Finite propagation speed for Leibenson's equation on Riemannian manifolds

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Abstract

We consider on arbitrary Riemannian manifolds the Leibenson equation

$$\partial_t u = \Delta_p u^q.$$

This equation comes from hydrodynamics where it describes filtration of a turbulent compressible liquid in porous medium. We prove that that, under optimal restrictions on p and q, weak subsolutions to this equation have finite propagation speed.

Contents

1 Introduction

We are concerned here with a non-linear evolution equation

$$\partial_t u = \Delta_p u^q \tag{1.1}$$

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where $p>1$, $q>0$, $u=u(x,t)$ is an unknown non-negative function and Δ_p is the p-Laplacian

$$\Delta_p v = \operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right).$$

Equation (1.1) was introduced by L. S. Leibenson [27, 28] in order to describe filtration of turbulent compressible fluid through a porous medium. The physical meaning of u is the *volumetric moisture content,* i.e. the (infinitesimal) fraction of volume of the medium taken by the liquid. Parameter p characterizes the turbulence of a flow while $q-1$ is the index of *polytropy* of the liquid, which determines the relation $PV^{q-1}=const$ between volume V and pressure P. The equation (1.1) is frequently referred to as a *doubly non-linear parabolic equation.*

The physically interesting values of the parameters p and q are as follows: $\frac{3}{2}\le p\le 2$ and $q\ge 1$. The case $p=2$ corresponds to laminar flow (=absence of turbulence). In this case (1.1) becomes a *porous medium equation* $\partial_t u=\Delta u^q$, if $q>1$, and the classical heat equation $\partial_t u=\Delta u$ if $q=1$.

However, from the mathematical point of view, the entire range $p>1, q>0$ is interesting. For this range, G. I. Barenblatt [6] constructed spherically symmetric self-similar solutions of (1.1) in $\mathbb{R}^n$, that are nowadays called *Barenblatt solutions.*

Assume first that $q(p-1)>1$. Then the Barenblatt solution is given by

$$u(x,t)=\frac{1}{t^{n/\beta}}\left(C-\kappa\left(\frac{|x|}{t^{1/\beta}}\right)\right)_+^{\gamma}, \tag{1.2}$$

where $C>0$ is any constant, and

$$\beta=p+n[q(p-1)-1],\quad \gamma=\frac{p-1}{q(p-1)-1},\quad \kappa=\frac{q(p-1)-1}{pq}\beta^{-\frac{1}{p-1}}. \tag{1.3}$$

The parameter β determines the space/time scaling and is analogous to the notion of a *walk dimension*, known for diffusions on fractals.

Clearly, for the Barenblatt solution (1.2), we have

$$u(x,t)=0 \quad \text{whenever} \quad |x|>ct^{1/\beta},$$

where c is a large enough constant; thus, $u(\cdot,t)$ has a bounded support for any $t>0$. One says in this case that u has a *finite propagation speed.*

Assume now that $q(p-1)<1$. In this case $\gamma,\kappa<0$, and the Barenblatt solution is given by a similar formula

$$u(x,t)=\frac{1}{t^{n/\beta}}\left(C+|\kappa|\left(\frac{|x|}{t^{1/\beta}}\right)^{\frac{p}{p-1}}\right)^{\gamma}.$$

In the borderline case $q(p-1)=1$, the Barenblatt solution is given by

$$u(x,t)=\frac{1}{t^{n/p}}\exp\left(-\zeta\left(\frac{|x|}{t^{1/p}}\right)^{\frac{p}{p-1}}\right),$$

where $\zeta=(p-1)^2p^{-\frac{p}{p-1}}$. Hence, if $q(p-1)\le 1$, then $u(x,t)>0$ for all $x\in\mathbb{R}^n$ and $t>0$, that is, u has an *infinite propagation speed.*

In the present paper, we prove the finite propagation speed for solutions of the Leibenson equation (1.1) on arbitrary Riemannian manifolds, under the optimal assumption

$$q(p-1)>1. \tag{1.4}$$

We understand solutions in a certain weak sense (see Section 2 for the definition). It is worth mentioning that the existence results for weak solutions of (1.1) were obtained in various settings in [5, 26, 30, 33] and in [4, 8, 9, 37].

The main result of the present paper (Theorem 5.1) is as follows. Let M be an arbitrary Riemannian manifold. Assume that (1.4) is satisfied and let u be a bounded non-negative solution to (1.1) in $M \times \mathbb{R}_+$ with an initial function $u_0 = u(\cdot, 0)$. If u_0 vanishes in a precompact geodesic ball B_0 of radius R then

$$u = 0 \quad \text{in } \frac{1}{2} B_0 \times [0, t_0],$$

where

$$t_0 = \eta R^p ||u_0||_{L^\infty(M)}^{-[q(p-1)-1]},$$

and $\eta > 0$ depends on the intrinsic geometry of B_0. Hence, the solution u has a finite propagation speed inside B_0, and the speed of propagation is determined by the geometry of B_0 via the constant η.

Next, assume that $K = \operatorname{supp} u_0$ is compact. Then there exists an increasing continuous function $r : (0, T) \to \mathbb{R}_+$ for some $T \in (0, \infty]$ such that

$$\operatorname{supp} u(\cdot, t) \subset K_{r(t)} \quad \text{for all } t \in (0, T)$$

(Corollary 5.2). The function $r(t)$ is called the *propagation rate* of u. Hence, u has a finite propagation speed up to time T.

Let us emphasize that these results are valid for an arbitrary Riemannian manifold, and the property of finite propagation speed depends on the *local* structure of the manifold. However, in order to obtain a more detailed quantitative information about the propagation rate $r(t)$, one has to impose some restrictions on the global geometry of M. Indeed, we prove that if M is geodesically complete and if the Ricci curvature of M is non-negative then one can take $r(t) = Ct^{1/p}$ for all $t \in (0, \infty)$ (Corollary 5.3). In particular, in this case the solution has a finite propagation speed for all $t \in (0, \infty)$

Let us recall some previous results about finite propagation speed of solutions of (1.1). Consider first the special case $q = 1$ when (1.1) becomes the parabolic p-Laplace equation

$$\partial_t u = \Delta_p u. \tag{1.5}$$

In this case the condition (1.4) amounts to $p > 2$. The aforementioned results of Theorem 5.1 and Corollaries 5.2, 5.3 were proved for the equation (1.5) by S. Dekkers [13]. In fact, the finite propagation speed was deduced in [13] from a certain non-linear version of the mean value inequality for solutions. We have borrowed this approach from [13], although the proof of the crucial mean value inequality in our case is carried out in an entirely different way. Related results from the theory of the p-Laplace equation can be found, for instance, in [14, 15, 16, 23, 24].

Consider now another special case $p = 2$ when (1.1) becomes the porous medium equation

$$\partial_t u = \Delta u^q. \tag{1.6}$$

The condition (1.4) amounts in this case to $q > 1$. A finite propagation speed for solutions of (1.6) in hyperbolic spaces was proved by Vazquez [39], and in Cartan-Hadamard manifolds by Grillo and Muratori [20]. Some related qualitative properties of solutions of (1.6) were proved in [11] in the setting of compact Riemannian manifolds, in [3, 7, 11] for solutions in $\mathbb{R}^n$, and in [17, 38] for solutions in bounded domains in $\mathbb{R}^n$ with Dirichlet boundary condition.

In the general case, when $p > 1$ and $q > 0$ satisfy (1.4), a finite propagation speed for solutions of (1.1) was proved by Andreucci and Tadeev [2], under the hypothesis that the underlying manifold M satisfies a certain isoperimetric inequality; for example, the latter is the case when M is a Cartan-Hadamard manifold. However, the hypothesis about isoperimetric inequality fails on general manifolds of non-negative Ricci curvature that are covered by our Corollary 5.3.

See also [31, 34, 36] for other results about the asymptotic behaviour of solutions of (1.1).

The structure of the paper is as follows. In Section 2, we define the notion of a weak solution of the Leibenson equation (1.1) and introduce the time mollification, which is then used to prove a *Caccioppoli type inequality* for weak subsolutions (Lemma 2.6). This inequality is one of the ingredients of the proof of the central technical result of this paper $-$ the *mean value inequality* for subsolution that is proved in Section 4 (Lemma 4.3). Another ingredient for the proof of the mean value inequality is introduced in Section 3 (Lemma 3.1)

Using Lemma 4.3, we prove in Section 5 our aforementioned results about finite propagation speed.

