(x,y) \in M \times M$,

$$J(x,y) \geq \frac{C}{V(x,y)\,W(x,y)}, \tag{2.19}$$

where $C > 0$ is a constant independent of x, y.

Definition 2.11 (Lower bound of heat kernel)**.** We say that condition (LE) is satisfied if the heat kernel $p_t(x, y)$ exists and, for any $C_0 \geq 1$, there exists a constant $C > 0$ such that for $\mu \times \mu$-almost all $(x, y) \in M \times M$ and any $t < C_0(W(x, \overline{R}) \wedge W(y, \overline{R}))$,

$$p_t(x, y) \geq C \left(\frac{1}{V(x, W^{-1}(x, t))} \wedge \frac{t}{V(x, y)W(x, y)} \right). \tag{2.20}$$

We say that condition (sLE) is satisfied if condition (LE) is satisfied and the function $p_t(x, y)$ has a version satisfying

$$p_{t+s}(x, y) = \int_M p_t(x, z) p_s(z, y) dz$$

for any $t, s > 0,\ x, y \in M$ and satisfying (2.20) for all $x, y \in M$ and $t < C_0(W(x, \overline{R}) \wedge W(y, \overline{R}))$.

Denote by (C) the condition that the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is *conservative*, that is

$$P_t 1 = 1 \ \text{ in } M \text{ for each } t > 0.$$

The second main result of this paper is following theorem.

Theorem 2.12. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2 without killing part. If conditions (VD), (RVD) and (TJ) hold and the jump kernel $J(x, y)$ exists, then*

$$(\mathrm{J}_{\geq}) + (\mathrm{Gcap}) + (\mathrm{C}) \ \Rightarrow \ (\mathrm{sLLE}) + (\mathrm{sLE}) \ \Rightarrow \ (\mathrm{LLE}) + (\mathrm{LE}) \Rightarrow (\mathrm{J}_{\geq}) + (\mathrm{Gcap}).$$

We will prove Theorem 2.12 in Section 8.

Corollary 2.13. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2 without killing part. Assume that conditions (VD), (RVD) and (TJ) hold, and the jump kernel $J(x, y)$ exists. If*

$$\inf_{z \in M} W(z, \overline{R}) > 0, \tag{2.21}$$

then

$$(\mathrm{J}_{\geq}) + (\mathrm{Gcap}) \ \Leftrightarrow \ (\mathrm{sLLE}) + (\mathrm{sLE}) \ \Leftrightarrow \ (\mathrm{LLE}) + (\mathrm{LE}).$$

Moreover, under these hypotheses, $(\mathcal{E}, \mathcal{F})$ is conservative.

Remark 2.14. The assumption that (2.21) can be easily verified in the following two cases:

(1) $\overline{R} = \operatorname{diam} M$;

(2) the function $W(z, R)$ is, in fact, independent of the space variable z.

Indeed, in the case (2) the condition (2.21) is obvious. In the case (1), if $\overline{R} = \infty$ then $W(z, \overline{R}) = \infty$ so that (2.21) is again trivially satisfied. Consider the case when $\overline{R} = \operatorname{diam} M < \infty$. Then we have by (2.8), for any $x, y \in M$,

$$\frac{W(x, \overline{R})}{W(y, \overline{R})} \leq \frac{W(x, d(x, y) + \overline{R})}{W(y, \overline{R})} \leq C \left(\frac{d(x, y) + \overline{R}}{\overline{R}} \right)^{\beta_2} \leq 2^{\beta_2} C. \tag{2.22}$$

Hence, for an arbitrary fixed point $x \in M$,

$$\inf_{y \in M} W(y, \overline{R}) \geq c W(x, \overline{R}) > 0$$

for some $c > 0$, which proves (2.21).

In the paper [14], the authors study the upper bound of heat kernel under mild assumptions on a doubling space. Combining the results in [14] and the results in this paper, we can obtain the two-sided heat kernel estimates (see Theorem 2.19 and Corollary 2.20). To state these results, let us introduce more conditions.

For a given number $1 \leq q \leq \infty$, let q' be the *Hölder conjugate* of q, that is,

$$q' := \frac{q}{q - 1}$$

so that $q' = 1$ if $q = \infty$, and $q' = \infty$ if $q = 1$.

Definition 2.15 (L^q-tail estimate of jump kernel)**.** For $q \in [1, \infty]$, we say that condition (TJ$_q$) is satisfied if $(\mathcal{E}, \mathcal{F})$ has the jump kernel $J(x, y)$ such that, for all $x \in M$ and $R > 0$,

$$\|J(x, \cdot)\|_{L^q(B(x,R)^c)} \leq \frac{C}{V(x, R)^{1/q'} W(x, R)}, \tag{2.23}$$

where $C \in [0, \infty)$ is a constant independent of x, R.

Of course, if $q < \infty$ then we have

$$\|J(x, \cdot)\|_{L^q(B(x,R)^c)} = \left(\int_{B(x,R)^c} J(x, y)^q d\mu(y) \right)^{1/q},$$

while for $q = \infty$

$$\|J(x, \cdot)\|_{L^q(B(x,R)^c)} = \operatorname*{esup}_{B(x,R)^c} J(x, \cdot).$$

Definition 2.16 (Upper bound of jump kernel)**.** We say that condition (J$_\leq$) is satisfied if $(\mathcal{E}, \mathcal{F})$ has the jump kernel $J(x, y)$ such that, for $(\mu \times \mu)$-almost all (x, y) in $M \times M$,

$$J(x, y) \leq \frac{C}{V(x, y) W(x, y)}, \tag{2.24}$$

where $C \in [0, \infty)$ is a constant independent of x, y.

Note that if $q = \infty$ then $q' = 1$ and, hence, condition (TJ$_\infty$) coincides with (J$_\leq$), that is,

$$(\mathrm{J}_\leq) = (\mathrm{TJ}_\infty).$$

If further $V(x, R) \asymp R^\alpha$ and $W(x, R) \asymp R^\beta$, then (2.24) becomes

$$J(x, y) \leq \frac{C}{d(x, y)^{\alpha+\beta}},$$

which was a starting point in a lot of literature, see for example [6], [10] and the references therein.

Let us recall the notion of a regular $\mathcal{E}$-*nest* (cf. [8, Section 2.1, p.66-69]). For an open set $U \subset M$, let

$$\mathrm{Cap}_1(U) := \inf\{\mathcal{E}(u) + \|u\|_2^2 : u \in \mathcal{F} \text{ and } u \geq 1 \ \mu\text{-almost everywhere on } U\} \tag{2.25}$$

(noting that $\mathrm{Cap}_1(U) = \infty$ if the set $\{u \in \mathcal{F} : u \geq 1 \text{ on } U\}$ is empty). An increasing sequence of closed subsets $\{F_k\}_{k=1}^\infty$ of M is called an $\mathcal{E}$-*nest* of M if

$$\lim_{k\to\infty} \mathrm{Cap}_1(M \setminus F_k) = 0.$$

An $\mathcal{E}$-nest $\{F_k\}$ is said to be *regular* with respect to μ if for each k,

$$\mu(U(x) \cap F_k) > 0 \quad \text{for any } x \in F_k \text{ and any open neighborhood } U(x) \text{ of } x.$$

For an $\mathcal{E}$-nest $\{F_k\}_{k=1}^\infty$, denote by

$$C(\{F_k\}) := \{u \text{ is a function on } M : u|_{F_k} \text{ is continuous for each } k\}. \tag{2.26}$$

A function $u : M \mapsto \mathbb{R} \cup \{\infty\}$ is said to be *quasi-continuous* if and only if $u \in C(\{F_k\})$ for some $\mathcal{E}$-nest $\{F_k\}_{k=1}^\infty$.

We introduce condition (TP$_q$) for $q \in [1, \infty]$ that means a certain L^q-*estimate* of the *tail* $P_t \mathbf{1}_{B^c}$ of the heat semigroup $\{P_t\}$ outside ball.

Definition 2.17 (L^q-tail bound of heat semigroup)**.** We say that condition (TP$_q$) holds for a given number $1 \leq q \leq \infty$ if the heat kernel $p_t(x, y)$ exists on $(0, \infty) \times M \times M$, and there exists a regular $\mathcal{E}$-nest $\{F_k\}$ such that the following two statements are true:

(1) for every x in M and every $t > 0$,

$$p_t(x, \cdot) \in C(\{F_k\}),$$

(2) for any ball $B := B(x, R)$ with $R \in (0, \overline{R})$ and any $0 < t < W(x, \overline{R})$,

$$\|p_t(x, \cdot)\|_{L^q(B^c)} \le C \left(\frac{1}{V(x, W^{-1}(x,t))^{1/q'}} \wedge \frac{t}{V(x,R)^{1/q'} W(x,R)} \right), \tag{2.27}$$

where C is a positive constant independent of B, t.

For any $q \in [1, \infty]$, define condition (UE$_q$) that is an *off-diagonal upper estimate* of the heat kernel.

Definition 2.18 (L^q-upper bound of heat kernel)**.** For a given $1 \le q \le \infty$, we say that condition (UE$_q$) is satisfied if there exists a pointwise defined heat kernel $p_t(x, y)$ for a regular $\mathcal{E}$-nest $\{F_k\}$ such that, for all $x, y \in M$ and all $0 < t < W(x, \overline{R}) \wedge W(y, \overline{R})$,

$$\begin{aligned} p_t(x,y) \le C &\left(\frac{1}{V(x, W^{-1}(x,t))^{1/q'}} \wedge \frac{t}{V(x,y)^{1/q'} W(x,y)} \right) \\ &\times \left(\frac{1}{V(x, W^{-1}(x,t))^{1/q}} + \frac{1}{V(y, W^{-1}(y,t))^{1/q}} \right), \end{aligned} \tag{2.28}$$

for some positive constant C independent of t, x, y.

For $q = \infty$, we write (UE) for (UE$_\infty$), by omitting the subscript ∞.

The following theorem provides a two-sided estimate of the heat kernel.

Theorem 2.19. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2 without killing part. Assume that $\overline{R} = \operatorname{diam} M$. If conditions* (VD)*,* (RVD) *hold, then for any $q \in [2, \infty]$,*

$$\begin{aligned} (\text{PI}) + (\text{Gcap}) + (\text{TJ}_q) &\Leftrightarrow (\text{TP}_q) + (\text{sLLE}) \\ &\Leftrightarrow (\text{TP}_q) + (\text{LLE}) \\ &\Rightarrow (\text{UE}_q) + (\text{NLE}) + (C). \end{aligned}$$

We will prove Theorem 2.19 in Section 8.

Combining Theorem 2.19 and Theorem 2.12, we immediately obtain the following.

Corollary 2.20. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2 without killing part. Assume that $\overline{R} = \operatorname{diam} M$. If conditions* (VD)*,* (RVD) *hold, then for any $q \in [2, \infty]$,*

$$\begin{aligned} (\text{J}_\ge) + (\text{Gcap}) + (\text{TJ}_q) &\Leftrightarrow (\text{TP}_q) + (\text{sLE}) + (C) \\ &\Leftrightarrow (\text{TP}_q) + (\text{LE}) + (C) &(2.29) \\ &\Leftrightarrow (\text{TP}_q) + (\text{LE}) &(2.30) \\ &\Rightarrow (\text{UE}_q) + (\text{LE}) + (C). &(2.31) \end{aligned}$$

In particular, if $q = \infty$ (so that (TP$_\infty$) $\Leftrightarrow$ (UE$_\infty$) $=$ (UE)*), then we have*

$$(\text{Gcap}) + (\text{J}) \Leftrightarrow (\text{UE}) + (\text{LE}). \tag{2.32}$$

The equivalence in (2.32) generalizes the results in [6], [10] (see also [5]). Let us emphasize that the scaling function $W(x, r)$ in the present paper may depend on the variable x (which causes serious difficulties in the proof), while in [6] and [10] the scaling function does not depend on x.

Example 2.21. Assume that the measure μ is α-regular for some $\alpha > 0$, that is,

$$V(x, r) \simeq r^\alpha$$

for all $x \in M$ and $r > 0$. Then the both conditions (VD), (RVD) are trivially satisfied with $\overline{R} = \infty$. Set $W(x, r) = r^\beta$ for some $\beta > 0$ and all $x \in M$ and $r > 0$. The condition (TJ) means in this case that

$$\int_{B(x,R)^c} J(x, dy) \le \frac{C}{R^\beta}. \tag{2.33}$$

The Poincaré inequality (PI) means that, there exist $C > 0$ and $\kappa \in (0, 1]$ such that, for any ball $B := B(x_0, R)$ with $0 < R < \overline{R}$ and any function $u \in \mathcal{F}' \cap L^\infty$,

$$\int_{\kappa B} |u - u_{\kappa B}|^2 d\mu \le C R^\beta \int_B d\Gamma_B(u). \tag{2.34}$$

The generalized capacity condition (Gcap) means that, for any $u \in \mathcal{F}' \cap L^\infty$ and for any pair of concentric balls $B_0 := B(x_0, R)$, $B := B(x_0, R + r)$ with $x_0 \in M$ and $0 < R < R + r < \overline{R}$, there exists some $\phi \in \overline{\kappa}\text{-cutoff}(B_0, B)$ such that

$$\mathcal{E}(u^2\phi, \phi) \le \frac{C}{r^\beta} \int_B u^2 d\mu. \tag{2.35}$$

Theorem 2.9 says in this case that, under the standing assumption (2.33), the conditions (2.34), (2.35) are equivalent to the following localized lower estimate: for any ball $B = B(x, r)$ and for any $t < (\delta r)^\beta$, the Dirichlet heat kernel in B exists and satisfies the inequality

$$p_t^B(x, y) \ge \frac{c}{t^{\alpha/\beta}} \quad \text{if} \ \ y \in B(x, \delta t^{1/\beta}).$$

Consequently, the global heat kernel exists and satisfies the near-diagonal lower estimate

$$p_t(x, y) \ge \frac{c}{t^{\alpha/\beta}} \ \ \text{if} \ d(x, y) \le \delta t^{1/\beta}.$$

The condition ($J_\ge$) means in this case that

$$J(x, y) \ge \frac{C}{d(x, y)^{\alpha+\beta}}. \tag{2.36}$$

Hence, under the hypotheses (2.33), (2.34), (2.35) and (2.36), Theorem 2.12 yields the full lower estimate:

$$p_t(x, y) \ge C\left(\frac{1}{t^{\alpha/\beta}} \wedge \frac{t}{d(x, y)^{\alpha+\beta}}\right) \simeq \frac{C}{t^{\alpha/\beta}}\left(1 + \frac{d(x, y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}.$$

3. Energy measure

In this section we collect some elementary properties on energy measures, which will be used later on.

Let $(\mathcal{E}, \mathcal{F})$ be any regular Dirichlet form in L^2 with the Beurling-Deny decomposition (2.5). Let

$$\mathcal{F}_{\text{loc}} := \{u : \forall\ U \Subset M, \text{ there exists } v \in \mathcal{F} \text{ so that } v = u \ \mu\text{-a.e. on } U\}.$$

Since $(\mathcal{E}, \mathcal{F})$ is regular, the constant function $1 \in \mathcal{F}_{\text{loc}}$, so that $\mathcal{F}' \subset \mathcal{F}_{\text{loc}}$. It is known that for any $u \in \mathcal{F}_{\text{loc}} \cap L^\infty$, there exists a unique Radon measure $d\Gamma^{(L)}(u) := d\Gamma^{(L)}(u, u)$ such that

$$\mathcal{E}^{(L)}(u, u) = \int_M d\Gamma^{(L)}(u, u),$$

see for example [8, Lemma 3.2.3, and the first two paragraphs on p.130] wherein the symbol

$$d\mu^c_{\langle u\rangle} = 2d\Gamma^{(L)}(u, u)$$

is used instead. Moreover, these measures satisfy the following properties: for any $u, v, w \in \mathcal{F}_{\text{loc}} \cap L^\infty$,

- the *product rule* ([8, Lemma 3.2.5, and the second paragraph on p.130]):

$$d\Gamma^{(L)}(uv, w) = u d\Gamma^{(L)}(v, w) + v d\Gamma^{(L)}(u, w); \tag{3.1}$$

- the *chain rule* ([8, Theorem 3.2.2, and the second paragraph on p.130]):
$$d\Gamma^{(L)}(\Phi(u),v) = \Phi'(u)d\Gamma^{(L)}(v,w) \tag{3.2}$$
for any $\Phi \in C^1(\mathbb{R})$ (one does not need to assume $\Phi(0) = 0$);
- the *Cauchy-Schwarz inequality*: for any $f \in L^2(M, \Gamma^{(L)}(u))$, $g \in L^2(M, \Gamma^{(L)}(v))$
$$\int |fg| d\Gamma^{(L)}(u,v) \leq \left(\int f^2 d\Gamma^{(L)}(u)\right)^{1/2} \left(\int g^2 d\Gamma^{(L)}(v)\right)^{1/2} \tag{3.3}$$
(cf. [21, on p. 390]).

Moreover, for any $u \in \mathcal{F}_{\mathrm{loc}} \cap L^\infty$, we have
$$d\Gamma^{(L)}(|u|) = d\Gamma^{(L)}(u) \tag{3.4}$$
(see [12, Eq. (5.6)]).

For a Borel measurable subset Ω of M and $u \in \mathcal{F}'$, define the *measure* $d\Gamma_\Omega(u)$ by
$$d\Gamma_\Omega(u)(x) := d\Gamma^{(L)}(u)(x) + \int_M \mathbf{1}_\Omega(y)(u(x) - u(y))^2 dj(x,y). \tag{3.5}$$
Here we let the measure $j = 0$ on $\{x = y\}$ for convenience. Such a measure $d\Gamma_\Omega(u)$ is well-defined for any $u \in \mathcal{F}'$ and $\Omega \subset M$. Clearly, for any three sets A, B, Ω with $A \subset B$, any $u \in \mathcal{F}'$ and any measurable function $f \geq 0$,
$$\int_\Omega f d\Gamma_A(u) \leq \int_\Omega f d\Gamma_B(u), \tag{3.6}$$
and
$$\int_\Omega f d\Gamma_B(u \wedge 1) \leq \int_\Omega f d\Gamma_B(u). \tag{3.7}$$

4. Change of metric

4.1. **A new metric.** In [14], the authors introduce a new metric d_* on M, which is comparable with the original metric d. In particular, under this new metric d_*. the measure μ still keeps the doubling property (or the reverse doubling property). More importantly, the scaling function $W(x,R)$ becomes independent of point x. Let us recall some properties of this new metric which will be used in this paper.

For any $x, y \in M$, set $W(x,y) := W(x, d(x,y))$. Let
$$D(x,y) := W(x,y) + W(y,x), \tag{4.1}$$
Clearly, the quantity $D(x,y) = 0$ if and only if $x = y$, and is symmetric: $D(x,y) = D(y,x)$. The following proposition shows that $D(x,y)$ is a quasi-metric on M.

Proposition 4.1 ([14, Proposition 5.1])**.** *There exists a constant $C_1 \geq 1$ such that for all x, y, z in M,*
$$D(x,y) \leq C_1(D(x,z) + D(z,y)). \tag{4.2}$$
Consequently, there exist two constants β, $C_2 > 0$ and a metric d_* on M such that
$$C_2^{-1} d_*(x,y)^\beta \leq D(x,y) \asymp W(x,y) \leq C_2 d_*(x,y)^\beta \tag{4.3}$$
for all x, y in M.

In the rest of the paper, β will be always refer to as the constant from Proposition 4.1.

Define the function F by
$$F(x,R) := W(x,R)^{1/\beta}, \quad x \in M,\ R > 0, \tag{4.4}$$

where β comes from (4.3). Clearly, such a function $F(x,\cdot)$ is strictly increasing on $[0,\infty]$ for any $x \in M$, since so is $W(x,\cdot)$. Moreover, by (4.3)

$$L^{-1}d_*(x,y) \leq F(x,d(x,y)) = W(x,y)^{1/\beta} \leq Ld_*(x,y) \quad \text{for all } x,y \text{ in } M \tag{4.5}$$

for some constant $L \geq 1$. For $x \in M$, let $F^{-1}(x,\cdot)$ be the inverse of the function $t \mapsto F(x,t)$, and then

$$F^{-1}(x,t) = W^{-1}(x,t^\beta), \quad t > 0.$$

For any $r > 0$, let

$$B_*(x,r) := \{y \in M : d_*(y,x) < r\}$$

be an open ball under the new metric d_*.

Proposition 4.2 ([14, Proposition 5.2])**.** *There exists a number L_0 with $L_0 \geq L^2 > 1$, where $L > 1$ is the same constant as in (4.5), such that the following properties are true.*

(1) *For all x in M and all $r > 0$,*

$$B_*(x, L_0^{-1}r) \subset B(x, F^{-1}(x, L^{-1}r)) \subset B_*(x,r), \tag{4.6}$$

where $F^{-1}(x,\cdot)$ is the inverse of $F(x,\cdot)$ with $F(x,\cdot)$ defined by (4.4).

(2) *For all x in M and all $R > 0$,*

$$B(x, L_0^{-1}R) \subset B_*(x, L^{-1}F(x,R)) \subset B(x,R). \tag{4.7}$$

Consequently, a subset of M is open under the metric d_* if and only if it is also open under d.

Proposition 4.3. *For any $\eta > 0$ and for any $x,z \in M$ with $d_*(x,z) < \eta W(x,\overline{R})^{1/\beta}$, there exists a constant $C > 0$ such that*

$$C^{-1}W(z,\overline{R}) \leq W(x,\overline{R}) \leq CW(z,\overline{R}). \tag{4.8}$$

Proof. Fix $x,z \in M$ with $d_*(x,z) < \eta W(x,\overline{R})^{1/\beta}$. It suffices to consider the case when $\overline{R} < \infty$.

By the second inequality in (4.5) and the left inequality in (2.8), we have

$$W(x,d(x,z)) = F(z,d(z,y))^\beta \leq (Ld_*(x,z))^\beta < (L\eta)^\beta W(x,\overline{R}) \leq W(x,c\overline{R}),$$

for some $c > 0$. This implies that

$$d(x,z) \leq c\overline{R}.$$

This together with the right inequality in (2.8) implies that

$$\frac{W(x,\overline{R})}{W(z,\overline{R})} \leq \frac{W(x,d(x,z)+\overline{R})}{W(z,\overline{R})} \leq C\left(\frac{d(x,z)+\overline{R}}{\overline{R}}\right)^{\beta_2} \leq C(c+1)^{\beta_2}.$$

Similarly, we also have

$$\frac{W(z,\overline{R})}{W(x,\overline{R})} \leq C(c+1)^{\beta_2}.$$

By renaming the constant C, we finish the proof. □

For any x in M and any $r > 0$, let $V_*(x,r)$ be the volume of a ball $B_*(x,r)$ under the metric d_*, that is,

$$V_*(x,r) := \mu(B_*(x,r)).$$

Note that $\overline{R}$ is the diameter of M in [14], while in this paper, it can be smaller than diam M. Following the proof of [14, Proposition 5.4] and using Proposition 4.3 instead of [14, Proposition 5.3], we can prove that the reverse doubling condition (RVD_*) under the new metric d_* holds true for all $x \in M$ and $r < W(x,\overline{R})$ (see Proposition 4.4(2)).

Proposition 4.4 ([14, Proposition 5.4])**.** *Assume that* (VD) *is satisfied. Then the following statements are true.*

(1) *Condition* (VD$_*$) *holds true: there exists a constant* $C > 0$ *such that, for all* x *in* M *and all* $r > 0$,

$$V_*(x, 2r) \leq CV_*(x, r). \tag{4.9}$$

Consequently, there exists a constant $\alpha_* > 0$ *such that for all* $x, y \in M$ *and all* $0 < s \leq r$ *with* $d_*(x, y) \leq r$,

$$\frac{V_*(x, r)}{V_*(y, s)} \leq C\left(\frac{r}{s}\right)^{\alpha_*}.$$

(2) *Assume in addition that* (RVD) *is satisfied. Then condition* (RVD$_*$) *holds true: there exists a constant* $\alpha'_* > 0$ *such that for all* $x \in M$ *and all* $0 < s \leq r < W(x, \overline{R})^{1/\beta}$,

$$\frac{V_*(x, r)}{V_*(x, s)} \geq C^{-1}\left(\frac{r}{s}\right)^{\alpha'_*}. \tag{4.10}$$

4.2. **Some conditions under the new metric.** In this subsection, we will rephrase the conditions (TJ), (PI), (Cap$_\leq$) under the new metric d_*. Besides, we will use conditions (Nash$_*$) and (FK$_*$) under d_*, that are called the *Nash inequality* and *Faber-Krahn inequality*, respectively. These conditions will be used to derive the weak Harnack inequality.

Definition 4.5. We say that condition (TJ$_*$) is satisfied if there exists a non-negative kernel J on $M \times \mathcal{B}(M)$ such that

$$dj(x, y) = J(x, dy)d\mu(x) \quad \text{in } M \times M,$$

and for any point x in M and any $r > 0$,

$$J(x, B_*(x, r)^c) \leq \frac{C}{r^\beta}, \tag{4.11}$$

where $C \in [0, \infty)$ is a constant independent of x, r with $C = 0$ when $J \equiv 0$.

It is proved in [14, Proposition 6.4(3)] that

$$(\text{TJ}) \Rightarrow (\text{TJ}_*). \tag{4.12}$$

Definition 4.6. We say that the *Poincaré inequality* (PI$_*$) is satisfied if there exist two positive constants C_*, κ_* with $\kappa_* \leq 1$ such that, for all $B_* := B_*(x_0, r)$ with $r \in (0, W(x_0, \overline{R})^{1/\beta})$ and all functions $u \in \mathcal{F}' \cap L^\infty$,

$$\int_{\kappa_* B_*} |u - u_{\kappa_* B_*}|^2 d\mu \leq C_* r^\beta \int_{B_*} d\Gamma_{B_*}(u). \tag{4.13}$$

It known that for any ball B,

$$\fint_B |u - u_B|^2 d\mu = \inf_{a \in \mathbb{R}} \fint_B |u - a|^2 d\mu. \tag{4.14}$$

Indeed, for any $a \in \mathbb{R}$, we have

$$\begin{aligned}
\fint_B |u - a|^2 d\mu &= \fint_B |u - u_B + u_B - a|^2 d\mu \\
&= \fint_B \left(|u - u_B|^2 + 2(u_B - a)(u - u_B) + (u_B - a)^2\right) d\mu \\
&= \fint_B |u - u_B|^2 d\mu + 2(u_B - a)(u_B - u_B) + (u_B - a)^2 \\
&= \fint_B |u - u_B|^2 d\mu + (u_B - a)^2 \geq \fint_B |u - u_B|^2 d\mu,
\end{aligned}$$

which implies that

$$\inf_{a \in \mathbb{R}} \fint_B |u - a|^2 d\mu \geq \fint_B |u - u_B|^2 d\mu.$$

The other direction is trivial, thus showing that (4.14) is true.

Proposition 4.7. *Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form in L^2, and let d_* be the new metric defined in Proposition 4.1. Assume that condition (VD) holds. Then*

$$(\mathrm{PI}) \quad \Leftrightarrow \quad (\mathrm{PI}_*). \tag{4.15}$$

Proof. Assume that condition (PI) holds with the constant $\kappa \le 1$.

Fix a ball $B_* := B_*(x,r)$ with $r < W(x,\overline{R})^{1/\beta}$. Set

$$R := F^{-1}(x, L^{-1}r) < F^{-1}(x,r) < \overline{R}.$$

Note that by (4.4)

$$W(x,R) = F(x,R)^\beta = (L^{-1}cr)^\beta. \tag{4.16}$$

By (4.6), we have

$$B_*(x, L_0^{-1}r) \subset B(x,R) \subset B_*(x,r) = B_*. \tag{4.17}$$

It follows that

$$\begin{aligned}
\fint_{\kappa L_0^{-1}B_*} |u - u_{\kappa L_0^{-1}B_*}|^2 d\mu &= \inf_{a\in\mathbb{R}} \fint_{\kappa B_*(x,L_0^{-1}r)} |u-a|^2 d\mu \quad (\text{by } (4.14))\\
&\le \inf_{a\in\mathbb{R}} \frac{V_*(x,r)}{V_*(x,\kappa L_0^{-1}r)} \fint_{\kappa B(x,R)} |u-a|^2 d\mu \quad (\text{by } (4.17))\\
&\le C \inf_{a\in\mathbb{R}} \fint_{\kappa B(x,R)} |u-a|^2 d\mu \quad (\text{by } (\mathrm{VD}_*))\\
&= C \fint_{\kappa B(x,R)} |u - u_{\kappa B(x,R)}|^2 d\mu \quad (\text{by } (4.14) \text{ again})\\
&\le C' W(x,R) \int_{B(x,R)} d\Gamma_{B(x,R)}(u) \quad (\text{by } (\mathrm{PI}))\\
&\le C'(L^{-1}cr)^\beta \int_{B_*} d\Gamma_{B*}(u) \quad (\text{by } (4.16), (4.17), (3.6)),
\end{aligned}$$

thus showing condition (PI_*) by setting $\kappa_* = \kappa L_0^{-1}$.

Similarly, one can show $(\mathrm{PI}_*) \Rightarrow (\mathrm{PI})$. Indeed, let $B := B(x,R)$ with $R < \overline{R}$ and set

$$r := L^{-1}F(x,R) < W(x,\overline{R})^{1/\beta}.$$

Note that

$$W(x,R) = F(x,R)^\beta = (Lr)^\beta. \tag{4.18}$$

By (4.7), we have

$$B(x, L_0^{-1}R) \subset B_*(x,r) \subset B(x,R) = B. \tag{4.19}$$

It follows that

$$\begin{aligned}
\fint_{\kappa_* L_0^{-1}B} |u - u_{\kappa_* L_0^{-1}B}|^2 d\mu &= \inf_{a\in\mathbb{R}} \fint_{\kappa_* B(x,L_0^{-1}R)} |u-a|^2 d\mu \quad (\text{by } (4.14))\\
&\le \inf_{a\in\mathbb{R}} \frac{V(x,R)}{V(x,\kappa_* L_0^{-1}R)} \fint_{\kappa_* B_*(x,r)} |u-a|^2 d\mu \quad (\text{by } (4.19))\\
&\le C \inf_{a\in\mathbb{R}} \fint_{\kappa_* B_*(x,r)} |u-a|^2 d\mu \quad (\text{by } (\mathrm{VD}))\\
&\le C' r^\beta \int_{B_*(x,r)} d\Gamma_{B_*(x,r)}(u) \quad (\text{by } (4.14) \text{ and } (\mathrm{PI}_*))\\
&\le C' L^{-\beta} W(x,R) \int_B d\Gamma_B(u) \quad (\text{by } (4.18), (4.19), (3.6)),
\end{aligned}$$

thus showing that condition (PI) holds by setting $\kappa = \kappa_* L_0^{-1} \le 1$. □

We look at condition (Nash_*).

Definition 4.8. We say the *Nash inequality* (Nash$_*$) holds for $(\mathcal{E},\mathcal{F})$ if there exist two positive constants C,ν such that, for all $B_*:=B_*(x_0,r)$ with $r>0$ and all $u\in\mathcal{F}(B_*)$

$$\|u\|_2^{2(1+\nu)}\leq\frac{Cr^\beta}{V_*(x_0,r)^\nu}\left(\mathcal{E}(u,u)+W(x_0,\overline{R})^{-1}\|u\|_2^2\right)\|u\|_1^{2\nu}. \tag{4.20}$$

We remark that both constants C,ν above are independent of $\overline{R}$.

Lemma 4.9. *Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form in L^2, and let d_* be the new metric defined in Proposition 4.1. Then*

$$(\text{VD}_*)+(\text{RVD}_*)+(\text{PI}_*)\Rightarrow(\text{Nash}_*). \tag{4.21}$$

Proof. Fix a ball $B_*:=B_*(x_0,r)$ of radius $r>0$. We divide the proof into three steps.

Step 1. We show that for any $s>0$ and $u\in\mathcal{F}(B_*)$,

$$\|u_s\|_2^2\leq\sup_{x\in B_*}\frac{C}{V_*(x,s)}\|u\|_1^2, \tag{4.22}$$

where C is a positive constant independent of s,u,B_*, and

$$u_s(x):=\fint_{B_*(x,s)}u(z)d\mu(z)$$

is the average of u over the ball $B_*(x,s)$.

Indeed, we have by condition (VD$_*$) that for all $s>0$,

$$\begin{aligned}\|u_s\|_1&\leq\int_M\frac{1}{V_*(x,s)}\int_{B_*(x,s)}|u(z)|d\mu(z)d\mu(x)\\&=\int_M|u(z)|d\mu(z)\int_M\frac{\mathbf{1}_{B_*(x,s)}(z)}{V_*(x,s)}d\mu(x)\\&=\int_M|u(z)|d\mu(z)\int_M\frac{V_*(z,s)}{V_*(x,s)}\frac{\mathbf{1}_{B_*(z,s)}(x)}{V_*(z,s)}d\mu(x)\\&\leq C\int_M|u(z)|d\mu(z)\int_M\frac{\mathbf{1}_{B_*(z,s)}(x)}{V_*(z,s)}d\mu(x)=\ C\|u\|_1.\end{aligned}$$

On the other hand, since $u=0$ outside $B_*=B_*(x_0,r)$, the function $u_s=0$ outside the set $\cup_{y\in B_*}B(y,s)$. It follows that

$$\begin{aligned}\|u_s\|_\infty&=\sup_{x\in\cup_{y\in B_*}B(y,s)}\frac{1}{V_*(x,s)}\int_{B_*(x,s)}|u(z)|d\mu(z)\\&\leq\sup_{y\in B_*}\sup_{x\in B_*(y,s)}\frac{V_*(y,s)}{V_*(x,s)}\frac{1}{V_*(y,s)}\int_{B_*(x,s)}|u(z)|d\mu(z)\\&\leq\sup_{y\in B_*}\frac{C}{V_*(y,s)}\|u\|_1.\quad(\text{by }(\text{VD}_*))\end{aligned}$$

Therefore,

$$\|u_s\|_2^2\leq\|u_s\|_\infty\|u_s\|_1\leq\sup_{x\in B_*}\frac{C^2}{V_*(x,s)}\|u\|_1^2,$$

thus showing (4.22).

Step 2. We show that for any $s\in(0,\frac{\kappa_*}{6}W(x_0,\overline{R})^{1/\beta})$ and any $u\in\mathcal{F}$,

$$\|u-u_s\|_2^2\leq Cs^\beta\mathcal{E}(u,u), \tag{4.23}$$

where C is some positive constant independent of s,u, and the constant κ_* comes from condition (PI$_*$).

Indeed, fix a function $u \in \mathcal{F}$ and $s \in (0, \frac{\kappa_*}{6} W(x_0, \overline{R})^{1/\beta})$. Since M is separable, there is a countable sequence of balls $\{B_i := B_*(x_i, s), i \geq 1\}$ such that $B_i \cap B_j = \emptyset$ if $i \neq j$ and $M \subset \cup_{i=1}^{\infty} 5B_i$, see for example [19, Theorem 1.2, p. 2]. Thus,

$$\begin{aligned}\|u - u_s\|_2^2 &\leq \sum_{i=1}^{\infty} \int_{5B_i} |u - u_s|^2 d\mu \\ &\leq 2 \sum_{i=1}^{\infty} \int_{5B_i} |u - u_{6B_i}|^2 d\mu + 2 \sum_{i=1}^{\infty} \int_{5B_i} |u_{6B_i} - u_s|^2 d\mu. \end{aligned} \tag{4.24}$$

Note that by condition (PI$_*$),

$$\int_{5B_i} |u - u_{6B_i}|^2 d\mu \leq \int_{6B_i} |u - u_{6B_i}|^2 d\mu \leq C(6\kappa_*^{-1} s)^\beta \int_{(6\kappa_*^{-1} B_i)} d\Gamma_{(6\kappa_*^{-1} B_i)}(u), \tag{4.25}$$

from which, it follows from (VD$_*$) and Cauchy-Schwarz inequality that

$$\begin{aligned}\int_{5B_i} |u_{6B_i} - u_s|^2 d\mu &\leq \int_{5B_i} \frac{1}{V_*(x,s)} \int_{B_*(x,s)} |u_{6B_i} - u(z)|^2 d\mu(z) d\mu(x) \\ &\leq \int_{5B_i} \frac{1}{V_*(x,s)} \int_{6B_i} |u_{6B_i} - u(z)|^2 d\mu(z) d\mu(x) \\ &\leq \int_{5B_i} \frac{d\mu(x)}{V_*(x,s)} \cdot C(6\kappa_*^{-1} s)^\beta \int_{(6\kappa_*^{-1} B_i)} d\Gamma_{(6\kappa_*^{-1} B_i)}(u) \\ &= \int_{5B_i} \frac{V_*(x_i, 5s)}{V_*(x,s)} \frac{d\mu(x)}{\mu(5B_i)} \cdot C(6\kappa_*^{-1} s)^\beta \int_{(6\kappa_*^{-1} B_i)} d\Gamma_{(6\kappa_*^{-1} B_i)}(u) \\ &\leq C' s^\beta \int_{(6\kappa_*^{-1} B_i)} d\Gamma_{(6\kappa_*^{-1} B_i)}(u). \end{aligned} \tag{4.26}$$

Therefore, plugging (4.26), (4.25) into (4.24), we obtain

$$\|u - u_s\|_2^2 \leq \; C s^\beta \sum_{i=1}^{\infty} \int_{(6\kappa_*^{-1} B_i)} d\Gamma_{(6\kappa_*^{-1} B_i)}(u) \leq \; C s^\beta \sum_{i=1}^{\infty} \int_{(6\kappa_*^{-1} B_i)} d\Gamma(u).$$

On the other hand, by the doubling property (VD$_*$), there is an integer $N_0 \geq 1$, depending only on the constant in (VD$_*$), such that every point $x \in M$ is contained at most N_0 number of sets $6\kappa_*^{-1} B_j$. Then, we obtain

$$\|u - u_s\|_2^2 \leq \; C s^\beta \sum_{i=1}^{\infty} \int_{(6\kappa_*^{-1} B_i)} d\Gamma(u) = \; C s^\beta \int_M \sum_{i=1}^{\infty} \mathbf{1}_{(6\kappa_*^{-1} B_i)} d\Gamma(u) \leq N_0 C s^\beta \mathcal{E}(u),$$

thus showing (4.23).

