COOPERATION, CORRELATION AND COMPETITION IN ERGODIC N-PLAYER GAMES AND MEAN-FIELD GAMES OF SINGULAR CONTROLS: A CASE STUDY

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ABSTRACT. We consider ergodic symmetric N-player and mean-field games of singular control in both cooperative and competitive settings. The state process dynamics of a representative player follow geometric Brownian motion, controlled additively through a nondecreasing process. Agents aim to maximize a long-time average reward functional with instantaneous profit of power type. The game shows strategic complementarities, in that the marginal profit function is increasing with respect to the dynamic average of the states of the other players, when $N < \infty$, or with respect to the stationary mean of the players' distribution, in the mean-field case. In the mean-field formulation, we explicitly construct the solution to the mean-field control problem associated with central planner optimization, as well as Nash and coarse correlated equilibria (with singular and regular recommendations). Among our findings, we show that coarse correlated equilibria may exist even when Nash equilibria do not. Additionally, we show that a coarse correlated equilibrium with a regular (absolutely continuous) recommendation can outperform a Nash equilibrium where the equilibrium policy is of reflecting type (thus singularly continuous). Furthermore, we prove that the constructed mean-field control and mean-field equilibria can approximate the cooperative and competitive equilibria, respectively, in the corresponding game with N players when N is sufficiently large. To the best of our knowledge, this paper is the first to characterize coarse correlated equilibria, construct the explicit solution to an ergodic mean-field control problem, and provide approximation results for the related N-player game in the context of singular control games.

Keywords: mean-field games; N-player games; singular stochastic control; ergodic reward; Nash equilibrium; Pareto efficiency; coarse correlated equilibrium; strategic complementarities.

MSC2020 subject classification: 91A11, 91A15, 91A16, 49N80.

1. INTRODUCTION

In this paper, we investigate ergodic stochastic games of singular control in both competitive and cooperative settings, considering scenarios with a finite number N of players as well as in the mean-field limit. In the formulation with N players, each symmetric player, indexed by $i = 1, 2, ..., N$, seeks to maximize a long-term average reward functional. The instantaneous profit at time $t \geq 0$ is given by $\pi(X_t^i, \theta_t^N) = (X_t^i)^\alpha (\theta_t^N)^\beta$, where $\alpha, \beta \in (0, 1)$. Here, X_t^i represents the current level of the state variable for agent i, and $\theta_t^N = \frac{1}{N-1} \sum_{j \neq i} X_t^j$ denotes the empirical average of the state processes of all the other $N - 1$ agents. Each agent i can control the geometric dynamics of X^i by increasing its level through a nondecreasing control process ν^i, with the cost of control being proportional to the effort expended. We introduce the concepts of coarse correlated equilibrium, Nash equilibrium, and Pareto efficiency for this game. The notion of Pareto efficiency is associated with the problem of a central planner who seeks to maximize the average of the rewards of all N agents.

The game under study shows strategic complementarities (since the marginal profit is increasing in its second variable; cf. [49]) and finds natural applications in dynamic oligopolies, such as in Cournot oligopoly with complementary products or in advertising games (see, e.g., [50, 51, 52]). In this regard, the state variable of each agent can be the output or the goodwill stock, which is increased by irreversible investment or advertising, respectively. The resulting

Date: April 24, 2024.

payoff is then derived through an isoelastic inverse demand function, depending on the aggregate level of production or goodwill in the entire market. In particular, the ergodic structure of the reward functional we consider is relevant in the context of investment into public goods, in which it might be important to take care of the payoffs received by successive generations.

Constructing equilibria for N-player games in continuous-time and space is a challenging problem. The theory of mean-field games, developed independently by [38] and [42], provides approximation results for equilibria of symmetric games with finite players. Indeed, it is typically possible to prove that mean-field equilibria define ε-equilibria for the related N-player games. In this paper, we introduce the mean-field version of the previously described stochastic game and explicitly construct the mean-field Nash equilibrium and the solution to the mean-field central planner control problem. We also determine sufficient conditions for the existence of coarse correlated equilibria (based on suitable recommendations of the moderator). In the mean-field game, the representative agent reacts to the long-term average of the distribution of the population, which is represented by a scalar parameter θ. The stationary one-dimensional setting of the mean-field game and control problem allows for explicit characterizations of the equilibria (see also [9, 16, 17, 22] and references therein in the context of singular/impulse control games).

Our contributions. Despite the specific setting in which the game is formulated (geometric Brownian dynamics and profit function of power type), the analysis reveals a rich structure of the solution while also requiring technical results and arguments. Among these, we highlight the derivation of novel first-order conditions for optimality in ergodic singular stochastic control problems (see Lemma 4.3 below), which are of independent interest, as well as the probabilistic representation of the Lagrange multiplier employed in the analysis of the mean-field central planner control problem (see Lemma 5.2).

Our main results are as follows. First, to the best of our knowledge, this is the first paper that constructs the explicit solution to an ergodic mean-field singular stochastic control problem (Theorem 5.1) and proves that its solution can approximate the solution to a central planner problem aiming to achieve Pareto efficiency in the game with N players (see Theorem 5.3). We construct the mean-field solution using a Lagrange-multiplier approach, which transforms the original McKean-Vlasov control problem into a two-stage optimization problem, in which one first optimizes over the admissible control variables and then over the mean-field parameter. It is noteworthy that the probabilistic representation of the Lagrange multiplier as the derivative of the mean-field control problem's value function with respect to the mean-field parameter (see Lemma 5.2 and Remark 6 below) is the key ingredient for suitably applying the Law of Large Numbers and completing the proof of the approximation result in Theorem 5.3.

Secondly, we completely characterize the Nash equilibria in the ergodic mean-field game and prove that their existence and uniqueness depend on the strength of the strategic complementarity, measured by the parameter $\beta \in (0, 1)$ (see Theorem 6.8 below). In particular, if $0 < \beta < 1-\alpha$ or $1 - \alpha < \beta < 1$, then a unique Nash mean-field equilibrium exists, where the state process is reflected upwards à la Skorohod at an explicitly given barrier. On the other hand, if $\beta = 1 - \alpha$, either infinitely many equilibria exist, each of reflecting type, or none exist. The existence of multiple equilibria is related to the fact that the mean-field game under study faces strategic complementarities (see the seminal [49] and also [1, 2, 3, 28, 29, 26, 27] for contributions on mean-field games).

The potential presence of multiple Nash equilibria leads to the question of how players can coordinate towards one of them. As the final main contribution of this paper, we consider coarse correlated equilibria. The concept of coarse correlated equilibria offers an alternative to Nash equilibria, expanding on the latter by incorporating a mediator, or correlation device, that enables agents to adopt correlated strategies without any cooperation. We determine sufficient conditions for the existence of coarse correlated equilibria, with singular and regular (absolutely continuous) recommendations for the competitive mean-field game of singular controls (see Propositions 6.3

and 6.6 below). Since there conditions are non-linear in moments of the correlation device, and present intricate dependence on the model's parameters, we consider a specific choice of parameters, and through a numerical analysis we show that, in the case in which the correlation device has Gamma distribution, there are infinitely many instances of Gamma distributions under which coarse correlated equilibria may exist even when Nash equilibria do not. Additionally, we show that a coarse correlated equilibrium with a recommendation in the form of a regular control can outperform the Nash equilibrium, whose equilibrium policy is instead of a reflecting type (and thus singularly continuous), highlighting a feature of coarse correlated equilibria well known in standard game theory literature (see, e.g., [31, 43, 47]).

Related literature. Our paper contributes to various streams of literature. Firstly, we add to the literature on mean-field games with singular stochastic controls, a field where a still limited but rapidly increasing number of contributions has focused on abstract results regarding the existence and uniqueness of equilibria (see [23, 25, 29, 35, 34]), as well as on explicit characterizations of the Nash solution (see [14, 17, 16, 30, 36]). Particularly relevant to our paper is [16], where, in the context of a mean-field game with singular controls, stationary discounted and ergodic Nash equilibria are explicitly constructed and related via the vanishing discount factor method.

Secondly, we contribute to the literature studying games with strategic complementarities (also known as submodular/supermodular games) and the potential emergence of multiple equilibria. This class of games has garnered significant attention in Economics. As Xavier Vives asserts in [52], "Complementarities are intimately linked to multiple equilibria and have a deep connection with strategic situations, and the concept of strategic complementarity is at the center stage of game-theoretic analyses." Among the myriad contributions, we refer to the deterministic games considered in [49, 50, 51], as well as to the dynamic stochastic formulations presented in [1, 2, 3, 21, 27] and references therein. To the best of our knowledge, our paper is the first to provide a comprehensive analysis of a stationary mean-field singular stochastic game with strategic complementarities by explicitly characterizing its Pareto efficient outcome, as well as its Nash and coarse correlated equilibria.

Finally, we connect to recent works dealing with correlated and coarse correlated equilibria in mean-field games. While well-known in game theory as generalizations of Nash equilibria ([6, 37, 44]), correlated and coarse correlated equilibria have been considered in the mean-field games literature very recently. We refer to [10, 15] and [45, 46] for discrete-time finite-state mean-field games, and to [12, 13] for continuous-time mean-field games. The latter two papers are particularly relevant in our context, since we build on the definition and intuitions developed therein.

Structure of the paper. The rest of the paper is organized as follows: Section 2 presents the N-player game, and Section 3 introduces the corresponding mean-field formulation. Section 4 details the assumptions and includes some auxiliary control-theoretic results. In Section 5, the mean-field control problem is solved, and its connection with the central planner's optima are established. Section 6 characterizes both coarse correlated equilibria and Nash equilibria in the mean-field game, and discusses their relationships with the equilibria in the N-player game. Section 7 numerically illustrates the findings from previous sections. Finally, the Appendix contains technical proofs and lemmata.

2. The N-player Game

Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0^-}, \mathbb{P})$ be a filtered probability space satisfying the usual assumptions. Let $(W^i)_{i\geq 1}$, W be a sequence of independent $\mathbb{F}$-Brownian motions, and let ξ, $(\xi^i)_{i\geq 1}$ be a sequence of i.i.d. positive random variables with distribution $\mu_0 \in \mathcal{P}(\mathbb{R}_+)$. We assume that they are independent from W and $(W^i)_{i\geq 1}$, and they are $\mathcal{F}_{0^-}$-measurable.

We consider the following set of strategies, to be subject to further restrictions in the following:

$$\begin{aligned}\mathcal{A} :=&\{\nu : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+,\ \mathbb{F}\text{-adapted and such that } t \mapsto \nu_t \text{ is a.s.}\\ &\quad \text{nondecreasing, right-continuous, } \nu_{0^-} = 0 \text{ and } \mathbb{E}[\nu_T] < \infty\ \forall T > 0\}.\end{aligned}$$

Let $N \geq 2$. We denote a vector of strategies $(\nu^1, \dots, \nu^N) \in \mathcal{A}^N$ by $\boldsymbol{\nu}^N$. We refer to $\boldsymbol{\nu}^N \in \mathcal{A}^N$ as a strategy profile. We denote by $\boldsymbol{\nu}^{-i,N} = (\nu^1, \dots, \nu^{i-1}, \nu^{i+1}, \dots, \nu^N)$ the vector of strategies of players $j \neq i$, and we denote the vector of strategies $\boldsymbol{\nu}^N$ also by $(\nu^i, \boldsymbol{\nu}^{-i,N})$.

Let δ, σ be in $\mathbb{R}_+$. For any strategy profile $\boldsymbol{\nu} \in \mathcal{A}^N$, we consider the following dynamics:

$$dX_t^{\nu^i} = -\delta X_t^{\nu^i} dt + \sigma X_t^{\nu^i} dW_t^i + d\nu_t^i, \quad X_{0^-}^{\nu^i} = \xi^i, \tag{2.1}$$

for any $i = 1, \dots, N$. Observe that, for any $\boldsymbol{\nu}^N \in \mathcal{A}^N$, there exists a unique strong solution to (2.1) (see, e.g., [48, Theorem 7, Chapter V]). Actually, one has

$$X_t^{\nu^i} = X_t^{i,0} \left(\xi^i + \int_0^t \frac{d\nu_s^i}{X_s^{i,0}} \right),$$

where $X^0 = (X^{1,0}, \dots, X^{N,0})$ denotes the uncontrolled solution of (2.1), that is, the one associated to $\nu^i \equiv 0$. Moreover, for any $i = 1, \dots, N$, we define the flow of empirical averages of players $j \neq i$ by

$$\theta_t^{N,\boldsymbol{\nu}^{-i,N}} := \frac{1}{N-1} \sum_{j \neq i} X_t^{\nu^j}, \quad t \geq 0^-.$$

Let α, β in $(0,1)$, $q > 0$. Each player is associated with the following reward functional:

$$\mathsf{J}^N(\nu^i, \boldsymbol{\nu}^{-i,N}) = \varliminf_{T\uparrow\infty} \frac{1}{T} \mathbb{E}\left[\int_0^T (X_t^{\nu^i})^\alpha \big(\theta_t^{N,\boldsymbol{\nu}^{-i,N}}\big)^\beta dt - q\nu_T^i \right], \tag{2.2}$$

which can possibly be infinite. Occasionally, we use the notation $\pi(x, \theta) = x^\alpha \theta^\beta$, for any $(x, \theta) \in \mathbb{R}_+^2$, and we write $\pi_x(x,\theta) = \partial_x \pi(x,\theta)$ and analogously $\pi_\theta(x,\theta) = \partial_\theta \pi(x,\theta)$.

When dealing with N-player games, we consider open-loop strategies. Roughly speaking, we allow each player to observe the noises of all players, as well as their initial position. To this extent, denote by $\mathbb{F}^N = (\mathcal{F}_t^N)_{t \geq 0^-}$ be the $\mathbb{P}$-augmentation of the filtration generated by the Brownian motions $(W^i)_{i=1}^N$ and initial data $(\xi^i)_{i=1}^N$.

Definition 1 (Open-loop strategies for the N-player game)**.** We say that a process $\nu \in \mathcal{A}$ is an open-loop strategy for the N-player game if ν is $\mathbb{F}^N$-progressively measurable. We denote the set of open-loop strategies for the N-player game by $\mathcal{A}_N$.

We are interested in different kinds of equilibria in the N-player system. We deal both with the cooperative case and the competitive framework.

Cooperative framework. In the competitive case, we look for Pareto efficient strategy profiles, according to the following definition:

Definition 2 (Pareto efficiency)**.** Let $\mathcal{C} \subseteq \mathcal{A}_N^N$. A strategy profile $\hat{\boldsymbol{\nu}} \in \mathcal{C}$ is Pareto efficient in the class $\mathcal{C}$ if there does not exist any other $\boldsymbol{\nu} \in \mathcal{C}$ so that

$$\begin{aligned}&\mathsf{J}^N(\nu^j, \boldsymbol{\nu}^{-j}) \geq \mathsf{J}^N(\hat{\nu}^j, \hat{\boldsymbol{\nu}}^{-j}), \quad \forall j = 1, \dots, N,\\ &\mathsf{J}^N(\nu^i, \boldsymbol{\nu}^{-i}) > \mathsf{J}^N(\hat{\nu}^i, \hat{\boldsymbol{\nu}}^{-i}), \quad \text{for some } i.\end{aligned}$$

In other words, a strategy profile is Pareto efficient in $\mathcal{C}$ if there does not exist any other strategy profile in $\mathcal{C}$ which makes each player at least as well off and one player strictly better off. To search for Pareto efficient strategy profiles, we associate to the dynamics (2.1) and payoff functionals (2.2) an N-dimensional control problem. We consider the following functional

$$\bar{\mathsf{J}}^N(\boldsymbol{\nu}) := \frac{1}{N} \sum_{i=1}^N \mathsf{J}^N(\nu^i, \boldsymbol{\nu}^{-i}), \quad \boldsymbol{\nu} \in \mathcal{A}_N^N, \tag{2.3}$$

which can be regarded as a welfare utility for the N-player system.

Definition 3. Let $\varepsilon \geq 0$, $\mathcal{C} \subseteq \mathcal{A}_N^N$. A strategy profile $\hat{\boldsymbol{\nu}} = (\hat{\nu}^1, \ldots, \hat{\nu}^N) \in \mathcal{C}$ is ε-optimal for the central planner optimization problem within the set of strategy profiles $\mathcal{C}$ if

$$\bar{\mathsf{J}}^N(\hat{\boldsymbol{\nu}}) \geq \bar{\mathsf{J}}^N(\boldsymbol{\nu}) - \varepsilon, \quad \forall \boldsymbol{\nu} \in \mathcal{C}.$$

If $\varepsilon = 0$, we say that the strategy profile $\hat{\boldsymbol{\nu}}$ is optimal for the central planner within the set of strategy profiles $\mathcal{C}$.

When dealing with the central planner's optimization problem, the players are referred to as agents, since there is no competition between them: the central planner picks herself a strategy for each player in order to maximize the welfare utility functional $\bar{\mathsf{J}}^N$. As a consequence, agents are not allowed to unilaterally deviate from the central planner's strategy profile. It can be easily show that if a strategy profile is an optimum of the central planner maximization problem, it is Pareto efficient as well.

Competitive framework. We consider the notion of coarse correlated equilibria in the N-player game, which allows for correlation between players' strategies, in the sense described below. It comprehends the more common notion of Nash equilibria as the particular case in which players' strategies are not correlated.

We assume the following structural condition on the σ-algebra $\mathcal{F}_{0-}$:

Assumption U. *The σ-algebra $\mathcal{F}_{0-}$ is large enough to support a $\mathcal{F}_{0-}$-measurable uniform random variable independent of the initial data ξ, $(\xi^i)_{i\geq 1}$ and the noises W, $(W^i)_{i\geq 1}$.*

Next, we introduce correlation between players' strategies.

Definition 4 (Correlating device)**.** A correlation device is any random variable $Z : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ so that Z is $\mathcal{F}_{0-}$-measurable and independent of ξ, W, $(\xi^i)_{i\geq 1}$ and $(W^i)_{i\geq 1}$.

Definition 5 (Correlated strategy profile)**.** We define a correlated strategy profile as a pair $(Z, \boldsymbol{\lambda})$ so that

(i) Z is a correlation device;
(ii) $\boldsymbol{\lambda} = (\lambda^i)_{i=1}^N$ is an admissible recommendation to the N players; that is, for each $i = 1, \ldots, N$, $\lambda^i = (\lambda^i_t)_{t\geq 0-}$ belongs to $\mathcal{A}$ and it is progressively measurable with respect to the $\mathbb{P}$-augmentation of the filtration $(\sigma(Z) \vee \mathcal{F}^N_t)_{t\geq 0-}$.

We now assign dynamics and rewards of each player. To do so, we must distinguish two cases: Suppose that each player i follows the recommendation λ^i. Then, players' state dynamics are given by (2.1), and each player gets the reward $\mathsf{J}^N(\lambda^i, \boldsymbol{\lambda}^{-i})$, with J^N given by (2.2). Suppose player i deviates, while other players stick to the correlated strategy profile $\boldsymbol{\lambda}^{-i}$. The deviating player will pick instead an open loop strategy $\nu \in \mathcal{A}_N$. Her dynamics are given by (2.1) and her reward is given by $\mathsf{J}^N(\nu^i, \boldsymbol{\lambda}^{-i})$, with J^N given by (2.2).

