State Dependent Utility and Ambiguity

Lasse Mononen[†]

Abstract

Models of choice under uncertainty study choice behavior when outcomes depend on the realized state of the world. The typical assumption is that utilities of outcomes do not depend on the realized state and are state independent. Without this simplifying assumption, it is difficult to separately identify utilities and beliefs. This paper provides novel general foundations for models with state dependent utilities: once we depart from expected utility, it is often possible to uniquely identify utilities and beliefs. Specifically, we show that with general models of non-expected utility under ambiguity we have complete identification of utilities and probabilities under full-dimensional uncertainty. Additionally, we offer novel axiomatizations for state dependent dual-self variational expected utility and dual-self expected utility.

Keywords: State dependent utility, belief identification, ambiguity, multiple priors, dual-self expected utility

JEL codes: C81, C90, D80, D81, D82, D83.

1 Introduction

Decision making under uncertainty studies choice behavior when outcomes depend on the realized state of the world. Traditionally, it is assumed that the utilities of outcomes are *state independent* and do not depend on the realized state. This independence simplifies the identification of utilities and beliefs. However, as observed

[†]Center for Mathematical Economics, University of Bielefeld, PO Box 10 01 31, 33 501 Bielefeld, Germany: lasse.mononen@uni-bielefeld.de

The author thanks Pietro Ortoleva and Faruk Gul for valuable guidance and discussions and Artur Dolgopolov, Marc Fleurbaey, Jay Lu, Nick Netzer, Wolfgang Pesendorfer, Frank Riedel. Evgenii Safonov, Denis Shishkin, Leeat Yariv and seminar participants at RUD and University of Bielefeld for useful comments and suggestions. This work was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) — Project-ID 317210226 — SFB 1283.

by Aumann (1971) in many situations outcomes and their utilities might be *state dependent*. Known examples are when the state of the world is the health of the decision maker such as when choosing health insurance (Arrow, 1971). Assuming state independent utilities in these situations can lead to inaccurate identification of beliefs and incorrect model predictions overall.

This paper reconsiders state dependent utilities in the standard framework and shows that, under general conditions, it is possible to separately identify the state dependent utilities and probabilities whenever we have full-dimensional uncertainty about the state of the world. Additionally, we offer novel axiomatizations for state dependent dual-self variational expected utility and dual-self expected utility (Chandrasekher et al., 2022).

Before moving on to the results, we highlight the importance of identifying utilities and probabilities with a concrete example. Consider a government that wants to change people's behavior with a public health campaign, but people find the change difficult or inconvenient e.g. reducing smoking or increasing the use of seat belts. Here the choice of an effective campaign depends critically on if the lack of change reflects a taste-based reason (quitting smoking is difficult, seat belts are uncomfortable) or if it reflects a belief-based reason (only heavy smokers get cancer or only reckless drivers get into accidents). In the first case, an effective campaign would make the change of behavior easier by increasing the availability of nicotine replacement products and making smoking socially less acceptable or redesigning seat belts to be more comfortable and convenient. In the second case, an effective campaign would be an information campaign on the effects of behavior changes and risks associated with the current behavior. Here, it is crucial to separate tastes from beliefs in order to choose an effective campaign.

This paper characterizes the identification of utilities and probabilities in general state dependent models under uncertainty. First, we consider general *dual-self expected utility* (Chandrasekher et al., 2022) with state dependent utility. In this context, we show our main identification result: when the uncertainty about the states of the world is full-dimensional, the probabilities and the state dependent utilities are fully identified. This shows that the well-known impossibility of identification under expected utility is only a knife-edge case due to the linearity of expected utility. Once we depart from such linearity, we regain the identification. Especially, this shows the identification for many special cases such as maxmin expected utility (Gilboa and Schmeidler, 1989), Choquet expected utility (Schmeidler, 1989), α -maxmin expected utility (Ghirardato et al., 2004), and invariant biseparable preferences (Ghirardato et al., 2004; Amarante, 2009).

