NUMERICAL APPROXIMATION OF DYNKIN GAMES WITH ASYMMETRIC INFORMATION

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Abstract. We propose an implementable, feedforward neural network-based structure preserving probabilistic numerical approximation for a generalized obstacle problem describing the value of a zero-sum differential game of optimal stopping with asymmetric information. The target solution depends on three variables: the time, the spatial (or state) variable, and a variable from a standard \((I - 1)\)-simplex which represents the probabilities with which the \(I\) possible configurations of the game are played. The proposed numerical approximation preserves the convexity of the continuous solution as well as the lower and upper obstacle bounds. We show convergence of the fully-discrete scheme to the unique viscosity solution of the continuous problem and present a range of numerical studies to demonstrate its applicability.

1. INTRODUCTION

We consider the generalized obstacle problem

\[
\begin{aligned}
\max \left\{ \max \left\{ \min \left\{ (-\partial_t - \mathcal{L}) u, u - p^T f \right\}, u - p^T h \right\}, -\lambda(p, D_p^2 u) \right\} = 0,
\end{aligned}
\]

where \( u : [0, T] \times \mathbb{R}^d \times \Delta(I) \to \mathbb{R}, T > 0 \) and \( \Delta(I) = \{ p = (p_1, \ldots, p_I) \in [0, 1]^I; \sum_{i=1}^I p_i = 1 \} \) is the set of \( \mathbb{R}^I \)-valued vectors of probabilities (see below for a detailed explanation). Furthermore, \( \mathcal{L} u := \frac{1}{2} \text{Tr} \left[ aa^T D_x^2 u \right] + b \cdot D_x u \), where the coefficient functions \( a : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times d}, b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \), as well as the barrier functions \( f, h : [0, T] \times \mathbb{R}^d \to \mathbb{R}^I \) and the terminal condition \( g : \mathbb{R}^d \to \mathbb{R}^I \) are given. The mapping \( \lambda : \Delta(I) \times \mathbb{S}^d \to \mathbb{R} \) enforces the convexity of the solution \( u = u(t, x, p) \) with respect to the probability variable \( p \) and is given as

\[
\lambda(p, A) := \min_{z \in T_{\Delta(I)}(p) \setminus \{0\}} \frac{z^T A z}{\|z\|^2},
\]

where \( \mathbb{S}^{d \times d} \) denotes the set of \( d \times d \) matrices that are symmetric and \( T_{\Delta(I)}(p) \) denotes the tangent cone to \( \Delta(I) \) at \( p \); i.e., \( T_{\Delta(I)}(p) := \bigcup_{t \geq 0} \mathcal{C}^t(\Delta(I) - p) \).

Under suitable assumptions on the problem’s data, it is shown in [14, Theorem 3.4] that (1.1) admits a unique viscosity solution in the class of bounded, uniformly continuous functions that are Lipschitz-continuous and convex with respect to \( p \). The viscosity solution identifies with the value of a zero-sum game of optimal stopping (Dynkin game) with asymmetric information. More precisely, given a stochastic basis \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\), with the augmented canonical filtration \( \mathbb{F} := \{ \mathcal{F}_s : t \leq s \leq T \} \) generated by an \( \mathbb{R}^d \)-valued Brownian motion \( B := \{ B_s : t \leq s \leq T \} \), operator \((-\partial_t - \mathcal{L})\) is the so-called Dynkin operator associated to the Itô-process \( \{ X_{t,s}^{t,x} : 0 \leq t \leq s \leq T \} \) defined on \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) and satisfying

\[
X_{t,s}^{t,x} = x + \int_t^s b(r, X_{r,s}^{t,x})dr + \int_t^s a(r, X_{r,s}^{t,x})dB_r.
\]

Consider then the zero-sum game of optimal stopping in which \(-\) under a given scenario \( i \in I \) selected randomly with probability \( p_i \), at time 0 - two players decide to stop the evolution of the process \( X_{t,s}^{t,x} \) in order to optimize the performance criterion:

\[
\begin{aligned}
\mathbb{E} \left[ f_i(\sigma, X_{\tau,\sigma}^{t,x}) \mathbb{I}_{\sigma < \tau, \sigma < T} + h_i(\tau, X_{\tau}^{t,x}) \mathbb{I}_{\tau \leq \sigma, \tau < T} + g_i(X_{T}^{t,x}) \mathbb{I}_{\tau = \sigma = T} \right], \quad i \in \{1, 2, \ldots, I\}.
\end{aligned}
\]

Exercising a stopping rule \( \tau \), Player 1 pays Player 2 the random amount \( h_i(\tau, X_{\tau}^{t,x}) \), while stopping at time \( \sigma \), Player 2 receives the random payoff \( f_i(\sigma, X_{\sigma}^{t,x}) \). Finally, if neither of the players has stopped before the final maturity \( T \), then both players receive the amount \( g_i(X_{T}^{t,x}) \). Clearly, Player 1 aims at minimizing (1.3), while Player 2 at maximizing (1.3). The asymmetric information feature of the game arises because only one player knows the exact scenario under which the game is played - and therefore
the realized values of the game’s payoffs \( f_i, h_i, g_i \) — while the other player is informed only about the probability according to which each of the possible scenarios is selected.

Zero-sum games of optimal stopping have been proposed by E.B. Dynkin in 1967 (see [10]) as an extension of problems of optimal stopping. Since their introduction a large number of contributions using probabilistic and/or analytic techniques arose, with the main aim of proving existence of a value for the game and a characterization of the (Stackelberg) equilibrium stopping times. We refer to the introduction of [14] for a detailed literature review. In particular, Dynkin games have received increasing attention in the mathematical finance literature since the work by Y. Kifer [17], where it is shown that zero-sum games of optimal stopping provide the fair value of the so-called Game or Israeli Options.

The literature on Dynkin games with asymmetric information is quite recent and still only counts a very limited number of contributions. Other than [14], we like to refer to [21], where the value and the equilibrium strategies of a zero-sum game of optimal stopping have been constructed in the case in which both players have different knowledge about the occurrence of a default. In [11], it is considered a zero-sum game of optimal stopping where the payoffs depend on two independent continuous-time Markov chains, the first Markov chain being observed only by Player 1, the second Markov chain being observed only by Player 2. This in particular implies asymmetric information, in the sense that players employ stopping rules that are stopping times with respect to different filtrations. More recently, in a very general not necessarily Markovian setting, it is proved in [9] that continuous-time zero-sum Dynkin games with partial and asymmetric information admit a value in randomized stopping times. As a byproduct, existence of equilibrium strategies for both players are also shown to exist. Finally, explicit results via free-boundary methods have been obtained in [8] for a class of games in which the players have asymmetric information regarding the drift of the one-dimensional diffusion underlying the game.

Concerning the numerical approximation of games with asymmetric information we refer to [3], [13] and the references therein. Furthermore, we mention [16] which studies probabilistic neural network approximations schemes for obstacle problems arising from game theory (with complete and symmetric information).

In the current paper we propose a structure-preserving probabilistic numerical approximation of (1.1). Following [3], we combine the time-discretization with a convexity-preserving discretization in the probability variable. Furthermore, we introduce a neural network approximation in the spatial variable to obtain an implementable numerical scheme and show its convergence to the viscosity solution to (1.1). We present numerical studies where we compare the the proposed neural network-based algorithm to a semi-Lagrangian scheme with piecewise linear interpolation in the spatial variable. Furthermore, we demonstrate the ability of the proposed numerical approximation to capture the expected structure of free boundaries arising in the problem of pricing Israeli \( \delta \)-penalty put option (cf. [18], [19] among others) with asymmetric information.

The paper is organized as follows. The notation and preliminaries are introduced in Section 2. In Section 3 we introduce the semi-discrete and the fully-discrete probabilistic numerical approximation schemes and show their convergence to the viscosity solution of (1.1). Sections 4 and 5 contain auxiliary results needed for the convergence of the semi-discrete scheme: in Section 4 we discuss regularity properties of the semi-discrete numerical solution and in Section 5 we show that the accumulation points of the semi-discrete numerical approximation satisfy the viscosity sub- and supersolution properties. Numerical experiments are presented in Section 6.

2. Notation and preliminaries

2.1. Notation. We list here some notation in complement to those introduced in Section 1:

- Let \( m \in \mathbb{N} \). For any \( x = (x_1, \ldots, x_m)^\top \in \mathbb{R}^m \), we denote the Euclidean norm of \( x \) by
  \[ |x| := \sqrt{|x_1|^2 + \cdots + |x_m|^2}. \]

- For any Lipschitz continuous function \( \varphi : [0, T] \times \mathbb{R}^d \to \mathbb{R}^m \), we denote the Lipschitz coefficient by
  \[ \left[ \varphi \right] := \sup_{(t,x),\varphi(s,y)} \frac{|\varphi(t,x) - \varphi(s,y)|}{|t-s| + |x-y|}. \]

- A continuous function \( \varphi : [0, T] \times \mathbb{R}^d \times \Delta(I) \to \mathbb{R}^m \) is said to be bounded if the norm
  \[ ||\varphi||_\infty := \sup \{ ||\varphi(t,x,p)|| : (t,x,p) \in [0, T] \times \mathbb{R}^d \times \Delta(I) \} \]
  is finite.
For $t \in E \subseteq [0, T]$, we denote by $\mathcal{T}_E$ the set of $\mathbb{F}$-stopping times with values in $E$. The subset of stopping rules will be denoted by

$$\mathcal{T}_E^I := \{ \tau \in \mathcal{T}_E : \tau \text{ is independent of } \mathcal{F}_t \}.$$ 

We will frequently use the fact that the maps $(x, y) \mapsto \max\{x, y\}$ and $(x, y) \mapsto \min\{x, y\}$ satisfy for every $(x, x')$ and $(y, y')$ in $\mathbb{R}^2$ the following inequalities

\begin{align}
&\max\{x, x'\} - \max\{y, y'\} \leq \max\{|x - y|, |x' - y'|\}, \\
&\min\{x, x'\} - \min\{y, y'\} \leq \max\{|x - y|, |x' - y'|\}.
\end{align}

### 2.2. Assumptions and viscosity solution

The following assumptions will be assumed to hold for the data in (1.1):

(A1) The functions $a : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, $f : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, $g : \mathbb{R}^d \to \mathbb{R}^d$, and $h : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ are uniformly Lipschitz continuous and uniformly bounded.

(A2) Furthermore, for all $(x, p) \in \mathbb{R}^d \times \Delta(I)$, we have that

$$p^T f(T, x) \leq p^T g(x) \leq p^T h(T, x).$$

(A3) We assume that, $P$-a.s., $X^{t, x} \in \mathcal{D}$, for some bounded domain $\mathcal{D}$ of $\mathbb{R}^n$.

We recall the definition of the viscosity solution (1.1) below, cf. [14].

**Definition 2.1** (Viscosity solution). A function $w : [0, T] \times \mathbb{R}^d \times \Delta(I) \to \mathbb{R}$ is a viscosity solution to (1.1) if the following two properties hold:

(i) $w$ is a viscosity subsolution to (1.1) if for all $(\bar{t}, \bar{x}, \bar{p}) \in [0, T] \times \mathbb{R}^d \times \text{Int}(\Delta(I))$ and for any smooth test function $\varphi : [0, T] \times \mathbb{R}^d \times \text{Int}(\Delta(I)) \to \mathbb{R}$ such that $(w - \varphi)$ has a strict maximum at $(\bar{t}, \bar{x}, \bar{p})$ with $(w - \varphi)(\bar{t}, \bar{x}, \bar{p}) = 0$ it holds that

\begin{equation}
\max \left\{ \max \left\{ \min \left\{ (-\partial_t - \mathcal{L})\varphi, -\varphi - p^T f \right\}, \varphi - p^T h \right\}, -\lambda(p, D^2_p \varphi) \right\} \leq 0,
\end{equation}

at $(\bar{t}, \bar{x}, \bar{p})$.

(ii) $w$ is a viscosity supersolution to (1.1) if for all $(\bar{t}, \bar{x}, \bar{p}) \in [0, T] \times \mathbb{R}^d \times \Delta(I)$ and for any smooth test function $\varphi : [0, T] \times \mathbb{R}^d \times \Delta(I) \to \mathbb{R}$ such that $(w - \varphi)$ has a strict minimum at $(\bar{t}, \bar{x}, \bar{p})$ with $(w - \varphi)(\bar{t}, \bar{x}, \bar{p}) = 0$ it holds that

\begin{equation}
\max \left\{ \min \left\{ (-\partial_t - \mathcal{L})\varphi, \varphi - p^T f \right\}, \varphi - p^T h \right\}, -\lambda(p, D^2_p \varphi) \right\} \geq 0,
\end{equation}

at $(\bar{t}, \bar{x}, \bar{p})$.

Under the assumptions (A1),(A2), the existence of a unique viscosity solution to (1.1) in the sense of the above definition is guaranteed by the next theorem, cf. [14, Theorem 3.4].

**Theorem 2.2.** There exists a unique viscosity solution to (1.1) in the class of bounded uniformly continuous functions, which are convex and uniformly Lipschitz in $p$.

### 3. Numerical approximation

In Section 3.1 we introduce a probabilistic numerical numerical scheme which is discrete in time and in the probability variable. We combine the semi-discrete scheme with a neural network approximation in the spatial variable to obtain a fully discrete numerical approximation in Section 3.2.

#### 3.1. Semi-discrete numerical approximation in $t$ and $p$

For $N \in \mathbb{N}$ we consider an equidistant partition $\Pi^\Delta = \{ t_n = n\Delta t, \ n = 0, \ldots, N \}$ of the time interval $[0, T]$ with time-step $\Delta t := T/N$. We define the Euler approximation of the stochastic process $X^{t, x}$ (1.2) over the time interval $(t_n, t_{n+1})$ as

\begin{equation}
X^{n, x}_{n+\ell} = X_n + \sum_{j=n}^{n+\ell-1} \left( b(t_j, X^{n, x}_j) \Delta t + a(t_j, X^{n, x}_j) \xi_j \sqrt{\Delta t} \right),
\end{equation}

where $\{\xi^n\}_{n \in \mathbb{N}}$ are $\mathbb{R}^d$-valued i.i.d. random variables with zero mean and unit variance.

For the discretization with respect to the probability variable $p$ we adopt the approach of [3]. To this end we consider a family of regular partitions $\{ \mathcal{M}_{\Delta p} \}_{\Delta p > 0}$ of $\Delta(I)$ into open simplices $K$ with mesh-size $\Delta p := \max_{K \in \mathcal{M}_{\Delta p}} \{ \text{diam}(K) \}$ such that $\Delta(I) = \cup_{K \in \mathcal{M}_{\Delta p}} K$. The set of vertices of all $K \in \mathcal{M}_{\Delta p}$ is denoted by $N_{\Delta p} := \{ p_1, \ldots, p_M \}$.

Given the step size $\Delta := (\Delta t, \Delta p)$ we introduce the following semi-discrete numerical scheme:
For $m = 1, \ldots, M$ define the map $x \mapsto \bar{u}^\Delta(t_N, x, p_m)$ as
\begin{equation}
\bar{u}^\Delta(t_N, x, p_m) = p_m^T g(x).
\end{equation}

For $n = N - 1, \ldots, 0$ proceed as follows:

- For $m = 1, \ldots, M$ define the map $x \mapsto \bar{Y}^{n,x,p_m}_\Delta$ as
\begin{equation}
\bar{Y}^{n,x,p_m}_\Delta := \min \left\{ \max \left\{ \mathbb{E}[\bar{u}^\Delta(t_{n+1}, X_{n+1}^{n,x}), p_m^T f(t_n, x)], p_m^T h(t_n, x) \right\} \right\}.
\end{equation}

- For $m = 1, \ldots, M$, determine the map $p_m \mapsto \bar{u}^\Delta(t_n, x, p_m)$ as
\begin{equation}
\bar{u}^\Delta(t_n, x, p_m) = \text{Vexp} \left[ \bar{Y}^{n,x,p_1}_\Delta, \ldots, \bar{Y}^{n,x,p_M}_\Delta \right](p_m).
\end{equation}

We note that for $p \in \Delta(I)$, the lower convex envelope in (3.4) is the solution of the minimization problem (cf. [6])
\begin{equation}
\text{Vexp} \left[ \bar{Y}^{n,x,p_1}_\Delta, \ldots, \bar{Y}^{n,x,p_M}_\Delta \right](p) = \min \left\{ \sum_{i=1}^M \lambda_i \bar{Y}^{n,x,p_i}_\Delta : \sum_{i=1}^M \lambda_i = 1, \lambda_i \geq 0, p = \sum_{i=1}^M \lambda_i p_i \right\}.
\end{equation}

Following [3] we consider the (convexity preserving) data-dependent simplicial partition $\mathcal{M}^{n,x}_\Delta$ of $\Delta(I)$ with nodes $\mathcal{N}^{n,x}_\Delta := \{p_1^{n,x}, \ldots, p_{M}^{n,x}\} \subseteq \mathcal{N}_\Delta$, such that the piecewise linear interpolant of the data values at the nodes $\mathcal{N}^{n,x}_\Delta$ over the partition $\mathcal{M}^{n,x}_\Delta$ (for precise definition see (3.7) below) agrees with the discrete data values $\{p_m, \bar{u}^\Delta(t_n, x, p_m)\}_{m=1}^M$. We note that the partition $\mathcal{M}^{n,x}_\Delta$ does not necessarily coincide with the original mesh $\mathcal{M}_\Delta$.

