# SDE DRIVEN BY CYLINDRICAL $\alpha$-STABLE PROCESS WITH DISTRIBUTIONAL DRIFT AND APPLICATION

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Abstract. For $\alpha \in (1,2)$, we study the following stochastic differential equation driven by a non-degenerate symmetric $\alpha$-stable process in $\mathbb{R}^d$:

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_{t-})\mathrm{d}L_t^{(\alpha)}, \quad X_0 = x \in \mathbb{R}^d,$$

where $b$ belongs to $L^\infty(\mathbb{R}_+; \mathbf{B}^{-\beta}_{\infty,\infty}(\mathbb{R}^d))$ with some $\beta \in (0, \frac{\alpha-1}{2})$, and $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ is a $d \times d$ matrix-valued measurable function. We point out that the noise could be a cylindrical $\alpha$-stable process. We first show the generalized martingale problems and then establish the stability estimates of solutions. As an application, we give the weak convergence rate of the Euler scheme for additive noises with drift coefficient $b = b(x)$.

## 1. Introduction

In this paper, we consider the following stochastic differential equation (abbreviated as SDE) in $\mathbb{R}^d$ ($d \geqslant 1$):

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_{t-})\mathrm{d}L_t^{(\alpha)}, \quad X_0 = x \in \mathbb{R}^d, \tag{1.1}$$

where $L_t^{(\alpha)}$ is a symmetric $\alpha$-stable process with $\alpha \in (1,2)$ on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$, and $b$ belongs to $L^\infty(\mathbb{R}_+; \mathbf{B}^{-\beta}_{\infty,\infty}(\mathbb{R}^d))$ with some $\beta \in (0, \frac{\alpha-1}{2})$ (here, $\mathbf{B}^{-\beta}_{\infty,\infty}$ are Besov spaces, see Definition 2.2 below), and $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ is a $d \times d$ matrix-valued measurable function satisfying the following assumption:

**($\mathbf{H}^\sigma$)** $\sigma$ is Lipschitz continuous in $x$ uniformly in $t$. There is a constant $c_0 > 1$ such that for all $t \geqslant 0$ and $x, \xi \in \mathbb{R}^d$,

$$c_0^{-1}|\xi| \leqslant |\sigma(t,x)\xi| \leqslant c_0|\xi|, \quad \text{and} \quad |\nabla\sigma(t,x)| \leqslant c_0. \tag{1.2}$$

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1.1. **Well-posedness of generalized martingale problems.** The first goal of this paper is to establish the well-posedness of the generalized martingale solution for SDE (1.1) including the stability (see Theorem 1.1 and Theorem 4.6). A $d$-dimentional $\alpha$-stable process with $\alpha \in (0,2)$ is a purely jumped Lévy process with Lévy measure (called the $\alpha$-stable measure)

$$\nu^{(\alpha)}(A) = \int_0^\infty \left( \int_{\mathbb{S}^{d-1}} \frac{1_A(r\theta)\Sigma(\mathrm{d}\theta)}{r^{1+\alpha}} \right) \mathrm{d}r, \quad A \in \mathscr{B}(\mathbb{R}^d),$$

where $\Sigma$ is a finite measure over the unit sphere $\mathbb{S}^{d-1}$ (called spherical measure of $\nu^{(\alpha)}$). In this paper, we consider $\alpha$-stable measures that satisfy the following *non-degenerate* assumption:

(**ND**) For each $\theta_0 \in \mathbb{S}^{d-1}$,

$$\int_{\mathbb{S}^{d-1}} |\theta \cdot \theta_0| \Sigma(\mathrm{d}\theta) > 0.$$

The most common $\alpha$-stable process is the standard (or strictly) $\alpha$-stable process whose Lévy measure is absolutely continuous with the Lebesgue measure and given by $\frac{c}{|z|^{d+\alpha}}\mathrm{d}z$ with some constant $c > 0$, and the infinitesimal generator is the fractional Laplace operator $\Delta^{\alpha/2}$. From the point of view of Fourier analysis, (fractional) derivatives are characterized by Fourier multipliers. It is well-known that the symbol of the fractional Laplacian is given by $|\xi|^\alpha$ with $\xi \in \mathbb{R}^d$. If we consider $\alpha = 2$, the situation becomes the usual Laplacian $\Delta$, i.e. the infinitesimal generator of the $d$-dimensional standard Brownian motion

$$L^{(2)} := (W^1, W^2, \dots, W^d)$$

where $\{W^i\}_{i=1}^d$ are independent 1-dimensional Brownian motions. Under the distributional drift assumption, there have been several pieces of literature on Brownian motion cases. The earliest works are [4] and [20], both focusing on the one-dimensional time-independent multiplicative noise. We also refer to the works [21], [14], [24], [15], and [25] for the one-dimensional case, and [19], [46], [8], and [47] for the multi-dimensional case.

In recent years, researchers have paid more and more attention to the jumped case (for example [1], [10], [17], [30], [33] and so on). However, these studies consider only additive $\alpha$-stable noises except for [33]. Although the authors of [33] study the SDE (1.1) with multiplicative Lévy noises, their results do not cover the singular case such as cylindrical $\alpha$-stable noises. We say an $\alpha$-stable process

$$L^{(\alpha)} := (L^1, L^2, \dots, L^d)$$

is cylindrical, if its components are independent one-dimensional standard $\alpha$-stable processes. In fact, components of a multi-dimensional standard $\alpha$-stable process are not jointly independent, which is different from the Brownian case. Nevertheless, in many models, the joint independence of $\{L^i\}_{i=1}^d$ plays a vital role. For instance, in the following $N$-particle system:

$$\mathrm{d}X_t^{N,i} = \frac{1}{N}\sum_{i\neq i} K(X_t^{N,i} - X_t^{N,j})\mathrm{d}t + \mathrm{d}L_t^i,$$

$K : \mathbb{R}^d \to \mathbb{R}^d$ is the interaction kernel, and $\{L^i\}_{i=1}^N$ is a family of independent $\alpha$-stable processes, which stands for the random phenomenon like collisions between two particles (see [9] and references therein for examples). On the other hand, the Lévy measure of a cylindrical $\alpha$-stable process (called the cylindrical $\alpha$-stable measure) is given by

$$\nu^{(\alpha),c}(\mathrm{d}z) := \sum_{k=1}^d \delta_0(\mathrm{d}z_1)\cdots\delta_0(\mathrm{d}z_{i-1})\frac{\mathrm{d}z_i}{|z_i|^{1+\alpha}}\delta_0(\mathrm{d}z_{i+1})\cdots\delta_0(\mathrm{d}z_d),$$

where $\delta_0$ is the Dirac measure at zero in $\mathbb{R}^1$. Then the symbol of the associated infinitesimal generator is

$$\sum_{i=1}^d |\xi_i|^\alpha, \ \ \xi = (\xi_1, \cdots, \xi_d) \in \mathbb{R}^d,$$

which is more singular than the symbol of the standard $\alpha$-stable process since $|\xi|^\alpha$ is only not smooth at origin, but $\sum_{i=1}^d |\xi_i|^\alpha$ is not smooth on all axes $\cup_{i=1}^d\{\xi_i = 0\}$. That is why we say the cylindrical one is singular. In [33], Ling and Zhao obtained the well-posedness of generalized martingale problems relying on the following change of variables:

$$\begin{aligned}\mathscr{L}_\sigma^{(\alpha),s} f(x) &= \int_{\mathbb{R}^d} (f(x+\sigma(x)z) - f(x) - \nabla f(x)\cdot\sigma(x)z^{(\alpha)})\frac{\mathrm{d}z}{|z|^{d+\alpha}} \\ &= \int_{\mathbb{R}^d} (f(x+z) - f(x) - \nabla f(x)\cdot z^{(\alpha)})\frac{\mathrm{d}z}{|\sigma^{-1}(x)z|^{d+\alpha}|\det\sigma(x)|},\end{aligned} \tag{1.3}$$

where $z^{(\alpha)} := z\mathbf{1}_{|z|\leqslant 1}\mathbf{1}_{\alpha=1} + z\mathbf{1}_{\alpha\in(1,2)}$, the symbol $\nabla$ denotes the gradient operator, and the diffusion coefficient $\sigma$ is time-independent. It is obvious that such a change of variables method in [33] might not be valid in general cases, since, for example, the infinitesimal generator of a cylindrical one,

$$\mathscr{L}_\sigma^{(\alpha),c} f(x) = \sum_{i=1}^d \int_{\mathbb{R}} (f(x+(\sigma(x)z)_i) - f(x) - \nabla f(x)\cdot(\sigma(x)z^{(\alpha)})_i)\frac{\mathrm{d}z_i}{|z_i|^{1+\alpha}}, \tag{1.4}$$

where $(\sigma z)_i := (\sigma_{1i}z_i, ..., \sigma_{di}z_i)$ for any $\sigma = (\sigma_{ij}) \in \mathbb{R}^d\otimes\mathbb{R}^d$, is essentially different from the standard one.

The preceding theorem is our first main result:

**Theorem 1.1.** *Assume that $\alpha \in (1, 2)$ and $(\mathbf{H}^\sigma)$ holds with constant $c_0$. Let $b \in L^\infty([0,T]; \mathbf{B}^{-\beta}_{\infty,\infty})$ with some $\beta \in (0, \frac{\alpha-1}{2})$. Then there is a unique generalized martingale solution $\mathbf{Q} \in \mathcal{M}_{b,\sigma}(x)$ in the sense of Definition 4.2 below.*

**Remark 1.2.** *It is worth pointing out that the definition of martingale solutions presented in* [33] *looks different from Definition 4.2, but verifying their equivalence seems not a challenging task. We prove the solution to the approximating equation, wherein these two definitions coincide, converges to the solution in Theorem 1.1.*

In order to explore martingale solutions as defined in Definition 4.2, we consider the following forward partial differential equation (PDE for short):

$$\partial_t u = \mathscr{L}_\sigma^{(\alpha)} u + b \cdot \nabla u + f, \quad u(0) = u_0, \tag{1.5}$$

where $\alpha \in (1,2)$, $b, f \in L^\infty([0,T]; \mathbf{B}^{-\beta}_{\infty,\infty})$ with some $\beta \in (0, \frac{\alpha-1}{2})$, and

$$\mathscr{L}_\sigma^{(\alpha)} g(t,x) := \int_{\mathbb{R}^d} \Big( g(t, x + \sigma_t(x)z) - g(t,x) - \sigma_t(x)z \cdot \nabla g(t,x) \Big) \nu^{(\alpha)}(\mathrm{d}z) \tag{1.6}$$

with a non-degenerate symmetric $\alpha$-stable measure $\nu^{(\alpha)}$. Note that in [33], the assumption on $\sigma$, where $\sigma \in L^\infty([0,T]; \mathbf{B}^{\beta+\varepsilon}_{\infty,\infty})$, is weaker than ours. As a matter of fact, it seems to be open to give the well-posedness of martingale (or weak) solutions for SDEs driven by cylindrical $\alpha$-stable processes with distributional drifts when $\sigma$ is just Hölder continuous with respect to the space variable. Specifically, the key ingredient of reaching our first goal is the boundedness of operator $\mathscr{L}_\sigma^{(\alpha)} : \mathbf{B}^{\alpha-\beta}_{\infty,\infty} \to \mathbf{B}^{-\beta}_{\infty,\infty}$ (see Lemma 3.1). Regarding the operator $\mathscr{L}_\sigma^{(\alpha)}$ as a natural extension of the operator $\kappa(x)\Delta^{\frac{\alpha}{2}}$ as we illustrated in (1.3), the authors in [33] achieved the boundedness when $\nu^{(\alpha)}$ is absolutely continuous with respect to the Lebesgue measure, where the function $\kappa$ possesses the same Besov-Hölder regularity as that of $\sigma$. Thus, the boundedness of $\mathscr{L}_\sigma^{(\alpha)}$ in [33] can be attributed to the product law: (see Lemma 2.4

$$\|\kappa \Delta^{\frac{\alpha}{2}} u\|_{\mathbf{B}^{-\beta}_{\infty,\infty}} \leqslant c \|\Delta^{\frac{\alpha}{2}} u\|_{\mathbf{B}^{-\beta}_{\infty,\infty}} \|\kappa\|_{\mathbf{B}^{\beta+\varepsilon}_{\infty,\infty}},$$

where $c > 0$ is a constant only depends on $\beta, \varepsilon > 0$; and this is the reason they only need $\sigma$ belongs to $L^\infty([0,T]; \mathbf{B}^{\beta+\varepsilon}_{\infty,\infty})$. Nevertheless, the focus of this paper is on compound operators, as exemplified by $u(x) \to u(\sigma(x))$. And then it is imperative that $\sigma$ is Lipschitz in order to ensure the Besov-Hölder regularity of $u$. Moreover, we require a distinct technique to that presented in [33] for handling PDE (1.5). See more details in Section 3.

1.2. **Weak convergence rate of Euler scheme.** Our second aim is to investigate the Euler scheme for a toy model of SDE (1.1), where $\sigma \equiv \mathbb{I}_{d\times d}$ and $b = b(x)$ belongs to the negative order Besov space $\mathbf{B}^{-\beta}_{\infty,\infty}$. To study such a singular stochastic model, a natural problem is how to define the distributional drift term for Euler scheme. Modifying $b(x)$ in the sense of

$$b_m := b * \mathcal{K}_m, \quad m > 0,$$

where $\mathcal{K}_m(x) := m^d \mathcal{K}(mx)$ with some good kernel $\mathcal{K} \in C_0^\infty(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \mathcal{K}(x)\mathrm{d}x = 1$, we consider the following modified Euler scheme for SDE (1.1): for any $n \in \mathbb{N}$,

$$X_t^{m,n} = x + \int_0^t b_m(X^{m,n}_{\phi_n(t)})\mathrm{d}t + L_t^{(\alpha)}, \quad x \in \mathbb{R}^d$$

where $\phi_n(t) := k/n$ when $t \in [k/n, (k+1)/n)$ with $k = 0, 1, 2, ...., n$. Set $\mathbf{P}_{m,n}(t) := \mathbb{P} \circ (X_t^{m,n})^{-1}$.

In this paper, we study the so-called moderate case, i.e. $m = n^\gamma$ with some $\gamma > 0$. Here is our second main result, the weak convergence of this Euler scheme.

**Theorem 1.3.** *Let $\alpha \in (1, 2)$. Assume $(\mathbf{H}^\sigma)$ holds with constant $c_0$, $b \in \mathbf{B}^{-\beta}_{\infty,\infty}$ with some $\beta \in (0, \frac{\alpha-1}{2})$ and $m = n^\gamma$ with some $\gamma > 0$. Take the probability measure $\mathbf{P}$ be the unique solution of the generalized martingale problem $\mathcal{M}_{b,\sigma}(x)$ (see Definition 4.2). Then, for any $\gamma \in (0, \frac{\alpha-1}{2\alpha\beta})$ and $T > 0$, there are constants $c = c(d, \alpha, \beta, \mathcal{K}, T, \delta) > 0$ and $\delta = \delta(\alpha, \beta, \gamma) > 0$ such that for any $n \in \mathbb{N}$,*

$$\sup_{t \in [0,T]} \|\mathbf{P}_{m,n}(t) - \mathbf{P}(t)\|_{\text{var}} \leqslant c n^{-\delta},$$

*where $\|\cdot\|_{\text{var}}$ denotes the total variance of measures.*

**Remark 1.4.** *In the current paper, for simplicity, we only consider the additive noise case. Combining with a classical Levi's freezing coefficients method, our method might work as well for the case of uniformly elliptic and Lipschitz diffusion coefficients.*

So far, the rate of convergence for the Euler scheme of SDEs with regular or irregular drift coefficients has been widely studied. Whatever the Brownian case or the jumped case, people are used to adopting the Yamada-Watanabe approximation technique and the upper bound of heat kernels to prove the convergence rate. One could see the introduction section of [3] for more discussion of the Yamada-Watanabe approximation; see [26] or [43] for an example of using heat kernel pointwise estimates in this direction. It is worth emphasizing that the density of the standard $\alpha$-stable is equivalent with (cf. [5, Theorem 2.1])

$$\frac{t}{(t^{1/\alpha} + |x|)^{d+\alpha}},$$

but the cylindrical one has an asymptotic behavior as follows:

$$\prod_{i=1}^{d} \frac{t}{(t^{1/\alpha} + |x_i|)^{1+\alpha}}.$$

More generally, when the spherical measure $\Sigma$ of $\alpha$-stable measure $\nu^{(\alpha)}$ is not equivalent to the uniform measure, though a smooth density exists (cf. [11]), it is so far away to get an analytic expression for it (see [42] for more details). The best we know is just an integral-type estimate (see subsection 2.3), not a pointwise estimate. Moreover, for more irregular coefficient cases, the methods described above do not deal well with the problem, which leads to the search for more powerful tools.

Thanks to the deep connection between Kolmogorov equations and SDEs, to the best of our knowledge, in [34] Menoukeu Pamen and Taguchi first exploited the so-called Itô-Tanaka trick to get the strong rate of the Euler-Maruyama approximation of SDEs driven by a Wiener process or a truncated symmetric $\alpha$-stable process, with Hölder drift continuous drift coefficients. Meanwhile, concerning standard $\alpha$-stable processes, inspired by [18], Mikulevičius and Xu in [35] obtained the following strong convergence under $\alpha \in (1,2)$ and $\beta$-Hölder drift with $\beta < -(1-\alpha/2)$:

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X_t^{m,n} - X_t|^p\right] \lesssim n^{-p\beta/\alpha},$$

where $p \in (0,\alpha)$, relying on the Itô-Tanaka trick as well.

Nowdays, it has been gained much attention on SDEs with irregular coefficients driven by singular noises. In [32], Kühn and Schilling studied the strong convergence of the Euler-Maruyama approximation for a class of Lévy-driven SDEs with some Hölder assumption on the drift coefficient through the Itô-Tanaka approach. Moreover, under irregular $\beta$-Hölder drift, instead of using the Itô-Tanaka trick, in [7] Butkovsky, Dareiotis, and Gerencsér showed the strong rate of convergence of the Euler scheme for SDEs driven by a variety of Lévy noises, based on a new extended stochastic sewing lemma, where the rate can be uniformly with respect to $\beta$ and any moment $p > 2$.

On the other hand, observing the equivalence between norms of Hölder spaces and positive integer order Besov spaces (cf. [41]), it is natural to think about negative order cases. In addition, singular SDEs with distributional drifts actually appear in the context of many stochastic models (cf. [30]). In the present paper, partially inspired by mentioned works above, we study the weak convergence rate of the Euler approximation by applying the Itô-Tanaka trick to estimate the difference from $\mathbb{P}\circ(X_t^{m,n})^{-1} - \mathbb{P}\circ(X_t)^{-1}$ to $\mathbb{P}\circ(X_t^{m,n})^{-1} - \mathbb{P}\circ(X_{\phi_n(t)}^{m,n})^{-1}$, where the later one is obtained by heat kernel estimates (see Section 5 for more details). It is worth noting that before being adopted to prove the strong rate in [34], the Itô-Tanaka technique has traditionally been used to obtain weak convergence rates for Euler schemes which has important applications in stochastic financial theory (cf. [37]).

1.3. **Outline of paper.** The rest of this paper is organized as follows. In Section 2, we introduce some basic concepts and estimates. In Section 3, we establish Schauder's estimate and obtain the well-posedness for the non-local parabolic equation with singular $\alpha$-stable measure and distributional drift term. In Section 4, we show the first main result of this paper, Theorem 1.1, which gives the well-posedness of the generalized martingale solution to SDE (1.1) for any $b \in L^\infty([0,T]; \mathbf{B}^{-\beta}_{\infty,\infty})$ with some $\beta \in (0, \frac{\alpha-1}{2})$. As a result, we prove the stability estimate Theorem 4.6. In Section 5, based on the results in Section 3 and 4, we show the weak convergence rate of the Euler scheme.