Definition 6 (ε-coarse correlated equilibrium within the set of strategies $\mathcal{B}$)**.** Let $\varepsilon \geq 0$, $\mathcal{B} \subseteq \mathcal{A}_N$. A correlated strategy profile $(Z, \boldsymbol{\lambda})$ is an ε-coarse correlated equilibrium (ε-CCE) of the ergodic N-player game within the set of strategies $\mathcal{B}$, if for any $i = 1, \ldots, N$, we have

$$\mathsf{J}^N(\lambda^i, \boldsymbol{\lambda}^{-i}) \geq \mathsf{J}^N(\nu, \boldsymbol{\lambda}^{-i}) - \varepsilon, \quad \forall \nu \in \mathcal{B}.$$

If $\varepsilon = 0$, we say that the correlated strategy profile $(Z, \boldsymbol{\lambda})$ is a coarse correlated equilibrium (CCE) of the ergodic N-player game within the set of strategies $\mathcal{B}$.

We give the following interpretation of correlation devices and the correlated strategy profiles: A correlation device or a mediator runs a lottery over open loop strategies. The extraction of the strategy happens before the game starts and it is independent of the of the initial data and idiosyncratic shocks that determine the random evolution of players' states. These features are captured by the fact that the random variable Z is $\mathcal{F}_{0-}$-measurable and it is independent of

$(\xi^i)_{i\geq 1}$ and $(W^i)_{i\geq 1}$. The correlation device Z introduces extra randomness in the game, but is not exogenous in the sense of a *common noise* (see, e.g., [20]): Indeed, it is picked by the moderator as part of her recommendation to the players. Note that existence of the correlation device Z is guaranteed by Assumption **U**.

We interpret deviations in the following way: Each player must decide whether to commit to moderator's lottery before the extraction happens, only by relying on the information given by the law of the correlated strategy $(Z, \boldsymbol{\lambda})$, which is assumed to be common knowledge between the players. If a player does not commit, she will pick a strategy without any information on the outcome of the extraction. Notice that, since the deviating player has only knowledge of the law of the correlated strategy profile $(Z, \boldsymbol{\lambda})$, she will use a strategy $\nu \in \mathcal{A}_N$, which is, in particular, independent of the correlation device Z; consequently, her state process is independent of Z. We refer to [12, 13] for more comments.

We observe that the definition of ε-coarse correlated equilibria for the N-player game extends the one of Nash equilibria, that we recall:

Definition 7 (ε-Nash equilibrium within the set of strategies $\mathcal{B}$)**.** Let $\varepsilon \geq 0$, $\mathcal{B} \subseteq \mathcal{A}_N$. A strategy profile $\boldsymbol{\nu}^* = (\nu^{1,*}, \dots, \nu^{N,*}) \in \mathcal{B}^N$ is an ε-Nash equilibrium (ε-NE) of the ergodic N-player game within the set of strategies $\mathcal{B}$, if for any $i = 1, \dots, N$, we have

$$\mathsf{J}^N(\nu^{i,*}, \boldsymbol{\nu}^{-i,*}) \geq \mathsf{J}^N(\nu, \boldsymbol{\nu}^{-i,*}) - \varepsilon, \quad \forall\, \nu \in \mathcal{B}.$$

If $\varepsilon = 0$, we say that the strategy profile $\boldsymbol{\nu}^*$ is a Nash equilibrium (NE) of the ergodic N-player game within the set of strategies $\mathcal{B}$.

Every ε-CCE $(Z, \boldsymbol{\lambda})$ with deterministic correlation device Z is an ε-NE: It is enough to notice that, since Z is deterministic, the correlated strategy profile $\boldsymbol{\lambda}$ reduces to an open-loop strategy profile $(\nu^1, \dots, \nu^N)$ in $\mathcal{A}_N^N$. Conversely, any ε-NE induces an ε-CCE with deterministic correlation device.

3. The Ergodic Mean-field Game

In order to determine ε-optimal solutions to the central planner problem and ε-equilibria in the competitive setting, we consider the mean-field counterparts of the optimization problem and game considered before. We will then show in Sections 5 and 6 the relation between mean-field solutions to the N-player cooperative and competitive problems respectively.

We work on the same probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ defined in the previous section. Given a strategy $\nu \in \mathcal{A}$, we consider the following dynamics:

$$dX_t^\nu = -\delta X_t^\nu dt + \sigma X_t^\nu dW_t + d\nu_t, \quad X_{0^-} = \xi. \tag{3.1}$$

For any $\mathcal{F}_{0^-}$-measurable non-negative random variable θ, possibly degenerate, we consider the following payoff functional to be maximized:

$$\mathsf{J}(\nu, \theta) = \varliminf_{T\uparrow\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T (X_t^\nu)^\alpha \theta^\beta dt - q\nu_T\right]. \tag{3.2}$$

Let $\mathbb{F}^{\xi,W} = (\mathcal{F}_t^{\xi,W})_{t\geq 0^-}$ be the $\mathbb{P}$-augmentation of the filtration generated by ξ and W. Analogously to Definition 1, we consider the following strategies:

Definition 8 (Open-loop strategy for the ergodic MFG)**.** A process $\nu \in \mathcal{A}$ is an open-loop strategy if it is progressively measurable with respect to the filtration $\mathbb{F}^{\xi,W}$. We denote the set of open-loop strategies by $\mathcal{A}_{mf}$.

Cooperative framework. We address the mean-field counterpart of the central planner's maximization problem, which is given by the mean-field control (MFC) maximization problem. Roughly speaking, this problem consists in maximizing the reward (3.2) under the additional constraint $\theta = \lim_{t\to\infty} \mathbb{E}[X_t^\nu] =: \mathbb{E}[X_\infty^\nu]$, for every control choice ν.

In order to properly define the reward, we need to restrict the class of admissible controls.

Definition 9. We say that a strategy ν is admissible for the mean-field control problem if $\nu \in \mathcal{A}_{mf}$ and the process $(X_t^\nu)_{t\geq 0^-}$ admits a unique stationary distribution $p_\infty^\nu \in \mathcal{P}(\mathbb{R}_+)$. We denote the set of admissible strategies for the stationary MFC problem by $\mathcal{A}_{MFC}$.

For any $\nu \in \mathcal{A}_{MFC}$, denote by $\mathbb{E}[X_\infty^\nu]$ the first moment of the corresponding limit measure p_∞^ν. The payoff functional associated to a strategy $\nu \in \mathcal{A}_{MFC}$ is given by

$$\mathsf{J}(\nu, \mathbb{E}[X_\infty^\nu]) = \varliminf_{T\uparrow\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T (X_t^\nu)^\alpha (\mathbb{E}[X_\infty^\nu])^\beta dt - q\nu_T\right].$$

Definition 10. An admissible control $\hat{\nu} \in \mathcal{A}_{MFC}$ is an optimal control for the mean-field control problem if

$$\mathsf{J}(\hat{\nu}, \mathbb{E}[X_\infty^{\hat{\nu}}]) \geq \mathsf{J}(\nu, \mathbb{E}[X_\infty^\nu]), \quad \forall\, \nu \in \mathcal{A}_{MFC}.$$

The study of the central planner's optimization problem and its relation with the N-agent system is the content of Section 5: We show in Theorem 5.1 that it is possible to completely characterize solutions of the MFC problem. Then, in Theorem 5.3, we use the solution of the MFC problem to build a sequence of approximate central planner's optima in the underlying N-agent system, with vanishing approximating error.

Competitive framework. We consider the mean field analogues of CCEs and NEs in the N-player game. To this extent, we give the following definition:

Definition 11 (Correlated stationary strategy)**.** We define a correlated stationary strategy as a triple $(Z, \theta_\infty, \lambda)$ so that the following holds:

(i) Z is a correlation device;
(ii) θ_∞ is a $\sigma(Z)$-measurable non-negative random variable;
(iii) $\lambda = (\lambda_t)_{t\geq 0^-}$ belongs to $\mathcal{A}$ and it is progressively measurable with respect to the $\mathbb{P}$-augmentation of the filtration $(\sigma(Z) \vee \mathcal{F}_t^{\xi,W})_{t\geq 0^-}$.

In the following, we will denote the law of θ_∞ by $\rho \in \mathcal{P}(\mathbb{R}_+)$.

Let $(Z, \theta_\infty, \lambda)$ be a correlated stationary strategy. We now assign dynamics and payoff functional. We distinguish the following two cases: If the representative player decides to trust the mediator and so to follow her recommendation λ, then the dynamics is given by (3.1) with λ instead of ν and the payoff is given by $\mathsf{J}(\lambda, \theta_\infty)$, with J defined by (3.2). If instead the representative player chooses to deviate, she uses a strategy $\nu \in \mathcal{A}$, her dynamics is given by (3.1), and she gets the reward $\mathsf{J}(\nu, \theta_\infty)$. Observe that, when the representative player deviates, her strategy ν is $\mathbb{F}^{\xi,W}$-progressively measurable and therefore independent of θ_∞, since she has no information on the outcome of moderator's lottery. As in the N-player game, the deviating player can only use her knowledge of the law of the correlated stationary strategy $(Z, \lambda, \theta_\infty)$, which is assumed to be publicly known. Nevertheless, θ_∞ still appears in her payoff.

Definition 12 (Coarse correlated Equilibrium for the ergodic MFG)**.** A correlated stationary triple $(Z, \lambda, \theta_\infty)$ is a coarse correlated equilibrium (CCE) for the ergodic MFG if the following holds:

(1) $\mathsf{J}(\lambda, \theta_\infty) \geq \mathsf{J}(\nu, \theta_\infty)$ for any $\nu \in \mathcal{A}$,
(2) The process X^λ admits a stationary distribution and it holds

$$\theta_\infty = \int_{\mathbb{R}_+} x p_\infty(dx, \theta_\infty), \tag{3.3}$$

where p_∞ is the stochastic kernel so that $\mu_\infty(dx, d\theta) = p_\infty(dx, \theta)\rho(d\theta)$ with $\rho = \mathbb{P} \circ \theta_\infty^{-1}$ and $\mu_\infty = \lim_{t\to\infty} \mathbb{P} \circ (X_t^\lambda, \theta_\infty)^{-1}$ in the weak sense.

We will refer to CCEs for the ergodic MFG as mean-field CCEs as well.

Remark 1. Property (2) in Definition 12 is equivalent to

$$\theta_\infty \sim w - \lim_{t\to\infty} \mathbb{E}[X_t^\lambda | \theta_\infty]. \tag{3.4}$$

The consistency condition (2) in Definition 12 should be read in the following way: the mediator imagines what the stationary mean θ_∞ will be, before the game starts, and gives a recommendation to each player according to her idea. Since θ_∞ is expected to be stochastic only as the result of the mediator's randomization, we request it to be measurable with respect to the correlation device Z that the moderator uses to generate both the recommendation λ and the random stationary mean θ_∞ itself. If all players commit to the mediator's lottery for generating recommendations, then the long-time average should be consistent with what imagined by the mediator.

The notion of CCE for the ergodic MFG extends the notion of notion of Nash equilibrium for the ergodic MFG, that we borrow from [16]:

Definition 13 (Nash equilibrium of the ergodic MFG)**.** A pair $(\nu^*, \theta^*) \in \mathcal{A}_{mf} \times \mathbb{R}_+$ is said to be a Nash equilibrium of the ergodic MFG if

(1) $\mathsf{J}(\nu^*, \theta^*) \geq \mathsf{J}(\nu, \theta^*)$, for any $\nu \in \mathcal{A}_{mf}$;
(2) The optimally controlled process X^{ν^*} admits a limiting distribution $p_\infty^* \in \mathcal{P}(\mathbb{R}_+)$ satisfying

$$\theta^* = \int_{\mathbb{R}_+} x p_\infty^*(dx).$$

We will refer to NEs for the ergodic MFG as mean-field NEs as well. We stress that, differently from mean-field CCEs, when looking for Nash equilibria, θ^* is assumed to be deterministic.

As in the N-player game, and actually by exactly the same reasoning, every CCE for the ergodic MFG $(Z, \theta_\infty, \lambda)$ with deterministic correlation device Z is an Nash equilibrium for the ergodic MFG as well. Conversely, any Nash equilibrium for the ergodic MFG (ν^*, θ^*) induces a mean-field CCE with deterministic correlation device.

The study of the existence of CCEs in the ergodic MFG and its relation with the N-player game is the content of Section 6: We identify specific classes of correlated stationary strategies for which it is possible to state a sufficient condition to be a CCE. Then we use CCEs in those classes to build a sequence of approximate CCEs the underlying N-player system, with vanishing approximating error. As a byproduct, we get the same approximation result for NEs as well.

4. Assumptions and Preliminary Results

On top of Assumption **U**, we assume the following structural condition on the coefficients of the SDE:

Assumption D. *The parameters δ and σ satisfy the following condition:*

$$2\delta - \sigma^2 > 0.$$

Moreover, μ_0 admits a finite second moment.

Remark 2. Let X^0 be the solution of (3.1) when the policy ν is identically equal to 0. Assumption **D** is a dissipativity assumption on the square of X^0: indeed, by Itô's formula, we have

$$(X_t^0)^2 = \xi^2 + \int_0^t (-\delta + \frac{1}{2}\sigma^2)(X_s^0)^2 ds + \int_0^t \sigma X_s^2 dW_s,$$

which is then square-integrable and dissipative. Notice that the same assumption is assumed in [16, Section 6].

We state some important properties of the diffusion X^{ν^a} reflected upwards at some positive barrier a, which will be used through the whole manuscript.

Lemma 4.1. *i) For any $a > 0$, let $p_a \in \mathcal{P}(\mathbb{R}_+)$ be given by*

$$p_a(dx) = \frac{2\delta + \sigma^2}{2} a^{\frac{2\delta}{\sigma^2}+1} x^{-\frac{2\delta}{\sigma^2}-2} \mathbb{1}_{\{x \geq a\}} dx. \tag{4.1}$$

For any $0 \leq k \leq 2$, the measure p_a admits finite k-moment. In particular, it holds

$$\int_{\mathbb{R}_+} x p_a(dx) = \frac{2\delta + \sigma^2}{2\delta} a. \tag{4.2}$$

Moreover, the map $\mathbb{R}_+ \times \mathcal{B}_{\mathbb{R}_+} \ni (a, B) \mapsto p_a(B)$ defines a stochastic kernel from $\mathbb{R}_+$ to $\mathcal{B}_{\mathbb{R}_+}$.

ii) For any $a > 0$, there exists a unique strong solution $(X_t^{\nu^a}, \nu_t^a)_{t \geq 0^-}$ of the Skorohod reflection problem at the barrier a, i.e. a pair of processes so that equation (3.1) is satisfied for any t, $\nu_{0^-} = 0$, $t \mapsto \nu_t$ is non-decreasing, and $\int_0^\infty \mathbb{1}_{\{X_s^{\nu^a} > a\}} d\nu_s^a = 0$, $\mathbb{P}$-a.s.. The process ν^a is given by

$$\nu_t^a = \int_0^t X_s^0 \, d\left(\sup_{0 \leq u \leq s} \left(\frac{a - X_u^0}{X_u^0} \right)^+ \right), \quad \nu_{0^-}^a = 0.$$

Moreover, the process X^{ν^a} is positively recurrent with stationary measure given by p_a.

iii) There exists a positive constant c so that

$$\sup_{t \geq 0} \mathbb{E}[(X_t^{\nu^a})^2] \leq c(1 + a^2), \quad \sup_{T > 0} \mathbb{E}\left[\left(\frac{1}{T} \nu_T^a \right)^2 \right] \leq c(1 + a^2).$$

The proof is postponed to the Appendix. For later use, we introduce the real function $C(a, p)$, for $(a, p) \in \mathbb{R}_+^2$, given by

$$C(a, p) = (2\delta + \sigma^2) \left(\frac{p}{2\delta + \sigma^2(1 - \alpha)} a^\alpha - \frac{q}{2} a \right). \tag{4.3}$$

Let ν^a is the policy that reflects the process X^{ν^a} solution to (3.1) upwards à la Skorohod at the level $a > 0$. By [5, Lemma 2.1], it holds

$$C(a, p) = \lim_{T \uparrow \infty} \frac{1}{T} \mathbb{E}\left[\int_0^T p \cdot (X_t^{\nu^a})^\alpha dt - q\nu_T^a \right] = \underline{\lim}_{T \uparrow \infty} \frac{1}{T} \mathbb{E}\left[\int_0^T p \cdot (X_t^{\nu^a})^\alpha dt - q\nu_T^a \right]$$

In the following, we will need to solve several ergodic optimization problems of singular controls. The existence of a unique solution to such problems is ensured by following Lemma:

Lemma 4.2. *Let $\lambda \in \mathbb{R}$ so that $q\delta - \lambda > 0$ and $p > 0$. Define the following function*

$$g(x, p, \lambda) := x^\alpha p + \lambda x, \tag{4.4}$$

and consider the reward functional

$$\tilde{\mathsf{J}}(\nu, p, \lambda) := \underline{\lim}_{T \uparrow \infty} \frac{1}{T} \mathbb{E}\left[\int_0^T g(X_t^\nu, p, \lambda) dt - q\nu_T \right], \tag{4.5}$$

where $X^\nu = (X_t^\nu)_{t \geq 0^-}$ evolves accordingly to (3.1). Then, there exists a unique optimal control $\nu^* \in \mathcal{A}$ so that

$$\tilde{\mathsf{J}}(\nu^*, p, \lambda) = \sup_{\nu \in \mathcal{A}} \tilde{\mathsf{J}}(\nu, p, \lambda).$$

Moreover, the process ν^* reflects the state process upwards à la Skorohod at the barrier $a^*(p, \lambda)$ given by

$$a^*(p, \lambda) = \left(\frac{2\alpha\delta}{2\delta + \sigma^2(1 - \alpha)} \frac{p}{q\delta - \lambda} \right)^{\frac{1}{1-\alpha}}. \tag{4.6}$$

The proof is postponed to the Appendix.

With respect to a smaller class of strategies, we state a first order optimality condition for a control $\hat{\nu}$, inspired by [32] (see also [8]). Although only the necessary part will be needed, for the sake of completeness, we also show that it is sufficient under additional assumptions.

Lemma 4.3 (First order optimality condition). *Let $p > 0$, $\lambda > q\delta$. Let $1 \leq q, q' \leq \infty$ be Young conjugates, and define the set $\mathcal{A}_{2q}$ as the set of controls $\nu \in \mathcal{A}$ so that*

$$\sup_{T>0} \frac{1}{T}\mathbb{E}\left[\int_0^T |X_t^\nu|^{2q} dt\right] < \infty. \tag{4.7}$$

Let $\hat{\nu} \in \mathcal{A}_{2q}$ so that

$$\sup_{T>0} \frac{1}{T}\mathbb{E}\left[\int_0^T |(X_t^{\hat{\nu}})^{\alpha-2}|^{q'}\right] < \infty, \tag{4.8}$$

if $q' < \infty$, and so that

$$\sup_{T>0} \inf \Big\{ C \geq 0 : \, (X_t^{\hat{\nu}})^{\alpha-2} \leq C \text{ for } dt \otimes d\mathbb{P}\text{-a.e. } (t,\omega) \in [0,T] \times \Omega \Big\} < \infty, \tag{4.9}$$

if $q' = \infty$.