We consider the set of piecewise linear Lagrange basis functions $\{\psi_i^{n,x}, i = 1, \ldots, M^{n,x}\}$ associated with the set of nodes $\mathcal{N}^{n,x}_\Delta$ of the partition $\mathcal{M}^{n,x}_\Delta$. We recall the following properties of the Lagrange basis functions which will be frequently used throughout the paper: (a) $\psi_i^{n,x}(p_i^{n,x}) = \delta_{ij}$, where $\delta_{ij}$ is the Kronecker delta, and (b) $\sum_{i=1}^{M^{n,x}} \psi_i^{n,x}(p) = 1$ for $p \in \Delta(I)$. We note that (a) implies that at any point $p \in \Delta(I)$ there are at most $I$ basis functions with nonzero value; hence the sum in (b) reduces to $\sum_{i=1}^I$.

For $n = 0, \ldots, N$, $x \in \mathbb{R}^d$, we define the piecewise linear interpolant $p \mapsto \bar{u}^\Delta(t_n, x, p)$ of the discrete lower convex envelope $\{\bar{u}^\Delta(t_n, x, p_1), \ldots, \bar{u}^\Delta(t_n, x, p_M)\}$ over the convexity preserving partition $\mathcal{M}^{n,x}_\Delta$ as
\begin{equation}
\bar{u}^\Delta(t_n, x, p) := \sum_{i=1}^{M^{n,x}} \bar{u}^\Delta(t_n, x, p_{m_i}^{n,x}) \psi_i^{n,x}(p),
\end{equation}
where $m_i(p_i^{n,x}) \in \mathbb{N}$ is the index of $p_i^{n,x}$ in $\mathcal{N}_\Delta$: i.e., $p_i^{n,x} := p_{m_i(p_i^{n,x})}$ for some $p_{m_i(p_i^{n,x})} \in \mathcal{N}_\Delta$, cf. [3].

Note that for each $p \in \Delta(I)$ there exist at most $I$ non-zero basis functions (i.e., the basis functions associated with the $I$ vertices $\{\pi_i^{n,x}\}_{i=1}^I$ of the simplex $K = \text{conv}(\pi_i^{n,x}, i = 1, \ldots, I) \in \mathcal{M}^{n,x}_\Delta$ for which $p \in \overline{K}$) with nonzero value which we denote by $\{\lambda_1^{n,x}, \ldots, \lambda_I^{n,x}\}$ such that (3.6) is equivalent to
\begin{equation}
\bar{u}^\Delta(t_n, x, p) = \sum_{i=1}^I \bar{u}^\Delta(t_n, x, \pi_i^{n,x}(p)) \lambda_i^{n,x}(p),
\end{equation}
moreover
\begin{equation}
\sum_{i=1}^I \lambda_i^{n,x}(p) = 1 \quad \text{and} \quad p = \sum_{i=1}^I \pi_i^{n,x}(p) \lambda_i^{n,x}(p).
\end{equation}

Next, we define for every $t \in [t_n, t_{n+1}]$
\begin{equation}
\tilde{u}^\Delta(t, x, p) = \bar{u}^\Delta(t_n, x, p) + \frac{\bar{u}^\Delta(t_{n+1}, x, p) - \bar{u}^\Delta(t_n, x, p)}{\Delta t} (t - t_n).
\end{equation}

**Theorem 3.1.** Assume $(A_1)$ and $(A_2)$. Then the sequence $\{\tilde{u}^\Delta\}_\Delta$ converges uniformly on every compact subsets of $[0, T] \times \mathbb{R}^d \times \Delta(I)$, i.e.,
\begin{equation}
\left. \lim_{\Delta t \to 0} \tilde{u}^\Delta(s, y, q) \right|_{(s,y) \to (t,x)} = u(t, x, p),
\end{equation}
where $u$ is the unique viscosity solution to (1.1) in the class of bounded uniformly continuous functions, which are uniformly Lipschitz and convex in $p$.

**Proof.** We verify that the sequence $\{\tilde{u}^\Delta\}_\Delta$ satisfies the assumptions of the Arzelà–Ascoli theorem, cf. [26, Section III.3]. The equiboundedness follows by Lemma 4.1 and the equicontinuity by Lemmas 4.1, 4.3 and 4.4. Hence, up to a subsequence, on every compact subsets of $[0, T] \times \mathbb{R}^d \times \Delta(I)$, the sequence $\{\tilde{u}^\Delta\}_\Delta$ converges uniformly to a limit $w$ that is bounded, uniformly continuous and is convex and uniformly Lipschitz continuous in $p$. 

By Proposition 5.1 (resp. Proposition 5.4), \( u \) satisfies the viscosity sub- and super-solution property, hence it is a viscosity solution of (1.1) in the sense of Definition 2.1.

By Theorem 2.2, the viscosity solution of (1.1) is unique in the class of bounded, uniformly continuous functions, which are convex and uniformly Lipschitz in \( p \). Since every limit of \( \{w^\Delta\}_\Delta \) is bounded, uniformly continuous and convex and uniformly Lipschitz in \( p \), the limit is unique.

\section{Fully discrete approximation and convergence.} Below we describe a modification of the numerical scheme (3.2)-(3.4) which employs a feedforward neural network approximation in the state variable \( x \). The resulting fully-discrete algorithm (3.15)-(3.18) is practically implementable.

To describe the neural network approximation scheme we loosely follow the exposition of [4, 16]. Typically a continuous function \( x \mapsto \Phi(x)\) can be expanded in terms of a linear combination of fixed nonlinear basis functions \( \varphi_j(x) \) and take the form

\begin{equation}
\Phi(x; \theta) = \sum_j w_j \varphi_j(x),
\end{equation}

where \( x = (x_1, \ldots, x_d)^\top \in \mathbb{R}^d \) are called input variables, \( \theta = (w_j)_j \) is a set of numbers called parameters, and \( \Phi \) the target variables (the function we want to predict or approximate). The idea behind neural networks is to extend (3.10) with basis functions that depend on a linear combination of the inputs, where the coefficients in the linear combination are adaptive parameters.

Depending on the nature of the input and the assumed distribution of the target variables it is possible to generate a number of neural networks expansions of different characteristics. We are interested on the so-called Feedforward Neural Network for standard regression problems. It can be described as a series of layers of functional transformations. Each subsequent layer has a connection from the previous layer. In the first layer or input layer, we take the image of the input variable \( x \) through an affine transformation that takes the form

\begin{equation}
w^{(1)}_1x + b^{(1)}.
\end{equation}

We shall refer to the parameter \( w^{(1)}_1 \in M_{\kappa,d} \) as the weight matrix and the parameter \( b^{(1)}_1 \in \mathbb{R}^\kappa \) as the bias vector. The superscript (1) indicates that the corresponding parameters are in the first layer of the network. Next, a differentiable, nonlinear activation function \( \rho : \mathbb{R} \mapsto \mathbb{R} \) acts component-wise on the activation vector (3.11) and transforms it into a vector \( \rho(w^{(1)}_1x + b^{(1)}) \) of hidden units that constitutes the second layer of the neural network also know as hidden layer. Finally, we take the image of the vector of hidden unit vector through another affine transformation to get a scalar-valued output variable

\begin{equation}
\Phi_\kappa(x; \theta) = w^{(2)}_1 \cdot \rho(w^{(1)}_1x + b^{(1)}) + b^{(2)}_1
\end{equation}

with \( w^{(2)}_1 \in \mathbb{R}^\kappa \) and \( b^{(2)}_1 \in \mathbb{R} \). The set of all weight and bias parameters have been grouped together into \( \theta := \{w^{(1)}_1, b^{(1)}_1, w^{(2)}_1, b^{(2)}_1\} \).

We shall consider neural networks (3.12) with total variation smaller than \( \gamma_\kappa \), and activation functions with bounded derivatives. Hence, we consider a class of neural networks that is then represented by the parametric set of functions

\begin{equation}
\mathcal{N}^\kappa_{d,1,2,\kappa}(\Theta^{\gamma_\kappa}) := \left\{x \in \mathbb{R}^d \mapsto \Phi_\kappa(x; \theta) = w^{(2)}_1 \cdot \rho(w^{(1)}_1x + b^{(1)}_1) + b^{(2)}_1, \theta := \{w^{(1)}_1, b^{(1)}_1, w^{(2)}_1, b^{(2)}_1\}^2_{l=1} \in \Theta^{\gamma_\kappa}\right\},
\end{equation}

with

\begin{equation}
\Theta^{\gamma_\kappa} := \left\{\theta := \{w^{(1)}_1, b^{(1)}_1, \theta^2_{l=1} : \|w^{(1)}_1\|_{\infty} \leq \gamma_\kappa, \|w^{(2)}_1\|_1 \leq \gamma_\kappa\right\}
\end{equation}

for some sequence \((\gamma_k)_{\kappa}\) such that for all \( \kappa \in \mathbb{N} \) it holds that

\begin{equation}
\gamma_{2^\kappa} \leq \ell ip_{\kappa},
\end{equation}

where \( \ell ip_{\kappa} := \max \{1[\bar{f}], [\bar{g}], [\bar{h}]\} e^{([\bar{b}]+\frac{1}{2}([\bar{a}])^2)}T \). The condition (3.14) can be imposed by using a regularizer (so-called weight decay) into the training function, see e.g. [4, Section 3.1.4].

In addition to the family of partition \( \{M_{\Delta p}\}_{\Delta p=0} \) introduced in Section 3 we consider a family of regular partition \( \{M_{\Delta x}\}_{\Delta x=0} \) of the spatial domain \( D \subset \mathbb{R}^d \) (see Assumption (A_3)) into open simplices \( T \) with the mesh size \( \Delta x := \operatorname{max}_{T \in M_{\Delta x}} \operatorname{diam}(T) \) and denote the set of vertices of all \( T \in M_{\Delta x} \) as \( \mathcal{N}_{\Delta x} := \{x_1, \ldots, x_{2^d}\} \).

The fully-discrete numerical solution \( \tilde{u}^\Delta_{n,\Delta}(t_n, x, p) \approx \tilde{u}^\Delta(t_n, x, p) \) is constructed by employing the above neural network approximation in the spatial variable in the semi-discrete scheme (3.2)-(3.4). The resulting algorithm reads as:
For $m = 1, \ldots, M$ and $\ell = 1, \ldots, L$ define
\begin{equation}
\tilde{u}_{n,\ell}^\Delta(t_n, x, t; p_m) = p_m^\ell g(x). 
\end{equation}

For $n = N - 1, \ldots, 0$ proceed as follows:
\begin{itemize}
  \item For $m = 1, \ldots, M$ compute:
  \begin{equation}
  \theta_n(p_m) = \arg \min_{\theta \in \Theta_n} \frac{1}{L} \sum_{i=1}^L |\Phi_n(x; \theta) - \tilde{u}_{i,\ell}^\Delta(t_{n+1}, x, t; p_m)|^2, 
  \end{equation}

  and define the map $x \mapsto \tilde{Y}_{n,x}^{\ell,p_m}$ by
  \begin{equation}
  \tilde{Y}_{n,x}^{\ell,p_m} = \min \left\{ \max \left\{ \mathbb{E} \left[ \Phi_n(\tilde{X}_{n+1}; \theta_n(p_m)) \right], p_m^T f(t_n, x) \right\}, p_m^T h(t_n, x) \right\}.
  \end{equation}

  \item For $m = 1, \ldots, M$ and $\ell = 1, \ldots, L$, determine the map $p_m \mapsto \tilde{u}_{n,\ell}^\Delta(t_n, x, t; p_m)$ as
  \begin{equation}
  \tilde{u}_{n,\ell}^\Delta(t_n, x, t; p_m) = \text{Vexp}_p \left[ \tilde{Y}_{n,x}^{\ell,p_1}, \ldots, \tilde{Y}_{n,x}^{\ell,p_M} \right](p_m).
  \end{equation}
\end{itemize}

For $n = 0, \ldots, N$, $\ell = 1, \ldots, L$, we denote the (data dependent) convexity preserving simplicial partition of $\Delta(I)$ as $\tilde{\mathcal{M}}_{\Delta_p}^{n,x,\ell}$ and $\hat{\mathcal{N}}_{\Delta_p}^{n,x,\ell} = \{\hat{p}_1^{n,x,\ell}, \ldots, \hat{p}_{\lambda_n^{n,x,\ell}}^{n,x,\ell}\} \subseteq \mathcal{N}_{\Delta_p}$ the set of its nodes such that the piecewise linear interpolant of the data values at the nodes $\hat{\mathcal{N}}_{\Delta_p}^{n,x,\ell}$ over the partition $\tilde{\mathcal{M}}_{\Delta_p}^{n,x,\ell}$ agrees with the values $\{p_m, \tilde{u}_{n,\ell}^\Delta(t_n, x, t; p_m)\}_{m=1}^M$. Furthermore, we note the respective counterparts of (3.7), (3.8) as
\begin{equation}
\tilde{u}_{n,\ell}^\Delta(t_n, x, t; p) = \sum_{i=1}^L \tilde{u}_{n,\ell}^\Delta(t_n, x, t, \tilde{p}_{\lambda_i^{n,x,\ell}}(p)) \hat{\lambda}_{\lambda_i^{n,x,\ell}}(p),
\end{equation}
and
\begin{equation}
\sum_{i=1}^L \hat{\lambda}_{\lambda_i^{n,x,\ell}}(p) = 1, \quad p = \sum_{i=1}^L \tilde{p}_{\lambda_i^{n,x,\ell}}(p) \hat{\lambda}_{\lambda_i^{n,x,\ell}}(p).
\end{equation}

We set for every $n = 0, \ldots, N - 1$, we define
\begin{equation}
\varepsilon_{n,\ell}^\Delta := \max_{m=1,\ldots,M} \sup_{x \in \tilde{\mathcal{M}}_{\Delta_p}^{n,x,\ell}} |\Phi_n(x; \theta_n(p_m)) - \tilde{u}_{n,\ell}^\Delta(t_{n+1}, x, t; p_m)|,
\end{equation}
where $x \mapsto \tilde{u}_{n,\ell}^\Delta(t_{n+1}, x, t; p_m)$ is the piecewise linear interpolant over the mesh $\mathcal{M}_{\Delta_p}$ (i.e. in the spatial variable) of the discrete solution in (3.18).

**Remark 3.2.** Assumption (A3) together with the additional assumption that the random variables $\{\xi_n\}_{n \in \mathbb{N}}$ satisfy $\|\xi_n\| < R < \infty$ P-a.s. (which is satisfied if one considers a random walk approximation where for instance $\xi_n = \pm 1$ with probability 1/2) ensure that the Euler approximation (3.1) satisfies $\tilde{X}_{n+1}^{x,\ell} \in \mathcal{D}^\varepsilon \text{-P.a.s.}$ for some bounded domain $\mathcal{D}^\varepsilon$ with $\mathcal{D} \subset \mathcal{D}^\varepsilon$. In the proofs below we assume without loss of generality that $\mathcal{D}^\varepsilon = \mathcal{D}$.