**Conventions and notations.** Throughout this paper, we use the following conventions and notations: As usual, we use $:=$ as a way of definition. Define $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$ and $\mathbb{R}_+ := [0,\infty)$. The letter $c = c(\cdots)$ denotes an unimportant constant, whose value may change in different places. We use $A \asymp B$ and $A \lesssim B$ to denote $c^{-1}B \leqslant A \leqslant cB$ and $A \leqslant cB$, respectively, for some unimportant constant $c \geqslant 1$. We also use $A \lesssim_c B$ to denote $A \leqslant cB$ when we want to emphasize the constant. Denote by Beta functions and Gamma functions, respectively,

$$\mathrm{B}(s_1,s_2):=\int_0^1 x^{s_1-1}(1-x)^{s_2-1}\mathrm{d}x,\ \ \forall s_1,s_2>0 \tag{1.7}$$

and

$$\Gamma(s):=\int_0^\infty x^{s-1}\mathrm{e}^{-x}\mathrm{d}x,\ \ \forall s>0. \tag{1.8}$$

- Let $\mathbb{M}^d$ be the space of all real $d\times d$-matrices, and $\mathbb{M}^d_{non}$ the set of all non-singular matrices. Denote the identity $d\times d$-matrix by $\mathbb{I}_{d\times d}$.
- For every $p\in[1,\infty)$, we denote by $L^p$ the space of all $p$-order integrable functions on $\mathbb{R}^d$ with the norm denoted by $\|\cdot\|_p$.
- For a Banach space $\mathbb{B}$ and $T>0$, $q\in[1,\infty]$, we denote by

$$L^q_T\mathbb{B}:=L^q([0,T];\mathbb{B}),\ \ L^q_T:=L^q([0,T]\times\mathbb{R}^d).$$

## 2. Preliminary

2.1. **Besov spaces.** In this subsection, we introduce Besov spaces. Let $\mathscr{S}(\mathbb{R}^d)$ be the Schwartz space of all rapidly decreasing functions on $\mathbb{R}^d$, and $\mathscr{S}'(\mathbb{R}^d)$ the dual space of $\mathscr{S}(\mathbb{R}^d)$ called Schwartz generalized function (or tempered distribution) space. Given $f\in\mathscr{S}(\mathbb{R}^d)$, the Fourier transform $\hat f$ and the inverse Fourier transform $\check f$ are defined by

$$\hat f(\xi):=(2\pi)^{-d/2}\int_{\mathbb{R}^d}\mathrm{e}^{-i\xi\cdot x}f(x)\mathrm{d}x,\quad \xi\in\mathbb{R}^d,$$

$$\check f(x):=(2\pi)^{-d/2}\int_{\mathbb{R}^d}\mathrm{e}^{i\xi\cdot x}f(\xi)\mathrm{d}\xi,\quad x\in\mathbb{R}^d.$$

For every $f\in\mathscr{S}'(\mathbb{R}^d)$, the Fourier and the inverse transforms are defined by

$$\langle\hat f,\varphi\rangle:=\langle f,\hat\varphi\rangle,\qquad \langle\check f,\varphi\rangle:=\langle f,\check\varphi\rangle,\ \ \forall\varphi\in\mathscr{S}(\mathbb{R}^d).$$

Let $\chi:\mathbb{R}^d\to[0,1]$ be a radial smooth function with

$$\chi(\xi)=\begin{cases}1, & |\xi|\leqslant 1,\\ 0, & |\xi|>3/2.\end{cases}$$

For $\xi\in\mathbb{R}^d$, define $\psi(\xi):=\chi(\xi)-\chi(2\xi)$ and for $j\in\mathbb{N}_0$,

$$\psi_j(\xi):=\psi(2^{-j}\xi).$$

Let $B_r:=\{\xi\in\mathbb{R}^d\mid|\xi|\leqslant r\}$ for $r>0$. It is easy to see that $\psi\geqslant0$, $\mathrm{supp}\psi\subset B_{3/2}/B_{1/2}$, and

$$\chi(2\xi)+\sum_{j=0}^k\psi_j(\xi)=\chi(2^{-k}\xi)\to1,\ \ \text{as}\ \ k\to\infty. \tag{2.1}$$

Since $\check\psi_j(y)=2^{jd}\check\psi(2^jy)$, $j\geqslant0$, we have

$$\int_{\mathbb{R}^d}|x|^\theta|\nabla^k\check\psi_j|(x)\mathrm{d}x\leqslant c2^{(k-\theta)j},\ \ \theta>0,\ \ k\in\mathbb{N}_0,$$

where the constant $c$ is equal to $\int_{\mathbb{R}^d} |x|^\theta |\nabla^k \check{\psi}|(x)\mathrm{d}x$ and $\nabla^k$ stands for the $k$-order gradient. The block operators $\mathcal{R}_j$, $j \geqslant 0$ are defined on $\mathscr{S}'(\mathbb{R}^d)$ by

$$\mathcal{R}_j f(x) := (\psi_j \hat{f})\check{}(x) = \check{\psi}_j * f(x) = 2^{jd} \int_{\mathbb{R}^d} \check{\psi}(2^j y) f(x-y)\mathrm{d}y, \tag{2.2}$$

and $\mathcal{R}_{-1} f(x) := (\chi \hat{f})\check{}(x) = \check{\chi} * f(x)$. Then by (2.1),

$$f = \sum_{j \geqslant -1} \mathcal{R}_j f. \tag{2.3}$$

**Remark 2.1.** *For $j \geqslant -1$, by definitions, it is easy to see that*

$$\mathcal{R}_j = \mathcal{R}_j \widetilde{\mathcal{R}}_j, \quad \text{where } \widetilde{\mathcal{R}}_j := \sum_{\ell=-1}^{1} \mathcal{R}_{j+\ell} \text{ with } \mathcal{R}_{-2} := 0,$$

*and $\mathcal{R}_j$ is symmetric in the sense of*

$$\int_{\mathbb{R}^d} \mathcal{R}_j f(x) g(x)\mathrm{d}x = \int_{\mathbb{R}^d} f(x) \mathcal{R}_j g(x)\mathrm{d}x, \quad f \in \mathscr{S}'(\mathbb{R}^d),\ g \in \mathscr{S}(\mathbb{R}^d).$$

Now we state the definitions of Besov spaces.

**Definition 2.2** (Besov spaces)**.** *For every $s \in \mathbb{R}$ and $p, q \in [1,\infty]$, the Besov space $\mathbf{B}^s_{p,q}(\mathbb{R}^d)$ is defined by*

$$\mathbf{B}^s_{p,q}(\mathbb{R}^d) := \Big\{ f \in \mathscr{S}'(\mathbb{R}^d) \,\Big|\, \|f\|_{\mathbf{B}^s_{p,q}} := \Big[ \sum_{j \geqslant -1} \big( 2^{sj} \|\mathcal{R}_j f\|_p \big)^q \Big]^{1/q} < \infty \Big\}.$$

*If $p = q = \infty$, it is in the sense*

$$\mathbf{B}^s_{\infty,\infty}(\mathbb{R}^d) := \Big\{ f \in \mathscr{S}'(\mathbb{R}^d) \,\Big|\, \|f\|_{\mathbf{B}^s_{\infty,\infty}} := \sup_{j \geqslant -1} 2^{sj} \|\mathcal{R}_j f\|_\infty < \infty \Big\}.$$

This definition and Young's inequality ensure that for any $s_i \in \mathbb{R}$, $i = 0, 1, 2$, with $s_0 < s_1 < s_2$ and $\kappa > 0$, there is a constant $c_\kappa > 0$ such that

$$\|f\|_{\mathbf{B}^{s_1}_{\infty,\infty}} \leqslant \kappa \|f\|_{\mathbf{B}^{s_2}_{\infty,\infty}} + c_\kappa \|f\|_{\mathbf{B}^{s_0}_{\infty,\infty}},$$

In addition, for any $s_2 > s_1 > 0$,

$$\|f\|_{\mathbf{B}^{s_1}_{\infty,\infty}} \leqslant \kappa \|f\|_{\mathbf{B}^{s_2}_{\infty,\infty}} + c_\kappa \|f\|_\infty. \tag{2.4}$$

Recall the following Bernstein's inequality (cf. [2, Lemma 2.1]).

**Lemma 2.3** (Bernstein's inequality)**.** *For every $k \in \mathbb{N}_0$, there is a constant $c = c(d,k) > 0$ such that for all $j \geqslant -1$ and $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$,*

$$\|\nabla^k \mathcal{R}_j f\|_{p_2} \lesssim_c 2^{(k + d(\frac{1}{p_1} - \frac{1}{p_2}))j} \|\mathcal{R}_j f\|_{p_1}.$$

*In particular, for any $s \in \mathbb{R}$ and $1 \leqslant p, q \leqslant \infty$,*

$$\|\nabla^k f\|_{\mathbf{B}^s_{p,q}} \lesssim_c \|f\|_{\mathbf{B}^{s+k}_{p,q}}. \tag{2.5}$$

It is worth discussing here the equivalence between the Besov and Hölder spaces, which will be used in various contexts in this paper without much explanation. For $s > 0$, let $\mathbf{C}^s(\mathbb{R}^d)$ be the classical $s$-order Hölder space consisting of all measurable functions $f : \mathbb{R}^d \to \mathbb{R}$ with

$$\|f\|_{\mathbf{C}^s} := \sum_{j=0}^{[s]} \|\nabla^j f\|_\infty + [\nabla^{[s]} f]_{\mathbf{C}^{s-[s]}} < \infty,$$

where $[s]$ denotes the greatest integer not more than $s$, and

$$\|f\|_\infty := \sup_{x\in\mathbb{R}^d} |f(x)|, \quad [f]_{\mathbf{C}^\gamma} := \sup_{h\in\mathbb{R}^d} \frac{\|f(\cdot + h) - f(\cdot)\|_\infty}{|h|^\gamma}, \ \gamma \in (0, 1).$$

If $s > 0$ and $s \notin \mathbb{N}$, we have the following equivalence between $\mathbf{B}^s_{\infty,\infty}(\mathbb{R}^d)$ and $\mathbf{C}^s(\mathbb{R}^d)$: (cf. [41])

$$\|f\|_{\mathbf{B}^s_{\infty,\infty}} \asymp \|f\|_{\mathbf{C}^s}.$$

However, for any $n \in \mathbb{N}_0$, we only have one side control that is $\|f\|_{\mathbf{B}^n_{\infty,\infty}} \lesssim \|f\|_{\mathbf{C}^n}$.

At the end of this subsection, we introduce the following product law (cf. [22, Lemma 2.1] or [6]) and interpolation inequality (cf. [2, Theorem 2.80]).

**Lemma 2.4** (Product laws). *For any $s > 0$ and $\varepsilon > 0$, there is a constant $c = c(s, \varepsilon) > 0$ such that*

$$\|fg\|_{\mathbf{B}^{-s}_{\infty,\infty}} \lesssim_c \|f\|_{\mathbf{B}^{s+\varepsilon}_{\infty,\infty}} \|g\|_{\mathbf{B}^{-s}_{\infty,\infty}}. \tag{2.6}$$

**Lemma 2.5** (Interpolation inequality). *Let $s_1, s_2 \in \mathbb{R}$ with $s_2 > s_1$. For any $p \in [1, \infty]$ and $\theta \in (0, 1)$, there is a constant $c = c(s_1, s_2, p) > 0$ such that*

$$\|f\|_{B^{\theta s_1+(1-\theta)s_2}_{p,1}} \lesssim_c \|f\|^\theta_{\mathbf{B}^{s_1}_{p,\infty}} \|f\|^{1-\theta}_{\mathbf{B}^{s_2}_{p,\infty}}.$$

*Furthermore, for any $s_2 > 0 > s_1$,*

$$\|f\|_\infty \lesssim_c \|f\|^\theta_{\mathbf{B}^{s_1}_{\infty,\infty}} \|f\|^{1-\theta}_{\mathbf{B}^{s_2}_{\infty,\infty}}, \tag{2.7}$$

*where $\theta = s_2/(s_2 - s_1)$.*

2.2. **$\alpha$-stable processes.** We call a $\sigma$-finite positive measure $\nu$ on $\mathbb{R}^d$ a Lévy measure if

$$\nu(\{0\}) = 0, \quad \int_{\mathbb{R}^d} (1 \wedge |z|^2)\nu(\mathrm{d}z) < +\infty. \tag{2.8}$$

Fix $\alpha \in (0, 2)$. Let $L^{(\alpha)}_t$ be a $d$-dimentional $\alpha$-stable process with Lévy measure (or $\alpha$-stable measure)

$$\nu^{(\alpha)}(A) = \int_0^\infty \left( \int_{\mathbb{S}^{d-1}} \frac{1_A(r\theta)\Sigma(\mathrm{d}\theta)}{r^{1+\alpha}} \right) \mathrm{d}r, \quad A \in \mathscr{B}(\mathbb{R}^d), \tag{2.9}$$

where $\Sigma$ is a finite measure over the unit sphere $\mathbb{S}^{d-1}$ (called spherical measure of $\nu^{(\alpha)}$). We say an $\alpha$-stable measure $\nu^{(\alpha)}$ is non-degenerate, if the assumption **(ND)** holds. Note that $\alpha$-stable process $L_t^{(\alpha)}$ has the scaling property,

$$(L_t^{(\alpha)})_{t\geqslant 0} \overset{(d)}{=} (\lambda^{-1/\alpha} L_{\lambda t}^{(\alpha)})_{t\geqslant 0}, \quad \forall\, \lambda > 0, \tag{2.10}$$

and for any $\gamma_2 > \alpha > \gamma_1 \geqslant 0$,

$$\int_{|z|\leqslant 1} |z|^{\gamma_2} \nu^{(\alpha)}(\mathrm{d}z) + \int_{|z|>1} |z|^{\gamma_1} \nu^{(\alpha)}(\mathrm{d}z) < \infty. \tag{2.11}$$

Moreover, it is easy to see that for any $\lambda > 0$ and $p \geqslant 2$,

$$\int_{\mathbb{R}^d} (1 \wedge |\lambda z|^p) \nu^{(\alpha)}(\mathrm{d}z) = \lambda^\alpha \int_{\mathbb{R}^d} (1 \wedge |z|^p) \nu^{(\alpha)}(\mathrm{d}z). \tag{2.12}$$

Let $N(\mathrm{d}r, \mathrm{d}z)$ be the associated Poisson random measure defined by

$$N((0,t] \times A) := \sum_{s\in(0,t]} \mathbf{1}_A(L_s^{(\alpha)} - L_{s-}^{(\alpha)}), \quad A \in \mathscr{B}(\mathbb{R}^d \setminus \{0\}), t > 0.$$

By Lévy-Itô's decomposition (cf. [38, Theorem 19.2]), one sees that

$$L_t^{(\alpha)} = \lim_{\varepsilon\downarrow 0} \int_0^t \int_{\varepsilon<|z|\leqslant 1} z\tilde{N}(\mathrm{d}r, \mathrm{d}z) + \int_0^t \int_{|z|>1} zN(\mathrm{d}r, \mathrm{d}z),$$

where $\tilde{N}(\mathrm{d}r, \mathrm{d}z) := N(\mathrm{d}r, \mathrm{d}z) - \nu^{(\alpha)}(\mathrm{d}z)\mathrm{d}r$ is the compensated Poisson random measure. In the sequel, we always assume that $\nu^{(\alpha)}$ is symmetric. Hence, we can write

$$L_t^{(\alpha)} = \int_0^t \int_{|z|\leqslant c} z\tilde{N}(\mathrm{d}r, \mathrm{d}z) + \int_0^t \int_{|z|>c} zN(\mathrm{d}r, \mathrm{d}z), \quad \forall c > 0. \tag{2.13}$$

The following moment estimate is taken from [12, Lemma 2.4] with some slight modification. The proof is similar, so we omit it.

**Lemma 2.6.** *Let $T, \delta > 0$. Assume that $0 \leqslant \tau_1 < \tau_2 \leqslant \tau_1 + \delta \leqslant T$ are two bounded stopping times and $p \in (0, \alpha)$. Let $g : \mathbb{R}_+ \times \Omega \to \mathbb{M}^d_{non}$ be a bounded predictable process, where $\mathbb{M}^d_{non}$ is the set of all non-singular $d \times d$ matrices. Then, there is a constant $c = c(d, \alpha, p, T, \nu^{(\alpha)}) > 0$ such that*

$$\mathbb{E}\left|\int_{\tau_1}^{\tau_2} g(r)\mathrm{d}L_r^{(\alpha)}\right|^p \lesssim_c \delta^{p/\alpha} \mathbb{E}\|g\|^p_{L^\infty([0,T])}. \tag{2.14}$$

2.3. **Time-dependent Lévy-type operator.** Fix $\alpha \in (1, 2)$. We start with the following time-inhomogeneous Lévy process: for $0 \leqslant t < \infty$,

$$L_t^\sigma := \int_0^t \sigma_r \mathrm{d}L_r^{(\alpha)} = \int_0^t \int_{|z|\leqslant 1} \sigma_r z\tilde{N}(\mathrm{d}r, \mathrm{d}z) + \int_0^t \int_{|z|>1} \sigma_r zN(\mathrm{d}r, \mathrm{d}z), \tag{2.15}$$

where $\tilde{N}(\mathrm{d}r, \mathrm{d}z) := N(\mathrm{d}r, \mathrm{d}z) - \nu^{(\alpha)}(\mathrm{d}z)\mathrm{d}r$ is the compensated Poisson random measure, and $\sigma : \mathbb{R}_+ \to \mathbb{M}^d_{non}$ is a bounded measurable function. Define

$$P^\sigma_{s,t} f(x) := \mathbb{E}(x + \int_s^t \sigma_r \mathrm{d}L_r^{(\alpha)})$$

for all $f\in C_b^2(\mathbb{R}^d)$. By Itô's formula (cf. [27, Theorem 5.1 of Chapter II]), one sees that

$$\partial_t P^\sigma_{s,t} f(x)=\mathscr{L}^{(\alpha)}_{\sigma_t} P^\sigma_{s,t} f(x),$$

where

$$\mathscr{L}^{(\alpha)}_{\sigma_t} f(x):=\int_{\mathbb{R}^d}\Big(f(x+\sigma_t z)-f(x)-\sigma_t z\mathbf{1}_{|z|\leqslant 1}\cdot\nabla f(x)\Big)\nu^{(\alpha)}(\mathrm{d}z).$$

Below, we always make the following assumption in this subsection:

(**H0**) There is a constant $a_0>1$ such that

$$a_0|\xi|\leqslant|\sigma_t\xi|\leqslant a_0^{-1}|\xi|,\quad \forall (t,\xi)\in\mathbb{R}_+\times\mathbb{R}^d.$$

Under the assumption (**H0**), owing to Lévy-Khintchine's formula (cf. [38, Theorem 8.1]) and (2.9), for all $|\xi|\geqslant 1$, we have

$$\begin{aligned}|\mathbb{E}\mathrm{e}^{i\xi\cdot L^\sigma_t}|&\leqslant\exp\left(t\int_{\mathbb{R}^d}(\cos(\xi\cdot\sigma_t z)-1)\nu^{(\alpha)}(\mathrm{d}z)\right)\\&\leqslant\exp\left(-t|\xi|^\alpha\int_0^\infty\int_{\mathbb{S}^{d-1}}\frac{1-\cos(\xi/|\xi|\cdot\sigma_t r\theta)}{r^{1+\alpha}}\Sigma(\mathrm{d}\theta)\mathrm{d}r\right)\leqslant\mathrm{e}^{-ct|\xi|^\alpha},\end{aligned}$$

where the constant $c>0$ depends only on $\alpha$ and $\Sigma(\mathbb{S}^{d-1})$. Hence, by [38, Proposition 28.1], the random variable $L^\sigma_t$ defined by (2.15) admits a smooth density $p^\sigma(t,x)$ given by Fourier's inverse transform

$$p^\sigma(t,x)=(2\pi)^{-d/2}\int_{\mathbb{R}^d}\mathrm{e}^{-ix\cdot\xi}\mathbb{E}\mathrm{e}^{i\xi\cdot L^\sigma_t}\mathrm{d}\xi,\quad\forall t>0,$$

and the partial derivatives of $p^\sigma(t,\cdot)$ at any orders tend to 0 as $|x|\to\infty$. Furthermore, by [11, Lemma 3.2] or [12, Lemma 2.5], for each $0\leqslant t<\infty$, $p^\sigma(t,x)$ satisfies that for any $k\in\mathbb{N}_0$ and $0\leqslant\beta<\alpha$,

$$\int_{\mathbb{R}^d}|x|^\beta|\nabla^k p^\sigma(t,x)|\mathrm{d}x\lesssim_c t^{-\frac{k-\beta}{\alpha}},\tag{2.16}$$

where $c=c(a_0,k,d,\alpha,\beta)>0$.

