Expected Utility Without Linearity: Distinguishing Between Prospect Theory and Cumulative Prospect Theory

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Abstract

We reconsider the foundations of expected utility without assuming the linearity of the independence axiom. We consider a decision-maker who cancels out common outcomes when comparing a pair of lotteries with the same probability tree. We show that if the decision-maker is consistent with first-order stochastic dominance or topological continuity in weak convergence, then the decision-maker is an expected utility maximizer. This offers a simple method to differentiate behavior between prospect theory, canceling out common outcomes in pairwise comparisons, and cumulative prospect theory, satisfying first-order stochastic dominance. Additionally, this offers a novel method to test technical continuity assumptions based on their behavioral content that rules out, e.g., prospect theory.

Keywords: Expected utility, branch cancellation, first-order stochastic dominance, testing axioms, prospect theory, cumulative prospect theory.

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1 Introduction

Prospect theory (Kahneman and Tversky, 1979) is one of the most used alternatives to expected utility in choice under risk. This theory accounts for many observed violations of expected utility. One of the central tenants of prospect theory is extending the expected utility by a probability weighting function. This captures the assumption that the decision-maker cancels out common outcomes when comparing a pair of lotteries. However, prospect theory does not satisfy first-order stochastic dominance that has been seen as problematic based on normative and descriptive grounds (e.g. Quiggin, 1982). This led to the development of cumulative prospect theory (Tversky and Kahneman, 1992) which instead applies the probability weighting to the cumulative distribution function and avoids first-order stochastic dominance violations.

Recently, there has been a discussion (Bernheim and Sprenger, 2020; Abdellaoui et al., 2020) on testing and comparing prospect theory and cumulative prospect theory. We provide a perspective to this discussion by offering a novel characterization for the expected utility that clarifies the key differences between these two models. We consider a decision-maker who cancels out common outcomes when comparing a pair of lotteries with the same probability tree, as in prospect theory. We show that if this decision-maker is consistent with first-order stochastic dominance, as in cumulative prospect theory, then the decision-maker is an expected utility maximizer. This incompatibility shows that these two key differences can be used to differentiate behavior between prospect theory and cumulative prospect theory. Additionally, this cancellation of common consequences formalizes the concept that Bernheim and Sprenger (2020) tested and found empirical evidence to support.

In our second result, we offer a novel method to test technical assumptions. We show that under our previous cancellation axiom, topological continuity in weak convergence is equivalent to the expected utility. Especially, people, such as prospect theory maximizers, who satisfy our cancellation axiom but are not expected utility maximizers violate the topological continuity.

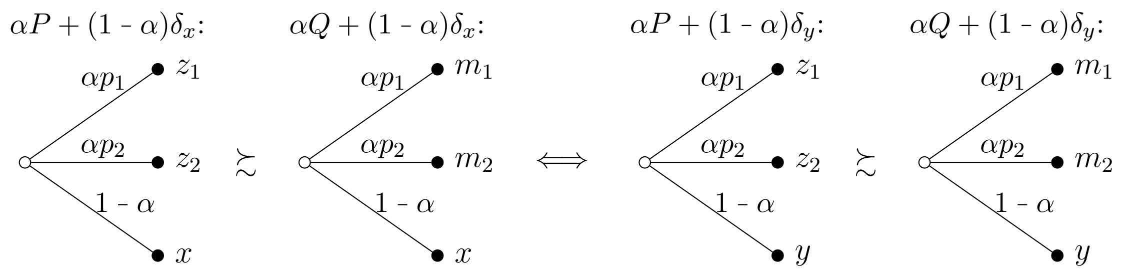

Figure 1. An example of restricted branch cancellation, Axiom 2. When the lotteries $\alpha P+(1-\alpha)\delta_x$ and $\alpha Q+(1-\alpha)\delta_x$ share the same probability tree, then the common prize x with the common probability $1-\alpha$ is canceled out and it can be changed to y without affecting the comparison.

More specifically, we consider a decision-maker who satisfies the restricted branch cancellation axiom illustrated in Figure 1. In the figure, we have two lotteries $\alpha P+(1-\alpha)\delta_x$ and $\alpha Q+(1-\alpha)\delta_x$ that share the same probability tree αp_i with different prizes, z_i and m_i respectively, and that have a common prize x with a common probability $1-\alpha$. We assume that the decision-maker cancels out the common outcome x with the common probability when comparing the lotteries. Then especially changing it to some other common outcome y, that is not a part of P or Q, does not affect the comparison.

This condition relaxes the independence axiom in three ways. First, by making only simple comparisons between lotteries with the same probability tree. Second, by assuming that the canceled prizes x and y are different from the other prizes and not a part of P or Q. Third, after canceling the common prize, the decision-maker does not evaluate the lotteries without the canceled common outcomes by using the conditional lotteries without the canceled prizes. This relaxes the linearity of the independence axiom. These relaxations allow for a limited understanding of probabilities and only require that the possible prizes are understood as disjoint and hence can be evaluated independently of each other.

However, we show that restricted branch cancellation is equivalent to the independence axiom if the preferences satisfy first-order stochastic dominance or a joint continuity in prizes and probabilities, that is continuity in weak convergence. In this case, the decision-maker is an expected utility maximizer. This characterization shows the necessity of relaxing stochastic dominance and joint continuity for a model of non-expected utility when the outcomes are separable within the probability tree.

First, our characterization contributes to the recent discussion on comparing and testing prospect theory and cumulative prospect theory (Bernheim and Sprenger, 2020; Abdellaoui et al., 2020) and to the literature on axiomatizing them. A stronger non-restricted version of the restricted branch cancellation axiom was suggested by Kahneman and Tversky (1979) as a cancellation operation in the editing phase of lotteries in prospect theory. First-order stochastic dominance was one of the motivating features in the development of cumulative prospect theory. Our characterization shows that these two properties are the key differences between the two versions and can be used to differentiate behavior between them. The restricted branch cancellation axiom formalizes the concept that Bernheim and Sprenger (2020) tested and found empirical evidence to support. In contrast, Birnbaum (2018) reviewed experimental evidence on violations of restricted branch cancellation and first-order stochastic dominance.

Second, based on experimental evidence, Bernheim and Sprenger (2020) suggest that a unified theory of choice under risk should capture rank independence of lottery outcome evaluations and strong event splitting effect. The event splitting effect is the change in the valuation of the lottery after splitting a prize x with a probability p into two almost equal prizes x and $x + \varepsilon$ each with the probability $p/2$. Our characterization shows that these two effects are connected: If the decision-maker cancels common consequences with common probabilities rank independently, then there must be a discontinuous event splitting effect or the decision-maker is an expected utility maximizer.[1] This suggests there might be a common underlying behavioral foundation for these two different empirical findings.

Third, our characterization contributes to the literature on testing technical assumptions and offers a novel method for testing them. Our characterization shows

[1]Our proof uses a weaker continuity axiom than continuity in weak convergence that essentially rules out event splitting effects and assumes a continuous utility for prizes.

that under the restricted branch cancellation axiom, the independence axiom and continuity in weak convergence are equivalent and so it has testable behavioral implications. In the literature, there is less evidence for violations of the restricted branch cancellation axiom than evidence for violations of the independence axiom suggesting that many people violate the topological continuity axiom.

Fourth, our characterization contributes to the literature on characterizing the expected utility and highlights the crucial role of restricted branch cancellation and first-order stochastic dominance in understanding violations of the expected utility. Under first-order stochastic dominance, non-expected utility models must violate restricted branch cancellation. That is the decision-maker's evaluation of possible prizes must depend on the other possible prizes as for example with rank-dependent expected utility. On the other hand, if the decision-maker evaluates each possible prize separably within the probability tree, then we must violate stochastic dominance for a non-expected utility model. The previous characterizations of the expected utility have assumed the linearity of the independence axiom (von Neumann and Morgenstern, 1947; Arrow, 1951; Luce and Raiffa, 1957; Wakker, 2010; Segal, 2023).

Finally, our characterizations contribute to the literature studying first-order stochastic dominance in non-expected utility models. Fishburn (1978) showed that expected utility with subjectively weighted probabilities satisfies stochastic dominance only if it is expected utility. We generalize this result for a full characterization of the expected utility by considering preferences that satisfy the restricted branch cancellation axiom which especially does not restrict how lotteries that do not share the same probability tree are compared.

The remainder of the paper proceeds as follows: We begin, in Section 2.1, by introducing the setting. Sections 2.2.1 and 2.2.2 show our main results and characterizes the expected utility first by first-order stochastic dominance and then by weak convergence continuity. Section 2.2.3 illustrates the theoretical tractability of restricted branch cancellation and sketches the proof. Section 3 applies our results to differentiating behavior between prospect theory and cumulative prospect theory and to the experiment by Bernheim and Sprenger (2020) on testing these theories. Finally, Section 4 concludes. The appendix extends the expected utility characterization to general prize intervals and Borel lotteries. The proof is included in the appendix.

2 Preliminaries and Expected Utility

2.1 Preliminaries

We consider a standard setting in choice under risk with monetary prizes without compound lotteries. The prizes are monetary prizes on an open interval $X = (m_*, m^*)$ where $m_*, m^* \in \mathbb{R} \cup \{\infty, -\infty\}$ and $m_* < m^*$. The set of (simple) lotteries on X is denoted by $\Delta(X)$. We consider preferences $\succsim$ over lotteries $P \in \Delta(X)$. $\operatorname{supp} P$ denotes the support of the lottery P.[2] We endow the set of lotteries $\Delta(X)$ with the topology of weak convergence.[3]

We define mixtures of lotteries prizewise which assumes that compound lotteries are reduced to single-stage lotteries: Define for all $\alpha \in [0,1]$, $P, Q \in \Delta(X)$, and $x \in X$,

$$\Big(\alpha P + (1-\alpha)Q\Big)(x) = \alpha P(x) + (1-\alpha)Q(x).$$

We discuss this definition at the end of this section.

Our approach is to focus only on comparing lotteries with the same induced probability tree over outcomes. This is illustrated in Figure 2. Here, we have lotteries P and Q with the same induced probability tree (p_1, p_2, p_3) and prizes (x_1, x_2, x_3) and (y_1, y_2, y_3) respectively where for each $i \neq j$, $x_i \neq x_j$ and $y_i \neq y_j$. Formally:

[2]For simple lotteries, $\operatorname{supp} P = \{x \in X | P(x) > 0\}$.

[3]For simple lotteries, weak convergence simplifies to the convergence of cumulative distribution functions outside the support of the limit lottery: A sequence of lotteries $(P_n)_{n=1}^{\infty} \subseteq \Delta(X)$ converges weakly to $P \in \Delta(X)$ if for all $a \in \mathbb{R}$ with $a \notin \operatorname{supp} P$,

$$\sum_{\substack{x \in \operatorname{supp} P_n \\ x \leq a}} P_n(x) \to \sum_{\substack{x \in \operatorname{supp} P \\ x \leq a}} P(x) \text{ as } n \to \infty.$$

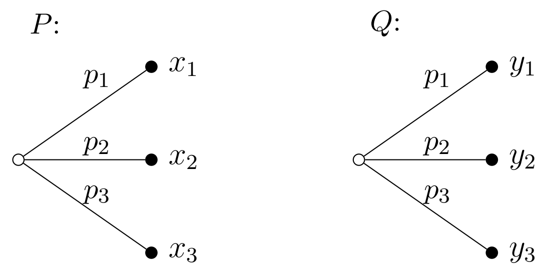

Figure 2. An example of lotteries $P = \sum_{i=1}^{3} p_i \delta_{x_i}$, $Q = \sum_{i=1}^{3} p_i \delta_{y_i}$ with the same induced probability tree where for all $i \neq j$, $x_i \neq x_j$ and $y_i \neq y_j$.

Definition Lotteries P and Q *have the same induced probability tree* if there exist $p \in \Delta(\mathbb{N})$ and $(x_i)_{i \in \operatorname{supp} p}, (y_i)_{i \in \operatorname{supp} p} \in X^{\operatorname{supp} p}$ such that for all $i \neq j$, $x_i \neq x_j$ and $y_i \neq y_j$, $P = \sum_{i \in \operatorname{supp} p} p_i \delta_{x_i}$, and $Q = \sum_{i \in \operatorname{supp} p} p_i \delta_{y_i}$.

In this definition, the probability trees of the lotteries are only compared based on the observable probabilities for each outcome. That is after the reduction of compound lotteries and coalescing common prizes. For example, if in Figure 2 $y_1 = y_2$, then the lottery Q simplifies since the probabilities p_1 and p_2 with the same prize can be joined together simplifying the probability tree. Formally, this change follows from our definition of mixtures since the following two lotteries are identically the same

$$p_1 \delta_{y_1} + p_2 \delta_{y_1} + p_3 \delta_{y_3} = (p_1 + p_2) \delta_{y_1} + p_3 \delta_{y_3}.$$

Our definition of mixtures and lottery simplifications incorporate the idea of prospect theory's editing phase (Kahneman and Tversky, 1979). In this editing phase, the decision-maker combines probabilities with the same prizes and reduces multistage lotteries to a single stage. In the literature, the first property is known as the coalescing axiom (Starmer and Sugden, 1993; Humphrey, 1995) or combination operation of the editing phase (Kahneman and Tversky, 1979) and the second property as the reduction of compound lotteries (Segal, 1988; 1990).

2.2 Expected Utility

We move on to our main results. We provide a novel axiomatization for the expected utility by assuming that the decision-maker cancels out common consequences when doing simple comparisons and comparing lotteries with the same induced probability tree. We show that under a weak continuity in prizes and first-order stochastic dominance or topological continuity on lotteries, the decision-maker is an expected utility maximizer.

Our first axiom is the standard assumption that preferences are a weak order and monotonic in certain amounts of money.

Axiom 1 (Weak Order and Monotonicity) $\succsim$ is complete and transitive and for all $x > y$, $\delta_x \succ \delta_y$.

The next axiom is the main axiom for the expected utility. It assumes that if two lotteries share the same induced probability tree and a common prize with a common probability, then the common prize is canceled out when comparing the lotteries. Then especially, if the common prize is changed into something else, the comparison is not affected. This is illustrated in Figure 4.

Axiom 2 (Restricted Branch Cancellation) If two lotteries P, Q have the same induced probability tree, $x, y \notin \operatorname{supp} P \cup \operatorname{supp} Q$, and $\alpha \in (0, 1)$, then

$$\alpha P + (1-\alpha)\delta_x \succsim \alpha Q + (1-\alpha)\delta_x \iff \alpha P + (1-\alpha)\delta_y \succsim \alpha Q + (1-\alpha)\delta_y.$$

This axiom relaxes the independence axiom in three ways. First, this axiom does not assume the linearity of the independence axiom. After canceling the common prize with the common probability, the lotteries might not be viewed as conditional lotteries without the canceled prize. Second, the common consequence is only canceled in simple comparisons between lotteries with the same induced probability tree. Third, only common prizes with common probabilities are canceled and the decision-maker might not be able to split probabilities to cancel all common prizes. This axiom requires only a limited understanding of probabilities as disjoint possibilities for prizes. However, we show that this axiom is equivalent to the independence axiom under a continuity in shifting the prizes and first-order stochastic dominance or topological continuity on lotteries.