**Theorem 3.3.** Let (A1)-(A3) hold and let $\tilde{u}_{n,\ell}^\Delta$, $\tilde{u}_{n,\ell}^\Delta$ be the numerical solutions constructed by (3.15)-(3.18) and (3.2)-(3.4), respectively. For all $n = 0, \ldots, N$ and all $(x, t; p_m) \in \mathcal{N}_{\Delta_p} \times \tilde{\mathcal{M}}_{\Delta_p}$ it holds that
\begin{equation}
|\tilde{u}_{n,\ell}^\Delta(t_n, x, t; p_m) - \tilde{u}_{n,\ell}^\Delta(t_n, x, t, p_m)| \leq (\gamma_n^2 + \ell \varepsilon_n^\Delta) N \Delta x + \sum_{n=0}^{N-1} \varepsilon_n^\Delta,
\end{equation}
with $\gamma_n$ defined (3.14) and $\ell \varepsilon_n^\Delta := \max \{1, [\|f\|, [g], [h]]\} e^{(3+\frac{1}{2}L^2)} T$.

**Proof.** We fix $(n, \ell, m) \in \{0, \ldots, N\} \times \{1, \ldots, L\} \times \{1, \ldots, M\}$ and assume that, P-a.s., $\tilde{X}_{n+1}^{x,\ell} \in \mathcal{D}$. First we suppose that $0 \leq \tilde{u}_{n,\ell}^\Delta(t_n, x, t; p_m) - \tilde{u}_{n,\ell}^\Delta(t_n, x, t, p_m)$. By (3.7), (3.8), (3.19), (3.20) and the convexity of the map $p_m \mapsto \tilde{u}_{n,\ell}^\Delta(t_n, x, t; p_m)$ we have
\begin{equation}
\tilde{u}_{n,\ell}^\Delta(t_n, x, t; p_m) - \tilde{u}_{n,\ell}^\Delta(t_n, x, t, p_m) = \tilde{u}_{n,\ell}^\Delta(t_n, x, t; p_m) - \sum_{i=1}^L \tilde{u}_{n,\ell}^\Delta(t_n, x, t, \tilde{p}_{\lambda_i^{n,x,\ell}}(p_m)) \hat{\lambda}_{\lambda_i^{n,x,\ell}}(p_m)
\end{equation}
\begin{equation}
\leq \sum_{i=1}^L \left( \tilde{u}_{n,\ell}^\Delta(t_n, x, t, \tilde{p}_{\lambda_i^{n,x,\ell}}(p_m)) - \tilde{u}_{n,\ell}^\Delta(t_n, x, t, \tilde{p}_{\lambda_i^{n,x,\ell}}(p_m)) \right) \hat{\lambda}_{\lambda_i^{n,x,\ell}}(p_m).
\end{equation}
Since $F \mapsto \text{Vex}_\pi[F]$ is nonexpansive, it follows for every $\pi \in \{\pi^{n,x}_1(p_m), \ldots, \pi^{n,x}_{\ell}(p_m)\}$ that
\[
\hat{u}_n^{\Delta}(t_n, x_{\ell}, \pi) - \hat{u}_n^{\Delta}(t_n, x_{\ell}, \pi) \leq \max_{m=1,\ldots,M} |\hat{u}_n^{\Delta}(t_n, x_{\ell}, \pi^{n,x}_m(p_m)) - \hat{u}_n^{\Delta}(t_n, x_{\ell}, \pi^{n,x}_m(p_m))|
\]
\[
\leq \max_{m=1,\ldots,M} |\hat{Y}^{n,x}_n(p_m) - \hat{Y}^{n,x}_n(p_m)|.
\]
(3.23)

Recalling (3.3), (3.17), we express for $\pi \in \{\pi^{n,x}_1(p_m), \ldots, \pi^{n,x}_{\ell}(p_m)\}$
\[
|\hat{Y}^{n,x}_n(p_m) - \hat{Y}^{n,x}_n(p_m)| = \min \left\{ \max \left\{ E[\Phi_n(\hat{X}^{n,x}_{n+1}; \theta_n(\pi))], \pi^T f(t_n, x_{\ell}) \right\}, \pi^T h(t_n, x_{\ell}) \right\}
\]
\[
- \min \left\{ \max \left\{ E[\hat{u}_n^{\Delta}(t_{n+1}, \hat{X}^{n,x}_{n+1}, \pi)], \pi^T f(t_n, x_{\ell}) \right\}, \pi^T h(t_n, x_{\ell}) \right\}.
\]
Using (2.2) we get
\[
|\hat{Y}^{n,x}_n(p_m) - \hat{Y}^{n,x}_n(p_m)| \leq \left| \max \left\{ E[\Phi_n(\hat{X}^{n,x}_{n+1}; \theta_n(\pi))], \pi^T f(t_n, x_{\ell}) \right\} - \max \left\{ E[\hat{u}_n^{\Delta}(t_{n+1}, \hat{X}^{n,x}_{n+1}, \pi)], \pi^T f(t_n, x_{\ell}) \right\} \right|.
\]
and then, by (2.1),
\[
|\hat{Y}^{n,x}_n(p_m) - \hat{Y}^{n,x}_n(p_m)| \leq \left| E[\Phi_n(\hat{X}^{n,x}_{n+1}; \theta_n(\pi)) - \hat{u}_n^{\Delta}(t_{n+1}, \hat{X}^{n,x}_{n+1}, \pi)] \right|.
\]

By the above estimates we deduce from (3.23) that
\[
\hat{u}_n^{\Delta}(t_n, x_{\ell}, \pi) - \hat{u}_n^{\Delta}(t_n, x_{\ell}, \pi) \leq E[\Phi_n(\hat{X}^{n,x}_{n+1}; \theta_n(\pi)) - \hat{u}_n^{\Delta}(t_{n+1}, \hat{X}^{n,x}_{n+1}, \pi)].
\]
(3.24)

Below we consider $\mathcal{N}_{\Delta_k}$-valued random variables $\hat{x}_k$, $k = n + 1, \ldots, N$ which are defined as follows. Due to the assumption (A1), see also Remark 3.2 for $1 < k \leq N$, there exists a simplex $S = S(\omega) \in \mathcal{M}_{\Delta_k}^\pi$ such that $\hat{X}_k^{-1,\pi}(\omega) = S(\omega)$ for $\omega \in \Omega$. For $\omega \in \Omega$ we then set
\[
\hat{x}_k(\omega) = \arg \max_{i=0,\ldots,d} |\hat{u}_n^{\Delta}(t_k, v_i, \pi) - \hat{u}_n^{\Delta}(t_k, v_i, \pi)|,
\]
(3.25)
i.e., $\hat{x}_k(\omega)$ is the vertex of $S(\omega) = \text{conv}\{v_0, \ldots, v_d\}$ which realizes the greatest difference between the fully-discrete and semi-discrete numerical solution. Below we suppress the explicit dependence of $\hat{x}_k, S$ on $\omega \in \Omega$.

For any $x \in S$ we rewrite
\[
\left[ \Phi_n(\hat{X}^{n,x}_{n+1}; \theta_n(\pi)) - \hat{u}_n^{\Delta}(t_{n+1}, \hat{X}^{n,x}_{n+1}, \pi) \right]
\]
\[
= \left[ \Phi_n(\hat{X}^{n,x}_{n+1}; \theta_n(\pi)) - \Phi_n(x; \theta_n(\pi)) \right] + \left[ \Phi_n(x; \theta_n(\pi)) - \hat{u}_n^{\Delta}(t_{n+1}, v_i, \pi) \right]
\]
\[
+ \left[ \hat{u}_n^{\Delta}(t_{n+1}, v_i, \pi) - \hat{u}_n^{\Delta}(t_{n+1}, v_i, \pi) \right] + \left[ \hat{u}_n^{\Delta}(t_{n+1}, v_i, \pi) - \hat{u}_n^{\Delta}(t_{n+1}, \hat{X}^{n,x}_{n+1}, \pi) \right].
\]

We express $\hat{u}_n^{\Delta}(t_{n+1}, x, \pi) = \sum_{i=0}^d \hat{u}_n^{\Delta}(t_{n+1}, v_i, \pi)\chi_i(x)$, where $\chi_i(x)$ are the linear Lagrange basis functions on $S$, s.t., $\sum_{i=0}^d \chi_i = 1, \chi_i(v_j) = \delta_{ij}$. Hence, we express
\[
\left[ \Phi_n(\hat{X}^{n,x}_{n+1}; \theta_n(\pi)) - \hat{u}_n^{\Delta}(t_{n+1}, \hat{X}^{n,x}_{n+1}, \pi) \right] = \left[ \Phi_n(\hat{X}^{n,x}_{n+1}; \theta_n(\pi)) - \Phi_n(x; \theta_n(\pi)) \right]
\]
\[
+ \left[ \Phi_n(x; \theta_n(\pi)) - \hat{u}_n^{\Delta}(t_{n+1}, x, \pi) \right] + \sum_{i=0}^d \left[ \hat{u}_n^{\Delta}(t_{n+1}, v_i, \pi) - \hat{u}_n^{\Delta}(t_{n+1}, \hat{X}^{n,x}_{n+1}, \pi) \right]\chi_i(x)
\]
\[
+ \sum_{i=0}^d \left[ \hat{u}_n^{\Delta}(t_{n+1}, v_i, \pi) - \hat{u}_n^{\Delta}(t_{n+1}, \hat{X}^{n,x}_{n+1}, \pi) \right]\chi_i(x).
\]

We take the expectation in the above expression, note (3.25) and estimate
\[
E[\Phi_n(\hat{X}^{n,x}_{n+1}; \theta_n(\pi)) - \hat{u}_n^{\Delta}(t_{n+1}, \hat{X}^{n,x}_{n+1}, \pi)] \leq E[\Phi_n(\hat{X}^{n,x}_{n+1}; \theta_n(\pi)) - \Phi_n(x; \theta_n(\pi))]
\]
\[
+ E[\Phi_n(x; \theta_n(\pi)) - \hat{u}_n^{\Delta}(t_{n+1}, x, \pi)] + E[\hat{u}_n^{\Delta}(t_{n+1}, \hat{x}_{n+1}, \pi) - \hat{u}_n^{\Delta}(t_{n+1}, \hat{x}_{n+1}, \pi)]
\]
\[
+ \sum_{i=0}^d E[\hat{u}_n^{\Delta}(t_{n+1}, v_i, \pi) - \hat{u}_n^{\Delta}(t_{n+1}, \hat{X}^{n,x}_{n+1}, \pi)]\chi_i(x).
\]

Since $\Phi_n \in \mathcal{N}_{\Delta_{1,2}}^\pi(\Theta^\pi)$, the map $x \mapsto \Phi_n(x; \theta_n(\pi))$ is uniformly Lipschitz continuous with a Lipschitz constant $\gamma_k^2 \leq \ell_1p_{\pi}$ (see (3.13)). The map $x \mapsto \hat{u}_n^{\Delta}(t_{n+1}, x, \pi)$ is uniformly Lipschitz continuous with a
Lipschitz constant $\ell p x$ by Lemma 4.1. Hence, we deduce that
\[
\begin{align*}
&\left|E\left[\Phi_\kappa(x; \theta_n(x)) - \hat u^\Delta(t_n, x, \pi)\right]\right| \\
&\leq \gamma_k^2 E\left[|x - \hat X^k_n(x)|\right] + E\left[|\Phi_\kappa(x; \theta_n) - \hat u^\Delta_k(t_n, x, \pi)|\right] \\
&+ E\left[|\hat u^\Delta_k(t_n, \hat x_n, \pi) - \hat u^\Delta(t_n, \hat x_n, \pi)|\right] + \sum_{i=0}^d \ell p x E\left[|v_i - \hat X^k_n(x)|\right].
\end{align*}
\]

Since $\hat X^k_n(x, \pi)$ in $S$ it holds $p$-a.s. that $|v_i - \hat X^k_n(x)| \leq \Delta x, |\hat X^k_n(x)| \leq \Delta x$. Consequently, we estimate
\[
\begin{align*}
&\left|E\left[\Phi_\kappa(x; \theta_n(x)) - \hat u^\Delta(t_n, x, \pi)\right]\right| \\
&\leq \gamma_k^2 \Delta x + \max_{x \in D} E\left[|\hat u^\Delta_k(t_n, \hat x_n, \pi) - \hat u^\Delta_k(t_n, x, \pi)|\right] \\
&+ \sum_{i=0}^d \ell p x E\left[|v_i - \hat X^k_n(x)|\right].
\end{align*}
\]

Using the above inequality in (3.24), we obtain for $\pi \in \{\pi^1_1(x), \ldots, \pi^n_1(x)\}$ that
\[
\hat u^\Delta_k(t_n, x, \pi) - \hat u^\Delta(t_n, x, \pi) \leq (\gamma_k^2 + \ell p x) \Delta x + \varepsilon_k^\Delta + \sum_{i=0}^d \ell p x E\left[|v_i - \hat X^k_n(x)|\right].
\]

Corollary 3.4. Let the assumptions of Theorems 3.1 and 3.3 hold and assume in addition that $\Delta x = \Delta t^2$. Then the sequence $\{\hat u^\Delta_k\}$ converges uniformly on $[0, T] \times \mathcal{D} \times \Delta(I)$, i.e.,
\[
\lim_{(s, y, q) \to (t, x, p)} \lim_{\Delta t, \Delta p \to 0} \hat u^\Delta_k(s, y, q) = u(t, x, p),
\]

where $u$ is the unique viscosity solution to (1.1) in the class of bounded uniformly continuous functions, which are uniformly Lipschitz and convex in $p$.

Proof. For a fixed $(t, x, p) \in \mathcal{D}_{T, I} := [0, T] \times \mathcal{D} \times \Delta(I)$ we consider a sequence $(\bar t_k, \bar x_k, \bar p_k)_{k \in \mathbb{N}}$ such that $(\bar t_k, \bar x_k, \bar p_k) \to (t, x, p)$ for $k \to \infty$ where $t_k := n_k \Delta t_k \in \Pi \Delta t_k$, $n_k \in \mathbb{N}$, $\bar p_k \in \mathcal{N} \Delta p_k$, $\bar x_k \in \mathcal{N} \Delta x_k$ with $\Delta t_k := T/n_k$, $\Delta x_k := \Delta t^2_k$, s.t., $\Delta p_k \to 0$, $n_k \to \infty$ for $k \to \infty$.

By Theorem 3.3 we estimate by the triangle inequality
\[
\begin{align*}
&\left|\hat u^\Delta_k(\bar t_k, \bar x_k, \bar p_k) - u(t, x, p)\right| \\
&\leq T(\gamma_k^2 + \ell p x) \Delta x + \sum_{n=n}^{N_k-1} \varepsilon_k^\Delta + \left|\hat u^\Delta(\bar t_k, \bar x_k, \bar p_k) - u(t, x, p)\right|,
\end{align*}
\]

with
\[
\varepsilon_k^\Delta := \max_{m=1, \ldots, M} \sup_{x \in D} \left|\Phi_\kappa(x; \theta_n(p_m)) - \hat u^\Delta_k(t_n, x, p_m)\right|.
\]

By the universal approximation theorem, see [15, Theorem 2.1] and Remark 3.5, it holds that $\varepsilon_k^\Delta \to 0$ for $k \to \infty$. Consequently, $\varepsilon_k^\Delta \leq \mathcal{O}(\Delta t^2_k)$ for sufficiently large $k = \kappa(\Delta t_k, \Delta p_k, \Delta x_k)$. 