(a) *Suppose that $\hat{\nu}$ is optimal within the set $\mathcal{A}_{2q}$ for the control problem with dynamics* (3.1) *and reward* (4.5)*. Then, for every $\nu \in \mathcal{A}_{2q}$, it holds*

$$\overline{\lim_{T\uparrow\infty}} \frac{1}{T}\mathbb{E}\left[\int_0^T g_x(X_t^{\hat{\nu}}, p, \lambda)(X_t^{\hat{\nu}} - X_t^\nu) dt - q(\hat{\nu}_T - \nu_T)\right] \geq 0. \tag{4.10}$$

(b) *Suppose that either $\hat{\nu}$ satisfies* (4.10) *and*

$$\underline{\lim_{T\uparrow\infty}} \frac{1}{T}\mathbb{E}\left[\int_0^T g(X_t^{\hat{\nu}}, p, \lambda) dt - q\hat{\nu}_T\right] = \lim_{T\uparrow\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T g(X_t^{\hat{\nu}}, p, \lambda) dt - q\hat{\nu}_T\right], \tag{4.11}$$

or that $\hat{\nu}$ satisfies

$$\underline{\lim_{T\uparrow\infty}} \frac{1}{T}\mathbb{E}\left[\int_0^T g_x(X_t^{\hat{\nu}}, p, \lambda)(X_t^{\hat{\nu}} - X_t^\nu) dt - q(\hat{\nu}_T - \nu_T)\right] \geq 0. \tag{4.12}$$

Then $\hat{\nu}$ is optimal within the set $\mathcal{A}_{2q}$.

The proof is postponed to the Appendix.

Remark 3. Let $a > 0$ and let ν^a be the policy that reflects the process $X^{\hat{\nu}}$ upwards à la Skorohod at the barrier a. Then, the control ν^a is so that $\mathbb{P}((X_t^{\nu^a})^{\alpha-2} \leq c \,\forall t \geq 0) = 1$, for a constant $c > 0$, since the reflected process is so that $X_t^{\nu^a} \geq a$ for any t, and $X^{\hat{\nu}}$ belongs to $\mathcal{A}_2$ by Lemma 4.1. Therefore, (4.9) is satisfied, and we take $q = 1$ in (4.7). Assumption **D** and Lemma 4.1 imply that X^{ν^a} satisfies (4.7). By Lemma 4.1, (4.11) is satisfied as well.

Remark 4. We can restate Lemma 4.3 in terms of linear conditions involving optional projections. Consider the probability measure $\tilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ defined via the Radon-Nykodim derivative

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = e^{\sigma W_t - \frac{\sigma^2}{2}t}, \quad t \geq 0.$$

It can be shown that, for every $\nu \in \mathcal{A}$, the state process X^ν can be represented as

$$X_t^\nu = e^{-\delta t} M_t(\xi + \bar{\nu}_t),$$

where $\bar{\nu} = (\bar{\nu}_t)_{t\geq 0}$ is defined via the identity $\nu_t = \int_0^t e^{-\delta s} M_s d\bar{\nu}_s$. Denote by $\tilde{\mathbb{E}}$ the expectation with respect to $\tilde{\mathbb{P}}$. By taking advantage of Fubini's theorem and optional projections as in [24, Theorem 57, Chapter VI, p. 122], we can restate necessary condition (4.10) as

$$\overline{\lim_{T\uparrow\infty}} \frac{1}{T}\tilde{\mathbb{E}}\left[\int_0^T \left(\tilde{\mathbb{E}}\left[\int_s^T e^{-\delta t} g_x(X_t^{\hat{\nu}}, p, \lambda) dt \middle| \mathcal{F}_s\right] - q e^{-\delta s}\right) d(\hat{\bar{\nu}}_s - \bar{\nu}_s)\right] \geq 0, \quad \forall \nu \in \mathcal{A}_{2q},$$

and analogously holds for (4.12) and (4.11). While first order conditions for optimality are well known for both finite and infinite time horizon singular control problems (see, e.g., [7, 8, 32, 33] among others), to the best of our knowledge, this is the first time in the literature that first order conditions for singular control problems with ergodic reward functionals have been derived.

5. Cooperative Case: Mean-field Solution and Approximation

In this section, we tackle the mean-field solution of the central planner's optimization problem defined in Definition 10.

5.1. Mean-field solution.

The first result, Theorem 5.1, regards existence and uniqueness of optimal solutions of the MFC problem. In order to compute the optimal control, we use a Lagrangian multiplier type approach, to take care of the constraint on the stationary first order moment. We first restrict to strategies so that the corresponding stationary mean is equal to some prescribed level $\theta \geq 0$, we compute the optimal strategy within this smaller constrained set and finally we optimize over all possible values of the stationary mean. A somehow similar approach has been used in [22], for a MFC problem of impulse control.

Theorem 5.1. *If $\alpha + \beta < 1$, there exists a unique optimal control $\hat{\nu}$ for the MFC problem. Moreover, the process upwards à la Skorohod $\hat{\nu}$ reflects the state process at the barrier $\hat{a}$ given by*

$$\hat{a} = \left[\frac{2(\alpha+\beta)}{q(2\delta + \sigma^2(1-\alpha))}\left(\frac{2\delta+\sigma^2}{2\delta}\right)^{\beta}\right]^{\frac{1}{1-\alpha-\beta}}, \tag{5.1}$$

and the corresponding stationary mean is given by

$$\hat{\theta} = \left(\frac{2\delta+\sigma^2}{2\delta}\right)^{\frac{1-\alpha}{1-\alpha-\beta}}\left[\frac{2(\alpha+\beta)}{q(2\delta+\sigma^2(1-\alpha))}\right]^{\frac{1}{1-\alpha-\beta}}. \tag{5.2}$$

If instead $\alpha+\beta > 1$, the problem is ill-posed, in the sense that

$$\sup_{\nu\in\mathcal{A}_{MFC}} \mathsf{J}(\nu, \mathbb{E}[X_\infty^\nu]) = +\infty.$$

Finally, if $\alpha + \beta = 1$ and

$$\frac{2\delta+\sigma^2}{2\delta+\sigma^2(1-\alpha)}\left(\frac{2\delta}{2\delta+\sigma^2}\right)^{\alpha} < q\delta,$$

the null control $\nu \equiv 0$ is optimal; otherwise, the problem is ill-posed.

Proof. We note that

$$\sup_{\nu\in\mathcal{A}_{MFC}} \mathsf{J}(\nu,\mathbb{E}[X_\infty^\nu]) = \sup_{\theta>0}\sup_{\substack{\nu\in\mathcal{A}_{MFC}\\ \mathbb{E}[X_\infty^\nu]=\theta}} \mathsf{J}(\nu,\theta) = \sup_{\theta>0}\sup_{\substack{\nu\in\mathcal{A}_{MFC}\\ \mathbb{E}[X_\infty^\nu]=\theta}} \left(\mathsf{J}(\nu,\theta) + \lambda(\theta)(\mathbb{E}[X_\infty^\nu]-\theta)\right), \tag{5.3}$$

where $\lambda : \mathbb{R}_+ \to \mathbb{R}$ is any real function of the mean θ. Moreover, for $\nu \in \mathcal{A}_{MFC}$ so that $\mathbb{E}[X_\infty^\nu] = \theta$, we rewrite the right-hand side term of (5.3) using ergodicity:

$$\begin{aligned}
&\mathsf{J}(\nu,\theta) + \lambda(\theta)(\mathbb{E}[X_\infty^\nu]-\theta) = \mathsf{J}(\nu,\theta) + \lim_{T\uparrow\infty}\frac{1}{T}\int_0^T \mathbb{E}[\lambda(\theta)X_t^\nu - \lambda(\theta)\theta]dt \\
&= -\lambda(\theta)\theta + \lim_{T\uparrow\infty}\frac{1}{T}\mathbb{E}\left[\int_0^T \left((X_t^\nu)^\alpha\theta^\beta + \lambda(\theta)X_t^\nu\right)dt - q\nu_T\right] = -\lambda(\theta)\theta + \tilde{\mathsf{J}}(\nu,\theta^\beta,\lambda(\theta)),
\end{aligned} \tag{5.4}$$

where $\tilde{\mathsf{J}}$ is defined in (4.5). We split the problem in the following three steps:

1) For fixed θ and λ, show that there exists a unique optimal control of barrier type which minimizes $\tilde{J}(\nu, \theta^\beta, \lambda)$ over $\mathcal{A}_{mf}$. Denote by $\hat{a}(\theta, \lambda)$ the optimal reflection barrier.

2) Show that for any $\theta > 0$ there exists a real value $\lambda(\theta)$ so that $\theta = \mathbb{E}[X_\infty^{\nu^{\hat{a}(\theta,\lambda(\theta))}}]$ and deduce that

$$\sup_{\substack{\nu \in \mathcal{A}_{MFC} \\ \mathbb{E}[X_\infty^\nu]=\theta}} \varliminf_{T\uparrow\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T \Big((X^\nu)^\alpha \theta^\beta + \lambda(\theta) X_t^\nu\Big)\, dt - q\nu_T\right] = \tilde{\mathsf{J}}(\nu^{\hat{a}(\theta,\lambda(\theta))}, \theta^\beta, \lambda(\theta)).$$

3) Perform the optimization over $\theta \in \mathbb{R}_+$.

As for Step 1), we restrict to $\lambda < q\delta$, so that, by applying Lemma 4.2 with $p = \theta^\beta$, the unique optimal control is the reflection policy at the level $\hat{a}(\theta, \lambda) = a^*(\theta^\beta, \lambda)$ given by (4.6).

As for Step 2), we look for $\lambda(\theta)$ so that the stationary distribution satisfies $\mathbb{E}[X_\infty^{\nu^{\hat{a}(\theta,\lambda(\theta))}}] = \theta$ and the condition $q\delta - \lambda(\theta) > 0$ holds. In view of (4.2), this is equivalent to imposing

$$\frac{2\delta + \sigma^2}{2\delta}\left(\frac{2\delta + \sigma^2(1-\alpha)}{2\alpha\delta}\frac{q\delta - \lambda}{\theta^\beta}\right)^{\frac{1}{\alpha-1}} = \theta, \quad q\delta - \lambda(\theta) > 0$$

which are both satisfied by

$$\lambda(\theta) = q\delta - \left(\frac{2\delta + \sigma^2}{2\delta}\right)^{1-\alpha} \frac{2\delta\alpha}{2\delta + \sigma^2(1-\alpha)}\theta^{\alpha+\beta-1}. \tag{5.5}$$

Therefore, by choosing $\lambda(\theta)$ as the Lagrangian multiplier in (5.3), we have that

$$\sup_{\nu \in \mathcal{A}_{MFC}} \mathsf{J}(\nu, \mathbb{E}[X_\infty^\nu]) = \sup_{\theta > 0}\Bigg(-\lambda(\theta)\theta + \sup_{\substack{\nu \in \mathcal{A}_{MFC} \\ \mathbb{E}[X_\infty^\nu]=\theta}} \tilde{\mathsf{J}}(\nu; \theta, \lambda(\theta))\Bigg)$$

$$= \sup_{\theta>0}\Big(-\lambda(\theta)\theta + \tilde{\mathsf{J}}(\nu^{\hat{a}(\theta,\lambda(\theta))}; \theta, \lambda(\theta))\Big) = \sup_{\theta>0} \mathsf{J}(\nu^{\hat{a}(\theta,\lambda(\theta))}, \theta) = \sup_{\theta>0} \mathsf{J}(\nu^{\hat{a}(\theta,\lambda(\theta))}, \mathbb{E}[X_\infty^{\nu^{\hat{a}(\theta,\lambda(\theta))}}]).$$

We are left with performing the optimization over $\theta \in \mathbb{R}_+$. By exploiting (4.2) and (4.3), for every $\theta > 0$ we have

$$\mathsf{J}(\nu^{\hat{a}(\theta,\lambda(\theta))}, \mathbb{E}[X_\infty^{\nu^{\hat{a}(\theta,\lambda(\theta))}}]) = C(\hat{a}(\theta, \lambda(\theta)), \theta) = \frac{2\delta + \sigma^2}{2\delta + \sigma^2(1-\alpha)}\left(\frac{2\delta}{2\delta + \sigma^2}\right)^\alpha \theta^{\alpha+\beta} - q\delta\theta.$$

Set

$$f(\theta) := \frac{2\delta + \sigma^2}{2\delta + \sigma^2(1-\alpha)}\left(\frac{2\delta}{2\delta + \sigma^2}\right)^\alpha \theta^{\alpha+\beta} - q\delta\theta. \tag{5.6}$$

If $\alpha + \beta < 1$, we have

$$f'(\theta) = (\alpha + \beta)\left(\frac{2\delta}{2\delta + \sigma^2}\right)^\alpha \frac{2\delta + \sigma^2}{2\delta + \sigma^2(1-\alpha)}\theta^{\alpha+\beta-1} - q\delta,$$

so that $f''(\theta) < 0$ for every $\theta > 0$, i.e. f is strictly concave in $\mathbb{R}_+$. This implies that there exists a unique maximizer $\hat{\theta}$. By imposing $f'(\theta) = 0$, we find the expression of $\hat{\theta}$ given by (5.2), and by (4.2) we find the expression of $\hat{a}$ in (5.1).

If $\alpha + \beta > 1$, the function f defined in (5.6) is unbounded, and therefore the MFC problem does not admit a maximizer. Finally, suppose $\alpha + \beta = 1$. Then, the function f is just given by

$$f(\theta) = \left(\frac{2\delta + \sigma^2}{2\delta + \sigma^2(1-\alpha)}\left(\frac{2\delta}{2\delta + \sigma^2}\right)^\alpha - q\delta\right)\theta.$$

Since f is linear in θ, we either have $\sup_{\theta>0} f(\theta)$ equal to 0 or $+\infty$ depending on the sign of the coefficient. □

Remark 5. Assumptions **U** and **D** are not needed to prove Proposition 5.1. On the other hand, according to the previous result, restrictions on the parameters are needed in order to have existence of the optimal control. Notice that, if an optimal control exists, it is always unique, by strict concavity of the reward functional.

For the sake of completeness and for later use, we derive the relationship between the Lagrangian multiplier $\lambda(\theta)$ and the constraint parameter θ.

Lemma 5.2. *Let $\nu = \nu^{a(\theta)}$ be the strategy which reflect the process $X^{\nu^{a(\theta)}}$ upwards at the barrier $a(\theta) = \frac{2\delta}{2\delta+\sigma^2}\theta$. Let $\lambda(\theta)$ be given by* (5.5) *and f given by* (5.6)*. Then, for any $\theta > 0$, it holds*

$$f'(\theta) + \lambda(\theta) = \lim_{T\uparrow\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T \pi_\theta(X_t^{\nu^{a(\theta)}}, \theta)dt\right]. \tag{5.7}$$

The proof is postponed to the Appendix.

Remark 6. The same calculations of Lemma 5.2 show that the following relation holds:

$$\lambda(\hat\theta) = \partial_\theta \mathsf{J}(\hat\nu, \hat\theta) = \partial_\theta \left(\sup_{\nu\in\mathcal{A}_{MFC}} \mathsf{J}(\nu, \mathbb{E}[X_\infty^\nu]) \right).$$

In this sense, the Lagrange multiplier $\lambda(\hat\theta)$ can be regarded as the derivative of the value function with respect to the measure argument.

5.2. **Approximation.** We show that any solution to the MFC problem as given by Theorem 5.1 induces a sequence $(\hat{\boldsymbol{\nu}}^N)_{N\geq 1}$ of approximate optimal strategy profiles for the central planner, with vanishing error. Our approach is mainly inspired by [18, Section 6], although it uses different techniques due to the nature of our dynamics and payoff.

Let $N \geq 2$. We consider the following set $\mathcal{C} \subseteq \mathcal{A}_N^N$ of strategies for the central planner.

Definition 14 (Admissible strategies for the central planner optimization problem)**.** A strategy profile $\boldsymbol{\beta}^N = (\beta^{1,N}, \dots, \beta^{N,N})$ is an admissible strategy profile for the central planner optimization problem if $\beta^{i,N} \in \mathcal{A}_N$ for any $i = 1, \dots, N$, if they are exchangeable, in the sense that the triples $(\xi^i, \beta^{i,N}, W^i)_{i=1}^N$ are exchangeable random elements, and ergodic, in the sense that the resulting N-dimensional process $(X_t^{\beta^{1,N}}, \dots, X_t^{\beta^{N,N}})_{t\geq 0}$ is ergodic, where $X^{\beta^{i,N}}$ is given by (2.1) for every $i = 1, \dots, N$. Moreover, every strategy β^j is such that

$$\sup_{T>0} \frac{1}{T}\mathbb{E}\left[\int_0^T |X_t^{\beta^j}|^2 dt\right] \leq c, \quad \forall\, j = 1, \dots, N. \tag{5.8}$$

We denote the set of strategy profiles for the central planner by $\mathcal{A}_N^{cp,c}$.

Observe that the inclusion $\mathcal{A}_N^{cp,c} \subsetneq \mathcal{A}_N^N$ holds strictly. The exchangeability assumption is well known in the MFC literature, when dealing with approximation results. We refer to [19, Paragraph 6.1.3] for more comments. Observe that, in particular, exchangeability implies

$$\bar{\mathsf{J}}^N(\boldsymbol{\beta}^N) = \mathsf{J}^N(\beta^{i,N}, \boldsymbol{\beta}^{-i,N}),$$

for every $i = 1, \dots, N$, for any strategy profile $\boldsymbol{\beta}^{(N)} \in \mathcal{A}_N^{cp,c}$.

Theorem 5.3. *Let $\hat{\boldsymbol{\nu}}^N = (\hat\nu^1, \dots, \hat\nu^N)$ be the strategy profile that reflects each process $X^{\hat\nu^i}$ upwards à la Skorohod at the level $\hat a$ given by* (5.1)*. It holds*

$$\lim_{N\to\infty} \sup_{\boldsymbol{\beta}^{(N)}\in\mathcal{A}_N^{cp,c}} \bar{\mathsf{J}}^N(\boldsymbol{\beta}^{(N)}) = \lim_{N\to\infty} \bar{\mathsf{J}}^N(\boldsymbol{\nu}^{(N)}) = \mathsf{J}(\hat\nu, \mathbb{E}[X_\infty^{\hat\nu}]). \tag{5.9}$$

Proof. Notice that, for every $N \geq 2$, $\boldsymbol{\nu}^N$ belongs to $\mathcal{A}_N^{cp}$: since the sequence $(\hat\nu^i, X^{\hat\nu^i})_{i\geq 1}$ is i.i.d. as $(\hat\nu, X^{\hat\nu})$, the system is excheangeble. Moreover, by [39, Lemmata 23.17-19], the N-dimensional process $(X^{\hat\nu^i})_{i=1}^N$ is a positively recurrent regular diffusion with ergodic measure $\bigotimes_{i=1}^N \hat p_\infty(dx_i)$, where we set $\hat{p}_\infty = p_{\hat{a}}$, with p_a given by (4.1). Moreover, up to choosing c large enough, the processes $(X^{\hat{\nu}^i})_{i=1}^N$ satisfy (14) by point iii) of Lemma 4.1.