A stronger non-restricted version of this axiom was suggested by Kahneman and Tversky (1979) as a cancellation operation in the editing phase of lottery comparisons. This axiom is formally related to the weak c-independence axiom in choice under uncertainty (Maccheroni et al., 2006).[4]

2.2.1 Expected Utility and First-Order Stochastic Dominance

Our first characterization for expected utility shows that if the decision-maker satisfies restricted branch cancellation, first-order stochastic dominance, and a weak form of continuity in shifting all the prizes, then the decision-maker is an expected utility maximizer. Next, we define prize shifts for lotteries.

Definition For a lottery P and $\varepsilon \in \mathbb{R}$, define a lottery $P + \varepsilon$ by for all $x \in X$,

$$P + \varepsilon(x + \varepsilon) = P(x).$$

Here, we shift the prize distribution of P by increasing every prize by ε. That is the probability of the prize $x + \varepsilon$ is $P(x)$. Since X is an open interval, for a small enough ε, we have $P + \varepsilon \in \Delta(X)$. The next axiom is continuity in shifting all the prizes of a lottery. This continuity allows for example rank-dependent discontinuities.

Axiom 3 (Prize Shift Continuity) For all lotteries P, Q, if $P \succ Q$, then there exist $\varepsilon, \eta > 0$ such that $P - \varepsilon, Q + \eta \in \Delta(X)$ and

$$P - \varepsilon \succ Q + \eta.$$

The last axiom for expected utility is first-order stochastic dominance.

[4]This axiom is also called restricted branch independence in Birnbaum (2008) and restricted common branch substitution in Luce et al. (2008).

Axiom 4 (FOSD) For all lotteries P, Q, if P first-order stochastically dominates Q, then $P \succsim Q$.

The next result shows that under continuity in prize shifts and stochastic dominance, restricted branch cancellation is equivalent to the standard independence axiom.

Theorem 1 (EU and FOSD) The following two are equivalent:

(1) $\succsim$ satisfies Axioms 1-4.

(2) There exists continuous and strictly increasing $u : X \to \mathbb{R}$ such that for all lotteries $P, Q \in \Delta(X)$,

$$P \succsim Q \iff \sum_{x \in \operatorname{supp} P} P(x)u(x) \geq \sum_{x \in \operatorname{supp} Q} Q(x)u(x).$$

This result highlights the strength of first-order stochastic dominance. It gives a novel characterization for the expected utility with only consistency in evaluating prizes independently of each other within the induced probability tree, shift continuity in prizes, and monotonicity with stochastic dominance. This also highlights the prevalence of violations of stochastic dominance in models of non-expected utility that satisfy restricted branch cancellation such as subjectively weighted expected utility (Hong, 1983; Fishburn, 1983).

This result clarifies the key differences between prospect theory and cumulative prospect theory. Prospect theory satisfies the restricted branch cancellation axiom whereas cumulative prospect theory satisfies first-order stochastic dominance. Our result shows that these two assumptions are not compatible with each other and are the key differences between the two models. These two assumptions can be used to differentiate behavior between the two models.

Remark In Appendix A, we extend the result to general prize intervals and Borel lotteries. In this case, for the shift continuity, we use censored shifts that are censored to remain in the closure of the prize interval. We show that for a compact prize interval, Theorem 1 generalizes to Borel lotteries. Additionally, we show that with simple lotteries, Theorem 1 generalizes to any prize interval.

2.2.2 Expected Utility and Continuity

Our second characterization for the expected utility uses restricted branch cancellation with topological continuity on lotteries. The next axiom is joint continuity in prizes and probabilities using continuity in weak convergence. This is a standard assumption and has been used e.g. in Segal (1990), Border and Segal (1994), Castagnoli and Calzi (1996), Machina (2001), Cerreia-Vioglio et al. (2015; 2020), and Hara et al. (2019).

Axiom 5 (Weak Convergence Continuity) For all lotteries $(P_n)_{n=1}^{\infty}, P$ and Q such that P_n converges weakly to P as $n \to \infty$,

$$\text{if } P_n \succsim Q \text{ for all } n \in \mathbb{N}, \text{ then } P \succsim Q$$

and

$$\text{if } Q \succsim P_n \text{ for all } n \in \mathbb{N}, \text{ then } Q \succsim P.$$

Our next result shows that under continuity in weak convergence, restricted branch cancellation is equivalent to the independence axiom.

Theorem 2 (EU and Continuity) The following two are equivalent:

(1) $\succsim$ satisfies Axioms 1, 2, and 5.

(2) There exists continuous, strictly increasing, and bounded $u : X \to \mathbb{R}$ such that for all lotteries $P, Q \in \Delta(X)$,

$$P \succsim Q \iff \sum_{x \in \operatorname{supp} P} P(x)u(x) \geq \sum_{x \in \operatorname{supp} Q} Q(x)u(x).$$

This result highlights the strength of the joint continuity in prizes and probabilities. It gives a novel characterization for the expected utility with only consistency in evaluating prizes independently of each other within the induced probability tree and continuity in prizes and probabilities. This characterization shows that the standard technical assumption of weak convergence continuity has significant behavioral content. Especially, it rules out prospect theory.

These characterizations provide a novel perspective to the observed violations of the independence axiom. There is less evidence for violations of the restricted branch cancellation axiom and prize shift continuity is a weak assumption only on common shifts of prizes. Especially, these decision-makers who violate the expected utility but satisfy restricted branch cancellation and prize shift continuity must violate the first-order stochastic dominance and weak convergence continuity axioms. This highlights the critical role of technical assumptions in interpreting observed empirical violations of axioms. The weak convergence continuity axiom is regarded as a technical assumption but our characterizations show that it has testable behavioral implications when considered with other behavioral assumptions such as the restricted branch cancellation axiom. This highlights a novel method to test technical assumptions.

Remark In Appendix A, we generalize Theorem 2 to any interval and to Borel lotteries over any interval.

2.2.3 Proof Sketch

This section highlights the technical tractability of the restricted branch cancellation axiom. Our approach is to consider prizes separately from the probabilities. This makes the setting symmetrical to a Savagian subjective probability setting when the probability numbers of the induced probability tree are considered as events. This allows us to incorporate techniques from choice under uncertainty into the risk domain. This technical approach has been rarely used and it might be useful beyond characterizing the expected utility. More specifically, the proof follows in four steps:

Step 1: FOSD and prize shift continuity implies continuity in weak convergence for lotteries with the same induced probability tree: Assume by contradiction that there is a sequence of lotteries with the same induced probability tree, $(P_n)_{n=1}^{\infty}$, converging to

P with a discontinuous jump in the preferences downwards. Here, weak convergence simplifies to branch-wise convergence of the prizes on the probability tree. Now, we can continuously shift P slightly upwards to stay below the preference jump but to be above all high enough P_n violating FOSD.

Step 2: For each uniform probability tree, the preferences restricted to lotteries with this induced probability tree are separable and symmetric across the tree branches and hence have an additive representation with the same utility for each tree branch.

Step 3: The branch utility for the uniform probability tree with 2^n branches is twice the branch utility of the uniform probability tree with 2^{n+1} branches: For $n=1$, the lottery $(\frac{1}{2}, x; \frac{1}{2}, y)$ has an alternative representation as a uniform lottery with 4 branches as $(\frac{1}{4}, x; \frac{1}{4}, x+\varepsilon; \frac{1}{4}, y; \frac{1}{4}, y+\varepsilon)$ when taking $\varepsilon \to 0$ by the continuity from Step 1. By the uniqueness of additive representations, after normalization, these two representations have the same value which shows the claim. This step gives expected utility representation for lotteries with a dyadic uniform probability tree, i.e. 2^n branches for some n.

Step 4: The lotteries with a dyadic uniform probability tree can approximate any simple lottery. Hence, the expected utility representation can be extended to all simple lotteries.

3 Distinguishing Between Prospect Theory and Cumulative Prospect Theory

Our characterizations, Theorems 1 and 2, offer a novel perspective to the key differences between prospect theory and cumulative prospect theory and to the recent experiment by Bernheim and Sprenger (2020) on testing prospect theory and cumulative prospect theory. Especially, restricted branch cancellation formalizes the concept that Bernheim and Sprenger (2020) tested and found empirical evidence to support.

Prospect theory (Kahneman and Tversky, 1979)[5] extends the expected utility by a probability weighting function. Formally, prospect theory uses a probability weighting function $w : [0, 1] \to [0, 1]$ with $w(0) = 0$ and $w(1) = 1$ and a utility function $u : X \to \mathbb{R}$ to evaluate lotteries $\sum_{i=1}^{n} p_i \delta_{x_i}$ by

$$\sum_{i=1}^{n} w(p_i) u(x_i).$$

Here, the outcomes x_i are gains or losses relative to the reference point. Prospect theory satisfies the restricted branch cancellation axiom. However, it does not satisfy first-order stochastic dominance that led to the development of cumulative prospect theory (Tversky and Kahneman, 1992).

In contrast to prospect theory, cumulative prospect theory applies probability weighting to the cumulative distribution function to avoid FOSD violations. Formally, cumulative prospect theory uses probability weighting functions $w^+ : [0, 1] \to [0, 1]$ and $w^- : [0, 1] \to [0, 1]$ with $w^+(0) = w^-(0) = 0$ and $w^+(1) = w^-(1) = 1$ and a utility function $u : X \to \mathbb{R}$ with $u(0) = 0$ to evaluate lotteries $\sum_{i=1}^{n} p_i \delta_{x_i}$ with ordered prizes, for each i, $x_i < x_{i+1}$, by

$$\sum_{i=1}^{n} \pi_i u(x_i)$$

where the decision weights π_i are

$$\pi_i = \begin{cases} w^+(p_i + \cdots + p_n) - w^+(p_{i+1} + \cdots + p_n), & \text{if } \; x_i \geq 0 \\ w^-(p_1 + \cdots + p_i) - w^-(p_1 + \cdots + p_{i-1}), & \text{if } \; x_i < 0 \end{cases}$$

and the outcomes x_i are gains or losses relative to the reference point. Here, the decision weights depend on the rank of the prize and if the prize is a gain or a loss.

First, Theorem 1 clarifies the choice between prospect theory and cumulative prospect theory. The choice can be reduced to choosing between rank independent cancellation of common branches for lotteries with the same probability tree and the first-order stochastic dominance since these are the two key differences between the models.

[5] Following the extension to simple lotteries by Camerer and Ho (1994) and Fennema and Wakker (1996).

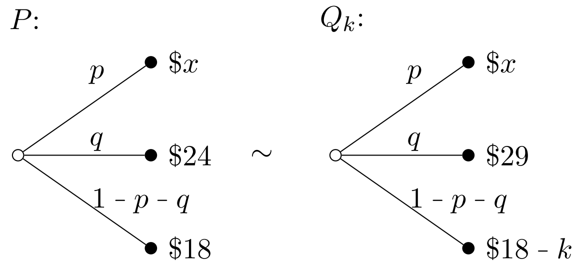

Figure 3. Bernheim and Sprenger's (2020) experiment testing the restricted branch cancellation axiom. Given p, q, and x, the value of k was elicited to make the lotteries indifferent. Bernheim and Sprenger found that the elicitated k does not depend on the value of x in support of the axiom and prospect theory.

Second, Theorem 2 highlights that the lack of event splitting effect in cumulative prospect theory is another key difference between it and prospect theory. The event splitting effect is illustrated in Figure 4. This effect captures the change in the valuation of the lottery after splitting the prize \$20 with probability 60% into two different prizes of $\$20 + \varepsilon$ and $\$20 - \varepsilon$ for a small ε by halving the probability. The weak convergence continuity rules out the discontinuous event splitting effect. This shows that empirical evidence on the event splitting effect that we discuss next also clarifies the differences between prospect theory and cumulative prospect theory.

Next, our Theorems 1 and 2 offer a perspective to the recent experiment by Bernheim and Sprenger (2020) on testing prospect theory and cumulative prospect theory. They compare the models by testing the restricted branch cancellation axiom. Their experiment is illustrated in Figure 3. In the experiment, the probabilities p and q[6] and the prize x were held constant, and the value of k that makes the two lotteries indifferent was elicited. This was repeated for different values of $x \in \{19, 21, 23, 30, 32, 34\}$. The authors found that the elicited k does not depend on the value or the rank of x as assumed by the restricted branch cancellation axiom in contrast to cumulative prospect theory.

Additionally, Bernheim and Sprenger (2020) found evidence of first-order stochastic dominance and weak convergence continuity violations in accord with Theorems 1

[6]The experiment was repeated for $(p, q) \in \{(0.6, 0.3), (0.4, 0.3), (0.1, 0.3)\}$.

and 2. In this experiment, the authors compared the certainty equivalents[7] of the two lotteries in Figure 3 for $\varepsilon = 0.5$. The authors found that the certainty equivalent for Q was on average \$0.47 lower (s.e. 0.11) than the certainty equivalent for P. This strong event splitting effect contradicts first-order stochastic dominance[8] and is suggestive of weak convergence continuity violation. This result provides further indirect evidence in support of the restricted branch cancellation axiom by Theorems 1 and 2.

Based on these findings, Bernheim and Sprenger (2020) suggest that a unified theory of choice under risk should capture rank independence of probability weighting and strong event splitting effect. Theorem 2 shows that these two effects are connected: If the decision-maker cancels common consequences with common probabilities rank independently, then there must be a discontinuous event splitting effect or the decision-maker is an expected utility maximizer.[9] This suggests there might be a common underlying behavioral foundation for these two different empirical findings.

These findings show empirical support for the restricted branch cancellation axiom. Furthermore, our proof sketch in Section 2.2.3 highlighted the technical tractability of the restricted branch cancellation axiom. These observations underscore the validity of models that generalize prospect theory and adhere to the restricted branch cancellation, as they not only find empirical support but also possess theoretical tractability. Consequently, these models warrant additional research and investigation.

4 Conclusion

This paper reconsidered the expected utility. We considered a decision-maker who cancels out common consequences when comparing a pair of lotteries with the same

[7]The certainty equivalent of a lottery P is $x \in X$ such that $\delta_x \sim P$.

[8]Under the weak assumption that increasing the prize \$19.5 with probability 30% by \$0.5 increases the certainty equivalent by less than \$0.47.

[9]Our proof for Theorem 2 uses a weaker continuity axiom, Axiom 6′, than the weak convergence continuity axiom that essentially rules out event splitting effects and has a continuous utility for prizes.