Second, we consider the identification of more general *dual-self variational expected utility* (Chandrasekher et al., 2022) with state dependent utility. In this case, under full-dimensional uncertainty, the probabilities and the intensities of utilities are fully identified. That is, the probabilities are uniquely identified and the state dependent utilities are identified up to a common positive transformation and addition of state dependent constants. This general result characterizes the identification for monotone mean-variance preferences (Maccheroni et al., 2009), multiplier preferences (Hansen and Sargent, 2001; Strzalecki, 2011), variational preferences (Maccheroni et al., 2006), monotonic mean-dispersion preferences (Grant and Polak, 2013), and vector expected utility (Siniscalchi, 2009).

Finally, we characterize the existence of these representations. We focus on standard preferences under ambiguity, that is continuous, monotonic, and risk independent preferences. First, we show that if these standard preferences have two acts with the same ambiguity, in a sense that will be made precise, then they admit a *dualself variational expected utility* representation (Chandrasekher et al., 2022) with state dependent utility.¹ Second, if these preferences additionally have an unambiguous

¹This result extends Chandrasekher et al.'s (2022) characterization. However, this extension is not entirely straightforward since under state-dependence, constant acts may no longer be unambiguous, so we need to find an alternative way to capture the characterizing properties of the representation. We will infer from the decision maker's behavior which acts are unambiguous and use this to capture the characterizing properties.

act, in a sense that will be made precise, then they admit a *dual-self expected utility* representation (Chandrasekher et al., 2022) with state dependent utility.

For practical use, we provide a behavioral elicitation method for state dependent utilities based on acts that have a constant utility for dual-self expected utility and acts that have a constant utility difference for dual-self variational expected utility. This allows the elicitation of state dependent utilities and beliefs based on standard choice data. This behavioral identification is illustrated in our example in Section 2.

Our identification results and the standard setting that we consider are very flexible and suitable for applications beyond just uncertainty. First, in the context of social choice theory, our identification results show the identification of interpersonal utility comparisons and the fairness of the society. This is formalized in Mononen (2024b). Here, we assume that we observe the preferences of a social planner over social alternatives. The planner aggregates members' utilities by giving a Pareto weight to each member. Here, the states of the world are interpreted as members of the society and the probabilities for the states are the planner's Pareto weights. Our identification result shows in this context that if any redistribution from one member to another affects the welfare under some utility distribution, then interpersonal utility comparisons and the fairness of the society, that is the Pareto weights, are identified.

Second, in the context of intertemporal choice, our results provide foundations and identification for evolving tastes over time when there is uncertainty about the discount factor following Wakai (2008) and Mononen (2024a). Here, the states of the world are interpreted to be time periods and we have time dependent utilities. The probabilities give the possible discount factors for each time period. In this context. our identification result shows that if there is uncertainty about the discount factor at each time period, then the time dependent utilities are identified.

This paper contributes to the literature studying the identification of state dependent utilities in models of non-expected utilities following Drèze (1987) and Chew and Wang (2020). Our identification results offer an alternative to using additional information (e.g. preferences conditional on signals with Bayesian updating, hypothetical lotteries of state-outcome pairs, or stochastic choice) for the identification of state dependent utilities as in Karni (2011a; 2011b), Tsakas (2023), Karni and Schmeidler (2016), and Lu (2019).

Second, our paper contributes to the literature on modeling choice under uncertainty that was recently discussed in Gilboa et al. (2020). We generalize the standard setting for state dependent consequences as has been suggested in the literature (Aumann, 1971; Shafer, 1986; Gilboa et al., 2020). However, we show that we can still uniquely identify the beliefs of the decision maker in this more general setting that does not contain simple bets on events.

The remainder of the paper proceeds as follows: We begin, in Section 2, by showing a simple example of the fundamental forces behind the identification result. Next, Section 3 studies the identifications of state dependent dual-self expected utility and state dependent dual-self variational expected utility. Section 3.3 shows the main result of the paper that full-dimensional uncertainty characterizes the identification of beliefs and shows how to behaviorally elicitate the state dependent utilities. Section 3.4 generalizes the results for the identification of beliefs for a single state and for the relative likelihoods between two states. Additionally, we characterize the general identification of beliefs and utilities. Section 4 axiomatically characterizes the existence of the representations. Finally, Section 5 discusses the related literature and Section 6 concludes. Proofs for all the results are in the Appendix.