We show that

$$\lim_{N\to\infty} \inf_{\boldsymbol{\beta}^{(N)}\in\mathcal{A}_N^{cp,c}} \left(\mathsf{J}^{MFC}(\hat{\nu}) - \bar{\mathsf{J}}^N(\boldsymbol{\beta}^{(N)})\right) \geq 0, \qquad \lim_{N\to\infty} \left(\mathsf{J}^{MFC}(\hat{\nu}) - \bar{\mathsf{J}}^N(\boldsymbol{\nu}^{(N)})\right) = 0.$$

Observe that, by Lemma 4.1, the inferior limit in the definition of $\mathsf{J}^{MFC}(\hat{\nu}^i)$ is actually a limit. Then, by using the inequalities $\overline{\lim}_n z_n - \underline{\lim}_n x_n \geq \overline{\lim}_n (z_n - y_n)$ and $\overline{\lim}_n (z_n + y_n) \geq \overline{\lim}_n z_n + \underline{\lim}_n (y_n)$ and concavity of $\pi(x,\theta) = x^\alpha \theta^\beta$ jointly in (x,θ), it holds

(5.10)

$$\begin{aligned}
&\mathsf{J}^{MFC}(\hat{\nu}^i) - \mathsf{J}^N(\beta^{i,N}, \boldsymbol{\beta}^{-i,N}) \geq \overline{\lim_{T\uparrow\infty}} \frac{1}{T}\mathbb{E}\left[\int_0^T \left(\pi(X_t^{\hat{\nu}^i}, \hat{\theta}) - \pi(X_t^{\beta^{i,N}}, \theta_t^{N,\boldsymbol{\beta}^{-i,N}})\right) dt - q(\hat{\nu}_T^i - \beta_T^{i,N})\right] \\
&\geq \overline{\lim_{T\uparrow\infty}} \frac{1}{T}\mathbb{E}\left[\int_0^T \left(\pi_x(X_t^{\hat{\nu}^i}, \hat{\theta}) + \lambda(\hat{\theta})\right)(X_t^{\hat{\nu}^i} - X_t^{\beta^{i,N}}) dt - q(\hat{\nu}_T^i - \beta_T^{i,N})\right] \\
&+ \underline{\lim_{T\uparrow\infty}} \frac{1}{T}\mathbb{E}\left[\int_0^T \left(-\lambda(\hat{\theta})(X_t^{\hat{\nu}^i} - X_t^{\beta^{i,N}}) + \pi_\theta(X_t^{\hat{\nu}^i}, \hat{\theta})(\hat{\theta} - \theta_t^{N,\boldsymbol{\beta}^{-i,N}})\right) dt\right] \\
&\geq \underline{\lim_{T\uparrow\infty}} \frac{1}{T}\mathbb{E}\left[\int_0^T -\lambda(\hat{\theta})(X_t^{\hat{\nu}^i} - X_t^{\beta^{i,N}}) dt\right] + \underline{\lim_{T\uparrow\infty}} \frac{1}{T}\mathbb{E}\left[\int_0^T \pi_\theta(X_t^{\hat{\nu}^i}, \hat{\theta})(\hat{\theta} - \theta_t^{N,\boldsymbol{\beta}^{-i,N}}) dt\right]
\end{aligned}$$

where we added and subtracted $\lambda(\hat{\theta})(X_t^{\hat{\nu}^i} - X_t^{\beta^i})$ inside the time integral. Last inequality follows from sublinearity of the $\underline{\lim}$ and from Lemma 4.3, as the pair $(X^{\hat{\nu}^i}, \hat{\nu}^i)$ has the same distribution as $(X^{\hat{\nu}}, \hat{\nu})$. Moreover, since $(X^{\hat{\nu}^i})_{i\geq 1}$ are i.i.d. copies of $X^{\hat{\nu}}$, by Lemma 5.2, it holds

$$\lambda(\hat{\theta}) = \lim_{T\uparrow\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T \pi_\theta(X_t^{\hat{\nu}^i}, \hat{\theta}) dt\right] \quad \forall i \geq 1.$$

Since the second $\underline{\lim}$ in the last line of (5.10) is actually a limit, by using ergodicity and the identical distribution of $(X^{\hat{\nu}^i})_{i=1}^N$ and $(X^{\beta^i})_{i=1}^N$ respectively, we deduce

(5.11)

$$\begin{aligned}
\mathsf{J}^{MFC}(\hat{\nu}^i) - \mathsf{J}^N(\beta^{i,N}, \boldsymbol{\beta}^{-i,N}) &\geq \underline{\lim_{T\uparrow\infty}} \frac{1}{T}\mathbb{E}\left[\int_0^T \left(\pi_\theta(X_t^{\hat{\nu}^i}, \hat{\theta}) - \lambda(\hat{\theta})\right)\left(\hat{\theta} - \theta_t^{N,\boldsymbol{\beta}^{-i,N}}\right) dt\right] \\
&= \frac{N}{N-1} \underline{\lim_{T\uparrow\infty}} \frac{1}{T}\mathbb{E}\bigg[\int_0^T \left(\pi_\theta(X_t^{\hat{\nu}^i}, \hat{\theta}) - \lambda(\hat{\theta})\right)\Big(\frac{1}{N}\sum_{j=1}^N (\hat{\theta} - X_t^{\beta^{j,N}}) - \frac{1}{N}(\hat{\theta} - X_t^{\beta^{i,N}})\Big) dt\bigg],
\end{aligned}$$

where we used the identity $\frac{1}{N-1}\sum_{j\neq 1} y_j = \frac{N}{N-1}(\frac{1}{N}\sum_{i=1}^N y_j - \frac{1}{N} y_i)$. By taking the average over N, we have

$$\begin{aligned}
\mathsf{J}^{MFC}(\hat{\nu}) - \bar{\mathsf{J}}^N(\boldsymbol{\beta}^{(N)}) &\geq \frac{N}{N-1} \underline{\lim_{T\uparrow\infty}} \frac{1}{T}\int_0^T \mathbb{E}\bigg[\Big(\frac{1}{N}\sum_{i=1}^N (\pi_\theta(X_t^{\hat{\nu}^i}, \hat{\theta}) - \lambda(\hat{\theta}))\Big)\Big(\frac{1}{N}\sum_{j=1}^N (\hat{\theta} - X_t^{\beta^{j,N}})\Big) \\
&- \frac{1}{N^2}\sum_{i=1}^N (\pi_\theta(X_t^{\hat{\nu}^i}, \hat{\theta}) - \lambda(\hat{\theta}))(\hat{\theta} - X_t^{\beta^{i,N}})\bigg] dt \\
&\geq -c\Bigg(\overline{\lim_{T\uparrow\infty}} \frac{1}{T}\mathbb{E}\int_0^T \bigg[\Big|\frac{1}{N}\sum_{i=1}^N (\pi_\theta(X_t^{\hat{\nu}^i}, \hat{\theta}) - \lambda(\hat{\theta}))\Big|\Big|\frac{1}{N}\sum_{j=1}^N (\hat{\theta} - X_t^{\beta^{j,N}})\Big|\bigg] dt \\
&+ \frac{1}{N}\overline{\lim_{T\uparrow\infty}} \frac{1}{T}\int_0^T \mathbb{E}\bigg[\Big|\frac{1}{N}\sum_{i=1}^N (\pi_\theta(X_t^{\hat{\nu}^i}, \hat{\theta}) - \lambda(\hat{\theta}))(\hat{\theta} - X_t^{\beta^{i,N}})\Big|\bigg] dt\Bigg).
\end{aligned}$$

We study separately the two $\overline{\lim}$. As for the the second one, by taking advantage the exchangeability of $(X^{\hat{\nu}^i})_{i\geq 1}$ and $(X^{\beta^{i,N}})_{i=1}^N$, we get

$$\begin{aligned}
&\frac{1}{N}\overline{\lim_{T\uparrow\infty}}\frac{1}{T}\int_0^T \mathbb{E}\Big[\Big|\frac{1}{N}\sum_{i=1}^N(\pi_\theta(X_t^{\hat{\nu}^i},\hat{\theta})-\lambda(\hat{\theta}))(\hat{\theta}-X_t^{\beta^{i,N}})\Big|\Big]dt\\
&\leq \frac{1}{N}\overline{\lim_{T\uparrow\infty}}\left(\frac{1}{T}\int_0^T\mathbb{E}\Big[\Big|\pi_\theta(X_t^{\hat{\nu}},\hat{\theta})-\lambda(\hat{\theta})\Big|^2\Big]dt\right)^{\frac{1}{2}}\left(\frac{1}{T}\int_0^T\mathbb{E}\Big[\Big|\hat{\theta}-X_t^{\beta^{1,N}}\Big|^2\Big]dt\right)^{\frac{1}{2}}\\
&\leq \frac{c}{N}\overline{\lim_{T\uparrow\infty}}\left(\frac{1}{T}\int_0^T\mathbb{E}\Big[\Big|\pi_\theta(X_t^{\hat{\nu}},\hat{\theta})-\lambda(\hat{\theta})\Big|^2\Big]dt\right)^{\frac{1}{2}},
\end{aligned}$$

that goes to 0 as $N\to\infty$, by definition of the set of strategy profiles $\mathcal{A}_N^{cp,c}$. As for the first term, by analogous computations, we get

$$\begin{aligned}
&\overline{\lim_{T\uparrow\infty}}\frac{1}{T}\mathbb{E}\int_0^T\Big[\Big|\frac{1}{N}\sum_{i=1}^N(\pi_\theta(X_t^{\hat{\nu}^i},\hat{\theta})-\lambda(\hat{\theta}))\Big|\Big|\frac{1}{N}\sum_{j=1}^N(\hat{\theta}-X^{\beta^{j,N}})\Big|\Big]dt\\
&\leq \overline{\lim_{T\uparrow\infty}}\left(\frac{1}{T}\mathbb{E}\int_0^T\Big[\Big|\frac{1}{N}\sum_{i=1}^N(\pi_\theta(X_t^{\hat{\nu}^i},\hat{\theta})-\lambda(\hat{\theta}))\Big|^2\Big]dt\right)^{\frac{1}{2}}\left(\frac{1}{T}\int_0^T\mathbb{E}\Big[\Big|\frac{1}{N}\sum_{j=1}^N(\hat{\theta}-X^{\beta^{j,N}})\Big|^2\Big]dt\right)^{\frac{1}{2}}\\
&\leq c\,\overline{\lim_{T\uparrow\infty}}\left(\frac{1}{T}\mathbb{E}\int_0^T\Big[\Big|\frac{1}{N}\sum_{i=1}^N(\pi_\theta(X_t^{\hat{\nu}^i},\hat{\theta})-\lambda(\hat{\theta}))\Big|^2\Big]dt\right)^{\frac{1}{2}},
\end{aligned}$$

for a constant c independent of N, by using the bound (5.8) and exchangeability of $\boldsymbol{\beta}^{(N)}\in\mathcal{A}_N^{cp,c}$. By the ergodic ratio theorem, it holds
(5.12)

$$\lim_{T\uparrow\infty}\frac{1}{T}\int_0^T\Big|\frac{1}{N}\sum_{i=1}^N\Big(\pi_\theta(X_t^{\hat{\nu}^i},\hat{\theta})-\lambda(\hat{\theta})\Big)\Big|^2dt=\int_{\mathbb{R}_+^N}\Big|\frac{1}{N}\sum_{i=1}^N\Big(\pi_\theta(x_i,\hat{\theta})-\lambda(\hat{\theta})\Big)\Big|^2\bigotimes_{i=1}^N\hat{p}_\infty(dx_i),$$

$\mathbb{P}$-a.s.. We show that the right hand-side is uniformly integrable. Take $r={}^1/_\alpha>1$. By Jensen inequality and identical distribution of $(X^{\hat{\nu}^i})_{i\geq1}$, we have

$$\begin{aligned}
&\mathbb{E}\left[\left(\frac{1}{T}\int_0^T\Big|\frac{1}{N}\sum_{i=1}^N\Big(\pi_\theta(X_t^{\hat{\nu}^i},\hat{\theta})-\lambda(\hat{\theta})\Big)\Big|^2dt\right)^r\right]\leq\frac{C}{T}\mathbb{E}\left[\int_0^T\Big|\Big(\frac{1}{N}\sum_{i=1}^N\pi_\theta(X_t^{\hat{\nu}^i},\hat{\theta})-\lambda(\hat{\theta})\Big)\Big|^{2r}dt\right]\\
&\leq C\left(|\lambda(\hat{\theta})|^{2r}+\frac{1}{N}\sum_{i=1}^N\frac{1}{T}\int_0^T\mathbb{E}\left[|\pi_\theta(X_t^{\hat{\nu}^i},\hat{\theta})|^{2r}\right]dt\right)\leq C\left(1+\frac{1}{T}\int_0^T\mathbb{E}\left[|\pi_\theta(X_t^{\hat{\nu}},\hat{\theta})|^{2r}\right]dt\right)\\
&\leq C\left(1+\frac{1}{T}\int_0^T\mathbb{E}\left[|(X_t^{\hat{\nu}})^\alpha|^{2r}\right]dt\right)=C\left(1+\frac{1}{T}\int_0^T\mathbb{E}\left[|(X_t^{\hat{\nu}})|^2\right]dt\right)\leq C(1+\hat{a}^2),
\end{aligned}$$

where we used Lemma 4.1 in the last estimate. Since last estimate holds for any $T>0$ and $r>1$, we deduce that the right hand-side of (5.12) is uniformly integrable. By, e.g., [39, Lemma 4.12]) this yields

$$\lim_{T\uparrow\infty}\frac{1}{T}\mathbb{E}\Big[\int_0^T\Big|\frac{1}{N}\sum_{i=1}^N\Big(\pi_\theta(X_t^{\hat{\nu}^i},\hat{\theta})-\lambda(\hat{\theta})\Big)\Big|^2dt\Big]=\int_{\mathbb{R}_+^N}\Big|\frac{1}{N}\sum_{i=1}^N\Big(\pi_\theta(x_i,\hat{\theta})-\lambda(\hat{\theta})\Big)\Big|^2\bigotimes_{i=1}^N\hat{p}_\infty(dx_i).$$

We conclude by invoking the law of large numbers: let $(X_i)_{i\geq1}$ be a sequence of i.i.d. random variables with law $\hat{p}_\infty(dx)$. Therefore, by Lemmata 4.1 and 5.2, the sequence $(Y_i)_{i\geq1}$ defined by

$Y_i = \pi_\theta(X_i, \hat{\theta}) - \lambda(\hat{\theta})$ is i.i.d., square integrable and centered, which yields

$$\int_{\mathbb{R}_+^{N-1}} \Big|\frac{1}{N}\sum_{i=1}^N \Big(\pi_\theta(x_i,\hat{\theta}) - \lambda(\hat{\theta})\Big)\Big|^2 \bigotimes_{i=1}^N \hat{p}_\infty(dx_i) = \mathbb{E}\left[\Big|\frac{1}{N}\sum_{i=1}^N Y_i\Big|^2\right] = \frac{\mathbb{E}[|Y_1|]^2}{N} \to 0.$$

Finally, $\lim_{N\to\infty} \bar{\mathsf{J}}^N(\boldsymbol{\nu}^{(N)}) = \mathsf{J}^{MFC}(\hat{\nu})$ follows form ergodicity of the processes $(X^{\hat{\nu}^i})_{i=1}^N$ and the law of large numbers, analogously as before. $\square$

6. Competitive Case: Mean-field Equilibria and Approximation

In this section, we focus on coarse correlated equilibria for the MFG, as defined by Definition 12. Since the set of CCEs is typically very wide and it is difficult to characterize in a continuous time setting, following the procedure outlined in [12], we restrict our analysis to specific classes of correlated stationary strategies for which we are able to state a sufficient condition for being a CCE for the ergodic MFG. With respect to [12], we move a step forward, and show that every CCE in these classes induces a sequence of approximate CCEs in the underlying N-player game with vanishing error. More specifically, we establish the following:

- We fix a correlated stationary strategy $(Z, \lambda, \theta_\infty)$. We suppose that the representative player decides to ignore the moderator's recommendation, and compute her best deviating strategy, i.e.
$$U^* = \arg\max_{\nu\in\mathcal{A}_{mf}} \mathsf{J}(\nu, \theta_\infty).$$
This is the content of Proposition 6.1.
- We define specific classes of correlated flows $(Z, \lambda, \theta_\infty)$ so that the consistency condition (2) is satisfied. This is established in Propositions 6.2 and 6.5.
- For $(Z, \lambda, \theta_\infty)$ in such classes, we express the optimality condition (1) as an inequality involving the law of θ_∞ only, thus deriving a sufficient condition for the existence of CCEs. This is the content of Propositions 6.3 and 6.6.
- We show that every mean-field CCEs in each class induces a sequence of aprroximate CCEs in the underlying N-player game with vanishing error. This is the content of Theorems 6.4 and 6.7.

We consider two classes of correlated stationary strategies: while in both classes the correlating device is the random mean θ_∞ itself, in the first class the recommendation λ^r is a $\sigma(\theta_\infty)$-measurable regular control, while in the second one, the recommendation λ^s is a policy of reflection type at a random barrier $a(\theta_\infty)$. Surprisingly, the sufficient condition of the two classes differ only by a constant. Moreover, in Theorem 6.8, we explicitly characterize existence and uniqueness of Nash equilibria. We find that they belong to class of CCEs with recommendation of reflection type, so that the same approximation result applies.

6.1. The Deviating Player Problem.

Suppose that the representative player decides to ignore the moderator's recommendation. By definition of CCE for the ergodic MFG, the deviating player must choose her strategy $\nu \in \mathcal{A}_{mf}$ only by knowing the joint law of the correlated stationary strategy $(Z, \lambda, \theta_\infty)$, which is assumed to be publicly known, and not by observing its realizations. Since $\nu \in \mathcal{A}_{mf}$, it follows that X^ν is $\mathbb{F}^{\xi,W}$-adapted as well and thus independent of the random variable θ_∞, which implies that deviating player's payoff can be written as:

$$\begin{aligned}\mathsf{J}(\nu,\theta_\infty) &= \varliminf_{T\uparrow\infty} \frac{1}{T}\left(\int_0^T \mathbb{E}\left[\mathbb{E}[(X_t^\nu)^\alpha \theta_\infty^\beta | \mathcal{F}_t^{\xi,W}]\right] - q\mathbb{E}\left[\mathbb{E}[\nu_T|\mathcal{F}_T^{\xi,W}]\right]\right) \\ &= \varliminf_{T\uparrow\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T (X_t^\nu)^\alpha \mathbb{E}[\theta_\infty^\beta]dt - q\nu_T\right].\end{aligned} \tag{6.1}$$

Observe that deviating player's payoff functional depends on θ_∞ only through its expectation.

Proposition 6.1. *There exists a unique optimal strategy for the deviating player $U^* \in \mathcal{A}_{mf}$ which reflects the process X^{U^*} upwards á la Skorohod at the level a^*, where a^* is given by*

$$a^* = \left(\frac{2\alpha}{q(2\delta + \sigma^2(1-\alpha))}\right)^{\frac{1}{1-\alpha}} (\mathbb{E}[\theta_\infty^\beta])^{\frac{1}{1-\alpha}}. \tag{6.2}$$

Proof. Since the payoff functional of the deviating player is given by (6.1), it is enough to apply Lemma 4.2 with $\lambda = 0$ and $p = \mathbb{E}[\theta_\infty^\beta]$. □

In the following, we set

$$K := \left(\frac{2\alpha}{q(2\delta + \sigma^2(1-\alpha))}\right)^{\frac{1}{1-\alpha}}, \tag{6.3}$$

so that the optimal policy of the deviating player is the reflection at the level $a^* = K(\mathbb{E}[\theta_\infty^\beta])^{1/1-\alpha}$.

6.2. **Regular recommendation.** Let $\mathcal{G}^r$ be the set of correlated stationary strategies $(Z, \lambda^r, \theta_\infty)$ so that $\theta_\infty \in L^2(\mathcal{F}_{0-})$ independent of ξ and W, $Z = \theta_\infty$ and

$$d\lambda_t^r = \delta\theta_\infty dt, \quad t \geq 0, \tag{6.4}$$

so that, in particular, the recommended strategy λ^r is a $\sigma(\theta_\infty)$-measurable regular control.