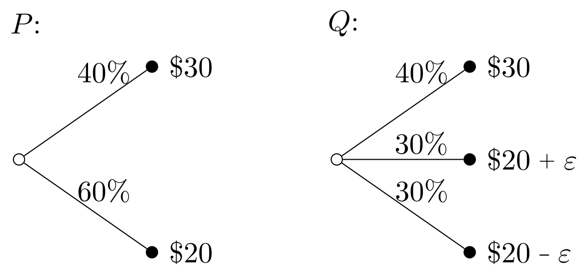

Figure 4. Example of the event splitting effect. The event splitting effect is the change in the valuation of the lottery after splitting the prize $20 into two different prizes of $20+\varepsilon$ and $20-\varepsilon$ by halving the probability. Bernheim and Sprenger (2020) found a strong event splitting effect for $\varepsilon = 0.5$ in contrast to first-order stochastic dominance and weak convergence continuity.

probability tree. We showed that if the preferences either satisfy first-order stochastic dominance or topological continuity on lotteries, then the decision-maker is an expected utility maximizer.

First, this result highlights the importance of understanding the restricted branch cancellation axiom and in which situation it would be satisfied. The models that generalize prospect theory and adhere to the restricted branch cancellation, not only have empirical support as documented by Bernheim and Sprenger (2020) but also are theoretically tractable as we highlighted. Consequently, these models warrant additional research and investigation.

Second, this result illustrates the tension of restricted branch cancellation and first-order stochastic dominance for a theory of non-expected utility since both cannot be satisfied. This highlights the difference between prospect theory and cumulative prospect theory where the first one satisfies restricted branch cancellation and the second one first-order stochastic dominance. Especially, these two assumptions can be used to differentiate behavior between these two models.

Third, this result suggests a potential common behavioral foundation for rank independent probability weighting and the event splitting effect. We showed that a model with rank independent probability weighting is not expected utility if and only if it has a discontinuous event splitting effect. This formalizes the experimental connection between rank independent probability weighting and event splitting effect found in Bernheim and Sprenger (2020). This empirically and theoretically supported connection warrants further research and investigation.

Lastly, we illustrated a novel method to test technical axioms. We showed that the topological continuity assumption has strong and testable behavioral implications under restricted branch cancellation as it rules out e.g. prospect theory. This highlights the critical role of technical assumptions in interpreting observed violations of axioms since they can have testable behavioral implications when combined with behavioral axioms.

Appendix to "Expected Utility Without Linearity: Distinguishing Between Prospect Theory and Cumulative Prospect Theory"

This appendix generalizes the characterization to Borel lotteries and to general prize intervals and includes the main proofs. The online appendix shows the results under weak convergence continuity and approximation of lotteries by dyadic lotteries. This appendix is organized as follows. In Appendix A, we extend our axioms to Borel lotteries and general prize intervals and provide the characterization results. Appendix B provides additional definitions used in the proof. The proof follows in several steps. First, Appendix B.1 shows that first-order stochastic dominance and restricted branch cancellation or weak convergence continuity gives on-diagonal continuity. Second, Appendix B.2 shows that the on-diagonal continuity and restricted branch cancellation axioms give strict first-order stochastic dominance for dyadic lotteries. Third, Appendix B.3 shows expected utility representation for dyadic lotteries. Finally, Appendix B.4 extends the representation for all lotteries under FOSD and shows Theorems 1 and 2. The online appendix shows extension to all lotteries under weak convergence continuity and Theorem 4.

A Expected Utility with Borel Lotteries and General Prize Interval

In this appendix, we reconsider the expected utility in a more general setup with Borel lotteries and general prize intervals. We allow $X \subseteq \mathbb{R}$ to be any interval. Denote by $\Delta^{\text{Bor}}(X)$ the set of all Borel probability measures on X. We consider preferences $\succsim$ on $\Delta^{\text{Bor}}(X)$ and endow $\Delta^{\text{Bor}}(X)$ with the topology of weak convergence. We define simple lotteries with the same induced probability tree as before. Our first axiom is as before that the preferences on the general lotteries are complete, transitive, and monotone on degenerate lotteries.

Axiom 1′ (Weak Order and Monotonicity) $\succsim$ is complete and transitive and for all $x > y$, $\delta_x \succ \delta_y$.

The restricted branch cancellation axiom is as before and only applies to simple lotteries.

Axiom 2′ (Restricted Branch Cancellation) If two simple lotteries $P, Q \in \Delta(X)$ have the same induced probability tree, $x, y \notin \operatorname{supp} P \cup \operatorname{supp} Q$, and $\alpha \in (0,1)$, then

$$\alpha P + (1-\alpha)\delta_x \succsim \alpha Q + (1-\alpha)\delta_x \Longrightarrow \alpha P + (1-\alpha)\delta_y \succsim \alpha Q + (1-\alpha)\delta_y.$$

Next, we generalize prize shifts. For the possibility of compact intervals, we use censored prize shifts that keep the shifted lottery within the closure of the prize interval by censoring. We consider shifting only simple lotteries.

Definition For a simple lottery $P \in \Delta(X)$ and $\varepsilon \in \mathbb{R}$, define $P +_{|X} \varepsilon$ by for all $x \in X$

$$P +_{|X} \varepsilon(x) = \begin{cases} P(x-\varepsilon), & \text{if } x \notin \{\inf X, \sup X\} \\ \displaystyle\sum_{y \in \operatorname{supp} P, y \geq x-\varepsilon} P(y), & \text{if } x = \sup X \\ \displaystyle\sum_{y \in \operatorname{supp} P, y \leq x-\varepsilon} P(y), & \text{if } x = \inf X. \end{cases}$$

Additionally, denote $P -_{|X} \varepsilon = P +_{|X} (-\varepsilon)$.

Here, we are shifting the probabilities of the prizes of P by ε. Hence, the probability of the prize x for $P +_{|X} \varepsilon$ after shifting is the probability of the prize $x - \varepsilon$ for P. Additionally, the shifted lottery is censored above $\sup X$ and below $\inf X$ by rectifying the shifted values at $\sup X$ and $\inf X$ respectively.

First, if X is an open interval, then for a small enough ε, we have $P + \varepsilon = P +_{|X} \varepsilon$. Second, for any interval X, we have for a small enough ε, $P +_{|X} \varepsilon \in \Delta(X)$. Third, for a small enough of ε, at most either the best prize or the worst prize of the lottery is censored. The next axiom generalizes our shift continuity to a general interval and Borel lotteries using the censored shifts for simple lotteries.

Axiom 3′ (Censored Prize Shift Continuity) For all simple lotteries $P \in \Delta(X)$ and Borel lotteries $Q \in \Delta^{\text{Bor}}(X)$, if $P \succ Q$, then there exists $\varepsilon > 0$ such that

$$P -_{|X} \varepsilon \in \Delta(X) \text{ and } P -_{|X} \varepsilon \succ Q$$

and if $Q \succ P$, then there exists $\varepsilon > 0$ such that

$$P +_{|X} \varepsilon \in \Delta(X) \text{ and } Q \succ P +_{|X} \varepsilon.$$

Without Borel lotteries, for an open interval of prizes, this is equivalent to our simple prize shift continuity, Axiom 3.

Finally, we generalize first-order stochastic dominance to apply for all the lotteries.

Axiom 4′ (FOSD) For all lotteries $P, Q \in \Delta^{\text{Bor}}(X)$, if P first-order stochastically dominates Q, then $P \succsim Q$.

These axioms provide the first characterization for the expected utility. For the second characterization, we generalize weak convergence continuity to apply for all the lotteries.

Axiom 5′ (Weak Convergence Continuity) For each $Q \in \Delta^{\text{Bor}}(X)$, the sets $\{P \in \Delta^{\text{Bor}}(X) | P \succsim Q\}$ and $\{P \in \Delta^{\text{Bor}}(X) | Q \succsim P\}$ are closed.

In this general context, our first result considers the case when X is a compact interval. In this case, weak order, restricted branch cancellation and either weak convergence continuity or censored prize shift continuity and FOSD are equivalent to preferences having an expected utility representation for all the lotteries. Since X is compact, the continuous utility u will be bounded.

Theorem 3 Assume that X is a compact interval. Then the following three conditions are equivalent:

(1) $\succsim$ satisfies Axioms 1′-4′.

(2) $\succsim$ satisfies Axioms 1′, 2′, and 5′.

(3) There exists continuous and strictly increasing $u : X \to \mathbb{R}$ such that for all lotteries $P, Q \in \Delta^{\text{Bor}}(X)$,

$$P \succsim Q \iff \int_X u\, dP \geq \int_X u\, dQ.$$

Second, we consider the case of a general interval. First, in this case, the results under weak convergence continuity remain but give a bounded utility function.

Theorem 4 Assume that X is an interval. Then the following two conditions are equivalent:

(1) $\succsim$ satisfies Axioms $1'$, $2'$, and $5'$.

(2) There exists continuous, strictly increasing, and bounded $u : X \to \mathbb{R}$ such that for all lotteries $P, Q \in \Delta^{\text{Bor}}(X)$,

$$P \succsim Q \iff \int_X u\, dP \geq \int_X u\, dQ.$$

Lastly, with FOSD and general interval, we only get a representation for the simple lotteries due to unbounded utilities.

Theorem 5 Assume that X is an interval. If $\succsim$ satisfies Axioms $1'$-$4'$, then there exists continuous and strictly increasing $u : X \to \mathbb{R}$ such that for all simple lotteries $P, Q \in \Delta(X)$,

$$P \succsim Q \iff \int_X u\, dP \geq \int_X u\, dQ.$$

B Proof for Expected Utility with Borel Lotteries and General Prize Interval

We start off with definitions. First, we say that a simple probability on indices $p \in \Delta(\mathbb{N})$ is *a probability tree.*

Second, for a probability tree $p \in \Delta(\mathbb{N})$, denote *the set of possible prizes on the probability tree* by

$$X^{\operatorname{supp} p} = \left\{ (x_i)_{i \in \operatorname{supp} p} \middle| \text{for all } i \in \operatorname{supp} p, x_i \in X \right\}.$$

We endow $X^{\operatorname{supp} p}$ with the Euclidean product topology. Third, for a probability tree $p \in \Delta(\mathbb{N})$, denote *the off-diagonal of* $X^{\operatorname{supp} p}$ by

$$\mathrm{NDiag}(X^p) = \{x \in X^{\operatorname{supp} p} | \, \forall\, i, j \in \operatorname{supp} p, i \neq j, x_i \neq x_j\}.$$

$\mathrm{NDiag}(X^p)$ is the set of all the possible prizes on the probability tree $x \in X^{\operatorname{supp} p}$ such that the induced probability tree of the lottery $\sum_{i \in \operatorname{supp} p} p_i \delta_{x_i}$ is p.

In our proof we show the expected utility first for dyadic lotteries. We say that a lottery P is *dyadic* if P can be represented with a uniform probability tree with 2^n outcomes for some n. That is, formally, for some $n \in \mathbb{N}_0$ and $x \in X^{2^n}$, $P = \sum_{i=1}^{2^n} 2^{-n} \delta_{x_i}$.

In our proof, we focus on the following on-diagonal continuity for lotteries with the same induced probability tree. In the online appendix, we show that this continuity is satisfied by restricted branch cancellation preferences under FOSD or topological continuity. This continuity unifies their proofs. This is continuity when the limit prizes might be on-diagonal but the converging sequence is off-diagonal. This rules out discontinuous event splitting effect and assumes continuous utility on prizes.

Axiom 6′ (On-Diagonal Continuity) For all lotteries $Q \in \Delta^{\mathrm{Bor}}(X)$ and probability trees $p \in \Delta(\mathbb{N})$, if $(x^i)_{i=1}^{\infty} \subseteq \mathrm{NDiag}(X^p)$ converges to $x \in X^{\operatorname{supp} p}$ and if for all $i \in \mathbb{N}$,

$$\sum_{l \in \operatorname{supp} p} p_l \delta_{x_l^i} \succsim Q, \text{ then } \sum_{l \in \operatorname{supp} p} p_l \delta_{x_l} \succsim Q$$

and if for all $i \in \mathbb{N}$,

$$Q \succsim \sum_{l \in \operatorname{supp} p} p_l \delta_{x_l^i}, \text{ then } Q \succsim \sum_{l \in \operatorname{supp} p} p_l \delta_{x_l}.$$

B.1 On-Diagonal Continuity

In this section, we show that our on-diagonal continuity, Axiom 6′, is implied by first-order stochastic dominance and prize shift continuity and it is implied by weak convergence continuity. Additionally, we show that in the on-diagonal continuity axiom, the off-diagonal converging sequence can be also on-diagonal.

First, we show that weak convergence continuity gives on-diagonal continuity. We omit this standard proof.

Lemma 6 If $\succsim$ satisfies Axioms $1'$ and $5'$, then $\succsim$ satisfies Axiom $6'$.

Next, we show that prize shift continuity and FOSD give on-diagonal continuity.

Lemma 7 If $\succsim$ satisfies Axioms $1'$, $3'$, and $4'$, then $\succsim$ satisfies Axiom $6'$.

Proof. We show the continuity of upper contour sets. The continuity for lower contour sets follows symmetrically.

Let $Q \in \Delta^{\mathrm{Bor}}(X), p \in \Delta(\mathbb{N}), (x^i)_{i=1}^{\infty} \subseteq \mathrm{NDiag}(X^p)$ be a convergent sequence to $x \in X^{\operatorname{supp} p}$ such that for all $i \in \mathbb{N}$,

$$\sum_{l \in \operatorname{supp} p} p_l \delta_{x_l^i} \succsim Q. \tag{1}$$

Assume, per contra, that $Q \succ \sum_{l \in \operatorname{supp} p} p_l \delta_{x_l}$. For each $l \in \operatorname{supp} p$, define an indicator $\chi(x_l < \sup X)$. By Axiom $3'$, there exists $\varepsilon > 0$ such that

$$\varepsilon < \inf\left\{ |x_l - \sup X| \middle| l \in \operatorname{supp} p, x_l < \sup X \right\} \text{ and } Q \succ \sum_{l \in \operatorname{supp} p} p_l \delta_{x_l + \chi(x_l < \sup X)\varepsilon}, \tag{2}$$

where $\inf \varnothing = \infty$, since $\operatorname{supp} p$ is finite.