Remark 3.5. For a fixed $\bar{\kappa}$, we deduce that
\[
\lim_{\Delta t_k, \Delta p_k \to 0} \limsup_{\kappa \to \infty} |\tilde{u}_{\kappa, D}(\bar{t}_k, \bar{x}_k, \bar{p}_k) - u(t, x, p)| \leq \lim_{\Delta t_k, \Delta p_k \to 0} T(\gamma_k^2 + \ell \rho_k) + \lim_{\Delta t_k, \Delta p_k \to 0} \limsup_{\kappa \to \infty} |\tilde{u}_{\kappa, D}(\bar{t}_k, \bar{x}_k, \bar{p}_k) - u(t, x, p)|.
\]
From Theorem 3.1 it follows that
\[
\lim_{\Delta t_k, \Delta p_k \to 0} \limsup_{\kappa \to \infty} |\tilde{u}_{\kappa, D}(\bar{t}_k, \bar{x}_k, \bar{p}_k) - u(t, x, p)| = 0.
\]
Consequently,
\[
\lim_{\Delta t_k, \Delta p_k \to 0} \limsup_{\kappa \to \infty} |\tilde{u}_{\kappa, D}(\bar{t}_k, \bar{x}_k, \bar{p}_k) - u(t, x, p)| = 0.
\]
\[\square\]

Remark 3.5. For a fixed $M \in \mathbb{N}$ we deduce from (3.16) that for each $1 < m \leq M$ and any $\theta \in \Theta_\kappa$ it holds that
\[
\frac{1}{L} \sum_{i=1}^{L} \Phi_n(x_i; \theta_n(p_m)) - \bar{u}_{\kappa, D}(t_{n+1}, x, p_m) \leq \frac{1}{L} \sum_{i=1}^{L} \Phi_n(x_i; \theta) - \bar{u}_{\kappa, D}(t_{n+1}, x, p_m) \leq \max_{\theta = \theta_{\kappa,m}} \Phi_n(x; \theta) - \bar{u}_{\kappa, D}(t_{n+1}, x, p_m) \leq \sup_{x \in \mathcal{D}} \Phi_n(x; \theta) - \bar{u}_{\kappa, D}(t_{n+1}, x, p_m) \leq 0.
\]
By the universal approximation theorem [15, Theorem 2.1] it is possible to choose $\theta = \theta_{\kappa,m}$ such that $\varepsilon_{n,L} \to 0$ for $\kappa \to \infty$. Hence it follows that
\[
\lim_{\kappa \to \infty} (\varepsilon_{n,L})^2 \leq \lim_{\kappa \to \infty} \max_{m=1, \ldots, M} \sup_{x \in \mathcal{D}} \Phi_n(x; \theta_{\kappa,m}) - \bar{u}_{\kappa, D}(t_{n+1}, x, p_m) \leq 0.
\]

4. Regularity properties of the semi-discrete approximation

Lemma 4.1 (Uniform Lipschitz continuity in $x$ and uniform boundedness). The map $x \mapsto \bar{u}_{\kappa}(t_n, x, p)$ is uniformly Lipschitz continuous, i.e., for every $n \in \{0, \ldots, N\}$, it holds that
\[
\forall x, y \in \mathbb{R}^d, p \in \Delta(I), \quad |\bar{u}_{\kappa}(t_n, x, p) - \bar{u}_{\kappa}(t_n, y, p)| \leq \ell \rho_k |x - y|,
\]
with $\ell \rho_k := \max \{\|f\|, \|g\|, \|h\|\} e(\beta + \frac{1}{2} a^2)^T$. Moreover, $(t_n, x, p) \mapsto \bar{u}_{\kappa}(t_n, x, p)$ is uniformly bounded with
\[
\|\bar{u}_{\kappa}\|_x \leq \max \{\|f\|_x, \|h\|_x\}.
\]

Proof. We fix $x, y \in \mathbb{R}^d, p \in \Delta(I)$ and proceed by induction for $n = N, N-1, \ldots, 0$ to show that $x \mapsto \bar{u}_{\kappa}(t_n, x, p)$ is uniformly Lipschitz continuous and uniformly bounded. By (3.2) and (A1), (A2), the base case $n = N$ holds. We assume that at time level $t_{n+1}$ there exist constant $\ell \rho_k^{n+1} > 0$ such that
\[
|\bar{u}_{\kappa}(t_{n+1}, x, p) - \bar{u}_{\kappa}(t_{n+1}, y, p)| \leq \ell \rho_k^{n+1} |x - y|,
\]
and that $\|\bar{u}_{\kappa}(t_{n+1}, \cdot, \cdot, \cdot, \|_x \leq \max \{\|f\|_x, \|h\|_x\}$.

Uniform Lipschitz continuity in $x$. Suppose that $0 \leq \bar{u}_{\kappa}(t_n, y, p) - \bar{u}_{\kappa}(t_n, x, p)$. By (3.7) and the convexity of $p \mapsto \bar{u}_{\kappa}(t_n, y, p)$, we have
\[
\bar{u}_{\kappa}(t_n, y, p) - \bar{u}_{\kappa}(t_n, x, p) = \bar{u}_{\kappa}(t_n, y, p) - \sum_{i=1}^{I} \bar{u}_{\kappa}(t_n, x, \pi^{I, x}(p)) \lambda_i^{I, x}(p)
\]
\[
\leq \sum_{i=1}^{I} \{\bar{u}_{\kappa}(t_n, x, \pi^{I, x}(p)) - \bar{u}_{\kappa}(t_n, x, \pi^{I, x}(p))\} \lambda_i^{I, x}(p)
\]
(4.2)
The next step consists to derive an estimate for the summands in the right-hand side of (4.2). Note that $\{\pi^{I, x}(p), \ldots, \pi^{I, x}(p)\} \in \{p_1, \ldots, \pi^{I, x}(p)\}$ and the map $F \mapsto \text{Vex}_p F$ is nonexpansive. By (3.4), it follows that
\[
\bar{u}_{\kappa}(t_n, y, \pi^{I, x}(p)) - \bar{u}_{\kappa}(t_n, x, \pi^{I, x}(p)) \leq \max \{|Y^{n,y,x} - Y^{n,x,x}| : x \in \{\pi^{I, x}(p), \ldots, \pi^{I, x}(p)\}\}
\]
(4.3) Let us derive an estimate for the term $|Y^{n,y,x} - Y^{n,x,x}|$ appearing in the right-hand side of (4.3). First we recall the following property:
Proposition 4.2 ([1, Proposition 2 (a)]). Assume that \( (A_1) \) holds and let \( \varphi : \mathbb{R}^d \rightarrow \mathbb{R} \) be a function that is Lipschitz continuous and bounded. It follows that the map \( x \in \mathbb{R}^d \mapsto \mathbb{E}[\varphi(\bar{X}_{n+1}^\pi)] \) is Lipschitz continuous with a Lipschitz coefficient \( \|\varphi\|_{Q_{\Delta t}} \), where \( Q_{\Delta t} := (1 + [(b \cdot s) + \frac{1}{2} |s|^2])\Delta t + O(\Delta t^2) \).

Recall that \( \bar{Y}_{n,x}^{\pi} \) is given by (3.3). Using the inequalities (2.1) and (2.2) and Proposition 4.2, it implies that
\[
|\bar{Y}_{n,y}^{\pi} - \bar{Y}_{n,x}^{\pi}| \leq \min \left\{ \max \left\{ \mathbb{E}[\bar{u}^\Delta(t_{n+1}, \bar{X}_{n+1}^y, \pi)], \pi^T f(t_n, y) \right\}, \pi^T h(t_n, y) \right\} - \min \left\{ \max \left\{ \mathbb{E}[\bar{u}^\Delta(t_{n+1}, \bar{X}_{n+1}^x, \pi)], \pi^T f(t_n, x) \right\}, \pi^T h(t_n, x) \right\}
\leq \max \left\{ \mathbb{E}[|\bar{u}^\Delta(t_{n+1}, \bar{X}_{n+1}^y, \pi) - \bar{u}^\Delta(t_{n+1}, \bar{X}_{n+1}^x, \pi)|], \|f(t_n, y) - f(t_n, x)|, \|h(t_n, y) - h(t_n, x)| \right\},
\]
(4.4)
where we used \((A_1)\) to estimate the maps \( x \mapsto f(t_n, x) \) and \( x \mapsto g(t_n, x) \), as well as Proposition 4.2, supported by (4.1), to estimate the maps \( x \mapsto \bar{u}^\Delta(t_{n+1}, x, \pi) \).

We insert (4.4) into the right-hand side of (4.3), which shows that
\[
\bar{u}^\Delta(t_n, y, \pi^y_x(p)) - \bar{u}^\Delta(t_n, x, \pi^y_x(p)) \leq \max \{ \ell \pi^{n+1}_x Q_{\Delta t}, [f], [h] \}|x - y|.
\]
If \( 0 \leq \bar{u}^\Delta(t_n, x, p) - \bar{u}^\Delta(t_n, y, p) \), then we commute the role of \( x \) and \( y \) in the previous steps to get
\[
|\bar{u}^\Delta(t_n, y, p) - \bar{u}^\Delta(t_n, x, p)| \leq \max \{ \ell \pi^{n+1}_x Q_{\Delta t}, [f], [h] \}|x - y|,
\]
which implies that the map \( x \mapsto \bar{u}^\Delta(t_n, x, p) \) is Lipschitz continuous with a Lipschitz coefficient \( \ell \pi^n_x = \max \{ \ell \pi^{n+1}_x Q_{\Delta t}, [f], [h] \} \). A recursion implies
\[
\ell \pi^n_x \leq \max \{ \ell \pi^{n+1}_x, [f], [h] \} Q_{\Delta t}
\leq \max \{ \ell \pi^{n+2}_x, [f], [h] \} Q_{\Delta t}^2
\]
\[\vdots\]
(4.6)
\[
\leq \max \{ \ell \pi^n_x, [f], [h] \} Q_{\Delta t}^{n-1} \leq \max \{ [f], [g], [h] \} e^{(b + \frac{1}{2} |s|^2)t} = \ell \pi_x.
\]
Hence, the map \( x \mapsto \bar{u}^\Delta(t_n, x, p) \) is uniformly Lipschitz continuous with a Lipschitz coefficient \( \ell \pi_x \) defined in (4.6).

Uniform boundedness. It follows immediately from (3.4), (3.5) and (3.7) that
\[
|\bar{u}^\Delta(t_n, x, p)| \leq \sum_{i=1}^{I} \min \left\{ \max \left\{ \mathbb{E}[\bar{u}^\Delta(t_{n+1}, \bar{X}_{n+1}^y, \pi^y_x(p))], \pi^y_x(p)^T f(t_n, x) \right\}, \pi^y_x(p)^T h(t_n, x) \right\} \|\pi^y_x(p)\|
\leq \max \left\{ \|\bar{u}^\Delta(t_{n+1}, \cdot, \cdot)\|_{x^*}, \|f\|_x, \|h\|_x \right\}.
\]
A recursion implies
\[
|\bar{u}^\Delta(t_n, x, p)| \leq \max \left\{ \|\bar{u}^\Delta(t_{n+1}, \cdot, \cdot)\|_{x^*}, \|f\|_x, \|h\|_x \right\}
\leq \max \left\{ \|\bar{u}^\Delta(t_{n+2}, \cdot, \cdot)\|_{x^*}, \|f\|_x, \|h\|_x \right\}
\]
\[\vdots\]
\[
\leq \max \left\{ \|\bar{u}^\Delta(t_N, \cdot, \cdot)\|_{x^*}, \|f\|_x, \|h\|_x \right\} = \max \left\{ \|g\|_x, \|f\|_x, \|h\|_x \right\}.
\]
Hence, the map \( (t_n, x, p) \mapsto \bar{u}^\Delta(t_n, x, p) \) is uniformly bounded. \( \square \)

Lemma 4.3 (Uniform Lipschitz continuity in p). Assuming that Lemma 4.1 holds, then the map \( p \mapsto \bar{u}^\Delta(t_n, x, p) \) is uniformly Lipschitz continuous, i.e. for every \( n \in \{0, \ldots, N\} \), it holds that
\[
\forall x \in \mathbb{R}^d, p, q \in \Delta(I), \quad |\bar{u}^\Delta(t_n, x, p) - \bar{u}^\Delta(t_n, x, q)| \leq \ell \pi_p|p - q|,
\]
with \( \ell \pi_p := \max \{ \|f\|_x, \|g\|_x, \|h\|_x \} \).
Proof. We fix $x \in \mathbb{R}^d$, $p, q \in \Delta(I)$ and proceed successively for $n = N, N - 1, \ldots, 0$ to show that $p \mapsto \tilde{u}^\Delta(t_n, x, p)$ is uniformly Lipschitz continuous. By (A1), the base case $n = N$ holds. By recursion hypothesis, we suppose that there exist $\ell i p_1^{n+1} > 0$ such that

$$
|\tilde{u}^\Delta(t_{n+1}, x, p) - \tilde{u}^\Delta(t_n, x, q)| \leq \ell i p_1^{n+1} |p - q|.
$$

(4.7)

We suppose that $0 \leq \tilde{u}^\Delta(t_n, x, q) - \tilde{u}^\Delta(t_n, x, p)$. By (3.7) we have

$$
\tilde{u}^\Delta(t_n, x, q) - \tilde{u}^\Delta(t_n, x, p) = \tilde{u}^\Delta(t_n, x, q) - \sum_{i=1}^{I} \tilde{u}^\Delta(t_n, x, \pi_i^{n,x}(p)) \lambda_{i}^{n,x}(p),
$$

(4.8)

and by [20, Lemma 8.2], there exists $\{\pi_1^{n,x}(p), \ldots, \pi_I^{n,x}(p)\} \in \Delta(I)$ such that

$$
q = \sum_{i=1}^{I} \pi_i^{n,x}(q) \lambda_{i}^{n,x}(x, q), \quad \text{and} \quad |p - q| = \sum_{i=1}^{I} |\pi_i^{n,x}(p) - \pi_i^{n,x}(q)| \lambda_{i}^{n,x}(p).
$$

(4.9)

Because the map $p \mapsto \tilde{u}^\Delta(t_n, x, p)$ is convex, it follows from (3.4), (4.8) and (4.9) that

$$
\tilde{u}^\Delta(t_n, x, q) - \tilde{u}^\Delta(t_n, x, p) \leq \sum_{i=1}^{I} \left( \tilde{u}^\Delta(t_n, x, \pi_i^{n,x}(q)) - \tilde{u}^\Delta(t_n, x, \pi_i^{n,x}(p)) \right) \lambda_{i}^{n,x}(p),
$$

(4.10)

$$
\leq \sum_{i=1}^{I} \left( \tilde{Y}_n^{n,x,p} - \tilde{Y}_n^{n,x,q} \right) \lambda_{i}^{n,x}(p),
$$

where $p \mapsto \tilde{Y}_n^{n,x,p}$ is defined by

$$
\tilde{Y}_n^{n,x,p} := \min \left\{ \max \left\{ \mathbb{E} \left[ \tilde{u}^\Delta(t_{n+1}, \tilde{X}_n^{n,x}, p) \right] \right\}, p^{I} f(t_n, x) \right\}. \quad \text{(4.10)}
$$

We apply the inequality (2.1) to the right-hand side of (4.10), then by (4.7), (A1), and (4.9) we obtain that

$$
\tilde{u}^\Delta(t_n, x, q) - \tilde{u}^\Delta(t_n, x, p) \leq \max \left\{ \ell i p_1^{n+1}, \|f\|_x, \|h\|_x \right\} \sum_{i=1}^{I} \left| \pi_i^{n,x}(p) - \pi_i^{n,x}(q) \right| \lambda_{i}^{n,x}(p),
$$

(4.11)

$$
= \max \left\{ \ell i p_1^{n+1}, \|f\|_x, \|h\|_x \right\} |p - q|.
$$

If $0 \leq \tilde{u}^\Delta(t_n, x, p) - \tilde{u}^\Delta(t_n, x, q)$, we commute the role of $p$ and $q$ in the above steps to get

$$
|\tilde{u}^\Delta(t_n, x, p) - \tilde{u}^\Delta(t_n, x, q)| \leq \ell i p_1^{n+1} |p - q|,
$$

with $\ell i p_1^n := \max \left\{ \ell i p_1^{n+1}, \|f\|_x, \|h\|_x \right\}$. It follows immediately by recursion that

$$
\ell i p_1^n \leq \max \left\{ \ell i p_1^{n+1}, \|f\|_x, \|h\|_x \right\} \leq \max \left\{ \ell i p_1^{n+2}, \|f\|_x, \|h\|_x \right\}
$$

$$
\vdots
$$

(4.12)

$$
\leq \max \left\{ \ell i p_1^N, \|f\|_x, \|h\|_x \right\} = \max \left\{ \|g\|, \|f\|_x, \|h\|_x \right\} =: \ell i p_1.
$$

Hence, the map $p \mapsto \tilde{u}^\Delta(t_n, x, p)$ is uniformly Lipschitz continuous with a Lipschitz coefficient $\ell i p_1$ defined by (4.12).

\[ Q.E.D. \]

Lemma 4.4 (Uniform almost Hölder continuity in $t$). Assuming that Lemma 4.1 holds, then the map $t \mapsto \tilde{u}^\Delta(t, x, p)$ is uniformly almost Hölder continuous, i.e. it holds that

$$
\forall s, t \in [0, T], (x, p) \in \mathbb{R}^d \times \Delta(I), |\tilde{u}^\Delta(s, x, p) - \tilde{u}^\Delta(t, x, p)| \leq h o l_t \left( \sqrt{t} + |s - t|^{\frac{1}{2}} \right),
$$

with $h o l_t := 2 \max \{\ell i p_1, |f|_x, |h|_x\} \sqrt{\|a\|_Z \|b\|_Z} + \|b\|_Z$.