We need the following heat kernel estimates in integral form with Littlewood-Paley's decomposition, which is obtained in [11, Lemma 3.3 ] (see also [23, Lemma 2.12]).

**Lemma 2.7** (Heat kernel estimates)**.** *Let $\alpha\in(0,2)$. Suppose that* (**H0**) *holds with constant $a_0\in(0,1)$. For any $\vartheta\geqslant 0$ and $\gamma\in[0,\alpha)$, there is a constant $c>0$ such that for all $0\leqslant s<t<\infty$ and $j\geqslant -1$,*

$$\int_{\mathbb{R}^d}|x|^\gamma|\mathcal{R}_j p^\sigma_{s,t}(x)|\mathrm{d}x\lesssim_c (t-s)^{-\frac{\vartheta-\gamma}{\alpha}}2^{-j\vartheta},\tag{2.17}$$

*where the block operators $\mathcal{R}_j$ are defined by* (2.2)*. In particular,*

$$\int_0^t\int_{\mathbb{R}^d}|x|^\gamma|\mathcal{R}_j p^\sigma_{s,t}(x)|\mathrm{d}x\mathrm{d}s\lesssim_c 2^{-j\alpha}.\tag{2.18}$$

In particular, when $\sigma_t$ is always equal to the identity matrix $\mathbb{I}_{d\times d}$, one has

$$\mathscr{L}^{(\alpha)}_{\mathbb{I}_{d\times d}} f(x) = \int_{\mathbb{R}^d} \Big(f(x+z) - f(x) - z\mathbf{1}_{|z|\leqslant 1}\cdot\nabla f(x)\Big)\nu^{(\alpha)}(\mathrm{d}z), \tag{2.19}$$

which is the infinitesimal generator of $\alpha$-stable process $L^{(\alpha)}$ (cf. [38, Theorem 31.5]). Consider the following equation:

$$\partial_t u = \mathscr{L}^{(\alpha)}_{\mathbb{I}_{d\times d}} u, \quad u(0) = f,$$

where $f \in C_b^\infty(\mathbb{R}^d)$. We have the following estimates.

**Lemma 2.8.** *Assume that $\alpha \in (0,2)$ and $T > 0$. For $k = 0, 1$, there is a constant $c > 0$ such that for all $0 < s < t \leqslant T$,*

$$\|\nabla^k \mathscr{L}^{(\alpha)}_{\mathbb{I}_{d\times d}} u(t)\|_\infty \lesssim t^{-\frac{k+\alpha}{\alpha}}\|f\|_\infty$$

*and*

$$\|\nabla^k u(t) - \nabla^k u(s)\|_\infty \lesssim \Big[s^{-\frac{k}{\alpha}} \wedge (s^{-\frac{k+\alpha}{\alpha}}(t-s))\Big]\|f\|_\infty. \tag{2.20}$$

*Proof.* By Itô's formula (cf. [27, Theorem 5.1 of Chapter II]), it is easy to check that

$$u(t,x) = \mathbb{E} f(x + L_t^{(\alpha)}) = (p_t * f)(x),$$

where $*$ stands for the convolution operation and $p_t(x) := p^{\mathbb{I}_{d\times d}}(t,x)$. Then for $k = 0, 1$, by (2.16),

$$\|\nabla^k u(t)\|_\infty \lesssim t^{-\frac{k}{\alpha}}\|f\|_\infty. \tag{2.21}$$

Moreover, by (2.3) and (2.17), we have

$$\begin{aligned}\|\nabla^k \mathscr{L}^{(\alpha)}_{\mathbb{I}_{d\times d}} u(t)\|_\infty &\lesssim \sum_{j\geqslant -1} \|\nabla^k \mathscr{L}^{(\alpha)}_{\mathbb{I}_{d\times d}} \mathcal{R}_j p_t * f\|_\infty \lesssim \sum_{j\geqslant -1} 2^{(k+\alpha)j}\|\mathcal{R}_j p_t\|_1\|f\|_\infty \\ &\lesssim \sum_{j\geqslant -1} 2^{(k+\alpha)j}\Big([2^{-(k+\alpha+1)j} t^{-\frac{k+\alpha+1}{\alpha}}]\wedge 1\Big)\|f\|_\infty \lesssim t^{-\frac{k+\alpha}{\alpha}}\|f\|_\infty,\end{aligned}$$

where we used the following estimate in the last step: for any $0 < \beta < \gamma$ and $\lambda > 0$,

$$\begin{aligned}\sum_{j\geqslant 0} 2^{\beta j}\Big([2^{-\gamma j}\lambda]\wedge 1\Big) &\leqslant \lambda\wedge 1 + \int_0^\infty 2^{\beta s}\left([2^{-\gamma s}\lambda]\wedge 1\right)\mathrm{d}s \\ &\lesssim \lambda\wedge 1 + \lambda^{\frac{\beta}{\gamma}}\int_{\lambda^{-1/\gamma}}^\infty r^{\beta-1}\left(r^{-\gamma}\wedge 1\right)\mathrm{d}r \lesssim \lambda^{\frac{\beta}{\gamma}}.\end{aligned}$$

Hence, for all $0 \leqslant s < t \leqslant T$,

$$\begin{aligned}|\nabla^k u(t,x) - \nabla^k u(s,x)| &= \left|\int_s^t \nabla^k \partial_r u(r,x)\mathrm{d}r\right| = \left|\int_s^t \nabla^k \mathscr{L}^{(\alpha)}_{\mathbb{I}_{d\times d}} u(r,x)\mathrm{d}r\right| \\ &\lesssim \|f\|_\infty \int_s^t r^{-\frac{k+\alpha}{\alpha}}\mathrm{d}r \lesssim s^{-\frac{k+\alpha}{\alpha}}(t-s)\|f\|_\infty.\end{aligned}$$

Combining this with (2.21), we deduce (2.20). □

## 3. Nonlocal equations with Singular Lévy measures

Fix $\alpha \in (1, 2)$. Let $\sigma(t, x) : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{M}^d_{non}$ be a bounded measurable function. In this section, we establish Schauder's estimate and obtain the well-posedness for the following non-local parabolic equation with time-dependent variable diffusion coefficient $\sigma_t(x) := \sigma(t, x)$:

$$\partial_t u = \mathscr{L}^{(\alpha)}_\sigma u + b \cdot \nabla u + f, \ \ u(0) = u_0, \tag{3.1}$$

where $b, f \in L^\infty_T \mathbf{B}^{-\beta}_{\infty,\infty}$ with some $\beta \in (0, \frac{\alpha-1}{2})$, $\sigma$ satisfies the condition **($\mathbf{H}^\sigma$)** with constant $c_0$, and $\mathscr{L}^{(\alpha)}_\sigma$ is given by (1.6):

$$\mathscr{L}^{(\alpha)}_\sigma g(t, x) := \int_{\mathbb{R}^d} \Big(g(t, x + \sigma_t(x)z) - g(t, x) - \sigma_t(x)z \cdot \nabla g(t, x)\Big)\nu^{(\alpha)}(\mathrm{d}z). \tag{3.2}$$

Here $\nu^{(\alpha)}$ is defined by (2.9) and satisfies condition **(ND)**.

It is well known that the best regularity of the solution $u(t)$ is in $\mathbf{B}^{\alpha-\beta}_{\infty,\infty}$, but the domain of the operator $\mathscr{L}^{(\alpha)}_{\sigma_t}$ is $\cup_{\varepsilon>0}\mathbf{B}^{\alpha+\varepsilon}_{\infty,\infty}$. Therefore, in order to define the solutions of PDE (3.1), we fisrt extend the domain of $\mathscr{L}^{(\alpha)}_{\sigma_t}$. Before that, we introduce some notations. For any $x, y, z \in \mathbb{R}^d$ with $|z| \leqslant \frac{1}{2}c_0^{-1}$, define

$$\sigma^y_t(x) := \sigma(t, x + y), \ \text{ and } \ \Lambda_{\sigma^y_t,z}(x) := x + \sigma^y_t(x)z.$$

By [45, Lemma2.1], it is easy to see that for any $t \geqslant 0$ and $x_1, x_2 \in \mathbb{R}^d$,

$$\frac{1}{2}|x_1 - x_2| \leqslant |\Lambda^y_{t,z}(x_1) - \Lambda^y_{t,z}(x_2)| \leqslant 2|x_1 - x_2|.$$

Define

$$\mathscr{D}_{\sigma^y_t,z} f(x) := f(x + \sigma^y_t(x)z) - f(x + \sigma^y_t(0)z) - (\sigma^y_t(x) - \sigma^y_t(0))z \cdot \nabla f(x).$$

Then, by [11, Lemma 2.2], for any $f \in C^1$, $g \in W^{2,1}$ and $\theta \in [0, 1]$, we have

$$|\langle \mathscr{D}_{\sigma^y_t,z} f, g\rangle| \leqslant c_{d,\theta}|z|^{1+\theta}\|f\|_{\mathbf{B}^\theta_{\infty,\infty}}\Big(\sum_{j=0}^{1}\mu_j(|\nabla^j g|) + \mu_{1+\theta}(|\nabla^2 g|)^\theta \mu_{1+\theta}(|\nabla g|)^{1-\theta}\Big), \tag{3.3}$$

where the constant $c_{d,\theta} > 0$ is independent of the variables $t, y, z$, and

$$\mu_\theta(\mathrm{d}x) := (|x| \wedge 1)^\theta \mathrm{d}x, \ \text{ and } \ \mu_\theta(f) := \int_{\mathbb{R}^d} f(x)\mu_\theta(\mathrm{d}x). \tag{3.4}$$

In particular,

$$|\langle \mathscr{D}_{\sigma^y_t,z} f, g\rangle| \leqslant c_d|z|^2\|f\|_{\mathbf{B}^1_{\infty,\infty}} \sum_{j=0}^{2}\mu_j(|\nabla^j g|). \tag{3.5}$$

The following lemma is crucial to give Definition 3.2.

**Lemma 3.1** (Boundedness of operator $\mathscr{L}_\sigma^{(\alpha)}$)**.** *Let $\alpha \in (1,2)$ and $\beta \in (0,1)$. Under the condition* $(\mathbf{H}^\sigma)$*, there is a constant $c = c(\alpha,\beta,c_0) > 0$ such that for any $u \in C_b^\infty(\mathbb{R}^d)$ and $t \geqslant 0$,*

$$\|\mathscr{L}_\sigma^{(\alpha)} u(t)\|_{\mathbf{B}^{-\beta}_{\infty,\infty}} \lesssim_c \|u(t)\|_{\mathbf{B}^{\alpha-\beta}_{\infty,\infty}}, \tag{3.6}$$

*where $\mathscr{L}_\sigma^{(\alpha)}$ is given by* (3.2).

*Proof.* For simplicity, we drop the time variable $t$ and the superscript $\alpha$ in $\nu^{(\alpha)}$ in the following proof. Observe that

$$\mathcal{R}_j \mathscr{L}_\sigma^{(\alpha)} u(x) = \mathscr{L}_\sigma^{(\alpha)} \mathcal{R}_j u(x) + [\mathcal{R}_j, \mathscr{L}_\sigma^{(\alpha)}] u(x),$$

where $[\mathscr{A}_1, \mathscr{A}_2] := \mathscr{A}_1\mathscr{A}_2 - \mathscr{A}_2\mathscr{A}_1$ called commutator operator. Define

$$u^x(y) := u(y+x).$$

Recall the definition (2.2) of block operators $\mathcal{R}_j$. By the change of variable, we have

$$\begin{aligned}&\int_{\mathbb{R}^d} \check{\psi}_j(x-y)\Big(u(y+\sigma(y)z) - u(y+\sigma(x)z) - (\sigma(y)-\sigma(x))z\cdot\nabla u(y)\Big)\mathrm{d}y\\ =&\int_{\mathbb{R}^d} \check{\psi}_j(-y)\Big(u^x(y+\sigma^x(y)z) - u^x(y+\sigma^x(0)z) - (\sigma^x(y)-\sigma^x(0))z\cdot\nabla u^x(y)\Big)\mathrm{d}y,\end{aligned}$$

which yields

$$[\mathcal{R}_j, \mathscr{L}_\sigma^{(\alpha)}]u(x) = \int_{\mathbb{R}^d} \langle \mathscr{D}_{\sigma^x,z} u^x, \check{\psi}_j(-\cdot)\rangle \nu^{(\alpha)}(\mathrm{d}z).$$

For $|z| \leqslant \delta \leqslant \frac{1}{2c_0}$, based on the fact

$$\mu_\delta(|\nabla^k \check{\psi}_j|) \lesssim 2^{(k-\delta)j}, \quad \delta \geqslant 0,\ k \in \mathbb{N}_0,$$

and (3.3) with $\theta \in (\alpha-1, \alpha-\beta)$, one sees that

$$\sup_x |\langle \mathscr{D}_{\sigma^x,z} u^x, \check{\psi}_j(-\cdot)\rangle| \lesssim |z|^{1+\theta}\|u\|_{\mathbf{B}^{\alpha-\beta}_{\infty,\infty}}\Big(1 + 2^{\theta(2-1-\theta)j}2^{(1-\theta)(1-1-\theta)j}\Big) \lesssim |z|^{1+\theta}\|u\|_{\mathbf{B}^{\alpha-\beta}_{\infty,\infty}}.$$

For $|z| > \delta$, by the mean-value theorem and (1.2), we get

$$\sup_x |\langle \mathscr{D}_{\sigma^x,z} u^x, \check{\psi}_j(-\cdot)\rangle| \leqslant 2c_0|z|\|\nabla u\|_\infty \|\check{\psi}\|_1 \lesssim |z|\|u\|_{\mathbf{B}^{\alpha-\beta}_{\infty,\infty}},$$

where we used the fact $\alpha - \beta > 1$. Hence, by (2.11), we obtain that

$$\|[\mathcal{R}_j, \mathscr{L}_\sigma^{(\alpha)}]u\|_\infty \leqslant \int_{\mathbb{R}^d} \sup_x |\langle \mathscr{D}_{\sigma^x,z} u^x, \check{\psi}_j(-\cdot)\rangle| \nu(\mathrm{d}z) \lesssim \|u\|_{\mathbf{B}^{\alpha-\beta}_{\infty,\infty}}. \tag{3.7}$$

On the other hand, through Bernstein's inequality Lemma 2.3, we have

$$\sup_x |\mathcal{R}_j u(x+\sigma(x)z) - \mathcal{R}_j u(x) - \sigma(x)z\cdot\nabla\mathcal{R}_j u(x)| \lesssim (|2^j z| \wedge |2^j z|^2)\|\mathcal{R}_j u\|_\infty,$$

which implies that

$$\|\mathscr{L}_\sigma^{(\alpha)} \mathcal{R}_j u\|_\infty \lesssim \|\mathcal{R}_j u\|_\infty \int_{\mathbb{R}^d} (|2^j z| \wedge |2^j z|^2)\nu(\mathrm{d}z) \lesssim 2^{\alpha j}\|\mathcal{R}_j u\|_\infty,$$

provided by the scaling property (2.10) and (2.11). Therefore, combining this with (3.7), we have

$$\|\mathcal{R}_j\mathscr{L}^{(\alpha)}_\sigma u\|_\infty \lesssim \|u\|_{\mathbf{B}^{\alpha-\beta}_{\infty,\infty}} + 2^{\alpha j}\|\mathcal{R}_j u\|_\infty \lesssim 2^{\beta j}\|u\|_{\mathbf{B}^{\alpha-\beta}_{\infty,\infty}},$$

which derives (3.6) by taking the supremum of $j$. The proof is completed. □

Based on Lemma 3.1, we extend the domain of the linear operator $\mathscr{L}^{(\alpha)}_{\sigma_t}$ from $\mathbf{B}^{\alpha+\varepsilon}_{\infty,\infty}$ to $\mathbf{B}^{\alpha-\beta}_{\infty,\infty}$ with $\beta \in (0,1)$. Now, we can state the definitions of solutions to PDE (3.1).

**Definition 3.2** (Solutions)**.** *Let $\alpha \in (1,2)$, $\beta \in (0,1)$ and assume $(\mathbf{H}^\sigma)$ holds with constant $c_0$. For any $T > 0$, $u_0 \in \mathbf{B}^{\alpha-\beta}_{\infty,\infty}$, and $b, f \in L^\infty_T\mathbf{B}^{-\beta}_{\infty,\infty}$, we call a function $u \in \cup_{\varepsilon>0}L^\infty_T\mathbf{B}^{1+\beta+\varepsilon}_{\infty,\infty}$ a solution to PDE (3.1) on $[0,T]$, if for any $t \in [0,T]$,*

$$u(t) = u_0 + \int_0^t \Big(\mathscr{L}^{(\alpha)}_\sigma u + b\cdot\nabla u + f\Big)(s)\mathrm{d}s, \tag{3.8}$$

*where $\mathscr{L}^{(\alpha)}_\sigma$ is defined by (3.2).*

**Remark 3.3.** *Notice that every term in (3.8) is well defined since we have (3.6) and $1+\beta > \alpha - 1$.*

Here is our main result in this section.

**Theorem 3.4.** *Let $\alpha \in (1,2)$, $T > 0$ and assume $(\mathbf{H}^\sigma)$ holds with constant $c_0$. For any $\beta \in (0, \frac{\alpha-1}{2})$, $\gamma > \frac{\alpha}{\alpha-1-2\beta}$, $u_0 \in \mathbf{B}^{\alpha-\beta}_{\infty,\infty}$, and $b, f \in L^\infty_T\mathbf{B}^{-\beta}_{\infty,\infty}$, there is a unique solution $u$ to PDE (3.1) in the sense of Definition 3.2 satisfying*

$$\|u\|_{L^\infty_T\mathbf{B}^{\alpha-\beta}_{\infty,\infty}} \lesssim_c (1+\|b\|_{L^\infty_T\mathbf{B}^{-\beta}_{\infty,\infty}})^\gamma\Big(\|u_0\|_{\mathbf{B}^{\alpha-\beta}_{\infty,\infty}} + \|f\|_{L^\infty_T\mathbf{B}^{-\beta}_{\infty,\infty}}\Big), \tag{3.9}$$

*where $c > 0$ is a constant only depending on $d,\alpha,\beta,T,c_0$. Moreover, for all $s,t \in [0,T]$,*

$$\|u(t)-u(s)\|_\infty \lesssim_c |t-s|^{\frac{\alpha-\beta}{\alpha}}(1+\|b\|_{L^\infty_T\mathbf{B}^{-\beta}_{\infty,\infty}})^{\frac{\alpha-\beta}{\alpha}+\gamma}\Big(\|u_0\|_{\mathbf{B}^{\alpha-\beta}_{\infty,\infty}} + \|f\|_{L^\infty_T\mathbf{B}^{-\beta}_{\infty,\infty}}\Big). \tag{3.10}$$

3.1. **$\lambda$-dissipative equation with $\lambda \geqslant 0$.** To prove Theorem 3.4, we introduce the following $\lambda$-dissipative equation with $\lambda \geqslant 0$:

$$\partial_t u^\lambda = \mathscr{L}^{(\alpha)}_\sigma u^\lambda - \lambda u^\lambda + b\cdot\nabla u^\lambda + f, \quad u^\lambda(0) = u^\lambda_0. \tag{3.11}$$

In this section, we establish a priori estimates, Lemma 3.7, for $\lambda$-dissipative PDE (3.11) under the case of smooth coefficients $b, f$.