Since $x^i \to x$ as $i \to \infty$, there exists $i^0 \in \mathbb{N}$ such that $\left\| x^{i^0} - x \right\| < \varepsilon$ and so for all $l \in \operatorname{supp} p$ such that $x_l < \sup X$, we have

$$x_l + \varepsilon = x_l^{i^0} + \varepsilon + x_l - x_l^{i^0} \geq x_l^{i^0} + \varepsilon - \left| x_l^{i^0} - x_l \right| \geq x_l^{i^0} + \varepsilon - \left\| x^{i^0} - x \right\| > x_l^{i^0}.$$

Additionally, for all $l \in \operatorname{supp} p$ such that $x_l = \sup X$, we have

$$x_l \geq x_l^{i^0}.$$

Thus, $\sum_{l \in \operatorname{supp} p} p_l \delta_{x_l + \chi_l(x_l < \sup X)\varepsilon}$ first-order stochastically dominates $\sum_{l \in \operatorname{supp} p} p_l \delta_{x_l^{i^0}}$ and so by Axiom $4'$,

$$\sum_{l \in \operatorname{supp} p} p_l \delta_{x_l + \chi_l(x_l < \sup X)\varepsilon} \succsim \sum_{l \in \operatorname{supp} p} p_l \delta_{x_l^{i^0}}.$$

Thus especially, we have

$$Q \overset{(2)}{\succ} \sum_{l \in \operatorname{supp} p} p_l \delta_{x_l + \chi(x_l < \sup X)\varepsilon} \succsim \sum_{l \in \operatorname{supp} p} p_l \delta_{x_l^{i^0}}$$

which contradicts (1). $\square$

The next result shows that on-diagonal continuity gives continuity in prizes for converging sequences that might be on-diagonal and do not have the same induced probability tree.

Lemma 8 Assume that $\succsim$ satisfies Axioms $1'$ and $6'$. For all lotteries $Q \in \Delta^{\text{Bor}}(X)$, probability trees $p \in \Delta(\mathbb{N})$, and convergent sequences $(x^i)_{i=1}^{\infty} \subseteq X^{\operatorname{supp} p}$ with limit $x \in X^{\operatorname{supp} p}$, if for all $i \in \mathbb{N}$,

$$\sum_{l \in \operatorname{supp} p} p_l \delta_{x_l^i} \succsim Q, \text{ then } \sum_{l \in \operatorname{supp} p} p_l \delta_{x_l} \succsim Q$$

and if for all $i \in \mathbb{N}$,

$$Q \succsim \sum_{l \in \operatorname{supp} p} p_l \delta_{x_l^i}, \text{ then } Q \succsim \sum_{l \in \operatorname{supp} p} p_l \delta_{x_l}.$$

Proof. We show the continuity of upper contour sets. The continuity for lower contour sets follows symmetrically. Assume that $p \in \Delta(\mathbb{N})$ and $(x^i)_{i=1}^{\infty} \subseteq X^{\operatorname{supp} p}$ with limit $x \in X^{\operatorname{supp} p}$ is such that for all $i \in \mathbb{N}$,

$$\sum_{l \in \operatorname{supp} p} p_l \delta_{x_l^i} \succsim Q. \tag{3}$$

For each $i \in \mathbb{N}$, define

$$A^i = \Big\{ \{ l \in \operatorname{supp} p | x_l^i = x_k^i \} \Big| k \in \operatorname{supp} p \Big\}.$$

Each A^i is a partition of $\operatorname{supp} p$. Thus there exists a subsequence $(x_{i_j})_{j=1}^{\infty}$ such that for each j, k, $A^{i_j} = A^{i_k}$. Denote $A^{i_1} = A$. Let $\pi : \{1, \dots, |A|\} \to A$ be a one-to-one mapping and $\theta : A \to \mathbb{N}$ be a selection function.

Define $q \in \Delta(\mathbb{N})$ by for all $1 \leq i \leq |A|$

$$q_i = \sum_{l \in \pi(i)} p_l$$

and 0 otherwise. Now for each $j \in \mathbb{N}$

$$\sum_{l \in \operatorname{supp} p} p_l \delta_{x_l^{i_j}} = \sum_{k \in \operatorname{supp} q} q_k \delta_{x^{i_j}_{\theta(\pi(k))}}$$

and $(x^{i_j}_{\theta(\pi(k))})_{k \in \operatorname{supp} q} \in \operatorname{NDiag}(X^q)$. Thus by Axiom $6'$ and (3)

$$\sum_{l \in \operatorname{supp} p} p_l \delta_{x_l} = \sum_{k \in \operatorname{supp} q} q_k \delta_{x_{\theta(\pi(k))}} \succsim Q.$$

□

Lastly, we show a simple lemma stating that we can take limits of both sides of preferences simultaneously. The proof is standard and omitted.

Lemma 9 Assume that $\succsim$ satisfies Axioms 1′ and 6′. For all probability trees $p \in \Delta(\mathbb{N})$ and convergent sequences $(x^i)_{i=1}^{\infty}, (y^i)_{i=1}^{\infty} \subseteq X^{\operatorname{supp} p}$ with limits $x, y \in X^{\operatorname{supp} p}$ respectively, if for all $i \in \mathbb{N}$,

$$\sum_{l \in \operatorname{supp} p} p_l \delta_{x_l^i} \succsim \sum_{l \in \operatorname{supp} p} p_l \delta_{y_l^i}$$

then

$$\sum_{l \in \operatorname{supp} p} p_l \delta_{x_l} \succsim \sum_{l \in \operatorname{supp} p} p_l \delta_{y_l}.$$

B.2 Monotonicity on Dyadic Lotteries

This section shows that under the monotonicity on certain amounts of money, on-diagonal continuity, and restricted branch cancellation axioms, Axioms 1′, 2′, and 6′, lotteries with a dyadic probability tree satisfy first-order stochastic dominance. We will split the proof of this result into several lemmas and prove it by induction.

Proposition 10 Assume that $\succsim$ satisfes Axioms 1′, 2′, and 6′ and $n \in \mathbb{N}$. For all $x, y \in \operatorname{int} X^{2^n}$, if for all $1 \leq i \leq 2^n$, $x_i \geq y_i$, then

$$\sum_{i=1}^{2^n} 2^{-n} \delta_{x_i} \succsim \sum_{i=1}^{2^n} 2^{-n} \delta_{y_i}$$

and additionally if for some $1 \leq i^0 \leq 2^n$, $x_{i^0} > y_{i^0}$, then

$$\sum_{i=1}^{2^n} 2^{-n} \delta_{x_i} \succ \sum_{i=1}^{2^n} 2^{-n} \delta_{y_i}.$$

Proof. We show the claim by induction on n. For $n = 0$, we have by Axiom 1′, for all $x, y \in \operatorname{int} X$, $x \geq y$ iff $\delta_x \succsim \delta_y$. Next assume that we have shown the claim for all $0 \leq m \leq n-1$. Denote $p = (2^{-n})_{i=1}^{2^n}$ and for all $x, y \in \operatorname{int} X^{2^n}$,

$$x \succsim^p y \iff \sum_{i=1}^{2^n} 2^{-n} \delta_{x_i} \succsim \sum_{i=1}^{2^n} 2^{-n} \delta_{y_i}.$$

We will show by a second induction that for each $k \in \{0, \dots, 2^{n-1}\}$, we have for all $x \in \operatorname{int} X^{2^n}$ such that for all $l \in \{k, \dots, 2^{n-1} - 1\}$,

$$x_{2l+1} = x_{2l+2},$$

and for all $i \in \{1, \dots, 2k\}, a_i, b_i \in \operatorname{int} X$ such that $a_i > b_i$,

$$(a_i, x_{-i}) \succ^p (b_i, x_{-i}). \tag{4}$$

First, for $k = 0$, the claim holds trivially. So assume that the claim holds for $0 \leq k - 1 \leq 2^{n-1}$.

We will split the induction into several lemmas. In the following lemmas, we maintain that $\succsim$ satisfies Axioms $1'$, $2'$, and $6'$, n and k are the induction indices, and the induction assumptions hold for all $j \leq k - 1$ and $m \leq n - 1$.

First, we show that $\succsim^p$ is symmetric.

Lemma 11 If $x \in \operatorname{int} X^{2^n}$, $a, b \in \operatorname{int} X$, and $i, j \in \{1, \dots, 2^n\}$, then

$$(a_i, b_j, x_{-i,j}) \sim^p (b_i, a_j, x_{-i,j}).$$

Proof. Follows from the identity

$$\sum_{l=1}^{2^n} 2^{-n} \delta_{(a_i, b_j, x_{-i,j})_l} = \sum_{l=1}^{2^n} 2^{-n} \delta_{(a_j, b_i, x_{-i,j})_l}.$$

□

Next, we show strict monotonicity in $2k - 1, 2k$.

Lemma 12 If $x \in \operatorname{int} X^{2^n}$ and $a, b \in \operatorname{int} X$ are such that for all $l \in \{k, \dots, 2^{n-1} - 1\}$, $x_{2l+1} = x_{2l+2}$ and $b > a$, then

$$(b_{2k-1,2k}, x_{-2k-1,2k}) \succ^p (a_{2k-1,2k}, x_{-2k-1,2k}).$$

Proof. First, if $k = 1$, then the claim follows by the induction assumption for $n - 1$. So assume that $k > 1$. Denote $c = x_{2k-3}$, $d = x_{2k-2}$, and $A = \{2k-3, 2k-2, 2k-1, 2k\}$.

We have by Lemma 11 and the induction assumption (4)

$$(c_{2k-3}, c_{2k-2}, b_{2k-1}, b_{2k}, x_{-A}) \sim^p (b_{2k-3}, b_{2k-2}, c_{2k-1}, c_{2k}, x_{-A})$$
$$\succ^p (a_{2k-3}, a_{2k-2}, c_{2k-1}, c_{2k}, x_{-A}) \sim^p (c_{2k-3}, c_{2k-2}, a_{2k-1}, a_{2k}, x_{-A}).$$

By Lemma 8 and since $c \in \operatorname{int} X$, there exists $\varepsilon > 0$ such that $c + \varepsilon \in \operatorname{int} X$ and

$$(c_{2k-3}, c_{2k-2}, b_{2k-1,2k}, x_{-A}) \succ^p ((c+\varepsilon)_{2k-3}, c_{2k-2}, a_{2k-1,2k}, x_{-A}).$$

Now let $(a^j)_{j=1}^\infty, (b^j)_{j=1}^\infty, (c^j)_{j=1}^\infty, (e^j)_{j=1}^\infty \subseteq \operatorname{int} X$ and $(x^j)_{j=1}^\infty$ be such that for all $j \in \mathbb{N}$ and $l \in \operatorname{supp} p \setminus A$,

$$\{x_l^j\} \cap \{a^j, b^j, c^j, e^j, c, d\} = \{a^j, b^j\} \cap \{c^j, e^j, c, d\} = \{c^j, e^j\} \cap \{c, d\} = \varnothing \tag{5}$$

and

$$\lim_{j\to\infty} a^j = a, \lim_{j\to\infty} b^j = b, \lim_{j\to\infty} c^j = c, \lim_{j\to\infty} e^j = c+\varepsilon, \lim_{j\to\infty} x^j = x.$$

By Lemma 9, there exists $j^0 \in \mathbb{N}$ such that for all $j > j^0$,

$$(c_{2k-3}^j, c_{2k-2}, b_{2k-1,2k}^j, x_{-A}^j) \succ^p (e_{2k-3}^j, c_{2k-2}, a_{2k-1,2k}^j, x_{-A}^j).$$

By Axiom 2′ and (5),

$$(c_{2k-3}^j, d_{2k-2}, b_{2k-1,2k}^j, x_{-A}^j) \succ^p (e_{2k-3}^j, d_{2k-2}, a_{2k-1,2k}^j, x_{-A}^j).$$

By Lemma 9, we have by taking $j \to \infty$,

$$(c_{2k-3}, d_{2k-2}, b_{2k-1,2k}, x_{-A}) \succsim^p ((c+\varepsilon)_{2k-3}, d_{2k-2}, a_{2k-1,2k}, x_{-A}).$$

Finally, by the induction assumption (4), we have the claim

$$(c_{2k-3}, d_{2k-2}, b_{2k-1,2k}, x_{-A}) \succ^p (c_{2k-3}, d_{2k-2}, a_{2k-1,2k}, x_{-A}).$$

□

The next result shows that $\succsim^p$ is weakly monotonic in a single coordinate.

Lemma 13 If $x \in \operatorname{int} X^{2^n}$, $i \in \{1, \ldots, 2k\}$, i odd, and $a, b, c \in \operatorname{int} X$ are such that for all $l \in \{k, \ldots, 2^{n-1} - 1\}$, $x_{2l+1} = x_{2l+2}$ and $c > b$, then

$$(a_i, c_{i+1}, x_{-i,i+1}) \succsim^p (a_i, b_{i+1}, x_{-i,i+1}).$$

Proof. First, by Lemma 12, we have after switching the indices $(i, i+1)$ and $(2k-1, 2k)$ by Lemma 11

$$(c_{i,i+1}, x_{-i,i+1}) \succ^p (b_{i,i+1}, x_{-i,i+1}). \tag{6}$$

Assume, per contra, that

$$(a_i, b_{i+1}, x_{-i,i+1}) \succ^p (a_i, c_{i+1}, x_{-i,i+1}).$$

Let $(b^j)_{j=1}^{\infty}, (c^j)_{j=1}^{\infty} \subseteq \operatorname{int} X$ and $(x^j)_{j=1}^{\infty} \subseteq \operatorname{NDiag}(X^p)$ be such that for all $j \in \mathbb{N}$ and $l \in \operatorname{supp} p \setminus \{i, i+1\}$,

$$\{x_l^j\} \cap \{b^j, c^j, a, b, c\} = \varnothing, \{b^j, c^j\} \cap \{a, b, c\} = \varnothing, \tag{7}$$

and

$$\lim_{j\to\infty} b^j = b, \lim_{j\to\infty} c^j = c, \lim_{j\to\infty} x^j = x.$$

By Lemma 9, there exists $j^0 \in \mathbb{N}$ such that for all $j > j^0$,

$$(a_i, b_{i+1}^j, x_{-i,i+1}^j) \succ^p (a_i, c_{i+1}^j, x_{-i,i+1}^j).$$

For all $j > j^0$, $(a_i, b_{i+1}^j, x_{-i,i+1}^j), (a_i, c_{i+1}^j, x_{-i,i+1}^j) \in \operatorname{NDiag}(X^p)$ and so by Axiom 2′ and (7),

$$(b_i, b_{i+1}^j, x_{-i,i+1}^j) \succ^p (b_i, c_{i+1}^j, x_{-i,i+1}^j) \,\&\, (c_i, b_{i+1}^j, x_{-i,i+1}^j) \succ^p (c_i, c_{i+1}^j, x_{-i,i+1}^j).$$

By Lemma 9, we have by taking $j \to \infty$,

$$(b_i, b_{i+1}, x_{-i,i+1}) \succsim^p (b_i, c_{i+1}, x_{-i,i+1}) \,\&\, (c_i, b_{i+1}, x_{-i,i+1}) \succsim^p (c_i, c_{i+1}, x_{-i,i+1}).$$

Finally, by Lemma 11,

$$(b_i, b_{i+1}, x_{-i,i+1}) \succsim^p (b_i, c_{i+1}, x_{-i,i+1}) \sim^p (c_i, b_{i+1}, x_{-i,i+1}) \succsim^p (c_i, c_{i+1}, x_{-i,i+1}).$$

But this contradicts (6). □

The next result shows strict monotonicity for a single coordinate when $n = 1$.