Step 3. We show the inequality (4.20) in condition (Nash$_*$).

Indeed, we have by (4.22), (4.23) that for any $s \in (0, \frac{\kappa_*}{6} W(x_0, \overline{R})^{1/\beta})$ and any $u \in \mathcal{F}(B_*)$,

$$\|u\|_2^2 \leq 2\|u - u_s\|_2^2 + 2\|u_s\|_2^2 \leq C_1 \left(s^\beta \mathcal{E}(u,u) + \frac{\|u\|_1^2}{\inf_{x \in B_*} V_*(x,s)} \right).$$

On the other hand, when $\overline{R} < \infty$, we have for any $s \in [\frac{\kappa_*}{6} W(x_0, \overline{R})^{1/\beta}, \infty)$

$$\|u\|_2^2 \leq \frac{s^\beta}{(6^{-1}\kappa_* W(x_0, \overline{R})^{1/\beta})^\beta} \|u\|_2^2 = (6\kappa_*^{-1})^\beta s^\beta W(x_0, \overline{R})^{-1} \|u\|_2^2.$$

Adding up the above two inequalities and setting $C_2 := C_1 \vee (6\kappa_*^{-1})^\beta$, we have for all $s > 0$

$$C_2^{-1} \|u\|_2^2 \leq s^\beta \left(\mathcal{E}(u) + W(x_0, \overline{R})^{-1} \|u\|_2^2 \right) + \sup_{x \in B_*} \frac{\|u\|_1^2}{V_*(x,s)}$$

$$= s^{\beta}\left(\mathcal{E}(u) + W(x_0,\overline{R})^{-1}\|u\|_2^2\right) + \frac{\|u\|_1^2}{V_*(x_0,r)}\sup_{x\in B_*}\frac{V_*(x_0,r)}{V_*(x,s)}.$$

From this and (VD$_*$), we have for all $s<r$,

$$C_2^{-1}\|u\|_2^2 \le s^{\beta}\left(\mathcal{E}(u) + W(x_0,\overline{R})^{-1}\|u\|_2^2\right) + C\left(\frac{r}{s}\right)^{\alpha_*}\frac{\|u\|_1^2}{V_*(x_0,r)}. \tag{4.27}$$

Whilst for all $s\ge r$,

$$C_2^{-1}\|u\|_2^2 \le s^{\beta}\left(\mathcal{E}(u) + W(x_0,\overline{R})^{-1}\|u\|_2^2\right) + C\left(\frac{r}{s}\right)^{\alpha_*'}\frac{\|u\|_1^2}{V_*(x_0,r)} \tag{4.28}$$

since, by using (VD$_*$) and (RVD$_*$),

$$\sup_{x\in B_*}\frac{V_*(x_0,r)}{V_*(x,s)} = \sup_{x\in B_*}\frac{V_*(x_0,r)}{V_*(x,r)}\frac{V_*(x,r)}{V_*(x,s)} \le C\left(\frac{r}{s}\right)^{\alpha_*'}.$$

Define two functions $f_1, f_2 : \mathbb{R}_+ \mapsto \mathbb{R}_+$ by

$$f_1(s) := \frac{s^{\beta}}{\beta}\left(\mathcal{E}(u) + W(x_0,\overline{R})^{-1}\|u\|_2^2\right) + \frac{s^{-\alpha_*}}{\alpha_*}\left(\frac{r^{\alpha_*}\|u\|_1^2}{V_*(x_0,r)}\right), \quad s>0,$$

$$f_2(s) := \frac{s^{\beta}}{\beta}\left(\mathcal{E}(u) + W(x_0,\overline{R})^{-1}\|u\|_2^2\right) + \frac{s^{-\alpha_*'}}{\alpha_*'}\left(\frac{r^{\alpha_*'}\|u\|_1^2}{V_*(x_0,r)}\right), \quad s>0.$$

It follows from (4.27), (4.28) that

$$C^{-1}\|u\|_2^2 \le \begin{cases} f_1(s), & \text{if } s<r, \\ f_2(s), & \text{if } s\ge r, \end{cases} \tag{4.29}$$

where $C>0$ is independent of u, B_*, $W(x_0,\overline{R})^{1/\beta}$, s, but may depend on α_*, α_*', β and on constants in conditions (VD$_*$), (RVD$_*$).

We will minimize the right hand side of (4.29) over $s\in(0,\infty)$. Indeed, a direct computation shows that $f_1(s)$ attains its minimum over $s\in(0,\infty)$ at

$$s_1 := r(F(u,r))^{\frac{1}{\alpha_*+\beta}},$$

where

$$F(u,r) := \frac{\|u\|_1^2}{V_*(x_0,r)r^{\beta}\left(\mathcal{E}(u,u) + W(x_0,\overline{R})^{-1}\|u\|_2^2\right)}.$$

Similarly, the function $f_2(s)$ attains its minimum over $s\in(0,\infty)$ at

$$s_2 := r(F(u,r))^{\frac{1}{\alpha_*'+\beta}}.$$

We distinguish two cases whether $F(u,r)<1$ or not.

Case 1 when $F(u,r)<1$. In this case,

$$s_1 = r(F(u,r))^{\frac{1}{\alpha_*+\beta}} < r.$$

Thus, applying (4.29) with $s=s_1$, we obtain

$$C^{-1}\|u\|_2^2 \le f_1(s_1) = C(\alpha_*,\beta)\left(\mathcal{E}(u) + W(x_0,\overline{R})^{-1}\|u\|_2^2\right)^{\frac{\alpha_*}{\alpha_*+\beta}}\left(\frac{r^{\alpha_*}\|u\|_1^2}{V_*(x_0,r)}\right)^{\frac{\beta}{\alpha_*+\beta}}$$

for a positive constant $C(\alpha_*,\beta)$ depending only on α_*,β. Raising to the power $1+\frac{\beta}{\alpha_*}$ on the both sides and then rearranging the terms, we obtain that

$$\|u\|_2^2\left(\frac{\|u\|_2^2 V_*(x_0,r)}{\|u\|_1^2}\right)^{\frac{\beta}{\alpha_*}} \le Cr^{\beta}\left(\mathcal{E}(u) + W(x_0,\overline{R})^{-1}\|u\|_2^2\right), \tag{4.30}$$

where C is a positive constant independent of u, B_*, $W(x_0,\overline{R})^{1/\beta}$.

Case 2 when $F(u,r) \geq 1$. In this case,

$$s_2 = r(F(u,r))^{\frac{1}{\alpha_*+\beta}} \geq r.$$

Thus, applying (4.29) with $s = s_2$, we obtain

$$\|u\|_2^2 \left(\frac{\|u\|_2^2 V_*(x_0,r)}{\|u\|_1^2} \right)^{\frac{\beta}{\alpha'_*}} \leq Cr^\beta \left(\mathcal{E}(u) + W(x_0,\overline{R})^{-1}\|u\|_2^2 \right) \tag{4.31}$$

for a positive constant C independent of u, B_*, $W(x_0,\overline{R})^{1/\beta}$.

On the other hand, since $u \in \mathcal{F}(B_*)$, we have by the Hölder inequality,

$$\|u\|_1^2 \leq \mu(B_*)\|u\|_2^2 = V_*(x_0,r)\|u\|_2^2.$$

That is,

$$\frac{\|u\|_2^2 V_*(x_0,r)}{\|u\|_1^2} \geq 1.$$

Hence, it follows from (4.30), (4.31) that

$$\|u\|_2^2 \left(\frac{\|u\|_2^2 V_*(x_0,r)}{\|u\|_1^2} \right)^{\nu} \leq Cr^\beta \left(\mathcal{E}(u) + W(x_0,\overline{R})^{-1}\|u\|_2^2 \right),$$

where $\nu := \frac{\beta}{\alpha_*} \wedge \frac{\beta}{\alpha'_*} = \frac{\beta}{\alpha_*}$ since $\alpha'_* \leq \alpha_*$, thus showing that

$$\|u\|_2^{2(1+\nu)} \leq \frac{Cr^\beta}{V_*(x_0,r)^\nu} \left(\mathcal{E}(u) + W(x_0,\overline{R})^{-1}\|u\|_2^2 \right) \|u\|_1^{2\nu}$$

for some $C > 0$ independent of $W(x_0,\overline{R})^{1/\beta}$. This proves that condition (Nash$_*$) holds. □

Let $\mathcal{L}^U$ be the generator of the Dirichlet form $(\mathcal{E},\mathcal{F}(U))$. Denote by $\lambda_1(U)$ the *bottom* of the spectrum of $\mathcal{L}^U$ in $L^2(U)$. It is known that

$$\lambda_1(U) = \inf_{u\in\mathcal{F}(U)\setminus\{0\}} \frac{\mathcal{E}(u)}{\|u\|_2^2}.$$

Definition 4.10. We say the *Faber-Krahn inequality* (FK$_*$) holds for $(\mathcal{E},\mathcal{F})$ if there exist three positive constants C,ν,σ_* with $\sigma_* \leq 1$ such that, for all $B_* := B_*(x,r)$ with $r \in (0,\sigma_* W(x_0,\overline{R}))$ and for all non-empty open subsets U_* of B_*

$$\lambda_1(U_*) \geq \frac{C^{-1}}{r^\beta} \left(\frac{\mu(B_*)}{\mu(U_*)} \right)^\nu. \tag{4.32}$$

The following derives the Faber-Krahn inequality from the Nash inequality, under the metric d_*.

Lemma 4.11. *Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form in L^2, and d_* be the new metric defined in Proposition 4.1. Assume that every ball has finite measure under the metric d_*. Then*

$$(\text{Nash}_*) \Rightarrow (\text{FK}_*).$$

Consequently, we have

$$(\text{VD}_*) + (\text{RVD}_*) + (\text{PI}_*) \Rightarrow (\text{FK}_*) \tag{4.33}$$

Proof. Let $\sigma_* \in (0,1/2)$ be a number to be determined later on. Fix a ball $B_* := B(x_0,r)$ of radius $r \in (0,\sigma_* W(x_0,\overline{R})^{1/\beta})$. Let $U_* \subset B_*$ be open. For any $u \in \mathcal{F}(U_*) \setminus \{0\}$, since

$$\|u\|_1^2 \leq \mu(U_*)\|u\|_2^2,$$

we have by condition (Nash$_*$) that

$$\|u\|_2^{2(1+\nu)} \leq \frac{Cr^\beta}{V_*(x_0,r)^\nu} \left(\mathcal{E}(u,u) + W(x_0,\overline{R})^{-1}\|u\|_2^2 \right) \|u\|_1^{2\nu}$$

$$\leq Cr^{\beta}\left(\mathcal{E}(u,u)+W(x_0,\overline{R})^{-1}\|u\|_2^2\right)\left(\frac{\mu(U_*)\|u\|_2^2}{\mu(B_*)}\right)^{\nu}$$
$$\leq C\left(r^{\beta}\mathcal{E}(u,u)+\sigma_*^{\beta}\|u\|_2^2\right)\left(\frac{\mu(U_*)}{\mu(B_*)}\right)^{\nu}\|u\|_2^{2\nu}.$$

Since $\mu(U_*)\leq\mu(B_*)$, it follows that

$$\|u\|_2^2\leq Cr^{\beta}\mathcal{E}(u,u)\left(\frac{\mu(U_*)}{\mu(B_*)}\right)^{\nu}+C\sigma_*^{\beta}\|u\|_2^2. \tag{4.34}$$

Taking $\sigma_*\in(0,1/2)$ to be so small that

$$C\sigma_*^{\beta}\leq\frac{1}{2},$$

we obtain from (4.34) that

$$\frac{1}{2}\|u\|_2^2\leq Cr^{\beta}\mathcal{E}(u,u)\left(\frac{\mu(U_*)}{\mu(B_*)}\right)^{\nu},$$

from which,

$$\lambda_1(U_*)=\inf_{u\in\mathcal{F}(U_*)\setminus\{0\}}\frac{\mathcal{E}(u,u)}{\|u\|_2^2}\geq\frac{1}{2Cr^{\beta}}\left(\frac{\mu(B_*)}{\mu(U_*)}\right)^{\nu},$$

thus showing that condition (FK$_*$) holds. □

Let us rephrase (FK$_*$) under the original metric d, and denote it by (FK).

Definition 4.12 (Faber-Krahn inequality)**.** We say that condition (FK) holds if there exist three numbers $\sigma\in(0,1]$ and $C,\nu>0$ such that, for all balls B with radius less than $\sigma\overline{R}$ and all non-empty open subsets U of B,

$$\lambda_1(U)\geq\frac{C^{-1}}{W(B)}\left(\frac{\mu(B)}{\mu(U)}\right)^{\nu}. \tag{4.35}$$

Proposition 4.13. *Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form in L^2, and d_* be the new metric defined in Proposition 4.1. Assume that condition* (VD) *holds. Then*

$$\text{(FK)}\quad\Leftrightarrow\quad\text{(FK}_*\text{)}. \tag{4.36}$$

Proof. Assume that condition (FK) holds. Fix a ball $B_*:=B_*(x,r)$ with $0<r<\sigma_*W(x,\overline{R})^{1/\beta}$, where σ_* is a positive constant to be determined later. Let U_* be an open subset of B_*. Note that U_* is also open under the metric d by using Proposition 4.2. Set

$$R:=F^{-1}(x,L^{-1}L_0r)$$

so that

$$W(x,R)=F(x,R)^{\beta}=(L^{-1}L_0r)^{\beta}<(L^{-1}L_0)^{\beta}\sigma_*^{\beta}W(x,\overline{R}).$$

By the right inequality in (2.8), one can choose σ_* to be so small that $R<\sigma\overline{R}$, where σ is the constant from condition (FK).

Using (4.6) with r replaced by L_0r, we have

$$B_*=B_*(x,r)\subset B(x,R)=:B,$$

from which, using condition (FK), it follows that

$$\lambda_1(U_*)\geq\frac{C^{-1}}{W(x,R)}\left(\frac{\mu(B)}{\mu(U_*)}\right)^{\nu}\geq\frac{C^{-1}}{W(x,R)}\left(\frac{\mu(B_*)}{\mu(U_*)}\right)^{\nu}=\frac{C^{-1}}{(L^{-1}L_0r)^{\beta}}\left(\frac{\mu(B_*)}{\mu(U_*)}\right)^{\nu},$$

thus showing that condition (FK$_*$) holds.

Finally, we show the converse implication (FK$_*$) $\Rightarrow$ (FK). Indeed, assume that condition (FK$_*$) holds. Fix a ball $B:=B(x,R)$ with $R<\sigma\overline{R}$, where $\sigma>0$ will be picked up later. Let U be an open subset of B, which is also open under the metric d_*. Set

$$r:=L^{-1}F(x,L_0R)$$

so that

$$W(x, L_0R) = F(x, L_0R)^\beta = (Lr)^\beta.$$

Using (4.7) with R replaced by L_0R, we have

$$B = B(x, R) \subset B_*(x, r) =: B_*. \tag{4.37}$$

By the left inequality in (2.8), one can choose σ to be so small that

$$r = L^{-1}W(x, L_0R)^{1/\beta} \leq L^{-1}W(x, L_0\sigma\overline{R})^{1/\beta} \leq \sigma_* W(x, \overline{R})^{1/\beta}.$$

Since $U \subset B \subset B_*$, it follows from condition (FK_*) that, using (4.37) and the right inequality in (2.8),

$$\begin{aligned}\lambda_1(U) &\geq \frac{C^{-1}}{r^\beta}\left(\frac{\mu(B_*)}{\mu(U)}\right)^\nu \geq \frac{C^{-1}}{r^\beta}\left(\frac{\mu(B)}{\mu(U)}\right)^\nu \\ &= \frac{C^{-1}}{L^{-\beta}W(x, L_0R)}\left(\frac{\mu(B)}{\mu(U)}\right)^\nu \geq \frac{C'}{W(x, R)}\left(\frac{\mu(B)}{\mu(U)}\right)^\nu,\end{aligned}$$

thus showing that condition (FK) holds. □

We introduce condition ($\mathrm{Cap}^*_\leq$).

Definition 4.14 (Condition ($\mathrm{Cap}^*_\leq$))**.** We say that condition ($\mathrm{Cap}^*_\leq$) is satisfied if there exists a positive constant C such that for all balls $B_*(x_0, r)$ with $r < W(x_0, \overline{R})^{1/\beta}$

$$\mathrm{cap}(\frac{1}{2}B_*, B_*) \leq C\frac{\mu(B_*)}{r^\beta}. \tag{4.38}$$

Lemma 4.15. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2, and d_* be the new metric defined in Proposition 4.1. Then*

$$(\mathrm{VD}) + (\mathrm{Cap}_\leq) \Rightarrow (\mathrm{Cap}_\leq{}^*).$$

Consequently, we have

$$(\mathrm{VD}) + (\mathrm{Gcap}) \Rightarrow (\mathrm{VD}) + (\mathrm{Cap}_\leq{}^*) \Rightarrow (\mathrm{Cap}^*_\leq).$$

Proof. Fix $B_* := B_*(x_0, r)$ with $r < W(x_0, \overline{R})^{1/\beta}$. Let C_W be the constant in (2.8), $L_0 > 1$ be the constant as in (4.6) and $\overline{C}$ be the constant in Proposition 4.3 with $\eta = 1$.

Step 1. We show that for any $z \in \frac{1}{2}B_*$,

$$\mathrm{cap}(B_*(z, ar), B_*(z, br)) \leq \frac{c\mu(B_*(z, br))}{r^\beta}, \tag{4.39}$$

where c is some positive constant independent of z, B_*, and a, b are given by

$$a := (2^{\beta_2/\beta+1}L_0C_W^{2/\beta}\overline{C}^{1/\beta})^{-1} \quad \text{and} \quad b = (2C_W^{1/\beta}\overline{C}^{1/\beta})^{-1}$$

so that $a < L_0^{-1}b < b < \frac{1}{2}$.

Indeed, let

$$R_1 := F^{-1}(z, L^{-1}L_0ar) \quad \text{and} \quad R_2 := F^{-1}(z, L^{-1}br)$$

so that $R_1 < R_2$ and

$$W(z, R_1) = F(z, R_1)^\beta = (L^{-1}L_0ar)^\beta \quad \text{and} \quad W(z, R_2) = F(z, R_2)^\beta = (L^{-1}br)^\beta.$$

Moreover, by (4.8) and the fact that $d_*(x_0, z) < \frac{1}{2}W(x_0, \overline{R})^{1/\beta}$, we have

$$W(z, R_2) = \frac{r^\beta}{(2L)^\beta C_W\overline{C}} < \frac{W(x_0, \overline{R})}{(2L)^\beta C_W\overline{C}} \leq \frac{\overline{C}W(z, \overline{R})}{(2L)^\beta C_W\overline{C}} \leq W(z, \overline{R}),$$

which shows that

$$R_2 < \overline{R}$$

By (2.8), we have

$$C_W\left(\frac{R_2}{R_1}\right)^{\beta_2} \geq \frac{W(z,R_2)}{W(z,R_1)} = \frac{(L^{-1}br)^{\beta}}{(L^{-1}L_0ar)^{\beta}} = 2^{\beta_2}C_W,$$

which implies that

$$2R_1 \leq R_2.$$

Also note that by (4.6),

$$B_*(z,ar) \subset B(z,R_1) \subset B(z,R_2) \subset B_*(z,br).$$

It follows from condition ($\mathrm{Cap}_{\leq}$) that

$$\begin{aligned}\mathrm{cap}(B_*(z,ar),B_*(z,br)) &\leq \mathrm{cap}(B(z,R_1),B(z,R_2)) \leq \mathrm{cap}(B(z,R_1),B(z,2R_1))\\ &\leq \frac{C\mu(B(z,2R_1))}{W(z,2R_1)} \leq \frac{C\mu(B_*(z,br))}{W(z,R_1)} = \frac{C\mu(B_*(z,br))}{(L^{-1}L_0ar)^{\beta}},\end{aligned}$$

thus showing (4.39).

Step 2. We show condition ($\mathrm{Cap}^*_{\leq}$).

Indeed, using condition (VD_*), there exist a finite number N of balls $\{B_*(z_i,ar)\}_{i=1}^N$ covering $\frac{1}{2}B_*$, where each center z_i lies in $\frac{1}{2}B_*$ and $\{B_*(z_i,ar/5)\}_{i=1}^N$ are disjoint, for some integer N independent of B_*. Since each ball $B_*(z_i,br)$ is contained in B_*, using the subadditivity and monotonicity of capacity, it follows from (4.39) that

$$\begin{aligned}\mathrm{cap}((1/2)B_*,B_*) &\leq \sum_{i=1}^{N}\mathrm{cap}(B(z_i,ar),B_*) \leq \sum_{i=1}^{N}\mathrm{cap}(B(z_i,ar),B_*(z,br))\\ &\leq \sum_{i=1}^{n}\frac{c\mu(B_*(z_i,br))}{r^{\beta}} \leq Nc\frac{\mu(B_*)}{r^{\beta}},\end{aligned}$$

thus proving (4.38), and so condition ($\mathrm{Cap}^*_{\leq}$) follows. □

Remark 4.16.

(1) Under (VD), conditions (TJ_*), (PI_*), ($\mathrm{Cap}^*_{\leq}$) follow from conditions (TJ), (PI), ($\mathrm{Cap}_{\leq}$) respectively. We emphasize that the number β appearing in (TJ_*), (PI_*), (Nash_*), (FK_*), ($\mathrm{Cap}^*_{\leq}$) keeps the same, which comes from (4.3) under the new metric d_*.

(2) By Propositions 4.4 and 4.7, Lemmas 4.9 and 4.11, and Proposition 4.13, we see that under the conditions (VD) and (RVD), condition (PI) implies (FK). Hence, most of results involving (FK) in [12], [13], [14] also hold true if (FK) is replaced by (PI).

5. Weak Harnack inequality and Oscillation inequality

In this section, we will derive the weak Harnack inequality and the oscillation inequality. The oscillation inequality gives rise to the local Hölder continuity of harmonic function, which is used to derive the lower bound of the heat kernel.

Definition 5.1. Let Ω be an open subset of M. We say that a function $u \in \mathcal{F}'$ is *subharmonic* (resp. *superharmonic*) in Ω if

$$\mathcal{E}(u,\varphi) \leq 0 \quad (\text{resp. } \mathcal{E}(u,\varphi) \geq 0) \tag{5.1}$$

for any $0 \leq \varphi \in \mathcal{F}(\Omega)$. A function $u \in \mathcal{F}'$ is called *harmonic* in Ω if it is both subharmonic and superharmonic in Ω.

For any function $v \in \mathcal{F}$ and any two subsets $U \subset V$ of M, define the *tail* of v outside V by

$$T_{U,V}(v) := \underset{x\in U}{\mathrm{esup}}\int_{V^c}|v(y)|J(x,dy). \tag{5.2}$$

Note that, since $v \in \mathcal{F}$ is quasi-continuous, the integral in (5.2) is well defined.

5.1. **Lemma of growth under new metric.** We introduce condition (LG$_*$) that is called a *lemma of growth.*

Definition 5.2 (Lemma of growth under new metric)**.** We say that condition (LG$_*$) holds if there exist three constants $\varepsilon_0 > 0, \sigma_* \in (0,1), \nu > 0$ such that the following is true: if a function $u \in \mathcal{F}' \cap L^\infty$ is superharmonic, non-negative in $2B_*$, where $B_* := B_*(x_0, r)$ with radius $r \in (0, 2^{-1}\sigma_* W(x_0, \overline{R})^{1/\beta})$, and if

$$\frac{\mu(B_* \cap \{u < a\})}{\mu(B_*)} \leq \varepsilon_0 \left(1 + \frac{r^\beta T_{B_*,2B_*}(u_-)}{a}\right)^{-1/\nu}, \tag{5.3}$$

for some $a > 0$, where tail function $T_{B_*,2B_*}(u_-)$ is defined by (5.2), then

$$\operatorname*{einf}_{\frac{1}{2}B_*} u \geq \frac{a}{2} \tag{5.4}$$

(see Figure 3).

We remark that the three constants $\varepsilon_0, \nu, \sigma$ appearing in (5.3) are all independent of $B_*, u, \overline{R}_*$.

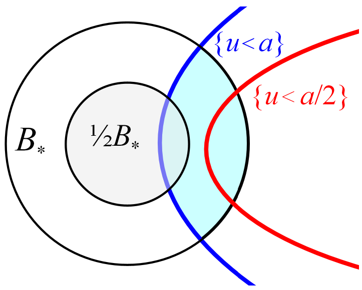

FIGURE 3. Illustration to condition (LG$_*$)

Before proving (LG$_*$), we need more conditions, say, (ABB$_*$), (EP$_*$). We first introduce condition (ABB$^*_\zeta$) for a number $\zeta \geq 0$.

Definition 5.3. Given a number $\zeta \geq 0$, we say that condition (ABB$^*_\zeta$) is satisfied if for any $u \in \mathcal{F}' \cap L^\infty$ and any three concentric balls $B_0^* := B_*(x_0, R)$, $B_* := B_*(x_0, R + r)$ and $\Omega := B_*(x_0, R')$ with $0 < R < R + r < R' < W(x_0, \overline{R})^{1/\beta}$, there exists some $\phi \in \text{cutoff}(B_0^*, B_*)$ such that

$$\int_\Omega u^2 d\Gamma_\Omega(\phi) \leq \zeta \int_{B_*} \phi^2 d\Gamma_{B_*}(u) + \frac{C}{r^\beta}\left(\frac{R'}{r}\right)^{C_1} \int_\Omega u^2 d\mu, \tag{5.5}$$

where $C > 0, C_1 \geq 0$ are two constants independent of function u and three balls B_0^*, B_*, Ω, and $d\Gamma_\Omega$, $d\Gamma_{B_*}$ are defined by (3.5). We say that condition (ABB$_*$) holds if condition (ABB$^*_\zeta$) holds for some $\zeta \geq 0$ and $C_1 \geq 0$.

We mention that the value of the exponent C_1 in (5.5) is unimportant.

The following result is an analogue of [12, Lemma 6.2], which derives condition (ABB$_*$).

Lemma 5.4. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2, and d_* be the new metric defined in Proposition in 4.1. Then*

$$(\text{VD}_*) + (\text{Gcap}) + (\text{TJ}_*) \Rightarrow (\text{ABB}_*).$$

Proof. Let $C_W, L_0 \geq 1$ be the constants in (2.8) and (4.6) respectively. Fix three concentric balls $B_0^* := B_*(x_0, R)$, $B_* := B_*(x_0, R + r)$ and $\Omega := B_*(x_0, R')$ with $0 < R < R + r < R' < W(x_0, \overline{R})^{1/\beta}$.

We divide the proof into two steps.

Step 1. We will show that for any $u \in \mathcal{F}' \cap L^\infty$ and any two concentric balls $B_*(z, ar) \subset B_*(z, r/2)$ with $z \in B_0^* = B_*(x_0, R)$ and with

$$a := (2^{\beta_2/\beta+1} L_0 C_W^{1/\beta})^{-1} \in (0, 1/2), \tag{5.6}$$

there exists some $\phi \in \text{cutoff}(B_*(z, ar), B_*(z, r/2))$ such that

$$\int_\Omega u^2 d\Gamma_\Omega(\phi) \leq 4\overline{\kappa} \int_{B_*(z,r)} \phi^2 d\Gamma_{B_*(z,r)}(u) + \frac{C}{r^\beta} \int_\Omega u^2 d\mu, \tag{5.7}$$

where $\overline{\kappa}$ is the constant in condition (Gcap) and $C > 0$ is independent of u, B_0^*, B_*, Ω and the point z.

Indeed, let

$$R_1 := F^{-1}(z, L^{-1} L_0 ar) \ \text{ and } \ R_2 := F^{-1}(z, L^{-1} r/2)$$

so that, using definition (4.4),

$$W(z, R_1) = F(z, R_1)^\beta = (L^{-1} L_0 ar)^\beta \ \text{ and } \ W(z, R_2) = F(z, R_2)^\beta = (L^{-1} r/2)^\beta. \tag{5.8}$$

With the choice of a in (5.6), we have

$$2R_1 \leq R_2, \tag{5.9}$$

since, by (2.8),

$$C_W \left(\frac{R_2}{R_1}\right)^{\beta_2} \geq \frac{W(z, R_2)}{W(z, R_1)} = \frac{(L^{-1} r/2)^\beta}{(L^{-1} L_0 ar)^\beta} = 2^{\beta_2} C_W.$$

We may assume that $r < C_0^{-1} W(x_0, \overline{R})^{1/\beta}$ for some sufficiently large constant $C_0 \geq 1$ so that

$$R_2 < \overline{R}, \tag{5.10}$$

otherwise, one can replace r by $C_0^{-1} r$, which is less than $C_0^{-1} W(x_0, \overline{R})^{1/\beta}$, and then runs the same argument in the sequel. In fact, the value of C_0 can be chosen in the following way: by (4.8)

$$\begin{aligned} W(z, R_2) &= (L^{-1} r/2)^\beta < (L^{-1}/2)^\beta \left(C_0^{-1} W(x_0, \overline{R})^{1/\beta}\right)^\beta \\ &= (L^{-1}/2)^\beta (C_0^{-1})^\beta C W(z, \overline{R}) < W(z, \overline{R}) \end{aligned}$$

provided C_0 is chosen such that $(2LC_0)^{-\beta} C < 1$. With this choice of C_0, we see that $R_2 < \overline{R}$ by using the monotonicity of the function $W(z, \cdot)$.

Note that by (4.6),

$$B_*(z, ar) \subset B(z, R_1) \subset B(z, R_2) \subset B_*(z, r/2). \tag{5.11}$$

By condition (Gcap), there exists a function g such that

$$g \in \overline{\kappa}\text{- cutoff}(B(z, R_1), B(z, R_2)) \subset \overline{\kappa}\text{- cutoff}(B_*(z, ar), B_*(z, r/2)) \tag{5.12}$$

and that

$$\begin{aligned} \mathcal{E}(u^2 g, g) &\leq \sup_{x \in B(z,R_2)} \frac{C}{W(x, R_2 - R_1)} \int_{B(z,R_2)} u^2 d\mu \\ &\leq \sup_{x \in B(z,R_2)} \frac{C}{W(x, R_1)} \int_{B(z,R_2)} u^2 d\mu \quad \text{(using (5.9))} \\ &= \sup_{x \in B(z,R_2)} \frac{W(z, R_1)}{W(x, R_1)} \frac{C}{W(z, R_1)} \int_{B(z,R_2)} u^2 d\mu \\ &\leq \frac{C}{(L^{-1} L_0 ar)^\beta} \int_{B_*(z,r/2)} u^2 d\mu \qquad \text{(using (2.8), (5.8) and (5.11)).} \end{aligned}$$

From this and applying [12, Eq. (5.9)] with Ω being replaced by $B_*(z,r)$, we obtain

$$\begin{aligned}\int_{B_*(z,r)} u^2 d\Gamma_{B_*(z,r)}(g) &\le 4\int_{B_*(z,r)} g^2 d\Gamma_{B_*(z,r)}(u) + 2\mathcal{E}(u^2 g, g)\\ &\le 4\int_{B_*(z,r)} g^2 d\Gamma_{B_*(z,r)}(u) + \frac{C}{r^\beta}\int_{B_*(z,r/2)} u^2 d\mu. \end{aligned}\tag{5.13}$$

On the other hand, since the function g is supported in $B_*(z,r/2) \subset B_*(z,r) \subset \Omega = B_*(x_0,R')$, we have

$$\begin{aligned}\iint_{\Omega\times\Omega} u^2(x)(g(x)-g(y))^2 dj &= \Bigg(\iint_{B_*(z,r)\times B_*(z,r)} + \iint_{B_*(z,r)\times(\Omega\setminus B_*(z,r))} + \iint_{(\Omega\setminus B_*(z,r))\times B_*(z,r)}\\ &\quad + \iint_{(\Omega\setminus B_*(z,r))\times(\Omega\setminus B_*(z,r))}\Bigg)\cdots\\ &= \iint_{B_*(z,r)\times B_*(z,r)} u^2(x)(g(x)-g(y))^2 dj\\ &\quad + \iint_{B_*(z,\frac{r}{2})\times(\Omega\setminus B_*(z,r))} u^2(x)g^2(x)dj\\ &\quad + \iint_{(\Omega\setminus B_*(z,r))\times B_*(z,\frac{r}{2})} u^2(x)g^2(y)dj. \end{aligned}\tag{5.14}$$

We estimate the last two integrals in (5.14). Indeed, for any

$$(x,y)\in B_*(z,\frac{r}{2})\times(\Omega\setminus B_*(z,r)),$$

we have $d_*(x,y)\ge \frac{r}{2}$. It follows from (TJ$_*$) that

$$\begin{aligned}\iint_{B_*(z,r/2)\times(\Omega\setminus B_*(z,r))} u^2(x)g^2(x)dj &\le \int_{B_*(z,r/2)} u^2(x)d\mu(x) \operatorname*{esup}_{x\in B_*(z,r/2)}\int_{B_*(x,r/2)^c} J(x,dy)\\ &\le \frac{c}{r^\beta}\int_{B_*(z,r/2)} u^2(x)d\mu(x).\end{aligned}$$

Similarly, for any $(x,y)\in(\Omega\setminus B_*(z,r))\times B_*(z,\frac{r}{2})$, we see that $d_*(x,y)\ge\frac{r}{2}$. Thus by (TJ$_*$),

$$\begin{aligned}\iint_{(\Omega\setminus B_*(z,r))\times B_*(z,\frac{r}{2})} u^2(x)g^2(y)dj &\le \int_{\Omega\setminus B_*(z,r)} u^2(x)d\mu(x)\operatorname*{esup}_{x\in\Omega\setminus B_*(z,r)}\int_{B_*(x,r/2)^c} g^2(y)J(x,dy)\\ &\le \int_{\Omega\setminus B_*(z,r)} u^2(x)d\mu(x)\operatorname*{esup}_{x\in\Omega\setminus B_*(z,r)}\int_{B_*(x,r/2)^c} J(x,dy)\\ &\le \frac{c}{r^\beta}\int_{\Omega\setminus B_*(z,r)} u^2(x)d\mu(x).\end{aligned}$$

Plugging the above two inequalities into (5.14), we have

$$\iint_{\Omega\times\Omega} u^2(x)(g(x)-g(y))^2 dj \le \iint_{B_*(z,r)\times B_*(z,r)} u^2(x)(g(x)-g(y))^2dj + \frac{c}{r^\beta}\int_\Omega u^2 d\mu.\tag{5.15}$$

Therefore, using the fact that $\mathrm{supp}(g)\subset B_*(z,\frac{r}{2})$, we have by (5.15) that

$$\begin{aligned}\int_\Omega u^2 d\Gamma_\Omega(g) &= \int_\Omega u^2 d\Gamma^{(L)}(g) + \iint_{\Omega\times\Omega} u^2(x)(g(x)-g(y))^2 dj\\ &= \int_{B_*(z,r)} u^2 d\Gamma^{(L)}(g) + \iint_{\Omega\times\Omega} u^2(x)(g(x)-g(y))^2 dj\\ &\le \int_{B_*(z,r)} u^2 d\Gamma^{(L)}(g) + \iint_{B_*(z,r)\times B_*(z,r)} u^2(x)(g(x)-g(y))^2 dj + \frac{c}{r^\beta}\int_\Omega u^2 d\mu\end{aligned}$$

$$
\begin{aligned}
&= \int_{B_*(z,r)} u^2 d\Gamma_{B_*(z,r)}(g) + \frac{c}{r^\beta}\int_\Omega u^2 d\mu \\
&\leq 4\int_{B_*(z,r)} g^2 d\Gamma_{B_*(z,r)}(u) + \frac{C}{r^\beta}\int_{B_*(z,r/2)} u^2 d\mu + \frac{c}{r^\beta}\int_\Omega u^2 d\mu \quad \text{(by (5.13))} \\
&\leq 4\int_{B_*(z,r)} g^2 d\Gamma_{B_*(z,r)}(u) + \frac{c'}{r^\beta}\int_\Omega u^2 d\mu. \qquad (5.16)
\end{aligned}
$$

By (5.11) and (5.12), we have

$$\phi := g\wedge 1 \in \text{cutoff}(B(z,R_1), B(z,R_2)) \subset \text{cutoff}(B_*(z,ar), B_*(z,r/2)).$$

Since $g \leq \overline{\kappa}\phi$ in M, we obtain by (3.7) and (5.16) that

$$
\begin{aligned}
\int_\Omega u^2 d\Gamma_\Omega(\phi) \leq \int_\Omega u^2 d\Gamma_\Omega(g) &\leq 4\int_{B_*(z,r)} g^2 d\Gamma_{B_*(z,r)}(u) + \frac{c'}{r^\beta}\int_\Omega u^2 d\mu \\
&\leq 4\overline{\kappa}\int_{B_*(z,r)} \phi^2 d\Gamma_{B_*(z,r)}(u) + \frac{c'}{r^\beta}\int_\Omega u^2 d\mu,
\end{aligned}
$$

thus showing (5.7), as desired.

Step 2. We show condition (ABB$_*$).