2 An Example of Identification

We begin with a simple example illustrating that with state dependent subjective expected utility the intensities of preferences and the probabilities cannot be separated. However, this is only an unidentified knife-edge case. In the second part of the example, we show that under ambiguity these can be separated and identified. The idea for the lack of identification is simple and can be illustrated with purchasing a health insurance. In here, we would want to quantify the likelihood of getting injured and the subjective cost of the injury. However, separating these two quantities is difficult if we only observe insurance purchase decisions. The difficulty comes from the fact that a purchase of a health insurance can always be explained equally well by either having a high likelihood of getting injured or having a low likelihood of injury but a large subjective cost for it. These two possibilities cannot be separated.

Formally, we consider preferences for health insurance contracts (f_I, f_H) that pays out f_I in the case of injury in the state I (injury), and f_H without injury in the state H (healthy). Assume that the preferences have a state dependent subjective expected utility representation with a probability of 0.5 for injury. The value of the insurance contract (f_I, f_H) is

$$0.5u_I(f_I) + 0.5u_H(f_H)$$

where $u_I, u_H : \Delta(X) \to \mathbb{R}$ are state dependent utilities.

Now these preferences have an alternative state dependent subjective expected utility representation with any probability $p \in (0,1)$ for state 1 since $0.5u_I(f_I)+0.5u_H(f_H)=p\left(\frac{0.5}{p}u_I(f_I)\right)+(1-p)\left(\frac{0.5}{1-p}u_H(f_H)\right)=p\tilde{u}_I(f_I)+(1-p)\tilde{u}_H(f_H),$ where the terms inside the parentheses define new state dependent utility functions $\tilde{u}_I, \tilde{u}_H.$

In this alternative representation, we have replaced the probability of an injury with an intensity of preference for money in the case of an injury. This highlights the impossibility of identifying the state dependent subjective expected utility since the intensities of preferences are inseparable from the probabilities.

However, once there is uncertainty about the subjective probability of the injury, then identification is possible. We illustrate this in the simplest case of extreme uncertainty aversion. We consider first the identification of the state dependent maxmin expected utility. Here, there exist (increasing) state dependent utility functions (u_I, u_H) such that the value of the health insurance contract (f_I, f_H) is

$$\min_{p \in [0,1]} p u_I(f_I) + (1-p) u_H(f_H)$$

Assume that we observe a purchase for the insurance amount $c \in \mathbb{R}$ at the cost of π at wealth w for the insurance contract that pays out $(w + c, w - \pi c)$. Under the extreme uncertainty aversion, the purchase provides full insurance and the purchase reveals that the state dependent utilities $u_I(w + c)$ and $u_H(w - \pi c)$ are equal. This illustrates the underlying mechanism for observing the state dependent utilities across the states from the choices of the decision maker.

Second, we consider identification for the state dependent variational expected utility with a linear ambiguity index function. Here, there exist (increasing) state dependent utility functions (u_I, u_H) and $A_I, A_H \in \mathbb{R}$ such that the value of the health insurance contract (f_I, f_H) is

$$\min_{p \in [0,1]} p u_I(f_I) + (1-p) u_H(f_H) + p A_I + (1-p) A_H$$

where $pA_I + (1-p)A_H$ is the linear ambiguity index of p.

In this case, the purchase for the insurance amount c at the cost of π at wealth w does not reveal that the utilities for the states are equal since the same choices can be explained by equal utility or by a linear cost function. Formally, we have for an insurance contract (f_I, f_H) ,

$$\min_{p \in [0,1]} p u_I(f_I) + (1-p) u_H(f_H) + p A_I + (1-p) A_H = \min_{p \in [0,1]} p \left(u_I(f_I) + A_I \right) + (1-p) \left(u_H(f_H) + A_H \right)$$
$$= \min_{p \in [0,1]} p \widetilde{u}_I(f_I) + (1-p) \widetilde{u}_H(f_H),$$

where the terms inside the parentheses define new state dependent utility functions $(\tilde{u}_I, \tilde{u}_H)$. This illustrates the lack of identification of utility levels with state dependent dual-self variational expected utility.

However, if we observe another purchase for the insurance amount c' at a higher wealth w', then under a linear ambiguity index, the decision maker will fully hedge the utility improvement across the states and distributes the utility improvement equally across the states: $u_I(w+c) - u_I(w'+c') = u_H(w-\pi c) - u_H(w'-\pi c')$. This illustrates the underlying mechanism for observing the intensities of preferences, that is utility differences, across the states from the choices. In our main results, we generalize both of the identifications to non-convex preferences and to full-dimensional uncertainty.