Let us make some comments on the mean value inequality of the key Lemma 4.3. It says the following. Let $q(p-1) \geq 1$ and let u be a non-negative bounded subsolution of (1.1) in a cylinder

$$Q = B \times [0, t]$$

where B is a precompact geodesic ball in M. Assume that $u(\cdot, 0) = 0$ in B. Then, for the cylinder

$$Q' = \frac{1}{2}B \times [0, t]$$

and for any large enough constant $\sigma > 0$, we have

$$\|u\|_{L^\infty(Q')} \leq \left(\frac{CS_B}{R^{p(1+\nu)}}\right)^{\frac{1}{\sigma\nu}} \|u\|_{L^\infty(Q)}^{\frac{q(p-1)-1}{\sigma}} \|u\|_{L^\sigma(Q)},$$

where $C = C(p, q, \nu, \sigma)$. Here S_B and ν are positive constants that depend on the intrinsic geometry of the ball B, namely, on the Sobolev inequality in B (see Section 3).

Although the proof of Lemma 4.3 follows the classical Moser iteration argument [32], it has certain peculiarities due to the non-linearity of the equation, which is worth mentioning here. We consider a shrinking sequence of cylinders $\{Q_k\}_{k=0}^\infty$ interpolating between $Q_0 = Q$ and $Q_\infty = Q'$, and first prove that

$$\int_{Q_{k+1}} u^{\sigma(1+\nu)} \leq C(\cdots) \left(\int_{Q_k} u^\sigma\right)^{1+\nu}, \tag{1.7}$$

for some $\sigma > 1$ and $\nu > 0$, where ν come from the Sobolev inequality in B and "$\cdots$" stands for some terms that are unimportant for the present discussion (see Corollary 4.2 for details). In the classical Moser argument, one proves (1.7) first for $\sigma = 2$ and then applies this inequality also to $u^{\sigma/2}$ with any $\sigma > 2$ because $u^{\sigma/2}$ is also a subsolution. This allows to set in (1.7) $\sigma = 2(1+\nu)^k$, reiterate (1.7) and to reach in the limit $\|u\|_{L^\infty(Q')}$ as $k \to \infty$.

However, in our case this trick does not work as the powers of a subsolution are *not* necessarily subsolutions. Hence, we need to prove (1.7) directly for *any* σ and to compute carefully the constant $C = C(\sigma)$ in (1.7). It turns out that $C \simeq \sigma^{(2-p)\nu}$ and, surprisingly enough, this power growth of C with σ still allows to complete the iteration argument and to obtain (1.7).

Note also that the proof of a similar mean value inequality in [13] for subsolutions of (1.5) was carried out in an entirely different way by using instead of the powers of u the functions

$(u-a)_+$ that are subsolutions of (1.5) for any $a>0$. However, this approach does not work for the general equation (1.1) because $(u-a)_+$ is not a subsolution in this case.

For mean value inequalities in various settings see also [1, 19, 21].

2 Weak subsolutions

2.1 Definition and basic properties

We consider in what follows the following evolution equation on a Riemannian manifold M:

$$\partial_t u = \Delta_p u^q. \tag{2.8}$$

By a *subsolution* of (2.8) we mean a non-negative function v satisfying

$$\partial_t u \leq \Delta_p u^q \tag{2.9}$$

in a certain weak sense as explained below.

We assume throughout that

$$p>1 \quad \text{and} \quad q>0.$$

Set

$$\delta = (p-1)q-1.$$

Later we will assume that $\delta>0$.

Let μ denote the *Riemannian measure* on M. For simplicity of notation, we omit in almost all integrations the notation of measure. All integration in M is done with respect to $d\mu$, and in $M\times\mathbb{R}$ – with respect to $d\mu dt$, unless otherwise specified.

Definition 2.1. Let Ω be an open subset of M and set $\Omega_T=\Omega\times[0,T)$, $T>0$. Then we call a non-negative function $u=u(x,t)$ a *weak subsolution* of (2.8) in Ω_T, if

$$u\in\mathcal{S}_{p,q}(\Omega_T) = C\left([0,T);L^2(\Omega)\right)\cap\left\{u^q\in L^p_{loc}\left([0,T);W^{1,p}(\Omega)\right)\right\} \tag{2.10}$$

and (2.9) holds weakly in Ω_T, which means that for all $0\leq t_1<t_2<T$, and all non-negative functions

$$\psi\in\mathcal{T}_{p,q}(\Omega_T) = W^{1,2}_{loc}\left([0,T);L^2(\Omega)\right)\cap L^p_{loc}\left([0,T);W^{1,p}_0(\Omega)\right), \tag{2.11}$$

we have

$$\left[\int_\Omega u\psi\right]_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_\Omega -u\partial_t\psi + |\nabla u^q|^{p-2}\langle\nabla u^q,\nabla\psi\rangle \leq 0. \tag{2.12}$$

Weak supersolutions and *weak solutions* of (2.8) are defined analogously. Note that the notion of weak solutions is standard (see [15, 22]).

If $u\in\mathcal{S}_{p,q}(\Omega_T)$, we define

$$\nabla u := \begin{cases} q^{-1}u^{1-q}\nabla(u^q), & u>0,\\ 0, & u=0.\end{cases}$$

Remark 2.2. Note that it follows from (2.10) and (2.11) that the integrals in (2.12) are finite. Indeed, we have by Hölder's inequality

$$\begin{aligned}\int_{t_1}^{t_2}\int_\Omega |\nabla u^q|^{p-2}\langle\nabla u^q,\nabla\psi\rangle &\leq \int_{t_1}^{t_2}\int_\Omega |\nabla u^q|^{p-1}|\nabla\psi| \\ &\leq \left(\int_{t_1}^{t_2}\int_\Omega (|\nabla u^q|)^p\right)^{\frac{p-1}{p}}\left(\int_{t_1}^{t_2}\int_\Omega |\nabla\psi|^p\right)^{\frac{1}{p}}.\end{aligned}$$

Definition 2.3. Let $u = u(x,t)$ be a measurable function in Ω_T and $u(\cdot,0) = u_0$. Then we define, for $h \in (0,T)$,

$$u^h(\cdot,t) = \frac{1}{h}\int_0^t e^{(s-t)/h}u(\cdot,s)ds$$

and

$$u_h(\cdot,t) = e^{-t/h}u_0 + \frac{1}{h}\int_0^t e^{(s-t)/h}u(\cdot,s)ds.$$

The properties of u^h and u_h in the following Lemma are proved in Lemma 2.2 in [25] and in Lemma B.1 and Lemma B.2 in [10].

Lemma 2.4. *Let $p \geq 1$ and suppose that $u \in L^p(\Omega_T)$. Then*

$$||u^h||_{L^p(\Omega_T)} \leq ||u||_{L^p(\Omega_T)}$$

and

$$||u_h||_{L^p(\Omega_T)} \leq ||u||_{L^p(\Omega_T)} + h^{1/p}||u_0||_{L^p(\Omega)},$$

Moreover, $u^h \to u$ and $u_h \to u$ in $L^p(\Omega_T)$ as $h \to 0$ and

$$\partial_t u_h = \frac{1}{h}(u - u_h) \in L^p(\Omega_T). \tag{2.13}$$

Lemma 2.5. *Let $u = u(x,t)$ be a weak subsolution of (2.8) in Ω_T. Then, for all $\psi \in \mathcal{T}_{p,q}(\Omega_T)$ and $\tau \in (0,T)$,*

$$\int_0^\tau\int_\Omega (\partial_t u_h)\psi + \langle[|\nabla u^q|^{p-2}\nabla u^q]^h, \nabla\psi\rangle \leq 0. \tag{2.14}$$

Proof. Fix some $s \in (0,\tau)$. By (2.12) with $t_1 = 0$, $t_2 = \tau - s$ and $\psi = \psi(x, t+s)$, we have

$$\left[\int_\Omega u(x,t)\psi(x,t+s)\right]_0^{\tau-s} + \int_0^{\tau-s}\int_\Omega -u\psi_t + |\nabla u^q|^{p-2}\langle\nabla u^q, \nabla\psi\rangle \leq 0.$$

Multiplying both sides by $h^{-1}e^{-s/h}$ and integrating over $[0,\tau]$ with respect to s, we get

$$\begin{aligned}&\frac{1}{h}\int_0^\tau\int_\Omega e^{-s/h}u(x,\tau-s)\psi(x,\tau)d\mu ds - \frac{1}{h}\int_0^\tau\int_\Omega e^{-s/h}u_0(x)\psi(x,s)d\mu ds\\ &+ \frac{1}{h}\int_0^\tau\int_s^\tau\int_\Omega e^{-s/h}(-u(x,t-s)\psi_t + |\nabla u(x,t-s)^q|^{p-2}\langle\nabla u(x,t-s)^q, \nabla\psi\rangle) \leq 0.\end{aligned}$$