Indeed, by condition (VD$_*$), there exist a finite number of balls $\{B_*(z_i, ar)\}_{i=1}^N$ with $z_i \in B_0^*$ such that $\{B_*(z_i, ar)\}_{i=1}^N$ cover $B_0^* = B_*(x_0, R)$ and $\{B_*(z_i, ar/5)\}_{i=1}^N$ are disjoint, where

$$N \leq c\left(\frac{R+r}{r}\right)^{\alpha_*}. \qquad (5.17)$$

By (5.7), for each $1 \leq i \leq N$, there exists some $\phi_i \in \text{cutoff}(B_*(z_i, ar), B_*(z_i, r/2))$ such that

$$\int_\Omega u^2 d\Gamma_\Omega(\phi_i) \leq 4\overline{\kappa}\int_{B_*(z_i,r)} \phi_i^2 d\Gamma_{B_*(z_i,r)}(u) + \frac{c}{r^\beta}\int_\Omega u^2 d\mu. \qquad (5.18)$$

Define

$$\phi := \max\{\phi_1, \phi_2, \cdots, \phi_N\}.$$

Since $\{B_*(z_i, ar)\}_{i=1}^N$ cover B_0^* and

$$\bigcup_{i=1}^N B_*(z_i, r/2) \subset B_*(x_0, R + r/2),$$

the function ϕ belongs to cutoff(B_0^*, B_*). On the other hand, for any $f, g \in \mathcal{F}'$,

$$d\Gamma_\Omega(f\vee g) + d\Gamma_\Omega(f\wedge g) \leq d\Gamma_\Omega(f) + d\Gamma_\Omega(g).$$

It follows from (5.18) that, using (5.17) and (3.6),

$$
\begin{aligned}
\int_\Omega u^2 d\Gamma_\Omega(\phi) &\leq \sum_{i=1}^N \int_\Omega u^2 d\Gamma_\Omega(\phi_i) \\
&\leq 4\overline{\kappa}\sum_{i=1}^N \int_{B_*(z_i,r)} \phi_i^2 d\Gamma_{B_*(z_i,r)}(u) + \sum_{i=1}^N \frac{c}{r^\beta}\int_\Omega u^2 d\mu \\
&\leq 4\overline{\kappa}\sum_{i=1}^N \int_{B_*(z_i,r)} \phi^2 d\Gamma_{B_*}(u) + \frac{c'}{r^\beta}\left(\frac{R+r}{r}\right)^{\alpha_*}\int_\Omega u^2 d\mu \\
&\leq 4\overline{\kappa}\int_{B_*}\left(\sum_{i=1}^N \mathbf{1}_{B_*(z_i,R+r)}\right)\phi^2 d\Gamma_{B_*}(u) + \frac{c'}{r^\beta}\left(\frac{R'}{r}\right)^{\alpha_*}\int_\Omega u^2 d\mu.
\end{aligned}
$$

By condition (VD$_*$), there exists an integer $N_0 \geq 1$, independent of balls B_0^*, B_*, Ω, such that each point y in B_* belongs to at most N_0 balls like $B_*(z_i, R+r)$. Therefore, we conclude from above that

$$\int_\Omega u^2 d\Gamma_\Omega(\phi) \leq \ 4\overline{\kappa} N_0 \int_{B_*} \phi^2 d\Gamma_{B_*}(u) + \frac{c'}{r^\beta}\left(\frac{R'}{r}\right)^{\alpha_*} \int_\Omega u^2 d\mu,$$

thus proving (ABB$_*$). □

The following condition (EP$_*$) is the counterpart of condition (EP) in [12] under the new metric d_*.

Definition 5.5 (Condition (EP$_*$))**.** We say that the condition (EP$_*$) is satisfied if there exist two universal constants $C > 0, C_1 \geq 0$ such that, for any three concentric balls $B_0^* := B_*(x_0, R)$, $B_* := B_*(x_0, R+r)$ and $\Omega := B_*(x_0, R')$ with $0 < R < R+r < R' < W(x_0, \overline{R})^{1/\beta}$, and for any $u \in \mathcal{F}' \cap L^\infty$, there exists some $\phi \in \text{cutoff}(B_0^*, B_*)$ such that

$$\mathcal{E}(u\phi, u\phi) \leq \frac{3}{2}\mathcal{E}(u, u\phi^2) + \frac{C}{r^\beta}\left(\frac{R'}{r}\right)^{C_1} \int_\Omega u^2 d\mu + 3\iint_{\Omega\times\Omega^c} u(x)u(y)\phi^2(x) dj.$$

The following is a parallel version of [12, Eq. (8.3)] under the new metric d_*.

Proposition 5.6. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2 without killing part. Assume that every metric ball has finite measure. Then*

$$(\text{TJ}_*) + (\text{ABB}_*) \Rightarrow (\text{EP}_*). \tag{5.19}$$

Consequently,

$$(\text{VD}) + (\text{Gcap}) + (\text{TJ}) \Rightarrow (\text{EP}_*). \tag{5.20}$$

Proof. Note that conditions (TJ$_*$) and (ABB$_*$) are analogous to conditions (TJ) and (ABB) in [12] respectively under the new metric d_* (although there is a ratio term $\left(\frac{R'}{r}\right)^{C_1}$ in (ABB$_*$) and there is no such term in (ABB) in [12]). Hence, one can follow the same arguments in the proof of [12, Eq. (8.3)] to obtain the first implication (5.19).

The second implication (5.20) follows from Proposition 4.4(1), (4.12), Lemma 5.4 and (5.19). □

The following lemma plays an important in the proof of Lemma 5.8. For simplicity of notation, fix some $x_0 \in M$ and set for any $r > 0$

$$B_r := B_*(x_0, r).$$

Let us also recall the notion of "*quasi-everywhere*" (see the last paragraph in [7, p. 68]). Let E be a subset of M. A statement depending on $x \in E$ is said to hold q.e. on E if there exists a set $N \subset E$ with $\text{Cap}_1(N) = 0$ (see (2.25) for the definition of Cap_1) such that the statement is true for every $x \in E \setminus N$. "q.e." is an abbreviation of "quasi-everywhere". We also write $\mathcal{E}$-q.e. to emphasize the notion q.e. for the Dirichlet form $(\mathcal{E}, \mathcal{F})$. In particular, one can introduce the notion of q.e. for other regular Dirichlet forms.

Lemma 5.7. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2 without killing part. Assume that conditions (VD$_*$), (EP$_*$), (FK$_*$), and (TJ$_*$) are all satisfied. Let a function $u \in \mathcal{F}' \cap L^\infty$ be superharmonic, non-negative in a ball B_{2r} with $r \in (0, 2^{-1}\sigma_* W(x_0, \overline{R})^{1/\beta})$, where constant σ_* comes from condition (FK$_*$). Let $0 < a < b$, $r_1 < r_2 < r$ be some numbers and set*

$$m_1 = \frac{\mu(B_{r_1} \cap \{u < a\})}{\mu(B_{r_1})} \quad \textit{and} \quad m_2 = \frac{\mu(B_{r_2} \cap \{u < b\})}{\mu(B_{r_2})}.$$

Then, with the same constants α_*, ν and C_1 as in conditions (VD$_*$), (FK$_*$) and (EP$_*$),

$$m_1 \leq \ CA\left(\frac{b}{b-a}\right)^2 \left(\frac{r_2}{r_1}\right)^{\alpha_*} \left(\frac{r_2}{r_2 - r_1}\right)^{\beta + C_1} m_2^{1+\nu}, \tag{5.21}$$

where the positive constant C depends only on the hypotheses and

$$A := 1 + \frac{(r_2 - r_1)^\beta T_{B_{(r_1+r_2)/2}, B_{r_2}}(u_-)}{b}.$$

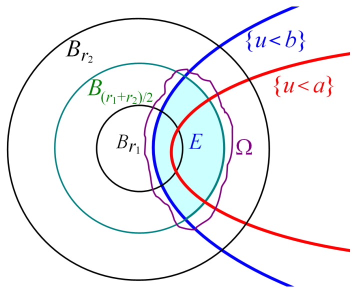

FIGURE 4. Illustration to Lemma 5.7

Proof. Note that any function $u \in \mathcal{F}$ admits a *quasi-continuous* version $\widetilde{u}$ (cf. [8, Theorem 2.1.3 on p.71]), and that for any $u \in \mathcal{F}$ and any open subset Ω of M, we have $u \in \mathcal{F}(\Omega)$ if and only if $\widetilde{u} = 0$ q.e. in Ω^c, (cf. [8, Corollary 2.3.1 on p.98]).

Let us fix a quasi-continuous modification of a given function u in $\mathcal{F}$ and denote it by the same letter u. Set $v := (b-u)_+$ and

$$\widetilde{m}_1 := \mu(B_{r_1} \cap \{u < a\}), \quad \widetilde{m}_2 := \mu(B_{r_2} \cap \{u < b\}).$$

Taking $\phi \in \text{cutoff}(B_{r_1}, B_{\frac{1}{2}(r_1+r_2)})$, we have

$$\widetilde{m}_1 = \int_{B_{r_1} \cap \{u<a\}} \phi^2 d\mu \le \int_{B_{r_1}} \phi^2 \underbrace{\left(\frac{(b-u)_+}{b-a}\right)^2}_{\ge 1 \text{ on } \{u<a\}} d\mu = \frac{1}{(b-a)^2} \int_{B_{r_1}} (\phi v)^2 d\mu. \tag{5.22}$$

Consider the set

$$E := B_{\frac{1}{2}(r_1+r_2)} \cap \{u < b\}.$$

By the outer regularity of μ, for any $\varepsilon > 0$, there is an open set Ω such that $E \subset \Omega \subset B_{r_2}$ and

$$\mu(\Omega) \le \mu(E) + \varepsilon \le \widetilde{m}_2 + \varepsilon. \tag{5.23}$$

On the other hand, since $\phi = 0$ q.e. outside $B_{\frac{1}{2}(r_1+r_2)}$ and $v = 0$ outside $\{u < b\}$, we see that $\phi v = 0$ q.e. in E^c. Since $\phi v \in \mathcal{F}$ and $\phi v = 0$ q.e. in $\Omega^c \subset E^c$, we have

$$\phi v \in \mathcal{F}(\Omega), \tag{5.24}$$

from which, by the definition of $\lambda_1(\Omega)$,

$$\int_\Omega (\phi v)^2 d\mu \le \frac{\mathcal{E}(\phi v, \phi v)}{\lambda_1(\Omega)}.$$

Therefore, using $\phi v = 0$ outside Ω again, it follows from (5.22) that

$$\widetilde{m}_1 \le \frac{1}{(b-a)^2} \int_\Omega (\phi v)^2 d\mu \le \frac{\mathcal{E}(\phi v, \phi v)}{(b-a)^2 \lambda_1(\Omega)}. \tag{5.25}$$

By condition (FK$_*$) and (5.23), we have

$$\lambda_1(\Omega) \ge \frac{c}{r_2^\beta} \left(\frac{\mu(B_{r_2})}{\mu(\Omega)}\right)^\nu \ge \frac{c}{r_2^\beta} \left(\frac{\mu(B_{r_2})}{\widetilde{m}_2 + \varepsilon}\right)^\nu, \tag{5.26}$$

where ν is the constant from (FK$_*$).

Let us estimate $\mathcal{E}(\phi v, \phi v)$ from above. Since u is superharmonic in B_{2r}, the function $b - u$ is subharmonic in B_{2r} and so, the function $v = (b-u)_+$ is also subharmonic in B_{2r} by using

[12, Lemma 9.3]. By Proposition 9.2(iii) in Appendix and using (5.24), we see $v\phi^2 = v\phi \cdot \phi \in \mathcal{F}(\Omega) \subset \mathcal{F}(B_{2r})$, which gives that

$$\mathcal{E}(v, v\phi^2) \leq 0.$$

Applying (EP$_*$) to the triple $B_{r_1}, B_{(r_1+r_2)/2}, B_{r_2}$ and to$= v$, there exists $\phi \in \text{cutoff}(B_{r_1}, B_{(r_1+r_2)/2})$ such that

$$\mathcal{E}(\phi v, \phi v) \leq \frac{3}{2}\mathcal{E}(v, v\phi^2) + \frac{c}{\overline{r}^\beta}\left(\frac{r_2}{\overline{r}}\right)^{C_1}\int_{B_{r_2}} v^2 d\mu + 3\iint_{B_{r_2}\times B_{r_2}^c} v(x)v(y)\phi^2(x)dj,$$

where $\overline{r} := r_2 - r_1$. Combining the above two inequalities and using the fact that $\phi = 0$ outside $B_{(r_1+r_2)/2}$, we obtain

$$\begin{aligned}
\mathcal{E}(v\phi, v\phi) &\leq \frac{c}{\overline{r}^\beta}\left(\frac{r_2}{\overline{r}}\right)^{C_1}\int_{B_{r_2}} v^2 d\mu + 3\int_{B_{(r_1+r_2)/2}} v(x)d\mu(x) \cdot \underset{x\in B_{(r_1+r_2)/2}}{\text{esup}} \int_{B_{r_2}^c} v(y)J(x,dy) \\
&\leq \frac{c}{\overline{r}^\beta}\left(\frac{r_2}{\overline{r}}\right)^{C_1}\int_{B_{r_2}} v^2 d\mu + 3\int_{B_{r_2}} v d\mu \cdot \underset{x\in B_{(r_1+r_2)/2}}{\text{esup}} \int_{B_{r_2}^c} (b + u_-(y))J(x,dy) \\
&\leq \frac{cb^2}{\overline{r}^\beta}\left(\frac{r_2}{\overline{r}}\right)^{C_1}\mu(B_{r_2}\cap\{u<b\}) + 3\,b\mu(B_{r_2}\cap\{u<b\}) \qquad (\text{using } v \leq b\mathbf{1}_{\{u<b\}}) \\
&\quad \cdot \left(b \underset{x\in B_{(r_1+r_2)/2}}{\text{esup}} \int_{B(x,(r_2-r_1)/2)^c} J(x,dy) + T_{B_{(r_1+r_2)/2},B_{r_2}}(u_-)\right) \\
&\leq c\widetilde{m}_2\frac{b^2}{\overline{r}^\beta}\left(\left(\frac{r_2}{\overline{r}}\right)^{C_1} + \frac{\overline{r}^\beta}{(\overline{r}/2)^\beta} + \frac{\overline{r}^\beta T_{B_{(r_1+r_2)/2},B_{r_2}}(u_-)}{b}\right) \quad (\text{by definition of } \widetilde{m}_2 \text{ and (TJ}_*)) \\
&\leq c\widetilde{m}_2\frac{b^2}{\overline{r}^\beta}\left(\frac{r_2}{\overline{r}}\right)^{C_1}\left(1 + \frac{\overline{r}^\beta T_{B_{(r_1+r_2)/2},B_{r_2}}(u_-)}{b}\right) \qquad (\text{by the fact } \frac{r_2}{\overline{r}} \geq 1) \\
&= c\widetilde{m}_2\frac{b^2}{\overline{r}^\beta}\left(\frac{r_2}{\overline{r}}\right)^{C_1} A, \qquad (5.27)
\end{aligned}$$

where in the fourth line we have used the fact that, for any point $x \in B_{(r_1+r_2)/2}$,

$$B_{r_2}^c \subset B_*(x, \overline{r}/2)^c.$$

Combining (5.25), (5.26), (5.27) and letting $\varepsilon \to 0$, we obtain

$$\widetilde{m}_1 \leq c\left(\frac{b}{b-a}\right)^2 \frac{(\widetilde{m}_2)^{1+\nu}}{\mu(B_{r_2})^\nu}\left(\frac{r_2}{\overline{r}}\right)^{\beta+C_1} A.$$

Dividing this inequality by $\mu(B_{r_1})$ and observing that

$$m_1 = \frac{\widetilde{m}_1}{\mu(B_{r_1})} \quad \text{and} \quad m_2 = \frac{\widetilde{m}_2}{\mu(B_{r_2})},$$

we obtain by using (VD$_*$),

$$\begin{aligned}
m_1 &\leq c\left(\frac{b}{b-a}\right)^2 m_2^{1+\nu}\frac{\mu(B_{r_2})}{\mu(B_{r_1})}\left(\frac{r_2}{\overline{r}}\right)^{\beta+C_1} A \\
&\leq C\left(\frac{b}{b-a}\right)^2\left(\frac{r_2}{r_1}\right)^{\alpha_*}\left(\frac{r_2}{\overline{r}}\right)^{\beta+C_1} A \cdot m_2^{1+\nu},
\end{aligned}$$

thus showing (5.21). □

Now we prove (LG$_*$).

Lemma 5.8. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2 without killing part. Then*

$$(\text{VD}_*) + (\text{EP}_*) + (\text{TJ}_*) + (\text{FK}_*) \Rightarrow (\text{LG}_*). \qquad (5.28)$$

Consequently,

$$(\text{VD}) + (\text{Gcap}) + (\text{TJ}) + (\text{FK}_*) \Rightarrow (\text{LG}_*). \qquad (5.29)$$

Moreover, the constants ν, σ_* in (5.3) of condition (LG_*) *can be taken as the same as in condition* (FK_*).

Proof. Let $u \in \mathcal{F}' \cap L^\infty$ be superharmonic, non-negative in B_{2r} with $r \in (0, 2^{-1}\sigma_* W(x_0, \overline{R})^{1/\beta})$ and let $a > 0$. Consider the following two sequences of numbers

$$r_k := \frac{1}{2}(1+2^{-k})r \quad \text{and} \quad a_k := \frac{1}{2}(1+2^{-k})a \qquad \text{for } k = 0, 1, 2, \cdots .$$

Clearly, $r_0 = r$, $a_0 = a$ and $r_k \downarrow \frac{1}{2}r$, $a_k \downarrow \frac{1}{2}a$ as $k \to \infty$ (see Figure 5). Set

$$m_k := \frac{\mu(B_{r_k} \cap \{u < a_k\})}{\mu(B_{r_k})}.$$

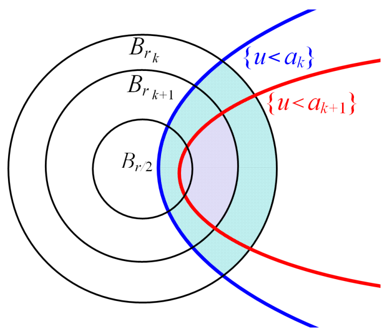

FIGURE 5. Sets $B_{r_k} \cap \{u < a_k\}$

Note that the hypothesis of Lemma 5.7 is satisfied. Applying inequality (5.21) with $a = a_k$, $b = a_{k-1}$, $r_1 = r_k$ and $r_2 = r_{k-1}$, we obtain for any $k \geq 1$,

$$m_k \leq C A_k \left(\frac{a_{k-1}}{a_{k-1} - a_k}\right)^2 \left(\frac{r_{k-1}}{r_k}\right)^{\alpha_*} \left(\frac{r_{k-1}}{r_{k-1} - r_k}\right)^{\beta + C_1} m_{k-1}^{1+\nu},$$

where

$$A_k := 1 + \frac{(r_{k-1} - r_k)^\beta T_{B_{(r_{k-1}+r_k)/2}, B_{r_{k-1}}}(u_-)}{a_{k-1}}.$$

Since u is non-negative in B_{2r} and $B_{r_{k-1}} \subset B_r \subset B_{2r}$, we see by definition (5.2)

$$T_{B_{(r_{k-1}+r_k)/2}, B_{r_{k-1}}}(u_-) = T_{B_{(r_{k-1}+r_k)/2}, B_{2r}}(u_-) \leq T_{B_r, B_{2r}}(u_-),$$

from which, using the fact that $a_{k-1} \geq \frac{1}{2}a$, it follows that

$$A_k \leq 2A,$$

where

$$A = 1 + \frac{r^\beta T_{B_r, B_{2r}}(u_-)}{a}.$$

Noting that

$$\frac{r_{k-1}}{r_k} \leq 2, \quad \frac{r_{k-1}}{r_{k-1} - r_k} = \frac{1 + 2^{-(k-1)}}{2^{-(k-1)} - 2^{-k}} \leq 2^{k+1}, \quad \frac{a_{k-1}}{a_{k-1} - a_k} \leq 2^{k+1},$$

we obtain that for all $k \geq 1$

$$m_k \leq C \cdot 2A \cdot 2^{2(k+1)} \cdot 2^{\alpha_*} \cdot 2^{(k+1)(\beta + C_1)} \cdot m_{k-1}^{1+\nu} = D\lambda^k m_{k-1}^{1+\nu},$$

where $D = 2^{3+\alpha_*+\beta+C_1} CA$, $\lambda = 2^{2+\beta+C_1}$. Thus, applying Proposition 9.3 in Appendix, we have

$$m_k \leq D^{-\frac{1}{\nu}} \left(D^{\frac{1}{\nu}} \lambda^{\frac{1+\nu}{\nu^2}} m_0\right)^{(1+\nu)^k} \to 0 \tag{5.30}$$

as $k \to \infty$, provided that

$$D^{\frac{1}{\nu}} \lambda^{\frac{1+\nu}{\nu^2}} m_0 = \left(2^{3+\alpha_*+\beta+C_1} CA\right)^{\frac{1}{\nu}} \left(2^{2+\beta+C_1}\right)^{\frac{1+\nu}{\nu^2}} m_0 \leq \frac{1}{2}. \tag{5.31}$$

Note that (5.31) is equivalent to

$$\begin{aligned} \frac{\mu(B_r \cap \{u < a\})}{\mu(B_r)} &= m_0 \leq \frac{1}{2} \left(2^{3+\alpha_*+\beta+C_1} C\right)^{-\frac{1}{\nu}} \left(2^{2+\beta+C_1}\right)^{-\frac{1+\nu}{\nu^2}} A^{-\frac{1}{\nu}} \\ &=: \varepsilon_0 A^{-\frac{1}{\nu}} = \varepsilon_0 \left(1 + \frac{r^\beta T_{B_r, B_{2r}}(u_-)}{a}\right)^{-\frac{1}{\nu}}, \end{aligned}$$

which is secured by the hypothesis (5.3) with

$$\varepsilon_0 := \frac{1}{2} \left(2^{3+\alpha_*+\beta+C_1} C\right)^{-\frac{1}{\nu}} \left(2^{2+\beta+C_1}\right)^{-\frac{1+\nu}{\nu^2}}.$$

With the choice of ε_0, we conclude by (5.30) that

$$\frac{\mu(B_{r/2} \cap \{u < \frac{a}{2}\})}{\mu(B_{r/2})} = 0,$$

thus showing (5.4). That is, we obtain (5.28).

It remains to prove the implication (5.29). Indeed, Proposition 4.4, we have that (VD_*) is true. By (5.20), we have that (EP_*) is true. By (4.12), we have that (TJ_*) is true. Therefore, implication (5.29) follows from implication (5.28). □

5.2. **Weak Harnack inequality.** In this subsection, we show the weak Harnack inequality, which will be used in deriving the oscillation inequality below.

We introduce condition (WHI_*), that is called the *weak Harnack inequality* .

Definition 5.9 (Weak Harnack inequality)**.** We say that condition (WHI_*) holds if there exist three numbers $\varepsilon, \kappa_*, \sigma_*$ in $(0, 1)$ such that the following is true: if a function $u \in \mathcal{F}' \cap L^\infty$ is superharmonic, non-negative in $2B_*$, where $B_* := B_*(x_0, r)$ has radius $r \in (0, 2^{-1}\sigma_* W(x_0, \overline{R})^{1/\beta})$, and if

$$\frac{\mu((\kappa_* B_*) \cap \{u \geq a\})}{\mu(\kappa_* B_*)} \geq \frac{1}{2} \tag{5.32}$$

and

$$r^\beta T_{\frac{3}{2}B_*, 2B_*}(u_-) \leq \varepsilon a \tag{5.33}$$

for some $a > 0$, then

$$\operatorname*{einf}_{\frac{\kappa_*}{2} B_*} u \geq \varepsilon a. \tag{5.34}$$

We remark that three constants $\varepsilon, \kappa_*, \sigma_*$ are all independent of B_*, u, a.

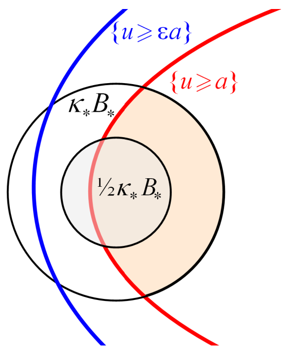

FIGURE 6. Illustration to condition (WHI_*)

To obtain the weak Harnack inequality, we need the following result on the average of logarithm function cut off by two constants, see (5.39) below. The following result is a stronger version of [12, Proposition 12.3] (see also [10, Lemma 3.7, p. 469] for the version of pure jump type Dirichlet form).

Proposition 5.10. *Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form in L^2 without killing part. Let a function $u\in\mathcal{F}'\cap L^\infty$ be non-negative in an open set $B\subset M$ and $\phi\in\mathcal{F}\cap L^\infty$ be such that $\phi=0$ in B^c. Fix any $\lambda>0$ and set $u_\lambda:=u+\lambda$. Then $\frac{\phi^2}{u_\lambda}\in\mathcal{F}\cap L^\infty$ and*

$$\mathcal{E}(u,\frac{\phi^2}{u_\lambda})\le 3\mathcal{E}(\phi,\phi)-\frac{1}{2}\int_M\frac{\phi^2}{u_\lambda^2}d\Gamma^{(L)}(u)-\frac{1}{2}\iint_{B\times B}\left(\phi^2(x)\wedge\phi^2(y)\right)\left|\ln\frac{u_\lambda(y)}{u_\lambda(x)}\right|^2dj$$
$$-2\iint_{B\times B^c}u_\lambda(y)\frac{\phi^2(x)}{u_\lambda(x)}dj. \tag{5.35}$$

Proof. It is already proved in [12, Proposition 12.3] that $\frac{\phi^2}{u_\lambda}\in\mathcal{F}\cap L^\infty$. We sketch its proof. Indeed, define the Lipschitz function $F(t):=\frac{1}{|t|+\lambda}$ with Lipschitz constant λ^{-2} on $\mathbb{R}$. Then, we have $\frac{\phi^2}{u_\lambda}=F(u)\phi^2$ on M. Moreover, by Proposition 9.2(i), (ii) in Appendix, we can prove

$$F(u)\in\mathcal{F}'\quad\text{and}\quad F(u)\phi^2\in\mathcal{F}\cap L^\infty.$$

That is, $\frac{\phi^2}{u_\lambda}\in\mathcal{F}\cap L^\infty$.

It remains to show (5.35). Indeed, it was proved in [10, Lemma 3.7, p. 469] that

$$\mathcal{E}^{(J)}\Big(u,\frac{\phi^2}{u_\lambda}\Big)\le 3\mathcal{E}^{(J)}(\phi,\phi)-\frac{1}{2}\iint_{B\times B}\left(\phi^2(x)\wedge\phi^2(y)\right)\left|\ln\frac{u_\lambda(y)}{u_\lambda(x)}\right|^2dj(x,y)$$
$$-2\iint_{B\times B^c}u_\lambda(y)\frac{\phi^2(x)}{u_\lambda(x)}dj(x,y), \tag{5.36}$$

(noting that the existence of the jump kernel J is not assumed therein).

On the other hand, by using the product and chain rules ((3.1) and (3.2)), and then using the Cauchy-Schwarz inequality (3.3), we have

$$\begin{aligned}\mathcal{E}^{(L)}(u,\frac{\phi^2}{u_\lambda})&=\int_M d\Gamma^{(L)}(u,\frac{\phi^2}{u_\lambda})=\int\frac{2\phi}{u_\lambda}d\Gamma^{(L)}(u,\phi)-\int\frac{\phi^2}{u_\lambda^2}d\Gamma^{(L)}(u,u_\lambda)\\&\le\frac{1}{2}\int\frac{\phi^2}{u_\lambda^2}d\Gamma^{(L)}(u)+2\int d\Gamma^{(L)}(\phi)-\int\frac{\phi^2}{u_\lambda^2}d\Gamma^{(L)}(u,u_\lambda)\\&=-\frac{1}{2}\int\frac{\phi^2}{u_\lambda^2}d\Gamma^{(L)}(u)+2\mathcal{E}^{(L)}(\phi,\phi).\end{aligned}$$

From this and using (5.36), we conclude that

$$\begin{aligned}\mathcal{E}\Big(u,\frac{\phi^2}{u_\lambda}\Big)&=\mathcal{E}^{(L)}\Big(u,\frac{\phi^2}{u_\lambda}\Big)+\mathcal{E}^{(J)}\Big(u,\frac{\phi^2}{u_\lambda}\Big)\\&\le-\frac{1}{2}\int\frac{\phi^2}{u_\lambda^2}d\Gamma^{(L)}(u)+2\mathcal{E}^{(L)}(\phi,\phi)+3\mathcal{E}^{(J)}(\phi,\phi)\\&\quad-\frac{1}{2}\iint_{B\times B}\left(\phi^2(x)\wedge\phi^2(y)\right)\left|\ln\frac{u_\lambda(y)}{u_\lambda(x)}\right|^2dj(x,y)\\&\quad-2\iint_{B\times B^c}u_\lambda(y)\frac{\phi^2(x)}{u_\lambda(x)}dj(x,y)\\&\le 3\mathcal{E}(\phi,\phi)-\frac{1}{2}\int\frac{\phi^2}{u_\lambda^2}d\Gamma^{(L)}(u)-\frac{1}{2}\iint_{B\times B}\left(\phi^2(x)\wedge\phi^2(y)\right)\left|\ln\frac{u_\lambda(y)}{u_\lambda(x)}\right|^2dj(x,y)\end{aligned}$$

$$- 2 \iint_{B\times B^c} u_\lambda(y) \frac{\phi^2(x)}{u_\lambda(x)} dj(x,y),$$

thus proving (5.35). ☐

Lemma 5.11. *Let $B \subset M$ be a ball and $u \in \mathcal{F}' \cap L^\infty$ be non-negative, superharmonic in $2B$ in M. Set $u_\lambda := u + \lambda$ for $\lambda > 0$. Then*

$$\int_B \frac{1}{u_\lambda^2} d\Gamma^{(L)}(u) + \iint_{B\times B} \left| \ln \frac{u_\lambda(y)}{u_\lambda(x)} \right|^2 dj \le 6 \operatorname{cap}\left(B, \frac{3}{2}B\right) + 4 \iint_{\frac{3}{2}B\times(2B)^c} \frac{(u_\lambda(y))_-}{u_\lambda(x)} dj. \quad (5.37)$$

Proof. Let ϕ be the potential for the pair $\left(B, \frac{3}{2}B\right)$ so that

$$\mathcal{E}(\phi,\phi) = \operatorname{cap}\left(B, \frac{3}{2}B\right). \quad (5.38)$$

By Proposition 9.2 in Appendix, one can show that

$$\frac{\phi^2}{u_\lambda} \in \mathcal{F}\left(\frac{3}{2}B\right) \cap L^\infty$$

because $0 \le \phi \in \mathcal{F}(\frac{3}{2}B) \cap L^\infty$. Since $u \in \mathcal{F}' \cap L^\infty$ is superharmonic in $2B$, we have

$$\mathcal{E}(u, \frac{\phi^2}{u_\lambda}) \ge 0.$$

Applying Proposition 5.10 in $2B$ instead of B, we obtain

$$\int_M \frac{\phi^2}{u_\lambda^2} d\Gamma^{(L)}(u) + \iint_{2B\times 2B} (\phi^2(x) \wedge \phi^2(y)) \left| \ln \frac{u_\lambda(y)}{u_\lambda(x)} \right|^2 dj \ \le 6\mathcal{E}(\phi,\phi)$$
$$- 4 \iint_{2B\times(2B)^c} u_\lambda(y) \frac{\phi^2(x)}{u_\lambda(x)} dj.$$

Applying (5.38) and observing that $\phi = 1$ on B and $\phi = 0$ outside $\frac{3}{2}B$, we obtain (5.37). ☐

Lemma 5.12. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2. Assume that conditions (VD$_*$), (Cap$_\le$*) and (PI$_*$) are all satisfied. Let $B_* := B_*(x_0, r)$ with $r < \frac{1}{2}W(x_0, \overline{R})^{1/\beta}$ and a function $u \in \mathcal{F}' \cap L^\infty$ be non-negative, superharmonic in the ball $2B_*$. Fix three positive numbers a, b, λ and consider the function:*

$$v := \left(\ln \frac{a}{u_\lambda} \right)_+ \wedge b,$$

where $u_\lambda := u + \lambda$. Then

$$\fint_{\kappa_* B_*} \fint_{\kappa_* B_*} (v(x) - v(y))^2 d\mu(x) d\mu(y) \le C \left(1 + \frac{r^\beta T_{\frac{3}{2}B_*, 2B_*}((u_\lambda)_-)}{\lambda} \right), \quad (5.39)$$

where $\kappa_* \in (0, 1]$ is the same constant as in (PI$_*$) and $C > 0$ is a universal constant depending only on constants in conditions (VD$_*$), (Cap$_\le$*) and (PI$_*$).

Proof. By the similar arguments in Proposition 5.10, we can prove that $v \in \mathcal{F} \cap L^\infty$. Moreover, by (3.4), (3.7) and (3.2), we obtain that

$$\int_{B_*} d\Gamma^{(L)}(v,v) \le \int_{B_*} d\Gamma^{(L)}(\ln \frac{a}{u_\lambda}, \ln \frac{a}{u_\lambda}) = \int_{B_*} d\Gamma^{(L)}(\ln u_\lambda, \ln u_\lambda)$$
$$= \int_{B_*} \frac{1}{u_\lambda^2} d\Gamma^{(L)}(u_\lambda, u_\lambda) = \int_{B_*} \frac{1}{u_\lambda^2} d\Gamma^{(L)}(u,u).$$

On the other hand, by the definition of v, we have for all $x, y \in B_*$,

$$|v(x) - v(y)| \le \left| \ln \frac{u_\lambda(y)}{u_\lambda(x)} \right|.$$

Combining the above two inequalities, we see that, using (5.37) with B being replaced by B_*,

$$\begin{aligned}
&\int_{B_*} d\Gamma^{(L)}(v,v) + \iint_{B_*\times B_*} (v(x)-v(y))^2 dj \\
&\le \int_{B_*} \frac{1}{u_\lambda} d\Gamma^{(L)}(u,u) + \iint_{B_*\times B_*} \left|\ln \frac{u_\lambda(y)}{u_\lambda(x)}\right|^2 dj \\
&\le 6\,\mathrm{cap}\left(B_*, \frac{3}{2}B_*\right) + 4\int_{(\frac{3}{2}B_*)\times(2B_*)^c} \frac{(u_\lambda(y))_-}{u_\lambda(x)} dj \\
&\le \frac{C\mu(\frac{3}{2}B_*)}{r^\beta} + 4\int_{\frac{3}{2}B_*} d\mu(x) \int_{(2B_*)^c} \frac{(u_\lambda(y))_-}{\lambda} J(x,dy) \quad (\text{by } (\mathrm{Cap}^*_{\le})) \\
&\le C'\left(\frac{\mu(B_*)}{r^\beta} + \frac{\mu(B_*)}{\lambda} \sup_{x\in\frac{3}{2}B_*} \int_{(2B_*)^c} (u_\lambda(y))_- J(x,dy)\right) \quad (\text{by } (\mathrm{VD}_*)) \\
&\le C' \frac{\mu(B_*)}{r^\beta}\left(1 + \frac{r^\beta T_{\frac{3}{2}B_*,2B_*}((u_\lambda)_-)}{\lambda}\right).
\end{aligned}$$

Using the above inequality and the following equality that holds for any open Ω in M,

$$\iint_{\Omega\times\Omega} (f(x)-f(y)^2 d\mu(x)d\mu(y) = 2\mu(\Omega)\int_\Omega (f-f_\Omega)^2 d\mu, \quad f\in L^2,$$

we obtain that

$$\begin{aligned}
\fint_{\kappa_*B_*}\fint_{\kappa_*B_*} (v(x)-v(y))^2 d\mu(x)d\mu(y) &= \frac{2}{\mu(\kappa_*B_*)} \int_{\kappa_*B_*} (v - v_{\kappa_*B_*})^2 d\mu \\
&\le \frac{2Cr^\beta}{\mu(\kappa_*B_*)}\left(\int_{B_*} d\Gamma^{(L)}(v,v) + \iint_{B_*\times B_*} (v(x)-v(y))^2 dj(x,y)\right) \\
&\le \frac{2Cr^\beta}{\mu(B_*)} \cdot C'\frac{\mu(B_*)}{r^\beta}\left(1 + \frac{r^\beta T_{\frac{3}{2}B_*,2B_*}((u_\lambda)_-)}{\lambda}\right) \\
&= 2CC'\left(1 + \frac{r^\beta T_{\frac{3}{2}B_*,2B_*}((u_\lambda)_-)}{\lambda}\right)
\end{aligned}$$

(where in the second line we have used (PI_*)), which proves (5.39). □

Lemma 5.13. *Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form in L^2 without killing part. Then*

$$(\mathrm{LG}_*) + (\mathrm{Cap}^*_{\le}) + (\mathrm{PI}_*) \Rightarrow (\mathrm{WHI}_*). \tag{5.40}$$

Moreover, the constants κ_*, σ_* in (WHI_*) can be taken as the same as in (PI_*) and (LG_*) respectively. Consequently,

$$(\mathrm{VD}) + (\mathrm{RVD}) + (\mathrm{TJ}) + (\mathrm{Gcap}) + (\mathrm{PI}) \Rightarrow (\mathrm{WHI}_*). \tag{5.41}$$

Proof. Consider a ball $B_* := B_*(x_0, r)$ with

$$r \in (0, 2^{-1}\sigma_* W(x_0, \overline{R})^{1/\beta}),$$

where σ_* is the constant from (LG_*). Assume that $u \in \mathcal{F}' \cap L^\infty$ is superharmonic, non-negative in $2B_*$. Let λ, b be two positive numbers to be determined later on. Let $u_\lambda := u + \lambda$ and

$$v := \left(\ln \frac{a+\lambda}{u_\lambda}\right)_+ \wedge b.$$

Note that $0 \le v \le b$ in M, and in $2B_*$

$$v = 0 \quad \Leftrightarrow \quad \frac{a+\lambda}{u_\lambda} \le 1 \quad \Leftrightarrow \quad u \ge a,$$

$$v = b \quad \Leftrightarrow \quad \frac{a+\lambda}{u_\lambda} \geq e^b \quad \Leftrightarrow \quad u_\lambda \leq (a+\lambda)e^{-b} =: q.$$

For simplicity, set

$$\omega := \frac{\mu((\kappa_* B_*) \cap \{u \geq a\})}{\mu(\kappa_* B_*)} = \frac{\mu((\kappa_* B_*) \cap \{v = 0\})}{\mu(\kappa_* B_*)}, \tag{5.42}$$

and

$$m_0 := \frac{\mu((\kappa_* B_*) \cap \{u_\lambda \leq q\})}{\mu(\kappa_* B_*)} = \frac{\mu((\kappa_* B_*) \cap \{v = b\})}{\mu(\kappa_* B_*)}. \tag{5.43}$$

Here k_* is the same constant as in (5.39).