3 Identification

In order to study identification in the most general setup, we study a state dependent version of dual-self variational expected utility (Chandrasekher et al., 2022). This includes as special cases monotone mean-variance preferences (Maccheroni et al., 2009), multiplier preferences (Hansen and Sargent, 2001; Strzalecki, 2011), variational preferences (Maccheroni et al., 2006), monotonic mean-dispersion preferences (Grant and Polak, 2013), and vector expected utility (Siniscalchi, 2009). Additionally, we provide stronger identification results for the special case of dual-self expected utility (Chandrasekher et al., 2022) that has as special cases maxmin expected utility (Gilboa and Schmeidler, 1989), Choquet expected utility (Schmeidler, 1989), and α -maxmin expected utility (Ghirardato et al., 2004) and is an alternative representation for invariant biseparable preferences (Ghirardato et al., 2004; Amarante, 2009). Our identification results encompass those for all the special cases.

3.1 Preliminaries and Notation

We consider the finite Anscombe-Aumann (1963) framework with state dependent consequences. S is a finite state space, for each $s \in S$, X_s is a set of state dependent consequences and $\Delta(X_s)$ is the set of (simple) lotteries on X_s . Acts are mappings from states to lotteries over state specific consequences and the set of acts is H = $\times_{s \in S} \Delta(X_s)$.² Our primitive is a binary relation \succeq on H. As usual, \succ and \sim denote the asymmetric and symmetric parts of \succeq respectively.

²This includes the standard state independent Anscombe-Aumann setting where $X_s = X$ for all s.

The following notation will be useful. $\Delta(S)$ is the set of probability measures on S. We endow $\Delta(S)$ with the Euclidean topology. $\mathcal{K}(\Delta(S))$ is the set of all closed, convex, and non-empty subsets of $\Delta(S)$ endowed with the Hausdorff topology. For $P \subseteq \Delta(S)$, denote the convex closure of P by $\overline{\operatorname{co}} P$. For $S' \subseteq S$ and $P \subseteq \Delta(S)$, denote the projection of P to S' by $\operatorname{pr}_{S'} P = \{(p_s)_{s \in S'} | p \in P\}$.

For $f \in H, s \in S, x_s \in \Delta(X_s)$, f_s denotes the consequence of the act f in the state s and (x_s, f_{-s}) denotes the act where the consequence in the state s is x_s and in the states $s' \in S \setminus \{s\}$, $f_{s'}$. Mixtures of acts are defined statewise: for all $f, g \in H, \alpha \in [0, 1], s \in S$, define $(\alpha f + (1 - \alpha)g)_s = \alpha f_s + (1 - \alpha)g_s$.

Our identification conditions are based on the unambiguous indifference as defined in Ghirardato et al. (2004).

Definition Acts f and g are unambiguously indifferent, denoted $f \sim^* g$, if for all acts h and $\alpha \in (0, 1)$

$$\alpha f + (1 - \alpha)h \sim \alpha g + (1 - \alpha)h.$$

If consequences in some state do not affect the preferences, then the utility for these consequences is unobservable. Hence our focus is on proper states:

Definition A state $s \in S$ is proper if there exist $x_s, y_s \in \Delta(X_s)$ and $f \in H$ such that

$$(x_s, f_{-s}) \not\sim (y_s, f_{-s}).$$

The collection of proper states is denoted $S^{\mathcal{P}}$.

We infer the preferences on consequences within each state as follows:

Definition For each $s \in S$, define \succeq_s on $\Delta(X_s)$ by for all $x_s, y_s \in \Delta(X_s)$,

$$x_s \succeq_s y_s \iff (x_s, f_{-s}) \succeq (y_s, f_{-s})$$
 for all $f \in H$.

Additionally, \succ_s and \sim_s denote the asymmetric and symmetric parts of \succeq_s respectively.