Proposition 6.2. *Let $(Z, \theta_\infty, \lambda^r) \in \mathcal{G}^r$. Define*

$$m'_{\infty,\theta}(x) = \frac{2}{\sigma^2} \exp\left(\frac{2\delta\theta}{\sigma^2}\right) x^{-\frac{2\delta}{\sigma^2}-2} \exp\left(-\frac{2\delta}{\sigma^2}\frac{\theta}{x}\right), \tag{6.5}$$

and let $p_\infty^r(dx, \theta)$ be the stochastic kernel from $\mathbb{R}_+$ to $\mathcal{B}_{\mathbb{R}_+}$ defined by

$$p_\infty^r(dx, \theta) = \frac{x^{-\frac{2\delta}{\sigma^2}-2} \exp\left(-\frac{2\delta}{\sigma^2}\frac{\theta}{x}\right)}{\int_0^\infty y^{-\frac{2\delta}{\sigma^2}-2} \exp\left(-\frac{2\delta}{\sigma^2}\frac{\theta}{y}\right) dy} dx. \tag{6.6}$$

Then, the triple $(Z, \theta_\infty, \lambda^r)$ is a correlated stationary strategy so that the consistency condition (3.3) is satisfied. In particular, it holds $\mu_\infty^r(dx, d\theta) = \lim_{t\to\infty} \mathbb{P} \circ (X_t^{\lambda^r}, \theta_\infty)^{-1} = p_\infty^r(dx, \theta)\rho(d\theta)$.

Proof. The measurability requirements on the triple $(Z, \theta_\infty, \lambda^r)$ are clearly satisfied. Let X^{λ^r} the state process controlled by λ^r, which satisfies

$$dX_t^{\lambda^r} = \delta(\theta_\infty - X_t^{\lambda^r})dt + \sigma X_t^{\lambda^r} dW_t, \quad X_0^{\lambda^r} = \xi. \tag{6.7}$$

We show that the joint law of $(X_t^{\lambda^r}, \theta_\infty)$ converges weakly to μ_∞^r as $t \to \infty$. To this extent, it is enough to verify that the regular conditional probability of $X_t^{\lambda^r}$ with respect to $\theta_\infty = \theta$, which we denote by $p_t^r(dx, \theta)$, converges weakly to $p_\infty^r(dx, \theta)$ as $t \to \infty$, for ρ-a.e. $\theta \in \mathbb{R}_+$. Conditionally to $\theta_\infty = \theta$, $X_t^{\lambda^r}$ satisfies the following equation:

$$dX_t^{\lambda^r,\theta} = \delta(\theta - X_t^{\lambda^r,\theta})dt + \sigma X_t^{\lambda^r,\theta} dW_t, \quad X_0^{\lambda^r,\theta} = \xi.$$

By Lemma 7.1, we have $\int_0^\infty m'_{\infty,\theta}(x)dx < \infty$, which implies that the diffusion $X^{\lambda^r,\theta}$ is positively recurrent for every $\theta > 0$. Thus, the measure $p_\infty^r(dx, \theta)$ is the unique stationary distribution and $p_t^r(dx, \theta) \to p_\infty^r(dx, \theta)$ in total variation norm (see, e.g., [11, Paragraph 36]). As for equality (3.3), define $\varphi = (\varphi_t(\theta))_{t\geq 0, \theta > 0}$ as

$$\varphi_t(\theta) = e^{-\delta t}\mathbb{E}[\xi] + \theta(1 - e^{\delta t}). \tag{6.8}$$

By Itô's formula, it follows that $\varphi_t(\theta_\infty) = \mathbb{E}[X_t^{\lambda^r}|\theta_\infty]$ for every $t \geq 0$, $\mathbb{P}$-a.s., which implies that $\theta_\infty = \lim_{t\to\infty} \mathbb{E}[X_t|\theta_\infty]$ $\mathbb{P}$-a.s., and therefore condition (3.4) is satisfied, and so (3.3). □

Proposition 6.3. *A correlated stationary strategy* $(Z,\theta_\infty,\lambda^r)$ *in the class* $\mathcal{G}^r$ *is a mean-field CCE if and only if the following inequality is satisfied:*

$$c_\beta(\mathbb{E}[\theta_\infty^\beta])^{\frac{1}{1-\alpha}} + c_1\mathbb{E}[\theta_\infty] \le c_{\alpha+\beta}\mathbb{E}[\theta_\infty^{\alpha+\beta}], \tag{6.9}$$

where c_β*,* $c_{\alpha+\beta}$ *and* c_1 *are positive constants defined by*

$$c_\beta := \frac{(2\delta+\sigma^2)q}{2}\left(\frac{2\alpha}{q(2\delta+\sigma^2(1-\alpha))}\right)^{\frac{1}{1-\alpha}}\frac{1-\alpha}{\alpha}, \quad c_1 := \delta q, \qquad c_{\alpha+\beta} := \left(\frac{2\delta}{\sigma^2}\right)^\alpha \frac{\Gamma(\frac{2\delta}{\sigma^2}+1-\alpha)}{\Gamma(\frac{2\delta}{\sigma^2}+1)}. \tag{6.10}$$

Proof. By Proposition 6.2, $(Z,\theta_\infty,\lambda^r)$ satisfies the consistency condition (3.3). Let U^* be the optimal control for the deviating player given by Proposition 6.1. Since $\mathsf{J}(U^*,\theta_\infty) = \max_{\nu\in\mathcal{A}_{mf}}\mathsf{J}(\nu,\theta_\infty)$, we just need to verify that the inequality $\mathsf{J}(\lambda^r,\theta_\infty)\ge\mathsf{J}(U^*,\theta_\infty)$ is equivalent to (6.9). Since U^* is a reflection policy at the level a^* given by (6.2), formulae (4.3) and (6.1) yield

$$\begin{aligned}\mathsf{J}(U^*,\theta_\infty) &= C(a^*,\mathbb{E}[\theta_\infty^\beta])\\ &= \frac{2\delta+\sigma^2}{2\delta+\sigma^2(1-\alpha)}\mathbb{E}[\theta_\infty^\beta]\left(K(\mathbb{E}[\theta_\infty^\beta])^{\frac{1}{1-\alpha}}\right)^\alpha - q\frac{2\delta+\sigma^2}{2}K(\mathbb{E}[\theta_\infty^\beta])^{\frac{1}{1-\alpha}}\\ &= (2\delta+\sigma^2)\left(\frac{1}{2\delta+\sigma^2(1-\alpha)}K^\alpha - q\frac{1}{2}K\right)(\mathbb{E}[\theta_\infty^\beta])^{\frac{1}{1-\alpha}} = c_\beta(\mathbb{E}[\theta_\infty^\beta])^{\frac{1}{1-\alpha}},\end{aligned} \tag{6.11}$$

by noticing that, with K given by (6.3), it holds

$$(2\delta+\sigma^2)\left(\frac{1}{2\delta+\sigma^2(1-\alpha)}K^\alpha - \frac{q}{2}K\right) = \frac{2\delta+\sigma^2}{2}qK\left(\frac{1-\alpha}{\alpha}\right) = c_\beta.$$

As for the payoff associated to the representative player, conditionally to $\theta_\infty=\theta$ and exploiting ergodicity, it holds

$$\begin{aligned}\lim_{T\uparrow\infty}\frac{1}{T}\mathbb{E}\left[\int_0^T (X_t^{\lambda^r})^\alpha\theta^\beta dt - q\lambda_T^r\middle|\theta_\infty=\theta\right] &= \int_0^\infty \theta^\beta x^\alpha p_\infty^r(dx,\theta) - \delta q\theta\\ &= \left(\frac{2\delta}{\sigma^2}\right)^\alpha\frac{\Gamma(\frac{2\delta}{\sigma^2}-\alpha+1)}{\Gamma(\frac{2\delta}{\sigma^2}+1)}\theta^{\alpha+\beta} - \delta q\theta,\end{aligned}$$

where last equality follows from the definition of $p_\infty^r(dx,\theta)$ and Lemma 7.1. Moreover, by (6.8), we have the bound

$$\begin{aligned}\frac{1}{T}\mathbb{E}\left[\int_0^T (X_t^{\lambda^r})^\alpha\theta^\beta dt - q\lambda_T^r\middle|\theta_\infty\right] &\le \frac{\theta^\beta}{T}\mathbb{E}\left[\int_0^T(1+X_t^{\lambda^r})dt\middle|\theta_\infty\right] + \delta q\theta_\infty\\ &\le \theta^\beta\left(1+\sup_{t\ge0}\mathbb{E}\left[X_t^{\lambda^r}\middle|\theta_\infty\right]\right) + \delta q\theta_\infty \le C(1+\theta_\infty^2),\end{aligned}$$

which is integrable by assumption. Therefore, by dominated convergence theorem, we can exchange limit and expectation to conclude

$$\begin{aligned}\mathsf{J}(\lambda^r,\theta_\infty) &= \lim_{T\uparrow\infty}\frac{1}{T}\mathbb{E}\left[\mathbb{E}\left[\int_0^T (X_t^{\lambda^r})^\alpha\theta^\beta dt - q\lambda_T^r\middle|\theta_\infty\right]\right] = \mathbb{E}\left[\lim_{T\uparrow\infty}\frac{1}{T}\mathbb{E}\left[\int_0^T (X_t^{\lambda^r})^\alpha\theta^\beta dt - q\lambda_T^r\middle|\theta_\infty\right]\right]\\ &= \left(\frac{2\delta}{\sigma^2}\right)^\alpha\frac{\Gamma(\frac{2\delta}{\sigma^2}-\alpha+1)}{\Gamma(\frac{2\delta}{\sigma^2}+1)}\int_0^\infty\theta^{\alpha+\beta}\rho(d\theta) - \delta q\int_0^\infty\theta\rho(d\theta),\end{aligned}$$

By comparing this equation with equation (6.11), we get equation (6.9). ☐

Remark 7. Observe that, if $\alpha+\beta\le1$, for this result to hold true it is enough to require θ_∞ to be in L^1; Assumption **D** is not needed as well.

Finally, we show how to use a CCE in the class $\mathcal{G}^r$ to build a sequence $\boldsymbol{\lambda}^N$ of ε_N-CCE in the N-player game with $\varepsilon_N \to 0$ as $N \to \infty$. To this extent, we consider the following set of strategies $\mathcal{B} \subseteq \mathcal{A}_N$:

Definition 15 (c-admissible strategies)**.** Let $c > 0$. A strategy ν be in $\mathcal{A}_N$ is c-admissible if

$$\overline{\lim_{T\uparrow\infty}} \frac{1}{T}\mathbb{E}\left[\int_0^T |X_t^\nu|^2 dt\right] \leq c. \tag{6.12}$$

We denote by $\mathcal{A}_{N,c}$ the set of c-admissible strategies for the N-player game.

Starting from a correlated stationary strategy $(Z, \theta_\infty, \lambda^r)$ in the class $\mathcal{G}^r$, we define the following correlated strategy profiles for the N-player game: take $Z = \theta_\infty$ as correlation device, and set, for any $i \geq 1$,

$$d\lambda_t^{i,r} = \delta\theta_\infty dt. \tag{6.13}$$

Then, when player i plays accordingly to moderator's suggestion, her dynamics are hold by the following equation:

$$dX_t^{i,r} = \delta(\theta_\infty - X_t^{i,r})dt + \sigma X_t^{i,r} dW_t^i, \quad X_0^{i,r} = \xi^i. \tag{6.14}$$

Observe that, for each $i \geq 1$, the triple $(X^{i,r}, \lambda^{i,r}, \theta_\infty)$ has the same law as $(X, \lambda, \theta_\infty)$. Moreover, while not independent, the processes $(X^{i,r})_{i\geq 1}$ are conditionally independent given θ_∞.

Theorem 6.4 (Approximation of CCEs - regular case)**.** *Let $(Z, \theta_\infty, \lambda^r)$ be a CCE in the class $\mathcal{G}^r$. Let $\boldsymbol{\lambda}^N = (\lambda^{i,r})_{i=1}^N$, with $\lambda^{i,r}$ defined by (6.13). Then, for any $c > 0$, the correlated strategy profile $(\theta_\infty, \boldsymbol{\lambda}^N)$ defines an ε_N-CCE for the N-player game within the set of strategies $\mathcal{A}_{N,c}$, with $\varepsilon_N \to 0$ as $N \to \infty$.*

Proof. For every $N \geq 2$, set

$$\varepsilon_N := \sup_{\nu \in \mathcal{A}_{N,c}} \left(\mathsf{J}_N(\nu, \boldsymbol{\lambda}^{-i,N}) - \mathsf{J}_N(\lambda^{i,r}, \boldsymbol{\lambda}^{-i,N})\right).$$

Notice that, by symmetry, ε_N is independent of $i = 1, \dots, N$. Clearly $(\theta_\infty, \boldsymbol{\lambda}^N)$ is an ε_N-CCE for the N-player game within the set of strategies $\mathcal{A}_{N,c}$. We show that ε_N vanishes as N goes to ∞. Note that, for any ν in $\mathcal{A}_{N,c}$, we have

$$\begin{aligned} \mathsf{J}_N(\nu, \boldsymbol{\lambda}^{-i,N}) - \mathsf{J}_N(\lambda^{i,r}, \boldsymbol{\lambda}^{-i,N}) &= \left(\mathsf{J}_N(\nu, \boldsymbol{\lambda}^{-i,N}) - \mathsf{J}(\nu, \theta_\infty)\right) \\ &+ \left(\mathsf{J}(\nu, \theta_\infty) - \mathsf{J}(\lambda^{i,r}, \theta_\infty)\right) + \left(\mathsf{J}(\lambda^{i,r}, \theta_\infty) - \mathsf{J}_N(\lambda^{i,r}, \boldsymbol{\lambda}^{-i,N})\right) \end{aligned} \tag{6.15}$$

where J is defined by (3.2). We treat separately each of the three terms in the right-hand side of (6.15).

As for the first term, by Cauchy-Schwartz inequality and using the inequality $\underline{\lim}_{n\to\infty} \alpha_n - \underline{\lim}_{n\to\infty} \beta_n \leq \overline{\lim}_{n\to\infty}(\alpha_n - \beta_n)$, we have the following estimates:

(6.16)

$$\begin{aligned} &|\mathsf{J}(\nu, \theta_\infty) - \mathsf{J}_N(\nu, \boldsymbol{\lambda}^{-i,N})| \\ &\leq \overline{\lim_{T\uparrow\infty}} \left(\frac{1}{T}\int_0^T \mathbb{E}\left[(X_t^{i,\nu})^{2\alpha}\right] dt\right)^{\frac{1}{2}} \left(\frac{1}{T}\int_0^T \mathbb{E}\left[\left|(\theta_t^{N,\boldsymbol{\lambda}^{-i,N}})^\beta - \theta_\infty^\beta\right|^2\right] dt\right)^{\frac{1}{2}} \\ &\leq \left(1 + \overline{\lim_{T\uparrow\infty}} \frac{1}{T}\int_0^T \mathbb{E}\left[|X_t^{i,\nu}|^2\right] dt\right)^{\frac{1}{2}} \left(\overline{\lim_{T\uparrow\infty}} \frac{1}{T}\int_0^T \mathbb{E}\left[\mathbb{E}\left[\left|(\theta_t^{N,\boldsymbol{\lambda}^{-i,N}})^\beta - \theta_\infty^\beta\right|^2 \middle|\theta_\infty\right]\right] dt\right)^{\frac{1}{2}} \\ &\leq (1+c)^{\frac{1}{2}} \left(\mathbb{E}\left[\overline{\lim_{T\uparrow\infty}} \frac{1}{T}\int_0^T \mathbb{E}\left[\left|(\theta_t^{N,\boldsymbol{\lambda}^{-i,N}})^\beta - \theta_\infty^\beta\right|^2 \middle|\theta_\infty\right] dt\right]\right)^{\frac{1}{2}}, \end{aligned}$$

where in the last inequality we exchanged limsup and expectation by reverse Fatou's lemma with the integrable upper bound $C(1+\theta_\infty^2)$. Indeed, by recalling that $(X^j)_{j\neq i}$ are i.i.d. as X conditionally to θ_∞, we have the following estimates

$$\mathbb{E}\left[\left|(\theta_t^{N,\boldsymbol{\lambda}^{-i,N}})^\beta - \theta_\infty^\beta\right|^2\Big|\theta_\infty\right] \leq C\left(1+\theta_\infty^2 + \frac{1}{N-1}\sum_{j\neq i}\mathbb{E}\left[|X_t^j|^2\Big|\theta_\infty\right]\right) \leq C(1+\theta_\infty^2),$$

where last inequality holds thanks to Lemma 7.2. By Lemma 7.3, we then have

$$\begin{aligned}
&\overline{\lim_{T\uparrow\infty}}\frac{1}{T}\int_0^T \mathbb{E}\left[\left|(\theta_t^{N,\boldsymbol{\lambda}^{-i,N}})^\beta - \theta_\infty^\beta\right|^2\Big|\theta_\infty\right]dt \\
&\quad = \int_{\mathbb{R}_+^{N-1}}\left|\Big(\frac{1}{N-1}\sum_{j\neq i}x_j\Big)^\beta - \theta_\infty^\beta\right|^2 \bigotimes_{j\neq i} p_\infty^r(dx_j,\theta_\infty), \quad \mathbb{P}\text{-a.s.}
\end{aligned}$$

which converges to 0 in expectation as N goes to infinity by Lemma 7.4 and dominated convergence theorem, with $c(1+\theta_\infty^2)$ as the integrable upper bound.

As for the second term, we claim that $\mathsf{J}(\nu,\theta_\infty)-\mathsf{J}(\lambda^{i,r},\theta_\infty)\leq 0$ for any $\nu\in\mathcal{A}_{N,c}$. Indeed, observe that, since $\nu\in\mathcal{A}_N$ is independent of θ_∞ by definition of admissible strategies for the N-player game, the proof of Proposition 6.1 shows that

$$\sup_{\nu\in\mathcal{A}_N}\mathsf{J}(\nu,\theta_\infty) = \mathsf{J}(U^{i,*},\theta_\infty),$$

where $U^{i,*}$ is the policy that reflects the process $X^{i,U^{i,*}}$ upward à la Skorohod at the level a^* given by (6.2). In particular, $X^{i,U^{i,*}}$ as the same distribution of the process X^{U^*}. Therefore, we have

$$\sup_{\nu\in\mathcal{A}_{N,c}}\big(\mathsf{J}(\nu,\theta_\infty)-\mathsf{J}(\lambda^{i,r},\theta_\infty)\big) \leq \sup_{\nu\in\mathcal{A}_N}\mathsf{J}(\nu,\theta_\infty)-\mathsf{J}(\lambda,\theta_\infty)\leq \mathsf{J}(U^*,\theta_\infty)-\mathsf{J}(\lambda,\theta_\infty)\leq 0$$

where we used the inclusion $\mathcal{A}_{N,c}\subseteq\mathcal{A}_N$, the fact that $(X^{i,r},\lambda^{i,r},\theta_\infty)$ has the same distribution as $(X,\lambda,\theta_\infty)$ and the optimality property (1) of CCE of the ergodic MFG.