3.1.1. *The zero-drift case.* In this part, we assume $b \equiv 0$ and investigate the following $\lambda$-dissipative equation with $\lambda \geqslant 0$:

$$\partial_t w = \mathscr{L}^{(\alpha)}_\sigma w - \lambda w + f_1 + f_2, \quad w(0) = w_0, \tag{3.12}$$

where $f_1, f_2, w_0$ are smooth functions.

We first show the following result with $\lambda$ large enough.

**Lemma 3.5.** *Fix $T > 0$. Let $\alpha \in (1,2)$, $\beta \in (0, \frac{\alpha-1}{2})$, and $f_1, f_2, w_0$ be smooth functions. Assume* $(\mathbf{H}^\sigma)$ *holds with constant $c_0$. If $w$ is a classical solution to PDE (3.12), then for any $\theta \in [0, \alpha-\beta]$, there are constants $\lambda_0 > 1$, and $c > 0$ depending on $d, \alpha, \beta, \theta, c_0, T$ such that for all $\lambda \geqslant \lambda_0$,*

$$\|w\|_{L^\infty_T \mathbf{B}^{\alpha-\beta-\theta}_{\infty,\infty}} \lesssim_c \|w_0\|_{\mathbf{B}^{\alpha-\beta-\theta}_{\infty,\infty}} + (\lambda+1)^{-\frac{\theta}{\alpha}} \left( \|w_0\|_{\mathbf{B}^{\alpha-\beta}_{\infty,\infty}} + \|f_1\|_{L^\infty_T \mathbf{B}^{-\beta}_{\infty,\infty}} \right) + (\lambda+1)^{-\frac{\beta+\theta}{\alpha}} \|f_2\|_{L^\infty_T}, \tag{3.13}$$

*and in particular, when $f_2 = 0$,*

$$\|w\|_{L^\infty_T \mathbf{B}^0_{\infty,1}} \lesssim_c \|w_0\|_{\mathbf{B}^0_{\infty,1}} + (\lambda+1)^{-\frac{\alpha-\beta}{\alpha}} \left( \|w_0\|_{\mathbf{B}^{\alpha-\beta}_{\infty,\infty}} + \|f_1\|_{L^\infty_T \mathbf{B}^{-\beta}_{\infty,\infty}} \right). \tag{3.14}$$

To prove this result, we only need to establish a priori estimates (3.13) and (3.14). For simplicity, we assume that $w_0 = 0$. If $w_0 \neq 0$, we shall substitute $w$ and $f$ in (3.12) by $\bar{w}(t) := w(t) - \mathrm{e}^{-\lambda t} w_0$ and $\bar{f}_1 := f_1 + \mathrm{e}^{-\lambda t} \mathscr{L}^{(\alpha)}_\sigma w_0$, respectively. Denote $f := f_1 + f_2$. Fix $y \in \mathbb{R}^d$. For any function $h$, let $h^y(x) := h(x+y)$. Then, we have

$$\partial_t w^y = \mathscr{L}^{(\alpha)}_y w^y - \lambda w^y + \mathscr{A} w^y + f^y,$$

where $\mathscr{A} := \mathscr{L}^{(\alpha)}_{\sigma^y} - \mathscr{L}^{(\alpha)}_y$ and

$$\mathscr{L}^{(\alpha)}_y g(t,x) := \mathscr{L}^{(\alpha)}_{\sigma^y(0)} g(t,x) = \int_{\mathbb{R}^d} (g(t, x+\sigma^y_t(0)z) - g(t,x) - \sigma^y_t(0)z \cdot \nabla g(t,x)) \nu^{(\alpha)}(\mathrm{d}z).$$

By subsection 2.3 and [11, Section 3], one sees that the operator $\partial_t - (\mathscr{L}^\alpha_y - \lambda)$ associates to a semigroup $\mathrm{e}^{-\lambda(t-s)} P^{\sigma^y(0)}_{s,t}$, i.e.

$$w^y(t,x) = \int_0^t \mathrm{e}^{-\lambda(t-s)} P^{\sigma^y(0)}_{s,t} \Big( \mathscr{A} w^y + f^y \Big)(s,x) \mathrm{d}s,$$

and $p^{\sigma^y(0)}_{s,t}$ is the heat kernel of $P^{\sigma^y(0)}_{s,t}$. For the sake of simplicity, in the sequel, we write $P^{(y)}_{s,t} = P^{\sigma^y(0)}_{s,t}$ and $p^{(y)}_{s,t} = p^{\sigma^y(0)}_{s,t}$. Thus, for $j \geqslant -1$, acting on both sides of the above equations by $\mathcal{R}_j$, we get that

$$\mathcal{R}_j w(t,y) = \mathcal{R}_j w^y(t,0) = \int_0^t \mathrm{e}^{-\lambda(t-s)} \mathcal{R}_j P^{(y)}_{s,t} \Big( \mathscr{A} w^y + f^y_1 + f^y_2 \Big)(s,0) \mathrm{d}s. \tag{3.15}$$

Let us separately estimate the terms on the right-hand side of (3.15).

**Lemma 3.6.** *Fix $T > 0$. Suppose that $\alpha \in (1,2)$ and $\beta \in (0, \frac{\alpha-1}{2})$. Assume* $(\mathbf{H}^\sigma)$ *holds with constant $c_0$. Let $\beta_1 = \beta$, $\beta_2 = 0$, and $i = 1,2$. For any $\vartheta \in [0, \alpha]$, there is a constant $c > 0$ such that*

$$\int_0^t \mathrm{e}^{-\lambda(t-s)} \left| \mathcal{R}_j P^{(y)}_{s,t} f^y_i(s,0) \right| \mathrm{d}s \lesssim_c 2^{-(\alpha-\beta_i-\vartheta)j} (\lambda+1)^{-\frac{\vartheta}{\alpha}} \|f_i\|_{L^\infty_T \mathbf{B}^{-\beta_i}_{\infty,\infty}}, \tag{3.16}$$

*and for any $0 \leqslant \eta < \alpha - \varepsilon$ with $\alpha > \varepsilon > 0$,*

$$\int_0^t \mathrm{e}^{-\lambda(t-s)} \left| \mathcal{R}_j P^{(y)}_{s,t} \mathscr{A} w^y(s,0) \right| \mathrm{d}s \lesssim_c 2^{-\eta j} (\lambda+1)^{-\frac{\alpha-\eta-\varepsilon}{\alpha}} \|w\|_{L^\infty_T \mathbf{B}^{\alpha-1+\varepsilon}_{\infty,\infty}}. \tag{3.17}$$

*Proof.* For the first one, by Remark 2.1 and (2.17), we have that for any $\vartheta \in (0, \alpha]$,

$$\begin{aligned}\int_0^t \mathrm{e}^{-\lambda(t-s)}\left|\mathcal{R}_j P^{(y)}_{s,t} f^y_i(s,0)\right|\mathrm{d}s &\leqslant \int_0^t \mathrm{e}^{-\lambda(t-s)}\int_{\mathbb{R}^d}|\mathcal{R}_j p^{(y)}_{s,t}(x)\widetilde{\mathcal{R}}_j f_i(s,x+y)|\mathrm{d}x\mathrm{d}s\\ &\lesssim 2^{\beta_i j}\|f_i\|_{L^\infty_T\mathbf{B}^{-\beta_i}_{\infty,\infty}}\int_0^t \mathrm{e}^{-\lambda(t-s)}(t-s)^{-\frac{\alpha-\vartheta}{\alpha}}2^{-j(\alpha-\vartheta)}\mathrm{d}s\\ &\lesssim 2^{-(\alpha-\beta_i-\vartheta)j}(\lambda+1)^{-\frac{\vartheta}{\alpha}}\|f_i\|_{L^\infty_T\mathbf{B}^{-\beta_i}_{\infty,\infty}},\end{aligned}$$

where the last inequality is provided by a change of variables and the definitions (1.8) of Gamma functions; and for $\vartheta = 0$, similarly, by Remark 2.1 and (2.18),

$$\begin{aligned}\int_0^t \mathrm{e}^{-\lambda(t-s)}\left|\mathcal{R}_j P^{(y)}_{s,t} f^y_i(s,0)\right|\mathrm{d}s &\lesssim 2^{\beta_i j}\|f_i\|_{L^\infty_T\mathbf{B}^{-\beta_i}_{\infty,\infty}}\int_0^t\int_{\mathbb{R}^d}|\mathcal{R}_j p^{(y)}_{s,t}(x)|\mathrm{d}x\mathrm{d}s\\ &\lesssim 2^{-(\alpha-\beta_i)j}\|f_i\|_{L^\infty_T\mathbf{B}^{-\beta_i}_{\infty,\infty}}.\end{aligned}$$

For the second one, applying [11, Lemma 4.4] to $\mathrm{e}^{-\lambda(t-s)}w(s)$, we have that for any $T > 0$, $\eta \in [0, \alpha-\varepsilon)$ with $\alpha > \varepsilon > 0$,

$$\begin{aligned}\int_0^t \mathrm{e}^{-\lambda(t-s)}\left|\mathcal{R}_j P^{(y)}_{s,t}\mathscr{A} w^y(s,0)\right|\mathrm{d}s &\lesssim 2^{-\eta j}\int_0^t \mathrm{e}^{-\lambda(t-s)}(t-s)^{-\frac{\eta+\varepsilon}{\alpha}}\|w(s)\|_{\mathbf{B}^{\alpha-1+\varepsilon}_{\infty,\infty}}\mathrm{d}s\\ &\lesssim 2^{-\eta j}(\lambda+1)^{-\frac{\alpha-\eta-\varepsilon}{\alpha}}\|w\|_{L^\infty_T\mathbf{B}^{\alpha-1+\varepsilon}_{\infty,\infty}},\end{aligned}$$

where we used a change of variables and the definitions (1.8) of Gamma functions in the second inequality.

The proof is finished. □

Now, we give the

*Proof of Lemma 3.5.* Notice that, by (3.16), for $\theta \in [0, \alpha-\beta]$,

$$\int_0^t \mathrm{e}^{-\lambda(t-s)}\left|\mathcal{R}_j P^{(y)}_{s,t} f^y_1(s,0)\right|\mathrm{d}s \lesssim 2^{-(\alpha-\beta-\theta)j}(\lambda+1)^{-\frac{\theta}{\alpha}}\|f_1\|_{L^\infty_T\mathbf{B}^{-\beta}_{\infty,\infty}}$$

and

$$\int_0^t \mathrm{e}^{-\lambda(t-s)}\left|\mathcal{R}_j P^{(y)}_{s,t} f^y_2(s,0)\right|\mathrm{d}s \lesssim 2^{-(\alpha-(\beta+\theta))j}(\lambda+1)^{-\frac{\beta+\theta}{\alpha}}\|f_2\|_{L^\infty_T\mathbf{B}^{0}_{\infty,\infty}}.$$

Moreover, by (3.17) with $\eta = \alpha-\beta-\theta$, we have that for any $\beta+\theta > \varepsilon > 0$,

$$\int_0^t \mathrm{e}^{-\lambda(t-s)}\left|\mathcal{R}_j P^{(y)}_{s,t}\mathscr{A} w^y(s,0)\right|\mathrm{d}s \lesssim 2^{-(\alpha-\beta-\theta)j}(\lambda+1)^{-\frac{\beta+\theta-\varepsilon}{\alpha}}\|w\|_{L^\infty_T\mathbf{B}^{\alpha-1+\varepsilon}_{\infty,\infty}}.$$

Thus, taking supremum of $y$ in (3.15), we have

$$\begin{aligned}\|w(t)\|_{\mathbf{B}^{\alpha-\beta-\theta}_{\infty,\infty}} = \sup_{j\geqslant 1} 2^{(\alpha-\beta-\theta)j}\|\mathcal{R}_j w(t)\|_\infty &\lesssim (\lambda+1)^{-\frac{\theta}{\alpha}}\|f_1\|_{L^\infty_T\mathbf{B}^{-\beta}_{\infty,\infty}} + (\lambda+1)^{-\frac{\theta+\beta}{\alpha}}\|f_2\|_{L^\infty_T}\\ &\quad + (\lambda+1)^{-\frac{\beta+\theta-\varepsilon}{\alpha}}\|w\|_{L^\infty_T\mathbf{B}^{\alpha-1+\varepsilon}_{\infty,\infty}}.\end{aligned} \tag{3.18}$$

On the other hand, when $f_2 = 0$, from (3.15), by (3.17) with $\eta = \varepsilon < \alpha/2$, taking $\vartheta = 0, \alpha$ in (3.16), we have

$$\|\mathcal{R}_j w(t)\|_\infty \lesssim (\lambda+1)^{-\frac{\alpha-2\varepsilon}{\alpha}} 2^{-\varepsilon j} \|w\|_{L^\infty_T \mathbf{B}^{\alpha-1+\varepsilon}_{\infty,\infty}} + \Big[(2^{\beta j}(\lambda+1)^{-1}) \wedge 2^{-(\alpha-\beta)j}\Big] \|f_1\|_{L^\infty_T \mathbf{B}^{-\beta}_{\infty,\infty}},$$

which derives that

$$\|w\|_{L^\infty_T \mathbf{B}^0_{\infty,1}} = \sum_{j\geqslant -1} \|\mathcal{R}_j w(t)\|_\infty \lesssim (\lambda+1)^{-\frac{\alpha-2\varepsilon}{\alpha}} \|w\|_{L^\infty_T \mathbf{B}^{\alpha-1+\varepsilon}_{\infty,\infty}} + (\lambda+1)^{-\frac{\alpha-\beta}{\alpha}} \|f_1\|_{L^\infty_T \mathbf{B}^{-\beta}_{\infty,\infty}}. \tag{3.19}$$

Now we estimate the term $\|w\|_{L^\infty_T \mathbf{B}^{\alpha-1+\varepsilon}_{\infty,\infty}}$. Similarly, taking $\eta = \varepsilon < 1/2$ in (3.17), and using (3.16) with $\vartheta = 1-\beta-\varepsilon$ when $i = 1$ and $\vartheta = 1-\varepsilon$ when $i = 2$, we obtain that there is a constant $c > 0$ such that

$$\|w\|_{L^\infty_T \mathbf{B}^{\alpha-1+\varepsilon}_{\infty,\infty}} \lesssim_c (\lambda+1)^{-\frac{1-2\varepsilon}{\alpha}} \|w\|_{L^\infty_T \mathbf{B}^{\alpha-1+\varepsilon}_{\infty,\infty}} + (\lambda+1)^{-\frac{1-\beta-\varepsilon}{\alpha}} \|f_1\|_{L^\infty_T \mathbf{B}^{-\beta}_{\infty,\infty}} + (\lambda+1)^{-\frac{1-\varepsilon}{\alpha}} \|f_2\|_{L^\infty_T},$$

which yields that for any $\lambda \geqslant \lambda_0 := (2c)^{\alpha/(1-2\varepsilon)}$,

$$\|w\|_{L^\infty_T \mathbf{B}^{\alpha-1+\varepsilon}_{\infty,\infty}} \lesssim (\lambda+1)^{-\frac{1-\beta-\varepsilon}{\alpha}} \|f_1\|_{L^\infty_T \mathbf{B}^{-\beta}_{\infty,\infty}} + (\lambda+1)^{-\frac{1-\varepsilon}{\alpha}} \|f_2\|_{L^\infty_T}.$$

Substituting the above inequality into (3.18) and (3.19) with $0 < \varepsilon < 1/3$, we have that for any $\theta \in [0, \alpha-\beta]$,

$$\|w\|_{L^\infty_T \mathbf{B}^{\alpha-\beta-\theta}_{\infty,\infty}} \lesssim (\lambda+1)^{-\frac{\theta}{\alpha}} \|f_1\|_{L^\infty_T \mathbf{B}^{-\beta}_{\infty,\infty}} + (\lambda+1)^{-\frac{\theta+\beta}{\alpha}} \|f_2\|_{L^\infty_T},$$

and when $f_2 = 0$,

$$\|w\|_{L^\infty_T \mathbf{B}^0_{\infty,1}} \lesssim (\lambda+1)^{-\frac{\alpha-\beta}{\alpha}} \|f_1\|_{L^\infty_T \mathbf{B}^{-\beta}_{\infty,\infty}}.$$

These complete the proof. □

3.1.2. *Distributional drift case.* Backing to (3.11), we establish the following a priori estimates.