Therefore, applying (LG_*) to the function $u_\lambda \in \mathcal{F}' \cap L^\infty$, which is superharmonic, non-negative in a ball $2\kappa_* B_*$, we have that, if

$$m_0 \leq \varepsilon_0 \left(1 + \frac{(\kappa_* r)^\beta T_{\kappa_* B_*, 2\kappa_* B_*}((u_\lambda)_-)}{q}\right)^{-1/\nu}, \tag{5.44}$$

then

$$\operatorname*{einf}_{\frac{\kappa_*}{2} B_*} u_\lambda \geq \frac{q}{2}. \tag{5.45}$$

We need to verify condition (5.44).

Since $(u_\lambda)_- \leq u_-$ in M and u is non-negative in $2B_*$, we have

$$A := r^\beta T_{\frac{3}{2}B_*, 2B_*}(u_-) = r^\beta T_{\frac{3}{2}B_*, 2\kappa_* B_*}(u_-) \geq (\kappa_* r)^\beta T_{\kappa_* B_*, 2\kappa_* B_*}\left((u_\lambda)_-\right),$$

from which, in order to guarantee (5.44), it suffices to ensure that

$$m_0 \leq \varepsilon_0 \left(1 + \frac{A}{q}\right)^{-1/\nu}. \tag{5.46}$$

By Lemma 5.12 and using definitions (5.42) and (5.43), we see that

$$\begin{aligned}
b^2 m_0 \omega &= \frac{1}{\mu(\kappa_* B_*)^2} \int_{(\kappa_* B_*) \cap \{v=0\}} \int_{(\kappa_* B_*) \cap \{v=b\}} b^2 d\mu(x) d\mu(y) \\
&= \frac{1}{\mu(\kappa_* B_*)^2} \int_{(\kappa_* B_*) \cap \{v=0\}} \int_{(\kappa_* B_*) \cap \{v=b\}} (v(x) - v(y))^2 d\mu(x) d\mu(y) \\
&\leq \fint_{\kappa_* B_*} \fint_{\kappa_* B_*} (v(x) - v(y))^2 d\mu(x) d\mu(y) \\
&\leq c \left(1 + \frac{r^\beta T_{\frac{3}{2}B_*, 2B_*}((u_\lambda)_-)}{\lambda}\right) \qquad \text{(by (5.39))} \\
&= c \left(1 + \frac{A}{\lambda}\right).
\end{aligned}$$

It follows that

$$m_0 \leq \frac{c}{b^2 \omega} \left(1 + \frac{A}{\lambda}\right) \leq \frac{2c}{b^2} \left(1 + \frac{A}{\lambda}\right),$$

where we have used the fact that $\omega \geq 1/2$, which is true by assumption (5.32). Hence, the condition (5.46) will be satisfied provided that

$$\frac{2c}{b^2} \left(1 + \frac{A}{\lambda}\right) \leq \varepsilon_0 \left(1 + \frac{A}{q}\right)^{-1/\nu},$$

which is equivalent to

$$b^2 \geq \frac{2c}{\varepsilon_0} \left(1 + \frac{A}{\lambda}\right) \left(1 + \frac{A}{q}\right)^{1/\nu}. \tag{5.47}$$

Fix a number $\varepsilon > 0$ whose value will be determined later. We pick up the parameters λ, b by

$$\lambda := \varepsilon a, \quad b := \ln \frac{1+\varepsilon}{4\varepsilon}.$$

Then we have

$$q = (a+\lambda)e^{-b} = 4\varepsilon a,$$

and the inequality (5.47) is equivalent to

$$\left(\ln \frac{1+\varepsilon}{4\varepsilon}\right)^2 \geq \frac{2c}{\varepsilon_0}\left(1+\frac{A}{\varepsilon a}\right)\left(1+\frac{A}{4\varepsilon a}\right)^{1/\nu}. \tag{5.48}$$

Since $A \leq \varepsilon a$ by assumption (5.33), the inequality (5.48) will follow if

$$\left(\ln \frac{1+\varepsilon}{4\varepsilon}\right)^2 \geq \frac{4c}{\varepsilon_0}\left(\frac{5}{4}\right)^{1/\nu},$$

which can be achieved by choosing ε to be sufficiently small. With this choice of ε, we conclude that (5.45) holds, which implies that

$$\operatorname*{einf}_{\frac{\kappa_*}{2}B_*} u \geq \frac{q}{2} - \lambda = 2\varepsilon a - \varepsilon a = \varepsilon a,$$

thus showing (5.34).

Finally, the implication (5.41) follows directly from Propositions 4.4 and 4.7, (4.33), Lemmas 5.8 and 4.15, and the implication (5.40). □

5.3. **Oscillation inequalities.** In this subsection, we will show the oscillation inequality for harmonic function in a ball. The oscillation inequality will be used to derive lower bound of the heat kernel. We will frequently use the notion $B_r := B_*(x_0, r)$, without mentioning its center x_0, nor the new metric d_*.

We introduce condition (OSC_*) that is called the *oscillation inequality*.

Definition 5.14 (Oscillation inequality)**.** We say that condition (OSC_*) holds if there exist three constants σ_*, ε, κ_* in $(0,1)$ such that the following is true: for any ball $B_r = B_*(x_0, r)$ with $r \in (0, \sigma_* W(x_0, \overline{R})^{1/\beta})$ and any function $u \in \mathcal{F}' \cap L^\infty$ that is harmonic in B_r, we have that either

$$\operatorname*{osc}_{B_{\kappa_* r/4}} u \leq (1-\varepsilon) \operatorname*{osc}_{B_r} u, \tag{5.49}$$

or

$$\operatorname*{osc}_{B_r} u \leq \varepsilon^{-1} r^\beta T_{B_{\frac{3}{4}r}, B_r}((u-m)_- + (M-u)_-), \tag{5.50}$$

where

$$m = \operatorname*{einf}_{B_r} u \ \text{ and } \ M = \operatorname*{esup}_{B_r} u.$$

We mention that the constants $\sigma_*, \varepsilon, \kappa_*$ are all independent of $B_r, u, \overline{R}$.

Lemma 5.15. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2 without killing part. Then*

$$(\mathrm{WHI}_*) \Rightarrow (\mathrm{OSC}_*).$$

Consequently, we have by (5.41)

$$(\mathrm{VD}) + (\mathrm{RVD}) + (\mathrm{TJ}) + (\mathrm{Gcap}) + (\mathrm{PI}) \Rightarrow (\mathrm{WHI}_*) \Rightarrow (\mathrm{OSC}_*). \tag{5.51}$$

Proof. Let $B_r = B_*(x_0, r)$ with $r \in (0, \sigma_* W(x_0, \overline{R})^{1/\beta})$, where σ_* is the positive constant in (WHI_*). For simplicity, denote by

$$A = r^\beta T_{B_{\frac{3}{4}r}, B_r}((u-m)_- + (M-u)_-).$$

Consider the function $u - m \in \mathcal{F}' \cap L^\infty$, which is non-negative, harmonic in B_r. Applying the weak Harnack inequality (WHI$_*$) in the ball $\frac{1}{2}B_r$ for the function $u - m$, we obtain that if

$$\frac{\mu(B_{\kappa_* r/2} \cap \{u - m \geq a\})}{\mu(B_{\kappa_* r/2})} \geq \frac{1}{2} \tag{5.52}$$

where $a = \frac{M-m}{2}$ and if

$$(r/2)^\beta T_{B_{\frac{3}{4}r}, B_r}((u-m)_-) \leq A_1 := r^\beta T_{B_{\frac{3}{4}r}, B_r}((u-m)_-) \leq \varepsilon' a \tag{5.53}$$

then

$$\operatorname*{einf}_{B_{\kappa_* r/4}} (u - m) \geq \varepsilon' a,$$

for some constant $\varepsilon' \in (0, 1)$.

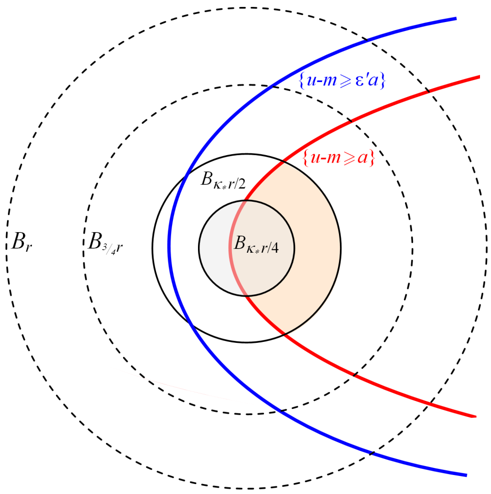

FIGURE 7. Level sets of the function $u - m$

It follows that

$$\operatorname*{osc}_{B_{\kappa_* r/4}} u = \operatorname*{osc}_{B_{\kappa_* r/4}} (u - m) \leq (M - m) - \varepsilon' a = \left(1 - \frac{\varepsilon'}{2}\right)(M - m) = \left(1 - \frac{\varepsilon'}{2}\right) \operatorname*{osc}_{B_r} u,$$

which implies that (5.49) holds with $\varepsilon = \varepsilon'/2$.

Similarly, if both conditions

$$\frac{\mu(B_{\kappa_* r/2} \cap \{M - u \geq a\})}{\mu(B_{\kappa_* r/2})} \geq \frac{1}{2} \tag{5.54}$$

and

$$(r/2)^\beta T_{B_{\frac{3}{4}r}, B_r}((M-u)_-) \leq A_2 := r^\beta T_{B_{\frac{3}{4}r}, B_r}((M-u)_-) \leq \varepsilon' a \tag{5.55}$$

are satisfied, then

$$\operatorname*{einf}_{B_{\kappa_* r/4}} (M - u) \geq \varepsilon' a,$$

so that

$$\operatorname*{osc}_{B_{\kappa_* r/4}} u \leq M - m - \varepsilon' a = \left(1 - \frac{\varepsilon'}{2}\right)(M - m) = \left(1 - \frac{\varepsilon'}{2}\right) \operatorname*{osc}_{B_r} u,$$

which again implies that (5.49) holds with $\varepsilon = \varepsilon'/2$.

Observing that

$$u - m \geq a \quad \Leftrightarrow \quad u \geq \frac{M + m}{2},$$

$$M - u \geq a \quad \Leftrightarrow \quad u \leq \frac{M+m}{2},$$

we see that either (5.52) or (5.54) is satisfied. Hence, if both (5.53) and (5.55) are satisfied, we conclude that (5.49) is true. However, if one of (5.53) and (5.55) is not satisfied, then

$$A \geq \frac{1}{2}(A_1 + A_2) \geq \varepsilon' a = \frac{\varepsilon'}{2}\frac{M-m}{2} = \frac{\varepsilon'}{4}\operatorname*{osc}_{B_r} u,$$

which is equivalent to (5.50) with $\varepsilon = \varepsilon'/4$. Hence, we finish the proof by setting $\varepsilon = \frac{\varepsilon'}{4}$. □

We introduce condition (IOS$_*$) that is called the *iterated oscillation inequality* for the harmonic function.

Definition 5.16 (Iterated oscillation inequality)**.** We say that condition (IOS$_*$) holds if there exist constants $\sigma_*, \gamma \in (0,1)$ and $q, C_0 > 1$ such that the following is true: for any ball $B_r := B_*(x_0, r)$ with $r \in (0, \sigma_* W(x_0, \overline{R})^{1/\beta})$ and any function $u \in \mathcal{F}' \cap L^\infty$, which is harmonic in B_r, we have, for all $k \geq 0$,

$$\operatorname*{osc}_{B_{q^{-k}r}} u \leq C_0 q^{-\gamma k} A\,, \tag{5.56}$$

where

$$A := r^\beta T_{B_{\frac{3}{4}r}, B_r}(u) + \|u\|_{L^\infty(B_r)}.$$

In what follows we set $r_k := q^{-k} r$ and $Q_k := \operatorname{osc}_{B_{r_k}} u$ so that (5.56) means that

$$Q_k \leq C_0 q^{-\gamma k} A. \tag{5.57}$$

Lemma 5.17. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2 without killing part. Then*

$$(\mathrm{OSC}_*) + (\mathrm{TJ}_*) \Rightarrow (\mathrm{IOS}_*).$$

Consequently, we have by (5.51)

$$(\mathrm{VD}) + (\mathrm{RVD}) + (\mathrm{TJ}) + (\mathrm{Gcap}) + (\mathrm{PI}) \Rightarrow (\mathrm{IOS}_*). \tag{5.58}$$

Proof. We will prove (5.57) by induction in k. For $k = 0, 1$, it is trivial, since

$$Q_1 \leq Q_0 = \operatorname*{osc}_{B_r} u \leq 2\|u\|_{L^\infty(B_r)} \leq 2A = 2q^\gamma (q^{-\gamma} A),$$

so that (5.57) holds, provided that

$$C_0 \geq 2q^\gamma.$$

In the sequel, we will choose three constants q, γ, C_0 in the following order: first choosing a large number q, then specifying a small number γ, and finally picking up a large constant C_0.

Assume that (5.57) holds up to some integer k with $k \geq 1$. We show that it also holds for $k+1$. To see this, let $q \geq 4/\kappa_*$ where κ_* comes from (OSC$_*$). Applying condition (OSC$_*$) over B_{r_k}, we obtain that

$$\text{either} \quad Q_{k+1} \leq (1-\varepsilon) Q_k \quad \text{or} \quad Q_k \leq \varepsilon^{-1} A_k, \tag{5.59}$$

where $\varepsilon \in (0,1)$ is the constant from (OSC$_*$), and A_k is given by

$$A_k = r_k^\beta T_{\frac{3}{4}B_{r_k}, B_{r_k}}((u - m_k)_- + (M_k - u)_-) = r_k^\beta T_{\frac{3}{4}B_{r_k}, B_{r_k}}(v) \tag{5.60}$$

with $v := (u - m_k)_- + (M_k - u)_-$, and

$$m_k := \operatorname*{einf}_{B_{r_k}} u, \qquad M_k := \operatorname*{esup}_{B_{r_k}} u.$$

In the first case in (5.59), that is, when

$$Q_{k+1} \leq (1-\varepsilon) Q_k,$$

we obtain by induction hypothesis that

$$Q_{k+1} \leq (1-\varepsilon) C_0 q^{-\gamma k} A = (1-\varepsilon) q^\gamma C_0 q^{-\gamma(k+1)} A \leq C_0 q^{-\gamma(k+1)} A,$$

provided that

$$(1-\varepsilon)q^{\gamma} \le 1. \tag{5.61}$$

In the second the case in (5.59) when

$$Q_k \le \varepsilon^{-1} A_k, \tag{5.62}$$

we will prove that

$$Q_k \le C_0 q^{-\gamma(k+1)} A, \tag{5.63}$$

by choosing the suitable values of the constants q, γ, C_0. Since $Q_{k+1} \le Q_k$, this will conclude the proof of the induction step.

In order to prove (5.64), we will estimate A_k (defined in (5.60)) by using the induction hypothesis

$$Q_j \le C_0 q^{-\gamma j} A \text{ for } j = 0, 1, \dots, k. \tag{5.64}$$

Indeed, let us decompose $T_{\frac{3}{4}B_{r_k}, B_{r_k}}(v)$ as follows:

$$\begin{aligned} T_{\frac{3}{4}B_{r_k}, B_{r_k}}(v) &= \operatorname*{esup}_{x \in \frac{3}{4}B_{r_k}} \int_{B^c_{r_k}} v(y) J(x, dy) \\ &= \operatorname*{esup}_{x \in \frac{3}{4}B_{r_k}} \left(\sum_{i=0}^{k-1} \int_{B_{r_i} \setminus B_{r_{i+1}}} v(y) J(x, dy) + \int_{B^c_r} v(y) J(x, dy) \right). \end{aligned} \tag{5.65}$$

Observe that, for any $0 \le i \le k$,

$$v = (u - m_k)_- + (M_k - u)_- \le Q_i - Q_k \text{ in } B_{r_i}. \tag{5.66}$$

Indeed, if $m_k \le u \le M_k$ in B_{r_i}, then $v = 0$ in B_{r_i}, and (5.66) is trivial by using the fact that $Q_i - Q_k \ge 0$ for any $i \le k$. If $u < m_k$ in B_{r_i}, we have in B_{r_i}

$$v = m_k - u \le m_k - m_i \le m_k - m_i + M_i - M_k = Q_i - Q_k,$$

thus showing that (5.66) holds. Similarly, if $u > M_k$ in B_{r_i}, then

$$v = u - M_k \le M_i - M_k \le M_i - M_k + (m_k - m_i) = Q_i - Q_k \text{ in } B_{r_i},$$

showing that (5.66) holds again.

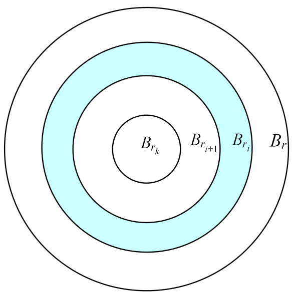

FIGURE 8. Balls B_{r_k} and B_{r_i}

Therefore, by Proposition 9.6 in Appendix and using condition (TJ_*), we obtain from (5.66) that for any $0 \le i < k$,

$$\begin{aligned} \operatorname*{esup}_{x \in \frac{3}{4}B_{r_k}} \int_{B_{r_i} \setminus B_{r_{i+1}}} v(y) J(x, dy) &\le (Q_i - Q_k) \operatorname*{esup}_{x \in \frac{3}{4}B_{r_k}} \int_{B_{r_i} \setminus B_{r_{i+1}}} J(x, dy) \\ &\le (Q_i - Q_k) \operatorname*{esup}_{x \in \frac{3}{4}B_{r_k}} \int_{B(x, (r_i - r_{i+1})/4)^c} J(x, dy) \\ &\le \frac{c(Q_i - Q_k)}{((r_i - r_{i+1})/4)^{\beta}}, \end{aligned}$$

where $c > 1$ is the constant from condition (TJ$_*$).

On the other hand, by Proposition 9.5 in Appendix, we have

$$v(y) \leq |u(y)| + \max(|m_k|, |M_k|) \leq |u(y)| + \|u\|_{L^\infty(B_r)} \text{ for every } y \in M.$$

Using (TJ$_*$), we obtain that, for any $k \geq 1$,

$$\begin{aligned}\operatorname*{esup}_{x \in \frac{3}{4}B_{r_k}} \int_{B_r^c} v(y)J(x,dy) &\leq \operatorname*{esup}_{x \in B_{3r/4}} \int_{B_r^c} (|u(y)| + \|u\|_{L^\infty(B_r)})J(x,dy) \\ &\leq T_{\frac{3}{4}B_r, B_r}(u) + \|u\|_{L^\infty(B_r)} \operatorname*{esup}_{x \in B_{3r/4}} \int_{B(x,r/4)^c} J(x,dy) \\ &\leq T_{\frac{3}{4}B_r, B_r}(u) + \frac{c\|u\|_{L^\infty(B_r)}}{(r/4)^\beta} \leq \frac{4^\beta c A}{r^\beta}.\end{aligned}$$

Therefore, it follows from (5.65) that

$$T_{\frac{3}{4}B_{r_k}, B_{r_k}}(v) \leq c4^\beta \sum_{i=0}^{k-1} \frac{Q_i - Q_k}{(r_i - r_{i+1})^\beta} + \frac{4^\beta c A}{r^\beta},$$

which together with (5.60) and the fact that $q \geq 4/\kappa_*$ implies that

$$\begin{aligned}A_k = (r_k)^\beta T_{\frac{3}{4}B_{r_k}, B_{r_k}}(v) &\leq c4^\beta \sum_{i=0}^{k-1} \left(\frac{r_k}{r_i - r_{i+1}}\right)^\beta (Q_i - Q_k) + 4^\beta c \left(\frac{r_k}{r}\right)^\beta A \\ &\leq C \sum_{i=0}^{k-1} q^{\beta(i+1-k)}(Q_i - Q_k) + Cq^{-k\beta}A. \qquad (5.67)\end{aligned}$$

Assuming that $\gamma < \beta$ and using the induction hypothesis (5.64), we obtain

$$\begin{aligned}\sum_{i=0}^{k-1} q^{\beta(i+1-k)} Q_i &\leq \sum_{i=0}^{k-1} q^{\beta(i+1-k)} \cdot C_0 q^{-\gamma i} A = C_0 A q^{-(k-1)\gamma} \sum_{i=0}^{k-1} q^{(\beta-\gamma)(i+1-k)} \\ &= C_0 A q^{-(k-1)\gamma} \sum_{j=0}^{k-1} q^{-(\beta-\gamma)j} \leq C_0 A \frac{q^{-(k-1)\gamma}}{1 - q^{-(\beta-\gamma)}}.\end{aligned}$$

Noting also that

$$\sum_{i=0}^{k-1} q^{\beta(i+1-k)} \geq 1$$

for $k \geq 1$, it follows from (5.67) that

$$A_k \leq CC_0 A \frac{q^{-(k-1)\gamma}}{1 - q^{-(\beta-\gamma)}} - CQ_k + Cq^{-k\beta}A.$$

This together with (5.62) implies that

$$Q_k \leq \frac{A_k}{\varepsilon} \leq \frac{CC_0 A}{\varepsilon} \frac{q^{-(k-1)\gamma}}{1 - q^{-(\beta-\gamma)}} - \frac{C}{\varepsilon} Q_k + \frac{C}{\varepsilon} q^{-k\beta} A,$$

which gives that

$$Q_k \leq \frac{C}{C+\varepsilon} \left(C_0 \frac{q^{-(k-1)\gamma}}{1 - q^{-(\beta-\gamma)}} + q^{-k\beta} \right) A.$$

To ensure (5.63), it suffices to have

$$\frac{C}{C+\varepsilon} \left(C_0 \frac{q^{-(k-1)\gamma}}{1 - q^{-(\beta-\gamma)}} + q^{-k\beta} \right) A \leq C_0 q^{-(k+1)\gamma} A,$$

that is,

$$\frac{q^{2\gamma}}{1-q^{-(\beta-\gamma)}}+\frac{1}{C_0}q^{\gamma-k(\beta-\gamma)}\leq 1+\frac{\varepsilon}{C}. \tag{5.68}$$

Now first choose $q \geq 4/\kappa_*$ to be so large that

$$\frac{1}{1-q^{-\frac{\beta}{2}}}<1+\frac{\varepsilon}{C},$$

and then choose $\gamma \in (0, \beta/2)$ so small that both (5.61) and

$$\frac{q^{2\gamma}}{1-q^{-\beta/2}}<1+\frac{\varepsilon}{C}$$

are satisfied. Since $\beta-\gamma>\beta/2$, it follows that

$$\frac{q^{2\gamma}}{1-q^{-(\beta-\gamma)}}<\frac{q^{2\gamma}}{1-q^{-\beta/2}}<1+\frac{\varepsilon}{C}.$$

Finally, we choose C_0 so big that (5.68) is satisfied. □

We introduce condition (OSL$_*$) that is called the *oscillation lemma* for harmonic function on a ball. This condition says that any harmonic function is locally uniformly Hölder continuous.

Definition 5.18 (Oscillation lemma)**.** We say that condition (OSL$_*$) holds if there exist three positive constants $\sigma_* \in (0,1)$ and γ, C such that for any ball $B_r := B_*(x_0, r)$ with $r \in (0, \sigma_* W(x_0, \overline{R})^{1/\beta})$ and any function $u \in \mathcal{F}' \cap L^\infty$, which is harmonic in B_r, we have for any $\rho \in (0, r]$,

$$\operatorname*{osc}_{B_\rho} u \leq C\left(\frac{\rho}{r}\right)^\gamma\left(r^\beta T_{\frac{3}{4}B_r,B_r}(|u|)+\|u\|_{L^\infty(B_r)}\right). \tag{5.69}$$

We mention that constants σ_*, γ and C are independent of B_r, u, ρ.

Lemma 5.19. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2. Then*

$$(\text{IOS}_*) \Rightarrow (\text{OSL}_*).$$

Moreover, then the constant γ in (5.69) of condition (OSL$_*$) *can be taken as the same as in* (5.57) of condition (IOS$_*$). *Consequently, we have by (5.58) that*

$$(\text{VD})+(\text{RVD})+(\text{TJ})+(\text{Gcap})+(\text{PI}) \Rightarrow (\text{OSL}_*).$$

Proof. Let C_0, γ, q be the same constants as in *(5.57) of condition* (IOS$_*$). As $\rho \in (0, r]$, there exists an integer $k \geq 0$ such that

$$q^{-(k+1)}<\frac{\rho}{r}\leq q^{-k}.$$

It follows from (IOS$_*$) that

$$\operatorname*{osc}_{B_\rho} u \leq \operatorname*{osc}_{B_{q^{-k}r}} u \leq C_0 q^{-k\gamma}A = C_0 q^\gamma\left(q^{-(k+1)}\right)^\gamma A \leq C_0 q^\gamma\left(\frac{\rho}{r}\right)^\gamma A,$$

thus showing (5.69) with $C = C_0 q^\gamma$. □

6. Hölder continuity

In this section, we show the Hölder continuity of the heat solution, including the harmonic function.

6.1. **Green operator and mean exit time.** Let $\Omega \subset M$ be a non-empty open set. Note that if

$$G^{\Omega}1 := \int_0^{\infty} P_t^{\Omega}\mathbf{1}_{\Omega}dt \in L^{\infty}(\Omega)$$

then G^{Ω} can be extended to a bounded operator on $L^2(\Omega)$ that satisfies the identity $G^{\Omega} = (\mathcal{L}^{\Omega})^{-1}$, see for example [18, Lemma 3.2, p.1232][1]. The function $G^{\Omega}1$ is called the *mean exit time* from the set Ω.

Lemma 6.1 ([10, Lemma 5.1])**.** *If $G^{\Omega}1 \in L^{\infty}(\Omega)$, then the function $u = G^{\Omega}f$, for any $f \in L^2(\Omega)$, belongs to $\mathcal{F}(\Omega)$ and satisfies the identity*

$$\mathcal{E}(u,\varphi) = (f,\varphi) \ \ \textit{for any } \varphi \in \mathcal{F}(\Omega).$$

If in addition $f \geq 0$, then u is superharmonic in Ω.

Definition 6.2. We say that condition ($E^*_{\leq}$) holds, if there exist two constants δ_*, $C > 0$ such that for all balls $B_* := B(x_0, r)$ of radius $r < \delta_* W(x_0, \overline{R})^{1/\beta}$,

$$\left\|G^{B_*}1\right\|_{\infty} \leq Cr^{\beta}.$$

We say that condition ($E^*_{\geq}$) holds, if there exists a constant $C > 0$ such that for all balls $B_* := B(x_0, r)$ of radius $r < W(x_0, \overline{R})^{1/\beta}$,

$$\inf_{\frac{1}{4}B_*} G^{B_*}1 \geq C^{-1}r^{\beta}.$$

We say that condition (E_*) holds if both conditions ($E^*_{\leq}$) and ($E^*_{\geq}$) are satisfied.

Definition 6.3 (condition (S_*))**.** We say that condition (S_*) holds, if there exist two small constant ε, δ_* in $(0, 1)$ such that for all metric balls $B_* = B_*(x, r)$ of radius $r < W(x, \overline{R})^{1/\beta}$,

$$P_t^{B_*}\mathbf{1}_{B_*} \geq \varepsilon \ \text{ in } \frac{1}{4}B_*, \tag{6.1}$$

provided that $t^{1/\beta} \leq \delta_* r$.

Lemma 6.4. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2. Then*

$$(\mathrm{FK}_*) \Rightarrow (\mathrm{E}^*_{\leq}). \tag{6.2}$$

and

$$(\mathrm{VD}_*) + (\mathrm{LG}_*) + (\mathrm{Cap}^*_{\leq}) \Rightarrow (\mathrm{E}^*_{\geq}). \tag{6.3}$$

Consequently,

$$(\mathrm{VD}) + (\mathrm{RVD}) + (\mathrm{Gcap}) + (\mathrm{PI}) + (\mathrm{TJ}) \Rightarrow (\mathrm{E}_*) \Rightarrow (\mathrm{S}_*). \tag{6.4}$$

Proof. One can follow the proof of [12, Lemma 12.2] to obtain the first implication (6.2), and follow the proof of [12, Lemma 12.4] to obtain the second implication (6.3).

Proposition 4.4 shows that (VD) + (RVD) implies (VD_*) + (RVD_*). Proposition 4.7 shows that (PI) implies (PI_*). Hence, it is shown in (4.33) that (FK_*) holds true. Then, Lemma 5.8 shows that (LG_*) holds true. On the other hand, condition ($Cap^*_{\leq}$) follows from (2.12) and Lemma 4.15. Finally, the first implication in (6.4) follows from the implications (6.2) and (6.3). The second implication in (6.4) follows from the same arguments in the proof of [12, Proposition 13.4]. □

[1]Although this lemma was stated for the local Dirichlet form, its proof also holds for the regular Dirichlet form.

Remark 6.5. One can also use [12, Theorem 14.1] to obtain condition (S_*). Indeed, as mentioned in the proof of Lemma 6.4, we have

$$(\mathrm{VD}) + (\mathrm{RVD}) + (\mathrm{PI}) \Rightarrow (\mathrm{FK}_*).$$

Then, by Proposition 4.13, condition (FK) holds true, and hence, it follows from [12, Theorem 14.1] that condition (S) (see Definition 7.13) holds rue. Finally, by Proposition 4.3 and by the method in the proof of [14, Proposition 6.4(2)], we can obtain condition (S_*) from (S) (note that condition (S_*) in [14, Proposition 6.4(2)] looks different from that in this paper, because $\overline{R} = \operatorname{diam} M$ in [14, Proposition 6.4(2)] and $\overline{R} \le \operatorname{diam} M$ in this paper. However, the method in [14, Proposition 6.4(2)] still works in the case when $\overline{R} \le \operatorname{diam} M$).

6.2. **Oscillation inequality for solutions of Poisson equation.** We study the oscillation of the weak solution of the Poisson-type equation on domain Ω by using Lemma 5.19 in Subsection 5.3. This property will be used to show the Hölder continuity of the heat kernel later on.

For a non-empty open set $\Omega \subset M$ and $f \in L^2(\Omega)$, we say that a function $u \in \mathcal{F}$ solves weakly the equation (called the *Poisson-type* equation)

$$\mathcal{L}u = f \quad \text{in } \Omega, \tag{6.5}$$

if for any $\phi \in \mathcal{F}(\Omega)$,

$$\mathcal{E}(u, \phi) = (f, \phi).$$

Proposition 6.6. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2. Assume that $u \in \mathcal{F}$ solves weakly the equation (6.5) for some $f \in L^2(\Omega)$. Let B be a non-empty open subset of Ω.*

1. *If $v \in \mathcal{F}$ solves weakly the equation $\mathcal{L}v = f$ in B, then $u - v$ is harmonic in B.*
2. *If $\|G^B 1\|_{L^\infty} < \infty$, then $u - G^B f$ is harmonic in B.*

Proof. (a) By the definition of weak solution, we see for any $\phi \in \mathcal{F}(B) \subset \mathcal{F}(\Omega)$

$$\mathcal{E}(u, \phi) = (f, \phi) \quad \text{and} \quad \mathcal{E}(v, \phi) = (f, \phi),$$

from which, it follows that

$$\mathcal{E}(u - v, \phi) = 0,$$

thus showing that $u - v$ is harmonic in B.

(b) If $\|G^B 1\|_{L^\infty} < \infty$, then by Lemma 6.1, the function $v = G^B f$ belongs to $\mathcal{F}(B)$ and satisfies

$$\mathcal{E}(v, \phi) = (f, \phi)$$

for any $\phi \in \mathcal{F}(B)$, that is, the function v solves weakly the equation $\mathcal{L}v = f$ in B. We conclude by (a) that $u - G^B f$ is harmonic in B. □

The following gives the oscillation of the weak solution of the Poisson-type equation.

Lemma 6.7. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2 without killing part. Assume that conditions (FK_*), (TJ_*), (OSL_*) are all satisfied. Let $C_0 \ge 1$ and Ω be any open subset of M containing a ball $B_* := B_*(x_0, r)$ of radius $r \in (0, C_0 W(x_0, \overline{R})^{1/\beta})$. If the function $u \in \mathcal{F}(\Omega) \cap L^\infty$ solves weakly the equation (6.5) for $f \in L^2 \cap L^\infty(\Omega)$, then for any $0 < \rho \le r$*

$$\operatorname*{osc}_{B_*(x_0,\rho)} u \le C \left(\frac{\rho}{r}\right)^\gamma \|u\|_{L^\infty(\Omega)} + C r^\beta \|f\|_{L^\infty(B_*)}, \tag{6.6}$$

where γ is the constant from condition (OSL_*) and C is independent of $B_*, u, \Omega, \rho, f, \overline{R}$. Consequently,

$$(\mathrm{VD}) + (\mathrm{RVD}) + (\mathrm{TJ}) + (\mathrm{Gcap}) + (\mathrm{PI}) \Rightarrow \text{ inequality (6.6)}. \tag{6.7}$$

Proof. We consider two cases. Let δ_* be the constant from ($E_{\leq}^*$) and σ_* be the constant from (OSL_*).

Case 1. $r < (\delta_* \wedge \sigma_*)W(x_0, \overline{R})^{1/\beta}$. Note that condition ($FK_*$) implies condition ($E_{\leq}^*$) by (6.2), that is, we have

$$\|G^{B_*}1\|_{L^\infty} \leq Cr^\beta.$$

From this, we see that

$$\|G^{B_*}f\|_{L^\infty(B_*)} \leq \|G^{B_*}1\|_{L^\infty}\|f\|_{L^\infty(B_*)} \leq Cr^\beta\|f\|_{L^\infty(B_*)}. \tag{6.8}$$

In particular, we have $\|G^{B_*}1\|_{L^\infty} < \infty$. Consider the function

$$v := u - G^{B_*}f.$$

Clearly, we see that $v \in \mathcal{F}(\Omega) \cap L^\infty$. By Proposition 6.6, the function v is harmonic in B_*. It follows from condition (OSL_*) that for any $0 < \rho \leq r$

$$\operatorname*{osc}_{B_*(x_0,\rho)} v \leq C\left(\frac{\rho}{r}\right)^\gamma \left(r^\beta T_{\frac{3}{4}B_*,B_*}(|v|) + \|v\|_{L^\infty(B_*)}\right). \tag{6.9}$$

Since $u = G^{B_*}f = 0$ in Ω^c, using Proposition 9.5 in Appendix, we obtain by condition (TJ_*) that

$$\begin{aligned}
T_{\frac{3}{4}B_*,B_*}(v) &\leq \operatorname*{esup}_{x\in\frac{3}{4}B_*}\int_{B_*^c}(|u(y)| + |G^{B_*}f(y)|)J(x,dy) \\
&\leq \left(\|u\|_{L^\infty(\Omega)} + \|G^{B_*}f\|_{L^\infty(B_*)}\right)\operatorname*{esup}_{x\in\frac{3}{4}B_*}\int_{B_*^c}J(x,dy) \\
&\leq \left(\|u\|_{L^\infty(\Omega)} + \|G^{B_*}f\|_{L^\infty(B_*)}\right)\operatorname*{esup}_{x\in\frac{3}{4}B_*}\int_{B_*(x,r/4)^c}J(x,dy) \\
&\leq \left(\|u\|_{L^\infty(\Omega)} + \|G^{B_*}f\|_{L^\infty(B_*)}\right)\frac{C}{(r/4)^\beta}.
\end{aligned}$$

Substituting this into (6.9) and using the fact that

$$\|v\|_{L^\infty(B_*)} \leq \|u\|_{L^\infty(B_*)} + \|G^{B_*}f\|_{L^\infty(B_*)},$$

we obtain

$$\operatorname*{osc}_{B_*(x_0,\rho)} v \leq C\left(\frac{\rho}{r}\right)^\gamma \left(\|u\|_{L^\infty(\Omega)} + \|G^{B_*}f\|_{L^\infty(B_*)}\right). \tag{6.10}$$

Therefore, we conclude by (6.10), (6.8) that

$$\begin{aligned}
\operatorname*{osc}_{B_*(x_0,\rho)} u &\leq \operatorname*{osc}_{B_*(x_0,\rho)} v + \operatorname*{osc}_{B_*(x_0,\rho)} G^{B_*}f \\
&\leq C\left(\frac{\rho}{r}\right)^\gamma \left(\|u\|_{L^\infty(\Omega)} + \|G^{B_*}f\|_{L^\infty(B_*)}\right) + 2\|G^{B_*}f\|_{L^\infty(B_*)} \\
&\leq C\left(\frac{\rho}{r}\right)^\gamma \|u\|_{L^\infty(\Omega)} + Cr^\beta\|f\|_{L^\infty(B_*)},
\end{aligned}$$

thus showing (6.6).