Noticing that

$$\frac{1}{h}\int_0^\tau e^{-s/h}u(\cdot,\tau-s)ds = u^h(\cdot,\tau)$$

and

$$\frac{1}{h}\int_0^\tau\int_s^\tau e^{-s/h}u(\cdot,t-s)dtds = \int_0^\tau u^h(\cdot,t)dt,$$

we deduce

$$\begin{aligned}&\int_\Omega u_h(x,\tau)\psi(x,\tau) - \int_\Omega e^{-\tau/h}u_0(x)\psi(x,\tau) - \int_\Omega u_0(x)\left(\frac{1}{h}\int_0^\tau e^{-s/h}\psi(x,s)ds\right)\\ &\qquad + \int_0^\tau\int_\Omega e^{-t/h}u_0\partial_t\psi + \int_0^\tau\int_\Omega -u_h\partial_t\psi + \langle[|\nabla u^q|^{p-2}\nabla u^q]^h, \nabla\psi\rangle \leq 0.\end{aligned}$$

By partial integration and using $u_h(\cdot,0)=u_0$, we have

$$\int_\Omega u_h(x,\tau)\psi(x,\tau)-\int_0^\tau\int_\Omega u_h\partial_t\psi=\int_\Omega u_0(x)\psi(x,0)+\int_0^\tau\int_\Omega(\partial_t u_h)\psi$$

and

$$\begin{aligned}\int_0^\tau\int_\Omega e^{-t/h}u_0\partial_t\psi&=\left[\int_\Omega e^{-t/h}u_0(x)\psi(x,t)\right]_0^\tau+\frac{1}{h}\int_0^\tau\int_\Omega e^{-t/h}u_0(x)\psi(x,t)\\&=\int_\Omega e^{-\tau/h}u_0(x)\psi(x,\tau)-\int_\Omega u_0(x)\psi(x,0)+\int_\Omega u_0(x)\left(\frac{1}{h}\int_0^\tau e^{-t/h}\psi(x,t)dt\right),\end{aligned}$$

which implies (2.14). ∎

2.2 Caccioppoli type inequality

Let Ω be an open subset of M and $T>0$.

Lemma 2.6. *Let $v=v(x,t)$ be a bounded non-negative subsolution to (2.8) in a cylinder Ω_T. Let $\eta(x,t)$ be a locally Lipschitz non-negative bounded function in Ω_T such that $\eta(\cdot,t)$ has compact support in Ω for all $t\in[0,T)$. Fix some real λ such that*

$$\lambda\geq\max(2,1+q)\tag{2.1}$$

and set

$$\sigma=\lambda+\delta\qquad\text{and}\qquad\alpha=\frac{\sigma}{p}.\tag{2.2}$$

Choose $0\leq t_1<t_2<T$ and set $Q=\Omega\times[t_1,t_2]$. Then

$$\left[\int_\Omega v^\lambda\eta^p\right]_{t_1}^{t_2}+c_1\int_Q|\nabla(v^\alpha\eta)|^p\leq\int_Q\left[pv^\lambda\eta^{p-1}\partial_t\eta+c_2v^\sigma|\nabla\eta|^p\right],\tag{2.3}$$

where c_1,c_2 are positive constants depending on p, q, λ.

In particular, if η does not depend on t, then

$$\left[\int_\Omega v^\lambda\eta^p\right]_{t_1}^{t_2}+c_1\int_Q|\nabla(v^\alpha\eta)|^p\leq c_2\int_Q v^\sigma|\nabla\eta|^p.\tag{2.4}$$

Proof. Consider the function $\Phi_\alpha(u)=u^{\frac{\alpha}{q}}$. It follows from $\lambda\geq1+q$, that $\frac{\alpha}{q}\geq1$, whence Φ_α is a Lipschitz function on $[0,\sup v]$ and we obtain that $v^\alpha(\cdot,t)=\Phi_\alpha(v^q)(\cdot,t)\in W^{1,p}(\Omega)$ for all $t\in(0,T)$. Also, note that $\sigma\geq1+q+(p-1)q-1=pq>0$, so that all integrals in (2.3) are well-defined. Since v is a weak subsolution of (2.8), we obtain by (2.14),

$$\int_0^\tau\int_\Omega(\partial_t v_h)\psi+\langle[|\nabla v^q|^{p-2}\nabla v^q]^h,\nabla\psi\rangle\leq0,\tag{2.5}$$

for all $h\in(0,T)$, $\tau\in(0,T)$ and $\psi\in\mathcal{T}_{p,q}(\Omega_T)$.

Claim:

$$\left[\int_\Omega v^\lambda\eta^p\right]_{t_1}^{t_2}\leq\int_Q-\lambda\langle|\nabla v^q|^{p-2}\nabla v^q,\nabla(v^{\lambda-1}\eta^p)\rangle+pv^\lambda\eta^{p-1}\partial_t\eta.\tag{2.6}$$

Let us consider, for $\nu < \frac{1}{4}(t_2 - t_1)$, the function

$$\theta_\nu(t) = \begin{cases} 0, & t < t_1, \\ \frac{1}{\nu}(t - t_1), & t_1 \le t < t_1 + \nu, \\ 1, & t_1 + \nu \le t < t_2 - \nu, \\ \frac{1}{\nu}(t_2 - t), & t_2 - \nu \le t < t_2, \\ 0, & t \ge t_2 \end{cases}$$

(cf. [29]). We want to show that, for all $t \in [0, T)$,

$$v^{\lambda-1}(\cdot, t)\eta^p(\cdot, t)\theta_\nu(t) \in L^2(\Omega) \cap W_0^{1,p}(\Omega), \tag{2.7}$$

which will make this function admissible as a test function in (2.5).

Using the function $\Phi_{\lambda-1}(u) = u^{\frac{\lambda-1}{q}}$, $\lambda \ge 1 + q$ and the same argumentation as above, we obtain that $v^{\lambda-1} \in W^{1,p}(\Omega)$ and

$$\nabla(v^{\lambda-1}) = \Phi'_{\lambda-1}(v^q)\nabla(v^q) = (\lambda - 1)q^{-1}v^{\lambda-(q+1)}\nabla(v^q) = (\lambda - 1)v^{\lambda-2}\nabla v.$$

Similarly, by $\lambda \ge 2$, we also get $v^{\lambda-1} \in L^2(\Omega)$ so that (2.7) holds. Hence, using this test function in (2.5),

$$\int_Q \partial_t v_h v^{\lambda-1}\eta^p\theta_\nu + \langle[|\nabla v^q|^{p-2}\nabla v^q]^h, \nabla(v^{\lambda-1}\eta^p)\rangle\theta_\nu \le 0.$$

Let us write

$$\int_Q \partial_t v_h v^{\lambda-1}\eta^p\theta_\nu = \int_Q \partial_t v_h v_h^{\lambda-1}\eta^p\theta_\nu + \int_Q \partial_t v_h(v^{\lambda-1} - v_h^{\lambda-1})\eta^p\theta_\nu.$$

By (2.13), we see that

$$\int_Q \partial_t v_h(v^{\lambda-1} - v_h^{\lambda-1})\eta^p\theta_\nu = \frac{1}{h}\int_Q (v - v_h)(v^{\lambda-1} - v_h^{\lambda-1})\eta^p\theta_\nu \ge 0,$$

whence we obtain

$$\int_Q \partial_t v_h v_h^{\lambda-1}\eta^p\theta_\nu + \langle[|\nabla v^q|^{p-2}\nabla v^q]_h, \nabla(v^{\lambda-1}\eta^p)\rangle\theta_\nu \le 0. \tag{2.8}$$

By using

$$\lambda\int_Q \partial_t v_h v_h^{\lambda-1}\eta^p\theta_\nu = \int_Q \partial_t v_h^\lambda\eta^p\theta_\nu = \left[\int_\Omega v_h^\lambda\eta^p\theta_\nu\right]_{t_1}^{t_2} - p\int_Q v_h^\lambda\eta^{p-1}\partial_t\eta\theta_\nu - \int_Q v_h^\lambda\eta^p\partial_t\theta_\nu,$$

we get, since $\theta_\nu(t_1) = \theta_\nu(t_2) = 0$,

$$-\int_Q v_h^\lambda\eta^p\partial_t\theta_\nu \le \int_Q -\lambda\langle[|\nabla v^q|^{p-2}\nabla v^q]_h, \nabla(v^{\lambda-1}\eta^p)\rangle\theta_\nu + pv_h^\lambda\eta^{p-1}\partial_t\eta\theta_\nu. \tag{2.9}$$

We now want to let $h \to 0$ in (2.9) and apply Lemma 2.4 and then let $\nu \to 0$ to obtain (2.6). Note that $|\nabla v^q|^{p-1} \in L^{\frac{p}{p-1}}(Q)$, so that by Lemma 2.4, for $h \to 0$,

$$[|\nabla v^q|^{p-2}\nabla v^q]^h \to |\nabla v^q|^{p-2}\nabla v^q \quad \textit{in } L^{\frac{p}{p-1}}(Q).$$