Lemma 14 If $n = 1$ and $a, b, c \in \operatorname{int} X$ are such that $c > b$, then

$$(a_1, c_2) \succ^p (a_1, b_2).$$

Proof. First, we have by Axiom 1′,

$$(c_{1,2}) \succ^p (b_{1,2}). \tag{8}$$

Assume, per contra, that

$$(a_1, b_2) \succsim^p (a_1, c_2).$$

By Lemma 13, we have

$$(a_1, b_2) \sim^p (a_1, c_2). \tag{9}$$

Let $(c^j)_{j=1}^{\infty}, (b^j)_{j=1}^{\infty} \subseteq \operatorname{int} X$ be such that for all $j \in \mathbb{N}$, $c > c^j > b^j > b$,

$$c^j \neq b^j, b^j \neq a \neq c^j,$$

$\lim_{j\to\infty} b^j = b$, and $\lim_{j\to\infty} c^j = c$. By Lemma 13, we have

$$(a_1, c_2) \succsim^p (a_1, c_2^j) \succsim^p (a_1, b_2^j) \succsim^p (a_1, b_2)$$

and so by (9)

$$(a_1, b_2^j) \sim^p (a_1, c_2^j). \tag{10}$$

By Axiom 2′ and (10), we have for all $j \in \mathbb{N}$, by

$$(b_1, b_2^j) \sim^p (b_1, c_2^j) \text{ and } (c_1, b_2^j) \sim^p (c_1, c_2^j)$$

By taking $j \to \infty$, we have by Lemma 9,

$$(b_1, b_2) \sim^p (b_1, c_2) \text{ and } (c_1, b_2) \sim^p (c_1, c_2).$$

So by Lemma 11,

$$(b_1, b_2) \sim^p (b_1, c_2) \sim^p (c_1, b_2) \sim^p (c_1, c_2),$$

which contradicts (8). □

If $n = 1$, this completes the induction for (4). We assume for the rest of the proof that $n > 1$.

Next, we show strict monotonicity in uniform improvements for all except two indices in $\{1, \ldots, 2k\}$ with the same value.

Lemma 15 If $n > 1$, $x \in \operatorname{int} X^{2^n}$, $i \in \{1, \ldots, 2k\}$, i odd, $a \in \operatorname{int} X$, and $\varepsilon > 0$ are such that for all $l \in \{k, \ldots, 2^{n-1} - 1\}$, $x_{2l+1} = x_{2l+2}$ and $x + \varepsilon \in \operatorname{int} X^{2^n}$, then

$$\left(a_{i,i+1}, (x + \varepsilon)_{-i,i+1}\right) \succ^p (a_{i,i+1}, x_{-i,i+1}).$$

Proof. First, if $k = 1$, then by (4) the induction assumption for $n - 1$ shows the claim directly.

Second, assume that $k > 1$. Denote $A = \{1, \ldots, 2k\} \setminus \{i, i+1\}$. By Lemma 11, (4), and $k > 1$, we have if $i \neq 2k-1$,

$$
\begin{aligned}
&\Big(a_{i,i+1}, (x+\varepsilon)_{-i,i+1}\Big) \\
&\sim^p \Big((x_{2k-1}+\varepsilon)_i, (x_{2k}+\varepsilon)_{i+1}, a_{2k-1,2k}, (x+\varepsilon)_{A\setminus\{2k,2k-1\}}, (x+\varepsilon)_{-A,2k-1,2k,i,i+1}\Big) \\
&\succ^p (x_{2k-1})_i, (x_{2k})_{i+1}, a_{2k-1,2k}, x_{A\setminus\{2k,2k-1\}}, (x+\varepsilon)_{-A,2k-1,2k,i,i+1}\Big) \\
&\sim^p \Big(a_{i,i+1}, x_A, (x+\varepsilon)_{-A,i,i+1}\Big)
\end{aligned}
$$

the case $i = 2k-1$ follows directly from (4).

Next, by Lemmas 12 and 13, for all $2k < j \leq 2^n$, j odd, we have if $i \neq 2k-1$,

$$
\begin{aligned}
&(a_{i,i+1}, x_A, (x+\varepsilon)_{-i,i+1,A}) \\
&\sim^p \Big((x_{2k-1})_i, (x_{2k})_{i+1}, a_{j,j+1}, (x_j+\varepsilon)_{2k-1}, (x_{j+1}+\varepsilon)_{2k}, x_A, (x+\varepsilon)_{-i,i+1,j,j+1,A}\Big) \\
&\succ^p \Big((x_{2k-1})_i, (x_{2k})_{i+1}, a_{j,j+1}, (x_j)_{2k-1}, (x_{j+1})_{2k}, x_A, (x+\varepsilon)_{-i,i+1,j,j+1,A}\Big) \\
&\sim^p (a_{i,i+1}, x_{A,j,j+1}, (x+\varepsilon)_{-i,i+1,j,j+1,A}).
\end{aligned}
$$

The case $i = 2k-1$ follows symmetrically by directly switching the positions of $\{2k-1, 2k\}$ and $\{j, j+1\}$.

By repeating this for all $k \leq j \leq 2^n$, j odd, we have

$$\Big(a_{i,i+1}, (x+\varepsilon)_{-i,i+1}\Big) \succsim^p \Big(a_{i,i+1}, x_{-i,i+1}\Big).$$

□

Next, we show weak monotonicity in uniform improvements for all except two indices in $\{1, \ldots, 2k\}$ with different values.

Lemma 16 If $n > 1$, $x \in \operatorname{int} X^{2^n}$, $i \in \{1, \ldots, 2k\}$, i odd, $a, b \in \operatorname{int} X$, and $\varepsilon > 0$ are such that for all $l \in \{k, \ldots, 2^{n-1}-1\}$, $x_{2l+1} = x_{2l+2}$ and $x + \varepsilon \in \operatorname{int} X^{2^n}$, then

$$\Big(b_i, a_{i+1}, (x+\varepsilon)_{-i,i+1}\Big) \succsim^p (b_i, a_{i+1}, x_{-i,i+1}).$$

Proof. Next, let $(a^j)_{j=1}^{\infty}, (\tilde{a}^j)_{j=1}^{\infty}, (b^j)_{j=1}^{\infty} \subseteq \operatorname{int} X$ be such that for all $j \in \mathbb{N}$ and $l \in \operatorname{supp} p \setminus \{i, i+1\}$,

$$\{a^j, \tilde{a}^j, b^j\} \cap \{x_l, x_l + \varepsilon\} = \varnothing, \tilde{a}^j \neq b^j, \tag{11}$$

$$\lim_{j\to\infty} a^j = \lim_{j\to\infty} \tilde{a}^j = a, \text{ and } \lim_{j\to\infty} b^j = b.$$

By Lemmas 13 and 15,

$$(a_i, a_{i+1}, (x+\varepsilon)_{-i,i+1}) \succ^p (a_i, a_{i+1}, x_{-i,i+1}).$$

By Lemma 9, there exist $j^0 \in \mathbb{N}$ such that for all $j > j^0$,

$$(a_i^j, \tilde{a}_{i+1}^j, (x+\varepsilon)_{-i,i+1}) \succ^p (a_i^j, \tilde{a}_{i+1}^j, x_{-i,i+1}).$$

For each $j > j^0$,

$$\sum_{i=1}^{2^n} 2^{-n} \delta_{(a_i^j, \tilde{a}_{i+1}^j, (x+\varepsilon)_{-i,i+1})_i} \text{ and } \sum_{i=1}^{2^n} 2^{-n} \delta_{(a_i^j, \tilde{a}_{i+1}, x_{-i,i+1})_i}$$

have the same induced probability tree by (11) and since $(x+\varepsilon)_{-i,i+1}$ is a shift of $x_{-i,i+1}$ so by Axiom 2$'$ and (11),

$$(b_i^j, \tilde{a}_{i+1}^j, (x+\varepsilon)_{-i,i+1}) \succ^p (b_i^j, \tilde{a}_{i+1}^j, x_{-i,i+1}).$$

Finally, by Lemma 9, we have by taking $j \to \infty$,

$$(b_i, a_{i+1}, (x+\varepsilon)_{-i,i+1}) \succsim^p (b_i, a_{i+1}, x_{-i,i+1}).$$

□

Next, we show that the previous weak increase in uniform improvements for all except two indices are strict increases.

Lemma 17 If $n > 1$, $x \in \operatorname{int} X^{2^n}$, $i \in \{1, \ldots, 2k\}$, i odd, $a, b \in \operatorname{int} X$, and $\varepsilon > 0$ are such that for all $l \in \{k, \ldots, 2^{n-1} - 1\}$, $x_{2l+1} = x_{2l+2}$ and $x + \varepsilon \in \operatorname{int} X^{2^n}$, then

$$(b_i, a_{i+1}, (x+\varepsilon)_{-i,i+1}) \succ^p (b_i, a_{i+1}, x_{-i,i+1}).$$

Proof. Assume, per contra, by Lemma 16,

$$(a_i, b_{i+1}, (x+\varepsilon)_{-i,i+1}) \sim^p (a_i, b_{i+1}, x_{-i,i+1}).$$

By Lemma 16, for all $0 \le \varepsilon' \le \varepsilon$, we have

$$(a_i, b_{i+1}, (x+\varepsilon)_{-i,i+1}) \succsim^p (a_i, b_{i+1}, (x+\varepsilon')_{-i,i+1}) \succsim^p (a_i, b_{i+1}, x_{-i,i+1}). \quad (12)$$

Especially, there exists $0 < \varepsilon' < \varepsilon'' < \varepsilon$ such that for all $l \in \operatorname{supp} p \setminus \{i, i+1\}$,

$$\{a, b\} \cap \{x_l + \varepsilon', x_l + \varepsilon''\} = \varnothing, \quad (13)$$

Let $(b^j)_{j=1}^{\infty} \subseteq \operatorname{int} X$ be such that for all $j \in \mathbb{N}$ and $l \in \operatorname{supp} p \setminus \{i, i+1\}$,

$$\{b^j\} \cap \{a, b, x_l + \varepsilon', x_l + \varepsilon''\} = \varnothing \text{ and } \lim_{j \to \infty} b^j = b. \tag{14}$$

Next,

$$\sum_{l=1}^{2^n} 2^{-n} \delta_{(a_i, b_{i+1}, (x+\varepsilon'')_{-i,i+1})_l} \text{ and } \sum_{l=1}^{2^n} 2^{-n} \delta_{(a_i, b_{i+1}, (x+\varepsilon')_{-i,i+1})_l}$$

have the same induced probability tree since $(x + \varepsilon'')_{-i,i+1}$ is a shift of $(x + \varepsilon')_{-i,i+1}$ and by (13). So by Axiom 2′ and (12,14), for all $j \in \mathbb{N}$,

$$\Big(b_i^j, b_{i+1}, (x + \varepsilon'')_{-i,i+1}\Big) \sim^p \Big(b_i^j, b_{i+1}, (x + \varepsilon')_{-i,i+1}\Big).$$

By taking $j \to \infty$, we have by Lemma 9,

$$\Big(b_{i,i+1}, (x + \varepsilon'')_{-i,i+1}\Big) \sim^p \Big(b_{i,i+1}, (x + \varepsilon')_{-i,i+1}\Big).$$

However, this contradicts Lemma 15. □

Next, we finally show strict monotonicity for a single coordinate.

Lemma 18 If $n > 1$, $x \in \operatorname{int} X^{2^n}$ and $i \in \{1, \dots, 2k\}$, i odd, and $a, b, c \in \operatorname{int} X$ are such that for all $l \in \{k, \dots, 2^{n-1} - 1\}$, $x_{2l+1} = x_{2l+2}$ and $c > b$, then

$$(a_i, c_{i+1}, x_{-i,i+1}) \succ^p (a_i, b_{i+1}, x_{-i,i+1}).$$

Proof. We consider two cases. First, assume that

$$(c_i, c_{i+1}, x_{-i,i+1}) \succ^p (b_i, c_{i+1}, x_{-i,i+1}).$$

By Lemma 8 and since $x \in \operatorname{int} X^{2^n}$, there exists $\varepsilon > 0$ such that for all $l \in \operatorname{supp} p$, $x_l + \varepsilon \in \operatorname{int} X$ and

$$(c_i, c_{i+1}, x_{-i,i+1}) \succ^p (b_i, c_{i+1}, (x + \varepsilon)_{-i,i+1}).$$

Let $(b^j)_{j=1}^{\infty}, (c^j)_{j=1}^{\infty} \subseteq \operatorname{int} X$ and $(x^j)_{j=1}^{\infty}, (y^j)_{j=1}^{\infty} \subseteq \operatorname{int} X^{2^n}$ be such that

$$(x^j)_{j=1}^{\infty}, (y^j)_{j=1}^{\infty} \subseteq \operatorname{NDiag}(X^p),$$

for all $j \in \mathbb{N}$ and $l \in \operatorname{supp} p \setminus \{i, i+1\}$,

$$\{x_l^j, y_l^j\} \cap \{b^j, c^j, a, b, c\} = \varnothing, \{b^j, c^j\} \cap \{a, b, c\} = \varnothing, \tag{15}$$

and

$$\lim_{j\to\infty} b^j = b, \lim_{j\to\infty} c^j = c_i, \lim_{j\to\infty} x^j = x, \lim_{j\to\infty} y^j = x + \varepsilon.$$

By Lemma 9, there exists $j^0 \in \mathbb{N}$ such that for all $j > j^0$,

$$(c_i^j, c_{i+1}, x_{-i,i+1}^j) \succ^p (b_i^j, c_{i+1}, y_{-i,i+1}^j).$$

For all $j > j^0$, $(c_i^j, c_{i+1}, x_{-i,i+1}^j), (b_i^j, c_{i+1}, y_{-i,i+1}^j) \in \mathrm{NDiag}(X^p)$ and so by Axiom 2′ and (15),

$$(c_i^j, a_{i+1}, x_{-i,i+1}^j) \succ^p (b_i^j, a_{i+1}, y_{-i,i+1}^j).$$

By Lemma 9, we have by taking $j \to \infty$,

$$(c_i, a_{i+1}, x_{-i,i+1}) \succsim^p (b_i, a_{i+1}, (x+\varepsilon)_{-i,i+1}).$$

By Lemma 17,

$$(b_i, a_{i+1}, (x+\varepsilon)_{-i,i+1}) \succ^p (b_i, a_{i+1}, x_{-i,i+1})$$

and so by Lemma 11,

$$\begin{aligned}(a_i, c_{i+1}, x_{-i,i+1}) &\sim^p (c_i, a_{i+1}, x_{-i,i+1}) \succsim^p (b_i, a_{i+1}, (x+\varepsilon)_{-i,i+1}) \\ &\succ^p (b_i, a_{i+1}, x_{-i,i+1}) \sim^p (a_i, b_{i+1}, x_{-i,i+1})\end{aligned}$$

that shows the claim.