Proof. We fix $(x, p) \in \mathbb{R}^d \times \Delta(I)$. We start the proof with $s, t \in \Pi t$. For every $n \in \{0, \ldots, N - 1\}$ and $n' \in \{0, \ldots, N - n\}$, we have that

$$
|\tilde{u}^\Delta(t_n, x, p) - \tilde{u}^\Delta(t_{n+n'}, x, p)| \leq |\tilde{u}^\Delta(t_n, x, p) - \mathbb{E}[\tilde{u}^\Delta(t_{n+n'}, \tilde{X}_{n+n'}^{n,x}, p)]| + |\mathbb{E}[\tilde{u}^\Delta(t_{n+n'}, \tilde{X}_{n+n'}^{n,x}, p)] - \tilde{u}^\Delta(t_{n+n'}, x, p)|,
$$

(4.13)
where the discrete process \( \tilde{X}^{n,x}_{n+n'} \) is given by (3.1). By Lemma 4.1 and (A1) it is straightforward to get

\[
|\mathbb{E}[\tilde{u}^{\Delta}(t_{n+n'}, \tilde{X}^{n,x}_{n+n'}, p)] - \tilde{u}^{\Delta}(t_{n+n'}, x, p)| \leq \epsilon t \mathbb{E}\left[ \sum_{j=n}^{n+n'-1} \left( b(t_j, \tilde{X}^{n,x}) \Delta t + a(t_j, \tilde{X}^{n,x}) \xi_j \sqrt{\Delta t} \right) \right]
\]

(4.14)

It remains to derive an estimate for the first term on the right-hand side of (4.13). We proceed in two steps to show that

\[
|\tilde{u}^{\Delta}(t_n, x, p) - \mathbb{E}[\tilde{u}^{\Delta}(t_{n+n'}, \tilde{X}^{n,x}_{n+n'}, p)]| \leq \max\{||f||, ||A||\} \sqrt{||a||^2 + ||b||^2} \sqrt{t_{n+n'} - t_n}.
\]

**Step 1:** We prove that

\[
\tilde{u}^{\Delta}(t_n, x, p) - \mathbb{E}[\tilde{u}^{\Delta}(t_{n+n'}, \tilde{X}^{n,x}_{n+n'}, p)] \leq -[h] \mathbb{E}\left[ \|a\|^2 + \|b\|^2 \right] \sqrt{t_{n+n'} - t_n}.
\]

Let \( p \in \{p_1, \ldots, p_M\} \subset \Delta(I) \). The formula (3.4) implies that

\[
\tilde{u}^{\Delta}(t_n, x, p) = \min_{\mu \in T^N_{(n, n')}} \mathbb{E}[I_{\mu}(t_{n+n'}, \tilde{X}^{n,x}_{n+n'}, p)]
\]

(4.17)

which by induction yields

\[
\tilde{u}^{\Delta}(t_n, x, p) \geq \min_{\mu \in T^N_{(n, n')}} \mathbb{E}[I_{\mu}(t_{n+n'}, \tilde{X}^{n,x}_{n+n'}, p)]
\]

(4.18)

where in (4.18) we used the fact that \( \tilde{u}^{\Delta}(t_{n+n'}, \tilde{X}^{n,x}_{n+n'}, p) \leq \mathbb{E}[h](t_{n+n'}, \tilde{X}^{n,x}_{n+n'}, p) \). Thus by (A1), we have for all \( p \in \{p_1, \ldots, p_M\} \subset \Delta(I) \),

\[
\tilde{u}^{\Delta}(t_n, x, p) \leq -[h] \mathbb{E}\left[ \|a\|^2 + \|b\|^2 \right] \sqrt{t_{n+n'} - t_n} + \mathbb{E}[\tilde{u}^{\Delta}(t_{n+n'}, \tilde{X}^{n,x}_{n+n'}, p)].
\]

Now let \( p \in \Delta(I) \setminus \{p_1, \ldots, p_M\} \) and use (3.7). Since \( \{\pi_{1,n}(p), \ldots, \pi_{n,n}(p)\} \subset \{p_1, \ldots, p_M\} \), we can apply (4.19) to get

\[
\tilde{u}^{\Delta}(t_n, x, \pi_{i,n}(p)) \leq -[h] \mathbb{E}\left[ \|a\|^2 + \|b\|^2 \right] \sqrt{t_{n+n'} - t_n} + \mathbb{E}[\tilde{u}^{\Delta}(t_{n+n'}, \tilde{X}^{n,x}_{n+n'}, \pi_{i,n}(p))].
\]

Next, we multiply both sides of (4.20) by \( \lambda_{i,n}(p) \), then sum for \( i = 1, \ldots, I \). It follows by (3.7) and the convexity of \( p \mapsto \tilde{u}^{\Delta}(t_{n+n'}, \tilde{X}^{n,x}_{n+n'}, p) \) that

\[
\tilde{u}^{\Delta}(t_n, x, p) \leq -[h] \mathbb{E}\left[ \|a\|^2 + \|b\|^2 \right] \sqrt{t_{n+n'} - t_n} + \mathbb{E}\left[ \sum_{i=1}^{I} \tilde{u}^{\Delta}(t_{n+n'}, \tilde{X}^{n,x}_{n+n'}, \pi_{i,n}(p)) \lambda_{i,n}(p) \right]
\]

(4.21)

\[
\tilde{u}^{\Delta}(t_n, x, p) \leq -[h] \mathbb{E}\left[ \|a\|^2 + \|b\|^2 \right] \sqrt{t_{n+n'} - t_n} + \mathbb{E}\left[ \sum_{i=1}^{I} \tilde{u}^{\Delta}(t_{n+n'}, \tilde{X}^{n,x}_{n+n'}, \pi_{i,n}(p)) \right].
\]

This proves (4.16).

**Step 2:** We prove that

\[
\tilde{u}^{\Delta}(t_n, x, p) - \mathbb{E}[\tilde{u}^{\Delta}(t_{n+n'}, \tilde{X}^{n,x}_{n+n'}, p)] \leq [h] \mathbb{E}\left[ \|a\|^2 + \|b\|^2 \right] \sqrt{t_{n+n'} - t_n}.
\]

Let \( p \in \{p_1, \ldots, p_M\} \subset \Delta(I) \). The formula (3.4) implies that

\[
\tilde{u}^{\Delta}(t_n, x, p) \leq \max_{\mu \in T^N_{(n, n')}} \mathbb{E}[I_{\mu}(t_{n+n'}, \tilde{X}^{n,x}_{n+n'}, p)]
\]

(4.22)

which by induction yields

\[
\tilde{u}^{\Delta}(t_n, x, p) \leq \max_{\mu \in T^N_{(n, n')}} \mathbb{E}[I_{\mu}(t_{n+n'}, \tilde{X}^{n,x}_{n+n'}, p)]
\]

(4.23)
where in (4.23) we used the fact that $p^T f(t_{n+n'}, X_{n+n'}^{x}) \leq \bar{u}^\Delta(t_{n+n'}, X_{n+n'}^{x}, p)$. Thus, by (A1), we have for all $p \in \{p_1, \ldots, p_M\} \subset \Delta(I)$

$$
\bar{u}^\Delta(t_n, x, p) \leq [f] \sqrt{\|a\|_{Z_0}^2 + \|b\|_{Z_0}^2} \sqrt{t_{n+n'} - t_n} + \mathbb{E}[\bar{u}^\Delta(t_{n+n'}, X_{n+n'}^{x}, p)]
$$

Now, let $p \in \Delta(I) \setminus \{p_1, \ldots, p_M\}$. By (3.7) and because of $\{\pi_i^{n_{N}, x_{n+n'}}(p), \ldots, \pi_i^{n_{N}, x_{n+n'}}(p)\} \subset \{p_1, \ldots, p_M\}$, we can apply (4.24) to get

$$
\bar{u}^\Delta(t_n, x, \pi_i^{n_{N}, x_{n+n'}}(p)) \leq [f] \sqrt{\|a\|_{Z_0}^2 + \|b\|_{Z_0}^2} \sqrt{t_{n+n'} - t_n} + \mathbb{E}[\bar{u}^\Delta(t_{n+n'}, X_{n+n'}^{x}, \pi_i^{n_{N}, x_{n+n'}}(p))].
$$

Next, we multiply both sides of (4.25) by $\lambda_{i}^{n_{N}, x_{n+n'}}(p)$, then sum for $i = 1, \ldots, I$. It follows by (3.7) and the convexity of $p \mapsto \bar{u}^\Delta(t_{n+n'}, x, p)$ that

$$
\sum_{i=1}^{I} \bar{u}^\Delta(t_n, x, \pi_i^{n_{N}, x_{n+n'}}(p)) \lambda_{i}^{n_{N}, x_{n+n'}}(p) \geq \bar{u}^\Delta(t_n, x, p).
$$

This proves (4.21).

Now, we combine the estimates (4.14) and (4.15), which provides that

$$
|\bar{u}^\Delta(t_n, x, p) - \bar{u}^\Delta(t_{n+n'}, x, p)| \leq \max\{\epsilon_1p_x, \|f\|, [f]\} \sqrt{\|a\|_{Z_0}^2 + \|b\|_{Z_0}^2} (\sqrt{t_{n+n'} - t_n} + \sqrt{t_{n+n'} - t_t}).
$$

We conclude the proof by considering the case where $t \in [t_n, t_{n+1})$ and $s \in [t_{n+n'}, t_{n+n'+1})$. We estimate the term $\sqrt{t_{n+n'} - t_t}$ appearing in the right hand side of (4.26) as follows. We use the inequality $\sqrt{x+y+z} \leq \sqrt{x} + \sqrt{y} + \sqrt{z}$ for any positive number $x, y, z$; and since $t \in [t_n, t_{n+1})$ and $s \in [t_{n+n'}, t_{n+n'+1})$, we get

$$
\sqrt{t_{n+n'} - t_t} = \sqrt{(t-s) + (s-t)} \leq 2\sqrt{t_s - t_t}.
$$

It follows immediately that

$$
\bar{u}^\Delta(t_n, x, p) - \bar{u}^\Delta(t_{n+n'}, x, p) \leq 2 \max\{\epsilon_1 p_x, \|f\|, [f]\} \sqrt{\|a\|_{Z_0}^2 + \|b\|_{Z_0}^2} (\sqrt{t_{n+n'} - t_t} + \sqrt{s-t}).
$$

Finally, by (3.7) and (4.27), it implies that

$$
|\bar{u}^\Delta(s, x, p) - \bar{u}^\Delta(t, x, p)| \leq 2 \max\{\epsilon_1 p_x, \|f\|, [f]\} \sqrt{\|a\|_{Z_0}^2 + \|b\|_{Z_0}^2} (\sqrt{t_s - t_t} + \sqrt{s-t}).
$$

$\square$

5. Viscosity solution property

In this section, we show the second part of Theorem 3.1. Here, $w$ denotes the limit of a subsequence of $\{\bar{u}^\Delta\}$ on arbitrary compact subsets of $[0, T] \times \mathbb{R}^d \times \Delta(I)$ obtained by the Arzelá-Ascoli theorem. As a limit of $\bar{u}^\Delta$, which is uniformly continuous and bounded function due to Lemma 4.1, Lemma 4.3, Lemma 4.4, $w$ is bounded and uniformly continuous. As a limit of $\bar{u}^\Delta$, which is convex in $p$ by construction due to (3.7) and (3.8), $w$ is convex in $p$.

5.1. Viscosity subsolution property of $w$.

**Proposition 5.1.** The limit $w$ is a viscosity subsolution of (1.1) on $[0, T] \times \mathbb{R}^d \times \text{Int}(\Delta(I))$.

**Proof.** Let $\varphi : [0, T] \times \mathbb{R}^d \times \Delta(I) \to \mathbb{R}$ be a smooth test function such that $w - \varphi$ has a strict global maximum at $(\bar{t}, \bar{x}, \bar{p})$. We have to show that $w$ satisfies (2.3) at $(\bar{t}, \bar{x}, \bar{p})$, which equivalently consists of showing that

1. $\lambda(\bar{p}, D^2_{\bar{t}}w(\bar{t}, \bar{x}, \bar{p})) \geq 0$, and
2. $w(\bar{t}, \bar{x}, \bar{p}) \leq \bar{p}^T h(\bar{t}, \bar{x})$, and
3. $w(\bar{t}, \bar{x}, \bar{p}) > \bar{p}^T f(\bar{t}, \bar{x})$ implies $(-\bar{c}_\ell - L)w(\bar{t}, \bar{x}, \bar{p}) \leq 0$.

As a limit of a convex function, $w$ is also a convex function in $p$ and (see [24, Theorem 1]) since $\bar{p} \in \Delta(I)$, thus (P1) holds. By (3.4), $w(\bar{n}, \bar{x}, \bar{p}) \in \{0, \ldots, N\} \times \mathbb{R}^d \times \Delta(I)$, it holds that $\bar{u}^\Delta(t_n, x, p) \leq p^T h(t_n, x)$. Thus by continuity, we obtain (P2).

It remains to show (P3). For $\Delta_k = \Delta_k N_k$, $\Delta_k p_k$, s.t., $N_k \to \infty$ (i.e. $\Delta_k \to 0$), $\Delta_k p_k \to 0$ for $k \to \infty$ we consider a sequence $(\bar{t}_k, \bar{x}_k, \bar{p}_k) \in \mathbb{N}$ with $\bar{t}_k = n_k \Delta k, \bar{n}_k \in \mathbb{N}, \bar{p}_k \in \mathbb{N} \Delta k$ such that $(\bar{t}_k, \bar{x}_k, \bar{p}_k) \to (\bar{t}, \bar{x}, \bar{p})$ for $k \to \infty$ and s.t. $\bar{u}^\Delta - \varphi$ admits a global maximum at $(\bar{t}_k, \bar{x}_k, \bar{p}_k)$ with maximum
value equal to 0, cf. [2, Lemma 2.4]. Define \( \varphi^\Delta := \varphi + (\tilde{u}^\Delta - \varphi)(\tilde{t}_k, \tilde{x}_k, \tilde{p}_k) \). Then, for all \((x, p) \in \mathbb{R} \times \Delta(I)\) we have
\[
(\tilde{u}^\Delta - \varphi^\Delta)(\tilde{t}_k + \Delta t_k, x, p) \leq (\tilde{u}^\Delta - \varphi^\Delta)(\tilde{t}_k, \tilde{x}_k, \tilde{p}_k) = 0. \tag{5.1}
\]

To prove \((P3)\) we use arguments similar to [25, Chapter 3], i.e. we assume in contrario that
\[
w(t, \tilde{x}, \tilde{p}) > \tilde{p}_k^1 f(t, \tilde{x}) \quad \text{and} \quad (-\tilde{c}_t - \mathcal{L})w(t, \tilde{x}, \tilde{p}) > 0,
\]
and work toward a contradiction on the dynamic programming principle, which implies that
\[
\tilde{u}^\Delta(\tilde{t}_k, \tilde{x}_k, \tilde{p}_k) \leq \max_{\nu \in T_{n_k}^k \setminus \left\{ n_k \right\}} \mathbb{E}\left[\tilde{u}^\Delta(\tilde{t}_k + \Delta t_k, \tilde{X}_{n_k+1}^{\tilde{x}_k}, \tilde{p}_k) \mathbb{I}_{\nu \geq n_k + 1} + \tilde{p}_k^1 f(t, \tilde{x}, \tilde{p}) \mathbb{I}_{\nu < n_k + 1}\right]. \tag{5.2}
\]