Together with $|\nabla(v^{\lambda-1}\eta^p)|\theta_\nu \in L^p(Q)$, we obtain

$$\lim_{h\to 0}\int_Q -\lambda\langle[|\nabla v^q|^{p-2}\nabla v^q]^h, \nabla(v^{\lambda-1}\eta^p)\rangle\theta_\nu = \int_Q -\lambda\langle|\nabla v^q|^{p-2}\nabla v^q, \nabla(v^{\lambda-1}\eta^p)\rangle\theta_\nu.$$

For the convergence of the remaining terms in (2.9), we will use the boundedness of v. Note that by assumption $v \in L^2(Q)$ whence Lemma 2.4 implies that $v_h \to v$ in $L^2(Q)$. Since the function $u \mapsto u^\lambda$ is Lipschitz on $[0, \sup v]$ we get $v_h^\lambda \to v^\lambda$ in $L^2(Q)$ and thus,

$$\lim_{h\to 0} \int_Q p v_h^\lambda \eta^{p-1} \partial_t \eta \theta_\nu = \int_Q p v^\lambda \eta^{p-1} \partial_t \eta \theta_\nu.$$

The convergence

$$\lim_{h\to 0} \int_Q v_h^\lambda \eta^p \partial_t \theta_\nu = \int_Q v^\lambda \eta^p \partial_t \theta_\nu$$

follows by the same arguments. Hence,

$$-\int_Q v^\lambda \eta^p \partial_t \theta_\nu \le \int_Q -\lambda \langle [|\nabla v^q|^{p-2} \nabla v^q], \nabla (v^{\lambda-1}\eta^p)\rangle \theta_\nu + p v^\lambda \eta^{p-1} \partial_t \eta \theta_\nu.$$

Sending now $\nu \to 0$, we deduce (2.6).

We have

$$\nabla (v^{\lambda-1}\eta^p) = (\lambda - 1)\eta^p v^{\lambda-2} \nabla v + p\eta^{p-1} v^{\lambda-1} \nabla \eta. \tag{2.10}$$

Therefore, by (2.6) and (2.10), we obtain

$$\begin{aligned} \left[\int_\Omega v^\lambda \eta^p\right]_{t_1}^{t_2} &\le \int_Q -\lambda(\lambda-1) v^{\lambda-2+(q-1)(p-1)} \eta^p |\nabla v|^p + \lambda p v^{\lambda-1+(q-1)(p-1)} |\nabla v|^{p-1} |\nabla \eta| \eta^{p-1} \\ &\quad + \int_Q p v^\lambda \eta^{p-1} \partial_t \eta \\ &= \int_Q -\lambda(\lambda-1) v^{p(\alpha-1)} \eta^p |\nabla v|^p + \lambda p v^{p(\alpha-1)+1} |\nabla v|^{p-1} |\nabla \eta| \eta^{p-1} + p v^\lambda \eta^{p-1} \partial_t \eta. \end{aligned} \tag{2.11}$$

Then by Young's inequality we have, for all $\varepsilon > 0$,

$$\begin{aligned} v^{p(\alpha-1)+1} |\nabla v|^{p-1} |\nabla \eta| \eta^{p-1} &= \left(v^{p(\alpha-1)\frac{p-1}{p}} |\nabla v|^{p-1} \eta^{p-1}\right) (v^\alpha |\nabla \eta|) \\ &\le \varepsilon^{p'} v^{p(\alpha-1)} |\nabla v|^p \eta^p + \frac{1}{\varepsilon^p} v^{\alpha p} |\nabla \eta|^p, \end{aligned} \tag{2.12}$$

where $p' = \frac{p}{p-1}$. Combining this with (2.11), we deduce

$$\left[\int_\Omega v^\lambda \eta^p\right]_{t_1}^{t_2} \le \int_Q -\lambda(\lambda - 1 - p\varepsilon^{p'}) v^{p(\alpha-1)} |\nabla v|^p \eta^p + \frac{p}{\varepsilon^p} v^{\alpha p} |\nabla \eta|^p + p v^\lambda \eta^{p-1} \partial_t \eta.$$

Also,

$$|\nabla (v^\alpha \eta)|^p = \left|\alpha v^{\alpha-1} \eta \nabla v + v^\alpha \nabla \eta\right|^p \le 2^{p-1} \alpha^p |\nabla v|^p v^{p(\alpha-1)} \eta^p + 2^{p-1} v^{\alpha p} |\nabla \eta|^p,$$

which implies that

$$|\nabla v|^p v^{p(\alpha-1)} \eta^p \ge 2^{1-p} \alpha^{-p} |\nabla (v^\alpha \eta)|^p - \alpha^{-p} v^{\alpha p} |\nabla \eta|^p.$$

Therefore,

$$\left[\int_\Omega v^\lambda \eta^p\right]_{t_1}^{t_2} \le \int_Q -\lambda(\lambda - 1 - p\varepsilon^{p'}) 2^{1-p} \alpha^{-p} |\nabla (v^\alpha \eta)|^p$$

$$+\int_Q \lambda\left(\left(\lambda-1-p\varepsilon^{p'}\right)\alpha^{-p}+\frac{p}{\varepsilon^p}\right)v^{\alpha p}|\nabla\eta|^p+pv^\lambda\eta^{p-1}\partial_t\eta$$
$$=-c_1\int_Q|\nabla\left(v^\alpha\eta\right)|^p+c_2\int_Q v^{\alpha p}|\nabla\eta|^p+\int_Q pv^\lambda\eta^{p-1}\partial_t\eta,$$

where

$$c_1=\lambda\left(\lambda-1-p\varepsilon^{p'}\right)2^{1-p}\alpha^{-p}$$

and

$$c_2=\lambda\left(\left(\lambda-1-p\varepsilon^{p'}\right)\alpha^{-p}+\frac{p}{\varepsilon^p}\right).$$

Hence, choosing ε small enough so that $c_1>0$, that is

$$p\varepsilon^{p'}<\lambda-1,$$

we obtain (2.3). Finally, let us specify c_1 and c_2. Let us choose ε so that

$$p\varepsilon^{p'}=\frac{1}{2}\left(\lambda-1\right),$$

that is

$$c_1=\lambda\left(\lambda-1\right)2^{-p}\alpha^{-p}. \tag{2.13}$$

It follows that

$$\begin{aligned} c_2 &= \frac{1}{2}\lambda\left(\lambda-1\right)\alpha^{-p}+\lambda\frac{p}{\varepsilon^p} \\ &= \frac{1}{2}\lambda\left(\lambda-1\right)\alpha^{-p}+\frac{p}{\left(\frac{1}{2}\left(\lambda-1\right)/p\right)^{p/p'}} \\ &= \frac{1}{2}\lambda\left(\lambda-1\right)\alpha^{-p}+\lambda\frac{2^{p/p'}p^{1+p/p'}}{\left(\lambda-1\right)^{p/p'}}. \end{aligned}$$

Since

$$\frac{p}{p'}+1=\frac{p}{p/(p-1)}+1=p$$

we have

$$c_2=\frac{1}{2}\lambda\left(\lambda-1\right)\alpha^{-p}+\frac{\lambda 2^{p-1}p^p}{\left(\lambda-1\right)^{p-1}}. \tag{2.14}$$

which finishes the proof. ∎

Remark 2.7. For the future we need the ratio $\frac{c_2}{c_1}$. It follows from (2.13) and (2.14) that

$$\begin{aligned} \frac{c_2}{c_1} &= 2^{p-1}+\lambda\frac{2^{p-1}p^p}{\left(\lambda-1\right)^{p-1}\lambda\left(\lambda-1\right)2^{-p}\alpha^{-p}} \\ &= 2^{p-1}+\frac{2^{2p-1}\sigma^p}{\left(\lambda-1\right)^p}, \end{aligned}$$

where we have used that $\alpha p=\sigma$. Since $\sigma=\lambda+\delta$, we obtain

$$\frac{c_2}{c_1}=2^{p-1}+\frac{2^{2p-1}\left(\lambda+\delta\right)^p}{\left(\lambda-1\right)^p}.$$

It follows that, for all $\lambda\geq 2$,

$$\frac{c_2}{c_1}\leq C_{p,\delta},$$

where $C_{p,\delta}$ depend only on p and δ and does not depend on λ.