Case 2. $(\delta_* \wedge \sigma_*)W(x_0, \overline{R})^{1/\beta} \leq r < C_0W(x_0, \overline{R})^{1/\beta}$ with $C_0 \geq 1$ when $\overline{R} < \infty$.

If $\rho < \frac{1}{3}(\delta_*\wedge\sigma_*)W(x_0, \overline{R})^{1/\beta}$, then, applying the result in Case 1 for $r = \frac{1}{2}(\delta_*\wedge\sigma_*)W(x_0, \overline{R})^{1/\beta}$, we obtain that

$$\begin{aligned}
\operatorname*{osc}_{B_*(x_0,\rho)} u &\leq C\left(\frac{\rho}{\frac{1}{2}(\delta_* \wedge \sigma_*)W(x_0, \overline{R})^{1/\beta}}\right)^\gamma \|u\|_{L^\infty(\Omega)} \\
&\quad + C\left(\frac{1}{3}(\delta_* \wedge \sigma_*)W(x_0, \overline{R})^{1/\beta}\right)^\beta \|f\|_{L^\infty(B_*(x_0,\frac{1}{2}(\delta_*\wedge\sigma_*)W(x_0,\overline{R})^{1/\beta}))}
\end{aligned}$$

$$\leq (2(\delta_* \wedge \sigma_*)^{-1}C_0)^\gamma C \left(\frac{\rho}{C_0 W(x_0,\overline{R})^{1/\beta}}\right)^\gamma \|u\|_{L^\infty(\Omega)} + Cr^\beta \|f\|_{L^\infty(B_*)}$$
$$\leq (2(\delta_* \wedge \sigma_*)^{-1}C_0)^\gamma C \left(\frac{\rho}{r}\right)^\gamma \|u\|_{L^\infty(\Omega)} + Cr^\beta \|f\|_{L^\infty(B_*)},$$

which is (6.6).

If $\rho \geq \frac{1}{3}(\delta_* \wedge \sigma_*)W(x_0,\overline{R})^{1/\beta}$, then, $\frac{r}{\rho} \leq \frac{3C_0}{\delta_* \wedge \sigma_*}$. Hence,

$$\operatorname*{osc}_{B_*(x_0,\rho)} u = \left(\frac{r}{\rho}\right)^\gamma \left(\frac{\rho}{r}\right)^\gamma \operatorname*{osc}_{B_*(x_0,\rho)} u \leq 2\left(\frac{3C_0}{(\delta_* \wedge \sigma_*)}\right)^\gamma \left(\frac{\rho}{r}\right)^\gamma \|u\|_{L^\infty(\Omega)},$$

which implies (6.6).

Finally, the implication (6.7) follows directly from Lemma 5.19 by using the facts that (PI) $\Rightarrow$ (PI_*) $\Rightarrow$ (FK_*) (see (4.15) and (4.33)) and (TJ) $\Rightarrow$ (TJ_*) (see (4.12)). □

6.3. **Estimates for the heat semigroup solutions.** We derive the L^∞-estimate of the heat semigroup solutions on any open subset. The following gives the $L^1 \to L^\infty$ ultra-contractivity of the heat semigroup $\{P_t^{B_*}\}$ from condition (Nash_*) (cf. [2, Theorem 2.1]).

Lemma 6.8. *Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form in L^2. If condition (Nash_*) holds, then for any ball $B_* := B_*(x_0,r)$ of radius $r > 0$, the operator $P_t^{B_*}$ satisfies that for any $t > 0$*

$$\|P_t^{B_*}\|_{L^1\to L^\infty} \leq \|P_{t/2}^{B_*}\|^2_{L^1\to L^2} \leq \frac{C(\nu)r^{\beta/\nu}}{V_*(x_0,r)} \exp\left(tW(x_0,\overline{R})^{-1}\right) t^{-\frac{1}{v}}, \tag{6.11}$$

where $\nu > 0$ comes from condition (Nash_*). Consequently, we have for any $t > 0$

$$\|P_t^{B_*} f\|_{L^\infty} \leq \|P_t^{B_*}\|_{L^1\to L^\infty} \|f\|_{L^1(B_*)} \leq \frac{C(\nu)r^{\beta/\nu}}{V_*(x_0,r)} \exp\left(tW(x_0,\overline{R})^{-1}\right) t^{-\frac{1}{v}} \|f\|_{L^1(B_*)}. \tag{6.12}$$

Moreover, we have by (4.21)

$$(\text{VD}) + (\text{RVD}) + (\text{PI}) \Rightarrow (\text{Nash}_*) \Rightarrow \textit{inequalities (6.11) and (6.12)}.$$

Proof. Since $P_t^{B_*}$ is symmetric, we see by duality

$$\|P_t^{B_*}\|_{L^1\to L^2} = \|P_t^{B_*}\|_{L^2\to L^\infty},$$

from which, using the semigroup property, we see that

$$\|P_t^{B_*}\|_{L^1\to L^\infty} \leq \|P_{t/2}^{B_*}\|^2_{L^1\to L^2}.$$

By condition (Nash_*), for all $B_* := B_*(x_0,r)$ with $r > 0$ and all $u \in \mathcal{F}(B_*)$

$$\|u\|_2^{2(1+\nu)} \leq \frac{Cr^\beta}{V_*(x_0,r)^\nu} \left(\mathcal{E}(u,u) + W(x_0,\overline{R})^{-1}\|u\|_2^2\right) \|u\|_1^{2\nu} =: A\left(\mathcal{E}(u,u) + \delta\|u\|_2^2\right) \|u\|_1^{2\nu},$$

where A, δ are given by

$$A = \frac{Cr^\beta}{V_*(x_0,r)^\nu} \quad \text{and} \quad \delta = W(x_0,\overline{R})^{-1}.$$

Applying [2, Theorem 2.1 and its proof on Line 8 on p.252] wherein ν being replaced by $\frac{2}{\nu}$, it follows that

$$\exp(-2\delta t)\|P_t^{B_*} f\|_2^2 = u(t) \leq \left(\frac{2\nu t}{A}\right)^{-1/\nu} \|f\|_1^2$$

for any $t > 0$ and any non-negative f in $L^2(B_*) \cap L^1$, that is,

$$\|P_t^{B_*}\|^2_{L^1\to L^2} \leq \exp(2\delta t)\left(\frac{A}{2\nu t}\right)^{1/\nu} = \left(\frac{C}{2\nu}\right)^{1/\nu} \frac{r^{\beta/\nu}}{V_*(x_0,r)} \exp\left(2tW(x_0,\overline{R})^{-1}\right) t^{-\frac{1}{v}},$$

thus showing (6.11). □

Lemma 6.9. *Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form in L^2. Let $B_* := B_*(x_0,r) \subset M$ be a ball of radius $r>0$, and Ω an open subset of B_*. Set $u = P_t^\Omega f$ for $f \in L^1(\Omega)\cap L^2$. If condition (Nash$_*$) holds, then for any $t>0$,*

$$\|\partial_t u(\cdot,t)\|_{L^\infty(\Omega)} \leq \frac{Cr^{\beta/\nu}}{V_*(x_0,r)} \exp\left(tW(x_0,\overline{R})^{-1}\right) t^{-(1+\frac{1}{v})}\|f\|_{L^1(\Omega)}, \tag{6.13}$$

where $\partial_t u(\cdot,t)$ is the Fréchet derivative of the $L^2(\Omega)$-valued function $t\mapsto u(\cdot,t)$. Moreover, for all $t\geq s>0$,

$$\|u(\cdot,t)-u(\cdot,s)\|_{L^\infty} \leq C(t-s)\frac{r^{\beta/\nu}}{V_*(x_0,r)} \exp\left(sW(x_0,\overline{R})^{-1}\right) s^{-(1+\frac{1}{v})}\|f\|_{L^1(\Omega)}. \tag{6.14}$$

Here $\nu>0$ is the same as in condition (Nash$_*$) and constant C depends only on condition (Nash$_*$).

Proof. Let $f\in L^1(\Omega)\cap L^2$ be non-negative in M. Since P_t^Ω is contractive in L^2 and

$$P_t^\Omega f = P_s^\Omega P_{t-s}^\Omega f$$

for any $t>0$ and any $s\in(0,t)$, we have

$$\partial_t(P_t^\Omega f) = P_s^\Omega(\partial_t P_{t-s}^\Omega f).$$

From this and using the following general inequality (see [18, Lemma 5.4])

$$\|\partial_s(P_s^\Omega f)\|_{L^2(\Omega)} \leq \frac{2}{s}\|P_{s/2}^\Omega f\|_{L^2(\Omega)} \quad \text{for any } s>0,\ f\in L^2, \tag{6.15}$$

we obtain, since $\Omega\subset B_*$, that

$$\begin{aligned}\|\partial_t(P_t^\Omega f)\|_{L^\infty} &= \|P_s^\Omega\partial_t(P_{t-s}^\Omega f)\|_\infty \leq \|P_s^\Omega\|_{L^2\to L^\infty}\|\partial_t(P_{t-s}^\Omega f)\|_{L^2}\\ &\leq \|P_s^\Omega\|_{L^2\to L^\infty}\cdot\frac{2}{t-s}\|P_{(t-s)/2}^\Omega f\|_{L^2}\\ &\leq \frac{2}{t-s}\|P_s^{B_*}\|_{L^2\to L^\infty}\|P_{(t-s)/2}^{B_*}\|_{L^1\to L^2}\|f\|_{L^1(\Omega)}.\end{aligned}$$

Setting $s=t/2$ in the above inequality and using the fact that

$$\|P_t^{B_*}\|_{L^1\to L^2} = \|P_t^{B_*}\|_{L^2\to L^\infty}, \tag{6.16}$$

it follows from (6.11) that

$$\begin{aligned}\|\partial_t(P_t^\Omega f)\|_{L^\infty} &\leq \frac{4}{t}\|P_{t/2}^{B_*}\|_{L^1\to L^2}\|P_{t/4}^{B_*}\|_{L^1\to L^2}\|f\|_{L^1(\Omega)}\\ &\leq \frac{Cr^{\beta/\nu}}{V_*(x_0,r)}\exp\left(tW(x_0,\overline{R})^{-1}\right)t^{-1-\frac{1}{v}}\|f\|_{L^1(\Omega)},\end{aligned}$$

thus showing (6.13).

Finally, we show (6.14). For simplicity, let $t>s\geq 2\tau>0$. Then

$$\begin{aligned}\|P_t^\Omega f - P_s^\Omega f\|_{L^\infty(\Omega)} &= \|P_\tau^\Omega(P_{t-\tau}^\Omega f - P_{s-\tau}^\Omega f)\|_{L^\infty(\Omega)}\\ &\leq \|P_\tau^\Omega\|_{L^2\to L^\infty}\|P_{t-\tau}^\Omega f - P_{s-\tau}^\Omega f\|_{L^2(\Omega)}.\end{aligned}$$

By (6.15) and using the fact that $t\mapsto\|P_t^{B_*}f\|_{L^2(B_*)}$ is non-increasing in $(0,\infty)$, we see that

$$\begin{aligned}\|P_{t-\tau}^\Omega f - P_{s-\tau}^\Omega f\|_{L^2(\Omega)} &= \left\|\int_{s-\tau}^{t-\tau}\partial_\xi(P_\xi^\Omega f)d\xi\right\|_{L^2(\Omega)}\\ &\leq \int_{s-\tau}^{t-\tau}\frac{2}{\xi}\|P_{\xi/2}^\Omega f\|_{L^2}d\xi\\ &\leq (t-s)\frac{2}{\tau}\|P_{\tau/2}^\Omega f\|_{L^2} \leq (t-s)\frac{2}{\tau}\|P_{\tau/2}^\Omega\|_{L^1\to L^2}\|f\|_{L^1}.\end{aligned}$$

Therefore, using (6.16) and (6.11), we conclude that

$$\begin{aligned}\|P_t^\Omega f - P_s^\Omega f\|_{L^\infty(\Omega)} &\le (t-s)\frac{2}{\tau}\|P_\tau^\Omega\|_{L^2\to L^\infty}\|P_{\tau/2}^\Omega\|_{L^1\to L^2}\|f\|_{L^1(\Omega)} \\ &\le (t-s)\frac{2}{\tau}\|P_\tau^{B_*}\|_{L^1\to L^2}\|P_{\tau/2}^{B_*}\|_{L^1\to L^2}\|f\|_{L^1(\Omega)} \quad (\text{since } \Omega\subset B_*) \\ &\le C(t-s)\frac{r^{\beta/\nu}}{V_*(x_0,r)}\exp\left(\tau W(x_0,\overline{R})^{-1}\right)\tau^{-(1+\frac{1}{v})}\|f\|_{L^1(\Omega)},\end{aligned}$$

thus showing (6.14) by letting $\tau = \frac{s}{2}$. □

6.4. **Hölder continuity of the heat semigroup solutions.** We derive that the heat semigroup solutions are locally Hölder continuous.

Lemma 6.10. *Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form in L^2 without killing part. Let $C_0\ge 1$ and Ω be a non-empty open subset of a ball $B_*(x_0,R)$ with $R\in(0,C_0W(x_0,\overline{R})^{1/\beta})$, and let $u(x,t)=P_t^\Omega f(x)$ for $f\in L^1\cap L^2(\Omega)$. Assume that conditions (VD), (RVD), (Gcap), (TJ), (PI) are all satisfied. Then, for any x and $r>0$ so that $B_*(x,r)\subset\Omega$, and for any $t>0$ and $\rho>0$ so that $\rho^\beta\le t\wedge r^\beta$, we have*

$$\operatorname*{osc}_{B_*(x,\rho)} u(\cdot,t) \le \frac{C}{V_*(x_0,R)}\left(\frac{R^\beta}{t}\right)^{1/v}\exp\left(tW(x_0,\overline{R})^{-1}\right)\left(\frac{\rho}{t^{1/\beta}\wedge r}\right)^\theta\|f\|_{L^1(\Omega)}, \tag{6.17}$$

where C is a positive number depending only on constants in the hypothesis, and θ is given by

$$\theta = \frac{\gamma\beta}{\gamma+\beta}. \tag{6.18}$$

Here γ is the same as in Lemma 6.7, and ν comes from condition (Nash$_*$).

Proof. By (6.12), we see for any $t>0$

$$\|u(\cdot,t)\|_{L^\infty} \le \|P_t^{B_*(x_0,R)}f\|_{L^\infty} \le \frac{CR^{\beta/\nu}}{V_*(x_0,R)}\exp\left(tW(x_0,\overline{R})^{-1}\right)t^{-\frac{1}{v}}\|f\|_{L^1}. \tag{6.19}$$

For any $t>0$, the function $u(\cdot,t)$ belongs to $\mathrm{dom}(\mathcal{L}^\Omega)$, is Fréchet differentiable with respect to t in $L^2(\Omega)$, and satisfies weakly

$$\partial_t u(\cdot,t) = -\mathcal{L}^\Omega u(\cdot,t),$$

that is, for any $\phi\in\mathcal{F}(\Omega)$ and $t>0$,

$$\mathcal{E}(u(\cdot,t),\phi) = (\mathcal{L}^\Omega u(\cdot,t),\phi) = -(\partial_t u(\cdot,t),\phi).$$

By Lemma 6.9, we have $\partial_t u(\cdot,t)\in L^\infty(\Omega)$ for any $t>0$.

Let $\rho\le t^{1/\beta}\wedge r<r$, and $r'\in(\rho,r)$ be a number to be specified later on. Applying Lemma 6.7 for the ball $B_*(x,r')\subset\Omega$ and then using (6.13), (6.19), we have for any $t>0$

$$\begin{aligned}\operatorname*{osc}_{B_*(x_0,\rho)} u(\cdot,t) &\le C\left(\left(\frac{\rho}{r'}\right)^\gamma\|u\|_{L^\infty(\Omega)} + (r')^\beta\|\partial_t u\|_{L^\infty(\Omega)}\right) \\ &\le \frac{C}{V_*(x_0,R)}\left(\frac{R^\beta}{t}\right)^{1/v}\exp\left(tW(x_0,\overline{R})^{-1}\right)\left(\left(\frac{\rho}{r'}\right)^\gamma + \frac{(r')^\beta}{t}\right)\|f\|_{L^1} \\ &\le \frac{C}{V_*(x_0,R)}\left(\frac{R^\beta}{t}\right)^{1/v}\exp\left(tW(x_0,\overline{R})^{-1}\right)\left(\left(\frac{\rho}{r'}\right)^\gamma + \frac{(r')^\beta}{\tau}\right)\|f\|_{L^1},\end{aligned} \tag{6.20}$$

where

$$\tau = t\wedge r^\beta \ge \rho^\beta.$$

Now choose r' such that

$$\left(\frac{\rho}{r'}\right)^\gamma = \frac{(r')^\beta}{\tau},$$

that is,

$$r' = \rho^{\frac{\gamma}{\gamma+\beta}} \tau^{\frac{1}{\gamma+\beta}}.$$

With this choice of r', we have that $r' \in (\rho, r)$, as desired, since

$$r' = \rho^{\frac{\gamma}{\gamma+\beta}} \tau^{\frac{1}{\gamma+\beta}} \geq \rho^{\frac{\gamma}{\gamma+\beta}} \rho^{\frac{\beta}{\gamma+\beta}} = \rho,$$
$$r' = \rho^{\frac{\gamma}{\gamma+\beta}} \tau^{\frac{1}{\gamma+\beta}} \leq \tau^{\frac{\gamma}{(\gamma+\beta)\beta}} \tau^{\frac{1}{\gamma+\beta}} = \tau^{1/\beta} \leq r.$$

Noting that

$$\frac{(r')^\beta}{\tau} = \rho^{\frac{\gamma\beta}{\gamma+\beta}} \tau^{-\frac{\gamma}{\gamma+\beta}} = \left(\frac{\rho^\beta}{\tau}\right)^{\gamma/(\gamma+\beta)},$$

we see that (6.17) follows directly from (6.20). □

For any set $U \subset M$ and $r > 0$, denote by U_r by the open r-neighborhood of U, that is

$$U_r = \bigcup_{x \in U} B_*(x, r).$$

The following gives the locally Hölder continuity of the heat semigroup solutions in an open subset.

Lemma 6.11. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2 without killing part. Let $C_0 \geq 1$ and Ω be a non-empty open subset of a ball $B_*(x_0, R)$ with $R \in (0, C_0 W(x_0, \overline{R})^{1/\beta})$, and let $u(x,t) = P_t^\Omega f(x)$ for $f \in L^1 \cap L^2(\Omega)$. Assume that conditions (VD), (RVD), (Gcap), (TJ), (PI) are all satisfied. Then the following properties are true.*

- *(a) For any $t > 0$, the function $u(\cdot, t)$ has a locally Hölder continuous version $\widetilde{u}(\cdot, t)$ in Ω with the Hölder exponent θ. Moreover, the function $\widetilde{u}(x,t)$ is jointly continuous in $(x,t) \in \Omega \times (0, \infty)$.*
- *(b) For any open subset $U \subset \Omega$ and $r > 0$ with $U_r \subset \Omega$, we have for all $x, x' \in U$ and all $t > 0$*

$$\left|\widetilde{u}(x,t) - \widetilde{u}(x',t)\right| \leq \frac{C}{V_*(x_0, R)} \left(\frac{R^\beta}{t}\right)^{1/v} \exp\left(tW(x_0, \overline{R})^{-1}\right) \left(\frac{d_*(x,x')}{t^{1/\beta} \wedge r}\right)^\theta \|f\|_{L^1(\Omega)}. \tag{6.21}$$

Here θ is given by (6.18), and constant C depends only on constants in the hypothesis.

Proof. (a) By a standard argument, it follows from (6.17) that $u(\cdot, t)$ has a locally Hölder continuous version $\widetilde{u}(\cdot, t)$.

By (6.14), we have for all $t > s > \tau > 0$,

$$\sup_{x \in \Omega} |\widetilde{u}(x,t) - \widetilde{u}(x,s)| \leq C(t-s) \frac{R^{\beta/\nu}}{V_*(x_0, R)} \exp\left(tW(x_0, \overline{R})^{-1}\right) \tau^{-(1+\frac{1}{v})} \|f\|_{L^1(\Omega)},$$

from which, we see that the function $t \mapsto \widetilde{u}(x,t)$ is continuous in $t \in (0, \infty)$ uniformly in $x \in \Omega$. Since the function $x \mapsto \widetilde{u}(x,t)$ is continuous in $x \in \Omega$, we conclude that $\widetilde{u}(x,t)$ is jointly continuous in $(x,t) \in \Omega \times (0, \infty)$.

(b) Note that $B_*(x, r) \subset U_r$ for any $x \in U$. Set $\tau = t \wedge r^\beta$. By Lemma 6.10, we obtain

$$\operatorname*{osc}_{B_*(x,\rho)} \widetilde{u}(\cdot, t) \leq \frac{C}{V_*(x_0, R)} \left(\frac{R^\beta}{t}\right)^{1/\nu} \exp\left(tW(x_0, \overline{R})^{-1}\right) \left(\frac{\rho}{t^{1/\beta} \wedge r}\right)^\theta \|f\|_{L^1(\Omega)}, \tag{6.22}$$

provided that $\rho^\beta \leq \tau$. If $\rho^\beta \geq \tau$, we have by (6.12)

$$\operatorname*{osc}_{B_*(x,\rho)} \widetilde{u}(\cdot, t) \leq 2\|u(\cdot, t)\|_{L^\infty} \leq \frac{C}{V_*(x_0, R)} \left(\frac{R^\beta}{t}\right)^{1/\nu} \exp\left(tW(x_0, \overline{R})^{-1}\right) \|f\|_{L^1(\Omega)}$$
$$= \frac{C}{V_*(x_0, R)} \left(\frac{R^\beta}{t}\right)^{1/\nu} \exp\left(tW(x_0, \overline{R})^{-1}\right) \left(\frac{\rho}{\tau^{1/\beta}}\right)^{-\theta} \left(\frac{\rho}{\tau^{1/\beta}}\right)^\theta \|f\|_{L^1(\Omega)}$$

$$\le \frac{C}{V_*(x_0,R)}\left(\frac{R^\beta}{t}\right)^{1/\nu} \exp\left(tW(x_0,\overline{R})^{-1}\right)\left(\frac{\rho}{\tau^{1/\beta}}\right)^\theta \|f\|_{L^1(\Omega)}.$$

Hence, we see that (6.22) holds for all $\rho > 0$. Taking $\rho = d_*(x,x')$ in (6.22), we obtain (6.21). □

7. Near-diagonal lower estimates

In this section we study the regularity of the heat kernel, and then give its near-diagonal lower bound.

7.1. **Hölder continuity of the heat kernel.** For any non-empty open subset Ω of M, let $\{P_t^\Omega\}$ and $p_t^\Omega(x,y)$ be the heat semigroup and the heat kernel of the form $(\mathcal{E},\mathcal{F}(\Omega))$, respectively. In this subsection, we shall show that for any *bounded* open set $\Omega \subset M$, the heat kernel $p_t^\Omega(x,y)$ exists pointwise in $M \times M \times (0,\infty)$ and is locally, uniformly Hölder continuous. This property is used to derive near-diagonal lower bound of the heat kernel.

If conditions (VD), (RVD), (Gcap), (TJ), and (PI) are all satisfied, we see by Lemma 6.11, the function $P_t^\Omega f(\cdot)$ has a continuous version when $f \in L^1 \cap L^2(M)$, for any bounded open subset Ω of M. The following gives on-diagonal upper estimate and the Hölder continuity of the Dirichlet heat kernel $p_t^\Omega(x,y)$.

Lemma 7.1. *Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form in L^2 without killing part. Assume that conditions (VD), (RVD), (TJ), (Gcap), (PI) are all satisfied. Let $C_0 \ge 1$ and Ω be any non-empty open subset of a ball $B_*(x_0,R)$ with $R \in (0, C_0 W(x_0,\overline{R})^{1/\beta})$. Then the Dirichlet heat kernel $p_t^\Omega(x,y)$ exists and is locally Hölder continuous. Moreover, for each $t > 0$*

$$\sup_{x,y\in\Omega} p_t^\Omega(x,y) \le \frac{C}{V_*(x_0,R)}\left(\frac{R^\beta}{t}\right)^{1/\nu} \exp\left(tW(x_0,\overline{R})^{-1}\right), \tag{7.1}$$

and, for any non-empty open subset $U \subset \Omega$ and $r > 0$ with $U_r \subset \Omega$, and for all $x,x',y,y' \in U$, $t \ge s > 0$,

$$\begin{aligned}\left|p_t^\Omega(x,y) - p_s^\Omega(x',y')\right| \le{}& \frac{C}{V_*(x_0,R)}\left(\frac{R^\beta}{s}\right)^{1/\nu} \exp\left(tW(x_0,\overline{R})^{-1}\right)\\ &\times\left(\left(\frac{d_*(x,x')}{t^{1/\beta}\wedge r}\right)^\theta + \left(\frac{d_*(y,y')}{t^{1/\beta}\wedge r}\right)^\theta + \frac{t-s}{s}\right),\end{aligned} \tag{7.2}$$

where $\theta \in (0,1)$ is defined in (6.18) and $C > 0$ depends only on the constants in the hypothesis.

Proof. Fix an open subset U of Ω and fix a number $r > 0$ with $U_r \subset \Omega$. By Lemma 6.11, for any $f \in L^1 \cap L^2(M)$ and $t > 0$, the function $P_t^\Omega f$ is locally, uniformly Hölder continuous, that is, for all $x, x' \in U$ and all $t > 0$,

$$\left|P_t^\Omega f(x) - P_t^\Omega f(x')\right| \le \frac{C}{V_*(x_0,R)}\left(\frac{R^\beta}{t}\right)^{1/\nu} \exp\left(tW(x_0,\overline{R})^{-1}\right)\left(\frac{d_*(x,x')}{t^{1/\beta}\wedge r}\right)^\theta \|f\|_{L^1}, \tag{7.3}$$

where $C > 0$ depends only on the constants in the hypothesis, but is independent of $B_*(x_0,R)$, Ω, U, r, t, x, x', f, $\overline{R}$.

By (6.12), we have for all $t > 0$ and all $x \in M$,

$$\left|P_t^\Omega f(x)\right| \le \frac{C}{V_*(x_0,R)}\left(\frac{R^\beta}{t}\right)^{1/\nu} \exp\left(tW(x_0,\overline{R})^{-1}\right)\|f\|_{L^1(\Omega)}. \tag{7.4}$$

By (6.14), we have for all $t > s > 0$ and all $x \in M$,

$$\left|P_t^\Omega f(x) - P_s^\Omega f(x)\right| \le \frac{C}{V_*(x_0,R)}\left(\frac{R^\beta}{s}\right)^{1/\nu} \exp\left(sW(x_0,\overline{R})^{-1}\right)\frac{t-s}{s}\|f\|_{L^1(\Omega)}. \tag{7.5}$$

Since $P_t^\Omega f$ is continuous, using [11, Theorem 2.1 and Corollary 4.2], it follows from (7.4) that the following are true (see also [10, Lemma 5.13, p.506]):

(1) the Dirichlet heat kernel p_t^Ω exists pointwise for $(x,y,t) \in \Omega\times\Omega\times(0,\infty)$;
(2) both $p_t^\Omega(x,\cdot)$ and $p_t^\Omega(\cdot,x)$ are continuous in Ω for every $t>0$ and every $x\in\Omega$;
(3) the inequality (7.1) holds true.

We show (7.2). Indeed, fix a point y in U. Setting $f = p_t^\Omega(\cdot,y)$ in (7.3), we obtain for all $x,x'\in U$ and $t>0$,

$$\left|p_{2t}^\Omega(x,y) - p_{2t}^\Omega(x',y)\right| \le \frac{C}{V_*(x_0,R)}\left(\frac{R^\beta}{t}\right)^{1/\nu}\exp\left(tW(x_0,\overline{R})^{-1}\right)\left(\frac{d_*(x,x')}{t^{1/\beta}\wedge r}\right)^\theta.$$

Since the three points $x,x',y\in U$ are arbitrary in the above inequality, replacing x,x',y by y,y',x' respectively and using the symmetry $p_{2t}^\Omega(x,y) = p_{2t}^\Omega(y,x)$, we have

$$\left|p_{2t}^\Omega(x',y) - p_{2t}^\Omega(x',y')\right| \le \frac{C}{V_*(x_0,R)}\left(\frac{R^\beta}{t}\right)^{1/\nu}\exp\left(tW(x_0,\overline{R})^{-1}\right)\left(\frac{d_*(y,y')}{t^{1/\beta}\wedge r}\right)^\theta.$$

Summing up the above two inequalities and renaming $2t$ by t, we obtain

$$\left|p_t^\Omega(x,y) - p_t^\Omega(x',y')\right| \le \frac{C}{V_*(x_0,R)}\left(\frac{R^\beta}{t}\right)^{1/\nu}\exp\left(tW(x_0,\overline{R})^{-1}\right)\left(\left(\frac{d_*(x,x')}{t^{1/\beta}\wedge r}\right)^\theta + \left(\frac{d_*(y,y')}{t^{1/\beta}\wedge r}\right)^\theta\right).$$

Moreover, for $x',y'\in\Omega$ and $t\ge s>0$, applying (7.5) with t replaced by $t-\frac{s}{2}$, s by $\frac{s}{2}$, x by x' and $f = p_{s/2}^\Omega(\cdot,y')$, we obtain

$$\left|p_t^\Omega(x',y') - p_s^\Omega(x',y')\right| \le \frac{C}{V_*(x_0,R)}\left(\frac{R^\beta}{s/2}\right)^{1/\nu}\exp\left(2^{-1}sW(x_0,\overline{R})^{-1}\right)\frac{t-s}{s/2}.$$

Adding up the above two inequalities, we obtain (7.2). □

By Remark 4.16 and [13, Corollary 2.9], the following five conditions

$$(\text{VD}) + (\text{RVD}) + (\text{TJ}_q) + (\text{Gcap}) + (\text{PI}) \quad \text{for } 2\le q\le\infty$$

imply the existence of the pointwise defined heat kernel in [13, Definition 6.1]. While, the following gives a refinement of this result by reducing condition (TJ_q) for $2\le q\le\infty$ to condition (TJ), since condition (TJ_q) is stronger than condition (TJ) by [13, Proposition 3.1].

Proposition 7.2. *Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form in L^2. Assume that for any bounded open set $\Omega\subset M$, the Dirichlet heat kernel $p_t^\Omega(x,y)$ exists and is locally Hölder continuous in $(x,y)\in\Omega\times\Omega$ for any $t>0$. Then $(\mathcal{E},\mathcal{F})$ admits a pointwise defined heat kernel $p_t(x,y)$ satisfying the following properties.*

(1) *For any $t,s>0$,*

$$p_{t+s}(x,y) = \int_M p_t(x,z)p_s(z,y)d\mu(z), \quad \forall\, x,y\in M.$$

(2) *For any $t>0$,*

$$\int_M p_t(x,z)d\mu(z)\le 1, \quad \forall\, x\in M.$$

(3) *For any $t,s>0$,*

$$P_{t+s}f(x) = P_tP_sf(x), \quad \forall\, f\in L^2(M),\ x\in M,$$

where

$$P_tf(x) := \int_M p_t(x,y)f(y)d\mu(y), \quad f\in L^2(M),\ t>0.$$

(4) *For any bounded open set $\Omega\subset M$,*

$$p_t(x,y)\ge p_t^\Omega(x,y), \quad \forall\, t>0,\ x,y\in M.$$

Proof. Fix a point $x_0 \in M$. By assumption, for any ball $B_n := B(x_0, n)$ with $n \geq 1$, the locally Hölder continuous heat kernel $p_t^{B_n}$ exists. Since $p_t^{B_n}$ is increasing and non-negative, we can well define

$$p_t(x,y) := \lim_{n\to\infty} p_t^{B_n}(x,y), \quad t>0, x,y \in M.$$

Obviously, the function $p_t(x,y)$ is pointwise defined and is measurable on $M \times M \times (0,\infty)$. The rest proof is motivated by [11, Lemma 5.1].

Property (1) follows from the definition of $p_t(x,y)$, monotone convergence theorem and the identities:

$$p_{t+s}^{B_n}(x,y) = \int_M p_t^{B_n}(x,z) p_s^{B_n}(z,y) d\mu(z), \quad \forall\, t,s>0,\ x,y \in B_n.$$

Property (2) follows from the definition of $p_t(x,y)$, monotone convergence theorem and the inequality:

$$\int_M p_t^{B_n}(x,z) d\mu(z) \leq 1, \quad \forall\, t>0,\ x \in B_n.$$

Since each $p_t^{B_n}$ is locally Hölder continuous, for any $n \geq 1$, $t > 0$ and $f \in L^1 \cap L^2(M)$, the function $x \mapsto P_t^{B_n} f(x)$ is also locally Hölder continuous in B_n. Then, by monotone convergence theorem, we have for any $t > 0$, $0 \leq f \in L^1 \cap L^2(M)$ and $x \in M$,

$$\begin{aligned} P_t f(x) &:= \int_M p_t(x,y) f(y) d\mu(y) = \int_M \lim_{n\to\infty} p_t^{B_n}(x,y) f(y) d\mu(y) \\ &= \lim_{n\to\infty} \int_M p_t^{B_n}(x,y) f(y) d\mu(y) = \lim_{n\to\infty} P_t^{B_n} f(x). \end{aligned}$$

This together with the identities

$$P_{t+s}^{B_n} f(x) = P_t^{B_n} P_s^{B_n} f(x)$$

yields the property (3) for any $0 \leq f \in L^1 \cap L^2(M)$. Then, by the standard approximating arguments, we can extend it to all $f \in L^2(M)$.

Property (4) follows from the definition of $p_t(x,y)$, the monotonicity of $p_t^{B_n}$ in n and the continuities of $p_t^{B_n}$ and p_t^{Ω}. □

Corollary 7.3. *Under the hypothesis of Lemma 7.1, the form $(\mathcal{E}, \mathcal{F})$ admits a pointwise defined heat kernel $p_t(x,y)$ satisfying all properties in Proposition 7.2.*

In the rest of the paper, under the conditions (VD), (RVD), (TJ), (Gcap), (PI), the heat kernel $p_t(x,y)$ is always referred to as that obtained in Corollary 7.3.

7.2. **Derivation of the near-diagonal lower bounds.** In this subsection, we will derive the near diagonal lower estimate of the heat kernel. We introduce condition (LLE$_*$) that is called the *localized lower estimate* under the new metric d_*.

Definition 7.4. We say that condition (LLE$_*$) holds if the following two conditions are true.

1. For any bounded open set $\Omega \subset M$, the Dirichlet heat kernel $p_t^{\Omega}(x,y)$ exists.
2. There exist $c_* > 0$ and $\delta_* \in (0,1)$ such that, for any ball $B_* := B_*(x_0, r)$ with $r \in (0, W(x_0, \overline{R})^{1/\beta})$ and for any $t^{1/\beta} \leq \delta_* r$, we have

$$p_t^{B_*}(x,y) \geq \frac{c_*}{V_*(x_0, t^{1/\beta})}, \quad \mu\text{-a.a. } x,y \in B_*(x_0, \delta_* t^{1/\beta}). \tag{7.6}$$

We say that condition (sLLE$_*$) holds true if (LLE$_*$) holds true and for any non-empty bounded open set $\Omega \subset M$, the Dirichlet heat kernel $p_t^{\Omega}(x,y)$ is locally Hölder continuous in $(x,y,t) \in \Omega \times \Omega \times (0,\infty)$.

We introduce condition (NLE$_*$) that is called the *near-diagonal lower estimate* under the new metric d_*.

Definition 7.5. We say that condition (NLE$_*$) holds if the heat kernel $p_t(x,y)$ exists, and for any $C_0 \geq 1$, there exist two positive constants δ_*, C such that

$$p_t(x,y) \geq \frac{C^{-1}}{V_*(x,t^{1/\beta})} \tag{7.7}$$

for $\mu \times \mu$-almost all $(x,y) \in M \times M$ and all $t < C_0 W(x,\overline{R})$ satisfying

$$d_*(x,y) \leq \delta_* t^{1/\beta}.$$

We say that condition (sNLE$_*$) is satisfied if condition (NLE$_*$) is satisfied and the function $p_t(x,y)$ has a version satisfying $p_{t+s}(x,y) = \int_M p_t(x,z)p_s(z,y)dz$ for any $t,s>0,\ x,y \in M$ and satisfying (2.15) for any $x,y \in M$ and $t < C_0 W(x,\overline{R})$ with $d_*(x,y) \leq \delta_* t^{1/\beta}$.

Lemma 7.6. *Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form in L^2 without killing part. Then*

$$(\mathrm{VD}) + (\mathrm{RVD}) + (\mathrm{TJ}) + (\mathrm{Gcap}) + (\mathrm{PI}) \Rightarrow (\mathrm{VD}_*) + (\mathrm{sLLE}_*) \Rightarrow (\mathrm{sNLE}_*).$$

Proof. Assume that conditions (VD), (RVD), (TJ), (Gcap), (PI) are all satisfied. Note that the constant $C_0 \geq 1$ in Lemma 7.1 can be arbitrary. By Lemma 7.1, we see that for any bounded open subset Ω of M, the Dirichlet heat kernel $p_t^{\Omega}(x,y)$ exists and is locally Hölder continuous in $(x,y,t) \in \Omega \times \Omega \times (0,\infty)$. In particular, for any ball B_*, the Dirichlet heat kernel $p_t^{B_*}(x,y)$ exists and is jointly continuous.

Condition (VD$_*$) holds true by Proposition 4.4. To prove condition (sLLE$_*$) holds. It suffices to show (7.6).