As for the third term, taking advantage of the conditional independence and identical distribution of $(X^{\lambda^{i,r}})_{i\geq 1}$ and by analogous estimates as in (6.16), we have

$$\begin{aligned}
&|\mathsf{J}(\lambda^{i,r},\theta_\infty)-\mathsf{J}_N(\lambda^{i,r},\boldsymbol{\lambda}^{-i,N})| \\
&\quad\leq C(1+\mathbb{E}[\theta_\infty^2])^{\frac12}\left(\mathbb{E}\left[\overline{\lim_{T\uparrow\infty}}\frac{1}{T}\int_0^T\mathbb{E}\left[\left|(\theta_t^{N,\boldsymbol{\lambda}^{-i,N}})^\beta-\theta_\infty^\beta\right|^2\Big|\theta_\infty\right]dt\right]\right)^{\frac12}
\end{aligned}$$

and we conclude the proof by applying Lemmata 7.3 and 7.4 as before. □

Remark 8 (On the integrability condition)**.** It is worth to notice that the integrability condition (6.12) that defines c-admissible strategies can be weakened at the price of more integrability requirements on θ_∞, ξ and the diffusion X^0. Indeed, let $q={}^{\beta}/{1-\alpha}$, $k=1+\lceil q\rceil$, and suppose that $2\delta-(k-1)\sigma>0$, $\mathbb{E}[\xi^k]<\infty$ and $\mathbb{E}[\theta_\infty^k]<\infty$, which imply that the estimates in point iii) of Lemma 4.1 holds up to the k-th moment. Then, up to little modification of the proof, one could consider strategies $\nu\in\mathcal{A}_N$ so that

$$\overline{\lim_{T\uparrow\infty}}\frac{1}{T}\mathbb{E}\left[\int_0^T X_s^\nu ds\right]\leq c.$$

6.3. **Singular Recommendation.** We now look for a policy λ^s of reflection type at a random barrier $a(\theta_\infty)$. Let $\mathcal{G}^s$ be the set of correlated stationary strategies $(Z,\lambda^s,\theta_\infty)$ so that $Z=\theta_\infty$, $\theta_\infty\in L^2(\mathcal{F}_{0^-})$ independent of ξ and W, and λ^s is the control that reflects the process X^{λ^s} upwards à la Skorohod at the random level

$$a(\theta_\infty)=\frac{2\delta}{2\delta+\sigma^2}\theta_\infty. \tag{6.17}$$

Proposition 6.5. *Let $(Z, \lambda^s, \theta_\infty)$ in $\mathcal{G}^s$. Let $p^s_\infty(dx, \theta)$ be the stochastic kernel from $\mathbb{R}_+$ to $\mathcal{B}_{\mathbb{R}_+}$ defined by*

$$p^s_\infty(dx, \theta) = p_{a(\theta)}(dx), \tag{6.18}$$

where p_a is the family of measures defined by (4.1). Then, the triple $(Z, \theta_\infty, \lambda^s)$ is a correlated stationary strategy so that the consistency condition (3.3) is satisfied. In particular, it holds $\mu^s_\infty(dx, d\theta) = \lim_{t\to\infty} \mathbb{P} \circ (X^{\lambda^s}_t, \theta_\infty)^{-1} = p^s_\infty(dx, \theta)\rho(d\theta)$.

Proof. Since the map $\mathbb{R}_+ \times \mathcal{B}_{\mathbb{R}_+} \ni (a, B) \mapsto p_a(B)$ defines a stochastic kernel from $\mathbb{R}_+$ to $\mathcal{B}_{\mathbb{R}_+}$, the kernel (6.18) is well defined. As in the proof of Proposition 6.2, we show that the joint law of $X^{\lambda^s}_t$ and θ_∞ converges weakly to μ^s_∞ as $t \to \infty$. Indeed, conditionally to $\theta_\infty = \theta$, $X^{\lambda^s}_t$ satisfies the equation (3.1) with ν replaced by $\lambda^{s,\theta}$, where $\lambda^{s,\theta}$ reflects the process $X^{\lambda^s,\theta}$ upwards at the level $a(\theta)$, for ρ-a.e. $\theta \in \mathbb{R}_+$. By Lemma 4.1, the reflected process $X^{\lambda^s,\theta}$ admits $p_{a(\theta)}$ given by (4.1) as the unique invariant distribution. This implies that the regular conditional probability of $X^{\lambda^s}_t$ with respect to θ_∞, that we denote by $p^s_t(dx, \theta)$, converges weakly to $p^s_\infty(dx, \theta)$ as $t \to \infty$ for ρ-a.e. $\theta > 0$. Consistency condition (3.3) follows from the definition of $a(\theta_\infty)$ and (4.2). ☐

Proposition 6.6. *A correlated stationary strategy $(Z, \theta_\infty, \lambda^s)$ in $\mathcal{G}^s$ is a mean-field CCE for the ergodic MFG if and only if the following inequality is satisfied:*

$$c_\beta(\mathbb{E}[\theta^\beta_\infty])^{\frac{1}{1-\alpha}} + c_1\mathbb{E}[\theta_\infty] \leq \tilde{c}_{\alpha+\beta}\mathbb{E}[\theta^{\alpha+\beta}_\infty], \tag{6.19}$$

where c_1 and c_β are given by (6.10) and $\tilde{c}_{\alpha+\beta}$ is given by

$$\tilde{c}_{\alpha+\beta} := \frac{2\delta + \sigma^2}{2\delta + \sigma^2(1-\alpha)} \left(\frac{2\delta}{2\delta + \sigma^2}\right)^\alpha. \tag{6.20}$$

Proof. As in the proof of Proposition 6.3, it is enough to verify that the inequality $\mathsf{J}(\lambda^s, \theta_\infty) \geq \mathsf{J}(U^*, \theta_\infty)$ is equivalent to (6.19). By (6.11), the payoff of the deviating player is equal to

$$\mathsf{J}(U^*, \theta_\infty) = c_\beta(\mathbb{E}[\theta^\beta_\infty])^{\frac{1}{1-\alpha}}.$$

We turn our attention to $\mathsf{J}(\lambda^s, \theta_\infty)$. We note that, conditionally to $\theta_\infty = \theta$, it holds

$$\lim_{T\uparrow\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T (X^{\lambda^s}_t)^\alpha \theta^\beta dt - q\lambda^s_T \middle| \theta_\infty = \theta\right] = C(a(\theta), \theta^\beta), \quad \rho\text{-a.e. } \theta \in \mathbb{R}_+,$$

where $C(a, p)$ is given by (4.3). To see this, it is enough to recall that, by Proposition 6.5, for ρ-a.e. θ in $\mathbb{R}_+$, we have $p^s_t(dx, \theta) \to p^s_\infty(dx, \theta)$ weakly as $t \to \infty$, with $p^s_\infty(dx, \theta)$ given by (6.18). Since, conditionally to $\theta_\infty = \theta$, the control λ^s is a reflection at the barrier $a(\theta)$, we apply [5, Lemma 2.1] to get the equality above. By applying Lemma 4.1 with $a = a(\theta_\infty)$, and exploiting square-integrability of θ_∞, at any time $T > 0$ we can bound the left hand-side with $C(1 + \theta^2_\infty)$, for some positive constant C independent of θ_∞. Therefore, by dominated convergence theorem, we can exchange limit and expectation, to get

$$\begin{aligned}\mathsf{J}(\lambda^s, \theta_\infty) &= \lim_{T\uparrow\infty} \frac{1}{T}\mathbb{E}\left[\mathbb{E}\left[\int_0^T (X^{\lambda^s}_t)^\alpha \theta^\beta_\infty dt - q\lambda^s_T \middle| \theta_\infty\right]\right] = \mathbb{E}[C(a(\theta_\infty), \theta^\beta_\infty)] \\ &= \frac{2\delta + \sigma^2}{2\delta + \sigma^2(1-\alpha)} \left(\frac{2\delta}{2\delta + \sigma^2}\right)^\alpha \mathbb{E}[\theta^{\alpha+\beta}_\infty] - q\delta\mathbb{E}[\theta_\infty].\end{aligned}$$

By rearranging the terms, we have that $\mathsf{J}(U^*, \theta_\infty) \leq \mathsf{J}(\lambda, \theta_\infty)$ if and only if equation (6.19) is satisfied. ☐

Finally, consider a correlated stationary strategy $(Z, \theta_\infty, \lambda^s)$ in the family $\mathcal{G}^s$ and define a correlated strategy profile for the N-player game starting from it: we take $Z = \theta_\infty$ as correlation device, and, for any $i \geq 1$, we consider the policy $\lambda^{i,s} = (\lambda^{i,s}_t)_{t\geq 0^-}$ according to which the state $X^{i,s}$ is reflected upward at the random barrier $a(\theta_\infty)$ given by (6.17). As in the case of a regular recommendation, for each $i \geq 1$, the triple $(X^{i,s}, \lambda^{i,s}, \theta_\infty)$ has the same law as $(X, \lambda^s, \theta_\infty)$, and the processes $(X^{i,s})_{i\geq 1}$ are conditionally independent given θ_∞.

Theorem 6.7 (Approximation of CCEs - singular case)**.** *Let $(Z, \theta_\infty, \lambda^r)$ be a CCE in the class $\mathcal{G}^s$. Let $\boldsymbol{\lambda}^N = (\lambda^{i,s})_{i=1}^N$, with $\lambda^{i,s}$ the policy according to which the state is reflected upward at the random barrier $a(\theta_\infty)$ given by (6.17). Then, for any $c > 0$, the correlated strategy profile $(\theta_\infty, \boldsymbol{\lambda}^N)$ defines an ε_N-CCE for the N-player game within the set of strategies $\mathcal{A}_{N,c}$, with $\varepsilon_N \to 0$ as $N \to \infty$.*

We omit the proof since it is completely analogous to the proof of Theorem 6.4: it is enough to repeat the proof of Theorem 6.4, invoking Lemma 4.1 instead of Lemma 7.2.

Nash Equilibrium for the Ergodic Mean-field Game. Since the MFG considered here does not satisfy the assumptions of [16, Theorem 4.4], we cannot directly deduce existence and uniqueness of NE for the MFG can not be applied. Nevertheless, we have the following result:

Proposition 6.8. *If $\alpha + \beta \neq 1$, there exists a unique Nash equilibrium (ν^*, θ^*) of the ergodic MFG. Moreover, the process ν^* reflects the state process at a barrier a^*, and the pair (a^*, θ^*) is given by*

$$\begin{aligned} a^* &= \left(\frac{2\delta + \sigma^2}{2\delta}\right)^{\frac{\beta}{1-\alpha-\beta}} \left(\frac{2\alpha}{q(2\delta + \sigma^2(1-\alpha))}\right)^{\frac{1}{1-\alpha-\beta}}, \\ \theta^* &= \left(\frac{2\delta + \sigma^2}{2\delta}\right)^{\frac{1-\alpha}{1-\alpha-\beta}} \left(\frac{2\alpha}{q(2\delta + \sigma^2(1-\alpha))}\right)^{\frac{1}{1-\alpha-\beta}}. \end{aligned} \tag{6.21}$$

If $\alpha + \beta = 1$ and the relation

$$1 + \frac{\sigma^2}{2\delta} = \left(\frac{q\delta}{\alpha}\right)^{\frac{1}{1-\alpha}} \left(1 + (1-\alpha)\frac{\sigma^2}{2\delta}\right)^{\frac{1}{1-\alpha}} \tag{6.22}$$

holds, then there exist infinitely many mean-field Nash equilibria given by the pair $(\nu^{a(\theta)}, \theta)$, with ν a reflection at the level $a(\theta) = K\theta^\beta$ and $\theta > 0$; otherwise, it does not exist any Nash equilibrium for the MFG.

Proof. Fix $\theta > 0$. By applying Lemma 4.2 with $p = \theta^\beta$ and $\lambda = 0$, the payoff functional $\mathsf{J}(\nu, \theta)$ is maximized by the strategy ν^θ which reflect the process X^{ν^θ} upwards à la Skorohod at the point $a(\theta)$ given by

$$\hat{a}(\theta) = \left(\frac{2\alpha}{q(2\delta + \sigma^2(1-\alpha))}\right)^{\frac{1}{1-\alpha}} \theta^{\frac{\beta}{1-\alpha}}.$$

In view of (4.2), in order to get the consistency condition (2) satisfied, we impose

$$\theta^* = \frac{2\delta}{2\delta + \sigma^2} K (\theta^*)^{\frac{\beta}{1-\alpha}}. \tag{6.23}$$

If $\alpha + \beta \neq 1$, this is equivalent to

$$(\theta^*)^{\frac{1-\alpha-\beta}{1-\alpha}} = \frac{2\delta + \sigma^2}{2\delta} K.$$

If $\alpha + \beta \neq 1$, the function $g(\theta) := \theta^{\frac{1-\alpha-\beta}{1-\alpha}}$ is always non negative, strictly monotone and with image equal to $\mathbb{R}_+$, which implies that there exists a unique θ^* so that the above equality is verified; by direct computation, it can be verified that θ^* is given by (6.21). Finally, if $\alpha + \beta = 1$, condition (6.23) becomes

$$\theta^* = \frac{2\delta + \sigma^2}{2\delta} K \theta^*.$$

Thus, the MFG admits infinitely many Nash equilibria if $\frac{2\delta+\sigma^2}{2\delta} K = 1$, and none otherwise. By explicit calculations, this is equivalent to

$$\frac{2\delta + \sigma^2}{2\delta} = \left(\frac{q\delta}{\alpha}\right)^{\frac{1}{1-\alpha}} \left(\frac{2\delta + \sigma^2}{2\delta} - \alpha \frac{\sigma^2}{2\delta}\right)^{\frac{1}{1-\alpha}}.$$

By rearranging the terms, we get to (6.22). □

Remark 9. Analogous considerations as in Remark 5 hold for the MFC problem as well. On top of those considerations, we note that, when $\alpha+\beta=1$ and condition (6.22) is not satisfied, we do not have existence of a Nash equilibrium for the ergodic MFG, while the optimality conditions for both classes $\mathcal{G}^r$ and $\mathcal{G}^s$ are still valid. The ultimate reason is that the procedure outlined in Section 6 does not involve the usual two steps scheme used to compute mean-field NEs: first, optimize with a fixed flow of moments and, second, perform a fixed point argument to determine the flow. Actually, we first impose the consistency condition and then we restate the optimality condition.

We notice that the pair (ν^*,θ^*) is a correlated stationary strategy in $\mathcal{G}^s$ with deterministic correlation device. In particular, it satisfies the optimality condition (6.19). As a consequence of Theorem 6.7, we also deduce that every NE for the ergodic MFG induces a sequence of approximate Nash equilibria with vanishing error in the N-player game:

Corollary 6.8.1. *Let (ν^*,θ^*) be a Nash equilibrium for the MFG, as given by Proposition 6.8. For any $i\geq 1$, let $\nu^{i,*}$ be the policy according to which the state is reflected upward at the random barrier a^* given by (6.21). Then, for any $c>0$, the open-loop strategy profile $\boldsymbol{\nu}^{*,N}=(\nu^{i,*})_{i=1}^N$ defines an ε_N-NE for the N-player game within the set of strategies $\mathcal{A}_{N,c}$, with $\varepsilon_N\to 0$ as $N\to\infty$.*

Proof. It is enough to notice that, when starting from the NE (ν^*,θ^*) the recommendation $\boldsymbol{\lambda}^N$ defined in Theorem 6.7 is actually an open-loop strategy profile for the N-player game. □

7. Numerical Illustrations

In this Section, we numerically illustrate our previous findings. In particular, we exhibit possible choices of distribution of θ_∞ so that the correlated stationary strategies $(Z,\theta_\infty,\lambda^r)$ in $\mathcal{G}^r$ and $(Z,\theta_\infty,\lambda^s)$ in $\mathcal{G}^s$ are mean-field CCEs, i.e., according to Propositions 6.3 and 6.6, the inequalities (6.9) and (6.19) respectively are verified.

We suppose that θ_∞ is distributed as a Gamma with $u>0$ and scale parameter $v>0$. Then, for any $k\geq 0$ the k-th moment of $\theta_\infty\sim\Gamma(u,v)$ is given by

$$\mathbb{E}[\theta_\infty^k]=\frac{1}{\Gamma(u)v^u}\int_0^\infty x^k x^{u-1}e^{-\frac{x}{v}}dx=\frac{\Gamma(u+k)}{\Gamma(u)}v^k.$$

By assuming Gamma distribution on θ_∞, the optimality conditions (6.9) and (6.19) for regular and singular recommendation become, respectively,

$$c_\beta\left(\frac{\Gamma(\beta+u)}{\Gamma(u)}\right)^{\frac{1}{1-\alpha}}v^{\frac{\beta}{1-\alpha}}\leq c_{\alpha+\beta}\frac{\Gamma(\alpha+\beta+u)}{\Gamma(u)}v^{\alpha+\beta}-c_1uv, \tag{7.1}$$

$$c_\beta\left(\frac{\Gamma(\beta+u)}{\Gamma(u)}\right)^{\frac{1}{1-\alpha}}v^{\frac{\beta}{1-\alpha}}\leq \tilde{c}_{\alpha+\beta}\frac{\Gamma(\alpha+\beta+u)}{\Gamma(u)}v^{\alpha+\beta}-c_1uv, \tag{7.2}$$

Given the non-linear structure of the optimality inequalities and the intricate dependence on the parameters in the constants c_1, c_β, $c_{\alpha+\beta}$ and $\tilde{c}_{\alpha+\beta}$, we limit ourselves to a specific choice of the parameters. For the sake of illustrations, we fix $\delta=0.1$ and $\sigma=0.2$. Notice that this choice satisfies Assumption **D**. We also set $q=2$.

The case $\alpha+\beta<1$. In this case, there exist both a unique mean-field NE and a unique optimal control for the MFC problem. For the sake of comparison with the payoffs of the NE and the MFC solution, we are also interested in finding values of $(u,v)\in\mathbb{R}_+^2$ so that the reward of the associated CCE is higher than the reward of the NE. Therefore, we pair equations (7.1) and (7.2) with, respectively,

$$c_{\alpha+\beta}\frac{\Gamma(\alpha+\beta+u)}{\Gamma(u)}v^{\alpha+\beta}-c_1uv\geq\tilde{c}_{\alpha+\beta}(\theta^*)^{\alpha+\beta}-c_1\theta^*, \tag{7.3}$$

$$\tilde{c}_{\alpha+\beta}\frac{\Gamma(\alpha+\beta+u)}{\Gamma(u)}v^{\alpha+\beta}-c_1uv\geq\tilde{c}_{\alpha+\beta}(\theta^*)^{\alpha+\beta}-c_1\theta^*. \tag{7.4}$$

Figure 1 shows that there exist infinitely many mean-field CCEs both in $\mathcal{G}^r$ and $\mathcal{G}^s$ which yield an higher reward than the Nash equilibrium (ν^*,θ^*). Here, we set $\alpha=0.3$ and $\beta=0.5$.