Second, by Lemma 13,

$$(b_i, c_{i+1}, x_{-i,i+1}) \sim^p (c_i, c_{i+1}, x_{-i,i+1}),$$

then by Lemmas 11 and 12,

$$(c_i, c_{i+1}, x_{-i,i+1}) \succ^p (b_i, b_{i+1}, x_{-i,i+1})$$

and so

$$(b_i, c_{i+1}, x_{-i,i+1}) \succ^p (b_i, b_{i+1}, x_{-i,i+1}).$$

Then the claim follows symmetrically to the first case. □

Finally, Lemmas 11 and 18 show (4) and complete the induction on k.

Now for $k = 2^{n-1}$, by applying (4) inductively, this completes the induction on n. □

The next two results extend the first-order stochastic dominance for lotteries with a dyadic probability tree for prizes on the boundary of X. First, we show weak monotonicity.

Lemma 19 Assume that $\succsim$ satisfes Axioms 1′, 2′, and 6′ and $n \in \mathbb{N}$. For all $x, y \in X^{2^n}$, if for all $1 \leq i \leq 2^n$, $x_i \geq y_i$, then
$$\sum_{i=1}^{2^n} 2^{-n}\delta_{x_i} \succsim \sum_{i=1}^{2^n} 2^{-n}\delta_{y_i}.$$

Proof. Let $(x^j)_{j=1}^{\infty}, (y^j)_{j=1}^{\infty} \subseteq \operatorname{int} X^{\operatorname{supp} p}$ be such that for all $1 \leq l \leq 2^n$, $j \in \mathbb{N}$, $x_l^j \geq y_l^j$, $\lim_{j\to\infty} x^j = x$, and $\lim_{j\to\infty} y^j = y$. Now for all $j \in \mathbb{N}$, by Proposition 10,
$$\sum_{i=1}^{2^n} 2^{-n}\delta_{x_i^j} \succsim \sum_{i=1}^{2^n} 2^{-n}\delta_{y_i^j}.$$
Thus taking $j \to \infty$, by Lemma 9,
$$\sum_{i=1}^{2^n} 2^{-n}\delta_{x_i} \succsim \sum_{i=1}^{2^n} 2^{-n}\delta_{y_i}.$$
□

Second, we show strict monotonicity if at least a single coordinate is a strict improvement.

Lemma 20 Assume that $\succsim$ satisfes Axioms 1′, 2′, and 6′ and $n \in \mathbb{N}$. For all $x, y \in X^{2^n}$, if for all $1 \leq i \leq 2^n$, $x_i \geq y_i$, then
$$\sum_{i=1}^{2^n} 2^{-n}\delta_{x_i} \succsim \sum_{i=1}^{2^n} 2^{-n}\delta_{y_i}$$
and additionally if for some $1 \leq i^0 \leq 2^n$, $x_{i^0} > y_{i^0}$, then
$$\sum_{i=1}^{2^n} 2^{-n}\delta_{x_i} \succ \sum_{i=1}^{2^n} 2^{-n}\delta_{y_i}.$$

Proof. The first claim follows from Lemma 19. So let i^0 be such that $x_{i^0} > y_{i^0}$. Now there exist $\tilde{x}_{i^0}, \tilde{y}_{i^0} \in X$ such that $x_{i^0} > \tilde{x}_{i^0} > \tilde{y}_{i^0} > y_{i^0}$, and for all $i \in \operatorname{supp} p$, $\tilde{x}_{i^0} \neq x_i$, $\tilde{y}_{i^0} \neq y_i$. Denote

$$A = \{i \in \{1, \dots, 2^n\} | x_i = y_i = \inf X\}, B = \{i \in \{1, \dots, 2^n\} | x_i = y_i = \sup X\},$$

$$C = \{i \in \{1, \dots, 2^n\} \setminus \{i^0\} | x_i \neq y_i, y_i = \inf X\}.$$

Let $z^1, z^2, z^3 \in \operatorname{int} X$, be pairwise different and such that for all $1 \leq i \leq 2^n$,

$$\{z^1, z^2, z^3\} \cap \{x_i, y_i, \tilde{x}_{i^0}, \tilde{y}_{i^0}\} = \varnothing \quad \text{and} \quad \min\{x_i | i \in \{1, \dots, 2^n\} \setminus A\} > z^3$$

which exist since for all $1 \leq i \leq 2^n$, $x_i \geq y_i$. First, $i^0 \notin A \cup B \cup C$ and A, B, C are pairwise disjoint. Now we have $(\tilde{x}_{i^0}, z^1_A, z^2_B, z^3_C, y_{-A,B,C,i^0}), (\tilde{y}_{i^0}, z^1_A, z^2_B, z^3_C, y_{-A,B,C,i^0}) \in \operatorname{int} X^{\operatorname{supp} p}$. By Proposition 10, since $\tilde{x}_{i^0} > \tilde{y}_{i^0}$,

$$\sum_{i=1}^{2^n} 2^{-n} \delta_{(\tilde{x}_{i^0}, z^1_A, z^2_B, z^3_C, y_{-A,B,C,i^0})_i} \succ \sum_{i=1}^{2^n} 2^{-n} \delta_{(\tilde{y}_{i^0}, z^1_A, z^2_B, z^3_C, y_{-A,B,C,i^0})_i}.$$

By the above, $\sum_{i=1}^{2^n} 2^{-n} \delta_{(\tilde{x}_{i^0}, z^1_A, z^2_B, z^3_C, y_{-A,B,C,i^0})_i}$ and $\sum_{i=1}^{2^n} 2^{-n} \delta_{(\tilde{y}_{i^0}, z^1_A, z^2_B, z^3_C, y_{-A,B,C,i^0})_i}$ have the same induced probability tree. Thus by applying Axiom $2'$ twice, we have

$$\sum_{i=1}^{2^n} 2^{-n} \delta_{(\tilde{x}_{i^0}, (\inf X)_A, (\sup X)_B, z^3_C, y_{-A,B,C,i^0})_i} \succ \sum_{i=1}^{2^n} 2^{-n} \delta_{(\tilde{y}_{i^0}, (\inf X)_A, (\sup X)_B, z^3_C, y_{-A,B,C,i^0})_i}.$$

Finally, by the choice of z^3, $\tilde{x}_{i^0}$, $\tilde{y}_{i^0}$ and Lemma 19, we have

$$\begin{aligned} \sum_{i=1}^{2^n} 2^{-n} \delta_{x_i} &\succsim \sum_{i=1}^{2^n} 2^{-n} \delta_{(\tilde{x}_{i^0}, (\inf X)_A, (\sup X)_B, z^3_C, y_{-A,B,C,i^0})_i} \\ &\succ \sum_{i=1}^{2^n} 2^{-n} \delta_{(\tilde{y}_{i^0}, (\inf X)_A, (\sup X)_B, z^3_C, y_{-A,B,C,i^0})_i} \succsim \sum_{i=1}^{2^n} 2^{-n} \delta_{y_i}. \end{aligned}$$

□

B.3 Expected Utility for Dyadic Lotteries

This section shows an expected utility representation for dyadic lotteries. The expected utility follows in three steps. First, by restricted branch cancellation, the preferences are separable across the probability tree branches and hence have an additive utility representation across the branches. Second, for uniform probability tree each branch affects preferences symmetrically and hence the additive branch utility is the same for each branch. Third, dyadic probability tree branches correspond to halving current branches that allows us to link all the dyadic probability trees to each other giving the same utility for the prizes on all the dyadic probability trees.

Lemma 21 If $\succsim$ satisfies Axioms $1'$, $2'$, and $6'$ and $n \in \mathbb{N}$, then there exists a strictly increasing, continuous $u : X \to \mathbb{R}$ such that for all $x, y \in X^{2^n}$, we have

$$\sum_{i=1}^{2^n} 2^{-n}\delta_{x_i} \succsim \sum_{i=1}^{2^n} 2^{-n}\delta_{y_i} \iff \sum_{i=1}^{2^n} u(x_i) \geq \sum_{i=1}^{2^n} u(y_i).$$

Proof. First, assume that $n > 1$. Denote $p = (2^{-n})_{i=1}^{2^n}$. For all $x, y \in X^{\operatorname{supp} p}$, define $\succsim^p$ by

$$x \succsim^p y \iff \sum_{i \in \operatorname{supp} p} 2^{-n}\delta_{x_i} \succsim \sum_{i \in \operatorname{supp} p} 2^{-n}\delta_{y_i}.$$

By Lemma 8, $\succsim^p$ is continuous and by Axiom $1'$, complete and transitive.

We show that $\succsim^p$ is separable: for all $i \in \operatorname{supp} p$, $x, y \in X^{\operatorname{supp} p}$, $z_i, z_i' \in X$,

$$(z_i, x_{-i}) \succsim^p (z_i, y_{-i}) \iff (z_i', x_{-i}) \succsim^p (z_i', y_{-i}).$$

We will show the equivalent claim that for all $i \in \operatorname{supp} p$, $x, y \in X^{\operatorname{supp} p}$, $z_i, z_i' \in X$,

$$(z_i, x_{-i}) \succ^p (z_i, y_{-i}) \Longrightarrow (z_i', x_{-i}) \succ^p (z_i', y_{-i}).$$

First, by Lemma 20, there exists $i^0 \in \operatorname{supp} p \setminus \{i\}$ such that $\sup X > y_{i^0}$. By Lemma 8, there exists $\varepsilon > 0$ such that $y_{i^0} + \varepsilon \in X$ and

$$(z_i, x_{-i}) \succ^p \Big(z_i, (y_{i^0} + \varepsilon)_{i^0}, y_{-i,i^0}\Big).$$

Let $(x^j)_{j=1}^{\infty}, (y^j)_{j=1}^{\infty} \subseteq X^{\operatorname{supp} p}$ be such that for all $j \in \mathbb{N}$,

$$(z_i, x_{-i}^j), (z_i', x_{-i}^j), (z_i, x_{-i}^j), (z_i', y_{-i}^j) \in \operatorname{NDiag}(X^p), \tag{16}$$

$\lim_{j\to\infty} x^j = x$, and $\lim_{j\to\infty} y^j = \Big((y_{i^0} + \varepsilon)_{i^0}, y_{-i}\Big)$. By Lemma 9, there exists $j^0 \in \mathbb{N}$ such that for all $j > j^0$,

$$(z_i, x_{-i}^j) \succ^p (z_i, y_{-i}^j).$$

By Axiom $2'$ and (16), for all $j > j^0$,

$$(z_i', x_{-i}^j) \succ^p (z_i', y_{-i}^j).$$

Thus by taking $j \to \infty$, by Lemma 9,

$$(z_i', x_{-i}) \succsim^p \left(z_i', (y_{i^0} + \varepsilon)_{i^0}, y_{-i,i^0}\right).$$

Finally, by Lemma 20,

$$(z_i', x_{-i}) \succ^p (z_i', y_{-i}).$$

This shows that $\succsim^p$ is separable.

Additionally, by the definition of $\succsim^p$, $\succsim^p$ is symmetric. By Wakker (1988), since $n > 1$, there exists a continuous $u : X \to \mathbb{R}$ such that for all $x, y \in X^{\operatorname{supp} p}$,

$$x \succsim^p y \iff \sum_{i \in \operatorname{supp} p} u(x_i) \geq \sum_{i \in \operatorname{supp} p} u(y_i).$$

Additionally, u is strictly increasing since for $c > d$ by Axiom 1′,

$$\delta_c \succ \delta_d \iff (c)_{i \in \operatorname{supp} p} \succ^p (d)_{i \in \operatorname{supp} p} \iff 2^n u(c) > 2^n u(d).$$

Finally, for $n = 1$, we have for all $x, y \in X^2$ by the above representation for $n = 2$,

$$\begin{aligned} & \frac{1}{2}\delta_{x_1} + \frac{1}{2}\delta_{x_2} \succsim \frac{1}{2}\delta_{y_1} + \frac{1}{2}\delta_{y_2} \\ \iff \quad & \frac{1}{4}\delta_{x_1} + \frac{1}{4}\delta_{x_1} + \frac{1}{4}\delta_{x_2} + \frac{1}{4}\delta_{x_2} \succsim \frac{1}{4}\delta_{y_1} + \frac{1}{4}\delta_{y_1} + \frac{1}{4}\delta_{y_2} + \frac{1}{4}\delta_{y_2} \\ \iff \quad & 2u(x_1) + 2u(x_2) \geq 2u(y_1) + 2u(y_2). \end{aligned}$$

□

The next lemma shows that the different expected utility representations for different dyadic induced probability trees use the same utility as the binary dyadic expected utility.

Lemma 22 If $\succsim$ satisfies Axioms 1′, 2′, and 6′, then there exists a strictly increasing, continuous $u : X \to \mathbb{R}$ such that for all $n \geq 0$, $p = (2^{-n})_{i=1}^{2^n}$, $x, y \in X^{\operatorname{supp} p}$, we have

$$\sum_{i=1}^{2^n} 2^{-n}\delta_{x_i} \succsim \sum_{i=1}^{2^n} 2^{-n}\delta_{y_i} \iff \sum_{i=1}^{2^n} 2^{-n}u(x_i) \geq \sum_{i=1}^{2^n} 2^{-n}u(y_i).$$

Proof. Denote $p = (1/2, 1/2)$. By Lemma 21 for $n = 1$, there exists a strictly increasing, continuous $u : X \to \mathbb{R}$ such that for all $x, y \in X^2$, we have

$$\frac{1}{2}\delta_{x_1} + \frac{1}{2}\delta_{x_2} \succsim \frac{1}{2}\delta_{y_1} + \frac{1}{2}\delta_{y_2} \iff u(x_1) + u(x_2) \geq u(y_1) + u(y_2).$$

We show that for all $n \geq 0$, $x, y \in X^{2^n}$,

$$\sum_{i=1}^{2^n} 2^{-n}\delta_{x_i} \succsim \sum_{i=1}^{2^n} 2^{-n}\delta_{y_i} \iff \sum_{i=1}^{2^n} 2^{-n}u(x_i) \geq \sum_{i=1}^{2^n} 2^{-n}u(y_i). \tag{17}$$

First, for $n = 0$, (17) follows by Axiom $1'$ and since u is strictly increasing. Second, let $n \geq 1$. By Lemma 21, there exists a strictly increasing, continuous $u^n : X \to \mathbb{R}$ such that for all $x, y \in X^{2^n}$, we have

$$\sum_{i=1}^{2^n} 2^{-n}\delta_{x_i} \succsim \sum_{i=1}^{2^n} 2^{-n}\delta_{y_i} \iff \sum_{i=1}^{2^n} u^n(x_i) \geq \sum_{i=1}^{2^n} u^n(y_i).$$

Now we have for all $x_1, x_2, y_1, y_2 \in X$,

$$\begin{aligned} u(x_1) + u(x_2) \geq u(y_1) + u(y_2) &\iff \frac{1}{2}\delta_{x_1} + \frac{1}{2}\delta_{x_2} \succsim \frac{1}{2}\delta_{y_1} + \frac{1}{2}\delta_{y_2} \\ &\iff \sum_{i=1}^{2^{n-1}} 2^{-n}\delta_{x_1} + \sum_{i=1}^{2^{n-1}} 2^{-n}\delta_{x_2} \succsim \sum_{i=1}^{2^{n-1}} 2^{-n}\delta_{y_1} + \sum_{i=1}^{2^{n-1}} 2^{-n}\delta_{y_2} \\ &\iff 2^{n-1}u^n(x_1) + 2^{n-1}u^n(x_2) \geq 2^{n-1}u^n(y_1) + 2^{n-1}u^n(y_2), \end{aligned}$$

where the second equivalency follows as an identity of the lotteries.