Indeed, by (6.4), we see that condition (S$_*$) is true, that is, there exist two small constant ε, δ_1 in $(0,1)$ such that for all metric balls $B_*(z,r)$ of radius $r < W(z,\overline{R})^{1/\beta}$,

$$P_s^{B_*(z,r')} \mathbf{1}_{B_*(z,r)} \geq \varepsilon \quad \text{in } \frac{1}{4}B_*(z,r'), \tag{7.8}$$

provided that $s \leq (\delta_1 r')^\beta$. We split the proof of inequality (7.6) into two steps.

Step 1. Fix $t \in (0, \delta_1^\beta W(x_0,\overline{R}))$ and set

$$\rho := \delta_1^{-1} t^{1/\beta} < W(x_0,\overline{R})^{1/\beta} \quad \text{and} \quad B_* := B_*(x_0,\rho),$$

so that $t = (\delta_1\rho)^\beta$. We claim that there exist constants c, δ_2 in $(0,1)$ such that

$$p_t^{B_*}(x,y) \geq \frac{c}{V_*(x_0,\rho)} \tag{7.9}$$

for all points x,y in $B_*(x_0,\delta_2 t^{1/\beta})$.

Indeed, for any $x \in B_*$, we have by the Hölder inequality,

$$p_t^{B_*}(x,x) = \int_{B_*} p_{t/2}^{B_*}(x,y)^2 d\mu(y) \geq \frac{1}{\mu(B_*)} \left(\int_{B_*} p_{t/2}^{B_*}(x,y)d\mu(y) \right)^2 = \frac{\left(P_{t/2}^{B_*}\mathbf{1}_{B_*}(x)\right)^2}{\mu(B_*)}.$$

Since the function $P_{t/2}^{B_*}\mathbf{1}_{B_*}$ is continuous by Lemma 6.11 and $t/2 = (\delta_1\rho)^\beta/2 \leq (\delta_1\rho)^\beta$, it follows from (7.8) that,

$$p_t^{B_*}(x,x) \geq \frac{\left(P_{t/2}^{B_*}\mathbf{1}_{B_*}(x)\right)^2}{\mu(B_*)} \geq \frac{\varepsilon^2}{V_*(x_0,\rho)} \quad \text{for any } x \text{ in } \frac{1}{4}B_*. \tag{7.10}$$

On the other hand, since $\rho \in (0, W(x_0,\overline{R})^{1/\beta})$ and $t \in (0, \delta_1^\beta W(x_0,\overline{R}))$, applying (7.2) with $\Omega = B_*$, $U = \frac{1}{4}B_*$ and $r = \frac{3}{4}\rho$, we have for any $x,y \in \frac{1}{4}B_* = B_*(x_0, \frac{1}{4}\delta_1^{-1}t^{1/\beta})$,

$$\left|p_t^{B_*}(x,x) - p_t^{B_*}(x,y)\right| \leq \frac{C}{V_*(x_0,\rho)} \left(\frac{\rho^\beta}{t}\right)^{1/\nu} \exp\left(\frac{\delta_1^\beta W(x_0,\overline{R})}{W(x_0,\overline{R})}\right) \left(\frac{d_*(x,y)}{t^{1/\beta} \wedge ((3/4)\rho)}\right)^\theta$$

$$\le \frac{C}{V_*(x_0,\rho)}\left(\frac{\delta_1^{-\beta}t}{t}\right)^{1/\nu}\left(\frac{d_*(x,y)}{t^{1/\beta}}\right)^{\theta}$$
$$= \frac{C\delta_1^{-\beta/\nu}}{V_*(x_0,\rho)}\left(\frac{d_*(x,y)}{t^{1/\beta}}\right)^{\theta}.$$

Let δ_2 be a constant in $(0, \frac{1}{4}\delta_1^{-1})$ to be determined. If $x, y \in B_*(x_0, \delta_2 t^{1/\beta})$, then

$$\left|p_t^{B_*}(x,x) - p_t^{B_*}(x,y)\right| \le \frac{C'}{V_*(x_0,\rho)}\left(\frac{d_*(x,y)}{t^{1/\beta}}\right)^{\theta} \le \frac{C'(2\delta_2)^{\theta}}{V_*(x_0,\rho)}. \tag{7.11}$$

Combining this and (7.10), we have

$$p_t^{B_*}(x,y) \ge p_t^{B_*}(x,x) - |p_t^{B_*}(x,x) - p_t^{B_*}(x,y)| \ge \frac{\varepsilon^2 - C'(2\delta_2)^{\theta}}{V_*(x_0,\rho)}.$$

Choosing $\delta_2 = \frac{\varepsilon^{2/\theta}}{(2(2C')^{1/\theta})} \wedge (\frac{1}{4}\delta_1^{-1})$ so that

$$C'(2\delta_2)^{\theta} \le \varepsilon^2/2,$$

we obtain (7.9) with $c := \varepsilon^2/2$, thus proving our claim.

Step 2. Let $\delta_* := \delta_1 \wedge \delta_2$. Fix a ball $B_*(x_0, r)$ with $r \in (0, W(x_0, \overline{R})^{1/\beta})$ and some

$$t \le (\delta_* r)^{\beta} < \delta_1^{\beta} W(x_0, \overline{R}).$$

Then,

$$\rho = \delta_1^{-1} t^{1/\beta} \le \delta_*^{-1} t^{1/\beta} \le r,$$

so that $B_*(x_0,\rho) \subset B_*(x_0, r)$. By (7.9), we have, for all

$$x, y \in B_*(x_0, \delta_* t^{1/\beta}) \subset B(x_0, \delta_2 t^{1/\beta}),$$

that

$$p_t^{B_*(x_0,r)}(x,y) \ge p_t^{B_*(x_0,\rho)}(x,y) \ge \frac{c}{V_*(x_0,\rho)}.$$

Since $\rho = \delta_1^{-1} t^{1/\beta} > t^{1/\beta}$, we see by (VD$_*$) that

$$\frac{V_*(x_0,\rho)}{V_*(x_0,t^{1/\beta})} \le C\left(\frac{\delta_1^{-1}t^{1/\beta}}{t^{1/\beta}}\right)^{\alpha_*} = C\delta_1^{-\alpha_*}.$$

Therefore, it follows that for any $x, y \in B_*(x_0, \delta_* t^{1/\beta})$ and any $t \le (\delta_* r)^{\beta}$,

$$p_t^{B_*(x_0,r)}(x,y) \ge \frac{c'}{V_*(x_0,t^{1/\beta})},$$

thus showing (7.6). Therefore, condition (sLLE$_*$) holds.

It remains to show the implication (VD$_*$) + (sLLE$_*$) $\Rightarrow$ (sNLE$_*$).

Indeed, assume that condition (sLLE$_*$) holds true. Note that the hypothesis of Proposition 7.2 is satisfied. Then, the existence of the global heat kernel $p_t(x, y)$ follows from Proposition 7.2. In the sequel, we divide the proof of (7.7) into two steps.

Step 1. Let δ_* be the constant from (7.6). For any $x \in M$ and $s \in (0, \delta_*^{\beta} W(x, \overline{R}))$, set

$$r := \delta_*^{-1} s^{1/\beta} < W(x, \overline{R})^{1/\beta}$$

so that $s \le (\delta_* r)^{\beta}$. Then we have by (7.6) and Proposition 7.2(4) that

$$p_s(z,w) \ge p_s^{B_*(x,r)}(z,w) \ge \frac{c}{V_*(x,s^{1/\beta})}, \quad \forall\, z, w \in B_*(x, \delta_* s^{1/\beta}). \tag{7.12}$$

Step 2. Fix $C_0 \ge 1$ and take an integer $n > 1$ so that

$$n - 1 \le \frac{C_0}{\delta_*^{\beta}} < n.$$

Set

$$\delta_3 := \frac{\delta_*}{2n^{1/\beta}}.$$

Fix $x, y \in M$ and $t < C_0 W(x, \overline{R})$ with $d_*(x, y) \le \delta_3 t^{1/\beta}$. By the pointwise semigroup property in Proposition 7.2(1), we have

$$\begin{aligned} p_t(x,y) &= \int_{M^{n-1}} p_{t/n}(x,z_1)p_{t/n}(z_1,z_2)\cdots p_{t/n}(z_{n-1},y)dz_1dz_2\cdots dz_{n-1} \\ &\ge \int_{B(x,\delta_3 t^{1/\beta})^{n-1}} p_{t/n}(x,z_1)p_{t/n}(z_1,z_2)\cdots p_{t/n}(z_{n-1},y)dz_1dz_2\cdots dz_{n-1}. \end{aligned}$$

Since

$$B(x,\delta_3 t^{1/\beta}) \subset B(x,\delta_*(t/n)^{1/\beta}), \quad y \in B(x,\delta_*(t/n)^{1/\beta}) \ \text{ and } \ t/n < \delta_*^\beta W(x,\overline{R}),$$

we obtain by the above inequality and (7.12) with $s = t/n$ that

$$\begin{aligned} p_t(x,y) &\ge \int_{B(x,\delta_3 t^{1/\beta})^{n-1}} \left(\frac{c}{V_*(x,(t/n)^{1/\beta})}\right)^n dz_1dz_2\cdots dz_{n-1} \\ &= \left(\frac{c}{V_*(x,(t/n)^{1/\beta})}\right)^n \left(V_*(x,\delta_3 t^{1/\beta})\right)^{n-1} \\ &\ge \left(\frac{c}{V_*(x,t^{1/\beta})}\right)^n \left(V_*(x,\delta_3 t^{1/\beta})\right)^{n-1} = \frac{c}{V_*(x,t^{1/\beta})}\left(\frac{cV_*(x,\delta_3 t^{1/\beta})}{V_*(x,t^{1/\beta})}\right)^{n-1}. \end{aligned}$$

Moreover, by (VD$_*$),

$$\frac{V_*(x,\delta_3 t^{1/\beta})}{V_*(x,t^{1/\beta})} \ge C\left(\frac{\delta_3 t^{1/\beta}}{t^{1/\beta}}\right)^{\alpha_*} = c'.$$

Combining the above two inequalities, we obtain (7.7), thus proving (sNLE$_*$). □

Lemma 7.7. *Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form in L^2. If condition (VD) holds, then*

$$\begin{aligned} (\text{sLLE}_*) &\Leftrightarrow (\text{sLLE}), \qquad (7.13)\\ (\text{LLE}_*) &\Leftrightarrow (\text{LLE}),\\ (\text{sNLE}_*) &\Leftrightarrow (\text{sNLE}),\\ (\text{NLE}_*) &\Leftrightarrow (\text{NLE}). \end{aligned}$$

Proof. Let C_W be the constant from (2.8) and (2.9). If either (sLLE$_*$) or (sLLE) holds true, then for any bounded open set $\Omega \subset M$, the Dirichlet heat kernel $p_t^\Omega(x,y)$ exists and is locally Hölder continuous in $(x,y,t) \in \Omega\times\Omega\times(0,\infty)$.

(1). We show implication (sLLE$_*$) $\Rightarrow$ (sLLE).

Assume that conditions (VD), (sLLE$_*$) are satisfied. Fix a ball $B := B(x_0,R)$ with $R \in (0,\overline{R})$. It suffices to prove (2.17). Indeed, let c_*, δ_* be the constants from (7.6). Let

$$r := L^{-1}F(x_0,R) < W(x_0,\overline{R})^{1/\beta} \ \text{ so that } \ W(x_0,R) = F(x_0,R)^\beta = (Lr)^\beta.$$

By (4.7), we have

$$B_*(x_0,r) = B_*(x_0,L^{-1}F(x_0,R)) \subset B(x_0,R) = B.$$

Thus, by (7.6) and (VD$_*$), we have that for any

$$t \le (\delta_* r)^\beta = \delta_*^\beta L^{-\beta} W(x_0,R)$$

and for any x, y in $B_*(x_0,\delta_* t^{1/\beta})$,

$$p_t^B(x,y) \ge p_t^{B_*(x_0,r)}(x,y) \ge \frac{c}{V_*(x_0,t^{1/\beta})}. \qquad (7.14)$$

On the other hand, let

$$R_1 := F^{-1}(x_0, L^{-1}L_0 t^{1/\beta}) = W^{-1}(x_0, (L^{-1}L_0)^\beta t) \quad \text{so that} \quad W(x_0, R_1) = (L^{-1}L_0)^\beta t.$$

By (4.6),

$$B_*(x_0, t^{1/\beta}) \subset B(x_0, R_1) = B(x_0, W^{-1}(x_0, (L^{-1}L_0)^\beta t)),$$

from which, we see that

$$V_*(x_0, t^{1/\beta}) \le V(x_0, W^{-1}(x_0, (L^{-1}L_0)^\beta t)). \tag{7.15}$$

Let

$$R_2 := F^{-1}(x_0, L^{-1}\delta_* t^{1/\beta}) = W^{-1}(x_0, (L^{-1}\delta_*)^\beta t) \quad \text{so that} \quad W(x_0, R_2) = (L^{-1}\delta_*)^\beta t.$$

By (4.6), we obtain

$$B(x_0, W^{-1}(x_0, (L^{-1}\delta_*)^\beta t)) = B(x_0, R_2) \subset B_*(x_0, \delta_* t^{1/\beta}). \tag{7.16}$$

Combining (7.14)-(7.16), we obtain for any

$$t \le \delta_*^\beta L^{-\beta} W(x_0, R)$$

and for any x, y in $B(x_0, W^{-1}(x_0, (L^{-1}\delta_*)^\beta t))$,

$$p_t^B(x,y) \ge \frac{c}{V_*(x_0, t^{1/\beta})} \ge \frac{c}{V(x_0, W^{-1}(x_0, (L^{-1}L_0)^\beta t))}.$$

Moreover, by (2.8), we can choose $\delta \in (0,1)$ to be so small that

$$\frac{W(x_0, \delta R)}{\delta_*^\beta L^{-\beta} W(x_0, R)} \le \frac{C_W \delta^{\beta_1}}{\delta_*^\beta L^{-\beta}} \le 1,$$

and by (2.9) that

$$\frac{\delta W^{-1}(x_0, t)}{W^{-1}(x_0, (L^{-1}\delta_*)^\beta t)} \le \delta C_W^{1/\beta_1} \left(\frac{t}{(L^{-1}\delta_*)^\beta t} \right)^{1/\beta_1} = \delta C_W^{1/\beta_1} \left(\frac{1}{(L^{-1}\delta_*)^\beta} \right)^{1/\beta_1} \le 1.$$

With this choice of δ, we obtain for any $t \le W(x_0, \delta R)$ and for any x, y in $B(x_0, \delta W^{-1}(x_0, t))$,

$$p_t^B(x,y) \ge \frac{c}{V(x_0, W^{-1}(x_0, (L^{-1}L_0)^\beta t))}.$$

By (VD) and (2.9) and using the fact that $L^{-1}L_0 \ge L > 1$, we have

$$\frac{V(x_0, W^{-1}(x_0, (L^{-1}L_0)^\beta t))}{V(x_0, W^{-1}(x_0, t))} \le C \left(\frac{W^{-1}(x_0, (L^{-1}L_0)^\beta t)}{W^{-1}(x_0, t)} \right)^\alpha \le C \left(\frac{(L^{-1}L_0)^\beta t}{t} \right)^{\alpha/\beta_1} = C_1.$$

Therefore, it follows from above that for any $t \le W(x_0, \delta R)$ and for any x, y in $B(x_0, \delta W^{-1}(x_0, t))$

$$p_t^B(x,y) \ge \frac{c}{V(x_0, W^{-1}(x_0, (L^{-1}L_0)^\beta t))} \ge \frac{c'}{V(x_0, W^{-1}(x_0, t))},$$

thus showing (2.17). This proves the implication (sLLE_*) $\Rightarrow$ (sLLE).

To show the opposite implication (sLLE) $\Rightarrow$ (sLLE_*), assume that condition (sLLE) is true. Fix a ball $B_* := B_*(x_0, r)$ with some

$$r \in (0, W(x_0, \overline{R})^{1/\beta}).$$

It suffices to show that the inequality (7.6) is true. Indeed, let C, δ be the constants from (2.17). Let

$$R := F^{-1}(x_0, L^{-1}r) = W^{-1}(x_0, (L^{-1}r)^\beta) < W^{-1}(x_0, r^\beta) < W^{-1}(x_0, W(x_0, \overline{R})) = \overline{R}.$$

By (4.6), we have

$$B(x_0, R) = B(x_0, F^{-1}(x_0, L^{-1}r)) \subset B_*(x_0, r) = B_*.$$

Thus, by (2.17), we obtain that for any $t \le W(x_0, \delta R)$ and for any x, y in $B(x_0, \delta W^{-1}(x_0, t))$,

$$p_t^{B_*}(x,y) \ge p_t^{B(x_0,R)}(x,y) \ge \frac{C^{-1}}{V(x_0, W^{-1}(x_0,t))}. \tag{7.17}$$

On the other hand, set

$$r_1 := L^{-1}F(x_0, L_0W^{-1}(x_0,t)) = L^{-1}W(x_0, L_0W^{-1}(x_0,t))^{1/\beta}.$$

By (2.8),

$$r_1 \le L^{-1}C_W^{1/\beta}L_0^{\beta_2/\beta}W(x_0, W^{-1}(x_0,t))^{1/\beta} = L^{-1}C_W^{1/\beta}L_0^{\beta_2/\beta}t^{1/\beta},$$

from which, we see by (4.7) that

$$B(x_0, W^{-1}(x_0,t)) \subset B_*(x_0, r_1) \subset B_*(x_0, L^{-1}C_W^{1/\beta}L_0^{\beta_2/\beta}t^{1/\beta})$$

and hence,

$$V(x_0, W^{-1}(x_0,t)) \le V_*(x_0, L^{-1}C_W^{1/\beta}L_0^{\beta_2/\beta}t^{1/\beta}). \tag{7.18}$$

Let

$$r_2 := L^{-1}F(x_0, \delta W^{-1}(x_0,t)) = L^{-1}W(x_0, \delta W^{-1}(x_0,t))^{1/\beta}.$$

Then by (2.8),

$$r_2 \ge L^{-1}C_W^{-1/\beta}\delta^{\beta_2/\beta}W(x_0, W^{-1}(x_0,t))^{1/\beta} = L^{-1}C_W^{-1/\beta}\delta^{\beta_2/\beta}t^{1/\beta},$$

from which, we have by (4.7) that

$$B(x_0, \delta W^{-1}(x_0,t)) \supset B_*(x_0, r_2) \supset B_*(x_0, L^{-1}C_W^{-1/\beta}\delta^{\beta_2/\beta}t^{1/\beta}). \tag{7.19}$$

Combining (7.17)-(7.19), we obtain for any

$$t \le W(x_0, \delta R) = W(x_0, \delta W^{-1}(x_0, C_W^{-1}(L^{-1}r)^\beta))$$

and for any x, y in $B_*(x_0, L^{-1}C_W^{-1/\beta}\delta^{\beta_2/\beta}t^{1/\beta})$,

$$p_t^{B_*}(x,y) \ge \frac{c}{V_*(x_0, L^{-1}C_W^{1/\beta}L_0^{\beta_2/\beta}t^{1/\beta})}.$$

Note that by (2.8),

$$\frac{C_W^{-1}(L^{-1}r)^\beta}{W(x_0, \delta W^{-1}(x_0, C_W^{-1}(L^{-1}r)^\beta))} = \frac{W(x_0, W^{-1}(x_0, C_W^{-1}(L^{-1}r)^\beta))}{W(x_0, \delta W^{-1}(x_0, C_W^{-1}(L^{-1}r)^\beta))} \le C_W\delta^{-\beta_2}.$$

Therefore, taking $\delta_* \in (0,1)$ to be so small that $\delta_* \le L^{-1}C_W^{-2/\beta}\delta^{\beta_2/\beta}$ and

$$(\delta_* r)^\beta \le C_W^{-1}\delta^{\beta_2}C_W^{-1}(L^{-1}r)^\beta \le W(x_0, \delta W^{-1}(x_0, C_W^{-1}(L^{-1}r)^\beta)),$$

we obtain from above that for any $t \le (\delta_* r)^\beta$ and for any x, y in $B_*(x_0, \delta_* t^{1/\beta})$,

$$p_t^{B_*}(x,y) \ge \frac{c}{V_*(x_0, L^{-1}C_W^{1/\beta}L_0^{\beta_2/\beta}t^{1/\beta})},$$

thus showing (7.6) by using condition (VD_*). Equivalence (7.13) follows. Similarly, under (VD), we have (LLE_*) $\Leftrightarrow$ (LLE).

(2). We show (NLE_*) $\Rightarrow$ (NLE). Assume that (7.7) is satisfied. We need only to prove the inequality (2.15) is true.

Indeed, Let C, δ_* be the constants as in condition (NLE_*). For any $\delta > 0$, $t > 0$ and $x, y \in M$ with $d(x,y) \le \delta W^{-1}(x,t)$, by (4.5) and (2.8), we have

$$\begin{aligned} d_*(x,y) &\le LF(x, d(x,y)) \le LW(x, \delta W^{-1}(x,t))^{1/\beta} \\ &\le LC_W^{1/\beta}\delta^{\beta_1/\beta}W(x, W^{-1}(x,t))^{1/\beta} = LC_W^{1/\beta}\delta^{\beta_1/\beta}t^{1/\beta}. \end{aligned}$$

Then, we can take δ small enough such that

$$B(x, \delta W^{-1}(x,t)) \subset B_*(x, \delta_* t^{1/\beta}).$$

Moreover, by the second inclusion in (4.7) with $R = W^{-1}(x,t)$, we obtain

$$B_*(x, L^{-1}t^{1/\beta}) \subset B(x, W^{-1}(x,t)).$$

Therefore, using the above two inclusions and (VD$_*$), we obtain from condition (NLE$_*$) that

$$p_t(x,y) \geq \frac{C^{-1}}{V_*(x,t^{1/\beta})} \geq \frac{C^{-1}}{V_*(x,L^{-1}t^{1/\beta})} \geq \frac{C^{-1}}{V(x,W^{-1}(x,t))}$$

for all $x, y \in M$ and all $t < C_0 W(x, \overline{R})$ with $C_0 \geq 1$ satisfying

$$d(x,y) \leq \delta W^{-1}(x,t).$$

This shows the implication (NLE$_*$) $\Rightarrow$ (NLE).

To prove the opposite implication (NLE) $\Rightarrow$ (NLE$_*$), we need only to prove (7.7) holds true.

Indeed, let C, δ be the constants as in condition (NLE). For any $\delta_* > 0$ and $x, y \in M$ with $d_*(x,y) \leq \delta_* t^{1/\beta} W^{-1}(x,t)$, by (4.5), we have

$$W(x, d(x,y)) \leq (L d_*(x,y))^\beta \leq (L\delta_*)^\beta t.$$

Then, by (2.9), we can take δ_* small enough such that

$$d(x,y) \leq W^{-1}(x, (L\delta_*)^\beta t)) \leq \delta W^{-1}(x,t).$$

Moreover, by the second inclusion in (4.6) with $r = Lt^{1/\beta}$, we obtain

$$B(x, W^{-1}(x, t^{1/\beta})) \subset B_*(x, Lt^{1/\beta}).$$

Therefore, using the above two formulas and (VD$_*$), we obtain from condition (NLE) that

$$p_t(x,y) \geq \frac{C^{-1}}{V(x,W^{-1}(x,t^{1/\beta}))} \geq \frac{C^{-1}}{V_*(x,Lt^{1/\beta})} \geq \frac{C^{-1}}{V_*(x,t^{1/\beta})},$$

for all $x, y \in M$ and all $t < C_0 W(x, \overline{R})$ with $C_0 \geq 1$ satisfying

$$d(x,y) \leq \delta W^{-1}(x,t).$$

This shows the implication (NLE) $\Rightarrow$ (NLE$_*$).

Similarly, under (VD), we can prove (sNLE$_*$) $\Leftrightarrow$ (sNLE). □

7.3. **The reflected Dirichlet form.** In this subsection, we recall the general theory of the *reflected* Dirichlet form in $L^2(\Omega, \mu)$ for a non-empty open subset Ω of M. The reflected Dirichlet form will be used to derive the Poincaré inequality in the next subsection.

Let $(\mathcal{E}, \mathcal{F})$ be a general regular Dirichlet form defined in (2.5). Let U be a non-empty open subset of M.

- **The part $(\mathcal{E}, \mathcal{F}(U))$ of the Dirichlet form on $L^2(U)$.**

Let $\mathcal{F}(U)$ be a space defined by (2.16), that is $\mathcal{F}(U) = \overline{F \cap C_0(U)}^{\mathcal{E}_1}$. It is known (cf. [8, Corollary 2.3.1 on p.98]) that if $(\mathcal{E}, \mathcal{F})$ is regular, then $(\mathcal{E}, \mathcal{F}(U))$ is a regular Dirichlet form on $L^2(U)$, which is called the *part* of the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on U. Moreover, the two Dirichlet forms $(\mathcal{E}, \mathcal{F})$ in $L^2(M, \mu)$ and $(\mathcal{E}, \mathcal{F}(U))$ in $L^2(U)$ share the same set of quasi notions (cf. [8, Theorem 4.4.3(*ii*) on p.174]). Note that the energy $\mathcal{E}$ keeps the same expression (see for example [3, formula (3.3.1) on p.108]) but its domain $\mathcal{F}(U)$ becomes smaller than the original $\mathcal{F}$, so that $\mathcal{F}(U) \subset \mathcal{F}$.

For any $u \in \mathcal{F}(U)$, we have by (2.5)

$$\mathcal{E}(u,u) = \int_U d\Gamma^{(L)}(u,u) + \iint_{U\times U} (u(x) - u(y))^2 dj + \int_U u(x)^2 k_U(dx) \tag{7.20}$$

where $k_U(B) = k(B) + 2j(B \times M \setminus \mathrm{diag})$ for any Borel set $B \in \mathcal{B}(U)$ is the killing measure of the Dirichlet form $(\mathcal{E}, \mathcal{F}(U))$, see also [3, formula (5.2.29) on p.189].

- **The resurrected Dirichlet form $(\mathcal{E}^{U,\mathrm{res}}, \mathcal{F}^{U,\mathrm{res}})$ on $L^2(U)$.**

For any $u \in \mathcal{F}(U)$, we define

$$\mathcal{E}^{U,\text{res}}(u,u) = \int_U d\Gamma^{(L)}(u,u) + \iint_{U\times U} (u(x)-u(y))^2 dj, \tag{7.21}$$

that is, the energy $\mathcal{E}^{U,\text{res}}$ is defined through (7.20) by removing the killing part. By [3, the first paragraph on p.190], the form $(\mathcal{E}^{U,\text{res}}, \mathcal{F}\cap C_0(U))$ is *closable* under $\mathcal{E}_1^{U,\text{res}}$, where

$$\mathcal{E}_1^{U,\text{res}}(u,u) := \mathcal{E}^{U,\text{res}}(u,u) + (u,u)_{L^2(U)}.$$

Let $\mathcal{F}^{U,\text{res}}$ be the *closure* of $\mathcal{F}\cap C_0(U)$ in $\mathcal{E}_1^{U,\text{res}}$. By [3, Theorem 5.2.17], the form $(\mathcal{E}^{U,\text{res}}, \mathcal{F}^{U,\text{res}})$ is a regular Dirichlet form on $L^2(U)$. Moreover, the space $\mathcal{F}\cap C_0(U)$ is a core of $(\mathcal{E}^{U,\text{res}}, \mathcal{F}^{U,\text{res}})$, and the two Dirichlet forms $(\mathcal{E}, \mathcal{F}(U))$ and $(\mathcal{E}^{U,\text{res}}, \mathcal{F}^{U,\text{res}})$ in $L^2(U)$ share the same set of quasi notions (cf. [3, Theorem 5.2.17]).

Since $\mathcal{E}^{U,\text{res}}(u,u) \le \mathcal{E}(u,u)$ for any u in $\mathcal{F}\cap C_0(U)$, we have

$$\mathcal{F}(U) \subset \mathcal{F}^{U,\text{res}}. \tag{7.22}$$

The resurrected Dirichlet form $(\mathcal{E}^{U,\text{res}}, \mathcal{F}^{U,\text{res}})$ can be viewed as a modification of form $(\mathcal{E}, \mathcal{F}(U))$ on $L^2(U)$ by dropping its killing part and by enlarging its domain.

- **The reflected Dirichlet form $(\mathcal{E}^{U,\text{ref}}, \mathcal{F}^{U,\text{ref}})$ on $L^2(U)$.**

The *reflected* Dirichlet form $(\mathcal{E}^{U,\text{ref}}, \mathcal{F}^{U,\text{ref}})$ on $L^2(U)$ is a modification of the resurrected Dirichlet form $(\mathcal{E}^{U,\text{res}}, \mathcal{F}^{U,\text{res}})$ on $L^2(U)$ by adjusting its domain $\mathcal{F}^{U,\text{res}}$ while keeping the same expression of the energy $\mathcal{E}^{U,\text{res}}$ on the space $\mathcal{F}\cap L^\infty$.

Recall that a set $E \subset M$ is called $\mathcal{E}$-*quasi open* for the Dirichlet form $(\mathcal{E}, \mathcal{F})$ if for any $\varepsilon > 0$ there exists an open set U containing E with $\text{Cap}_1(U \setminus E) < \varepsilon$ (see (2.25) for the definition of Cap_1). So called $\mathcal{E}^{U,\text{res}}$-*quasi open* sets for the Dirichlet form $(\mathcal{E}^{U,\text{res}}, \mathcal{F}^{U,\text{res}})$ can be similarly defined. We follow the definition of the notion $\mathring{\mathcal{F}}_{\text{loc}}$ in [3, Eq. (4.3.31), p. 163] to define the function class $\mathring{\mathcal{F}}^{U,\text{ref}}_{\text{loc}}$ on U:

$$\mathring{\mathcal{F}}^{U,\text{res}}_{\text{loc}} = \big\{u : \text{there is an increasing sequence of } \mathcal{E}^{U,\text{res}}\text{-quasi open sets } \{D_n\} \text{ with } \cup_{n=1}^\infty D_n = U \\ \mathcal{E}^{U,\text{res}}\text{-q.e. and a sequence } \{u_n\} \subset \mathcal{F}^{U,\text{res}} \text{ such that } u = u_n \ \mu\text{-a.e. on } D_n\big\}.$$

That is, we replace $\mathcal{F}$, E appearing in [3, Eq. (4.3.31), p. 163] by $\mathcal{F}^{U,\text{res}}$, U respectively. Since $(\mathcal{E}^{U,\text{res}}, \mathcal{F}^{U,\text{res}})$ is a regular Dirichlet form on $L^2(U)$ by the above result, one can follow the arguments in the last two paragraphs in [3, p. 263] to prove that (7.21) is also well defined for all $u \in \mathring{\mathcal{F}}^{U,\text{res}}_{\text{loc}}$ (not only for $u \in \mathcal{F}(U)$). In this case, we write the formula in (7.21) by $\widehat{\mathcal{E}}^{U,\text{res}}(u,u)$ when $u \in \mathring{\mathcal{F}}^{U,\text{res}}_{\text{loc}}$. That is,

$$\widehat{\mathcal{E}}^{U,\text{res}}(u,u) = \int_U d\Gamma^{(L)}(u,u) + \iint_{U\times U} (u(x)-u(y))^2 dj, \qquad u \in \mathring{\mathcal{F}}^{U,\text{res}}_{\text{loc}}. \tag{7.23}$$

Again, since $(\mathcal{E}^{U,\text{res}}, \mathcal{F}^{U,\text{res}})$ is a regular Dirichlet form on $L^2(U)$, by [3, Defintion 6.4.4, p. 256], the reflected Dirichlet space $(\mathcal{E}^{U,\text{ref}}, \mathcal{F}^{U,\text{ref}})$ of $(\mathcal{E}^{U,\text{res}}, \mathcal{F}^{U,\text{res}})$ is defined by

$$\mathcal{F}^{U,\text{ref}} := \left\{u \in \mathring{\mathcal{F}}^{U,\text{res}}_{\text{loc}} : \widehat{\mathcal{E}}^{U,\text{res}}(u,u) < \infty\right\}, \tag{7.24}$$

and

$$\mathcal{E}^{U,\text{ref}}(u,v) = \widehat{\mathcal{E}}^{U,\text{res}}(u,v), \qquad u,v \in \mathcal{F}^{U,\text{ref}}. \tag{7.25}$$

Moreover, applying [3, Theorems 6.4.5, p. 266], with $(\mathcal{E}, \mathcal{F})$, $(\mathcal{E}^{\text{ref}}, \mathcal{F}^{\text{ref}})$ being replaced by $(\mathcal{E}^{U,\text{res}}, \mathcal{F}^{U,\text{res}})$, by $(\mathcal{E}^{U,\text{ref}}, \mathcal{F}^{U,\text{ref}})$ respectively, we see that the form $(\mathcal{E}^{U,\text{ref}}, \mathcal{F}_a^{U,\text{ref}})$ with

$$\mathcal{F}_a^{U,\text{ref}} := \mathcal{F}^{U,\text{ref}} \cap L^2(U) \tag{7.26}$$

is a Dirichlet form on $L^2(U)$ for any non-empty open subset U of M. This Dirichlet form is called the *reflected* Dirichlet form on U. It is not known in general whether the Dirichlet form

$(\mathcal{E}^{U,\mathrm{ref}}, \mathcal{F}_a^{U,\mathrm{ref}})$ is regular or not. In the following, we will construct a regular Dirichlet form on $L^2(U)$ such that the associated domain contains $\mathcal{F}|_U$:

$$\mathcal{F}|_U := \{u : \text{ there exists a function } f \text{ in } \mathcal{F} \text{ such that } u = f \text{ on } U\}.$$

We need the following lemma.

Lemma 7.8. *Let U be an open subset of M. Then*

$$\mathcal{F}|_U \subset \mathring{\mathcal{F}}_{\mathrm{loc}}^{U,\mathrm{res}}. \tag{7.27}$$

Proof. Let $\{U_n\}$ be a sequence of precompact open sets such that $U_n \uparrow U$ as $n \to \infty$ and $\overline{U_n} \subset U_{n+1}$ for $n \geq 1$. Fix a function u of its $\mathcal{E}$-quasi continuous version in $\mathcal{F}$. For any $n \geq 1$, let

$$D_n = \{x \in U : -n < u(x) < n\} \cap U_n.$$

Then each D_n is $\mathcal{E}$-quasi open, precompact with $\overline{D_n} \subset D_{n+1}$ for $n \geq 1$ and $\cup_{n=1}^{\infty} D_n = U$. Moreover, by definitions (7.20), (7.21), each D_n is also $\mathcal{E}^{U,\mathrm{res}}$-quasi open.

Let for $n \geq 1$

$$u_n := ((-n) \vee u) \wedge n.$$

Then each $u_n \in \mathcal{F} \cap L^\infty$ and $u_n = u$ on D_n. We show

$$u_n \in \mathcal{F}^{U,\mathrm{res}}.$$

Indeed, let $\phi \in \mathrm{cutoff}(D_n, D_{n+1})$ so that $\mathrm{supp}(\phi) \subset D_{n+1} \subset U$. We have by [8, Theorem 1.4.2(ii), p. 28] that $v_n := u_n \cdot \phi \in \mathcal{F} \cap L^\infty$, and so by [16, Proposition 2.8],

$$v_n \in \mathcal{F}(D_{n+1}).$$

From this, we have by (7.22)

$$v_n \in \mathcal{F}(D_{n+1}) \subset \mathcal{F}(U) \subset \mathcal{F}^{U,\mathrm{res}}.$$

Since $u = v_n$ μ-a.e. on D_n and $v_n \in \mathcal{F}^{U,\mathrm{res}}$, we see that $u|_U \in \mathring{\mathcal{F}}_{\mathrm{loc}}^{U,\mathrm{res}}$ by definition. □

For any open set $U \subset M$, it is easy to see that $\mathcal{F}|_U \subset L^2(U)$, and by the definitions (2.5), (7.23),

$$\widehat{\mathcal{E}}^{U,\mathrm{res}}(u|_U, u|_U) \leq \mathcal{E}(u,u) < \infty, \qquad u \in \mathcal{F}.$$

Hence, by (7.27) and the definitions (7.24), (7.26), we have

$$\mathcal{F}|_U \subset \mathcal{F}^{U,\mathrm{ref}} \cap L^2(U) = \mathcal{F}_a^{U,\mathrm{ref}}.$$

Assume in addition that $\mu(\partial U) = 0$. In this case, we have $L^2(U) = L^2(\overline{U})$. We will construct a regular Dirichlet form on $L^2(\overline{U})$. Indeed, all functions in $\mathcal{F}|_U$ or $\mathcal{F}_a^{U,\mathrm{ref}}$ can be identified as functions on $\overline{U}$. Moreover, note that $(\mathcal{E}^{U,\mathrm{ref}}, \mathcal{F}_a^{U,\mathrm{ref}})$ is a Dirichlet form on $L^2(\overline{U})$, and $\mathcal{F}|_U \subset \mathcal{F}_a^{U,\mathrm{ref}}$. We define

$$\mathcal{F}^U := \overline{\mathcal{F}|_U}^{\mathcal{E}_1^{U,\mathrm{ref}}}, \tag{7.28}$$

where

$$\mathcal{E}_1^{U,\mathrm{ref}}(u,u) := \mathcal{E}^{U,\mathrm{ref}}(u,u) + (u,u)_{L^2(U)}, \quad u \in \mathcal{F}^U.$$

Denote by $C_0(\overline{U})$ the space of continuous functions with compact supports in $\overline{U}$. Note that the space $C_0(\overline{U})$ is the same to the space of functions in $C_0(M)$ restricted to $\overline{U}$.