Remark 2.8. Let us obtain an upper bound of c_2. Using

$$\alpha = \frac{\sigma}{p} = \frac{\lambda + \delta}{p}$$

we obtain

$$c_2 = \frac{1}{2}\frac{\lambda(\lambda-1)}{(\lambda+\delta)^p}p^p + \frac{\lambda 2^{p-1}p^p}{(\lambda-1)^{p-1}}.$$

As $\lambda \geq 2$ and $\lambda + \delta \geq p > 1$, it follows that

$$c_2 \leq C_{p,\delta}\lambda^{2-p}. \tag{2.15}$$

Of course, if $p \geq 2$ then c_2 is uniformly bounded by a constant $C_{p,\delta}$ independently of λ, but if $p < 2$ then c_2 may grow with λ as in (2.15).

Lemma 2.9. *Let $v = v(x,t)$ be a bounded non-negative subsolution to (2.8) in M_T, and assume that M is complete. Then, for any $\gamma \geq \max(2, 1+q)$, the function*

$$t \mapsto \|v(\cdot,t)\|_{L^\gamma(M)}$$

is monotone decreasing.

Proof. Since M is complete, we have $W_0^{1,p}(M) = W^{1,p}(M)$, so that (2.7) holds true with $\eta \equiv 1$ and $\Omega = M$, that is, we can take $v^{\gamma-1}\theta_\nu$ as the test function in the proof of Lemma 2.6 when $\Omega = M$. Proceeding as in the proof of this Lemma, we obtain from (2.3), for any $0 \leq t_1 < t_2 < T$,

$$\left[\int_M v^\gamma\right]_{t_1}^{t_2} \leq -c_1 \int_M |\nabla(v^\alpha)|^p,$$

for some positive constant c_1. Therefore,

$$\|v(\cdot,t_2)\|_{L^\gamma(M)} \leq \|v(\cdot,t_1)\|_{L^\gamma(M)},$$

what was to be shown. ■

3 Sobolev and Moser inequalities

Let M be a connected Riemannian manifold of dimension n. Let d be the geodesic distance on M. For any $x \in M$ and $r > 0$, denote by $B(x,r)$ the geodesic ball of radius r centered at x, that is,

$$B(x,r) = \{y \in M : d(x,y) < r\}.$$

Let B be a precompact ball in M. The Sobolev inequality in B of order $p \geq 1$ says the following|: for any non-negative function $w \in W_0^{1,p}(B)$,

$$\left(\int_B w^{p\kappa}\right)^{1/\kappa} \leq S_B \int_B |\nabla w|^p, \tag{3.16}$$

where $\kappa > 1$ is some constant and S_B is called the *Sobolev constant* in B. The value of κ is independent of B and can be chosen as follows:

$$\kappa = \frac{n}{n-p} \quad \text{if } n > p,$$

and κ is an arbitrary real number > 1 if $n \leq p$.

We always assume that S_B is chosen to be minimal possible. In this case the function

$$B \mapsto S_B$$

is clearly monotone increasing with respect to inclusion of balls.

Dividing (3.16) by $\mu(B)$, we obtain

$$\left(\fint_B w^{p\kappa} \right)^{1/\kappa} \leq \mu(B)^{1/\kappa'} S_B \fint_B |\nabla w|^p \,, \tag{3.17}$$

where

$$\kappa' = \frac{\kappa}{\kappa - 1}$$

is the Hölder conjugate of κ.

Denoting by $r(B)$ the radius of B, let us define a new quantity

$$\iota(B) := \frac{1}{\mu(B)} \left(\frac{r(B)}{S_B} \right)^{\kappa'}$$

so that

$$S_B = \frac{r(B)^p}{(\iota(B)\mu(B))^{1/\kappa'}} \tag{3.18}$$

and

$$\left(\mu(B)^{1/\kappa'} S_B \right)^{1/p} = \frac{r(B)}{\iota(B)^{\frac{1}{p\kappa'}}}.$$

Hence, (3.17) can be rewritten in the form

$$\left(\fint_B |\nabla w|^p \right)^{1/p} \geq \frac{\iota(B)^{\frac{1}{p\kappa'}}}{r(B)} \left(\fint_B w^{p\kappa} \right)^{1/p\kappa}. \tag{3.19}$$

It is clear from (3.19) that the value of κ can be always reduced (by modifying the value of $\iota(B)$). It is only important that $\kappa > 1$. In fact, the exact value of κ does not affect the results, although various constants do depend on κ.

The constant $\iota(B)$ is called the *normalized Sobolev* constant in B. It is known that if M is complete and $Ricci_M \geq 0$ then, for all balls B, the normalized Sobolev constant $\iota(B)$ is bounded below by a positive constant (see [12], [18], [35]).

Let B be a precompact ball in M and $Q = B \times [0, T]$. Assume that the Sobolev inequality (3.19) holds in B with exponent $\kappa > 1$. Let κ' be its Hölder conjugate. Set

$$\nu = \frac{1}{\kappa'} = \frac{\kappa - 1}{\kappa}.$$

Lemma 3.1. *Let $w \in L^p\left([0,T]; W_0^{1,p}(B)\right)$ be a non-negative function. Then,*

$$\int_Q w^{p(1+\nu)} \leq S_B \left(\int_Q |\nabla w|^p \right) \sup_t \left(\int_B w^p \right)^{\nu}. \tag{3.20}$$

Proof. By the Hölder inequality, we have, for any $t \in [0, T]$

$$\begin{aligned} \int_B w^{p(1+\nu)} = \int_B w^p w^{p\nu} &\leq \left(\int_B w^{p\kappa} \right)^{1/\kappa} \left(\int_B w^{p\nu\kappa'} \right)^{1/\kappa'} \\ &= \left(\int_B w^{p\kappa} \right)^{1/\kappa} \left(\int_B w^p \right)^{\nu} \end{aligned}$$

$$\leq \left(\int_B w^{p\kappa}\right)^{1/\kappa} \sup_{t\in[0,T]} \left(\int_B w^p\right)^{\nu},$$

where we have used that $\nu\kappa' = 1$.

By the Sobolev inequality (3.16) we have

$$\left(\int_B w^{p\kappa}\right)^{1/\kappa} \leq S_B \int_B |\nabla w|^p .$$

It follows that

$$\int_B w^{p(1+\nu)} \leq S_B \left(\int_B |\nabla w|^p\right) \sup_t \left(\int_B w^p\right)^{\nu}.$$

Integrating this inequality in $t \in [0,T]$ gives (3.20). ■

4 Estimates of subsolutions

4.1 Comparison in two cylinders

Here we assume that

$$p > 1 \quad \text{and} \quad \delta := q(p-1) - 1 \geq 0.$$

Lemma 4.1. *Consider two balls $B = B(x,r)$ and $B' = B(x,r')$ with $0 < r' < r$, and two cylinders*

$$Q = B \times [0,T], \quad Q' = B' \times [0,T].$$

Assume that B is precompact. Let λ be any real such that

$$\lambda \geq \max(2, 1+q). \tag{4.21}$$

Set

$$\sigma = \lambda + \delta.$$

Let v be a non-negative bounded subsolution of (2.8) in $B \times [0,T')$ for some $T' > T$, such that

$$v(\cdot,0) = 0.$$

Then

$$\int_{Q'} v^{\sigma(1+\nu)} \leq \frac{C S_B \sigma^{(2-p)\nu}}{(r-r')^{p(1+\nu)}} \left(\int_Q v^\sigma\right)\left(\int_Q v^{\sigma+\delta}\right)^{\nu}, \tag{4.22}$$

where the constant C depends on p, δ and ν, but it is independent of σ.

Proof. As in Lemma 2.6, set $\alpha = \frac{\sigma}{p}$. Let η be a bump function of B' in B. Recalling the proof of Lemma 2.6, we see that $v^\alpha \eta \in L^p_{loc}\left([0,T'); W_0^{1,p}(B)\right)$. Applying (3.20) with

$$w = v^\alpha \eta$$

and using

$$w^p = v^\sigma \eta^p,$$

we obtain that, for any $t \in [0,T]$,

$$\int_Q v^{\sigma(1+\nu)} \eta^{p(1+\nu)} \leq S_B \left(\int_Q |\nabla(v^\alpha \eta)|^p\right) \sup_{t\in[0,T]} \left(\int_B v^\sigma \eta^p\right)^{\nu}.$$

By (2.4) we have

$$\int_Q |\nabla (v^\alpha \eta)|^p \le \frac{c_2}{c_1} \int_Q v^\sigma |\nabla \eta|^p$$

and

$$\sup_{t\in[0,T]} \left(\int_B v^\lambda \eta^p \right) \le c_2 \int_Q v^\sigma |\nabla \eta|^p .$$

Let us use the latter in the form

$$\sup_{t\in[0,T]} \left(\int_B v^{\lambda'} \eta^p \right) \le c_2' \int_Q v^{\sigma'} |\nabla \eta|^p ,$$

where

$$\lambda' = \sigma \quad \text{and} \quad \sigma' = \lambda' + \delta = \sigma + \delta.$$