**Lemma 3.7.** *Let $\alpha \in (1,2)$, $\beta \in (0, \frac{\alpha-1}{2})$, $T > 0$, $\gamma > \frac{\alpha}{\alpha-1-2\beta}$, and $b, f_1, f_2, u^\lambda_0$ be smooth functions. Assume* **(H$^\sigma$)** *holds with constant $c_0$. If $u^\lambda$ is a classical solution to PDE (3.11), then for any $\theta \in [0, \alpha-\beta]$, there is a constant $c = c(d, \alpha, \beta, \theta, c_0, T, \gamma, \|b\|_{L^\infty_T \mathbf{B}^{-\beta}_{\infty,\infty}}) > 0$ such that for all $\lambda \geqslant \lambda_0 := c(1 + \|b\|_{L^\infty_T \mathbf{B}^{-\beta}_{\infty,\infty}})^\gamma$,*

$$\|u^\lambda\|_{L^\infty_T \mathbf{B}^{\alpha-\beta-\theta}_{\infty,\infty}} \lesssim_c \|u^\lambda_0\|_{\mathbf{B}^{\alpha-\beta-\theta}_{\infty,\infty}} + (\lambda+1)^{-\frac{\theta}{\alpha}} \Big( \|u^\lambda_0\|_{\mathbf{B}^{\alpha-\beta}_{\infty,\infty}} + \|f_1\|_{L^\infty_T \mathbf{B}^{-\beta}_{\infty,\infty}} \Big) + (\lambda+1)^{-\frac{\beta+\theta}{\alpha}} \|f_2\|_{L^\infty_T}, \tag{3.20}$$

*and in particular when $f_2 = 0$,*

$$\|u^\lambda\|_{L^\infty_T \mathbf{B}^0_{\infty,1}} \lesssim_c \|u^\lambda_0\|_{\mathbf{B}^0_{\infty,1}} + (\lambda+1)^{-\frac{\alpha-\beta}{\alpha}} \Big( \|u^\lambda_0\|_{\mathbf{B}^{\alpha-\beta}_{\infty,\infty}} + \|f_1\|_{L^\infty_T \mathbf{B}^{-\beta}_{\infty,\infty}} \Big). \tag{3.21}$$

*Moreover, when $f_2 = 0$, for any $\lambda \geqslant 0$,*

$$\|u^\lambda\|_{L^\infty_T \mathbf{B}^{\alpha-\beta}_{\infty,\infty}} \lesssim_c \|u^\lambda_0\|_{\mathbf{B}^{\alpha-\beta}_{\infty,\infty}} + \|f_1\|_{L^\infty_T \mathbf{B}^{-\beta}_{\infty,\infty}} + (1 + \|b\|_{\mathbf{B}^{-\beta}_{\infty,\infty}})^{\gamma\frac{\alpha-\beta}{\alpha}} \|u^\lambda\|_{L^\infty_T}. \tag{3.22}$$

*Proof.* By Lemma 3.5, (2.6), and (2.5), for any $\theta \in [0, \alpha-\beta]$ and $\varepsilon > 0$, there is a constant $\widetilde{\lambda_0}$ such that for any $\lambda \geqslant \widetilde{\lambda_0}$,

$$\begin{aligned}\|u^\lambda\|_{L^\infty_T\mathbf{B}^{\alpha-\beta-\theta}_{\infty,\infty}} &\lesssim (\lambda+1)^{-\frac{\theta}{\alpha}}\left(\|u_0^\lambda\|_{\mathbf{B}^{\alpha-\beta}_{\infty,\infty}} + \|f_1 + b\cdot\nabla u^\lambda\|_{L^\infty_T\mathbf{B}^{-\beta}_{\infty,\infty}}\right) + \|u_0^\lambda\|_{\mathbf{B}^{\alpha-\beta-\theta}_{\infty,\infty}} + (\lambda+1)^{-\frac{\beta+\theta}{\alpha}}\|f_2\|_{L^\infty_T}\\ &\lesssim (\lambda+1)^{-\frac{\theta}{\alpha}}\left(\|u_0^\lambda\|_{\mathbf{B}^{\alpha-\beta}_{\infty,\infty}} + \|f_1\|_{L^\infty_T\mathbf{B}^{-\beta}_{\infty,\infty}} + \|b\|_{L^\infty_T\mathbf{B}^{-\beta}_{\infty,\infty}}\|u^\lambda\|_{L^\infty_T\mathbf{B}^{1+\beta+\varepsilon}_{\infty,\infty}}\right)\\ &\quad + \|u_0^\lambda\|_{\mathbf{B}^{\alpha-\beta-\theta}_{\infty,\infty}} + (\lambda+1)^{-\frac{\beta+\theta}{\alpha}}\|f_2\|_{L^\infty_T}. \end{aligned}\tag{3.23}$$

Hence, in particular, taking $\varepsilon \in (0, \alpha-1-2\beta]$ and $\theta = \alpha-1-2\beta-\varepsilon$ in (3.23), we infer that for any $\lambda \geqslant \widetilde{\lambda_0}$,

$$\begin{aligned}\|u^\lambda\|_{L^\infty_T\mathbf{B}^{1+\beta+\varepsilon}_{\infty,\infty}} \lesssim \|u_0^\lambda\|_{\mathbf{B}^{\alpha-\beta}_{\infty,\infty}} + \|f_1\|_{L^\infty_T\mathbf{B}^{-\beta}_{\infty,\infty}} + (\lambda+1)^{-\frac{\alpha-1-2\beta-\varepsilon}{\alpha}}\|b\|_{L^\infty_T\mathbf{B}^{-\beta}_{\infty,\infty}}\|u^\lambda\|_{L^\infty_T\mathbf{B}^{1+\beta+\varepsilon}_{\infty,\infty}}\\ + (\lambda+1)^{-\frac{\beta}{\alpha}}\|f_2\|_{L^\infty_T},\end{aligned}$$

which implies that for any $0 < \varepsilon < \alpha - 1 - 2\beta$, there is a constant $c > 0$ such that for any $\lambda_0 := c(\|b\|_{L^\infty_T\mathbf{B}^{-\beta}_{\infty,\infty}} + 1)^{\alpha/(\alpha-1-2\beta-\varepsilon)} \geqslant \widetilde{\lambda_0}$,

$$\|u^\lambda\|_{L^\infty_T\mathbf{B}^{1+\beta+\varepsilon}_{\infty,\infty}} \lesssim \|u_0^\lambda\|_{\mathbf{B}^{\alpha-\beta}_{\infty,\infty}} + \|f_1\|_{L^\infty_T\mathbf{B}^{-\beta}_{\infty,\infty}} + (\lambda+1)^{-\frac{\beta}{\alpha}}\|f_2\|_{L^\infty_T}.$$

Substituting this into (3.23), we obtain the desired estimates (3.20). As for (3.21), the proof is similar.

For (3.22), note that when $f_2 = 0$, for any $\lambda \geqslant 0$, we get that

$$\partial_t u^\lambda = \mathscr{L}^{(\alpha)}_\sigma u^\lambda - (\lambda+\lambda_0)u^\lambda + b\cdot\nabla u^\lambda + f_1 + \lambda_0 u^\lambda.$$

Taking $f_2 = \lambda_0 u^\lambda$ in (3.20) with $\theta = 0$, we deduce that

$$\begin{aligned}\|u^\lambda\|_{L^\infty_T\mathbf{B}^{\alpha-\beta}_{\infty,\infty}} &\lesssim \|u_0^\lambda\|_{\mathbf{B}^{\alpha-\beta}_{\infty,\infty}} + \|f_1\|_{L^\infty_T\mathbf{B}^{-\beta}_{\infty,\infty}} + (1+\lambda_0)^{1-\frac{\beta}{\alpha}}\|u^\lambda\|_{L^\infty_T}\\ &\lesssim \|u_0^\lambda\|_{\mathbf{B}^{\alpha-\beta}_{\infty,\infty}} + \|f_1\|_{L^\infty_T\mathbf{B}^{-\beta}_{\infty,\infty}} + (1+\|b\|_{L^\infty_T\mathbf{B}^{-\beta}_{\infty,\infty}})^{\gamma\frac{\alpha-\beta}{\alpha}}\|u^\lambda\|_{L^\infty_T},\end{aligned}$$

and the result follows. □

### 3.2. The proof of Theorem 3.4.

In this subsection, we prove the main result of this section.

Recalling PDE (3.1) (i.e. the equation (3.11) with $\lambda = 0$), we begin by establishing the following a priori estimates as a corollary of Lemma 3.7.

**Corollary 3.8.** *Let $\alpha \in (1,2)$, $\beta \in (0, \frac{\alpha-1}{2})$, $T > 0$, $\gamma > \frac{\alpha}{\alpha-1-2\beta}$ and $b, f$ be smooth functions. Assume* **(H$^\sigma$)** *holds with constant $c_0$. If $u$ is a classical solution to PDE (3.1), then there is a constant $c = c(d,\alpha,\beta,c_0,T,\gamma,\|b\|_{L^\infty_T\mathbf{B}^{-\beta}_{\infty,\infty}}) > 0$ such that*

$$\|u\|_{L^\infty_T\mathbf{B}^{\alpha-\beta}_{\infty,\infty}} \lesssim_c (1+\|b\|_{L^\infty_T\mathbf{B}^{-\beta}_{\infty,\infty}})^\gamma\left(\|u_0\|_{\mathbf{B}^{\alpha-\beta}_{\infty,\infty}} + \|f\|_{L^\infty_T\mathbf{B}^{-\beta}_{\infty,\infty}}\right). \tag{3.24}$$

*Proof.* Let $\lambda_0$ be the same constant in Lemma 3.7 and $u^{\lambda_0}$ be a classical solution to PDE (3.11) with $\lambda = \lambda_0$ and $u^{\lambda_0}(0) = u_0$. Denote by $\mathrm{v} := u - u^{\lambda_0}$. Then,

$$\partial_t \mathrm{v} = \mathscr{L}^{(\alpha)}_\sigma \mathrm{v} + b\cdot\nabla\mathrm{v} + \lambda_0 u^{\lambda_0}, \quad \mathrm{v}(0) = 0.$$

Thus, by maximum principle (see [23, Lemma 3.3]) and (3.21), we have

$$\|v\|_{L^\infty_T} \leqslant T\lambda_0\|u^{\lambda_0}\|_{L^\infty_T} \lesssim (1+\lambda_0)^{1-\frac{\alpha-\beta}{\alpha}}\left(\|u_0\|_{\mathbf{B}^{\alpha-\beta}_{\infty,\infty}} + \|f\|_{L^\infty_T\mathbf{B}^{-\beta}_{\infty,\infty}}\right)$$
$$\lesssim (1+\|b\|_{L^\infty_T\mathbf{B}^{-\beta}_{\infty,\infty}})^{\frac{\gamma\beta}{\alpha}}\left(\|u_0\|_{\mathbf{B}^{\alpha-\beta}_{\infty,\infty}} + \|f\|_{L^\infty_T\mathbf{B}^{-\beta}_{\infty,\infty}}\right),$$

which together with (3.22) yields

$$\|u\|_{L^\infty_T\mathbf{B}^{\alpha-\beta}_{\infty,\infty}} \lesssim \|u_0\|_{\mathbf{B}^{\alpha-\beta}_{\infty,\infty}} + \|f\|_{L^\infty_T\mathbf{B}^{-\beta}_{\infty,\infty}} + (1+\|b\|_{\mathbf{B}^{-\beta}_{\infty,\infty}})^{\gamma\frac{\alpha-\beta}{\alpha}}\|u\|_{L^\infty_T}$$
$$\lesssim \|u_0\|_{\mathbf{B}^{\alpha-\beta}_{\infty,\infty}} + \|f\|_{L^\infty_T\mathbf{B}^{-\beta}_{\infty,\infty}} + (1+\|b\|_{\mathbf{B}^{-\beta}_{\infty,\infty}})^{\gamma\frac{\alpha-\beta}{\alpha}}\left(\|v\|_{L^\infty_T} + \|u^{\lambda_0}\|_{L^\infty_T}\right)$$
$$\lesssim (1+\|b\|_{\mathbf{B}^{-\beta}_{\infty,\infty}})^{\gamma}\left(\|u_0\|_{\mathbf{B}^{\alpha-\beta}_{\infty,\infty}} + \|f\|_{L^\infty_T\mathbf{B}^{-\beta}_{\infty,\infty}}\right).$$

The proof is finished. □

Now we are in a position to give

*Proof of the Theorem 3.4.* The uniqueness is obvious and we only need to show the existence. Let $\rho_m(\cdot) := m^d\rho(m\cdot)$ be the usual mollifier with $\rho \in C^\infty_c(\mathbb{R}^d)$ and

$$b_m(t) := b(t) * \rho_m, \quad f_m(t) := f(t) * \rho_m.$$

Then, $b_m, f_m \in L^\infty_T C^\infty_c$ and for any $\gamma > \beta$,

$$\lim_{m\to\infty}\left(\|b_m - b\|_{L^\infty_T\mathbf{B}^{-\gamma}_{\infty,\infty}} + \|f_m - f\|_{L^\infty_T\mathbf{B}^{-\gamma}_{\infty,\infty}}\right) = 0.$$

Moreover,

$$\sup_m \|b_m\|_{L^\infty_T\mathbf{B}^{-\beta}_{\infty,\infty}} \leqslant \|b\|_{L^\infty_T\mathbf{B}^{-\beta}_{\infty,\infty}}, \quad \sup_m \|f_m\|_{L^\infty_T\mathbf{B}^{-\beta}_{\infty,\infty}} \leqslant \|f\|_{L^\infty_T\mathbf{B}^{-\beta}_{\infty,\infty}}.$$

Let $u_n$ be the classical solution of PDE (3.1) with $(b, f) = (b_m, f_m)$. Then, by Corollary 3.8 and Lemma 3.1, we have

$$\sup_m\left(\|u_m\|_{L^\infty_T\mathbf{B}^{\alpha-\beta}_{\infty,\infty}} + \|\mathscr{L}^{(\alpha)}_\sigma u_m\|_{L^\infty_T\mathbf{B}^{-\beta}_{\infty,\infty}}\right) \lesssim \|f\|_{L^\infty_T\mathbf{B}^{-\beta}_{\infty,\infty}} + \|u_0\|_{\mathbf{B}^{\alpha-\beta}_{\infty,\infty}}.$$

By a standard argument, we obtain the existence.

For (3.10), on the one hand, it follows from (3.8), (3.6), and (2.6) that

$$\|u(t)-u(s)\|_{\mathbf{B}^{-\beta}_{\infty,\infty}} \lesssim_c |t-s|\Big((1+\|b\|_{L^\infty_T\mathbf{B}^{-\beta}_{\infty,\infty}})\|u\|_{L^\infty_T\mathbf{B}^{\alpha-\beta}_{\infty,\infty}} + \|f\|_{L^\infty_T\mathbf{B}^{-\beta}_{\infty,\infty}}\Big)$$
$$\overset{(3.9)}{\lesssim_c} |t-s|(1+\|b\|_{L^\infty_T\mathbf{B}^{-\beta}_{\infty,\infty}})^{1+\gamma}\Big(\|u_0\|_{\mathbf{B}^{\alpha-\beta}_{\infty,\infty}} + \|f\|_{L^\infty_T\mathbf{B}^{-\beta}_{\infty,\infty}}\Big),$$

since $\alpha - 2\beta - 1 > 0$. On the other hand, it is easy to see that

$$\|u(t)-u(s)\|_{\mathbf{B}^{\alpha-\beta}_{\infty,\infty}} \leqslant 2\|u\|_{L^\infty_T\mathbf{B}^{\alpha-\beta}_{\infty,\infty}} \overset{(3.9)}{\lesssim_c} (1+\|b\|_{L^\infty_T\mathbf{B}^{-\beta}_{\infty,\infty}})^{\gamma}\Big(\|u_0\|_{\mathbf{B}^{\alpha-\beta}_{\infty,\infty}} + \|f\|_{L^\infty_T\mathbf{B}^{-\beta}_{\infty,\infty}}\Big).$$

Then by (2.7), we obtain (3.10) and complete the proof. □

3.3. **Backward Cauchy problem with $f \equiv 0$.** In the rest of this section, we set $f \equiv 0$ and consider the following backward Cauchy problem:

$$\partial_t u + \mathscr{L}_\sigma^{(\alpha)} u + b \cdot \nabla u = 0, \quad u(T) = \varphi,$$

where $t \in [0, T)$ and $\varphi \in C_b^\infty(\mathbb{R}^d)$. This subsection is devoted to give some estimates of $u(t)$ that play an important role in proving Theorem 4.6 and Theorem 5.1. Based on Theorem 3.4, one sees that there is a unique solution $u$. Furthermore, by [11, Theorem 1.1], there is a semigroup $P_{t,s}$ such that

$$u(t) = P_{t,T}\varphi + \int_t^T P_{t,s}(b \cdot \nabla u)(s)\mathrm{d}s, \tag{3.25}$$

and for any $\delta \in [0, \alpha + 1)$ and $\eta \in [-\delta, 1)$, there is a constant $c > 0$ such that for all $0 \leqslant t < s \leqslant T$ and $\varphi \in C_b^\infty(\mathbb{R}^d)$,

$$\|P_{t,s}\varphi\|_{\mathbf{B}^\delta_{\infty,\infty}} \lesssim_c (s-t)^{-\frac{\delta+\eta}{\alpha}} \|\varphi\|_{\mathbf{B}^{-\eta}_{\infty,\infty}}. \tag{3.26}$$

We state the following inequality for $u(t)$.

**Lemma 3.9.** *Let $T > 0$, $\alpha \in (1, 2)$, $\beta \in (0, \frac{\alpha-1}{2})$, and $b \in L_T^\infty C_b^\infty$. Assume $\sigma$ satisfies* **(H$^\sigma$)** *with constant $c_0$. For any $\gamma \in [0, \frac{\alpha-1}{2})$ and $\delta \in [0, \alpha - \beta]$, there is a constant $c = c(d, T, \alpha, c_0, \beta, \gamma, \delta, \|b\|_{L_T^\infty \mathbf{B}^{-\beta}_{\infty,\infty}}) > 0$ such that for any $\varphi \in C_b^\infty(\mathbb{R}^d)$ and $t \in [0, T)$,*

$$\|u(t)\|_{\mathbf{B}^\delta_{\infty,\infty}} \lesssim_c (T-t)^{-\frac{\delta+\gamma}{\alpha}} \|\varphi\|_{\mathbf{B}^{-\gamma}_{\infty,\infty}}. \tag{3.27}$$

*Proof.* Notice that by (2.6) and Bernstein's inequality (2.5) , we have that for any $\varepsilon > 0$,

$$\|b \cdot \nabla u(s)\|_{\mathbf{B}^{-\beta}_{\infty,\infty}} \lesssim \|b\|_{L_T^\infty \mathbf{B}^{-\beta}_{\infty,\infty}} \|u(s)\|_{\mathbf{B}^{1+\beta+\varepsilon}_{\infty,\infty}}, \tag{3.28}$$

which together with (3.25) and (3.26) implies that for any $\delta \in [0, \alpha + 1)$ and $\varepsilon > 0$,

$$\begin{aligned} \|u(t)\|_{\mathbf{B}^\delta_{\infty,\infty}} &\lesssim (T-t)^{-\frac{\delta+\gamma}{\alpha}} \|\varphi\|_{\mathbf{B}^{-\gamma}_{\infty,\infty}} + \int_t^T (s-t)^{-\frac{\delta+\beta}{\alpha}} \|b \cdot \nabla u(s)\|_{\mathbf{B}^{-\beta}_{\infty,\infty}} \mathrm{d}s \\ &\lesssim (T-t)^{-\frac{\delta+\gamma}{\alpha}} \|\varphi\|_{\mathbf{B}^{-\gamma}_{\infty,\infty}} + \int_t^T (s-t)^{-\frac{\delta+\beta}{\alpha}} \|u(s)\|_{\mathbf{B}^{1+\beta+\varepsilon}_{\infty,\infty}} \mathrm{d}s. \end{aligned} \tag{3.29}$$

Hence, particularly, taking $0 < \varepsilon < \alpha - 1 - 2(\beta \vee \gamma)$ and $\delta = 1 + \beta + \varepsilon$ in (3.29), we have

$$\|u(t)\|_{\mathbf{B}^{1+\beta+\varepsilon}_{\infty,\infty}} \lesssim (T-t)^{-\frac{1+\beta+\varepsilon+\gamma}{\alpha}} \|\varphi\|_{\mathbf{B}^{-\gamma}_{\infty,\infty}} + \int_t^T (s-t)^{-\frac{1+2\beta+\varepsilon}{\alpha}} \|u(s)\|_{\mathbf{B}^{1+\beta+\varepsilon}_{\infty,\infty}} \mathrm{d}s,$$

which yields

$$\begin{aligned} \|u(T-(T-t))\|_{\mathbf{B}^{1+\beta+\varepsilon}_{\infty,\infty}} \lesssim& \int_0^{T-t} (T-t-s)^{-\frac{1+2\beta+\varepsilon}{\alpha}} \|u(T-s)\|_{\mathbf{B}^{1+\beta+\varepsilon}_{\infty,\infty}} \mathrm{d}s \\ &+ (T-t)^{-\frac{1+\beta+\varepsilon+\gamma}{\alpha}} \|\varphi\|_{\mathbf{B}^{-\gamma}_{\infty,\infty}}, \end{aligned}$$

and then

$$\|u(t)\|_{\mathbf{B}^{1+\beta+\varepsilon}_{\infty,\infty}} \lesssim (T-t)^{-\frac{1+\beta+\varepsilon+\gamma}{\alpha}} \|\varphi\|_{\mathbf{B}^{-\gamma}_{\infty,\infty}} \tag{3.30}$$

since Gronwall's inequality of Volterra type (see [11, Lemma 2.6] or [44, Lemma 2.2]) with $\frac{1+\beta+\varepsilon+\gamma}{\alpha}, \frac{1+2\beta+\varepsilon}{\alpha} < 1$.