Theorem 7.9. *Let $U \subset M$ be a non-empty open set with $\mu(\partial U) = 0$. Then $(\mathcal{E}^{U,\mathrm{ref}}, \mathcal{F}^U)$ defined in (7.25), (7.28) is a regular Dirichlet form on $L^2(\overline{U})$. Moreover, $\mathcal{F}|_{\overline{U}} \cap C_0(\overline{U})$ is also the core of $(\mathcal{E}^{U,\mathrm{ref}}, \mathcal{F}^U)$.*

Proof. It is easy to see that $\mathcal{F}^U$ is dense in $L^2(\overline{U})$, since $\mathcal{F}|_U \subset \mathcal{F}^U$ and $\mathcal{F}|_U$ is dense in $L^2(\overline{U})$. Hence, $(\mathcal{E}^{U,\mathrm{ref}}, \mathcal{F}^U)$ is a Dirichlet form on $L^2(\overline{U})$. It remains to prove that $(\mathcal{E}^{U,\mathrm{ref}}, \mathcal{F}^U)$ is regular.

Indeed, Let us prove that $\mathcal{F}^U \cap C_0(\overline{U})$ is dense in $C_0(\overline{U})$. Since $\mathcal{F} \cap C_0(M)$ is dense in $C_0(M)$, we have that for any $u \in C_0(\overline{U})$ with $u = \widetilde{u}|_{\overline{U}}$ for some $\widetilde{u} \in C_0(M)$, there exists $\{\widetilde{u}_n\} \subset \mathcal{F} \cap C_0(M)$ such that

$$\sup_{x\in M} |\widetilde{u}_n(x) - \widetilde{u}(x)| < \frac{1}{n}, \quad n \geq 1.$$

Set $u_n := \widetilde{u}_n|_{\overline{U}} \in \mathcal{F}|_{\overline{U}} \cap C_0(\overline{U})$ for $n \geq 1$. Then

$$\sup_{x\in \overline{U}} |u_n(x) - u(x)| \leq \sup_{x\in M} |\widetilde{u}_n(x) - \widetilde{u}(x)| < \frac{1}{n}, \quad n \geq 1.$$

That is, $\mathcal{F}|_{\overline{U}} \cap C_0(\overline{U})$ is dense in $C_0(\overline{U})$. Since $\mathcal{F}|_{\overline{U}} \subset \mathcal{F}^U$, we have that $\mathcal{F}^U \cap C_0(\overline{U})$ is also dense in $C_0(\overline{U})$.

Let us prove that $\mathcal{F}^U \cap C_0(\overline{U})$ is dense in $\mathcal{F}^U$. By the definition of $\mathcal{F}^U$, for any $u \in \mathcal{F}^U$, there exists $\{u_n\} \subset \mathcal{F}|_{\overline{U}}$ such that

$$\mathcal{E}_1^{U,\mathrm{ref}}(u_n - u, u_n - u) < \frac{1}{n^2}, \quad n \geq 1. \tag{7.29}$$

Then there exists $\{\widetilde{u}_n\} \subset \mathcal{F}$ such that $u_n = \widetilde{u}_n|_{\overline{U}}$ for $n \geq 1$. Since $\mathcal{F} \cap C_0(M)$ is dense in $\mathcal{F}$, there exists $\{\widetilde{v}_n\} \subset \mathcal{F} \cap C_0(M)$ such that

$$\mathcal{E}_1(\widetilde{v}_n - \widetilde{u}_n, \widetilde{v}_n - \widetilde{u}_n) < \frac{1}{n^2}, \quad n \geq 1.$$

Set $v_n := \widetilde{v}_n|_{\overline{U}} \in \mathcal{F}|_{\overline{U}} \cap C_0(\overline{U})$ for $n \geq 1$. By (7.25) and (2.5), we have

$$\mathcal{E}_1^{U,\mathrm{ref}}(v_n - u_n, v_n - u_n) \leq \mathcal{E}_1(\widetilde{v}_n - \widetilde{u}_n, \widetilde{v}_n - \widetilde{u}_n) < \frac{1}{n^2}, \quad n \geq 1. \tag{7.30}$$

Combining (7.29) and (7.30), we have

$$\mathcal{E}_1^{U,\mathrm{ref}}(v_n - u, v_n - u) < \frac{4}{n^2}, \quad n \geq 1.$$

Since $v_n \in \mathcal{F}|_{\overline{U}} \cap C_0(\overline{U})$ for $n \geq 1$, $\mathcal{F}|_{\overline{U}} \cap C_0(\overline{U})$ is dense in $\mathcal{F}^U$. Moreover, since $\mathcal{F}|_{\overline{U}} \cap C_0(\overline{U}) \subset \mathcal{F}^U \cap C_0(\overline{U})$, we have that $\mathcal{F}^U \cap C_0(\overline{U})$ is dense in $\mathcal{F}^U$. Therefore, $(\mathcal{E}^{U,\mathrm{ref}}, \mathcal{F}^U)$ is regular. □

By Theorem 7.9, $(\mathcal{E}^{U,\mathrm{ref}}, \mathcal{F}^U)$ is a regular Dirichlet form on $L^2(\overline{U})$. Then, its part Dirichlet form $(\mathcal{E}^{U,\mathrm{ref}}, \mathcal{F}^U(U))$ is a regular Dirichlet form on $L^2(U)$, where

$$\mathcal{F}^U(U) := \overline{\mathcal{F}^U \cap C_0(U)}^{\mathcal{E}_1^{U,\mathrm{ref}}},$$

and $\mathcal{E}_1^{U,\mathrm{ref}}(u,u) = \mathcal{E}^{U,\mathrm{ref}}(u,u) + (u,u)_{L^2(U)}$, $u \in \mathcal{F}^U$.

Let $(\mathcal{E}, \mathcal{F}(U))$ be the part Dirichlet form on $L^2(U)$ of $(\mathcal{E}, \mathcal{F})$. If we identify the functions in $\mathcal{F} \cap C_0(U)$ as functions on U, then we have by (7.28),

$$\mathcal{F} \cap C_0(U) \subset \mathcal{F}^U(U) \cap C_0(U).$$

Moreover, since $\mathcal{E}^{U,\mathrm{ref}}(u|_U, u|_U) \leq \mathcal{E}(u,u)$, $u \in \mathcal{F}$, we have

$$\mathcal{F}(U) = \overline{\mathcal{F} \cap C_0(U)}^{\mathcal{E}_1} \subset \overline{\mathcal{F}^U \cap C_0(U)}^{\mathcal{E}_1^{U,\mathrm{ref}}} = \mathcal{F}^U(U). \tag{7.31}$$

We also have the following proposition.

Proposition 7.10. *Let $U \subset M$ be an open set with $\mu(\partial U) = 0$. Then, $\mathcal{F} \cap C_0(U)$ is weakly dense in $\mathcal{F}^U(U)$ with respect to $\mathcal{E}^{U,\mathrm{ref}}$ and L^2-norm.*

Proof. **Step 1.** Fix $u \in \mathcal{F}^U \cap C_0(U)$. Let $K := \mathrm{supp}(u) \subset U$ and choose $\varphi \in \mathrm{cutoff}(K, U) \subset \mathcal{F}^U$.

Since $u \in \mathcal{F}^U$, by Theorem 7.9, there exist $\{u_n\} \subset \mathcal{F}|_{\overline{U}} \cap C_0(\overline{U})$ such that

$$\mathcal{E}_1^{U,\mathrm{ref}}(u_n - u, u_n - u) < \frac{1}{n}, \quad n \geq 1.$$

Note that $u \in C_0(U)$ is bounded. Let $v_n := (-\|u\|_\infty) \vee u_n \wedge \|u\|_\infty \in \mathcal{F}|_{\overline{U}} \cap C_0(\overline{U})$ for $n \geq 1$. By [8, Theorem 1.4.2(v), p. 28] with $\phi(t) := (-\|u\|_\infty) \vee t \wedge \|u\|_\infty$, $t \in \mathbb{R}$, we have

$$\lim_{n\to\infty} \mathcal{E}_1^{U,\mathrm{ref}}(v_n - u, v_n - u) = 0.$$

Consequently, we have φv_n converges to $\varphi u = u$ as $n \to \infty$ in L^2-norm, and by [8, Theorem 1.4.2(ii), p. 28]

$$\begin{aligned}\sup_{n\geq 1} \mathcal{E}^{U,\mathrm{ref}}(\varphi v_n, \varphi v_n) &\leq \sup_{n\geq 1} 2\|\varphi\|_\infty \mathcal{E}^{U,\mathrm{ref}}(v_n, v_n) + \sup_{n\geq 1} 2\|v_n\|_\infty \mathcal{E}^{U,\mathrm{ref}}(\varphi, \varphi) \\ &\leq \sup_{n\geq 1} 2\mathcal{E}^{U,\mathrm{ref}}(v_n, v_n) + 2\|u\|_\infty \mathcal{E}^{U,\mathrm{ref}}(\varphi, \varphi) < \infty.\end{aligned}$$

Hence, by Lemma 9.4 in Appendix, there exist $\{\varphi v_{n_k}\} \subset \{\varphi v_n\}$ such that φv_{n_k} converges weakly to u in $\mathcal{E}^{U,\mathrm{ref}}$-norm. Moreover, since each $\varphi v_{n_k} \in \mathcal{F} \cap C_0(U)$, we see that $\mathcal{F} \cap C_0(U)$ is weakly dense in $\mathcal{F}^U \cap C_0(U)$ with respect to $\mathcal{E}^{U,\mathrm{ref}}$ and L^2-norm.

Step 2. Fix $u \in \mathcal{F}^U(U)$. There exist $\{u_n\} \subset \mathcal{F}^U \cap C_0(U)$ such that

$$\mathcal{E}_1^{U,\mathrm{ref}}(u_n - u, u_n - u) < \frac{1}{n}, \quad n \geq 1.$$

By step 1, for any $v \in \mathcal{F}^U$, there exist $\{v_n\} \subset \mathcal{F} \cap C_0(U)$ such that

$$|\mathcal{E}^{U,\mathrm{ref}}(u_n - v_n, v)| < \frac{1}{n}, \quad n \geq 1,$$

and

$$\|v_n - u_n\|_{L^2(U)} < \frac{1}{n}, \quad n \geq 1.$$

Consequently, we have for any $v \in \mathcal{F}^U$,

$$\lim_{n\to\infty} \mathcal{E}^{U,\mathrm{ref}}(v_n - u, v) = 0, \quad \lim_{n\to\infty} \|v_n - v\|_{L^2(U)} = 0.$$

The proof is complete. □

Let us recall the definition of subcaloric functions. Then, we will use the parabolic maximum principle [15, Proposition 4.11] to prove the following Proposition 7.11.

Let I be an interval in $\mathbb{R}$. A function $u : I \to L^2$ is said to be *weakly differentiable* at $t \in I$, if for any $\varphi \in L^2$, the function $(u(\cdot), \varphi)$ is differentiable at t, that is, the limit

$$\lim_{\varepsilon\to 0} \left(\frac{u(t+\varepsilon) - u(t)}{\varepsilon}, \varphi \right)$$

exists. In this case, by the principle of uniform boundedness, there is some $w \in L^2$ such that

$$\lim_{\varepsilon\to 0} \left(\frac{u(t+\varepsilon) - u(t)}{\varepsilon}, \varphi \right) = (w, \varphi)$$

for any $\varphi \in L^2$. The vector w is called the *weak derivative* of u at t, and we write $w = \frac{\partial u}{\partial t}$.

For an open subset $U \subset M$, a function $u : I \to \mathcal{F}$ is *subcaloric* (*caloric*) in $I \times U$ if u is weakly differentiable in L^2 at any $t \in I$ and if for any $t \in I$ and any non-negative $\varphi \in \mathcal{F}(U)$,

$$(\partial_t u, \varphi) + \mathcal{E}(u(t, \cdot), \varphi) \leq 0 \ (= 0).$$

One can prove that for any $f \in L^2(U)$, $P_t^U f$ is caloric in $(0, \infty) \times U$. Subcaloric (caloric) functions for other Dirichlet forms can be similarly defined.

Proposition 7.11. *Let $U \subset M$ be an open set with $\mu(\partial U) = 0$. Let $\{\widehat{P}_t\}_{t>0}$, $\{\widehat{P}_t^U\}_{t>0}$, $\{P_t^U\}_{t>0}$ be the heat semigroups of the Dirichlet forms $(\mathcal{E}^{U,\mathrm{ref}}, \mathcal{F}^U)$, $(\mathcal{E}^{U,\mathrm{ref}}, \mathcal{F}^U(U))$, $(\mathcal{E}, \mathcal{F}(U))$ respectively. Then for any $0 \le f \in L^2(U)$ and any $t > 0$,*

$$\widehat{P}_t f \ge \widehat{P}_t^U f \ge P_t^U f \quad \text{in } M. \tag{7.32}$$

Proof. Clearly, for any non-negative f in $L^2(U)$,

$$\widehat{P}_t f \ge \widehat{P}_t^U f \quad \text{in } M.$$

Since $\widehat{P}_t^U f = 0 = P_t^U f$ in U^c, we only need to show

$$\widehat{P}_t^U f \ge P_t^U f \quad \text{in } U.$$

To do this, note that both Dirichlet forms $(\mathcal{E}^{U,\mathrm{ref}}, \mathcal{F}^U(U))$ and $(\mathcal{E}, \mathcal{F}(U))$ in $L^2(U)$ are regular. Let for any $f \in L^2(U)$ and $t > 0$,

$$u(t, \cdot) := P_t^U f - \widehat{P}_t^U f.$$

We show that $u \le 0$ in $(0, \infty) \times U$ by using the parabolic maximum principle.

Indeed, it is easy to see that $u(t, \cdot) \to 0$ in $L^2(U)$ as $t \to 0$. Since

$$u(t, \cdot) \le P_t^U f(\cdot) \in \mathcal{F}(U),$$

we see by [15, Lemma 4.4] and by using (7.31) that

$$u_+(t, \cdot) \in \mathcal{F}(U) \subset \mathcal{F}^U(U).$$

Thus, the function u satisfies the initial and boundary conditions in $(0, \infty) \times U$ with respect to $(\mathcal{E}^{U,\mathrm{ref}}, \mathcal{F}^U(U))$.

We show that u is subcaloric in $(0, \infty) \times U$ with respect to the form $(\mathcal{E}^{U,\mathrm{ref}}, \mathcal{F}^U(U))$, that is, for any non-negative function $\varphi \in \mathcal{F}^U(U)$

$$(\frac{\partial u}{\partial t}, \varphi) + \mathcal{E}^{U,\mathrm{ref}}(u, \varphi) \le 0. \tag{7.33}$$

Indeed, using the definitions (7.20) and (7.25), we see for any $u, v \in \mathcal{F}(U)$

$$\begin{aligned}\mathcal{E}(u, v) &= \int_U d\Gamma^{(L)}(u, v) + \iint_{U\times U} (u(x) - u(y))(v(x) - v(y))dj + \int_U u(x)v(x)k_U(dx) \\ &= \mathcal{E}^{U,\mathrm{ref}}(u, v) + \int_U u(x)v(x)k_U(dx).\end{aligned}$$

From this, we have for any non-negative function $\varphi \in \mathcal{F}(U) \subset \mathcal{F}^U(U)$,

$$\begin{aligned}\left(\frac{\partial u}{\partial t}, \varphi\right) &= \left(\frac{\partial}{\partial t} P_t^U f - \frac{\partial}{\partial t}\widehat{P}_t^U f, \varphi\right) = -\mathcal{E}(P_t^U f, \varphi) + \mathcal{E}^{U,\mathrm{ref}}(\widehat{P}_t^U f, \varphi) \\ &= -\mathcal{E}^{U,\mathrm{ref}}(P_t^U f, \varphi) - \int_U P_t^U f(x)\varphi(x)k_U(dx) + \mathcal{E}^{U,\mathrm{ref}}(\widehat{P}_t^U f, \varphi) \\ &= -\mathcal{E}^{U,\mathrm{ref}}(u, \varphi) - \int_U P_t^U f(x)\varphi(x)k_U(dx) \le -\mathcal{E}^{U,\mathrm{ref}}(u, \varphi),\end{aligned} \tag{7.34}$$

that is, the inequality (7.33) holds for any $0 \le \varphi \in \mathcal{F}(U)$. Moreover, by Proposition 7.10, (7.33) holds for any $0 \le \varphi \in \mathcal{F}^U(U)$.

Therefore, by the parabolic maximum principle [15, Proposition 4.11], we have

$$u(t, \cdot) = P_t^U f - \widehat{P}_t^U f \le 0 \quad \text{in } (0, \infty) \times U.$$

The proof is complete. □

7.4. **Derivation of conditions (PI) and (S).** Recall that $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form without killing part defined by (2.7), and d_* is the metric defined by Proposition 4.1.

In this section we derive conditions (PI) and (S) by using condition (LLE). Here the condition (S) is called *survival estimate* which is defined below. This condition is used to derive the implication (LLE) $\Rightarrow$ (Gcap).

We start with the implication (LLE$_*$) $\Rightarrow$ (PI$_*$).

Lemma 7.12. *Assume that condition* (VD) *is satisfied. Then* (LLE$_*$) $\Rightarrow$ (PI$_*$).

Proof. Fix a ball $B_* := B_*(x_0, r)$ of radius $r \in (0, W(x_0, \overline{R})^{1/\beta})$.

Case 1. $\mu(\partial B_*) = 0$.

Let the Dirichlet forms $(\mathcal{E}^{B_*,\mathrm{ref}}, \mathcal{F}^{B_*})$, $(\mathcal{E}^{B_*,\mathrm{ref}}, \mathcal{F}^{B_*}(B_*))$, $(\mathcal{E}, \mathcal{F}(B_*))$ be as in Proposition 7.11 with $U = B_*$, whose heat semigroups are denoted by $\{\widehat{P}_t\}_{t>0}$, $\{\widehat{P}_t^{B_*}\}_{t>0}$, $\{P_t^{B_*}\}_{t>0}$ respectively.

By [17, formula (8.7)] and (7.32), we have for any $t > 0$ and for any $u \in \mathcal{F}^{B_*}$,

$$\begin{aligned}\mathcal{E}^{B_*,\mathrm{ref}}(u,u) &\geq \frac{1}{2t}(\widehat{P}_t(u(x)1-u)^2(x), 1(x)) \\ &\geq \frac{1}{2t}(\widehat{P}_t^{B_*}(u(x)1-u)^2(x), 1(x)) \\ &\geq \frac{1}{2t}(P_t^{B_*}(u(x)1-u)^2(x), 1(x)) \\ &= \frac{1}{2t}\int_{B_*}\int_{B_*} p_t^{B_*}(x,y)(u(x)-u(y))^2 d\mu(x)d\mu(y).\end{aligned}$$

Let $t = (\delta_* r)^\beta$ where δ_* is the constant from (LLE$_*$). By (LLE$_*$),

$$p_t^{B_*}(x,y) \geq \frac{c}{V_*(x_0, t^{1/\beta})} = \frac{c}{V_*(x_0, \delta_* r)} \geq \frac{c'}{\mu(B_*)}, \quad \mu\text{-a.a. } x, y \in B_*(x_0, \delta_* t^{1/\beta}) = \delta_*^2 B_*.$$

Combining the above two formulas and using condition (VD$_*$), it follows that for any $u \in \mathcal{F}^{B_*}$,

$$\begin{aligned}\mathcal{E}^{B_*,\mathrm{ref}}(u,u) &\geq \frac{c'}{2(\delta_* r)^\beta \mu(B_*)}\int_{\delta_*^2 B_*}\int_{\delta_*^2 B_*}(u(x)-u(y))^2 d\mu(x)d\mu(y) \\ &= \frac{c'\mu(\delta^2 B_*)}{(\delta_* r)^\beta \mu(B_*)}\int_{\delta_*^2 B_*}|u - u_{\delta_*^2 B_*}|^2 d\mu \geq \frac{c}{r^\beta}\int_{\delta_*^2 B_*}|u - u_{\delta_*^2 B_*}|^2 d\mu. \end{aligned} \tag{7.35}$$

Setting $\kappa_* := \delta_*^2$, we see by (7.35) and definition (7.25) that for any $u \in \mathcal{F}|_{B_*} \subset \mathcal{F}^{B_*}$,

$$\int_{\kappa_* B_*}|u - u_{\kappa_* B_*}|^2 d\mu \leq C r^\beta \mathcal{E}^{B_*,\mathrm{ref}}(u,u)(u,u) = C r^\beta \int_{B_*} d\Gamma_{B_*}(u,u),$$

thus showing that condition (PI$_*$) holds.

Case 2. $\mu(\partial B_*) > 0$.

It follows from (VD$_*$) that there exist at most countably many numbers $s \in (0, r)$ such that $\mu(\partial B_*(x_0, s)) > 0$. Then, we can take a sequence $\{r_n\}_{n\geq 1} \subset (0, r)$ such that $B_n := B_*(x_0, r_n) \uparrow B_*$ as $n \to \infty$ and $\mu(\partial B_n) = 0$ for all $n \geq 1$. Moreover, applying the result in Case 1 for each B_n and using (3.6), we obtain

$$\int_{\kappa_* B_n}|u - u_{\kappa_* B_n}|^2 d\mu(x) \leq C r_n^\beta \int_{B_n} d\Gamma_{B_n}(u) \leq C r^\beta \int_{B_*} d\Gamma_{B_*}(u).$$

Passing to the limit in the above inequality as $n \to \infty$, we obtain (4.13) for the ball B_*. That is, condition (PI$_*$) holds. □

Definition 7.13. We say that condition (S) holds if there exist two small constants $\varepsilon, \delta \in (0,1)$ such that for all balls B of radius less than $\overline{R}$,

$$P_t^B \mathbf{1}_B \geq \varepsilon \ \text{ in } \frac{1}{4}B \tag{7.36}$$

provided that $t \le \delta W(B)$.

We show that condition (LLE) implies condition (S).

Lemma 7.14. *Assume that condition (VD) is satisfied. Then (LLE) $\Rightarrow$ (S).*

Proof. Rename the constant δ in (LLE) by δ_0.

Step 1. Let $B_0 := B(x,R)$ with $x \in M$ and $R \in (0,\overline{R})$. By (LLE) and (VD), we obtain for any $t \le W(x,\delta_0 R)$ and μ-almost all $z \in B(x,\delta_0 W^{-1}(x,t))$,

$$\begin{aligned} P_t^{B_0}\mathbf{1}_{B_0}(z) &= \int_{B_0} p_t^{B_0}(z,y)d\mu(y) \ge \int_{B(x,\delta_0 W^{-1}(x,t))} p_t^{B_0}(z,y)d\mu(y) \\ &\ge \int_{B(x,\delta_0 W^{-1}(x,t))} \frac{C^{-1}}{V(x,W^{-1}(x,t))}d\mu(y) = C^{-1}\frac{V(x,\delta_0 W^{-1}(x,t))}{V(x,W^{-1}(x,t))} \\ &\ge \varepsilon \quad \text{(using (9.1))} \end{aligned} \tag{7.37}$$

for some positive constant c independent of B_0, t, z.

Step 2. Fix $B := B(x_0,R)$ with $x_0 \in M$ and $R \in (0,\overline{R})$. Let x be any point in $\frac{1}{4}B$ so that $B(x,\frac{R}{2}) \subset B$. Applying (7.37) with B_0 being replaced by $B(x,\frac{R}{2})$, it follows that for any $t \le W(x,\delta_0 R/2)$,

$$P_t^B\mathbf{1}_B \ge P_t^{B(x,R/2)}\mathbf{1}_{B(x,R/2)} \ge \varepsilon \quad \text{in } B(x,\delta_0 W^{-1}(x,t)). \tag{7.38}$$

Moreover, by the right inequality in (2.8), we have for any $x \in \frac{1}{4}B$

$$\frac{W(x_0,R)}{W(x,\delta_0 R/2)} \le c'\left(\frac{2}{\delta_0}\right)^{\beta_2} =: \delta^{-1},$$

that is, $\delta W(x_0,R) \le W(x,\delta_0 R/2)$. And, by the left inequality in (2.8), we have for any $x \in \frac{1}{4}B$ and $t < \delta W(x_0,R)$,

$$\begin{aligned} \frac{W^{-1}(x_0,t)}{W^{-1}(x,t)} &\le \frac{W^{-1}(x_0,t)+R}{W^{-1}(x,t)} \le c''\left(\frac{W(x_0,W^{-1}(x_0,t)+R)}{W(x,W^{-1}(x,t))}\right)^{1/\beta_1} \\ &= c''\left(\frac{W(x_0,W^{-1}(x_0,t)+R)}{t}\right)^{1/\beta_1} =: c(x_0,t,R)^{-1}, \end{aligned}$$

that is, $c(x_0,t,R)W^{-1}(x_0,t) \le W^{-1}(x,t)$. Here $c(x_0,t,R)$ is a positive constant depending on x_0, t, R.

Hence, by (7.38), we have that for any $x \in \frac{1}{4}B$ and $t < \delta W(x_0,R)$,

$$P_t^B\mathbf{1}_B \ge \varepsilon \quad \text{in } B(x,\delta_0 c(x_0,t,R)).$$

Since $\frac{1}{4}B$ can be covered by at most countable balls like $\left\{B(x,\delta_0 c(x_0,t,R)),\ x \in \frac{1}{4}B\right\}$, it follows that for any $t \le \delta W(x_0,R)$,

$$P_t^B\mathbf{1}_B \ge \varepsilon \quad \text{in } \frac{1}{4}B,$$

thus showing that (S) is true. □

7.5. **Proof of Theorem 2.9.**

Proof of Theorem 2.9. To prove (2.18), we follow a flowchart of the following results:

$$\text{(VD)} + \text{(RVD)} + \text{(Gcap)} + \text{(TJ)} + \text{(PI)} \Rightarrow (\text{LLE}_*) \quad \text{(Lemma 7.6)}$$
$$\text{(VD)} + (\text{LLE}_*) \Rightarrow \text{(sLLE)} \quad \text{(Lemma 7.7)}.$$

It is obvious that (sLLE) $\Rightarrow$ (LLE). To verify the opposition implication (LLE) $\Rightarrow$ (PI) + (Gcap), we use the following implications:

$$\text{(VD)} + \text{(LLE)} \Rightarrow (\text{LLE}_*) \quad \text{(Lemma 7.7)}$$

$$\begin{aligned}
(\text{VD}) + (\text{LLE}_*) &\Rightarrow (\text{PI}_*) \quad (\text{Lemma 7.12})\\
(\text{VD}) + (\text{PI}_*) &\Rightarrow (\text{PI}) \quad (\text{Proposition 4.7})\\
(\text{VD}) + (\text{LLE}) &\Rightarrow (\text{S}) \quad (\text{Lemma 7.14})\\
(\text{VD}) + (\text{S}) &\Rightarrow (\text{Gcap}) \quad ([12, \text{Lemma 13.5}]).
\end{aligned}$$

Hence, we have prove the first two equivalences.

To prove the implication (LLE) $\Rightarrow$ (sNLE), by the above equivalence, it suffices to prove (sLLE) $\Rightarrow$ (sNLE), which in turn follows from the following implications:

$$\begin{aligned}
(\text{VD}) + (\text{sLLE}) &\Rightarrow (\text{sLLE}_*) \quad (\text{Lemma 7.7})\\
(\text{VD}) + (\text{sLLE}_*) &\Rightarrow (\text{sNLE}_*) \quad (\text{Lemma 7.6})\\
(\text{VD}) + (\text{sNLE}_*) &\Rightarrow (\text{sNLE}) \quad (\text{Lemma 7.7}).
\end{aligned}$$

The rest is clear. The proof of Theorem 2.9 is complete. □

8. Full lower estimates of heat kernel

8.1. **Proof of Theorem 2.12.** In this subsection we prove Theorem 2.12. We start with derivation of (PI) from the lower bound of the jump kernel.

Lemma 8.1. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2 without killing part. Then*

$$(\text{VD}) + (\text{J}_{\geq}) \Rightarrow (\text{PI}).$$

Proof. Fix a ball $B := B(x_0, R)$ with $x_0 \in M$ and $R \in (0, \overline{R})$. It follows from conditions (VD), ($\text{J}_{\geq}$) that, for any $u \in \mathcal{F}$,

$$\begin{aligned}
\int_B d\Gamma_B(u) &\geq \iint_{B\times B} (u(x) - u(y))^2 J(x, y) d\mu(y) d\mu(x)\\
&\geq \iint_{B\times B} (u(x) - u(y))^2 \frac{c}{V(x, y)W(x, y)} d\mu(y) d\mu(x)\\
&\geq \iint_{B\times B} (u(x) - u(y))^2 \frac{c}{V(x, 2R)W(x, 2R)} d\mu(y) d\mu(x)\\
&\geq \frac{c'}{V(x_0, R)W(x_0, R)} \iint_{B\times B} (u(x) - u(y))^2 d\mu(y) d\mu(x).
\end{aligned}$$

From this and using the identity

$$\iint_{B\times B} (u(x) - u(y)^2 d\mu(y) d\mu(x) = 2\mu(B) \int_B (u - u_B)^2 d\mu,$$

we obtain

$$\int_B d\Gamma_B(u) \geq \frac{c'}{V(x_0, R)W(x_0, R)} \cdot 2\mu(B) \int_B (u - u_B)^2 d\mu = \frac{2c'}{W(x_0, R)} \int_B (u - u_B)^2 d\mu,$$

thus showing that (2.14) holds with $\kappa = 1$. □

We introduce condition ($\text{J}^*_{\geq}$).

Definition 8.2. We say that condition ($\text{J}^*_{\geq}$) holds if there exists a non-negative function J such that $dj(x, y) = J(x, y) d\mu(y) d\mu(x)$ in $M \times M$, and for $(\mu \times \mu)$-almost all (x, y) in $M \times M$,

$$J(x, y) \geq \frac{C}{V_*(x, y) d_*(x, y)^{\beta}}, \tag{8.1}$$

where $V_*(x, y) := V_*(x, d_*(x, y))$, $x, y \in M$, and C is a positive constant independent of x, y ($C = 0$ if $J \equiv 0$).

Then we have following.

Lemma 8.3. *The following implication is true:*

$$(\mathrm{VD}) + (\mathrm{J}_{\geq}) \Rightarrow (\mathrm{J}^*_{\geq}).$$

Proof. It suffices to show (8.1). Indeed, fix two points x, y in M. For any $z \in B(x, d(x, y))$, we have by (4.5) that

$$d_*(x, z) \leq LF(x, d(x, z)) \leq LF(x, d(x, y)) \leq L^2 d_*(x, y),$$

showing that

$$B(x, d(x, y)) \subset B_*(x, L^2 d_*(x, y)).$$

Since (VD_*) holds by Proposition 4.4, we have

$$V(x, y) = V(x, d(x, y)) \leq V_*(x, L^2 d_*(x, y)) \leq CV_*(x, d_*(x, y)) = CV_*(x, y).$$

One the other hand, using (4.5) again, we see

$$W(x, y) = W(x, d(x, y)) = F(x, d(x, y))^\beta \leq (Ld_*(x, y))^\beta.$$

Therefore, it follows from (2.19) that

$$J(x, y) \geq \frac{C}{V(x, y)W(x, y)} \geq \frac{C'}{V_*(x, y) d_*(x, y)^\beta},$$

thus showing (8.1). □

Proposition 8.4 ([9, Corollary 3.5])**.** *Assume that $(\mathcal{E}, \mathcal{F})$ is a regular conservative Dirichlet form in L^2. Let K be compact and U, V be open such that $K \subset U \subset V$. Then*

$$P_t \mathbf{1}_V \geq \left(1 - P_t^{K^c} \mathbf{1}_{K^c}\right) \inf_{0<s\leq t} \operatorname{einf}_U P_s^V \mathbf{1}_V \quad \text{in } M.$$

Proposition 8.5 ([9, Lemma 4.1])**.** *Assume that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form in L^2. Let Ω be an open subset of M and $f \in L^1 \cap L^2$ be non-negative. Let $\phi \in \mathcal{F}$ be such that $0 \leq \phi \leq 1$ in M and $\phi = 0$ in Ω. Then for any $t > 0$,*

$$(1 - P_t^\Omega \mathbf{1}_\Omega, f) \geq -\int_0^t \mathcal{E}(\phi, P_s^\Omega f) ds.$$

We introduce condition (LE_*), the full lower bound of the heat kernel under the metric d_*.

Definition 8.6. We say that condition (LE_*) holds if the heat kernel $p_t(x, y)$ exists and, for any $C_0 \geq 1$, there exists $C > 0$ such that for any x, y in M and $t < C_0(W(x, \overline{R}) \wedge W(y, \overline{R}))$,

$$p_t(x, y) \geq C \left(\frac{1}{V_*(x, t^{1/\beta})} \wedge \frac{t}{V_*(x, y) d_*(x, y)^\beta} \right), \tag{8.2}$$

where C is a positive constant independent of t, x, y.

The following gives a lower estimate of the heat kernel.

Lemma 8.7. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2 without killing part. Assume that $(\mathcal{E}, \mathcal{F})$ is conservative, and for any non-empty bounded open set $\Omega \subset M$, the Dirichlet heat kernel $p_t^\Omega(x, y)$ is locally Hölder continuous in $(x, y, t) \in \Omega \times \Omega \times (0, \infty)$. Then*

$$(\mathrm{VD}) + (\mathrm{J}^*_{\geq}) + (\mathrm{S}_*) + (\mathrm{sNLE}_*) \Rightarrow (\mathrm{VD}) + (\mathrm{LE}_*) \Rightarrow (\mathrm{sLE}).$$

Proof. By assumption, the hypothesis of Proposition 7.2 is satisfied, and for any non-empty bounded open set $\Omega \subset M$ and $t > 0$, the function $P_t^\Omega \mathbf{1}_\Omega$ is continuous.

We first show condition (LE_*) holds. The proof is motivated by that in [9, Theorem 4.8]. Let δ_1 be the constants from (6.1) in condition (S_*) and δ_2 from (7.7) in condition (sNLE_*). Let us fix $x, y \in M$ and $t < C_0(W(x, \overline{R}) \wedge W(y, \overline{R}))$. We consider two cases.

Case 1: $d_*(x, y) \leq \delta_2 t^{1/\beta}$. In this case, (8.2) follows directly from (7.7).

Case 2: $d_*(x, y) > \delta_2 t^{1/\beta}$. We divide the proof of this case into four steps.

Step 1. Let
$$\delta_3 := \frac{\delta_2}{2^{1+1/\beta}C_0} \quad \text{and} \quad \delta_4 := \frac{1}{2} \wedge (\delta_3\delta_1)^\beta.$$
Fix $s < C_0(W(x,\overline{R}) \wedge W(y,\overline{R}))$ with
$$d_*(x,y) > \delta_2 s^{1/\beta}. \tag{8.3}$$
Set $r := \delta_3 s^{1/\beta}$ so that
$$r < \frac{1}{2}(W(x,\overline{R}) \wedge W(y,\overline{R})),$$
and set
$$B_x := B_*(x,r) \quad \text{and} \quad B_y := B_*(y,r)$$
so that B_x and B_y are disjoint. By the pointwise semigroup property in Proposition 7.2(1), we have for any $z \in M$,
$$\begin{aligned} p_s(x,z) &= \int_M p_{(1-\delta_4)s}(x,w)p_{\delta_4 s}(w,z)d\mu(w) \\ &\geq \int_{B_x} p_{(1-\delta_4)s}(x,w)p_{\delta_4 s}(w,z)d\mu(w) \\ &\geq \operatorname*{einf}_{w\in B_x} p_{(1-\delta_4)s}(x,w)\int_{B_x} p_{\delta_4 s}(w,z)d\mu(w) \\ &= \operatorname*{einf}_{w\in B_x} p_{(1-\delta_4)s}(x,w)P_{\delta_4 s}\mathbf{1}_{B_x}(z). \end{aligned} \tag{8.4}$$
Since for any $w \in B_x$,
$$d_*(x,w) < r = \delta_3 s^{1/\beta} < \delta_2(s/2)^{1/\beta} \leq \delta_2((1-\delta_4)s)^{1/\beta},$$
we have by (sNLE$_*$) and (VD$_*$) that
$$\operatorname*{einf}_{w\in B_x} p_{(1-\delta_4)s}(x,w) \geq \operatorname*{einf}_{w\in B_x} \frac{C}{V_*(x,((1-\delta_4)s)^{1/\beta})} \geq \frac{C}{V_*(x,s^{1/\beta})}.$$
From this, we have by (8.4) that for any $z \in M$,
$$p_s(x,z) \geq \frac{C}{V_*(x,s^{1/\beta})}P_{\delta_4 s}\mathbf{1}_{B_x}(z). \tag{8.5}$$
We need to estimate $P_{\delta_4 s}\mathbf{1}_{B_x}$ from below.

Step 2. Let $K \subset \frac{1}{4}B_x$ be compact. Since $(\mathcal{E},\mathcal{F})$ is conservative, applying Proposition 8.4 with $V = B_x$, $U = \frac{1}{4}B_x$ and $t = \delta_4 s$, we obtain
$$P_{\delta_4 s}\mathbf{1}_{B_x} \geq \left(1 - P_{\delta_4 s}^{K^c}\mathbf{1}_{K^c}\right) \inf_{0<t'\leq\delta_4 s} \operatorname*{einf}_{\frac{1}{4}B_x} P_{t'}^{B_x}\mathbf{1}_{B_x} \quad \text{in } B_y. \tag{8.6}$$
Since
$$\delta_4 s = \delta_4(\delta_3^{-1}r)^\beta \leq (\delta_1 r)^\beta,$$
we see by condition (S$_*$) that
$$\operatorname*{einf}_{\frac{1}{4}B_x} P_{t'}^{B_x}\mathbf{1}_{B_x} \geq \varepsilon$$
where $\varepsilon \in (0,1)$ comes from (S$_*$). From this, we have by (8.6)
$$P_{\delta_4 s}\mathbf{1}_{B_x} \geq \varepsilon\left(1 - P_{\delta_4 s}^{K^c}\mathbf{1}_{K^c}\right) \quad \text{in } B_y. \tag{8.7}$$
We need to estimate $1 - P_{\delta_4 s}^{K^c}\mathbf{1}_{K^c}$ from below.