Then we have

$$\sup_t \left(\int_B v^\sigma \eta^p \right) \le c_2' \int_Q v^{\sigma'} |\nabla \eta|^p .$$

It follows that

$$\int_Q v^{\sigma(1+\nu)} \eta^{p(1+\nu)} \le S_B \frac{c_2}{c_1} \int_Q v^\sigma |\nabla \eta|^p \left(c_2' \int_Q v^{\sigma'} |\nabla \eta|^p \right)^\nu .$$

Using that $\eta = 1$ in B' and $|\nabla \eta| \le \frac{1}{r-r'}$ we obtain

$$\int_{Q'} v^{\sigma(1+\nu)} \le S_B \frac{c_2}{c_1} \frac{(c_2')^\nu}{(r_1 - r_2)^{p(1+\nu)}} \left(\int_Q v^\sigma \right) \left(\int_Q v^{\sigma'} \right)^\nu .$$

By Remark 2.7 we have

$$\frac{c_2}{c_1} \le C_{p,\delta},$$

and, by the estimate (2.15) of Remark 2.8,

$$c_2' \le C_{p,\delta} \left(\lambda'\right)^{2-p} = C_{p,\delta} \sigma^{2-p}.$$

Hence, (4.22) follows. ■

Corollary 4.2. *Under the hypotheses of* Lemma 4.1, *we have*

$$\int_{Q'} v^{\sigma(1+\nu)} \le \frac{C S_B \sigma^{(2-p)\nu} \|v\|_{L^\infty(Q)}^{\delta\nu}}{(r_1 - r_2)^{p(1+\nu)}} \left(\int_Q v^\sigma \right)^{1+\nu} , \tag{4.23}$$

where $C = C(p, \delta, \nu)$.

4.2 Mean value inequality

We assume here that

$$p > 1 \quad \text{and} \quad \delta \ge 0.$$

Lemma 4.3. *Let the ball* $B = B(x_0, r)$ *be precompact and* $T > 0$. *Let* u *be a non-negative bounded subsolution of* (2.8) *in* $B \times [0, T)$ *such that*

$$u(\cdot, 0) = 0 \text{ in } B.$$

For $0<t<T$*, set*

$$Q=B\times[0,t]$$

and let $\sigma>0$ *be large enough. Then, for the cylinder*

$$Q'=\frac{1}{2}B\times[0,t]\,,$$

(see Fig. 1) *we have*

$$\|u\|_{L^\infty(Q')}\leq\left(\frac{CS_B}{R^{p(1+\nu)}}\right)^{\frac{1}{\sigma\nu}}\|u\|^{\frac{\delta}{\sigma}}_{L^\infty(Q)}\|u\|_{L^\sigma(Q)}\,, \tag{4.24}$$

where $C=C(p,\delta,\nu,\sigma)$*.*

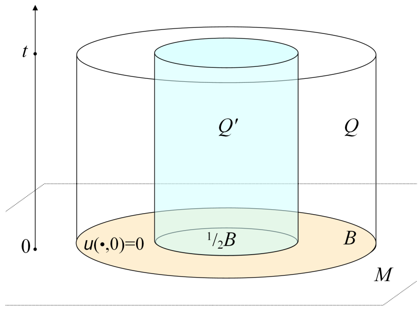

Figure 1: Cylinders Q and Q'

Proof. Consider a sequence of radii

$$r_k=\left(\frac{1}{2}+2^{-k-1}\right)R$$

so that $r_0=R$ and $r_k\searrow\frac{1}{2}R$ as $k\to\infty$. Set

$$B_k=B(x_0,r_k)\,,\qquad Q_k=B_k\times[0,t]$$

so that

$$B_0=B,\qquad Q_0=Q\qquad\text{and}\qquad Q_\infty:=\lim_{k\to\infty}Q_k=Q'$$

(see Fig. 2).

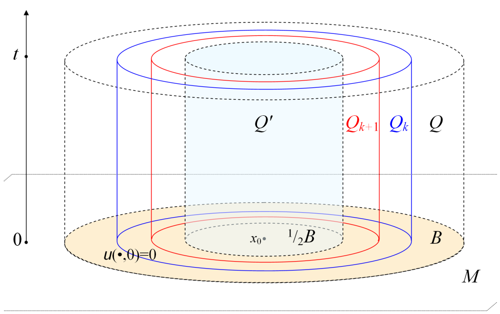

Figure 2: Cylinders Q_k

Set also

$$\sigma_k = \sigma\,(1+\nu)^k$$

and

$$J_k = \int_{Q_k} u^{\sigma_k}.$$

By (4.23) we have

$$\begin{aligned} J_{k+1} &\le \frac{CS_{B_k}\sigma_k^{(2-p)\nu}\,\|u\|_{L^\infty(Q_k)}^{\delta\nu}}{(r_k - r_{k+1})^{p(1+\nu)}} J_k^{1+\nu} \\ &\le \frac{C2^{kp(1+\nu)}\,(1+\nu)^{k(2-p)\nu}\,\sigma^{(2-p)\nu} S_B\,\|u\|_{L^\infty(Q)}^{\delta\nu}}{R^{p(1+\nu)}} J_k^{1+\nu} \\ &\le A^k \Theta^{-1} J_k^{1+\nu}, \end{aligned}$$

where

$$A = 2^{p(1+\nu)}\,(1+\nu)^{(2-p)_+\nu} \ge 1$$

and

$$\Theta^{-1} = \frac{CS_B\,\|u\|_{L^\infty(Q)}^{\delta\nu}}{R^{p(1+\nu)}},$$

where we have absorbed $\sigma^{(2-p)\nu}$ into C.

By Lemma 6.1 (see Appendix), we conclude that

$$\begin{aligned} J_k &\le \left(\left(A^{1/\nu}\Theta^{-1}\right)^{1/\nu} J_0\right)^{(1+\nu)^k} \left(A^{-1/\nu}\Theta\right)^{1/\nu} \\ &= A^{\frac{(1+\nu)^k-1}{\nu^2}}\,\Theta^{-\frac{(1+\nu)^k-1}{\nu}}\,J_0^{(1+\nu)^k}. \end{aligned}$$

It follows that

$$\left(\int_{Q_k} u^{\sigma_k}\right)^{1/\sigma_k} \le A^{\frac{1-(1+\nu)^{-k}}{\sigma\nu^2}}\,\Theta^{-\frac{1-(1+\nu)^{-k}}{\sigma\nu}} \left(\int_Q u^\sigma\right)^{1/\sigma}.$$

As $k\to\infty$, we obtain

$$\begin{aligned} \|u\|_{L^\infty(Q')} &\le A^{\frac{1}{\sigma\nu^2}}\,\Theta^{-\frac{1}{\sigma\nu}}\,\|u\|_{L^\sigma(Q)} \\ &\le A^{\frac{1}{\nu^2}} \left(\frac{CS_B\,\|u\|_{L^\infty(Q)}^{\delta\nu}}{R^{p(1+\nu)}}\right)^{\frac{1}{\sigma\nu}} \|u\|_{L^\sigma(Q)} \\ &= \left(\frac{CS_B}{R^{p(1+\nu)}}\right)^{\frac{1}{\sigma\nu}} \|u\|_{L^\infty(Q)}^{\frac{\delta}{\sigma}}\,\|u\|_{L^\sigma(Q)}, \end{aligned}$$

where A was absorbed into C. ■

Remark 4.4. Clearly, (4.24) implies

$$\|u\|_{L^\infty(Q')} \le \left(\frac{CS_B}{R^{p(1+\nu)}}\right)^{\frac{1}{\sigma\nu}} (t\mu(B))^{\frac{1}{\sigma}}\,\|u\|_{L^\infty(Q)}^{1+\frac{\delta}{\sigma}}. \tag{4.25}$$

5 Finite propagation speed

5.1 Propagation speed inside a ball

In this section we assume that M is complete. In particular, all balls are precompact. We assume here that

$$p > 1 \quad \text{and} \quad \delta > 0.$$

Theorem 5.1. *Let u be a bounded non-negative subsolution of* (2.8) *in M_T with the initial condition $u(\cdot, 0) = u_0$. Let $B_0 = B(x_0, R)$ be a precompact ball such that*

$$u_0 = 0 \ \textit{in} \ B_0$$

(see Fig. 3). *Set*

$$t_0 = \eta\iota(B_0)R^p \|u_0\|_{L^\infty(M)}^{-\delta} \wedge T, \tag{5.26}$$

where η is a sufficiently small positive constant depending only on p, δ, ν. Then

$$u = 0 \quad \textit{in} \ \ \frac{1}{2}B_0 \times [0, t_0].$$

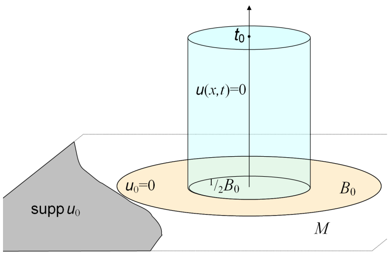

Figure 3: The support of u_0

Proof. Set $r = \frac{1}{2}R$ and fix for a while a point $x \in \frac{1}{2}B_0$ so that $B := B(x, r) \subset B_0$. Fix also some $t \in (0, T)$ and set

$$Q_k = 2^{-k}B \times [0, t] \quad \text{and} \quad J_k = \|u\|_{L^\infty(Q_k)}$$

(see Fig. 4).