i) *Case one:* $\delta \in [0, \alpha-\beta)$. Substituting (3.30) to (3.29), by a change of variables, we obtain that for any $0 < \varepsilon < \alpha - 1 - 2(\beta \vee \gamma)$,

$$\begin{aligned}\|u(t)\|_{\mathbf{B}^{\delta}_{\infty,\infty}} &\lesssim (T-t)^{-\frac{\delta+\gamma}{\alpha}}\|\varphi\|_{\mathbf{B}^{-\gamma}_{\infty,\infty}} + \|\varphi\|_{\mathbf{B}^{-\gamma}_{\infty,\infty}}\int_t^T (s-t)^{-\frac{\delta+\beta}{\alpha}}(T-s)^{-\frac{1+\beta+\varepsilon+\gamma}{\alpha}}\mathrm{d}s \\ &\lesssim (T-t)^{-\frac{\delta+\gamma}{\alpha}}\|\varphi\|_{\mathbf{B}^{-\gamma}_{\infty,\infty}}\left(1 + (T-t)^{\frac{\alpha-1-2\beta-\varepsilon}{\alpha}}\int_0^1 s^{-\frac{\delta+\beta}{\alpha}}(1-s)^{-\frac{1+\beta+\varepsilon+\gamma}{\alpha}}\mathrm{d}s\right) \\ &\lesssim (T-t)^{-\frac{\delta+\gamma}{\alpha}}\|\varphi\|_{\mathbf{B}^{-\gamma}_{\infty,\infty}},\end{aligned}$$

where we used the definitions of Beta functions with $0 < \frac{\delta+\beta}{\alpha}, \frac{1+\beta+\varepsilon+\gamma}{\alpha} < 1$.

ii) *Case two:* $\delta = \alpha - \beta$. By taking $\eta = \beta$ in (3.26), and $\delta = 0, \alpha$ respectively, we deduce that

$$\|\mathcal{R}_j P_{t,s}(b\cdot\nabla u(s))\|_\infty \lesssim \left[(2^{-\alpha j}(s-t)^{-\frac{\alpha+\beta}{\alpha}}) \wedge (s-t)^{-\frac{\beta}{\alpha}}\right]\|b\cdot\nabla u(s)\|_{\mathbf{B}^{-\beta}_{\infty,\infty}},$$

which derives that for any $0 < \varepsilon < \alpha - 1 - 2(\beta \vee \gamma)$,

$$\|\mathcal{R}_j P_{t,s}(b\cdot\nabla u(s))\|_\infty \lesssim \left[(2^{-\alpha j}(s-t)^{-\frac{1+\beta}{\alpha}}) \wedge (s-t)^{-\frac{\beta}{\alpha}}\right](T-s)^{-\frac{1+\beta+\varepsilon+\gamma}{\alpha}}\|\varphi\|_{\mathbf{B}^{-\gamma}_{\infty,\infty}}$$

provied by (3.28) and (3.30). Consequently, by Lemma A.1, one sees that

$$\begin{aligned}&\left\|\mathcal{R}_j\int_t^T P_{t,s}(b\cdot\nabla u(s))\mathrm{d}s\right\|_\infty \\ &\lesssim \|\varphi\|_{\mathbf{B}^{-\gamma}_{\infty,\infty}}\int_0^{T-t}\left[(2^{-\alpha j}s^{-\frac{\alpha+\beta}{\alpha}}) \wedge s^{-\frac{\beta}{\alpha}}\right](T-t-s)^{-\frac{1+\beta+\varepsilon+\gamma}{\alpha}}\mathrm{d}s \\ &\lesssim \|\varphi\|_{\mathbf{B}^{-\gamma}_{\infty,\infty}}2^{-j(\alpha-\beta)}(T-t)^{-\frac{1+\beta+\varepsilon+\gamma}{\alpha}} \lesssim \|\varphi\|_{\mathbf{B}^{-\gamma}_{\infty,\infty}}2^{-j(\alpha-\beta)}(T-t)^{-\frac{\alpha-\beta+\gamma}{\alpha}},\end{aligned}$$

which leads to the desired estimates trivially.

Combining these two cases, we complete the proof. □

## 4. Well-posedness of SDEs

Fix $\alpha \in (1,2)$. Let $(\Omega, \mathscr{F}, (\mathscr{F})_{t\geqslant 0}, \mathbb{P})$ be a filtered probability space satisfying the usual argument and $\{L_t^{(\alpha)}, t \geqslant 0\}$ be a non-degenerate symmetric $\alpha$-stable process on it. The expectation with respect to $\mathbb{P}$ denoted by $\mathbb{E}^{\mathbb{P}}$ or simply by $\mathbb{E}$ if there is no confusion possible. In this section, assuming that the Lévy measure of $L_t^{(\alpha)}$ satisfies condition **(ND)**, we focus on the well-posedness of SDEs with multiplicative noises and distributional drifts:

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_{t-})\mathrm{d}L_t^{(\alpha)},\ X_0 = x \in \mathbb{R}^d, \tag{4.1}$$

where the drift coefficient $b$ is a distribution belongs to the space $L_T^\infty\mathbf{B}^{-\beta}_{\infty,\infty}$ with $\beta \in (0, \frac{\alpha-1}{2})$, and the diffusion coefficient $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ satisfies the condition $(\mathbf{H}^\sigma)$.

Given $T > 0$, let $\mathbf{D} := D([0,T], \mathbb{R}^d)$ be the space of all càdlàg functions from $\mathbb{R}_+$ to $\mathbb{R}^d$. It is easy to see that, for any $\varepsilon, t > 0$, such a function has at most finitely many jumps of size greater than $\varepsilon$ before time $t$. In the sequel, $\mathbf{D}$ is equipped with Skorokhod $J_1$-topology, making $\mathbf{D}$ into a Polish space. Then we endow $\mathbf{D}$ with the usual Borel $\sigma$-algebra $\mathscr{B}(\mathbf{D})$. Denote all the probability measures over $\mathbf{D}$ by $\mathcal{P}(\mathbf{D})$. In this section, let the evaluation map $\pi_t : \mathbf{D} \to \mathbb{R}^d$ be the coordinate process defined by

$$\pi_t(\omega) = \omega_t, \quad \forall t \in [0,T],\ \omega \in \mathbf{D}. \tag{4.2}$$

Define the natural filtration

$$\mathcal{D}_t := \sigma(\pi_u, 0 \leqslant u \leqslant t).$$

By [16, Proposition 7.1 of Chapter 3] or [28, c) of Theorem 1.14, p.328]), one sees that $\mathcal{D}_t$ is equal to the Borel $\sigma$-algebra $\mathscr{B}(D([0,t], \mathbb{R}^d))$. For any probability measure $\mathbf{Q}$ on $\mathbf{D}$, we denote the expectation with respect to $\mathbf{Q}$ by $\mathbf{E}^{\mathbf{Q}}$ or simply by $\mathbf{E}$ if there is no risk of confusion.

4.1. **Generalized martingale problems.** In this subsection, we concentrate on showing our first main result Theorem 1.1. For fixed $T > 0$ and $f \in C_c^\infty(\mathbb{R}^d)$, we consider the following backward nonlocal parabolic equation:

$$\partial_t u + \mathscr{L}_\sigma^{(\alpha)} u + b \cdot \nabla u + f = 0, \quad u(T) = 0, \tag{4.3}$$

where $\mathscr{L}_\sigma^{(\alpha)}$ is defined by (1.6) with symmetric non-degenerate $\nu^{(\alpha)}$ given by (2.9).

**Remark 4.1.** *According to Theorem 3.4, the backward nonlocal parabolic partial differential equation (4.3) has a unique solution $u \in L_T^\infty \mathbf{B}_{\infty,\infty}^{\alpha-\beta}$ for any $f \in C_c^\infty(\mathbb{R}^d)$.*

**Definition 4.2** (Generalized martingale solution). *Let $\alpha \in (1,2)$ and $T > 0$. For $x \in \mathbb{R}^d$, a probability measure $\mathbf{Q}_x \in \mathcal{P}(\mathbf{D})$ is a generalized martingale solution of SDE (4.1) starting at $x$ with coefficients $b, \sigma$ if*

*(i) $\mathbb{P}(\pi_0 = x) = 1$;*
*(ii) For any $t \in [0,T]$ and $f \in C_c^\infty(\mathbb{R}^d)$, the process*

$$M_t := u(t, \pi_t) - u(0,x) + \int_0^t f(\pi_r) \mathrm{d}r$$

*is a martingale under $\mathbf{Q}_x$ with respect to filtration $(\mathcal{D}_t)_{t\in[0,T]}$, where $u$ is the unique solution of PDE (4.3).*

*The set of all the generalized martingale solutions is denoted by $\mathcal{M}_{b,\sigma}(x)$.*

First of all, we consider the following approximation SDE:

$$\mathrm{d}X_t^m = b_m(t, X_t^m)\mathrm{d}t + \sigma(t, X_{t-}^m)\mathrm{d}L_t^{(\alpha)}, \quad X_0^m = x \in \mathbb{R}^d, \tag{4.4}$$

where $b_m(t,x) := \varphi_m * b(t,x)$, $m \in \mathbb{N}$, is a sequence of smooth functions such that

$$\lim_{m\to\infty} \|b_m - b\|_{L_T^\infty \mathbf{B}_{\infty,\infty}^{-\vartheta}} = 0, \quad \forall \vartheta > \beta.$$

Obviously,

$$\|b_m\|_{L^\infty_T \mathbf{B}^{-\beta}_{\infty,\infty}} \lesssim \|b\|_{L^\infty_T \mathbf{B}^{-\beta}_{\infty,\infty}}.$$

It is well-known (for example, see [13, Theorem 1.1]) that there is a unique strong solution $X^m(x)$ for SDE (4.4) on $(\Omega, \mathscr{F}, (\mathscr{F})_{t\geqslant 0}, \mathbb{P})$. Denote the law of $X^m(x)$ by

$$\mathbf{Q}^m_x := \mathbb{P} \circ (X^m(x))^{-1}.$$

It is well-known that $\mathbf{Q}^m_x$ is also a martingale solution of SDE (4.4), denoted by $\mathbf{Q}^m_x \in \mathcal{M}_{b_m,\sigma}(x)$. Observe that for any good enough function $g : \mathbb{R}^d \to \mathbb{R}$,

$$\int_\Omega g(X^m_t(x, \omega))\mathbb{P}(\mathrm{d}\omega) = \int_{\mathbf{D}} g(\pi_t(\omega))\mathbf{Q}^m_x(\mathrm{d}\omega),$$

that is $\mathbb{E} g(X^m_t(x)) = \mathbf{E}^{\mathbf{Q}^m_x} g(\pi_t)$.

The following result is used to estimate the term $\int_0^\cdot b_m(r, X^m_r(x))\mathrm{d}r$, and then to show the tightness of $\{X^m(x), m \geqslant 1\}$ in Theorem 4.5. In the sequel, we use $X^m$ to denote $X^m(x)$ when there is no chance of confusion.

**Lemma 4.3.** *Let $T > 0$ and $\tau$ be a bounded stopping time satisfying $0 \leqslant \tau \leqslant \tau + \delta \leqslant T$ with some $\delta \in (0, 1)$. Assume that $f \in L^\infty_T C^\infty_c$ be a bounded and smooth function. For any $p > 0$ and $\beta \in (0, \frac{\alpha-1}{2})$, there is a constant $c > 0$ independent of $m, \tau$ such that*

$$\mathbb{E}\left|\int_\tau^{\tau+\delta} f(r, X^m_r)\mathrm{d}r\right|^p \lesssim_c \delta^{\frac{p(\alpha-\beta)}{\alpha}} \|f\|^p_{L^\infty_T \mathbf{B}^{-\beta}_{\infty,\infty}}. \tag{4.5}$$

*Proof.* Without loss of generality, we assume $p \geqslant 2$. When $p < 2$, it is directly from the case $p = 2$ and Jensen's inequality. Recalling PDE (3.1), let $u^\lambda_m$ be a classical solution of the backward equation

$$\partial_t u^\lambda_m + \left(\mathscr{L}^{(\alpha)}_\sigma - \lambda\right)u^\lambda_m + b_m \cdot \nabla u^\lambda_m + f = 0, \quad u^\lambda_m(T) = 0. \tag{4.6}$$

By Itô's formula (cf. [27, Theorem 5.1 of Chapter II]), we have that for any stopping time $\tilde\tau \leqslant T$,

$$\begin{aligned} u^\lambda_m(\tilde\tau, X^m_{\tilde\tau}) - u^\lambda_m(0, x) = &\int_0^{\tilde\tau} (\partial_r u_m)(r, X^m_r)\mathrm{d}r + \int_0^{\tilde\tau} b_m(r, X^m_r) \cdot \nabla u^\lambda_m(r, X^m_r)\mathrm{d}r \\ &+ \int_0^{\tilde\tau}\int_{\mathbb{R}^d} \left(u_m(r, X^m_{r-} + \sigma(r, X^m_{r-})z) - u^\lambda_m(r, X^m_{r-})\right)\tilde N(\mathrm{d}r, \mathrm{d}z) \\ &+ \int_0^{\tilde\tau}\int_{\mathbb{R}^d} \Big(u^\lambda_m(r, X^m_r + \sigma(r, X^m_{r-})z) - u^\lambda_m(r, X^m_r) \\ &\qquad - \sigma(r, X^m_{r-})z \cdot \nabla u^\lambda_m(r, X^m_r)\Big)\nu^{(\alpha)}(\mathrm{d}z)\mathrm{d}r, \end{aligned}$$

which together with (4.6) derives that

$$\int_\tau^{\tau+\delta} f(r, X^m_r)\mathrm{d}r = u^\lambda_m(\tau, X^m_\tau) - u^\lambda_m(\tau + \delta, X^m_{\tau+\delta}) + \lambda \int_\tau^{\tau+\delta} u^\lambda_m(r, X^m_r)\mathrm{d}r + M_{\tau,\tau+\delta} \tag{4.7}$$

where we substituted $\tau$ and $\tau+\delta$ for $\tilde\tau$, and

$$M_{\tau,\tau+\delta} := \int_\tau^{\tau+\delta}\int_{\mathbb{R}^d}\Big(u_m^\lambda(r, X_{r-}^m + \sigma(r, X_{r-}^m)z) - u_m^\lambda(r, X_{r-}^m)\Big)\tilde N(\mathrm{d}r, \mathrm{d}z).$$

Notice that, by the condition $(\mathbf{H}^\sigma)$, we have that

$$|u_m^\lambda(r, X_{r-}^m + \sigma(r, X_{r-}^m)z) - u_m^\lambda(r, X_{r-}^m)| \leqslant (2\|u_m^\lambda\|_{L_T^\infty}) \wedge (c_0\|\nabla u_m^\lambda\|_{L_T^\infty}|z|),$$

which together with Kunita's inequality(cf. [31, Theorem 2.11]) yields that for any $p \geqslant 2$,

$$\begin{aligned}\mathbb{E}|M_{\tau,\tau+\delta}|^p &\lesssim \mathbb{E}\Big[\Big(\int_\tau^{\tau+\delta}\int_{\mathbb{R}^d}|u_m^\lambda(r, X_{r-}^m + \sigma(r, X_{r-}^m)z) - u_m^\lambda(r, X_{r-}^m)|^2\nu^{(\alpha)}(\mathrm{d}z)\mathrm{d}r\Big)^{p/2}\Big]\\ &\quad + \mathbb{E}\int_\tau^{\tau+\delta}\int_{\mathbb{R}^d}|u_m^\lambda(r, X_{r-}^m + \sigma(r, X_{r-}^m)z) - u_m^\lambda(r, X_{r-}^m)|^p\nu^{(\alpha)}(\mathrm{d}z)\mathrm{d}r\\ &\lesssim \delta^{p/2}\left(\int_{\mathbb{R}^d}\left(\|u_m^\lambda\|_{L_T^\infty} \wedge (\|\nabla u_m^\lambda\|_{L_T^\infty})|z|\right)^2\nu^{(\alpha)}(\mathrm{d}z)\right)^{p/2}\\ &\quad + \delta\int_{\mathbb{R}^d}\left(\|u_m^\lambda\|_{L_T^\infty} \wedge (\|\nabla u_m^\lambda\|_{L_T^\infty})|z|\right)^p\nu^{(\alpha)}(\mathrm{d}z).\end{aligned}$$

Observe that by (2.7) and Bernstein's inequality,

$$\|\nabla u_m^\lambda\|_{L_T^\infty} \lesssim \|\nabla u_m^\lambda\|^{1/2}_{L_T^\infty\mathbf{B}^{-(\alpha-\beta)+1}_{\infty,\infty}}\|\nabla u_m^\lambda\|^{1/2}_{L_T^\infty\mathbf{B}^{(\alpha-\beta)-1}_{\infty,\infty}} \lesssim \|u_m^\lambda\|^{1/2}_{L_T^\infty\mathbf{B}^{2-(\alpha-\beta)}_{\infty,\infty}}\|u_m^\lambda\|^{1/2}_{L_T^\infty\mathbf{B}^{\alpha-\beta}_{\infty,\infty}}.$$

Hence, by (3.20) and (3.21), one sess that there is a constant $\lambda_0 > 1$ such that for any $\lambda > \lambda_0$,

$$\|\nabla u_m^\lambda\|_{L_T^\infty} \lesssim \lambda^{-\frac{(\alpha-\beta)-1}{\alpha}}\|f\|_{L_T^\infty\mathbf{B}^{-\beta}_{\infty,\infty}} \quad\text{and}\quad \|u_m^\lambda\|_{L_T^\infty} \leqslant \|u_m^\lambda\|_{\mathbf{B}^0_{\infty,1}} \lesssim \lambda^{-\frac{\alpha-\beta}{\alpha}}\|f\|_{L_T^\infty\mathbf{B}^{-\beta}_{\infty,\infty}},$$

where we used the fact $\beta \in (0, \frac{\alpha-1}{2})$. Consequently, from (4.7), we obtain that for any $\lambda > \lambda_0$,

$$\begin{aligned}\mathbb{E}\left|\int_\tau^{\tau+\delta} f(r, X_r^m)\mathrm{d}r\right|^p &\lesssim (2+\lambda\delta)^p\|u_m^\lambda\|^p_{L_T^\infty} + \delta^{p/2}\left(\int_{\mathbb{R}^d}\left(\|u_m^\lambda\|_{L_T^\infty} \wedge (\|\nabla u_m^\lambda\|_{L_T^\infty}|z|)\right)^2\nu^{(\alpha)}(\mathrm{d}z)\right)^{p/2}\\ &\quad + \delta\int_{\mathbb{R}^d}\left(\|u_m^\lambda\|_{L_T^\infty} \wedge (|\nabla u_m^\lambda\|_{L_T^\infty}|z|)\right)^p\nu^{(\alpha)}(\mathrm{d}z)\\ &\lesssim \Bigg[\lambda^{-\frac{\alpha-\beta}{\alpha}p}(1+(\lambda\delta)^p) + \delta^{p/2}\lambda^{-\frac{\alpha-\beta}{\alpha}p}\left(\int_{\mathbb{R}^d}\left(1 \wedge (\lambda^{\frac{2}{\alpha}}|z|^2)\right)\nu^{(\alpha)}(\mathrm{d}z)\right)^{p/2}\\ &\quad + \delta\lambda^{-\frac{\alpha-\beta}{\alpha}p}\int_{\mathbb{R}^d}\left(1 \wedge (\lambda^{\frac{p}{\alpha}}|z|^p)\right)\nu^{(\alpha)}(\mathrm{d}z)\Bigg]\|f\|^p_{L_T^\infty\mathbf{B}^{-\beta}_{\infty,\infty}}.\end{aligned}$$

Observe that $\|b_m\|_{L_T^\infty\mathbf{B}^{-\beta}_{\infty,\infty}} \lesssim \|b\|_{L_T^\infty\mathbf{B}^{-\beta}_{\infty,\infty}}$. Therefore, by (2.12), we have

$$\mathbb{E}\left|\int_\tau^{\tau+\delta} f(r, X_r^m)\mathrm{d}r\right|^p \lesssim \lambda^{-\frac{p(\alpha-\beta)}{\alpha}}(1+(\lambda\delta)^p + (\lambda\delta)^{p/2} + \lambda\delta)\|f\|^p_{L_T^\infty\mathbf{B}^{-\beta}_{\infty,\infty}}.$$

Taking $\lambda = \lambda_0\delta^{-1}$, we get the desired estimates. $\square$

**Remark 4.4.** *By* (4.5) *and Chebyshev's inequality, one sees that for any* $R > 0$,

$$\lim_{R\to\infty}\sup_m \mathbb{P}\left(\sup_{t\in[0,T]}\left|\int_0^t f(r,X_r^m)\mathrm{d}r\right| \geqslant R\right) = 0. \tag{4.8}$$

Now we give the tightness of $\{X^m, m \geqslant 1\}$.