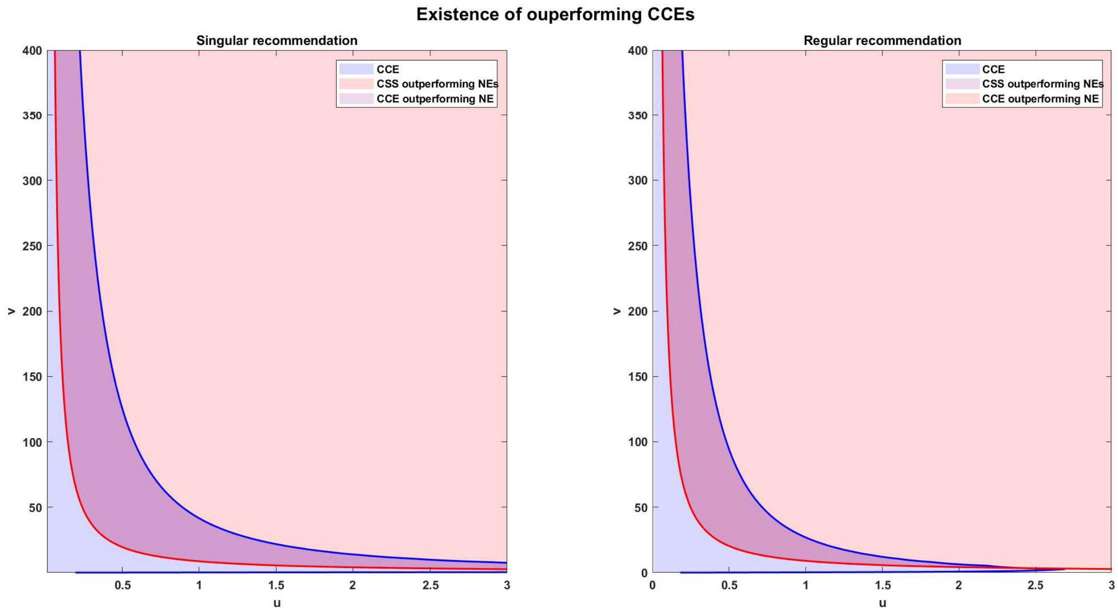

FIGURE 1. Values of the parameters $(u,v)\in\mathbb{R}^2_+$ so that the $(Z,\lambda^s,\theta_\infty)\in\mathcal{G}^s$ (on the left) and $(Z,\lambda^r,\theta_\infty)\in\mathcal{G}^r$ (on the right), $\theta_\infty\sim\Gamma(u,v)$ is a mean-field CCE outperforming the NE.

Figure 2 shows the reward associated to those mean-field CCEs in $\mathcal{G}^r$ and $\mathcal{G}^s$ that yield an higher reward than the Nash equilibrium (ν^*,θ^*). The improvement on the Nash equilibrium is $\approx 17\%$ of the reward yielded by the mean-field control solution $\hat{\nu}$ in the singular case, and $\approx 12\%$ in the regular case. We notice that the payoff associated to the Nash equilibrium for the ergodic MFG is strictly less than the reward given by the solution of the MFC problem. This can be directly deduced from the fact that the the stationary mean $\hat{\theta}$ associated to the MFC solution $\hat{\nu}$ is the unique maximizer of the function $f(\theta)$ given by (5.6), which can be equivalently expressed as $f(\theta)=\tilde{c}_{\alpha+\beta}\theta^{\alpha+\beta}-c_1\theta$. Since by Proposition 6.8 the value of the ergodic MFG at the Nash equilibrium (ν^*,θ^*) can be expressed as $f(\theta^*)$, we deduce

$$\mathsf{J}(\nu^*,\theta^*)=f(\theta^*)<f(\hat{\theta})=\mathsf{J}(\hat{\nu},\mathbb{E}[X^{\hat{\nu}}_\infty]). \tag{7.5}$$

The reward of the MFC solution $\mathsf{J}(\hat{\nu},\mathbb{E}[X^{\hat{\nu}}_\infty])$ appears to be an unattainable upper bound for the set of mean-field CCEs payoffs. While we limit to empirically observe this phenomenon, we point out that it is widely expected and that it is coherent with the findings of [12] for linear-quadratic mean-field games.

The case $\alpha+\beta=1$**.** In this case, by Theorem 6.8, either there does not exist any mean-field NE or there exist infinitely many, depending on whether the relation (6.22) is satisfied. As noticed in Remark 9, the optimality inequalities (6.9) and (6.19) are still valid. By imposing $\beta=1-\alpha$,

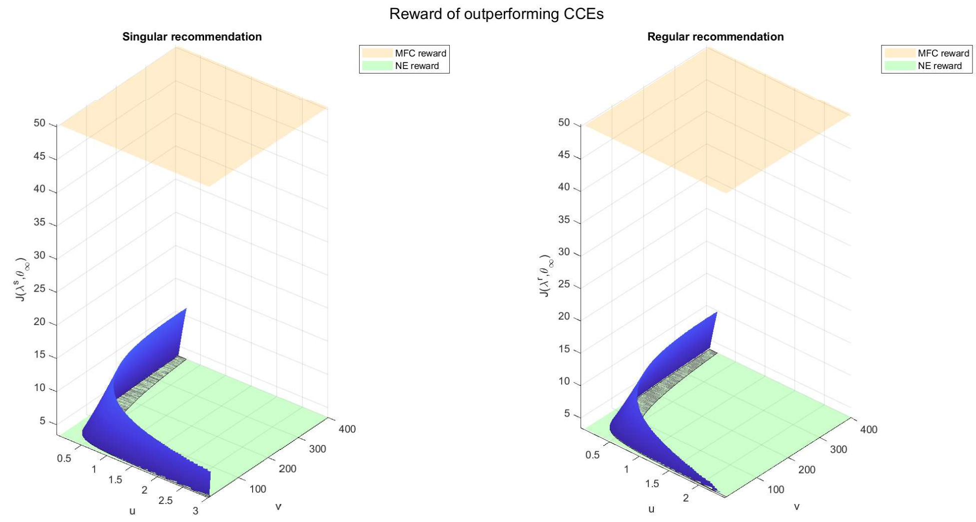

FIGURE 2. Reward associated to mean-field CCEs $(Z, \lambda^s, \theta_\infty) \in \mathcal{G}^s$ (on the left) and $(Z, \lambda^r, \theta_\infty) \in \mathcal{G}^r$ (on the right) which outperform the reward of the mean-field NE, when $\theta_\infty \sim \Gamma(u, v)$, $(u, v) \in \mathbb{R}^2_+$.

and noticing that the parameter v is not anymore relevant in the inequalities, equations (7.1) and (7.2) can be written in terms of u only:

$$c_\beta \left(\frac{\Gamma(1-\alpha+u)}{\Gamma(u)} \right)^{\frac{1}{1-\alpha}} \leq \left(c_{\alpha+\beta} \Gamma(u) - c_1 \right) u, \tag{7.6}$$

$$c_\beta \left(\frac{\Gamma(1-\alpha+u)}{\Gamma(u)} \right)^{\frac{1}{1-\alpha}} \leq \left(\tilde{c}_{\alpha+\beta} \Gamma(u) - c_1 \right) u. \tag{7.7}$$

We observe that there exist maximal values u_r^* and u_s^*, depending on α, so that the inequalities (7.6) and (7.7) are verified by any $0 < u \leq u_r^*$ and $0 < u \leq u_s^*$, respectively. Figure 3 plots such maximal values u_r^* and u_s^* as functions of $\alpha \in (0, 1)$. Therefore, we observe existence of mean-field CCEs even in the case in which there does not exist any mean-field NE. We notice that, when considering correlated stationary strategies $(Z, \lambda^s, \theta_\infty)$, there exist a value $\bar{\alpha}$ so that that the inequality (7.6) is verified for any $u > 0$, i.e. $u^* = \infty$. It can be numerically shown that such value $\bar{\alpha}$ is the unique solution of (6.22), for fixed δ, σ and q. We can explain this phenomenon as follows: for $\alpha = \bar{\alpha}$, by Theorem 6.8, for any $\theta > 0$ the pair $(\nu^{a(\theta)}, \theta)$ is a mean-field NE, where $\nu^{a(\theta)}$ is the policy that reflects the process $X^{\nu^{a(\theta)}}$ upwards at the level $a(\theta) = \frac{2\delta}{2\delta+\sigma^2}\theta$. Therefore, any correlated stationary strategy $(Z, \lambda^s, \theta_\infty) \in \mathcal{G}^s$ is just a randomization, or a mixture, of mean-field NE, since λ^s reflects the process X^{λ^s} at the same barrier $a(\theta_\infty) = \frac{2\delta}{2\delta+\sigma^2}\theta_\infty$. To put in other terms, the pair $(a(\theta_\infty), \theta_\infty)$ is supported on the set of mean-field NEs. This implies that the optimality condition is satisfied by any θ_∞ so that the optimality inequality (6.19) holds true, and so by any $(Z, \lambda^s, \theta_\infty) \in \mathcal{G}^s$.

The case $\alpha + \beta > 1$. The case $\alpha + \beta > 1$ is completely analogous to the case $\alpha + \beta < 1$. We just observe that, in this case, the MFC problem is ill-posed, in the sense of Theorem 5.1 and therefore a-priori we do not have any upper-bound on the set of mean-field CCEs payoffs.

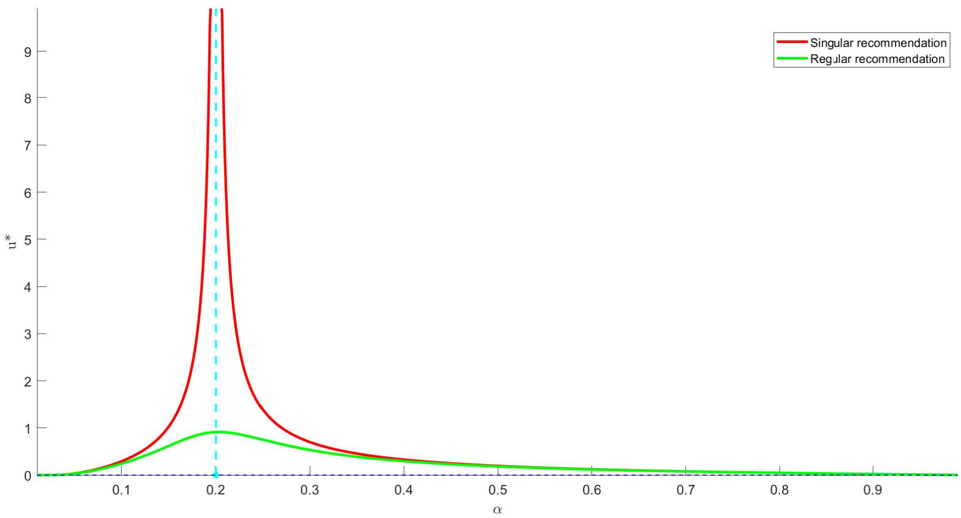

FIGURE 3. Value of u^* as α varies in $[0,1]$, both for regular and singular recommendations. The blue dashed line is located at the value of $\alpha = \bar{\alpha}$ which satisfies (6.22) for fixed δ, σ and q.

APPENDIX

Control-theoretic Results.

Proof of Lemma 4.1. Point i) follows from direct calculations. As for point ii), the solution of the Skorohod problem follows from standard arguments (see, e.g. [41, Proposition 3.6.16]). As for ergodicity, note that the derivative of the speed measure of the process X^{ν^a} is given by $m'(x) = \frac{2}{\sigma^2} x^{-\frac{2\delta}{\sigma^2}} \mathbb{1}_{[a,\infty)}(x)$, which is integrable over $[0,\infty)$ for any $a > 0$. By [11, Paragraph 36], the process X^{ν^a} is ergodic and admits $\frac{1}{m([a,\infty))} m'(x)dx = p_a(dx)$ given by (4.1) as the unique invariant distribution.

As for iii), set $L = 1 + a$ and observe that, since $a < L$ and the control ν^a never acts when the process lies in the region $\{x : \, x > a\}$, it holds $\text{supp}(d\nu^a) \cap \{X^{\nu^a} \geq L\} = \emptyset$, $\mathbb{P}$-a.s.. The proof of [16, Lemma 2] implies that

$$\mathbb{E}[|X_t^{\nu^a}|^2] \leq 2L^2 + \mathbb{E}[\xi^2].$$

By definition of L and Assumption **D**, the estimate follows. Finally, By (3.1), we have, for any $T > 0$

$$\nu_T^a = X_T^{\nu^a} - \xi + \delta \int_0^T X_s^{\nu^a} ds - \int_0^T \sigma X_s^{\nu^a} dW_s.$$

By taking the expectation, applying Jensen inequality and taking advantage of Itô isometry, we find

$$\frac{1}{T^2}\mathbb{E}[|\nu_T^a|] \leq \frac{c}{T^2}\left(1 + \mathbb{E}[|X_T^{\nu^a}|^2] + \mathbb{E}\left[\left(\int_0^T X_s^{\nu^a} ds\right)^2\right] + \mathbb{E}\left[\left(\int_0^T X_s^{\nu^a} dW_s\right)^2\right]\right)$$
$$\leq \frac{c}{T^2}\left(1 + (T + T^2)\sup_{T\geq 0}\mathbb{E}[(X_T^{\nu^a})^2]\right) \leq c(1 + a^2),$$

where last inequality follows from previous estimate. □

Proof of Lemma 4.2. We sketch the proof, which is essentially similar to the proof of [30, Theorem 2]. Recall from 4.4 the definition of the function $g(x, p, \lambda)$. Let $\mathcal{T}$ be the set of $\mathbb{F}$-stopping times. Consider the auxiliary optimal stopping problem

$$u(x,p,\lambda) := \inf_{\tau\in\mathcal{T}} \mathbb{E}_x\bigg[\int_0^\tau e^{-\delta t} g_x(\widehat{X}_t,p,\lambda)dt + q e^{-\delta\tau}\bigg], \tag{7.8}$$

where $\hat{X} = (\hat{X}_t)_{t\geq 0}$ is defined by

$$d\hat{X}_t = (-\delta+\sigma^2)\hat{X}_t dt + \sigma \hat{X}_t dW_t,$$

and $\mathbb{E}_x[\cdot]$ denotes $\mathbb{E}[\cdot|\widehat{X}_0 = x]$, $x\in\mathbb{R}_+$. Let $\hat{\phi}_0$ is the non-increasing fundamental solution of

$$\frac{1}{2}\sigma^2 x^2 u_{xx}(x) + (-\delta+\sigma^2)x u_x(x) - \delta u(x) = 0,$$

and let $\hat{m}'$ be the density of the speed measure $\hat{m}$ of the process $\hat{X}$. By the same reasoning of [4, Theorem 5], it can be shown that, if there exists a unique $a^* = a^*(p,\lambda) > 0$ solution to

$$\int_{a^*}^{+\infty} \hat{\phi}_0(y)\left(\alpha p y^{\alpha-1} - (q\delta-\lambda)\right)\hat{m}'(y)dy = 0, \tag{7.9}$$

then the value function $u(\cdot,p,\lambda)$ is $C^1(\mathbb{R}_+)$ with $u_{xx}(\cdot,p,\lambda)\in L^\infty_{\text{loc}}(\mathbb{R}_+)$, and that the optimal stopping time is given by $\hat{\tau}(x,p,\lambda) := \inf\{t\geq 0\,|\,\hat{X}_t \leq a^*(p,\lambda)\}$.

By explicit calculations, we have $\hat{\phi}_0(y) = y^{-1}$ and $\hat{m}'(y) = \frac{2}{\sigma^2}y^{-\frac{2\delta}{\sigma^2}}$, so that (7.9) becomes

$$\int_{a^*}^{+\infty} \left(\alpha p y^{\alpha-1} - (q\delta-\lambda)\right) y^{-\frac{2\delta}{\sigma^2}-1} dy = 0. \tag{7.10}$$

Since, by assumption, $q\delta - \lambda > 0$, there exists a unique solution $a^*(p,\lambda)$ given by (4.6). The proof can be then completed by the same methods of Step B.1 in [30, Appendix B]. ☐

Proof of Lemma 4.3. We deal with the case $q' < \infty$. The case $q' = \infty$ is completely analogous. We start by proving (a). Let $\hat{\nu}$ be optimal for the control problem with dynamics (3.1) and payoff functional $\tilde{J}(\cdot,p,\lambda)$. Recall the definition of the function $g(x,p,\lambda)$ in (4.4). For any $\nu\in\mathcal{A}_{mf}$, $\varepsilon\in(0,\frac{1}{2}]$, set $\nu^\varepsilon = \varepsilon\nu + (1-\varepsilon)\hat{\nu}$. Set

$$f(\varepsilon,T) = \frac{1}{\varepsilon}\left(\frac{1}{T}\mathbb{E}\bigg[\int_0^T g(X_t^{\nu^\varepsilon},p,\lambda)dt - q\nu_T^\varepsilon\bigg] - \frac{1}{T}\mathbb{E}\bigg[\int_0^T g(X_t^{\hat{\nu}},p,\lambda)dt - q\hat{\nu}_T\bigg]\right). \tag{7.11}$$

For any $\varepsilon\in(0,\frac{1}{2}]$ and $T>0$, it holds

$$\begin{aligned} f(\varepsilon,T) &= \frac{1}{\varepsilon}\frac{1}{T}\left(\mathbb{E}\bigg[\int_0^T\Big(\int_0^1 g_x(X_t^{\hat{\nu}} + \tau(X_t^{\nu^\varepsilon} - X_t^{\hat{\nu}}),p,\lambda)d\tau\Big)(X_t^{\nu^\varepsilon} - X_t^{\hat{\nu}})dt - q(\nu_T^\varepsilon - \hat{\nu}_T)\bigg]\right) \\ &= \frac{1}{T}\left(\mathbb{E}\bigg[\int_0^T\Big(\int_0^1 g_x(X_t^{\hat{\nu}} + \tau(X_t^{\nu^\varepsilon} - X_t^{\hat{\nu}}),p,\lambda)d\tau\Big)(X_t^{\nu} - X_t^{\hat{\nu}})dt - q(\nu_T - \hat{\nu}_T)\bigg]\right). \end{aligned}$$

We claim

$$\lim_{\varepsilon\downarrow 0} f(\varepsilon,T) = \frac{1}{T}\mathbb{E}\bigg[\int_0^T g_x(X_t^{\hat{\nu}},p,\lambda)(X_t^\nu - X_t^{\hat{\nu}})dt - q(\nu_T - \hat{\nu}_T)\bigg] \tag{7.12}$$

uniformly in T. Indeed,

$$\begin{aligned} &\left|f(\varepsilon,T) - \frac{1}{T}\mathbb{E}\bigg[\int_0^T g_x(X_t^{\hat{\nu}},p,\lambda)(X_t^\nu - X_t^{\hat{\nu}})dt - q(\nu_T - \hat{\nu}_T)\bigg]\right| \\ &\leq \left|\frac{1}{T}\mathbb{E}\bigg[\int_0^T (X_t^{\hat{\nu}} - X_t^\nu)\int_0^1\Big(g_x(X_t^{\nu^\varepsilon} + \tau(X_t^{\hat{\nu}} - X_t^{\nu^\varepsilon}),p,\lambda) - g_x(X_t^{\hat{\nu}},p,\lambda)\Big)d\tau dt\bigg]\right| \\ &\leq \frac{1}{T}\mathbb{E}\bigg[\int_0^T |X_t^{\hat{\nu}} - X_t^\nu|\int_0^1\Big|g_x(X_t^{\nu^\varepsilon} + \tau(X_t^{\hat{\nu}} - X_t^{\nu^\varepsilon}),p,\lambda) - g_x(X_t^{\hat{\nu}},p,\lambda)\Big|d\tau dt\bigg]. \end{aligned}$$

By continuity of $g_x(x,p,\lambda)$ in x, the inner integral converges to 0 as $\varepsilon \downarrow 0$ for any $t \geq 0$, $\mathbb{P}$-a.s.. By linearity of the dynamics (3.1), since X_t^ν is positive for any t, it holds

$$X_t^{\nu^\varepsilon} + \tau(X_t^{\hat{\nu}} - X_t^{\nu^\varepsilon}) = \tau X_t^{\hat{\nu}} + (1-\tau)X_t^{\nu^\varepsilon} \geq \frac{1}{2}X_t^{\nu^\varepsilon} = \frac{1}{2}(\varepsilon X_t^\nu + (1-\varepsilon)X_t^{\hat{\nu}} \geq \frac{1}{2}X_t^{\hat{\nu}}) \geq \frac{1}{4}X_t^{\hat{\nu}}.$$