Thus also $\left(2^{n-1}u^n, 2^{n-1}u^n\right)$ gives an additive representation on X^2. Since u is continuous, by the uniqueness of the additive representation (Krantz et al., 1971, Theorem 2), there exist $\alpha \in \mathbb{R}_{++}$, $\beta_1, \beta_2 \in \mathbb{R}$ such that

$$u = \alpha 2^{n-1}u^n + \beta_1 \text{ and } u = \alpha 2^{n-1}u^n + \beta_2.$$

This shows (17) as a positive affine transformation. □

The last result of this section shows that we have the expected utility representation across different dyadic probability trees since the dyadic lottery with more prizes contains the dyadic lottery with fewer prizes.

Lemma 23 If $\succsim$ satisfies Axioms $1'$, $2'$, and $6'$, then there exists a strictly increasing, continuous $u : X \to \mathbb{R}$ such that for all $n, m \geq 0$, $x \in X^{2^n}, y \in X^{2^m}$, we have

$$\sum_{i=1}^{2^n} 2^{-n}\delta_{x_i} \succsim \sum_{i=1}^{2^m} 2^{-m}\delta_{y_i} \iff \sum_{i=1}^{2^n} 2^{-n}u(x_i) \geq \sum_{i=1}^{2^m} 2^{-m}u(y_i).$$

Proof. Let $u : X \to \mathbb{R}$ be the utility from Lemma 22. Assume w.l.o.g. $m \geq n$. Define $x^* \in X^{2^m}$ by for all $j \in \{1, \ldots, 2^{m-n}\}$, $i \in \{1, \ldots, 2^n\}$, $x^*_{(j-1)\times 2^n+i} = x_i$. The following hold as identities by the definition of lottery mixtures and the definition of x^*,

$$\sum_{i=1}^{2^n} 2^{-n}\delta_{x_i} = \sum_{j=1}^{2^{m-n}} 2^{-(m-n)}\left(\sum_{i=1}^{2^n} 2^{-n}\delta_{x_i}\right) = \sum_{j=1}^{2^{m-n}}\sum_{i=1}^{2^n} 2^{-m}\delta_{x_i} = \sum_{j=1}^{2^{m-n}}\sum_{i=1}^{2^n} 2^{-m}\delta_{x^*_{(j-1)\times 2^n+i}} = \sum_{k=1}^{2^m} 2^{-m}\delta_{x^*_k}. \tag{18}$$

Additionally similarly, we have by the definition of x^*,

$$\sum_{k=1}^{2^m} 2^{-m}u(x^*_i) = \sum_{j=1}^{2^{m-n}}\sum_{i=1}^{2^n} 2^{-m}u(x^*_{(j-1)\times 2^n+i}) = \sum_{j=1}^{2^{m-n}}\sum_{i=1}^{2^n} 2^{-m}u(x_i) \tag{19}$$

$$= 2^{m-n}\sum_{i=1}^{2^n} 2^{-m}u(x_i) = \sum_{i=1}^{2^n} 2^{-n}u(x_i).$$

Thus since u represents lotteries with 2^m outcomes, we have

$$\sum_{i=1}^{2^n} 2^{-n}\delta_{x_i} \succsim \sum_{i=1}^{2^m} 2^{-m}\delta_{y_i} \quad \overset{(18)}{\iff} \quad \sum_{i=1}^{2^m} 2^{-m}\delta_{x^*_i} \succsim \sum_{i=1}^{2^m} 2^{-m}\delta_{y_i}$$

$$\overset{\text{Lemma 22}}{\iff} \sum_{i=1}^{2^m} 2^{-m}u(x^*_i) \geq \sum_{i=1}^{2^m} 2^{-m}u(y_i) \quad \overset{(19)}{\iff} \quad \sum_{i=1}^{2^n} 2^{-n}u(x_i) \geq \sum_{i=1}^{2^m} 2^{-m}u(y_i),$$

□

B.4 Expected Utility under FOSD

In this section, we consider the case with prize shift continuity, Axiom 3′, and FOSD. The proof with weak convergence continuity is similar and is shown in the online appendix.

We will only show the expected utility representation for a compact interval of prizes. The result for simple lotteries follows symmetrically.

Proposition 24 (EU under FOSD) Assume that X is a compact interval. If $\succsim$ satisfies Axioms 1′-4′, then there exists a strictly increasing, continuous $u : X \to \mathbb{R}$ such that for all lotteries $P, Q \in \Delta^{\text{Bor}}(X)$, we have

$$P \succsim Q \iff \int_X u\, dP \geq \int_X u\, dQ.$$

Proof. By Lemma 7, $\succsim$ satisfies Axiom 6′. So by Lemma 23, there exists a strictly increasing, continuous $u : X \to \mathbb{R}$ such that for all $n, m \geq 0$, $x \in X^{2^n}, y \in X^{2^m}$, we have

$$\sum_{i=1}^{2^n} 2^{-n}\delta_{x_i} \succsim \sum_{i=1}^{2^m} 2^{-m}\delta_{y_i} \iff \sum_{i=1}^{2^n} 2^{-n}u(x_i) \geq \sum_{i=1}^{2^m} 2^{-m}u(y_i).$$

We show the equivalent claim that for all lotteries P, Q,

$$\int_X u\, dP > \int_X u\, dQ \implies P \succ Q.$$

and

$$\int_X u\, dP = \int_X u\, dQ \implies P \sim Q.$$

First, we show that

$$\int_X u\, dP > \int_X u\, dQ \implies P \succ Q.$$

Since X is compact, by Lemma S.3, we can approximate the lottery by dyadic lotteries and so there exist $n, m \in \mathbb{N}$ and $x^* \in X^{2^n}, y^* \in X^{2^m}$ such that

$$P >_{\text{FOSD}} \sum_{i=1}^{2^n} 2^{-n}\delta_{x_i^*} \text{ and } \sum_{i=1}^{2^m} 2^{-m}\delta_{y_i^*} >_{\text{FOSD}} Q$$

and

$$\left|\sum_{i=1}^{2^n} 2^{-n}u(x_i^*) - \int_X u\, dP\right|, \left|\sum_{i=1}^{2^m} 2^{-m}u(y_i^*) - \int_X u\, dQ\right| < \frac{1}{2}\left(\int_X u\, dP - \int_X u\, dQ\right). \quad (20)$$

Thus we have

$$\sum_{i=1}^{2^n} 2^{-n}u(x_i^*) - \sum_{i=1}^{2^n} 2^{-n}u(y_i^*)$$
$$\geq \left[\int_X u\, dP - \int_X u\, dQ\right] - \left|\sum_{i=1}^{2^n} 2^{-n}u(x_i^*) - \int_X u\, dP\right| - \left|\int_X u\, dQ - \sum_{i=1}^{2^n} 2^{-n}u(y_i^*)\right| \overset{(20)}{>} 0.$$

So by Lemma 23 and Axiom 4′,

$$P \succsim \sum_{i=1}^{2^n} 2^{-n}\delta_{x_i^*} \succ \sum_{i=1}^{2^m} 2^{-m}\delta_{y_i^*} \succsim Q.$$

Thus $P \succ Q$.

Next, we show that

$$\int_X u\,dP = \int_X u\,dQ \implies P \sim Q.$$

Let $c^* \in X$ be such that

$$u(c^*) = \int_X u\,dP$$

that exists since X is connected and u is continuous. We will show that $\delta_{c^*} \sim P$ and $\delta_{c^*} \sim Q$ follows symmetrically. Now since X is compact, by Lemma S.3 for each $k \in \mathbb{N}$, there exist $n^k \in \mathbb{N}$ and $x^k, y^k \in X^{2^{n^k}}$ such that

$$\sum_{i=1}^{2^{n^k}} 2^{-n^k} \delta_{x_i^k} \geq_{\text{FOSD}} P \geq_{\text{FOSD}} \sum_{i=1}^{2^{n^k}} 2^{-n^k} \delta_{y_i^k}$$

and

$$\left| \sum_{i=1}^{2^{n^k}} 2^{-n^k} u(x_i^k) - \int_X u\,dP \right|, \left| \sum_{i=1}^{2^{n^k}} 2^{-n^k} u(y_i^k) - \int_X u\,dP \right| < \frac{1}{k}.$$

Since X is connected and u is continuous, for each $k \in \mathbb{N}$, there exist $c^k, d^k \in X$ such that

$$u(c^k) = \sum_{i=1}^{2^{n^k}} 2^{-n^k} u(x_i^k) \text{ and } u(d^k) = \sum_{i=1}^{2^{n^k}} 2^{-n^k} u(y_i^k).$$

Now by Lemma 23, we have for all $k \in \mathbb{N}$,

$$\delta_{c^k} \sim \sum_{i=1}^{2^{n^k}} 2^{-n^k} \delta_{x_i^k} \text{ and } \delta_{d^k} \sim \sum_{i=1}^{2^{n^k}} 2^{-n^k} \delta_{y_i^k}$$

and so by Axiom 4′, we have

$$\delta_{c^k} \sim \sum_{i=1}^{2^{n^k}} 2^{-n^k} \delta_{x_i^k} \succsim P \text{ and } P \succsim \sum_{i=1}^{2^{n^k}} 2^{-n^k} \delta_{y_i^k} \sim \delta_{d^k}.$$

Additionally, we have, for all $k \in \mathbb{N}$,

$$\left| u(c^k) - u(c^*) \right| = \left| \sum_{i=1}^{2^{n^k}} 2^{-n^k} u(x_i^k) - \int_X u\,dP \right| < \frac{1}{k}$$

and

$$\left| u(d^k) - u(c^*) \right| = \left| \sum_{i=1}^{2^{n^k}} 2^{-n^k} u(y_i^k) - \int_X u\,dP \right| < \frac{1}{k}.$$

Since u is strictly increasing and continuous, especially u^{-1} is continuous and strictly increasing (Austin, 1985). We have

$$\lim_{k\to\infty} c^k = c^* \text{ and } \lim_{k\to\infty} d^k = c^*.$$

Thus by Axiom $3'$, we have

$$\delta_{c^*} \succsim P \succsim \delta_{c^*}.$$

Symmetrically for Q, we have

$$\delta_{c^*} \succsim Q \succsim \delta_{c^*}$$

that shows the claim. □

The representation holds for a general interval for simple lotteries symmetrically to Proposition 24 since all simple lotteries have a compact support and so Lemma S.3 applies.

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Online Appendix to "Expected Utility Without Linearity: Distinguishing Between Prospect Theory and Cumulative Prospect Theory"

This online appendix includes additional technical proofs. First, Appendix S.1 shows the extension of expected utility representation from dyadic lotteries to all lotteries under weak convergence continuity. Second, Appendix S.2 shows approximation of lotteries by dyadic lotteries.

S.1 Expected Utility under Weak Convergence Continuity

In this section, we consider the case with weak convergence continuity, Axiom 5′. The proof is similar to FOSD considered in Appendix B.4.

First, we show the expected utility representation for a compact interval of prizes.

Proposition S.1 (EU under Continuity) Assume that X is a compact interval. If $\succsim$ satisfies Axioms 1′, 2′, and 5′, then there exists a strictly increasing, continuous $u : X \to \mathbb{R}$ such that for all lotteries $P, Q \in \Delta^{\mathrm{Bor}}(X)$, we have

$$P \succsim Q \iff \int_X u\, dP \geq \int_X u\, dQ.$$

Proof. By Lemma 6, $\succsim$ satisfies Axiom 6′. So by Lemma 23, there exists a strictly increasing, continuous $u : X \to \mathbb{R}$ such that for all $n, m \geq 0$, $x \in X^{2^n}, y \in X^{2^m}$, we have

$$\sum_{i=1}^{2^n} 2^{-n}\delta_{x_i} \succsim \sum_{i=1}^{2^m} 2^{-m}\delta_{y_i} \iff \sum_{i=1}^{2^n} 2^{-n}u(x_i) \geq \sum_{i=1}^{2^m} 2^{-m}u(y_i).$$

We show the equivalent claim that for all lotteries P, Q,

$$\int_X u\, dP > \int_X u\, dQ \implies P \succ Q.$$

and

$$\int_X u\,dP = \int_X u\,dQ \implies P \sim Q.$$

First, we show that

$$\int_X u\,dP > \int_X u\,dQ \implies P \succ Q.$$

Since u is bounded as a continuous function on a compact interval, $\int_X u\,dP, \int_X u\,dQ$ are finite. By Lemma S.4, for each $n \in \mathbb{N}$, there exist $x^n, y^n \in X^{2^n}$ such that

$$\sum_{i=1}^{2^n} 2^{-n} u(x_i^n) = \int_X u\,dP \text{ and } \sum_{i=1}^{2^n} 2^{-n} u(y_i^n) = \int_X u\,dQ$$

and

$$\sum_{i=1}^{2^n} 2^{-n} \delta_{x_i^n} \xrightarrow{w} P \text{ and } \sum_{i=1}^{2^n} 2^{-n} \delta_{y_i^n} \xrightarrow{w} Q.$$

Let $a, b \in X$ be such that

$$\int_X u\,dP > u(a) > u(b) > \int_X u\,dQ.$$

By Lemma 23, we have for all $n \in \mathbb{N}$,

$$\sum_{i=1}^{2^n} 2^{-n} \delta_{x_i^n} \succ \delta_a \succ \delta_b \succ \sum_{i=1}^{2^n} 2^{-n} \delta_{y_i^n}.$$

So by Axiom 5$'$,

$$P \succsim \delta_a \succ \delta_b \succsim Q.$$

that shows the claim.

Next, we show that

$$\int_X u\,dP = \int_X u\,dQ \implies P \sim Q.$$

Let $c^* \in X$ be such that

$$u(c^*) = \int_X u\,dP$$

that exists since X is connected, u is continuous, and $\int_X u\,dP$ is finite. We will show that $\delta_{c^*} \sim P$. $\delta_{c^*} \sim Q$ follows symmetrically. By Lemma S.4, for each $n \in \mathbb{N}$, there exists $x^n \in X^{2^n}$ such that

$$\sum_{i=1}^{2^n} 2^{-n} u(x_i^n) = \int_X u\,dP \text{ and } \sum_{i=1}^{2^n} 2^{-n} \delta_{x_i^n} \xrightarrow{w} P.$$

Now by Lemma 23, we have for all $n \in \mathbb{N}$,

$$\sum_{i=1}^{2^n} 2^{-n} \delta_{x_i^n} \sim \delta_{c^*}.$$

So by Axiom 5′,

$$P \sim \delta_{c^*}.$$

And symmetrically

$$Q \sim \delta_{c^*}$$

that shows the claim. □

Next, we show the representation for a general interval. In this case, the utility function is bounded.