By (5.1) and the continuity of \(\varphi\) and \(f\), we can find \(\delta > 0\) and \(k\) large enough (equivalently \(\Delta t_k\) small enough) such that
\[
\varphi^\Delta(\tilde{t}_k, \tilde{x}_k, \tilde{p}_k) \geq \tilde{p}_k^1 f(\tilde{t}, \tilde{x}) + \delta \quad \text{and} \quad (-\tilde{c}_t - \mathcal{L})\varphi^\Delta(\tilde{t}_k, \tilde{x}_k, \tilde{p}_k) \geq \delta,
\]
for \((\tilde{t}_k, \tilde{x}_k, \tilde{p}_k) \in \Delta t_k \mathcal{B}\), where \(\mathcal{B}\) is the unit ball of \([0, T] \times \mathbb{R}^d \times \Delta(I)\) centered at \((\tilde{t}, \tilde{x}, \tilde{p})\). We choose an arbitrary stopping rule \(\nu \in T_{n_k}^k \setminus \left\{ n_k \right\}\). By (5.1), we have that
\[
\mathbb{E}\left[\tilde{u}^\Delta((\tilde{t}_k + \Delta t_k) \wedge t, \tilde{X}_{n_k+1}^{\tilde{x}_k}, \tilde{p}_k) - \tilde{u}^\Delta(\tilde{t}_k, \tilde{x}_k, \tilde{p}_k) \right]
= \mathbb{E}\left[\varphi^\Delta((\tilde{t}_k + \Delta t_k) \wedge t, \tilde{X}_{n_k+1}^{\tilde{x}_k}, \tilde{p}_k) \right] + \mathbb{E}\left[\varphi^\Delta((\tilde{t}_k + \Delta t_k) \wedge t, \tilde{X}_{n_k+1}^{\tilde{x}_k}, \tilde{p}_k) \right] - \varphi^\Delta(\tilde{t}_k, \tilde{x}_k, \tilde{p}_k) \tag{5.4}
\]

Applying the Taylor formula to the term \(\varphi^\Delta((\tilde{t}_k + \Delta t_k) \wedge t, \tilde{X}_{n_k+1}^{\tilde{x}_k}, \tilde{p}_k)\) appearing in the right-hand side of (5.4), we get
\[
\mathbb{E}\left[\tilde{u}^\Delta((\tilde{t}_k + \Delta t_k) \wedge t, \tilde{X}_{n_k+1}^{\tilde{x}_k}, \tilde{p}_k) \right] - \tilde{u}^\Delta(\tilde{t}_k, \tilde{x}_k, \tilde{p}_k)
= \mathbb{E}\left[\tilde{u}^\Delta(\tilde{t}_k, \tilde{x}_k, \tilde{p}_k) \right] + \mathbb{E}\left[\varphi^\Delta(\tilde{t}_k, \tilde{x}_k, \tilde{p}_k) \right] + C \Delta t_k \mathcal{O}(\Delta t_k^{1/2}). \tag{5.5}
\]

For \(k\) large enough (equivalently \(\Delta t_k\) small enough), \((\tilde{t}_k, \tilde{x}_k, \tilde{p}_k) \in \Delta t_k \mathcal{B}\). Thus the assertion (5.3) applies to \((-\tilde{c}_t - \mathcal{L})\varphi^\Delta\). It follows directly from (5.5) that
\[
\mathbb{E}\left[\tilde{u}^\Delta((\tilde{t}_k + \Delta t_k) \wedge t, \tilde{X}_{n_k+1}^{\tilde{x}_k}, \tilde{p}_k) - \tilde{u}^\Delta(\tilde{t}_k, \tilde{x}_k, \tilde{p}_k) \right] \leq C \Delta t_k \mathcal{O}(\Delta t_k^{1/2}) + \mathbb{E}\left[\varphi^\Delta(\tilde{t}_k, \tilde{x}_k, \tilde{p}_k) \right] \tag{5.6}
\]

Note that if \(\nu < n_k + 1\), then \((t, \tilde{X}_{n_k+1}^{\tilde{x}_k}, \tilde{p}_k) \in \Delta t_k \mathcal{B}\). Therefore, on one hand, by (5.1) and the continuity of \((\tilde{u}^\Delta - \varphi^\Delta)\) it implies for the second term in the right-hand side of (5.6) that \((\tilde{u}^\Delta - \varphi^\Delta)(t, \tilde{X}_{n_k+1}^{\tilde{x}_k}, \tilde{p}_k) = 0\).

Thus we have
\[
\mathbb{E}\left[\tilde{u}^\Delta((\tilde{t}_k + \Delta t_k) \wedge t, \tilde{X}_{n_k+1}^{\tilde{x}_k}, \tilde{p}_k) - \tilde{u}^\Delta(\tilde{t}_k, \tilde{x}_k, \tilde{p}_k) \right] \leq -C \Delta t_k \mathcal{O}(\Delta t_k^{1/2}) - \gamma \mathbb{P}[\nu \geq n_k + 1],
\]
with
\[-\gamma = \max_{\Delta t_k \mathcal{E} \subseteq \mathcal{B}} (\tilde{u}^\Delta - \varphi) < 0.\]

On the other hand, the assertion (5.3) applies to \(\varphi^\Delta\). Thus we have
\[
\mathbb{E}\left[\tilde{u}^\Delta(\tilde{t}_k + \Delta t_k, \tilde{X}_{n_k+1}^{\tilde{x}_k}, \tilde{p}_k) \right]
= \mathbb{E}\left[\tilde{u}^\Delta(\tilde{t}_k + \Delta t_k, \tilde{X}_{n_k+1}^{\tilde{x}_k}, \tilde{p}_k) \mathbb{I}_{\nu \geq n_k + 1} + \tilde{u}^\Delta(t, \tilde{X}_{n_k+1}^{\tilde{x}_k}, \tilde{p}_k) \mathbb{I}_{\nu < n_k + 1} \right] \tag{5.8}
\]

Hence, combining (5.7) and (5.8), we arrive at
\[
\tilde{u}^\Delta(\tilde{t}_k, \tilde{x}_k, \tilde{p}_k) \geq \mathbb{E}\left[\tilde{u}^\Delta(\tilde{t}_k + \Delta t_k, \tilde{X}_{n_k+1}^{\tilde{x}_k}, \tilde{p}_k) \mathbb{I}_{\nu \geq n_k + 1} + \tilde{p}_k^1 f(t, \tilde{x}, \tilde{p}) \mathbb{I}_{\nu < n_k + 1} \right] + C \Delta t_k \mathcal{O}(\Delta t_k^{1/2}) + (\gamma \wedge \delta).
\]

Since \(\nu\) is arbitrary, the above inequality provides the desired contradiction to (5.2). □
5.2. Viscosity supersolution property of $w$. To establish the viscosity supersolution property of the candidate limit $w$, we construct martingale processes that satisfy a one-step-ahead dynamic programming principle.

We note that (3.7) implies

$$
\hat{u}^{\Delta}(t_n, x, p) = \sum_{i=1}^{I} \min \left\{ \max \left\{ \mathbb{E}\left[ \hat{u}^{\Delta}(t_{n+1}, X_{n+1}^{i}, \pi_i^{n,x}(p)) \right] \right\}, (\pi_i^{n,x}(p))^T f(t_n, x) \right\}, (\pi_i^{n,x}(p))^T h(t_n, x) \right\} \lambda_i^{n,x}(p).
$$

(5.9)

Similar to [3, 5, 13], it is now possible to construct the so called one-step a posteriori martingales, which start at $p$ and jump then to one of the support points of the convex hull $\pi_1^{n,x}(p), \ldots, \pi_I^{n,x}(p)$, $(n, x, p) \in \{0, \ldots, N\} \times \mathbb{R}^d \times \{p_1, \ldots, p_M\}$.

**Definition 5.2.** For all $i = 1, \ldots, I$ and $(n, x, p) \in \{0, \ldots, N\} \times \mathbb{R}^d \times \{p_1, \ldots, p_M\}$ we define the one-step feedbacks $p_{n+1}^{i,x,p}$ as $\{\pi_1^{n,x}(p), \ldots, \pi_I^{n,x}(p)\}$-valued random variables that are independent of $\sigma\{B_s : 0 \leq s \leq T\}$, such that

- for $n = 0, \ldots, N - 1$
  - if $p_i = 0$, set $p_{n+1}^{i,x,p} = p$.
  - if $p_i > 0$, choose $p_{n+1}^{i,x,p}$ among $\{\pi_1^{n,x}(p), \ldots, \pi_I^{n,x}(p)\}$ with probability

$$
\mathbb{P}\left[p_{n+1}^{i,x,p} = \pi_i^{n,x}(p) \mid (p_{n,y,q}^{i,x,p})_{y \in \{1, \ldots, I\}, (n,y,q) \in \{1, \ldots, k\} \times \mathbb{R}^d \times \{p_1, \ldots, p_M\}} = \frac{(\pi_i^{n,x}(p))}{p_i} \lambda_i^{n,x}(p).
$$

- for $k = N$, set $p_{N+1}^{i,x,p} = e_i$, where $\{e_i : i = 1, \ldots, I\}$ is the canonical basis of $\mathbb{R}^I$.

Moreover, we define $p_{n+1}^{x,p} = p_{n+1}^{i,x,p}$, where the index $i$ is a random variable with law $\mathbb{P}[i = i] = p_i$, independent of the processes $B$ and ($p_{n,x,p}^{i,x,p})_{y \in \{1, \ldots, I\}, (n,x,p) \in \{1, \ldots, N\} \times \mathbb{R}^d \times \{p_1, \ldots, p_M\}}$.

For all $(n, x, p) \in \{0, \ldots, N\} \times \mathbb{R}^d \times \{p_1, \ldots, p_M\}$, the process $p_{n+1}^{x,p}$ defined by Definition 5.2 is a martingale. Furthermore, it satisfies the following one-step dynamic programming principle.

**Lemma 5.3.** For all $(n, x, p) \in \{0, \ldots, N\} \times \mathbb{R}^d \times \{p_1, \ldots, p_M\}$ it holds that

$$
\hat{u}^{\Delta}(t_n, x, p) = \mathbb{E}\left[ \min \left\{ \hat{u}^{\Delta}(t_{n+1}, X_{n+1}^{i}, p_{n+1}^{i,x,p}), (p_{n+1}^{i,x,p})^T f(t_n, x) \right\}, (p_{n+1}^{i,x,p})^T h(t_n, x) \right\} \right].
$$

**Proof.** The proof is similar to the one of [13, Lemma 3.11].

We fix $(n, x) \in \{0, \ldots, N\} \times \mathbb{R}^d$, and define the map $p \in \{p_1, \ldots, p_M\} \mapsto F^{\prime}(p)$ by

$$
F(p) := \mathbb{E}\left[ \min \left\{ \hat{u}^{\Delta}(t_{n+1}, X_{n+1}^{i}, p_{n+1}^{i,x,p}), (p_{n+1}^{i,x,p})^T f(t_n, x) \right\}, (p_{n+1}^{i,x,p})^T h(t_n, x) \right\} \right].
$$

Assume $(p_i) > 0$ for all $i = 1, \ldots, I$. By Definition 5.2 it holds that

$$
\mathbb{E}\left[F(p_{n+1}^{x,p})\right] = \sum_{i=1}^{I} \mathbb{E}\left[ \mathbb{I}_{[i=i]} F(p_{n+1}^{i,x,p}) \right] = \sum_{i=1}^{I} p_i \mathbb{E}\left[F(p_{n+1}^{i,x,p})\right] = \sum_{i=1}^{I} p_i \sum_{i=1}^{I} \left( \frac{(\pi_i^{n,x}(p))}{p_i} \lambda_i^{n,x}(p) \right) f(\pi_i^{n,x}(p)) \lambda_i^{n,x}(p) = \sum_{i=1}^{I} F(\pi_i^{n,x}(p)) \lambda_i^{n,x}(p) \left( \sum_{i=1}^{I} (\pi_i^{n,x}(p)) \right) = \sum_{i=1}^{I} F(\pi_i^{n,x}(p)) \lambda_i^{n,x}(p),
$$

since $\sum_{i=1}^{I} (\pi_i^{n,x}(p)) = 1$. On noting (5.9), the statement follows immediately. \qed

**Proposition 5.4.** The limit $w$ is a viscosity supersolution of (1.1) on $[0, T] \times \mathbb{R}^d \times \Delta(I)$.

**Proof.** Let $\varphi : [0, T] \times \mathbb{R}^d \times \Delta(I) \to \mathbb{R}$ be a smooth test function, such that $(w - \varphi)$ has a strict global minimum at $(t, \bar{x}, \bar{p})$ with $(w - \varphi)(t, \bar{x}, \bar{p}) = 0$ and such that its derivatives are uniformly Lipschitz
continuous in \( p \). We have that \( w \) satisfies the viscosity supersolution property (2.4) at \( (\bar{t}, \bar{x}, \bar{p}) \), which equivalently consists to show that

\[
\lambda(\bar{p}, D^2_p \varphi(\bar{t}, \bar{x}, \bar{p})) \leq 0, \quad \text{or}
\]

\[
\varphi(\bar{t}, \bar{x}, \bar{p}) \geq \bar{p}^T f(\bar{t}, \bar{x}),
\]

and

\[
\varphi(\bar{t}, \bar{x}, \bar{p}) < \bar{p}^T h(\bar{t}, \bar{x}) \quad \text{implies } (-\bar{c}_t - \bar{L}) \varphi(\bar{t}, \bar{x}, \bar{p}) \geq 0.
\]

If (P5) holds, then (2.4) follows immediately. Let us assume that

\[
\lambda(\bar{p}, D^2_p \varphi(\bar{t}, \bar{x}, \bar{p})) > 0.
\]

By (3.4), for every \((n, x, p) \in \{0, \ldots, N\} \times \mathbb{R}^d \times \Delta(I)\), it holds that \( \bar{u}^\Delta(t_n, x, p) \geq \bar{p}^T f(t_n, x) \). Thus by continuity, we obtain (P5).

It remains to prove (P6). For \( \Delta t_k = T/N_k, \Delta p_k, \) s.t., \( \Delta t_k, \Delta p_k \to \infty \) for \( k \to \infty \) we consider a sequence \((\bar{t}_k, \bar{x}_k, \bar{p}_k)_{k \in \mathbb{N}}\) with \( \bar{t}_k = n_k \Delta t_k \in \mathbb{N}, n_k \in \mathbb{N}, \bar{p}_k \in \mathbb{N}\Delta p_k \) such that \((\bar{t}_k, \bar{x}_k, \bar{p}_k) \to (\bar{t}, \bar{x}, \bar{p})\) for \( k \to \infty \) and s.t. \( \bar{u}^\Delta - \varphi \) has a global minimum at \((\bar{t}_k, \bar{x}_k, \bar{p}_k)\). Define \( \varphi^\Delta := \varphi + (\bar{u}^\Delta - \varphi)(\bar{t}_k, \bar{x}_k, \bar{p}_k) \). Then, for all \((x, p) \in \mathbb{R} \times \Delta(I)\) we have

\[
(\bar{u}^\Delta - \varphi^\Delta)(\bar{t}_k + \Delta t_k, x, p) \geq (\bar{u}^\Delta - \varphi^\Delta)(\bar{t}_k, \bar{x}_k, \bar{p}_k) = 0.
\]

By (5.10), there exists a sequence \( \delta, \eta > 0 \) such that for s.l. \( k \) large enough we have

\[
\forall (t, x, p) \in \eta \bar{B}, \ z \in T_{\Delta(I)}(\bar{p}_k), z^T D^2_p \varphi(t, x, p) z > 4\delta |z|^2,
\]

where \( \bar{B} \) is the unit ball of \([0, T] \times \mathbb{R}^d \times \Delta(I)\) centered at \((\bar{t}_k, \bar{x}_k, \bar{p}_k)\).