Since $|g_{xx}(y,p,\lambda)| = \alpha(1-\alpha)px^{\alpha-2} \leq c|x|^{\alpha-2}$ for any $y \geq x$, $g_x(y,p,\lambda)$ is Lipschitz on $[x,+\infty)$ with Lipschitz constant $c|x|^{\alpha-2}$. Then, it follows
(7.13)

$$\begin{aligned}
&|f(\varepsilon,T) - g(T)| \leq \frac{c}{T}\mathbb{E}\left[\int_0^T |X_t^{\hat{\nu}} - X_t^\nu||X_t^{\hat{\nu}}|^{\alpha-2}\int_0^1 \left|X_t^{\nu^\varepsilon} + \tau(X_t^{\hat{\nu}} - X_t^{\nu^\varepsilon}) - X_t^{\hat{\nu}}\right| d\tau dt\right] \\
&\leq c\frac{1}{T}\mathbb{E}\left[\int_0^T |X_t^{\hat{\nu}} - X_t^\nu| \cdot |X_t^{\nu^\varepsilon} - X_t^{\hat{\nu}}||X_t^{\hat{\nu}}|^{\alpha-2}dt\right] \\
&\leq c\left(\sup_{T>0}\frac{1}{T}\mathbb{E}\left[\int_0^T |X_t^{\hat{\nu}}|^{2q}dt\right] + \sup_{T>0}\frac{1}{T}\mathbb{E}\left[\int_0^T |X_t^\nu|^{2q}dt\right]\right)\sup_{T>0}\left(\frac{1}{T}\mathbb{E}\left[\int_0^T |(X_t^{\hat{\nu}})^{\alpha-2}|^{q'}dt\right]\right)^{\frac{1}{q'}}\varepsilon,
\end{aligned}$$

where we used Hölder's inequality together with conditions (4.7) and (4.8). On the other hand, by taking the limit with respect to T, it holds

$$\begin{aligned}
&\varliminf_{T\uparrow\infty} f(\varepsilon,T) \leq \frac{1}{\varepsilon}\Bigg(\varliminf_{T\uparrow\infty}\frac{1}{T}\mathbb{E}\left[\int_0^T g(X_t^{\nu^\varepsilon},p,\lambda)dt - q\nu_T^\varepsilon\right] \\
&- \varliminf_{T\uparrow\infty}\frac{1}{T}\mathbb{E}\left[\int_0^T g(X_t^{\hat{\nu}},p,\lambda)dt - q\hat{\nu}_T\right]\Bigg) = \frac{1}{\varepsilon}(\tilde{\mathrm{J}}(\nu^\varepsilon) - \tilde{\mathrm{J}}(\hat{\nu})) \leq 0,
\end{aligned}$$

by using the inequality $\varliminf_n a_n - \varliminf_n b_n \geq \varliminf_n(a_n - b_n)$ and by optimality, for any $\varepsilon \in (0, \frac{1}{2}]$. Lemma 7.5 then implies

$$\varliminf_{T\uparrow\infty}\frac{1}{T}\mathbb{E}\left[\int_0^T g_x(X_t^{\hat{\nu}},p,\lambda)(X_t^\nu - X_t^{\hat{\nu}})dt - q(\nu_T - \hat{\nu}_T)\right] \leq 0.$$

This concludes the proof of point (a).

As for point (b), assume that both (4.11) and (a) hold. Then, by using the inequality $\varlimsup_n z_n - \varliminf_n x_n \geq \varlimsup_n(z_n - y_n)$ (see [40, Equation (4.25)]) and concavity of g jointly in x, it holds

$$\begin{aligned}
&\tilde{\mathrm{J}}(\hat{\nu},p,\lambda) - \tilde{\mathrm{J}}(\nu,p,\lambda) \geq \varlimsup_{T\uparrow\infty}\frac{1}{T}\mathbb{E}\left[\int_0^T \left(g(X_t^\nu,p,\lambda) - g(X_t^{\hat{\nu}},p,\lambda)\right)dt - q(\hat{\nu}_T - \nu_T)\right] \\
&\geq \varlimsup_{T\uparrow\infty}\frac{1}{T}\mathbb{E}\left[\int_0^T g_x(X_t^\nu,p,\lambda)(X_t^{\hat{\nu}} - X_t^\nu)dt - q(\hat{\nu}_T - \nu_T)\right] \geq 0,
\end{aligned}$$

which implies that $\hat{\nu}$ is optimal within $\mathcal{A}_s$. If instead we assume (4.12), the claim follows from sublinearity of the inferior limit. □

Proof of Lemma 5.2. We verify the identity by explicit calculations. First notice that

$$\lambda(\theta) = q_u\delta - \left(\frac{2\delta+\sigma^2}{2\delta}\right)^{1-\alpha}\frac{2\delta\alpha}{2\delta+\sigma^2(1-\alpha)}\theta^{\alpha+\beta-1} = q\delta - \alpha\varliminf_{T\uparrow\infty}\frac{1}{T}\mathbb{E}\left[\int_0^T (X_t^{\nu^{a(\theta)}})^\alpha\theta^{\beta-1}\right].$$

On the other hand, we have

$$\begin{aligned}
f'(\theta) &= -q\delta + (\alpha+\beta)\frac{2\delta+\sigma^2}{2\delta+\sigma^2(1-\alpha)}\left(\frac{2\delta}{2\delta+\sigma^2}\right)^\alpha\theta^{\alpha+\beta-1} \\
&= -q\delta + (\alpha+\beta)\varliminf_{T\uparrow\infty}\frac{1}{T}\mathbb{E}\left[\int_0^T (X_t^{\nu^{a(\theta)}})^\alpha\theta^{\beta-1}\right].
\end{aligned}$$

By summing the two terms, we find

$$f'(\theta)+\lambda(\theta)=\beta\lim_{T\uparrow\infty}\frac{1}{T}\mathbb{E}\left[\int_0^T (X_t^{\nu^{a(\theta)}})^\alpha\theta^{\beta-1}\right]=\lim_{T\uparrow\infty}\frac{1}{T}\mathbb{E}\left[\int_0^T \pi_\theta(X_t^{\nu^{a(\theta)}},\theta)dt\right].$$

□

Integrability Results.

Lemma 7.1. *Let $\theta>0$ and let $m'_{\infty,\theta}(x)$ be given by* (6.5)*. For any $k\geq 0$, the $\int_0^\infty x^k m'_{\infty,\theta}(x)dx$ is finite if and only if $2\delta-(k-1)\sigma^2>0$. If so, it holds*

$$\int_0^\infty x^k m'_{\infty,\theta}(x)dx=\theta^{k-\frac{2\delta}{\sigma^2}-1}\left(\frac{2\delta}{\sigma^2}\right)^{k-1-\frac{2\delta}{\sigma^2}}\Gamma\left(\frac{2\delta}{\sigma^2}-k+1\right). \tag{7.14}$$

Proof. Let $k\geq 0$ so that $2\delta-(k-1)\sigma^2>0$. By setting $z={x}/{\theta}$ in the integral in (7.14), we have:

$$\int_{\mathbb{R}_+} x^k x^{-\frac{2\delta}{\sigma^2}-2}\exp\left(-\frac{2\delta}{\sigma^2}\frac{\theta}{x}\right)dx=\theta^{k-\frac{2\delta}{\sigma^2}-1}\int_0^\infty z^{k-\frac{2\delta}{\sigma^2}-2}\exp\left(-\frac{2\delta}{\sigma^2}\frac{1}{z}\right)dz,$$

and, by making the change of variables $t={2\delta}/{\sigma^2}\cdot{1}/{z}$, we have

$$\int_0^\infty z^{k-\frac{2\delta}{\sigma^2}-2}\exp\left(-\frac{2\delta}{\sigma^2}\frac{1}{z}\right)dz=\left(\frac{2\delta}{\sigma^2}\right)^{k-1-\frac{2\delta}{\sigma^2}}\int_0^\infty t^{\frac{2\delta}{\sigma^2}-k}e^{-t}dt=\left(\frac{2\delta}{\sigma^2}\right)^{k-1-\frac{2\delta}{\sigma^2}}\Gamma\left(\frac{2\delta}{\sigma^2}-k+1\right),$$

which yields (7.14). □

Lemma 7.2. *For any $\theta>0$, let $X^\theta=(X_t^\theta)_{t\geq 0}$ be the solution of*

$$dX_t^\theta=\delta(\theta-X_t^\theta)dt+\sigma X_t^\theta dW_t,\quad X_0^\theta=\xi.$$

There exists a constant C independent of θ so that it holds

$$\sup_{t\geq 0}\mathbb{E}[(X_t^\theta)^2]\leq C(1+\theta^2).$$

Proof. By Itô formula, we have

$$d(X_t^\theta)^2=\left[-(2\delta-\sigma^2)(X_t^\theta)^2+2\delta\theta(X_t^\theta)\right]dt+2\sigma(X_t^\theta)^2dW_t,\quad (X_0^\theta)^2=\xi^2,$$

so that, by taking the expectation, we get

$$\mathbb{E}[(X_t^\theta)^2]=e^{-(2\delta-\sigma^2)t}\left(\mathbb{E}[\xi^2]+2\delta\theta\int_0^t\mathbb{E}[X_s^\theta]e^{-(2\delta-\sigma^2)s}ds\right).$$

By (6.8), we have $\mathbb{E}[X_s^\theta]\leq C(1+\theta)$ for any $s\geq 0$ and, by Assumption **D**, $2\delta-\sigma^2>0$. Thus, it holds

$$\begin{aligned}\mathbb{E}[(X_t^\theta)^2]&\leq e^{-(2\delta-\sigma^2)t}\left(\mathbb{E}[\xi^2]+C(1+\theta)2\delta\theta\int_0^t e^{(2\delta-(n-1)\sigma^2)s}ds\right)\\&\leq C(1+\theta^2)+e^{-(2\delta-\sigma^2)t}\left(\mathbb{E}[\xi^2]-C_1(1+\theta^2)\right)\leq C(1+\theta^2).\end{aligned}$$

This completes the proof. □

Auxiliary Results for the Backward Convergence Problem.

Lemma 7.3. *Suppose $\mathbb{E}[\theta_\infty^2]<\infty$. Let $(X^i)_{i=1}^N$ be i.i.d. as Y conditionally to θ_∞, with either $Y=X^{\lambda^r}$ or $Y=X^{\lambda^s}$, and let $\kappa(dx,\theta)$ be equal to $p^r_\infty(dx,\theta)$ given by* (6.6) *or equal to $p^s_\infty(dx,\theta)$ given by* (6.18)*. Let $\theta_t^{-i,N}=\frac{1}{N-1}\sum_{j\neq i}X_t^j$, for $t\geq 0$, $1\leq i\leq N$. Then, it holds*

$$\lim_{T\uparrow\infty}\frac{1}{T}\int_0^T\mathbb{E}\left[\left|(\theta_t^{-i,N})^\beta-\theta_\infty^\beta\right|^2\Big|\theta_\infty\right]dt=\int_{\mathbb{R}_+^{N-1}}\left|\Big(\frac{1}{N-1}\sum_{j\neq i}x_j\Big)^\beta-\theta_\infty^\beta\right|^2\bigotimes_{j\neq i}\kappa(dx_j,\theta_\infty),\ \mathbb{P}\text{-a.s.} \tag{7.15}$$

Proof. Suppose without loss of generality that $(\Omega, \mathcal{F}, \mathbb{P})$ is a Polish space (if it is not, we work on the canonical space). Consider the regular conditional probability of $\mathbb{P}$ given θ_∞. Denote the regular conditional probability of $\mathbb{P}$ given $\theta_\infty = \theta$ by $\mathbb{P}^\theta(\cdot) = \mathbb{P}(\cdot\,|\theta_\infty = \theta)$, and by $\mathbb{E}^\theta[\cdot]$ the expectation with respect to the probability measure $\mathbb{P}^\theta$. Conditionally to $\theta_\infty = \theta$, we have that $(X^j)_{j\neq i}$ are i.i.d. as Y; thus, in particular, the process $\theta^{-i,N}$ is a positively recurrent regular diffusion with ergodic measure $\bigotimes_{j\neq i} \kappa(dx_j, \theta)$ (see, e.g. [39, Lemmata 23.17-19]). By the ergodic ratio theorem, it holds

$$\lim_{T\uparrow\infty} \frac{1}{T} \int_0^T \left|\left(\theta_t^{-i,N}\right)^\beta - \theta^\beta\right|^2 dt = \int_{\mathbb{R}_+^{N-1}} \left|\Big(\frac{1}{N-1}\sum_{j\neq i} x_j\Big)^\beta - \theta^\beta\right|^2 \bigotimes_{j\neq i} \kappa(dx_j, \theta), \quad \mathbb{P}^\theta\text{-a.s..} \tag{7.16}$$

Therefore, convergence in probability with respect to the probability measure $\mathbb{P}^\theta$ holds as well. In order to get convergence in L^1 as well, we show that the family of random variables in the left hand-side of (7.16) is uniformly integrable. By, e.g., [39, Lemma 4.12], this implies that we can take the expectation with respect to $\mathbb{P}^\theta$ and exchange the limit and expectation, to get

$$\overline{\lim_{T\uparrow\infty}} \frac{1}{T} \int_0^T \mathbb{E}^\theta \left[\left|\left(\theta_t^{-i,N}\right)^\beta - \theta_\infty^\beta\right|^2\right] dt = \int_{\mathbb{R}_+^{N-1}} \left|\Big(\frac{1}{N-1}\sum_{j\neq i} x_j\Big)^\beta - \theta^\beta\right|^2 \bigotimes_{j\neq i} \kappa(dx_j, \theta),$$

holds for ρ-a.e. $\theta > 0$, which is equivalent to (7.15). To verify uniform integrability, take $r = 1/\beta > 1$. By standard estimates, we have

$$\begin{aligned}
\mathbb{E}^\theta\left[\left|\frac{1}{T}\int_0^T \left|\left(\theta_t^{-i,N}\right)^\beta - \theta^\beta\right|^2 dt\right|^r\right] &\leq \frac{2^{2r-1}}{T}\int_0^T \left(\theta^{2r\beta} + \mathbb{E}^\theta\left[\left(\theta_t^{-i,N}\right)^{2r\beta}\right]\right) dt \\
&\leq C\left(\theta^2 + \sup_{t\geq 0} \mathbb{E}^\theta[|Y_t|^2]\right) \leq C(1+\theta^2),
\end{aligned}$$

where last inequality holds true by Lemma 7.2 if $Y = X^{\lambda^r}$, and by Lemma 4.1 if $Y = X^{\lambda^s}$. This implies that such family is bounded in L^r-norm, thus, since $r > 1$, uniformly integrable. □

Lemma 7.4. *Let $\kappa(dx, \theta)$ be either equal to $p^r_\infty(dx, \theta)$ given by (6.6) or equal to $p^s_\infty(dx, \theta)$ given by (6.18). Then, for any $\theta > 0$, it holds*

$$\lim_{N\to\infty} \int_{\mathbb{R}_+^{N-1}} \left|\Big(\frac{1}{N-1}\sum_{j\neq i} x_j\Big)^\beta - \theta^\beta\right|^2 \bigotimes_{j\neq i} \kappa(dx_j, \theta) = 0. \tag{7.17}$$

Proof. Let $(Y_i)_{i\geq 1}$ be a sequence of i.i.d random variables with law $\kappa(dx, \theta)$, defined on some probability space $(X, \mathcal{X}, \mu)$. Up to reindexing, the integral in (7.17) can be expressed in terms of the expectation of a function of $\bar{Y}^n = 1/n \sum_{i=1}^n Y_i$. Since $(Y^i)_{i\geq 1}$ are i.i.d. as $\kappa(dx, \theta)$ and integrable, we have

$$(\bar{Y}^n)^\beta \to \theta^\beta \quad \mu\text{-a.s.}$$

by the strong law of large number and continuity of the function $x \mapsto x^\beta$. Therefore, convergence in probability holds as well. To conclude, we show that the sequence $(|(\bar{Y}^n)^\beta - \theta^\beta|^2)_{n\geq 1}$ is uniformly bounded in L^r-norm, for some $r > 1$, which implies that the sequence is uniformly integrable, and thus the convergence in L^2-norm holds by, e.g., [39, Proposition 4.12]. Let $r = 1/\beta > 1$. By standard estimates, we have

$$\mathbb{E}[(|(\bar{Y}^n)^\beta - \theta^\beta|^2)^r] \leq 2^{2r-1}\left(\theta^2 + \mathbb{E}[(\bar{Y}^n)^2]\right) \leq C\left(\theta^2 + \mathbb{E}[Y_1^2]\right),$$

where in the last inequality we used the identical distribution of the sequence $(Y_i)_{i\geq 1}$. The expectation of Y_1^2 is finite by Lemma 7.1 if $\kappa = p^r_\infty$ and by Lemma 4.1 if $\kappa = \tilde{p}$. This concludes the proof. □

A Technical Result on the Exchange of Limits.

Lemma 7.5. *Let* $(a_{n,m})_{n,m\geq 1}$ *be a real valued sequence. Suppose that the following holds:*

1. $\lim_{n\to\infty} a_{n,m} = b_m$ *uniformly in* m*, and*
2. $\underline{\lim}_{m\to\infty} a_{n,m} = c_n$ *for every* $n \geq 1$.

Then, it holds

$$\underline{\lim}_{m\to\infty} \lim_{n\to\infty} a_{n,m} \leq \lim_{n\to\infty} \underline{\lim}_{m\to\infty} a_{n,m}. \tag{7.18}$$

Proof. For any $n \geq 1$, consider a subsequence $(a_{n,m_k})_{n,k\geq 1}$ so that $c_n = \underline{\lim}_{m\to\infty} a_{n.m} = \lim_{k\to\infty} a_{n,m_k}$. Since 1. is satisfied by the subsequence $(a_{n,m_k})_{n,k\geq 1}$ as well, Moore-Osgood theorem implies that there exists $A \in \mathbb{R}$ so that

$$\lim_{n\to\infty} \lim_{k\to\infty} a_{n,m_k} = A = \lim_{k\to\infty} \lim_{n\to\infty} a_{n,m_k}. \tag{7.19}$$

In particular, (7.19) implies that $(b_{m_k})_{k\geq 1}$ is a convergent subsequence of $(b_m)_{m\geq 1}$. Therefore, we have

$$\underline{\lim}_{m\to\infty} \lim_{n\to\infty} a_{n,m} = \underline{\lim}_{m\to\infty} b_m \leq \lim_{k\to\infty} b_{m_k} = \lim_{k\to\infty} \lim_{n\to\infty} a_{n,m_k} = \lim_{n\to\infty} \lim_{k\to\infty} a_{n,m_k} = \lim_{n\to\infty} \underline{\lim}_{m\to\infty} a_{n,m},$$

where last equality holds by definition of $(a_{n,m_k})_{n,m\geq 1}$. This concludes the proof. □

Acknowledgements. The authors acknowledge the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) - Project-ID 317210226 - SFB 1283.

The first author acknowledges financial support under the National Recovery and Resilience Plan (NRRP), Mission 4, Component 2, Investment 1.1, Call for tender No. 1409 published on 14.9.2022 by the Italian Ministry of University and Research (MUR), funded by the European Union - NextGenerationEU– Project Title Probabilistic Methods for Energy Transition - CUP G53D23006840001 - Grant Assignment Decree No. 1379 adopted on 01/09/2023 by the Italian Ministry of Ministry of University and Research (MUR).

Disclosure statement. The authors declare that none of them has conflict of interests to mention.

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