**Theorem 4.5.** *The sequence* $\{X^m\}_{m\in\mathbb{N}}$ *in* $\mathbf{D}$ *is tight.*

*Proof.* Fix $T > 0$. Let $\tau$ be a bounded stopping time satisfying $0 \leqslant \tau \leqslant \tau+\delta \leqslant T$ with $\delta \in (0,1)$. By SDE (4.4), we have

$$X^m_{\tau+\delta} - X^m_\tau = \int_\tau^{\tau+\delta}\sigma(r,X^m_{r-})\mathrm{d}L_r^{(\alpha)} + \int_\tau^{\tau+\delta} b_m(r,X_r^m)\mathrm{d}r.$$

For $p \in [1,\alpha)$, by Chebyshev's inequality, Lemma 2.6, and Lemma 4.3, we have that for each $R > 0$,

$$\mathbb{P}(|X^m_{\tau+\delta} - X^m_\tau| \geqslant R) \leqslant R^{-p}\mathbb{E}|X^m_{\tau+\delta} - X^m_\tau|^p \lesssim \delta^{p/\alpha}(\|\sigma\|^p_{L^\infty_T} + \|b_m\|^p_{L^\infty_T\mathbf{B}^{-\beta}_{\infty,\infty}}), \tag{4.9}$$

where the implicit constant in the last inequality is independent of $m, \tau$, and $\delta$. Furthermore,

$$\lim_{\delta\downarrow 0}\sup_m\sup_{\tau\leqslant T}\sup_{h\in[0,\delta]}\mathbb{P}(|X^m_{\tau+h} - X^m_\tau| \geqslant R) = 0,$$

provided by $\|b_m\|_{L^\infty_T\mathbf{B}^{-\beta}_{\infty,\infty}} \leqslant \|b\|_{L^\infty_T\mathbf{B}^{-\beta}_{\infty,\infty}}$. Moreover, by Burkholder-Davis-Gundy's inequality for jump processes (see [39, Lemma 2.3] or [36, Theorem 1]) and (2.11), there is a constant $c > 0$ that only depends on $T, \|\sigma\|_{L^\infty_T}, \alpha, \nu^{(\alpha)}$ such that

$$\mathbb{E}\Big(\sup_{t\in[0,T]}\left|\int_0^t \sigma(r,X^m_{r-})\mathrm{d}L^{(\alpha)}_r\right|\Big) \lesssim \Big[\mathbb{E}\Big(\int_0^T\int_{|z|\leqslant 1}\left|\sigma(r,X^m_{r-})z\right|^2\nu^{(\alpha)}(\mathrm{d}z)\mathrm{d}r\Big)\Big]^{1/2}$$
$$+\mathbb{E}\Big(\int_0^T\int_{|z|>1}\left|\sigma(r,X^m_{r-})z\right|\nu^{(\alpha)}(\mathrm{d}z)\mathrm{d}r\Big) \leqslant c,$$

which together with (4.8), and by Chebyshev's inequality, yields that

$$\lim_{R\to\infty}\sup_m\mathbb{P}\Big(\sup_{t\leqslant T}|X_t^m| \geqslant R\Big) = 0. \tag{4.10}$$

Hence, by (4.9), (4.10) and Aldous's criterion for tightness (cf. [28, Theorem 4.5 of Chapter VI, p.356] and [29, Lemma 16.12]), we conclude the proof. □

Now we are in a position to give

*Proof of Theorem 1.1.* **(Existence)** Since the sequence $\{X^m(x)\}_{m\in\mathbb{N}}$ in $\mathbf{D}$ is tight, there are a subsequence $m_i$ and a probability measure $\mathbf{Q}_x$ on $\mathbf{D}$ such that $\mathbf{Q}_x^{m_i} := \mathbb{P}\circ(X^{m_i}(x))^{-1}$ converges to $\mathbf{Q}_x$ weakly. Below, for simplicity of notations, we still denote this subsequence by $\mathbf{Q}_x^m$. Next, we verify that the limit $\mathbf{Q}_x$ is a solution of the generalized martingale problem in the sense of Definition 4.2, and then the existence is obtained. Fix $T > 0$ and $f \in C_c^\infty(\mathbb{R}^d)$. Recalling PDE (3.1), by Theorem 3.4, there is a unique solution $u \in L^\infty_T\mathbf{B}^{\alpha-\beta}_{\infty,\infty}$ solves the backward nonlocal parabolic equation (4.3). Let $\pi_t : \mathbf{D} \to \mathbb{R}^d$ be the coordinate process defined by (4.2). Set

$$M_t(\omega) := u(t, \pi_t(\omega)) - u(0, x) + \int_0^t f(\pi_r(\omega))\mathrm{d}r, \quad \forall \omega \in \mathbf{D}. \tag{4.11}$$

Our aim is to prove that $M$ is a $(\mathcal{D}_t)_{t\in[0,T]}$-martingale under $\mathbf{Q}_x$.

Define

$$M_t^m(\omega) := u_m(t, \pi_t(\omega)) - u_m(0, x) + \int_0^t f(\pi_r(\omega))\mathrm{d}r, \quad \forall \omega \in \mathbf{D}, \tag{4.12}$$

where $u_m$ is a classical solution of the backward equation:

$$\partial_t u_m + \mathscr{L}_\sigma^{(\alpha)} u_m + b_m \cdot \nabla u_m + f = 0, \quad u_m(T) = 0. \tag{4.13}$$

We claim that $M^m$ is a $(\mathcal{D}_t)_{t\in[0,T]}$-martingale under $\mathbf{Q}_x^m$. Indeed, using Itô's formula (cf. [27, Theorem 5.1 of Chapter II]), one sees that

$$\begin{aligned} u_m(t, X_t^m) - u_m(0, x) = & \int_0^t (\partial_r u_m)(r, X_r^m)\mathrm{d}r + \int_0^t b_m(r, X_r^m) \cdot \nabla u_m(r, X_r^m)\mathrm{d}r \\ & + \int_0^t \int_{\mathbb{R}^d} \Big(u_m(r, X_{r-}^m + \sigma(r, X_{r-}^m)z) - u_m(r, X_{r-}^m)\Big)\tilde{N}(\mathrm{d}r, \mathrm{d}z) \\ & + \int_0^t \int_{\mathbb{R}^d} \Big(u_m(r, X_r^m + \sigma(r, X_{r-}^m)z) - u_n(r, X_r^m) \\ & \qquad - \sigma(r, X_{r-}^m)z \cdot \nabla u_n(r, X_r^m)\Big)\nu^{(\alpha)}(\mathrm{d}z)\mathrm{d}r. \end{aligned}$$

Thanks to (2.8), we have that the third term of the right hand of the above equality is a square-integrable $(\mathscr{F}_t)$-martingale under $\mathbb{P}$ (cf. [27]). Observe that every path of $X^m$ has at most countably many jumps on $\mathbb{R}_+$. Thus, we obtain that

$$M_t^m(\omega) \overset{(4.13)}{=} u_m(t, \pi_t(\omega)) - u_m(0, x) - \int_0^t (\partial_r + \mathscr{L}_\sigma^{(\alpha)} + b_m \cdot \nabla)u_m(r, \pi_r(\omega))\mathrm{d}r$$

is an $(\mathcal{D}_t)_{t\in[0,T]}$-martingale under $\mathbf{Q}_x^m$, which means $\mathbf{Q}_x^m \in \mathcal{M}_{b_m,\sigma}(x)$.

*Step 1.* We state that for every $0 \leqslant s \leqslant t$ and any $\mathcal{D}_s$-measurable bounded continuous function $\eta : \mathbf{D} \to \mathbb{R}$,

$$\lim_{m\to\infty} \mathbf{E}^{\mathbf{Q}_x^m}[(M_t - M_s)\eta] = 0. \tag{4.14}$$

In fact, by the definition of martingales, we have that for every $0 \leqslant s \leqslant t$,

$$\mathbf{E}^{\mathbf{Q}_x^m}[(M_t^m - M_s^m)\eta] = 0,$$

and then

$$\mathbf{E}^{\mathbf{Q}_x^m}[(M_t - M_s)\eta] = \mathbf{E}^{\mathbf{Q}_x^m}[(M_t - M_t^m)\eta] - \mathbf{E}^{\mathbf{Q}_x^m}[(M_s - M_s^m)\eta].$$

Since for any $\alpha-\beta>\varepsilon>0$, $u_m$ converges to $u$ in $L^\infty_T\mathbf{B}^{\alpha-\beta-\varepsilon}_{\infty,\infty}$ and $L^\infty([0,T]\times\mathbb{R}^d)$, by (4.11) and (4.12), we get

$$\sup_{\omega\in\mathbf{D}}|M_t(\omega)-M^m_t(\omega)|\leqslant 2\|u-u_m\|_{L^\infty_T}\to 0,\ \text{ as }\ m\to\infty,$$

which implies (4.14).

*Step 2.* In this step, we show that $M$ is an $(\mathcal{D}_t)_{t\in J(\pi)^c\cap[0,T]}$-martingale under $\mathbf{Q}_x$, where

$$\begin{aligned}J(\pi)&:=\{r>0:\mathbf{Q}_x(|\pi_r-\pi_{r-}|\neq 0)>0\}\\&=\{r>0:\mathbf{Q}_x(\omega:r\in J(\omega))>0\}\end{aligned}$$

with (the set of discontinuity points of $\omega\in\mathbf{D}$)

$$J(\omega):=\{t>0:|\omega_t-\omega_{t-}|\neq 0\},\quad\forall\omega\in\mathbf{D}.$$

By [28, 2.3 of Chapter VI, p.339], $\pi_r$ is continuous at every point $\omega\in\mathbf{D}$ such that $r\notin J(\omega)$. Thus, for any $t\notin J(\pi)$, $u(t,\pi_t)$ are continuous on $\mathbf{D}$ for almost all $\omega\in\mathbf{D}$. Moreover, for every $\omega\in\mathbf{D}$, letting $\{\omega^{(n)}\}$ be a sequence in $\mathbf{D}$ and converge to $\omega$ under Skorokhod $J_1$-topology, since $f\in C^\infty_c(\mathbb{R}^d)$, we also have

$$\lim_{n\to\infty}f(\pi_r(\omega^{(n)}))\mathbf{1}_{J(\omega)^c}(r)=f(\pi_r(\omega))\mathbf{1}_{J(\omega)^c}(r),$$

which together with the fact that $J(\omega)$ is at most countable and the dominated convergence theorem derives that the term $\int_0^t f(\pi_r(\cdot))\mathrm{d}r$ is continuous on $\mathbf{D}$ for any fixed $t$. Therefore, for any $t\notin J(\pi)$, by (4.12), the random variable $M_t:\mathbf{D}\to\mathbb{R}$ is bounded and almost everywhere continuous on $\mathbf{D}$. Consequently, by $\mathbf{Q}^m_x\xrightarrow{w}\mathbf{Q}_x$ and (4.14), we obtain that for every $s,t\notin J(\pi)$ and $\mathcal{D}_s$-measurable bounded continuous function $\eta:\mathbf{D}\to\mathbb{R}$,

$$\mathbf{E}^{\mathbf{Q}_x}[(M_t-M_s)\eta]=\lim_{m\to\infty}\mathbf{E}^{\mathbf{Q}^m_x}[(M_t-M_s)\eta]=0,\tag{4.15}$$

which implies our statement by approximation property of functions in $L^1(\mathbf{D},\mathcal{D}_s,\mathbf{Q}_x)$ (see [29, Lemma 1.35, p.18]) and the dominated convergence theorem.

*Step 3.* Due to the dominated convergence theorem, we obtain that (4.15) holds for all $s,t\in[0,T]$ since $M$ is a càdlàg process and $J(\pi)^c$ is dense in $[0,T]$ (cf. [28, Lemma 3.12 of Chapter VI] or [16, Lemma 7.7 of Chapter 3] ). Thus, $\mathbf{Q}_x\in\mathcal{M}_{b,\sigma}(x)$.

**(Uniqueness)** Let $\mathbf{Q}_1$ and $\mathbf{Q}_2$ be two solutions of the generalized martingale problem in the sense of Definition 4.2. Let $u$ be the unique solution of the backward nonlocal PDE (4.3) with $f\in C^\infty_c(\mathbb{R}^d)$. By the definition of generalized martingale solutions, the processes

$$M_t(\omega):=u(t,\pi_t(\omega))-u(0,x)-\int_0^t f(\pi_r(\omega))\mathrm{d}r,\quad\forall\omega\in\mathbf{D},$$

are $(\mathcal{D}_t)$-martingales under $\mathbf{Q}_i$, $i=1,2$. Combining with PDE (4.3), we have

$$\mathbf{E}^{\mathbf{Q}_1}\int_0^T f(\pi_r)\mathrm{d}r=u(0,x)=\mathbf{E}^{\mathbf{Q}_2}\int_0^T f(\pi_r)\mathrm{d}r,$$

which implies that $\mathbf{Q}_1$ and $\mathbf{Q}_2$ have the same one-dimensional marginal distributions. Hence, by [16, Corollary 4.4.3] (or [40, Theorem 6.2.3, p.147]), we get $\mathbf{Q}_1 = \mathbf{Q}_2$ through the standard induction approach.

The proof is finished. □

4.2. **Stability of SDE with $\mathbf{B}^{-\beta}_{\infty,\infty}$ drift.** By Theorem 1.1, for any $\sigma$ satisfying $(\mathbf{H}^\sigma)$ and $b_1, b_2 \in L^\infty_T \mathbf{B}^{-\beta}_{\infty,\infty}$ with some $\beta \in (0, \frac{\alpha-1}{2})$, there are unique $\mathbf{P}_1 \in \mathcal{M}_{b_1,\sigma}(x)$ and $\mathbf{P}_2 \in \mathcal{M}_{b_2,\sigma}(x)$ respectively. Denote by $\mathbf{P}_i(t) := \mathbb{P}_i \circ (\pi_t)^{-1}$. Here is the other main result in this section.

**Theorem 4.6** (Stability estimates). *Let $T > 0$ and $\alpha \in (1, 2)$. Assume $\sigma$ satisfies $(\mathbf{H}^\sigma)$ with constant $c_0$. Then, for any $\beta \in (0, \frac{\alpha-1}{2})$ and $b_1, b_2 \in L^\infty_T \mathbf{B}^{-\beta}_{\infty,\infty}$, there is a constant $c = (d, \alpha, \beta, T, c_0, \|b_1\|_{L^\infty_T \mathbf{B}^{-\beta}_{\infty,\infty}}, \|b_2\|_{L^\infty_T \mathbf{B}^{-\beta}_{\infty,\infty}}) > 0$ such that for any $t \in [0, T]$,*

$$\|\mathbf{P}_1(t) - \mathbf{P}_2(t)\|_{\rm var} \lesssim_c \|b_1 - b_2\|_{L^\infty_T \mathbf{B}^{-\beta}_{\infty,\infty}},$$

*where $\|\cdot\|_{\rm var}$ denotes the total variance of measures.*

*Proof.* Based on the uniqueness, we can assume $b_1, b_2 \in L^\infty_T C^\infty_b$. Then, for each $i = 1, 2$, it is well-known (see [13] for example) that, for any $x \in \mathbb{R}^d$, there is a unique solutions $X^i$, to the following classical SDE:

$$X^i_t = x + \int_0^t b_i(r, X^i_r) \mathrm{d}r + \int_0^t \sigma(r, X_{r-}) \mathrm{d}L^{(\alpha)}_r.$$

It suffices to estimate $\left|\mathbb{E}\varphi(X^2_t) - \mathbb{E}\varphi(X^1_t)\right|$ for any $\varphi \in C^\infty_b(\mathbb{R}^d)$ and $t \in (0, T]$. Using Itô-Tanaka trick, let $\varphi \in C^\infty_b(\mathbb{R}^d)$ be the terminal condition of the following backward PDE:

$$\partial_r u^t + \mathscr{L}^{(\alpha)}_\sigma u^t + b_1 \cdot \nabla u^t = 0, \quad u^t(t) = \varphi, \tag{4.16}$$

where $t \in (0, T]$, $u^t$ is the shifted function $u^t(r, x) := u(t - r, x)$, and $\mathscr{L}^{(\alpha)}_\sigma$ is defined by (1.6). It follows (3.27) that for every $\delta \in [0, \alpha - \beta]$,

$$\|u^t(r)\|_{\mathbf{B}^\delta_{\infty,\infty}} \lesssim (t - r)^{-\frac{\delta}{\alpha}} \|\varphi\|_\infty. \tag{4.17}$$

By Itô's formula (cf. [27, Theorem 5.1 of Chapter II]), we have that for $i = 1, 2$,

$$\begin{aligned} u^t(t, X^i_t) - u^t(0, x) = & \int_0^t (\partial_r u^t)(r, X^i_r) \mathrm{d}r + \int_0^t b_i(r, X^i_r) \cdot \nabla u^t(r, X^i_r) \mathrm{d}r \\ & + \int_0^t \int_{\mathbb{R}^d} \Big( u^t(r, X^i_{r-} + \sigma(r, X^i_{r-}) z) - u^t(r, X^i_{r-}) \Big) \tilde{N}(\mathrm{d}r, \mathrm{d}z) \\ & + \int_0^t \int_{\mathbb{R}^d} \Big( u^t(r, X^i_r + \sigma(r, X^i_{r-}) z) - u^t(r, X^i_r) \\ & \qquad\qquad - \sigma(r, X^i_{r-}) z \cdot \nabla u^t(r, X^i_r) \Big) \nu^{(\alpha)}(\mathrm{d}z) \mathrm{d}r. \end{aligned}$$

Observe that the third term of the right hand of the above equality is a martingale, and every path of $X^i$ has at most countably many jumps on $\mathbb{R}_+$. Thus, we obtain

$$\mathbb{E}u^t(t, X^i_t) = \int_0^t \Big( \partial_r + \mathscr{L}^{(\alpha)}_\sigma + b_i \cdot \nabla \Big) u^t(r, X^i_r) \mathrm{d}r + u^t(0, x)$$

$$\overset{(4.16)}{=} \int_0^t \Big((b_i - b_1)\cdot \nabla u^t\Big)(r, X_r^i)\mathrm{d}r + u^t(0,x),$$

which implies

$$\mathbb{E}\varphi(X_t^1) = \mathbb{E}u^t(t, X_t^1) = u^t(0,x), \tag{4.18}$$

and then

$$\mathbb{E}\varphi(X_t^2) - \mathbb{E}\varphi(X_t^1) = \mathbb{E}\int_0^t \Big((b_2 - b_1)\cdot \nabla u^t\Big)(r, X_r^2)\mathrm{d}r. \tag{4.19}$$

Now the key point is to estimate

$$\mathbb{E}((b_2 - b_1)\cdot \nabla u^t)(r,\cdot)(X_r^2).$$

Define $g_r(x) := ((b_2 - b_1)\cdot \nabla u^t)(r,x)$. Using the Itô-Tanaka trick again, we consider the following PDE:

$$\partial_s w + \mathscr{L}_\sigma^{(\alpha)} w + b_2 \cdot \nabla w = 0, \quad w(r) = g_r,$$

where $0 \leqslant s < r < T$, and $g$ belongs to $C_b^\infty(\mathbb{R}^d)$. Adopting the same argument as (4.18), by (3.27), one sees that for any $r \in [0,T)$, $\delta \in [0, \alpha-\beta]$, and $\beta \in [0, \frac{\alpha-1}{2})$,

$$|\mathbb{E}g_r(X_r^2)| = |\mathbb{E}w(r, X_r^2)| \leqslant \|w(0)\|_{\mathbf{B}^\delta_{\infty,\infty}} \lesssim r^{-\frac{\gamma+\delta}{\alpha}} \|g\|_{\mathbf{B}^{-\beta}_{\infty,\infty}},$$

which derives that

$$\Big|\mathbb{E}\varphi(X_t^2) - \mathbb{E}\varphi(X_t^1)\Big| \lesssim \int_0^t r^{-\frac{\beta+\delta}{\alpha}} \|((b_2 - b_1)\cdot \nabla u^t)(r)\|_{\mathbf{B}^{-\beta}_{\infty,\infty}} \mathrm{d}r.$$