Step 3. Indeed, let $\phi \in \mathcal{F}$ be such that $0 \leq \phi \leq \mathbf{1}_K$, and $0 \leq f \in L^1 \cap L^\infty$ be such that
$$\operatorname{supp}(f) \subset \frac{1}{4}B_y.$$

Since two functions ϕ and $P_\tau^{K^c} f$ have disjoint supports, we have for all $\tau \in (0, \delta_4 s)$

$$\begin{aligned}
-\mathcal{E}(\phi, P_\tau^{K^c} f) &= -\mathcal{E}^{(L)}(\phi, P_\tau^{K^c} f) - \mathcal{E}^{(J)}(\phi, P_\tau^{K^c} f) \\
&= -\mathcal{E}^{(J)}(\phi, P_\tau^{K^c} f) \\
&= -\int_M \int_M (\phi(z) - \phi(w))(P_\tau^{K^c} f(z) - P_\tau^{K^c} f(w)) dj \\
&= 2 \int_K \phi(z) \int_{K^c} P_\tau^{K^c} f(w) J(z,w) d\mu(w) d\mu(z) \\
&\geq 2 \operatorname*{einf}_{z \in B_x, w \in B_y} J(z,w) \int_K \phi(z) \int_{B_y} P_\tau^{K^c} f(w) d\mu(w) d\mu(z) \\
&= 2 \operatorname*{einf}_{z \in B_x, w \in B_y} J(z,w) \|\phi\|_{L^1} (P_\tau^{K^c} f, \mathbf{1}_{B_y}) \\
&\geq 2\, \varepsilon \operatorname*{einf}_{z \in B_x, w \in B_y} J(z,w) \|\phi\|_{L^1} \|f\|_{L^1(\delta_1 B_y)},
\end{aligned}$$

where we have used the fact that, by condition (S$_*$) and the fact that $\operatorname{supp}(f) \subset \frac{1}{4} B_y$

$$(P_\tau^{K^c} f, \mathbf{1}_{B_y}) = (f, P_\tau^{K^c} \mathbf{1}_{B_y}) \geq (f, P_\tau^{B_y} \mathbf{1}_{B_y}) \geq \varepsilon \|f\|_{L^1(\frac{1}{4} B_y)}.$$

Therefore, applying Proposition 8.5 with $\Omega = K^c$, we obtain from above that

$$\begin{aligned}
(1 - P_{\delta_4 s}^{K^c} \mathbf{1}_{K^c}, f) &\geq -\int_0^{\delta_4 s} \mathcal{E}(\phi, P_\tau^{K^c} f) ds \\
&\geq 2\varepsilon \delta_4 s \operatorname*{einf}_{z \in B_x, w \in B_y} J(z,w) \|\phi\|_{L^1} \|f\|_{L^1(\frac{1}{4} B_y)}.
\end{aligned} \tag{8.8}$$

Let us estimate $\operatorname{einf}_{z \in B_x, w \in B_y} J(z,w)$. By (8.3), we have that, for any $z \in B_x$ and $w \in B_y$,

$$d_*(z,w) \leq d_*(z,x) + d_*(x,y) + d_*(y,w) \leq 2r + d_*(x,y) \leq 2d_*(x,y),$$

which yields by the inequality (8.1) in condition (J$_\geq^*$) and (VD$_*$) that

$$\begin{aligned}
\operatorname*{einf}_{z \in B_x, w \in B_y} J(z,w) &\geq \operatorname*{einf}_{z \in B_x, w \in B_y} \frac{C}{V_*(z,w) d_*(z,w)^\beta} \\
&\geq \operatorname*{einf}_{z \in B_x, w \in B_y} \frac{C}{V_*(z, 2d_*(x,y))(2d(x,y))^\beta} \\
&\geq \frac{C'}{V_*(x,y) d(x,y)^\beta}.
\end{aligned}$$

Plugging the above inequality into (8.8) and using the arbitrariness of f, we obtain that

$$\begin{aligned}
1 - P_{\delta_4 s}^{K^c} \mathbf{1}_{K^c} &\geq 2\varepsilon \delta_4 s \operatorname*{einf}_{z \in B_x, w \in B_y} J(z,w) \|\phi\|_{L^1} \\
&\geq \frac{Cs}{V_*(x,y) d(x,y)^\beta} \|\phi\|_{L^1} \quad \text{in } \frac{1}{4} B_y,
\end{aligned}$$

Plugging the above inequality into (8.7), we have

$$P_{\delta_4 s} \mathbf{1}_{B_x} \geq \varepsilon \left(1 - P_{\delta_4 s}^{K^c} \mathbf{1}_{K^c}\right) \geq \frac{C\varepsilon s}{V_*(x,y) d(x,y)^\beta} \|\phi\|_{L^1} \ \text{ in } \frac{1}{4} B_y. \tag{8.9}$$

Step 4. Substituting (8.9) into (8.5), we obtain that for μ-almost all $z \in \frac{1}{4} B_y$,

$$p_s(x,z) \geq \frac{C}{V_*(x, s^{1/\beta})} P_{\delta_4 s} \mathbf{1}_{B_x}(z) \geq \frac{C'}{V_*(x, s^{1/\beta})} \frac{s \|\phi\|_{L^1}}{V_*(x,y) d(x,y)^\beta},$$

where C' is a positive constant depending only on constants in hypothesis.

Since ϕ is arbitrary with support in $K \subset \frac{1}{4}B_x$, it follows from above and (VD$_*$) that

$$p_s(x,z) \geq \frac{C's}{V_*(x,s^{1/\beta})} \frac{\mu(\frac{1}{4}B_x)}{V_*(x,y)d(x,y)^\beta} \geq \frac{Cs}{V_*(x,y)d(x,y)^\beta}, \quad \mu\text{-a.a. } z \in \frac{1}{4}B_y.$$

We emphasize that one can NOT set $z = y$ in the above inequality since it holds only for *μ-almost all* $z \in \frac{1}{4}B_y$. To overcome this difficulty, we set $\delta_5 := \left(\frac{\delta_1\delta_3}{4}\right)^\beta$. Multiplying the above inequality by $p_{\delta_5 s}(z,y)$, integrating it with respect to $d\mu(z)$, and using the inequality $p_{\delta_5 s}(z,y) \geq p^{\frac{1}{4}B_y}_{\delta_5 s}(z,y)$, $z \in M$ (by Proposition 7.2(4)), we have

$$\begin{aligned} p_{(1+\delta_5)s}(x,y) &= \int_M p_s(x,z)p_{\delta_5 s}(z,y)d\mu(z) \\ &\geq \int_{\frac{1}{4}B_y} p_s(x,z)p^{\frac{1}{4}B_y}_{\delta_5 s}(z,y)d\mu(z) \\ &\geq \frac{Cs}{V_*(x,y)d(x,y)^\beta}\int_{\frac{1}{4}B_y} p^{\frac{1}{4}B_y}_{\delta_5 s}(z,y)d\mu(z) \\ &= \frac{Cs}{V_*(x,y)d(x,y)^\beta} \cdot P^{\frac{1}{4}B_y}_{\delta_5 s}\mathbf{1}_{\frac{1}{4}B_y}(y). \end{aligned} \tag{8.10}$$

Moreover, note that

$$(\delta_5 s)^{1/\beta} = \delta_5^{1/\beta} \cdot \delta_3^{-1} r = \delta_1\left(\frac{r}{4}\right) \quad \text{and} \quad \frac{r}{4} < W(y,\overline{R}).$$

Since $P^{\frac{1}{4}B_y}_{\delta s}\mathbf{1}_{\frac{1}{4}B_y}$ is continuous by assumption, by condition (S$_*$), we have

$$P^{\frac{1}{4}B_y}_{\delta s}\mathbf{1}_{\frac{1}{4}B_y}(y) \geq \varepsilon.$$

Hence, it follows from (8.10) that for any $s < C_0(W(x,\overline{R}) \wedge W(y,\overline{R}))$ with $d_*(x,y) > \delta_2 s^{1/\beta}$,

$$p_{(1+\delta_5)s}(x,y) \geq \frac{Cs}{V_*(x,y)d(x,y)^\beta} \cdot P^{\frac{1}{4}B_y}_{\delta_5 s}\mathbf{1}_{\frac{1}{4}B_y}(y) \geq \frac{C\varepsilon s}{V_*(x,y)d(x,y)^\beta}.$$

Note that $t < C_0(W(x,\overline{R}) \wedge W(y,\overline{R}))$ and $d_*(x,y) > \delta_2 t^{1/\beta}$. Therefore, we can set $s = \frac{t}{1+\delta_5}$ in the above inequality and obtain

$$p_t(x,y) \geq \frac{C\varepsilon(1+\delta_5)^{-1}t}{V_*(x,y)d(x,y)^\beta} \geq C'\left(\frac{1}{V_*(x,t^{1/\beta})} \wedge \frac{t}{V_*(x,y)d_*(x,y)^\beta}\right).$$

Finally, combining the above two cases, we have proved condition (LE$_*$).

It remains to show the implication (VD) + (LE$_*$) $\Rightarrow$ (sLE). Assume that conditions (VD) and (LE$_*$) hold true. By Proposition 7.2(1), the heat kernel $p_t(x,y)$ satisfies the pointwise semigroup property. Hence, we need only to prove the inequality (2.20) in (sLE) is true for any $x,y \in M$ and $t < W(x,\overline{R}) \wedge W(y,\overline{R})$).

Indeed, fix $x,y \in M$ and $t < W(x,\overline{R}) \wedge W(y,\overline{R})$). Let

$$R := F^{-1}(x, L^{-1}L_0 t^{1/\beta}) = W^{-1}(x, L^{-\beta}L_0^\beta t).$$

By (4.6), we see $B_*(x,t^{1/\beta}) \subset B(x,R)$, and hence, by (VD) and (2.8),

$$\frac{1}{V_*(x,t^{1/\beta})} \geq \frac{1}{V(x,R)} = \frac{1}{V(x,W^{-1}(x,L^{-\beta}L_0^\beta t))} \geq \frac{C}{V(x,W^{-1}(x,t))}. \tag{8.11}$$

On the other hand, by (4.5) and (2.9), we have for any $z \in B_*(x,d_*(x,y))$,

$$F(x,d(x,z)) \leq Ld_*(x,z) \leq Ld_*(x,y) \leq L^2F(x,d(x,y)) \leq F(x,Cd(x,y)),$$

which gives that $d(x,z) \leq Cd(x,y)$ by the monotonicity of $F(x,\cdot) = W(x,\cdot)^{1/\beta}$. Thus

$$B_*(x,d_*(x,y)) \subset B(x,Cd(x,y)),$$

from which, we see by (VD)

$$V_*(x, d_*(x,y)) \le V(x, Cd(x,y)) \le CV(x, d(x,y)) = CV(x,y).$$

Note that by (4.5),

$$d_*(x,y)^\beta \le (LF(x, d(x,y)))^\beta = L^\beta W(x, d(x,y)) = L^\beta W(x,y).$$

Therefore, it follows from (8.11), the above two inequalities and condition (LE_*) that

$$p_t(x,y) \ge C\left(\frac{1}{V_*(x, t^{1/\beta})} \wedge \frac{t}{V_*(x,y) d_*(x,y)^\beta}\right) \ge C'\left(\frac{1}{V(x, W^{-1}(x,t))} \wedge \frac{t}{V(x,y)W(x,y)}\right)$$

thus showing (2.20). We have proved (sLE). □

We show that condition (LE) implies condition ($\mathrm{J}_\ge$).

Lemma 8.8. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2. If the jump kernel $J(x,y)$ exists, then*

$$(\mathrm{LE}) \Rightarrow (\mathrm{J}_\ge).$$

Proof. Let $U, V \subset M$ be bounded open sets such that $\mathrm{dist}(U,V) > 0$, and let $0 \le f, g \in \mathcal{F} \cap L^1$ be such that $\mathrm{supp}(f) \subset U$, $\mathrm{supp}(g) \subset V$. Since $\mathrm{supp}(f) \cap \mathrm{supp}(g) = \emptyset$, we have

$$-\mathcal{E}(f,g) = -\mathcal{E}^{(J)}(f,g) = 2\int_U \int_V f(x)g(y)J(x,y)d\mu(y)d\mu(x).$$

On the other hand, let us fix a point $x_0 \in U$. In the case when $\overline{R} < \infty$, by the right inequality in (2.8), we have for any $x \in U$

$$\frac{W(x_0, \overline{R})}{W(x, \overline{R})} \le \frac{W(x_0, \overline{R} + \mathrm{diam}\, U)}{W(x, \overline{R})} \le c\left(\frac{\overline{R} + \mathrm{diam}\, U}{\overline{R}}\right)^{\beta_2} := c_1^{-1}.$$

That is,

$$c_1 W(x_0, \overline{R}) \le W(x, \overline{R}), \quad x \in U.$$

Similarly, fix a point $y_0 \in V$ and there exists $c_2 = c_2(V, \overline{R}) > 0$ such that $c_2 W(y_0, \overline{R}) \le W(y, \overline{R})$ for all $y \in V$.

Hence by (LE), we have for $(\mu \times \mu)$-almost all (x,y) in $U \times V$ and for any $t < (c_1 W(x_0, \overline{R})) \wedge (c_2 W(y_0, \overline{R})) \le W(x, \overline{R}) \wedge W(y, \overline{R})$,

$$p_t(x,y) \ge C\left(\frac{1}{V(x, W^{-1}(x,t))} \wedge \frac{t}{V(x,y)W(x,y)}\right).$$

Consequently,

$$\begin{aligned}
-\mathcal{E}(f,g) &= \lim_{t\to 0} \frac{1}{t}(P_t f - f, g) = \lim_{t \to 0} \frac{1}{t}(P_t f, g) \\
&= \lim_{t\to 0} \frac{1}{t} \int_U \int_V p_t(x,y) f(x) g(y) d\mu(y) d\mu(x) \\
&\ge \liminf_{t \to 0} \frac{1}{t} \int_U \int_V \frac{Ct}{V(x,y)W(x,y)} f(x)g(y)d\mu(y)d\mu(x) \\
&= \int_U \int_V \frac{C}{V(x,y)W(x,y)} f(x)g(y)d\mu(y)d\mu(x).
\end{aligned}$$

Therefore, we obtain

$$\int_U \int_V f(x)g(y)J(x,y)d\mu(y)d\mu(x) \ge \frac{C}{2}\int_U \int_V \frac{f(x)g(y)}{V(x,y)W(x,y)} d\mu(y)d\mu(x).$$

Since $(\mathcal{E}, \mathcal{F})$ is regular, the functions

$$\sum_{i=1}^n f_i(x) g_i(y)$$

with $f_i, g_i \in \mathcal{F} \cap C_0(M)$ and $\operatorname{supp}(f_i) \cap \operatorname{supp}(g_i) = \emptyset$ for all $1 \le i \le n$, constitute a dense subalgebra of $C_0(M \times M \setminus \text{diag})$, see for example [8, Lemma 1.4.2 on p. 29]. It follows from above that

$$J(x,y) \geq \frac{C/2}{V(x,y)W(x,y)}$$

for $(\mu \times \mu)$-almost all (x,y) in $M \times M \setminus \text{diag}$, thus showing condition ($\text{J}_{\geq}$). □

We are now in a position to prove Theorem 2.12.

Proof of Theorem 2.12. Under the conditions (VD), (RVD) and (TJ), the implication

$$(\text{J}_{\geq}) + (\text{Gcap}) + (\text{C}) \;\Rightarrow\; (\text{sLLE}) + (\text{sLE})$$

follows from the following sequence of implications:

$$(\text{VD}) + (\text{J}_{\geq}) \Rightarrow (\text{PI}) \quad (\text{Lemma } 8.1)$$

$$(\text{VD}) + (\text{RVD}) + (\text{PI}) + (\text{Gcap}) + (\text{TJ}) \Rightarrow (\text{sLLE}) + (\text{NLE}) \quad (\text{Theorem } 2.9)$$

$$(\text{VD}) + (\text{RVD}) + (\text{PI}) + (\text{Gcap}) + (\text{TJ}) \Rightarrow (\text{S}_*) \quad (\text{by } (6.4))$$

$$(\text{VD}) + (\text{NLE}) \Rightarrow (\text{NLE}_*) \quad (\text{Lemma } 7.7)$$

$$(\text{VD}) + (\text{J}_{\geq}) \Rightarrow (\text{J}^*_{\geq}) \quad (\text{Lemma } 8.3)$$

$$(\text{VD}) + (\text{S}_*) + (\text{NLE}_*) + (\text{J}^*_{\geq}) + (\text{C}) + \text{Hölder continuity of } p_t^{\Omega} \Rightarrow (\text{sLE}) \quad (\text{Lemma } 8.7).$$

It is obvious that

$$(\text{sLLE}) + (\text{sLE}) \Rightarrow (\text{LLE}) + (\text{LE}).$$

Under the condition (VD) and the assumption that the jump kernel $J(x,y)$ exists, the implication

$$(\text{LLE}) + (\text{LE}) \;\Rightarrow\; (\text{J}_{\geq}) + (\text{Gcap})$$

follows from the following implications:

$$(\text{VD}) + (\text{LLE}) \Rightarrow (\text{S}) \quad (\text{Lemma } 7.14)$$

$$(\text{VD}) + (\text{S}) \Rightarrow (\text{Gcap}) \quad ([12, \text{Lemma } 13.5])$$

$$(\text{LE}) + \text{“the existence of jump kernel”} \Rightarrow (\text{J}_{\geq}) \quad (\text{Lemma } 8.8).$$

The proof of Theorem 2.12 is complete. □

The following conservativeness result is extracted from [9, Lemmas 4.5 and 4.6] and their proofs; see also [9, Remark 4.7].

Proposition 8.9. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2 without killing part. Let $r_0 > 0$, $T > 0$ and $\varepsilon \in (0,1)$. Assume the following conditions are true.*

1. *Every ball has finite measure;*
2. $\operatorname{cutoff}(B, 2B) \neq \emptyset$ *for every ball B of radius $> r_0$;*
3. *For any ball B of radius $R > r_0$ and any $t < T$,*

$$P_t \mathbf{1}_B \geq \varepsilon, \quad \mu\text{-a.e. in } \frac{1}{4}B.$$

Then $(\mathcal{E}, \mathcal{F})$ is conservative.

Corollary 8.10. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2 without killing part. Assume that*

$$\inf_{z \in M} W(z, \overline{R}) > 0$$

for some $c_0 > 0$. If one of the following conditions is satisfied, then $(\mathcal{E}, \mathcal{F})$ is conservative:

1. (S_*) *and every ball under the new metric d_* has finite measure;*
2. (S) *and every ball has finite measure;*

Proof. (1). Suppose that (S$_*$) is satisfied, and that every ball under the new metric d_* has finite measure.

Since $\inf_{z\in M} W(z,\overline{R}) > c_0$ for some $c_0 > 0$, condition (S$_*$) implies that for all metric balls $B_* = B_*(x,r)$ of radius $r < c_0^{1/\beta}$,

$$P_t^{B_*}\mathbf{1}_{B_*} \geq \varepsilon \ \text{ in } \frac{1}{4}B_*,$$

provided that $t^{1/\beta} \leq \delta_* r$. This is exactly the condition (S) in [9]. Hence, it follows from [9, Lemma 4.6, p. 3327] that $(\mathcal{E},\mathcal{F})$ is conservative.

(2). Suppose that (S) is satisfied, and that every ball has finite measure.

One can follow the method in the proof of [14, Proposition 6.4(2)] and use Proposition 4.3 to prove that (S) $\Rightarrow$ (S$_*$). On the other hand, it follows from (4.6) that every ball under the new metric d_* has finite measure. Hence, the conservativeness of $(\mathcal{E},\mathcal{F})$ follows from (1). □

Proof of Corollary 2.13. By Theorem 2.12, it suffices to prove the implication

$$(\text{VD}) + (\text{RVD}) + (\text{J}_{\geq}) + (\text{Gcap}) + (\text{TJ}) \Rightarrow (\text{C}).$$

under the assumption that $\overline{R} = \operatorname{diam} M$ or the function W in (2.8) is independent of the space variable. Indeed by the following two implications

$$(\text{VD}) + (\text{J}_{\geq}) \Rightarrow (\text{PI}) \quad (\text{Lemma 8.1})$$
$$(\text{VD}) + (\text{RVD}) + (\text{PI}) + (\text{Gcap}) + (\text{TJ}) \Rightarrow (\text{S}_*) \quad ((6.4))$$

we obtain (S$_*$). Therefore, $(\mathcal{E},\mathcal{F})$ is conservative by Corollary 8.10(1). □

8.2. **Two-sided estimates.** In this subsection we combine the upper bounds of heat kernels from [14] and the lower bounds obtained in this paper, in order to state two-sided estimates of the heat kernel.

Proof of Theorem 2.19. The first two equivalences

$$\begin{aligned}(\text{PI}) + (\text{Gcap}) + (\text{TJ}_q) &\Leftrightarrow (\text{TP}_q) + (\text{sLLE}) \\ &\Leftrightarrow (\text{TP}_q) + (\text{LLE})\end{aligned}$$

follow from the following implications:

$$(\text{VD}) + (\text{PI}) \Rightarrow (\text{PI}_*) \quad (\text{Proposition 4.7})$$
$$(\text{VD}) + (\text{RVD}) \Rightarrow (\text{VD}_*) + (\text{RVD}_*) \quad (\text{Proposition 4.4})$$
$$(\text{VD}_*) + (\text{RVD}_*) + (\text{PI}_*) \Rightarrow (\text{Nash}_*) \quad (\text{Lemma 4.9})$$
$$(\text{Nash}_*) \Rightarrow (\text{FK}_*) \quad (\text{Lemma 4.11})$$
$$(\text{VD}) + (\text{FK}_*) \Rightarrow (\text{FK}) \quad (\text{Proposition 4.13})$$
$$(\text{VD}) + (\text{RVD}) + (\text{FK}) + (\text{Gcap}) + (\text{TJ}_q) \Rightarrow (\text{TP}_q) \quad ([14, \text{Theorem 2.15}])$$
$$(\text{VD}) + (\text{TJ}_q) \Rightarrow (\text{TJ}) \quad ([14, \text{Proposition 3.1}])$$
$$(\text{VD}) + (\text{RVD}) + (\text{PI}) + (\text{Gcap}) + (\text{TJ}) \Rightarrow (\text{sLLE}) \quad \Rightarrow \quad (\text{LLE}) \quad (\text{Theorem 2.9})$$

and

$$(\text{VD}) + (\text{TP}_q) \Rightarrow (\text{TJ}_q) \quad ([14, \text{Theorem 9.1(2)}])$$
$$(\text{VD}) + (\text{LLE}) \Rightarrow (\text{LLE}_*) \quad (\text{Lemma 7.7})$$
$$(\text{VD}) + (\text{LLE}_*) \Rightarrow (\text{PI}_*) \quad (\text{Lemma 7.12})$$
$$(\text{VD}) + (\text{PI}_*) \Rightarrow (\text{PI}) \quad (\text{Proposition 4.7})$$
$$(\text{VD}) + (\text{LLE}) \Rightarrow (\text{S}) \quad (\text{Lemma 7.14})$$
$$(\text{VD}) + (\text{S}) \Rightarrow (\text{Gcap}) \quad ([12, \text{Lemma 13.5}])$$

The implication

$$(\mathrm{TP}_q) + (\mathrm{sLLE}) \Rightarrow (\mathrm{UE}_q) + (\mathrm{NLE}) + (\mathrm{C})$$

follows directly from the following implications:

$$(\mathrm{VD}) + (\mathrm{TP}_q) \Rightarrow (\mathrm{UE}_q) \quad ([14, \text{Lemma } 8.8])$$

$$(\mathrm{VD}) + (\mathrm{sLLE}) \Rightarrow (\mathrm{sLLE}_*) \quad (\text{Lemma } 7.7)$$

$$(\mathrm{VD}) + (\mathrm{sLLE}_*) \Rightarrow (\mathrm{sNLE}_*) \quad \Rightarrow \quad (\mathrm{NLE}_*) \quad (\text{Lemma } 7.6)$$

$$(\mathrm{VD}) + (\mathrm{NLE}_*) \Rightarrow (\mathrm{NLE}) \quad (\text{Lemma } 7.7)$$

$$(\mathrm{VD}) + (\mathrm{LLE}) \Rightarrow (\mathrm{S}) \quad (\text{Lemma } 7.14)$$

$$(\mathrm{VD}) + (\mathrm{S}) + \text{“}\overline{R} = \operatorname{diam} M\text{”} \Rightarrow (\mathrm{C}) \quad (\text{Corollary } 8.10(2) \text{ and Remark } 2.14).$$

The proof of Theorem 2.19 is complete. □

Denote the diameter of M under the metric d_* by

$$\overline{R}_* := \sup\{d_*(x,y)|\ x,y \in M\}.$$

Definition 8.11 (Condition (TP$_*$))**.** We say that condition (TP$_*$) is satisfied if for any $C_0 \geq 1$, there exists $C > 0$ such that for any ball $B_* := B_*(x,r)$ of radius $r \in (0, \overline{R}_*)$ and any $t < C_0(\overline{R}_*)^\beta$

$$P_t \mathbf{1}_{B_*^c} \leq \frac{Ct}{r^\beta} \quad \text{in } \frac{1}{4}B_*. \tag{8.12}$$

Lemma 8.12. *Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form in L^2 without killing part. Assume that $\overline{R} = \operatorname{diam} M$. Then*

$$(\mathrm{VD}_*) + (\mathrm{TP}_*) + (\mathrm{NLE}_*) \Rightarrow (\mathrm{S}_*).$$

Proof. The proof is motivated by [9, Lemma 4.10, p. 3334].

Fix a ball $B_* := B_*(x_0, r)$ with $r < \overline{R}_*$. Since every ball has finite measure by (VD$_*$), we can apply [16, Eq. (4.1), p. 2626] with

$$\Omega = M, \quad U = B_*, \ K = \left(\frac{3}{4}B_*\right)^c, \quad \text{and} \quad f = \frac{1}{2}B_*$$

and obtain that, for any $t > 0$,

$$P_t^{B_*}\mathbf{1}_{\frac{1}{2}B_*} \geq P_t\mathbf{1}_{\frac{1}{2}B_*} - \sup_{s\in(0,t)} \left\| P_s \mathbf{1}_{\frac{1}{2}B_*}\right\|_{L^\infty\left(\left(\frac{3}{4}B_*\right)^c\right)}, \quad \mu\text{-a.e. in } B_*. \tag{8.13}$$

Let δ_* be the constant from (NLE$_*$). Note that for any $x \in \frac{1}{4}B_*$ and $t < r^\beta$,

$$B_*(x, \frac{\delta_*}{4}t^{1/\beta}) \subset \frac{1}{2}B_*.$$

By (NLE$_*$) and (VD$_*$), we obtain, for any $t < r^\beta$ and μ-almost all $x \in \frac{1}{4}B_*$,

$$\begin{aligned} P_t\mathbf{1}_{\frac{1}{2}B_*}(x) &= \int_{\frac{1}{2}B_*} p_t(x,z)d\mu(z) \geq \int_{B_*(x,\frac{\delta_*}{4}t^{1/\beta})} p_t(x,z)d\mu(z) \\ &\geq \frac{c}{V_*(x,t^{1/\beta})}\int_{B_*(x,\frac{\delta_*}{4}t^{1/\beta})} d\mu(z) = \frac{cV_*(x,\frac{\delta_*}{4}t^{1/\beta})}{V_*(x,t^{1/\beta})} \geq c_0. \end{aligned} \tag{8.14}$$

On the other hand, note that for any $w \in \frac{1}{2}B_*$ and $z \in \left(\frac{3}{4}B_*\right)^c$,

$$d(z,w) \geq d(z,x_0) - d(x_0,w) > \frac{3}{4}r - \frac{1}{2}r = \frac{r}{4},$$

we have

$$\frac{1}{2}B_* \subset B_*(z, \frac{r}{4})^c.$$

Then, by (TP$_*$), we have, for any $s<t$,

$$P_s\mathbf{1}_{\frac{1}{2}B_*}\le P_s\mathbf{1}_{B_*(z,\frac{r}{4})^c}\le \frac{ct}{(r/4)^\beta}\quad \mu\text{-a.e. in } \frac{1}{4}B_*(z,\frac{r}{4})=B_*(z,\frac{r}{16}).$$

Covering $\left(\frac{3}{4}B_*\right)^c$ by at most countable balls like $B_*(z,\frac{r}{16})$ with $z\in\left(\frac{3}{4}B_*\right)^c$, we obtain from the above inequality that

$$\sup_{s\in(0,t)}\left\|P_s\mathbf{1}_{\frac{1}{2}B_*}\right\|_{L^\infty(\left(\frac{3}{4}B_*\right)^c)}\le\frac{ct}{r^\beta}. \tag{8.15}$$

Plugging (8.14) and (8.15) into (8.13), we obtain that for μ-almost all $x\in\frac{1}{4}B_*$ and $t<r^\beta$,

$$P_t^{B_*}\mathbf{1}_{\frac{1}{2}B_*}\ge c_0-\frac{ct}{r^\beta}.$$

Setting $\delta:=\left(\frac{c_0}{2c}\right)^{1/\beta}$, we have for μ-almost all $x\in\frac{1}{4}B_*$ and $t<(\delta r)^\beta$,

$$P_t^{B_*}\mathbf{1}_{\frac{1}{2}B_*}\ge c_0-\frac{ct}{r^\beta}=c_0-\frac{c_0}{2}=\frac{c_0}{2}.$$

To prove (S$_*$), it suffices to extend the radius of the ball B_* from $r<\overline{R}_*$ to $r<W(x_0,\overline{R})^{1/\beta}$ in the case when $\overline{R}<\infty$. Indeed, since $\overline{R}=\operatorname{diam} M$, by (2.22), we see that

$$\sup_{x\in M}W(x,\overline{R})\le CW(x_0,\overline{R})<\infty.$$

for some fixed $x_0\in M$. Moreover, by the standard covering arguments, we can extend the radius of the ball B_* from $r<\overline{R}_*$ to $r<a\overline{R}_*$ for any given $a\ge 1$. Taking a large enough so that $a\overline{R}_*>CW(x_0,\overline{R})$, we manage to extend the radius of the ball B_* from $r<\overline{R}_*$ to $r<W(x_0,\overline{R})^{1/\beta}$ in the case when $\overline{R}<\infty$. That is, we obtain (S$_*$). □

Proof of Corollary 2.20. We have the implication

$$(\text{VD})+(\text{J}_\ge)\Rightarrow(\text{PI})\qquad(\text{Lemma } 8.1)$$
$$(\text{VD})+(\text{RVD})+(\text{PI})+(\text{Gcap})+(\text{TJ}_q)\Rightarrow(\text{TP}_q)+(\text{C})\qquad(\text{Theorem } 2.19)$$
$$(\text{VD})+(\text{TJ}_q)\Rightarrow(\text{TJ})\qquad([14,\text{ Proposition } 3.1])$$
$$(\text{VD})+(\text{RVD})+(\text{J}_\ge)+(\text{Gcap})+(\text{TJ})+(\text{C})\Rightarrow(\text{sLE})\qquad(\text{Theorem } 2.12).$$

Combining the above four implications, we obtain the implication "⇒" in (2.29).

It is obvious that (sLE) ⇒ (LE).

To complete the circle in (2.29), we have the following implications:

$$(\text{Gcap})\Leftarrow(\text{VD})+(\text{TP}_q)+(\text{C})\qquad([14,\text{ Prop. } 9.1(3)\text{ and Eq. }(6.4)])$$
$$\text{“the existence of } J(x,y)\Leftarrow(\text{VD})+(\text{TP}_q)\qquad([14,\text{ Theorem } 9.1(2)])$$
$$(\text{J}_\ge)\Leftarrow(\text{LE})+\text{“the existence of } J(x,y)\ (\text{Lemma } 8.8),$$

where the existence of $J(x,y)$ is ensured by (TJ$_q$). Combining the above two formulas, we obtain the implication "⇐" in (2.29) and, hence, complete the circle.

To prove (2.30), it suffices to prove the implication

$$(\text{VD})+(\text{TP}_q)+(\text{LE})\quad\Rightarrow\quad(C).$$

This follows from the following implications:

$$(\text{VD})\Rightarrow(\text{VD}_*)\qquad(\text{Proposition } 4.4)$$
$$(\text{VD})+(\text{TP}_q)\Rightarrow(\text{TP}_*)\qquad([14,\text{ Prop. } 3.1\text{ and Eq. }(8.17)])$$
$$(\text{LE})\Rightarrow(\text{NLE})$$
$$(\text{VD})+(\text{NLE})\Rightarrow(\text{NLE}_*)\qquad(\text{Lemma } 7.7)$$
$$(\text{VD}_*)+(\text{TP}_*)+(\text{NLE}_*)\Rightarrow(\text{S}_*)\qquad(\text{Lemma } 8.12)$$

$$(\text{VD}) + (\text{S}) + \text{``}\overline{R} = \operatorname{diam} M\text{''} \Rightarrow (\text{C}) \quad \text{(Corollary 8.10(1) and Remark 2.14)}.$$

The implication (2.31) follows directly from (2.30) and the implication

$$(\text{VD}) + (\text{TP}_q) \Rightarrow (UE_q)$$

(cf. [14, Lemma 8.8]). Finally, the equivalence (2.32) follows from the equivalences (2.29), (2.30) and the following relations:

$$(\text{J}_{\leq}) = (\text{TP}_\infty) \Leftrightarrow (\text{UE}_\infty) \;=\; (\text{UE}).$$

□

9. Appendix

In this appendix, we collect some facts that have been used in this paper.

Proposition 9.1 ([14, Proposition 10.1 in Appendix])**.** *Assume that condition* (VD) *holds and W satisfies (2.8). Then there exists a constant $C > 0$ such that, for all $t > 0$ and all points x, y in M with $d(x, y) \leq W^{-1}(x, t) \vee W^{-1}(y, t)$,*

$$C^{-1} \leq \frac{V(x, W^{-1}(x, t))}{V(y, W^{-1}(y, t))} \leq C. \tag{9.1}$$

Proposition 9.2 ([12, Proposition 15.1 in Appendix])**.** *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2. Suppose that $u = w + a \in \mathcal{F}'$ with $w \in \mathcal{F}$ and $a \in \mathbb{R}$, $v \in \mathcal{F} \cap L^\infty$ and that $F : \mathbb{R} \mapsto \mathbb{R}$ is a Lipschitz function. Then the following statements are true.*

- *(i) The function $F(u) - F(a)$ belongs to $\mathcal{F}$, so that $F(u) \in \mathcal{F}'$.*
- *(ii) If in addition $F(u) \in L^\infty$, then $F(u)v \in \mathcal{F} \cap L^\infty$.*
- *(iii) Let Ω be an open subset of M. If in addition $v \in \mathcal{F}(\Omega)$, then $F(u)v \in \mathcal{F}(\Omega)$.*

The following iteration is elementary.

Proposition 9.3 ([12, Proposition 15.4 in Appendix])**.** *Let $\{a_k\}_{k=0}^\infty$ be a sequence of non-negative numbers such that*

$$a_k \leq D\lambda^k a_{k-1}^{1+\nu} \quad \text{for } k = 1, 2, \cdots$$

for some constants $D, \nu > 0$ and $\lambda \geq 1$. Then for any $k \geq 0$,

$$a_k \leq D^{-\frac{1}{\nu}} \left(D^{\frac{1}{\nu}} \lambda^{\frac{1+\nu}{\nu^2}} a_0 \right)^{(1+\nu)^k}.$$

The following was proved in [20, Lemma 2.12].

Lemma 9.4. *Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form in L^2. If*

$$f_n \xrightarrow{L^2} f, \quad \sup_n \mathcal{E}(f_n) < \infty,$$

then $f \in \mathcal{F}$, and there exists a subsequence, still denoted by $\{f_n\}$, such that $f_n \xrightarrow{\mathcal{E}} f$ weakly, that is,

$$\mathcal{E}(f_n, \varphi) \to \mathcal{E}(f, \varphi)$$

as $n \to \infty$ for any $\varphi \in \mathcal{F}$. And there exists a subsequence $\{f_{n_k}\}$ such that its Cesaro mean $\frac{1}{n}\sum_{k=1}^n f_{n_k}$ converges to f in $\mathcal{E}_1$-norm. Moreover, we have

$$\mathcal{E}(f) \leq \liminf_{n \to \infty} \mathcal{E}(f_n).$$

The notion of the μ-regular $\mathcal{E}$*-nest* $\{F_k\}$ is given in Section 2.

Proposition 9.5 ([12, Proposition 15.3 in Appendix])**.** *Let $\{F_k\}$ be a μ-regular $\mathcal{E}$-nest and $u \in C(\{F_k\})$. Then for any open set $U \subset M$*

$$\sup_{U \cap F} u = \operatorname{esup}_U u \quad \text{where } F := \bigcup_{k \geq 1} F_k.$$

Proposition 9.6 ([14, Proposition 10.6 in Appendix]). *Let $B_2 \subset B_1$ be two metric balls such that $B_1 \setminus B_2 \neq \emptyset$. Then for any quasi-continuous $v \in \mathcal{F}$,*

$$\int_{B_1\setminus B_2} v(y)J(x,dy) \leq \left(\operatorname*{esup}_{B_1} v\right)\int_{B_2^c} J(x,dy) \quad \textit{for } \text{q.e. } x \in B_2.$$

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