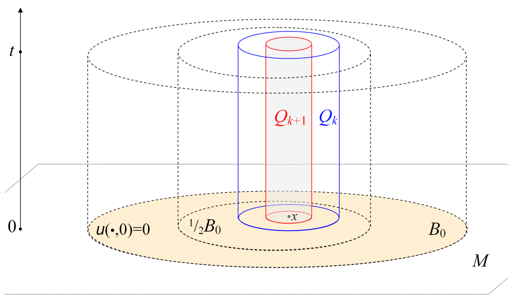

Figure 4: Cylinders Q_k

Choose and fix σ large enough as it is needed for Lemma 4.3. Then, by (4.25), we have

$$J_{k+1} \le \left(\frac{CS_{2^{-k}B}}{(2^{-k}R)^{p(1+\nu)}}\right)^{\frac{1}{\sigma\nu}} \left(t\mu(2^{-k}B)\right)^{\frac{1}{\sigma}} J_k^{1+\frac{\delta}{\sigma}}$$

$$\le 2^{k\frac{p(1+\nu)}{\sigma\nu}} \left(\frac{CS_B}{R^{p(1+\nu)}}\right)^{\frac{1}{\sigma\nu}} (t\mu(B))^{\frac{1}{\sigma}} J_k^{1+\frac{\delta}{\sigma}}.$$

Observe that, by (3.18) and $\frac{1}{\nu} = \kappa'$,

$$\left(\frac{S_B}{R^{p(1+\nu)}}\right)^{\frac{1}{\nu}} \mu(B) = \frac{R^{p/\nu}}{R^{p\frac{(1+\nu)}{\nu}} \iota(B)\mu(B)} \mu(B) = \frac{1}{\iota(B)R^p},$$

so that

$$\begin{aligned} J_{k+1} &\le 2^{k\frac{p(1+\nu)}{\sigma\nu}} \left(\frac{Ct}{\iota(B)R^p}\right)^{\frac{1}{\sigma}} J_k^{1+\frac{\delta}{\sigma}} \\ &= A^k \Theta^{-1} J_k^{1+\omega}, \end{aligned}$$

where

$$\omega = \frac{\delta}{\sigma}, \qquad A = 2^{\frac{p(1+\nu)}{\sigma\nu}}$$

and

$$\Theta^{-1} = \left(\frac{Ct}{\iota(B)R^p}\right)^{\frac{1}{\sigma}}.$$

By Lemma 6.1, if

$$\Theta^{-1} \le A^{-1/\omega} J_0^{-\omega} \tag{5.27}$$

then, for all $k \ge 0$,

$$J_k \le A^{-k/\omega} J_0. \tag{5.28}$$

The condition (5.27) is equivalent to

$$\left(\frac{Ct}{\iota(B)R^p}\right)^{\frac{1}{\sigma}} \le A^{-1/\omega} J_0^{-\omega}$$

that is, to

$$t \le C^{-1}\iota(B)R^p J_0^{-\delta}, \tag{5.29}$$

where A is absorbed to C. Since, by Lemma 2.9,

$$J_0 = \|u\|_{L^\infty(Q)} \le \|u_0\|_{L^\infty(M)}$$

the condition (5.29) is satisfied for $t = t_0$, where t_0 is determined by (5.26) with $\eta = C^{-1}$.

Hence, for $t = t_0$ we obtain from (5.28) that, for any k,

$$\|u\|_{L^\infty\left(2^{-k}B\times[0,t]\right)} \le A^{-k/\omega} \|u_0\|_{L^\infty}.$$

For any k, we cover the ball $\frac{1}{2}B_0$ by a countable (or even finite) sequence of balls $B\left(x_i, 2^{-k}r\right)$ with $x_i \in \frac{1}{2}B_0$. Since for all i

$$\|u\|_{L^\infty\left(B(x_i,2^{-k}r)\times[0,t]\right)} \le A^{-k/\omega} \|u_0\|_{L^\infty},$$

we obtain that

$$\|u\|_{L^\infty\left(\frac{1}{2}B_0\times[0,t]\right)} \le A^{-k/\omega} \|u_0\|_{L^\infty}.$$

Finally, letting $k \to \infty$, we obtain that $u = 0$ in $\frac{1}{2}B_0 \times [0,t]$, which was to be proved. ∎

5.2 Propagation speed of support

We assume here that, as above,

$$p > 1 \quad \text{and} \quad \delta > 0.$$

For any set $K \subset M$ and any $r > 0$, denote by K_r a closed r-neighborhood of K.

Corollary 5.2. *Let $u(x,t)$ be a non-negative bounded subsolution of* (2.8) *in $M \times \mathbb{R}_+$ with the initial function $u_0 = u(\cdot, 0)$. Assume that the support $K = \operatorname{supp} u_0$ is compact. Then there exists $T > 0$ and an increasing continuous function $\rho : (0,T) \to \mathbb{R}_+$ such that*

$$\operatorname{supp} u(\cdot, t) \subset K_{\rho(t)}$$

for all $t \in (0,T)$ (see Fig. 5).

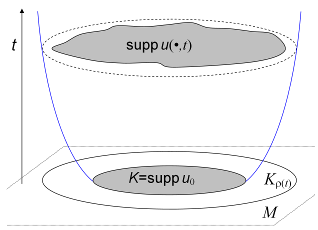

Figure 5: The support of $u(\cdot,t)$

Here both T and $\rho(t)$ may depend on u. The function $\rho(t)$ is called the *propagation rate* of u.

Proof. Let us fix a reference point $x_0 \in K$ and define the following function for all $r > 0$:

$$\varphi(r) = \frac{\eta}{4^{p+p/\nu}} \iota(B(x_0,r)) r^p \|u_0\|_{L^\infty(M)}^{-\delta}. \tag{5.30}$$

Denote $r_0 = \operatorname{diam} K$. Let us prove that that, for any $r \geq r_0$,

$$t \leq \varphi(3r + r_0) \Rightarrow \operatorname{supp} u(\cdot, t) \subset K_r,$$

that is,

$$u(\cdot,t) = 0 \quad \text{in} \quad M \setminus K_r.$$

Let us fix a point $x \in K_{2r} \setminus K_r$ (see Fig. 6). We have

$$d(x,K) \leq 2r \Rightarrow d(x,x_0) \leq 2r + r_0.$$

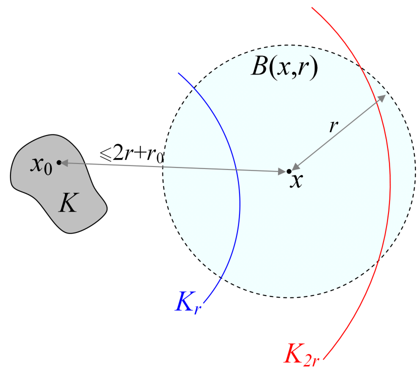

Figure 6: A point $x \in K_{2r} \setminus K_r$ and the ball $B(x,r)$

It follows that

$$B(x,r) \subset B(x_0, 3r + r_0) = B(x_0, R)$$

where

$$R := 3r + r_0.$$

The condition $r \geq r_0$ implies $R \leq 4r$. Since $B(x,r) \subset B(x_0,R)$, we have by the monotonicity of function (3.18) that

$$\frac{\iota(B(x,r))\mu(B(x,r))}{r^{p/\nu}} \geq \frac{\iota(B(x_0,R))\mu(B(x_0,R))}{R^{p/\nu}}.$$

It follows that

$$\begin{aligned}\iota(B(x,r))r^p &\geq \left(\frac{r}{R}\right)^{p+p/\nu} \iota(B(x_0,R))\frac{\mu(B(x_0,R))}{\mu(B(x,r))}R^p \\ &\geq \frac{1}{4^{p+p/\nu}}\iota(B(x_0,R))R^p.\end{aligned}$$

Therefore, the hypothesis $t \leq \varphi(R)$ implies that

$$t \leq \eta\iota(B(x,r)))r^p \left\|u_0\right\|_{L^\infty(M)}^{-\delta}.$$

Since $u(\cdot,0) = 0$ in $B(x,r)$, we conclude by Theorem 5.1 that

$$u(\cdot,t) = 0 \ \text{ in } \ B(x,r/2).$$

Since this is true for any $x \in K_{2r} \setminus K_r$, we obtain that

$$u(\cdot,t) = 0 \ \text{ in } K_{2r} \setminus K_r. \tag{5.31}$$

Let us show that also

$$u(\cdot,t) = 0 \ \text{ in } M \setminus K_r. \tag{5.32}$$

Fix some $s >> 2r$ and let $\eta(x)$ be a bump function of $K_s \setminus K_{2r}$ in $K_{2s} \setminus K_r$; that is, η is the following function of $|x| := d(x,K)$:

$$\eta(x) = \begin{cases} \left(\frac{|x|}{r} - 1\right)_+, & |x| \leq 2r, \\ 1, \ \ |x| \in [2r, s], \\ 2\left(1 - \frac{|x|}{2s}\right)_+, & |x| \geq s \end{cases}$$

(see Fig. 7).