From (2.6), Bernstein's inequality Lemma 2.3, and (4.17), we have that for any $0 < \delta < \frac{\alpha-1}{2} - \beta$,

$$\begin{aligned}
\Big|\mathbb{E}\varphi(X_t^2) - \mathbb{E}\varphi(X_t^1)\Big| &\lesssim \|b_1 - b_2\|_{\mathbf{B}^{-\beta}_{\infty,\infty}} \int_0^t r^{-\frac{\beta+\delta}{\alpha}} \|u^t(r)\|_{\mathbf{B}^{1+\beta+\delta}_{\infty,\infty}} \mathrm{d}r \\
&\lesssim \|b_1 - b_2\|_{\mathbf{B}^{-\beta}_{\infty,\infty}} \|\varphi\|_\infty \int_0^t r^{-\frac{\beta+\delta}{\alpha}} (t-r)^{-\frac{1+\beta+\delta}{\alpha}} \mathrm{d}r \\
&= |b_1 - b_2\|_{\mathbf{B}^{-\beta}_{\infty,\infty}} \|\varphi\|_\infty t^{1-\frac{1+2\beta+2\delta}{\alpha}} \int_0^1 r^{-\frac{\beta+\delta}{\alpha}} (1-r)^{-\frac{1+\beta+\delta}{\alpha}} \mathrm{d}r \\
&\lesssim \|b_1 - b_2\|_{\mathbf{B}^{-\beta}_{\infty,\infty}} \|\varphi\|_\infty,
\end{aligned}$$

where we used the definition of Beta function (1.7). This completes the proof. □

## 5. Euler approximation

5.1. **Euler scheme for SDE with bounded drift.** Fix $T > 0$. In this subsection, we assume $b(x)$ belongs to $L^\infty(\mathbb{R}^d)$ and consider the following SDE:

$$X_t = x + \int_0^t b(X_s)\mathrm{d}t + L_t^{(\alpha)}, \tag{5.1}$$

and its Euler scheme: $X^n_0 = X_0 = x$,

$$X^n_t = x + \int_0^t b(X^n_{\phi_n(s)})\mathrm{d}s + L^{(\alpha)}_t, \tag{5.2}$$

where $n \in \mathbb{N}$, and $\phi_n(t) := k/n$ for $t \in [k/n, (k+1)/n)$ with $k = 0, 1, 2, ...., n$. Define $\mathbf{P}(t) := \mathbb{P} \circ (X_t)^{-1}$, $\mathbf{P}_n(t) := \mathbb{P} \circ (X^n_t)^{-1}$. Note that, for any $p \in (0, \alpha)$,

$$\begin{aligned}\mathbb{E}[|X^n_r - X^n_{\phi_n(r)}|^p] &\leqslant \mathbb{E}\Big(\|b\|_{L^\infty} n^{-1} + |L^{(\alpha)}_r - L^{(\alpha)}_{\phi_n(r)}|\Big)^p \\ &\leqslant (2^{p-1} \vee 1)(2\|b\|^p_\infty n^{-p} + \mathbb{E}[|L^{(\alpha)}_r - L^{(\alpha)}_{\phi_n(r)}|^p]) \overset{(2.14)}{\lesssim} \|b\|^p_\infty n^{-p} + n^{-p/\alpha},\end{aligned} \tag{5.3}$$

where the implicit constant in inequality only depends on $d, \alpha, p, T, \nu^{(\alpha)}$.

This subsection is devoted to proving the following result.

**Theorem 5.1.** *Suppose that $T > 0$, $\alpha \in (1, 2)$, and $b \in L^\infty(\mathbb{R}^d)$. Assume $\sigma$ satisfies* ($\mathbf{H}^\sigma$) *with constant $c_0$. Let $\beta \in (0, (\alpha - 1)/2)$, and $\varepsilon \in (0, \alpha - 1)$, there is a constant $c = c(d, \alpha, T, c_0, \varepsilon, \beta, \|b\|_{L^\infty_T \mathbf{B}^{-\beta}_{\infty,\infty}}) > 0$ such that for any $n \in \mathbb{N}$,*

$$\|\mathbf{P}(t) - \mathbf{P}_n(t)\|_{\mathrm{var}} \lesssim_c \left(\|b\|^{\alpha-\beta}_\infty n^{-(\alpha-1-\beta)} + \|b\|_\infty n^{-\frac{\alpha-\beta-1}{\alpha}} + (1 + \|b\|^2_\infty) n^{-\frac{\alpha-1}{\alpha}}\right),$$

*where $\|\cdot\|_{\mathrm{var}}$ denotes the total variance of measures.*

*Proof.* By the weak uniqueness of SDEs (5.1) and (5.2), we assume $b \in C^\infty_b(\mathbb{R}^d)$. It suffices to estimate

$$\left|\mathbb{E}\varphi(X^n_t) - \mathbb{E}\varphi(X_t)\right|$$

for any $\varphi \in C^\infty_b(\mathbb{R}^d)$. Use Itô-Tanaka trick by considering the following backward PDE with terminal condition $\varphi \in C^\infty_b(\mathbb{R}^d)$:

$$\partial_s u^t + \mathscr{L}^{(\alpha)} u^t + b \cdot \nabla u^t = 0, \quad u^t_t = \varphi,$$

where $u^t$ is the shifted function $u^t(s, x) := u(t - s, x)$ with $0 \leqslant s < t \leqslant T$, and $\mathscr{L}^{(\alpha)}$ is the infinitesimal generator of $L^{(\alpha)}_t$ (see (2.19)). It follows (3.27) that for any $\beta \in (0, (\alpha - 1)/2)$ and $\delta \in [0, \alpha - \beta]$,

$$\|u^t(s)\|_{\mathbf{B}^\delta_{\infty,\infty}} \lesssim (t - s)^{-\frac{\delta}{\alpha}} \|\varphi\|_\infty, \tag{5.4}$$

where the implicit constant in the above inequality on depends on $d, \alpha, T, \delta, \beta, \|b\|_{L^\infty_T \mathbf{B}^{-\beta}_{\infty,\infty}}$. By the same argument as the one in the proof of Theorem 4.6, adopting Itô's formula (cf. [27, Theorem 5.1 of Chapter II]) to $u^t(s, X^n_s)$, we have

$$\begin{aligned}\mathbb{E}\varphi(X^n_t) - \mathbb{E}\varphi(X_t) =& \mathbb{E}\int_0^t \Big(b(X^n_{\phi_n(r)}) - b(X^n_r)\Big) \cdot \nabla u^t(r, X^n_r)\mathrm{d}r \\ =& \mathbb{E}\int_0^t b(X^n_{\phi_n(r)}) \cdot \Big(\nabla u^t(r, X^n_r) - \nabla u^t(r, X^n_{\phi_n(r)})\Big)\mathrm{d}r \\ &+ \int_0^t \Big(\mathbb{E}(b \cdot \nabla u^t(r))(X^n_{\phi_n(r)}) - \mathbb{E}(b \cdot \nabla u^t(r))(X^n_r)\Big)\mathrm{d}r\end{aligned}$$

$$=: \mathscr{I}_1 + \mathscr{I}_2.$$

• For $\mathscr{I}_1$, by Bernstein's inequality, (5.4) and (5.3), one sees that for any $\delta \in (0, \alpha - 1 - \beta]$,

$$|\mathscr{I}_1| \lesssim \|b\|_\infty \int_0^t \|\nabla u^t(r)\|_{\mathbf{B}^\delta_{\infty,\infty}} \mathbb{E}|X^n_{\phi_n(r)} - X^n_r|^\delta \mathrm{d}r$$
$$\lesssim \|b\|_\infty \int_0^t (t-r)^{-\frac{1+\delta}{\alpha}} \mathbb{E}|X^n_{\phi_n(r)} - X^n_r|^\delta \mathrm{d}r$$
$$\overset{(5.3)}{\lesssim} \|b\|_\infty(\|b\|^\delta_\infty n^{-\delta} + n^{-\delta/\alpha}) t^{\frac{\alpha-1-\delta}{\alpha}} \int_0^1 r^{\frac{\alpha-1-\delta}{\alpha}-1} \mathrm{d}r.$$

Consequently, we get that for for any $\delta \in (0, \alpha - 1 - \beta]$,

$$|\mathscr{I}_1| \lesssim \|b\|_\infty(\|b\|^\delta_\infty n^{-\delta} + n^{-\delta/\alpha}). \tag{5.5}$$

• As for $\mathscr{I}_2$, the estimate of

$$\left|\mathbb{E}(b \cdot \nabla u^t(r))(X^n_{\phi_n(r)}) - \mathbb{E}(b \cdot \nabla u^t(r))(X^n_r)\right|$$

is the key ingredient. Using the Itô-Tanaka trick again, we consider the following equation:

$$\partial_s w^r + \mathscr{L}^{(\alpha)} w^r = 0, \quad w^r(r) = f, \tag{5.6}$$

where $w^r(s,x) := w(r-s,x)$ is the shifted function with $0 \leqslant s < r \leqslant T$, $f \in C^\infty_b(\mathbb{R}^d)$, and $\mathscr{L}^{(\alpha)}$ is the infinitesimal generator of $L^{(\alpha)}_t$ given by (2.19). In the last, readers will see that the terminal condition $f$ is taken as $b \cdot \nabla u^t(r)$. Applying Itô's formula (cf. [27, Theorem 5.1 of Chapter II]) to $w^r(s, X^n_s)$ and by (5.6), we have

$$\mathbb{E}f(X^n_r) = \mathbb{E}w^r(r, X^n_r) = w^r(0,x) + \mathbb{E}\int_0^r b(X^n_{\phi_n(s)}) \cdot \nabla w^r(s, X^n_s)\mathrm{d}s,$$

which drives that for any $r_2 > r_1$,

$$\left|\mathbb{E}f(X^n_{r_2}) - \mathbb{E}f(X^n_{r_1})\right| \leqslant |w(r_2,x) - w(r_1,x)| + \left|\mathbb{E}\int_{r_1}^{r_2} b(X^n_{\phi_n(s)}) \cdot \nabla w(r_2 - s, X^n_s)\mathrm{d}s\right|$$
$$+ \left|\mathbb{E}\int_0^{r_1} b(X^n_{\phi_n(s)}) \cdot (\nabla w(r_2 - s, X^n_s) - \nabla w(r_1 - s, X^n_s))\mathrm{d}s\right|$$
$$\lesssim \|w(r_2) - w(r_1)\|_\infty + \|b\|_\infty \int_{r_1}^{r_2} \|\nabla w(r_2 - s)\|_\infty \mathrm{d}s$$
$$+ \|b\|_\infty \int_0^{r_1} \|\nabla w(r_2 - s) - \nabla w(r_1 - s)\|_\infty \mathrm{d}s$$

Furthermore, based on (2.20) and (2.21), we get that for all $0 \leqslant r_1 < r_2 \leqslant T$,

$$\left|\mathbb{E}f(X^n_{r_2}) - \mathbb{E}f(X^n_{r_1})\right| \lesssim \left[1 \wedge ((r_2 - r_1) r_1^{-1})\right] \|f\|_\infty + \|b\|_\infty \|f\|_\infty (r_2 - r_1)^{\frac{-1}{\alpha}+1}$$
$$+ \|b\|_\infty \|f\|_\infty \int_0^{r_1} \left[(r_1 - s)^{-\frac{1}{\alpha}} \wedge ((r_2 - r_1)(r_1 - s)^{-\frac{1+\alpha}{\alpha}})\right] \mathrm{d}s.$$

Noticing that

$$\int_0^{r_2-r_1} s^{-\frac{1}{\alpha}}\mathrm{d}s + (r_2-r_1)\int_{(r_2-r_1)\wedge r_1}^{r_1} s^{-\frac{1+\alpha}{\alpha}}\mathrm{d}s \lesssim (r_2-r_1)^{-\frac{1}{\alpha}+1},$$

one sees that

$$\left|\mathbb{E}f(X^n_{r_2})-\mathbb{E}f(X^n_{r_1})\right| \lesssim \|f\|_\infty\left(\left[1\wedge((r_2-r_1)r_1{}^{-1})\right]+\|b\|_\infty(r_2-r_1)^{-\frac{1}{\alpha}+1}\right). \tag{5.7}$$

Taking the place of $f(\cdot), r_1$, and $r_2$ by $(b\cdot\nabla u^t(r))(\cdot), \phi_n(r)$, and $r$ in (5.7) respectively, and noticing that, by (2.7) and Bernstein's inequality,

$$\begin{aligned}\|\nabla u^t(r)\|_\infty &\lesssim \|\nabla u^t(r)\|^{1/2}_{\mathbf{B}^{-(\alpha-\beta)+1}_{\infty,\infty}}\|\nabla u^t(r)\|^{1/2}_{\mathbf{B}^{(\alpha-\beta)-1}_{\infty,\infty}}\\ &\lesssim \|u^t(r)\|^{1/2}_{\mathbf{B}^{2-(\alpha-\beta)}_{\infty,\infty}}\|u^t(r)\|^{1/2}_{\mathbf{B}^{\alpha-\beta}_{\infty,\infty}} \overset{(5.4)}{\lesssim} (t-r)^{-\frac{1}{\alpha}},\end{aligned}$$

we infer that

$$\begin{aligned}|\mathscr{I}_2| &\lesssim \|b\|_\infty\int_0^t \|\nabla u^t(r)\|_\infty\left(\left[1\wedge(n^{-1}r^{-1})\right]+\|b\|_\infty n^{-\frac{\alpha-1}{\alpha}}\right)\mathrm{d}r\\ &\lesssim n^{-\frac{\alpha-1}{\alpha}}\|b\|_\infty\left(\int_0^t (t-r)^{-\frac{1}{\alpha}}r^{-\frac{\alpha-1}{\alpha}}\mathrm{d}r+\|b\|_\infty\right)\\ &\lesssim n^{-\frac{\alpha-1}{\alpha}}(\|b\|_\infty+\|b\|^2_\infty),\end{aligned}$$

where we used the fact $|\phi_n(r)-r|\leqslant 1/n$, and the definitions of Beta functions (1.7). Finally, combining the above calculations and taking $\delta=\alpha-1-\beta$ in (5.5), we establish the desired estimates. □

5.2. **Proof of Theorem 1.3.** Let $X^m_t$ be the solution to the following classical SDE:

$$X^m_t = x+\int_0^t b_m(X^m_s)\mathrm{d}s + L^{(\alpha)}_t.$$

Denote $\mathbf{P}_m(t) := \mathbb{P}\circ(X^m_t)^{-1}$. According to the stability estimates Theorem 4.6, for any $\delta\in(0,\frac{\alpha-1-2-\beta}{2})$, we have

$$\|\mathbf{P}_m(t)-\mathbf{P}(t)\|_{\mathrm{var}} \lesssim \|b-b_m\|_{\mathbf{B}^{-\frac{\alpha-1}{2}+\delta}_{\infty,\infty}} \lesssim m^{-\frac{\alpha-1-2\beta}{2}+\delta}\|b\|_{\mathbf{B}^{-\beta}_{\infty,\infty}}.$$

Noting $\|b_m\|_\infty \lesssim m^\beta\|b\|_{\mathbf{B}^{-\beta}_{\infty,\infty}}$ and $m=n^\gamma$, by Theorem 5.1, we have

$$\begin{aligned}\|\mathbf{P}_{m,n}(t)-\mathbf{P}(t)\|_{\mathrm{var}} &\leqslant \|\mathbf{P}_{m,n}(t)-\mathbf{P}_m(t)\|_{\mathrm{var}}+\|\mathbf{P}_m(t)-\mathbf{P}(t)\|_{\mathrm{var}}\\ &\lesssim \|b_m\|^{\alpha-\beta}_\infty n^{-(\alpha-1-\beta)}+\|b_m\|_\infty n^{-\frac{\alpha-\beta-1}{\alpha}}+(1+\|b_m\|^2_\infty)n^{-\frac{\alpha-1}{\alpha}}+m^{-\frac{\alpha-1-2\beta}{2}+\delta}\\ &\lesssim n^{-(\alpha-1-\beta)+\beta\gamma(\alpha-\beta)}+n^{-\frac{\alpha-\beta-1}{\alpha}+\beta\gamma}+n^{-\frac{\alpha-1}{\alpha}+2\beta\gamma}+n^{\gamma(-\frac{\alpha-1-2\beta}{2}+\delta)}\end{aligned}$$

which converges to 0 as $n\to\infty$ when

$$0<\gamma<\frac{1}{\beta}\left(\frac{\alpha-1-\beta}{\alpha-\beta}\wedge\frac{\alpha-\beta-1}{\alpha}\wedge\frac{\alpha-1}{2\alpha}\right)=\frac{\alpha-1}{2\alpha\beta}.$$

This completes the proof.

## Appendix A.

**Lemma A.1.** *Let $T > 0$. For any $0 < \beta, \gamma < 1 < \alpha$, there is a constant $c > 0$ such that for all $\lambda > 0$ and $t \in (0, T]$,*

$$\int_0^t \left[(\lambda s^{-\alpha}) \wedge s^{-\beta}\right](t-s)^{-\gamma}\mathrm{d}s \lesssim_c \lambda^{\frac{1-\beta}{\alpha-\beta}} t^{-\gamma}.$$

*Proof.* First of all, by a change of variable, we have

$$\int_0^t \left[(\lambda s^{-\alpha}) \wedge s^{-\beta}\right](t-s)^{-\gamma}\mathrm{d}s = t^{1-\gamma-\beta}\int_0^1 \left[(\lambda t^{-(\alpha-\beta)} s^{-\alpha}) \wedge s^{-\beta}\right](1-s)^{-\gamma}\mathrm{d}s.$$

Therefore, it is sufficient to show

$$\mathscr{I} := \int_0^1 \left[(\lambda s^{-\alpha}) \wedge s^{-\beta}\right](1-s)^{-\gamma}\mathrm{d}s \lesssim \lambda^{\frac{1-\beta}{\alpha-\beta}}.$$

Indeed, for any $0 < \beta, \gamma < 1 < \alpha$,

$$\begin{aligned}
\mathscr{I} &\lesssim \int_0^{\frac{1}{2}} \left[(\lambda s^{-\alpha}) \wedge s^{-\beta}\right]\mathrm{d}s + \left[(\lambda(\tfrac{1}{2})^{-\alpha}) \wedge (\tfrac{1}{2})^{-\beta}\right]\int_{\frac{1}{2}}^1 (1-s)^{-\gamma}\mathrm{d}s \\
&\lesssim \lambda^{\frac{1-\beta}{\alpha-\beta}} \int_0^{\frac{1}{2\lambda^{1/(\alpha-\beta)}}} r^{-\beta}\left[r^{-\alpha+\beta} \wedge 1\right]\mathrm{d}r + \lambda \wedge 1 \\
&\lesssim \lambda^{\frac{1-\beta}{\alpha-\beta}} \int_0^{\infty} r^{-\beta}\left[r^{-\alpha+\beta} \wedge 1\right]\mathrm{d}r + \lambda \wedge 1 \lesssim \lambda^{\frac{1-\beta}{\alpha-\beta}},
\end{aligned}$$

where we used the change of variable $s = \lambda^{1/(\alpha-\beta)} r$ in the second inequality. This completes the proof. □

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