Furthermore, we assume without loss of generality that outside of \( \eta \bar{B} \), \( \varphi^\Delta \) is still convex on \( \Delta(I) \). Thus for any \((s, x, p) \in [\bar{t}_k, T] \times \mathbb{R}^d \times \Delta(I)\) it holds that

\[
\bar{u}^\Delta(s, x, p) \geq \varphi^\Delta(s, x, p) \geq \varphi^\Delta(s, x, \bar{p}_k) + (p - \bar{p}_k)^T D_p \varphi^\Delta(s, x, \bar{p}_k).
\]

To prove (P6), we use also arguments similar to [25, Chapter 3]. We proceed by contradiction and assume that

\[
w(\bar{t}, \bar{x}, \bar{p}) < \bar{p}^T h(\bar{t}, \bar{x}) \quad \text{and } (-\bar{c}_t - \bar{L})w(\bar{t}, \bar{x}, \bar{p}) < 0.
\]

and work toward a contradiction on Lemma 5.3, which by (3.4) implies for all \((n, x, p) \in \{0, \ldots, N\} \times \mathbb{R}^d \times \{p_1, \ldots, p_M\}\) that

\[
\bar{u}^\Delta(t_n, x, p) \geq \min_{\mu \in T_{\Delta(I)}(\bar{p}_k)} \mathbb{E}\left[\bar{u}^\Delta(t_{n+1}, \bar{X}^n_{n+1}, p_{n+1}) \mathbf{1}_{\{\mu \geq n+1\}} + p_{n+1}^T h(t_n, x) \mathbf{1}_{\{\mu < n+1\}}\right],
\]

where \( p_{n+1} := \bar{p}_k \).

By (5.11) and the continuity of \( \varphi \) and \( h \), we can find \( \delta > 0 \) and \( k \) large enough (equivalently \( \Delta t_k \) small enough) such that

\[
\varphi^\Delta(\bar{t}_k, \bar{x}_k, \bar{p}_k) \leq \bar{p}_k^T h(\bar{t}_k, \bar{x}_k) - \delta \quad \text{and } (-\bar{c}_t - \bar{L})\varphi^\Delta(\bar{t}_k, \bar{x}_k, \bar{p}_k) \leq 0,
\]

for \((\bar{t}_k, \bar{x}_k, \bar{p}_k) \in \Delta t_k \bar{B}, \) where \( \bar{B} \) is now the unit ball of \([0, T] \times \mathbb{R}^d \times \Delta(I)\) centered at \((\bar{t}_k, \bar{x}_k, \bar{p}_k)\). We choose an arbitrary stopping rule \( \mu \in T_{\Delta(I)}(\bar{p}_k) \). By (5.11) and (5.13), we have that

\[
\mathbb{E}\left[\bar{u}^\Delta((\bar{t}_k + \Delta t_k) \wedge t_n, \bar{X}^n_{n+1} \wedge t_n, p_{n+1} \wedge t_n) - \bar{u}^\Delta(\bar{t}_k, \bar{x}_k, \bar{p}_k)\right]
\]

\[
\geq \mathbb{E}\left[\varphi^\Delta((\bar{t}_k + \Delta t_k) \wedge t_n, \bar{X}^n_{n+1} \wedge t_n, \bar{p}_k)\right] - \bar{u}^\Delta(\bar{t}_k, \bar{x}_k, \bar{p}_k)
\]

\[
+ \mathbb{E}\left[1_{\{p_{n+1} \geq n+1\}} - \bar{p}_k\right] \mathbb{E}\left[D_p \varphi^\Delta((\bar{t}_k + \Delta t_k) \wedge t_n, \bar{X}^n_{n+1} \wedge t_n, \bar{p}_k)\right]
\]

\[
= \mathbb{E}\left[\varphi^\Delta((\bar{t}_k + \Delta t_k) \wedge t_n, \bar{X}^n_{n+1} \wedge t_n, \bar{p}_k)\right] - \bar{u}^\Delta(\bar{t}_k, \bar{x}_k, \bar{p}_k).
\]

Applying the Taylor formula to the right-hand side of (5.16), we get

\[
\mathbb{E}\left[\bar{u}^\Delta((\bar{t}_k + \Delta t_k) \wedge t_n, \bar{X}^n_{n+1} \wedge t_n, \bar{p}_k)\right] - \bar{u}^\Delta(\bar{t}_k, \bar{x}_k, \bar{p}_k)
\]

\[
= \mathbb{E}\left[(\bar{u}^\Delta - \varphi^\Delta)((\bar{t}_k + \Delta t_k) \wedge t_n, \bar{X}^n_{n+1} \wedge t_n, \bar{p}_k)\right]
\]

\[
+ (-\bar{c}_t + \bar{L})\mathbb{E}\left[\varphi^\Delta((\bar{t}_k, \bar{x}_k, \bar{p}_k)\Delta t_k \wedge (t_n - \bar{t}_k))\right] + C \Delta t_k \mathcal{O}(\Delta t_k^{1/2}).
\]
For $k$ large enough (equivalently $\Delta$ small enough), $(\bar{t}_k, \bar{x}_k, \bar{p}_k) \in \Delta t_k \bar{B}$. Thus the assertion (5.15) applies to $(-\partial_t + \mathcal{L}) \varphi^\Delta$. It follows immediately from (5.17) that
\[
\mathbb{E}[\bar{u}^\Delta((\bar{t}_k + \Delta t_k) \wedge t_{\mu}, \bar{x}_{\mu}(n_{\mu+1}), \bar{p}_k)] - \bar{u}^\Delta(\bar{t}_k, \bar{x}_k, \bar{p}_k) \\
\geq \mathbb{E}[(\bar{u}^\mu - \varphi^\Delta)((\bar{t}_k + \Delta t_k) \wedge t_{\mu}, \bar{x}_{\mu}(n_{\mu+1}), \bar{p}_k)]
\]
(5.18)
\[
\geq \mathbb{E}[(\bar{u}^\Delta - \varphi^\Delta)(\bar{t}_k + \Delta t_k, \bar{x}_{\mu}(n_{\mu+1}), \bar{p}_k)I_{\{\mu \geq n_{\mu+1}\}} + (\bar{u}^\Delta - \varphi^\Delta)(t_{\mu}, \bar{x}_{\mu}(n_{\mu+1}), \bar{p}_k)I_{\{\mu < n_{\mu+1}\}}].
\]
Note that if $\mu < n_{\mu+1}$, then $(t_{\mu}, \bar{x}_{\mu}(n_{\mu+1}), \bar{p}_k) \in \Delta t_k \bar{B}$. Therefore, on one hand, by (5.11) and the continuity of $(\bar{u}^\Delta - \varphi^\Delta)$ it implies for the second term in the right-hand side of (5.18) that $(\bar{u}^\Delta - \varphi^\Delta)(t_{\mu}, \bar{x}_{\mu}(n_{\mu+1}), \bar{p}_k) = 0$. Thus we have
\[
\mathbb{E}[\bar{u}^\Delta((\bar{t}_k + \Delta t_k) \wedge t_{\mu}, \bar{x}_{\mu}(n_{\mu+1}), \bar{p}_k)] - \bar{u}^\Delta(\bar{t}_k, \bar{x}_k, \bar{p}_k) \geq \gamma \mathbb{P}[n \geq n_{k+1}],
\]
with
\[
\gamma = \min_{\Delta t_k \in \mathcal{B}} (\bar{u}^\Delta - \varphi^\Delta) > 0.
\]
On the other hand, the assertion (5.15) applies to $\varphi^\Delta$. Thus we have
\[
\mathbb{E}[\bar{u}^\Delta((\bar{t}_k + \Delta t_k) \wedge t_{\mu}, \bar{x}_{\mu}(n_{\mu+1}), \bar{p}_k)] \\
= \mathbb{E}[\bar{u}^\Delta((\bar{t}_k + \Delta t_k, \bar{x}_{\mu}(n_{\mu+1}), \bar{p}_k+1)I_{\{\mu \geq n_{\mu+1}\}} + \bar{u}^\Delta(t_{\mu}, \bar{x}_{\mu}(n_{\mu+1}), \bar{p}_k)I_{\{\mu < n_{\mu+1}\}}]
\]
(5.20)
\[
\leq \mathbb{E}[\bar{u}^\Delta((\bar{t}_k + \Delta t_k, \bar{x}_{\mu}(n_{\mu+1}), \bar{p}_k), \bar{p}_k+1)I_{\{\mu \geq n_{\mu+1}\}} + \bar{p}_k^\theta h(t_{\mu}, \bar{x}_k)I_{\{\mu < n_{\mu+1}\}}] - \delta \mathbb{P}[\mu < n_{k+1}].
\]
Hence, combining (5.19) and (5.20), we arrive at
\[
\bar{u}^\Delta(\bar{t}_k, \bar{x}_k, \bar{p}_k) \leq \mathbb{E}[\bar{u}^\Delta((\bar{t}_k + \Delta t_k, \bar{x}_{\mu}(n_{\mu+1}), \bar{p}_k+1)I_{\{\mu \geq n_{\mu+1}\}} + \bar{p}_k^\theta f(t_{\mu}, \bar{x}_k)I_{\{\mu < n_{\mu+1}\}}] - (\gamma + \delta).
\]
Since $\mu$ is arbitrary, the above inequality provides the desired contradiction to (5.14).

6. Numerical results

In the numerical experiments below we take $d = 1$, $T = 1$, $\mathcal{D} = [0, 1]$. We use the Quickhull algorithm for the computation of the discrete convex envelope in all experiments below. We compare the solution $\hat{u}_{\mathcal{D}}^\Delta$ computed using a Feedforward Neural Network and the solution $\tilde{u}_{\mathcal{D}}^\Delta$ computed using a semi-Lagrangian scheme.

**Numerical experiment 1.** In this experiment we determine experimental convergence rates of the numerical approximation for the linear PDE
\[
\partial_t u + a^2 \frac{1}{2} \partial^2 u + H = 0, \text{ in } [0, T] \times \mathcal{D},
\]
with a terminal condition $g(x) = u(T, x)$, where $a(x) = 0.2x(1-x)$, and $H(t, x) = 3\sin(3\pi t)\cos(3\pi x) + \frac{1}{2}(3\pi a(x))^2 \cos(3\pi t) \cos(3\pi x)$. For the given data, (6.1) has the solution $u(t, x) = \cos(3\pi t) \cos(3\pi x)$.

Due to the choice of the diffusion $a$, we may restrict the spatial domain to the interval $[0, 1]$, which is partitioned into uniform line segments $\mathcal{T} \sim \{(x_{t-1}, x_{t})\}_{t=1}^T$, $x_t = t\Delta x$ with the mesh size $\Delta x = 1/L$. The time interval $[0, T]$ is partitioned with time-step $\Delta t = 1/N$. We consider the Euler approximation of the diffusion process associated with the (6.1):
\[
\tilde{X}_{n+1}^\alpha = x_t + a(x_t)\xi^\alpha \sqrt{\Delta t} \quad n = 0, \ldots, N, \quad \ell = 0, \ldots, L,
\]
where $\{\xi^\alpha\}$ are i.i.d random variables that take values $\{-1, 1\}$ with equal probability (i.e., the expectation $\mathbb{E}X_{n+1}^\alpha$ is an average of the two possibilities $\xi^\alpha = \pm 1$).

For the computations we employ a feedforward neural network (FFN) algorithm Algorithm 6.1 and a semi-Lagrangian (SL) algorithm Algorithm 6.2 (cf., e.g. [3]) with piecewise linear interpolation in space. We use a feedforward neural network that contains only one hidden layer with 10 neurons, take $\tanh$ as the activation function for the hidden layers and choose identity function as activation function for the output layer. Furthermore, we use the Levenberg–Marquardt algorithm in Algorithm 6.1 to solve the least-square problem at each time-step.

**Algorithm 6.1 (FFN - Experiment 1).** Set $\hat{u}_{\mathcal{D}}^\Delta(t_n, x_\ell) = g(x_\ell)$ for $\ell = 0, \ldots, L$ and proceed for $n = N - 1, \ldots, 0$ as follows:
- **Compute:**
  \[
  \theta_n = \arg \min_{\theta \in \Theta_n} \frac{1}{L+1} \sum_{\ell=0}^L \left| \hat{u}^\Delta(\ell, x_\ell; \theta) - X_n(x_\ell; \theta) \right|^2,
  \]
\[ \Delta x \]

\[ \text{(6.2)} \]

\[ \text{Algorithm 6.2 (SL algorithm - Experiment 1). Set } \hat{u}_N(t,x) = g(x) \text{ for } x \in \{x_0, \ldots, x_L\} \text{ and proceed for } k = N - 1, \ldots, 0 \text{ as follows:} \]

\[ \text{o For } \ell = 0, \ldots, L \text{ set:} \]

\[ \hat{u}_N^\Delta(t_n,x_{\ell}) = E[\Phi_n(\tilde{X}_{n+1};\theta_n)] + \Delta t H(t_n,x_{\ell}). \]

We display the respective solutions computed by Algorithm 6.1 and Algorithm 6.2 in Figure 6.1 computed for \( \Delta x = \Delta t = 1.56 \times 10^{-2} \). With \( \Delta x = \Delta t = 1.56 \times 10^{-2} \), both numerical solutions are graphically similar, with noticeable differences at extremal points, i.e., around \( x = 0 \), between (0.6, 0.7), and around \( x = 1 \). We notice similar differences at \( t = 0, 0.3 \).

In Table 6.1 we examine convergence order of the error of respective algorithms; we measure the error in the maximum norm over the discrete space-time grid (MAX) and in the root mean square norm (RMS) for decreasing discretization parameters \( \Delta x \) and \( \Delta t \); we observe first order of convergence with respect to both discretization parameters.

**Numerical experiment 2.** In this experiment we consider a problem with incomplete information. We take \( I = 2 \) and eliminate one probability variable from the solution by parametrizing \( \Delta(2) = (p, 1 - p) \) for \( p \in [0, 1] \). Hence, we consider the generalized obstacle problem for the transformed solution

\[ \text{(6.2)} \]

\[ \max \left\{ \hat{c}_t u + a \frac{\partial^2 u}{\partial x^2} + H, -\lambda(\cdot, D_p^2 u) \right\} = 0 \text{ in } [0,T] \times \mathcal{D} \times [0,1], \]

with the terminal condition \( u(T,x) = 0 \); we take \( H(t,x,p) = \sin(\pi t) \cos(3\pi x) \sin(3\pi p) \) and \( a(x) = x(1 - x) \).

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<th>( \Delta )</th>
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<td>( 4.44 \times 10^{-3} )</td>
<td>( 7.58 \times 10^{-3} )</td>
</tr>
</tbody>
</table>

| \( 1.56 \times 10^{-1} \) | \( 9.31 \times 10^{-2} \) | \( 4.26 \times 10^{-2} \) | \( 5.67 \times 10^{-2} \) | \( 4.07 \times 10^{-2} \) |
| \( 7.81 \times 10^{-1} \) | \( 2.28 \times 10^{-2} \) | \( 1.44 \times 10^{-2} \) | \( 3.49 \times 10^{-2} \) | \( 2.14 \times 10^{-2} \) |
| \( 3.91 \times 10^{-1} \) | \( 1.32 \times 10^{-2} \) | \( 8.33 \times 10^{-3} \) | \( 1.54 \times 10^{-2} \) | \( 1.07 \times 10^{-2} \) |
| \( 1.95 \times 10^{-1} \) | \( 1.18 \times 10^{-2} \) | \( 4.75 \times 10^{-3} \) | \( 8.16 \times 10^{-3} \) | \( 5.47 \times 10^{-3} \) |
| \( 9.77 \times 10^{-1} \) | \( 1.29 \times 10^{-2} \) | \( 2.82 \times 10^{-3} \) | \( 4.09 \times 10^{-3} \) | \( 2.70 \times 10^{-3} \) |

**Figure 6.1.** Left: plot of the function \( u(t,x) = \cos(3\pi t) \cos(3\pi x) \). Middle: Graph of the solution at \( t = 0.25 \). Right: time evolution of the solution at \( x = 0.75 \). ((—) exact solution, (—) neural network approximation, (—) SL algorithm.)
As in the previous experiments we restrict the spatial domain to the interval $[0, 1]$ and approximate (6.2) on a space-time grid with (uniform) mesh sizes $\Delta x = 1/L$, $\Delta t = 1/N$ and define the discrete process $X_{n+1}^{x_{\ell}}$ analogously as in the previous experiment. The approximation in the probability variable $p \in [0, 1]$ is constructed on a uniform mesh with mesh size $\Delta p = 1/M$. We perform the computations using a feedforward neural network (FFN) algorithm Algorithm 6.3 and a semi-Lagrangian (SL) algorithm Algorithm 6.4 which employs piecewise linear interpolation in the spatial variable.