Proposition S.2 (General EU under Continuity) Assume that X is an interval. If $\succsim$ satisfies Axioms 1′, 2′, and 5′, then there exists a strictly increasing, continuous, and bounded $u : X \to \mathbb{R}$ such that for all lotteries $P, Q \in \Delta^{\mathrm{Bor}}(X)$, we have

$$P \succsim Q \iff \int_X u\, dP \geq \int_X u\, dQ.$$

Proof. By Lemma 6, $\succsim$ satisfies Axiom 6′. So by Lemma 23, there exists a strictly increasing, continuous $u : X \to \mathbb{R}$ such that for all $n, m \geq 0$, $x \in X^{2^n}, y \in X^{2^m}$, we have

$$\sum_{i=1}^{2^n} 2^{-n} \delta_{x_i} \succsim \sum_{i=1}^{2^m} 2^{-m} \delta_{y_i} \iff \sum_{i=1}^{2^n} 2^{-n} u(x_i) \geq \sum_{i=1}^{2^m} 2^{-m} u(y_i).$$

We will show that u is bounded. Assume, per contra, that u is unbounded. Assume w.l.o.g. that u is unbounded from above. Let $a, b \in X$, $a < b$. Now for each $n \in \mathbb{N}$, there exists $x^n \in X$ such that $u(x^n) \geq n2^n$. For each $n \in \mathbb{N}$, define $Q^n = 2^{-n}(2^n - 1)\delta_a + 2^{-n}\delta_{x^n}$. Now we show that $Q^n \xrightarrow{w} \delta_a$. Let $f : X \to \mathbb{R}$ be continuous and bounded function and $d \in \mathbb{R}_+$ such that for all $x \in X$, $|f(x)| < d$. Now we have

$$\left| \int_X f\, dQ^n - \int_X f\, d\delta_a \right| = \left| 2^{-n} f(x^n) - 2^{-n} f(a) \right| < 2 \times 2^{-n} \times d$$

and so

$$\int_X f\, dQ^n \to \int_X f\, d\delta_a \text{ as } n \to \infty.$$

Since f was arbitrary continuous and bounded function we have $Q^n \xrightarrow{w} \delta_a$.

But now there exists $n^0 \in \mathbb{N}$ such that $u(b)+|u(a)|<n^0$. So we have for all $n>n^0$,

$$2^{-n}(2^n-1)u(a)+2^{-n}u(x^n)\geq 2^{-n}(2^n-1)u(a)+n>2^{-n}(2^n-1)u(a)+u(b)+|u(a)|>u(b).$$

So by Lemma 23, for all $n > n^0$, $Q^n \succ \delta_b$. Thus by Axiom 5′, $\delta_a \succsim \delta_b$. But this contradicts Axiom 1′. Thus u is bounded.

Now the claim follows symmetrically to Proposition S.1

□

S.2 Approximation of Lotteries by Dyadic Lotteries

This section shows that lotteries can be approximated by dyadic lotteries. First, we show if a lottery has a compact support, then it can be approximated by dyadic lotteries from above and below in first-order stochastic dominance in such a way that the dyadic lotteries converge in expectation to the approximated lottery.

Lemma S.3 Assume that $u : X \to \mathbb{R}$ is a strictly increasing and continuous utility function, $P \in \Delta^{\text{Bor}}(X)$, and $\varepsilon > 0$. If there exist $A \subseteq X$ such that A is compact and $P(A) = 1$, then there exist $n \in \mathbb{N}$ and $x, y \in X^{2^n}$ such that

$$\left|\sum_{i=1}^{2^n} 2^{-n}u(x_i) - \int_X u\,dP\right|, \left|\sum_{i=1}^{2^n} 2^{-n}u(y_i) - \int_X u\,dP\right| < \varepsilon$$

and

$$\sum_{i=1}^{2^n} 2^{-n}\delta_{x_i} \geq_{\text{FOSD}} P \geq_{\text{FOSD}} \sum_{i=1}^{2^n} 2^{-n}\delta_{y_i}.$$

Proof. For all $p \in [0, 1]$, define

$$P^{-1}(p) = \inf\left\{c \in \mathbb{R}\middle|p < P\Big((-\infty, c)\Big)\right\} = \sup\left\{c \in \mathbb{R}\middle|p \geq P\Big((-\infty, c]\Big)\right\} \tag{21}$$

where the equality of infimum and supremum follows from P^{-1} being a quantile function.

Let $n \in \mathbb{N}$ be such that

$$2^{-n} < \frac{\varepsilon}{\Big(|u(\max A)| + |u(\min A)| + 1\Big)} \tag{22}$$

Now let $x, y \in X^{2^n}$ be such that for all $i \in \mathbb{N} \cap [1, 2^n]$,

$$x_i = P^{-1}(2^{-n} i) \text{ and } y_i = P^{-1}\Big(2^{-n}(i-1)\Big).$$

These are well-defined since for all $i \in \mathbb{N} \cap [1, 2^n]$, we have $x_i, y_i \in [\min A, \max A] \subseteq X$ since X is connected and A is compact. Now by (21)

$$\sum_{i=1}^{2^n} 2^{-n} \delta_{x_i} \geq_{\text{FOSD}} P \geq_{\text{FOSD}} \sum_{i=1}^{2^n} 2^{-n} \delta_{y_i}.$$

Thus especially since u is strictly increasing, we have

$$\sum_{i=1}^{2^n} 2^{-n} u(x_i) \geq \int_X u \, dP \geq \sum_{i=1}^{2^n} 2^{-n} u(y_i)$$

and so

$$\left| \sum_{i=1}^{2^n} 2^{-n} u(x_i) - \int_X u \, dP \right|, \left| \sum_{i=1}^{2^n} 2^{-n} u(y_i) - \int_X u \, dP \right| < \left| \sum_{i=1}^{2^n} 2^{-n} u(x_i) - \sum_{i=1}^{2^n} 2^{-n} u(y_i) \right|.$$

Finally, we have by the definitions of x and y and since u is strictly monotonic,

$$\begin{aligned} &\left| \sum_{i=1}^{2^n} 2^{-n} u(x_i) - \sum_{i=1}^{2^n} 2^{-n} u(y_i) \right| = \left| \sum_{i=1}^{2^n} 2^{-n} u(x_i) - \sum_{i=2}^{2^n} 2^{-n} u(x_{i-1}) + 2^{-n} u(y_1) \right| \\ &= \left| 2^{-n} u(x_{2^n}) - 2^{-n} u(y_1) \right| \leq 2^{-n} \left| u\Big(P^{-1}(1)\Big) \right| + 2^{-n} \left| u\Big(P^{-1}(0)\Big) \right| \\ &\leq 2^{-n} \Big(\left| u(\max A) \right| + \left| u(\min A) \right| \Big) \overset{(22)}{<} \varepsilon. \end{aligned}$$

□

Our second approximation result shows that for any bounded lottery can be approximated by dyadic lotteries in such a way that the expectation of the dyadic lotteries is the same as the approximated lotteries and they converge to the approximated lottery in weak convergence.

Lemma S.4 Assume that $u : X \to \mathbb{R}$ is a strictly increasing, continuous, and bounded utility function and $P \in \Delta^{\text{Bor}}(X)$. Then for each $n \in \mathbb{N}$, there exists $x^n \in X^{2^n}$ such that

$$\sum_{i=1}^{2^n} 2^{-n} u(x_i^n) = \int_X u \, dP$$

and

$$\sum_{i=1}^{2^n} 2^{-n} \delta_{x_i^n} \xrightarrow{w} P \text{ as } n \to \infty.$$

Proof. For all $p \in [0,1]$, define

$$P^{-1}(p) = \inf\Big\{c \in \mathbb{R}\Big| p < P\Big((-\infty, c)\Big)\Big\} = \sup\Big\{c \in \mathbb{R}\Big| p \geq P\Big((-\infty, c]\Big)\Big\} \tag{23}$$

where the equality of infimum and supremum follows from P^{-1} being a quantile function.

First, the standard proof that for all $p \in (0,1)$,

$$P^{-1}(p) \in X \tag{24}$$

is omitted.

For all $n \in \mathbb{N}$, define $x^n \in X^{2^n-1}$ by for all $1 \leq i \leq 2^n - 1$, $x_i^n = P^{-1}(2^{-n} * i)$ which is well-defined by (24). Let $n \in \mathbb{N}$. We will show that there exist $x^*, x_* \in X$ such that

$$\sum_{i=1}^{2^n-1} 2^{-n}u(x_i^n) + 2^{-n}u(x_*) \leq \int_X u\,dP \leq \sum_{i=1}^{2^n-1} 2^{-n}u(x_i^n) + 2^{-n}u(x^*).$$

If $\inf X \in X$, then we can take $x_* = \inf X$ and if $\sup X \in X$, then we can take $x^* = \sup X$, and these will show the claim by the definition of x_i^n and first-order stochastic dominance since u is increasing. First, assume that $\sup X \notin X$. As above, there exist $x^\dagger, x^* \in X$ such that

$$P\Big([x^\dagger, \sup X)\Big) < 2^{-n-1} \text{ and } u(x^\dagger) > \frac{1}{2}\Big(\sup u(X) + u(x_{2^n-1}^n)\Big)$$

and

$$u(x^*) > \frac{1}{2}\Big(\sup u(X) + u(x^\dagger)\Big)$$

since u is strictly increasing and bounded. Now we have

$$\begin{aligned}
&\int_{(x_{2^n-1}^n, \sup X)} u\,dP + u(x_{2^n-1}^n)\Big(2^{-n} - P(x_{2^n-1}^n, \infty)\Big) \\
&= \int_{[x^\dagger, \sup X)} u\,dP + \int_{(x_{2^n-1}^n, x^\dagger)} u\,dP + u(x_{2^n-1}^n)\Big(2^{-n} - P(x_{2^n-1}^n, \infty)\Big) \\
&\leq \sup u(X) P\Big([x^\dagger, \sup X)\Big) + u(x^\dagger)\Big(2^{-n} - P(x^\dagger, \sup X)\Big) \\
&\leq \sup u(X) 2^{-n-1} + u(x^\dagger)\Big(2^{-n} - 2^{-n-1}\Big) = 2^{-n}\frac{1}{2}\Big(\sup u(X) + u(x^\dagger)\Big) < 2^{-n}u(x^*).
\end{aligned}$$

And additionally, we have when we denote $x_0^n = -\infty$

$$\int_{(-\infty, x_{2^n-1}^n]} u\,dP - u(x_{2^n-1}^n)\Big(2^{-n} - P(x_{2^n-1}^n, \infty)\Big)$$

$$
\begin{aligned}
&= \int_{(-\infty, x^n_{2^n-1}]} u\, dP - u(x^n_{2^n-1})\Big(P(-\infty, x^n_{2^n-1}]\Big) - (1-2^{-n})\Big) \\
&= \int_{(-\infty, x^n_{2^n-1})} u\, dP + u(x^n_{2^n-1})\Big(2^{-n}(2^n-1) - P(-\infty, x^n_{2^n-1})\Big) \\
&= \sum_{i=1}^{2^n-1} \int_{(x^n_{i-1}, x^n_i)} u\, dP + u(x^n_{i-1})\Big(P(-\infty, x^n_{i-1}] - 2^{-n}(i-1)\Big) + u(x^n_i)\Big(2^{-n}i - P(-\infty, x^n_i)\Big) \\
&\le \sum_{i=1}^{2^n-1} u(x^n_i)P(x^n_{i-1}, x^n_i) + u(x^n_i)\Big(P(-\infty, x^n_{i-1}] - 2^{-n}(i-1)\Big) + u(x^n_i)\Big(2^{-n}i - P(-\infty, x^n_i)\Big) \\
&= \sum_{i=1}^{2^n-1} 2^{-n}u(x^n_i),
\end{aligned}
$$

where the second equality follows from separating the point $x^n_{2^n-1}$ from the expectation, the third equality follows from subtracting and adding $u(x^n_i)2^{-n}i$ and $P\Big(-\infty, x^n_i)\Big)$ for all $1 \le i \le 2^n - 2$ and from separating the point x^n_i from the expectation and joining it into $P\Big(-\infty, x^n_{i+1})\Big)$, and the inequality follows from the monotonicity of u.

Thus we have from the above inequalities.

$$
\int_X u\, dP \le \sum_{i=1}^{2^n-1} 2^{-n}u(x^n_i) + 2^{-n}u(x^*).
$$

Symmetrically, there exists $x_* \in X$ such that

$$
\sum_{i=1}^{2^n-1} 2^{-n}u(x^n_i) + 2^{-n}u(x_*) \le \int_X u\, dP.
$$

Finally, by the continuity of u and since X is connected, there exists $x^n_{2^n} \in X$ such that

$$
\sum_{i=1}^{2^n} 2^{-n}u(x^n_i) = \int_X u\, dP.
$$

Denote

$$
Q^n = \sum_{i=1}^{2^n} 2^{-n}\delta_{x^n_i}.
$$

Next, we show that

$$
Q^n \xrightarrow{w} P \text{ as } n \to \infty.
$$

By Portmanteau's Lemma (Vaart, 2000, Lemma 2.2), it suffices to show that for all $c \in \mathbb{R}$,

$$
Q^n\Big((-\infty, c]\Big) \to P\Big((-\infty, c]\Big) \text{ as } n \to \infty.
$$

Let $c \in \mathbb{R}$ and $n \in \mathbb{N}$. Denote $i^n = \inf\Big\{i \in \mathbb{N} | 2^{-n} \times i \geq P\Big((-\infty, c]\Big)\Big\}$. By the definition of x^n, we have for all $j < i^n$, $P^{-1}(2^{-n} * j) \leq c$ by (23) and $x_j^n \leq c$. Thus

$$Q^n\Big((-\infty, c]\Big) \geq 2^{-n}(i^n - 1) \geq P\Big((-\infty, c]\Big) - 2^{-n}.$$

On the other hand, we have for all $2^n - 1 \geq j > i^n$, $P^{-1}(2^{-n} * j) > c$ since $P\Big((-\infty, c]\Big)$ is a right continuous function of c and so $x_j^n > c$. Thus

$$Q^n\Big((-\infty, c]\Big) \leq 2^{-n}(i^n + 1) \leq 2^{-n+1} + P\Big((-\infty, c]\Big).$$

Hence

$$\Big|Q^n\Big((-\infty, c]\Big) - P\Big((-\infty, c]\Big)\Big| < 2^{-n+1}.$$

So especially for all $c \in \mathbb{R}$, $Q^n\Big((-\infty, c]\Big) \to P\Big((-\infty, c]\Big)$ as $n \to \infty$ and so $Q^n \xrightarrow{w} P$.

□