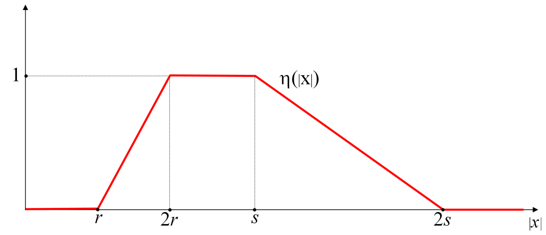

Figure 7: Function η

Applying the inequality (2.4) of Lemma 2.6 with some fixed λ, we obtain

$$\left[\int_M u^\lambda \eta^p\right]_0^t \le c_2 \int_0^t \int_M u^\sigma |\nabla \eta|^p . \tag{5.33}$$

Since $u(\cdot,0) = 0$ on $\operatorname{supp}\eta$ and $\eta = 1$ on $K_s \setminus K_{2r}$, the left hand side here is bounded below by

$$\int_{K_s\setminus K_{2r}} u^\lambda(\cdot,t).$$

Since $\eta = 0$ in K_r, $u(\cdot,\tau) = 0$ in $K_{2r}\setminus K_r$ for all $\tau \le t$ (by (5.31)), and $\nabla\eta = 0$ in $K_s \setminus K_{2r}$, the right hand side in (5.33) is equal to

$$c_2 \int_0^t \int_{M\setminus K_s} u^\sigma |\nabla\eta|^p .$$

Since

$$|\nabla\eta| \le \frac{1}{s} \text{ in } M\setminus K_s,$$

we obtain that

$$\int_{K_s\setminus K_{2r}} u^\lambda(\cdot,t) \le c_2 \int_0^t \int_{M\setminus K_s} u^\sigma |\nabla\eta|^p \le \frac{c_2}{s^p}\int_0^t \int_{M\setminus K_s} u^\sigma .$$

The right hand side goes to 0 as $s \to \infty$, which implies that $u(\cdot,t) = 0$ in $M \setminus K_{2r}$, thus proving (5.32).

Now let us define in $[r_0,\infty)$ a function

$$\psi(r) = \frac{1}{2}\sup_{r_0\le s\le r} \varphi(3s + r_0)$$

so that $\psi(r)$ is monotone increasing. If $t \le \psi(r)$ then $t \le \varphi(3s + r_0)$ for some $s \in [r_0, r]$, which implies by the first part of the proof that

$$u(\cdot,t) = 0 \text{ in } M \setminus K_s$$

and, hence,

$$u(\cdot,t) = 0 \text{ in } M \setminus K_r .$$

It is unclear whether ψ is continuous or not. As a monotone function, ψ may have only jump discontinuities. By subtracting all these jumps, we obtain a continuous monotone function $\widetilde{\psi} \le \psi$ with the same property|:

$$t \le \widetilde{\psi}(r) \Rightarrow u(\cdot,t) = 0 \text{ in } M \setminus K_r. \tag{5.34}$$

As a continuous monotone increasing function, $\widetilde{\psi}$ has an inverse $\rho = \widetilde{\psi}^{-1}$ on $[t_0, T)$ where

$$t_0 = \widetilde{\psi}(r_0) \text{ and } T = \sup\widetilde{\psi}.$$

Let us extend $\rho(t)$ to $t < t_0$ by setting $\rho(t) = \rho(t_0)$. Then $r = \rho(t)$ implies $t \le \widetilde{\psi}(r)$, and by (5.34)

$$u(\cdot,t) = 0 \text{ in } M \setminus K_r,$$

which was to be proved. ∎

5.3 Curvature and propagation rate

Corollary 5.3. *Let M be complete, non-compact and let $Ricci_M \geq 0$. Let Let u be a bounded non-negative subsolution in $M \times \mathbb{R}_+$ with the initial condition $u(\cdot, 0) = u_0$. Set $K = \operatorname{supp} u_0$. Then, for any $t \geq 0$,*

$$\operatorname{supp} u(\cdot, t) \subset K_{Ct^{1/p}}$$

where C depends on $\|u_0\|_{L^\infty}$, p, δ, n.

Proof. It is known that on such manifolds $\iota(B) \geq \text{const} > 0$ for all balls $B \subset M$ (see [12], [18], [35]). Let $B(x, R)$ be a ball that is disjoint with K. By Theorem 5.1, if

$$t \leq c\iota(B_0)R^p \|u_0\|_{L^\infty(M)}^{-\delta},$$

then

$$u(\cdot, t) = 0 \quad B(x, \frac{1}{2}R).$$

Hence, if

$$R \geq Ct^{1/p}$$

then

$$\operatorname{supp} u(\cdot, t) \cap B(x, \frac{1}{2}R) = \emptyset$$

and, hence,

$$\operatorname{supp} u(\cdot, t) \subset K_{\frac{1}{2}R},$$

whence the claim follows. ■

6 Appendix: an auxiliary lemma

The following lemma was used in Sections 4 and 5.

Lemma 6.1. *Let a sequence $\{J_k\}_{k=0}^\infty$ of non-negative reals satisfies*

$$J_{k+1} \leq \frac{A^k}{\Theta} J_k^{1+\omega} \quad \text{for all } k \geq 0. \tag{6.1}$$

where $A, \Theta, \omega > 0$. Then, for all $k \geq 0$,

$$J_k \leq \left(\left(A^{1/\omega}\Theta^{-1}\right)^{1/\omega} J_0\right)^{(1+\omega)^k} \left(A^{-k-1/\omega}\Theta\right)^{1/\omega}. \tag{6.2}$$

In particular, if

$$\Theta \geq A^{1/\omega} J_0^\omega, \tag{6.3}$$

then, for all $k \geq 0$,

$$J_k \leq A^{-k/\omega} J_0. \tag{6.4}$$

Proof. Consider the sequence

$$X_k = \left(\left(A^{1/\omega}\Theta^{-1}\right)^{1/\omega} J_0\right)^{(1+\omega)^k} \left(A^{-k-1/\omega}\Theta\right)^{1/\omega}.$$

Then we have

$$X_0 = \left(A^{1/\omega}\Theta^{-1}\right)^{1/\omega} J_0 \left(A^{-1/\omega}\Theta\right)^{1/\omega} = J_0$$

and

$$\begin{aligned}
\frac{A^k}{\Theta}X_k^{1+\omega} &= \frac{A^k}{\Theta}\left(\left(A^{1/\omega}\Theta^{-1}\right)^{1/\omega} J_0\right)^{(1+\omega)^{k+1}} \left(A^{-k-1/\omega}\Theta\right)^{\frac{1+\omega}{\omega}} \\
&= \left(\left(A^{1/\omega}\Theta^{-1}\right)^{1/\omega} J_0\right)^{(1+\omega)^{k+1}} A^k\Theta^{-1}\left(A^{-k-1/\omega}\Theta\right)\left(A^{-k-1/\omega}\Theta\right)^{\frac{1}{\omega}} \\
&= \left(\left(A^{1/\omega}\Theta^{-1}\right)^{1/\omega} J_0\right)^{(1+\omega)^{k+1}} A^{-1/\omega}\left(A^{-k-1/\omega}\Theta\right)^{1/\omega} \\
&= \left(\left(A^{1/\omega}\Theta^{-1}\right)^{1/\omega} J_0\right)^{(1+\omega)^{k+1}} \left(A^{-(k+1)-1/\omega}\Theta\right)^{1/\omega} = X_{k+1}.
\end{aligned}$$

Hence, by comparison we obtain $J_k \le X_k$, which was to be proved. ■

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