In Figure 6.2 we display the solution computed with $\Delta t = \Delta x = \Delta p = 1.56 \times 10^{-2}$ at time $t = 0$. To illustrate the effect of the convexity constraint we also display in Figure 6.2 the (nonconvex), analogously as in the previous experiment. The approximation in the probability variable $p \in [0, 1]$ and proceed for $\ell = 0, \ldots, L$, $m = 0, \ldots, M$ set $\tilde{u}_{\kappa}^\Delta(t_n, x_{\ell}, p_m) = 0$ and proceed for $n = N - 1, \ldots, 0$ as follows:

- For $m = 1, \ldots, M$ compute:

$$\theta_n(p_m) = \arg\min_{\theta \in \Theta_n^\kappa} \frac{1}{L + 1} \sum_{\ell=0}^{L} |\tilde{u}_{\kappa}^\Delta(t_{n+1}, x_{\ell}, p_m) - \Phi_n(x_{\ell}; \theta(p))|^2.$$

- For $\ell = 0, \ldots, L$, $m = 0, \ldots, M$ set:

$$y(t_n, x_{\ell}, p_m) = \mathbb{E}[\Phi_n(X_{n+1}, x_{\ell}, p_m)] + \Delta t H(t_n, x_{\ell}, p_m).$$

- For $\ell = 0, \ldots, L$, $m = 0, \ldots, M$ set:

$$\tilde{u}_{\kappa}^\Delta(t_n, x_{\ell}, p_m) = \text{Vex}_p[y(t_n, x_{\ell}, p_0), \ldots, y(t_n, x_{\ell}, p_M)](p_m).$$

**Algorithm 6.3 (FFN algorithm - Experiment 2).** For $\ell = 0, \ldots, L$, $m = 0, \ldots, M$ set $\tilde{u}_{\kappa}^\Delta(t_n, x_{\ell}, p_m) = 0$ and proceed for $n = N - 1, \ldots, 0$ as follows:

- For $m = 1, \ldots, M$ define the map $x \mapsto \tilde{u}_{\kappa}^\Delta(t_n+1, x, p_m)$ as the piecewise linear interpolant of $\left\{\tilde{u}_{\kappa}^\Delta(t_n+1, x_{\ell}, p_m)\right\}_{\ell=0}^{L}$, i.e., for $x \in [x_{\ell}, x_{\ell-1}]$ set

$$\tilde{u}_{\kappa}^\Delta(t_n+1, x, p_m) = \tilde{u}_{\kappa}^\Delta(t_n+1, x_{\ell-1}, p_m) + \frac{(\tilde{u}_{\kappa}^\Delta(t_n+1, x_{\ell}, p_m) - \tilde{u}_{\kappa}^\Delta(t_n+1, x_{\ell-1}, p_m))}{\Delta x}(x - x_{\ell-1}).$$

- For $\ell = 0, \ldots, L$, $m = 0, \ldots, M$ set:

$$y(t_n, x_{\ell}, p_m) = \mathbb{E}[\tilde{u}_{\kappa}^\Delta(t_{n+1}, X_{n+1})] + \Delta t H(t_n, x_{\ell}, p_m).$$

- For $\ell = 0, \ldots, L$, $m = 0, \ldots, M$ set:

$$\tilde{u}_{\kappa}^\Delta(t_n, x_{\ell}, p_m) = \text{Vex}_p[y(t_n, x_{\ell}, p_0), \ldots, y(t_n, x_{\ell}, p_M)](p_m).$$
Figure 6.3. Profiles of the solution computed using the FFN algorithm (+) and the SL algorithm (○) at (x, p) = (0.75, 0.25) (left), (t, p) = (0.25, 0.25) (middle), (t, x) = (0.25, 0.75) (right)

**Numerical experiment 3.** We consider the full problem (1.1) with I = 2. As in the previous experiment we eliminate one probability variable from the solution by parametrizing \(\Delta(2) = (p, 1 − p)\) for \(p ∈ (0, 1)\) and consider the following obstacle problem

\[
\max \left\{ \min \left\{ \delta_1 u + a \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + b \frac{\partial u}{\partial x}, \right\}, u - pf_1 - (1 - p)f_2 \right\}, u - ph_1 - (1 - p)h_2 \right\}, -\lambda(\delta u) \right\} = 0, \tag{6.3}
\]

in \([0, T] × \mathbb{R} × [0, 1]\) with a terminal condition \(u(T, x, p) = p^T g(x), \ g := (g_1, g_2)\) with \(g_1(x) = \bar{g}(x) := \max\{2 − \varepsilon^2, 0\}\), \(i = 1, 2\). The obstacles are chosen as \(h_1 := g_1(x) = \delta_1\) with \(\delta_1 = 1.25 \times 10^{-1}\), \(\delta_2 = 6.5 \times 10^{-2}\) and \(f_i(x) = g_i(x), i = 1, 2\), i.e., the lower obstacle in (6.3) does not depend on \(p\). Furthermore, we take \(b = (r − a^2/2), T = 1, a = 2 \times 10^{-1}, r = 3 \times 10^{-2}\). The above problem setup is motivated by the Dynkin game problem (without asymmetric information), considered in [18], which arises in the pricing of Israeli \(\delta\)-penalty put options. In this example, asymmetric information arises because the holder of the option does not know exactly how much the writer of the option is going to pay her upon termination of the contract, but she has only knowledge of the a priori probability according to which the additional payment \(\delta_i\) is initially chosen. Our numerical study provides the structure of the game’s state space and depict the equilibrium stopping boundaries, which is of importance for understanding the financial meaning of the equilibrium.

The discretization is performed as in the previous experiment. We restrict the spatial domain to the interval \([0, 1]\) and partition the solution domain \([0, T] × [0, 1]\) uniformly in each variable with the respective mesh sizes \(\Delta t = 1/N, \Delta x = 1/L, \Delta p = 1/M\). In the algorithm below we consider the following Euler approximation of the diffusion process associated with the problem (6.3):

\[
X_{n+1}^x = x + b\Delta t + a\xi_n^\nu \sqrt{\Delta t} \quad n = 0, \ldots, N, \ \ell = 0, \ldots, L,
\]

where \(\{\xi_n^\nu\}\) are i.i.d random variables that take values \(-1, 1\) with equal probability. The computations in this section are performed using the feedforward neural network algorithm Algorithm 6.5 where we employ a fully connected feedforward neural network that consists of one hidden layer with 50 neurons.

**Table 6.2.** Experiment 2 – convergence rate.

<table>
<thead>
<tr>
<th>(\Delta)</th>
<th>(\frac{\text{MAX-error rate}}{\text{RMS-error rate}})</th>
<th>(\frac{\text{MAX-error rate}}{\text{RMS-error rate}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conv. t</td>
<td>3.13 \times 10^{-2}</td>
<td>9.63 \times 10^{-3}</td>
</tr>
<tr>
<td>Conv. t</td>
<td>1.56 \times 10^{-2}</td>
<td>5.04 \times 10^{-3}</td>
</tr>
<tr>
<td>Conv. t</td>
<td>7.81 \times 10^{-3}</td>
<td>2.51 \times 10^{-3}</td>
</tr>
<tr>
<td>Conv. t</td>
<td>3.91 \times 10^{-3}</td>
<td>1.19 \times 10^{-3}</td>
</tr>
<tr>
<td>Conv. t</td>
<td>1.95 \times 10^{-3}</td>
<td>5.13 \times 10^{-4}</td>
</tr>
</tbody>
</table>

| Conv. t | 3.13 \times 10^{-2} | 7.77 \times 10^{-3} | 4.56 \times 10^{-3} | 1.13 \times 10^{-2} | 4.59 \times 10^{-3} |
| Conv. t | 1.56 \times 10^{-2} | 4.58 \times 10^{-3} | 2.51 \times 10^{-3} | 4.57 \times 10^{-3} | 1.46 \times 10^{-3} |
| Conv. t | 7.81 \times 10^{-3} | 2.40 \times 10^{-3} | 1.37 \times 10^{-3} | 3.33 \times 10^{-3} | 1.43 \times 10^{-3} |
| Conv. t | 3.91 \times 10^{-3} | 1.61 \times 10^{-3} | 6.58 \times 10^{-4} | 1.16 \times 10^{-3} | 4.11 \times 10^{-4} |
| Conv. t | 1.95 \times 10^{-3} | 5.07 \times 10^{-4} | 2.86 \times 10^{-4} | 5.05 \times 10^{-4} | 2.24 \times 10^{-4} |

| Conv. t | 3.13 \times 10^{-2} | 7.35 \times 10^{-3} | 3.34 \times 10^{-3} | 1.84 \times 10^{-3} |
| Conv. t | 1.56 \times 10^{-2} | 3.60 \times 10^{-3} | 1.56 \times 10^{-3} | 2.10 \times 10^{-3} |
| Conv. t | 7.81 \times 10^{-3} | 1.81 \times 10^{-3} | 1.65 \times 10^{-3} | 1.52 \times 10^{-3} |
| Conv. t | 3.91 \times 10^{-3} | 8.50 \times 10^{-4} | 7.53 \times 10^{-4} | 4.18 \times 10^{-4} |
| Conv. t | 1.95 \times 10^{-3} | 3.65 \times 10^{-4} | 3.28 \times 10^{-4} | 2.69 \times 10^{-4} |
We choose $\tanh$, i.e.
\[
\tanh : x \mapsto \frac{e^x - e^{-x}}{e^x + e^{-x}}.
\]
as the activation function for the hidden layers and identity function as activation function for the output layer. We use the limited-memory (BFGS) quasi-Newton algorithm \cite{nocedal2006numerical, george1981efficient} to solve the nonlinear least-squares problem at each time-step of the algorithm.

**Algorithm 6.5 (FFN - Experiment 3).** For $\ell = 0, \ldots, L$, $m = 1, \ldots, M$, we initialize $\tilde{u}_{\kappa,D}^\Delta(t_n, x_t, p_m) = p_m^\ell g(x_t)$ and proceed for $n = N - 1, \ldots, 0$ as follows:

- For $m = 1, \ldots, M$ compute:
  \[
  \theta_n(p_m) = \arg \min_{\theta \in \Theta_n^p} \frac{1}{L + 1} \sum_{\ell=0}^{L} \left| \tilde{u}_{\kappa,D}^\Delta(t_{n+1}, x_t, p_m) - \Phi_n(x_t; \theta) \right|^2.
  \]
- For $\ell = 0, \ldots, L$, $m = 0, \ldots, M$ set:
  \[
  \bar{v}_{\kappa,D}^\Delta(t_n, x_t, p_m) = \min \left\{ \max \left\{ g(t_n, x_t, p_m), p_m f_1(x_t) + (1 - p_m) f_2(x_t) \right\}, p_m h_1(x_t) + (1 - p_m) h_2(x_t) \right\}.
  \]
- For $\ell = 0, \ldots, L$, $m = 0, \ldots, M$ set:
  \[
  \bar{u}_{\kappa,D}^\Delta(t_n, x_t, p_m) = \exp \left[ \bar{v}_{\kappa,D}^\Delta(t_n, x_t, p_0), \ldots, \bar{v}_{\kappa,D}^\Delta(t_n, x_t, p_M) \right](p_m).
  \]

To illustrate the effect of the convexity constraint in (6.3) we compute the solution $v = v(t, x, p)$ for $p = 0, 0.5, 1$ of the non-constrained problem
\[
(6.4) \quad \max \left\{ \min \left\{ \hat{c}_v + \alpha^2 \frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x}, v - pf_1 - (1 - p)f_2 \right\}, v - ph_1 - (1 - p)h_2 \right\} = 0.
\]
In Figure 6.4 we compare the solution $\tilde{v}_{\kappa,D}^\Delta$ of (6.4) to the solution $\tilde{u}_{\kappa,D}^\Delta$ of (6.3). In the top row in Figure 6.4 we plot the difference $\frac{1}{2} (\tilde{v}_{\kappa,D}^\Delta(t, x, 0) - \tilde{u}_{\kappa,D}^\Delta(t, x, 0.5))$ which takes negative values since $\tilde{v}_{\kappa,D}^\Delta$ is not necessarily convex in the $p$ variable, the difference $\frac{1}{2} (\tilde{u}_{\kappa,D}^\Delta(t, x, 0) - \tilde{u}_{\kappa,D}^\Delta(t, x, 1.5))$ in the bottom row remains positive since $\tilde{u}_{\kappa,D}^\Delta$ is convex in $p$.

In Figure 6.5 we display space-time graphs of the numerical solution computed with $\Delta t = \Delta x = \Delta p = 1.56 \times 10^{-2}$ for different values of $p$ along with the regions where the two obstacles are active; the red color represents the active region of the lower obstacle $\{(t, x); \tilde{u}_{\kappa,D}^\Delta(t, x, p) = p f_1(x) + (1 - p) f_2(x) = \hat{g}(x)\}$ and the green color represents the active region of the upper obstacle $\{(t, x); \tilde{u}_{\kappa,D}^\Delta(t, x, p) = ph_1(x) + (1 - p)h_2(x)\}$. In all figures the regions were marked as active if the distance of the numerical solution to the obstacle at the nodes $x_k$ was below a tolerance $2 \times 10^{-5}$.

The results in Figure 6.5 show that the writer of the contract would exercise when the price of the underlying stock assumes value $\ln(K) \sim 0.7$. This is consistent with the results in \cite{cont2010}. Also, the buyer would stop when the price of the stock is sufficiently low. This is also consistent to the findings of \cite{cont2010}. Interestingly, we also see from Figure 6.5 that the waiting region (gray area) is not connected, a feature which is not observed in the symmetric information case of \cite{cont2010}. In particular, such a feature of the waiting region does not disappear when $p \downarrow 0$ or $p \uparrow 1$, i.e. in the two symmetric information cases. We repeated the experiment for different combinations of the discretization parameters but the obtained results were qualitatively very similar to those in Figure 6.5. We believe that the presence of a portion of the waiting region for small values of $x$ is a numerical artefact, depending on the accuracy of the nonlinear solver used for the neural network approximation. As a matter of fact, the waiting region is observed to be connected (for $p = 0$, i.e. in the symmetric information case) in Figure 6.6, the latter being produced by using a (BR) nonlinear solver or a semi-Lagrangian scheme, for which we have higher accuracy and better convergence. From this example we can thus conclude that sufficient accuracy of the nonlinear solver used for the neural network approximation of free boundaries should be required in order to obtain reliable results.

We repeat the simulation with the same discretization parameters as above $\Delta t = \Delta x = \Delta p = 1.56 \times 10^{-2}$ and employ a feedforward neural network with one hidden layer and 10 neurons (opposed to the 50 neurons used to compute Figure 6.5 above) with the difference that we solve least-squares problem in Algorithm 6.5 using a Bayesian regularization (BR) algorithm (see \cite{bishop1995, mackay1992}). The results displayed in Figure 6.6 for $p = 0$ reproduce the expected behavior of the free boundary. For comparison in Figure 6.6 we also display the numerical solution obtained by a semi-Lagrangian algorithm with piecewise linear
interpolation (i.e., a modification of Algorithm 6.5 which employs linear interpolation instead of the least-squares approximation, cf. Algorithm 6.4) for $p = 0$; the results are in good qualitative agreement.

For better illustration we display in Figure 6.7 the graph of the numerical solution $x \mapsto \bar{u}_{\kappa,D}(t, x, p)$ for fixed $t = 0.25$, $p = 0$. We observe that in the numerical solution the lower obstacle is active approximately between $x = 0$ and $x = 0.55$; and the upper obstacle is active approximately between $x = 0.55$ and $x = 0.8$.

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**References**


Figure 6.5. The 3d (left) and the top view (right) of the numerical solution \((t, x) \rightarrow \widehat{u}_{\kappa, D}^\Delta(t, x, p)\) for \(p = 0, 0.5, 1\) (row 1–3) and of the average \((t, x) \rightarrow \frac{1}{2}(\widehat{u}_{\kappa, D}^\Delta(t, x, 0) + \widehat{u}_{\kappa, D}^\Delta(t, x, 1))\) with obstacles respectively represented by \(\frac{1}{2}f_1(x) + \frac{1}{2}f_2(x), \frac{1}{2}h_1(x) + \frac{1}{2}h_2(x)\) (4th row).


Figure 6.6. (Top) Numerical solution with the (BFGS) nonlinear solver. (Middle) Numerical solution with the (BR) nonlinear solver. (Bottom) Numerical solution computed with a semi-Lagrangian scheme.

Figure 6.7. Graph of the solution $x \rightarrow \tilde{u}_{k_D}$ at $t = 0.25$, $p = 0$ with (BR) nonlinear solver.


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