Finally, for an ambiguity index for beliefs $c : \Delta(S) \to \mathbb{R} \cup \{\infty\}$, we denote the effective domain of c by dom $c = \{p \in \Delta(S) | c(p) \in \mathbb{R}\}.$

3.2 State Dependent Dual-Self Variational Expected Utility

The state independent dual-self variational expected utility was introduced by Chandrasekher et al. (2022) as a general model for preferences under ambiguity. We use a state dependent variation of it:

Definition (u, \mathbb{C}) is a state dependent dual-self variational expected utility for \succeq if for each $s \in S$, $u_s : \Delta(X_s) \to \mathbb{R}$ is affine, $\mathbb{C} \subseteq \{c : \Delta(S) \to \mathbb{R} \cup \{\infty\} | c \text{ is convex}\}$ is such that $\max_{c \in \mathbb{C}} \min_{p \in \Delta(S)} c(p) = 0$ and for each $c \in \mathbb{C}$, $p \in \text{dom } c$, and $s \notin S^{\mathcal{P}}$, $p_s = 0$, and for all $f, g \in H$,

$$f \succeq g \iff \max_{c \in \mathbb{C}} \min_{p \in \Delta(S)} \sum_{s \in S} p_s u_s(f_s) + c(p) \ge \max_{c \in \mathbb{C}} \min_{p \in \Delta(S)} \sum_{s \in S} p_s u_s(g_s) + c(p).$$

Here, $c \in \mathbb{C}$ is an index of ambiguity aversion where lower values capture higher uncertainty aversion (Maccheroni et al., 2006). This representation is the variational expected utility when \mathbb{C} is a singleton. Axiomatically, this representation is a nonconvex generalization of variational preferences and any uncertainty aversion and seeking of the preferences can be represented by the interplay of an ambiguity index (min) and the set of ambiguity indices (max).

As observed in Chandrasekher et al. (2022), the state independent dual-self variational expected utility is not unique. However, in the state independent case, the smallest convex closure of the effective domains of the ambiguity indices captures uniquely the beliefs of the decision maker. This gives us tight dual-self variational expected utility that we use to study the identification of the beliefs in the state dependent case.

Definition (u, \mathbb{C}) is a state dependent tight dual-self variational expected utility for \succeq , if (u, \mathbb{C}) is a state dependent dual-self variational expected utility for \succeq and if (u, \mathbb{C}) is another state dependent dual-self variational expected utility for \succeq , then $\overline{\mathrm{co}} \bigcup_{\widetilde{c} \in \mathbb{C}} \mathrm{dom} \, \widetilde{c} \supseteq \overline{\mathrm{co}} \bigcup_{c \in \mathbb{C}} \mathrm{dom} \, c.$

An important special case of dual-self variational expected utility is dual-self expected utility that corresponds to $0/\infty$ -valued ambiguity indices. We provide stronger identification results for this special case.

Definition (u, \mathbb{P}) is a state dependent dual-self expected utility for \succeq if for each $s \in S$, $u_s : \Delta(X_s) \to \mathbb{R}$ is affine and $\mathbb{P} \subseteq \mathcal{K}(\Delta(S))$ is compact and non-empty such that for each $P \in \mathbb{P}, p \in P$, and $s \notin S^{\mathcal{P}}, p_s = 0$, and for all $f, g \in H$,

$$f \succeq g \iff \max_{P \in \mathbb{P}} \min_{p \in P} \sum_{s \in S} p_s u_s(f_s) \ge \max_{P \in \mathbb{P}} \min_{p \in P} \sum_{s \in S} p_s u_s(g_s).$$

We define state dependent tight dual-self expected utility symmetrically.

Definition (u, \mathbb{P}) is a state dependent tight dual-self expected utility for \succeq if (u, \mathbb{P}) is a state dependent dual-self expected utility for \succeq and if $(u, \widetilde{\mathbb{P}})$ is another state dependent dual-self expected utility for \succeq , then $\overline{\operatorname{co}} \bigcup_{\widetilde{P} \in \widetilde{\mathbb{P}}} \widetilde{P} \supseteq \overline{\operatorname{co}} \bigcup_{P \in \mathbb{P}} P$.

3.3 Full Identification

This section provides the main result of the paper. We introduce a novel simple axiom stating that the decision maker has full-dimensional uncertainty. This axiom characterizes the full identification of probabilities and intensities of utilities in the state dependent dual-self variational expected utility. Additionally, the levels of utilities are identified in the state dependent dual-self expected utility. Furthermore, we show how the state dependent utilities can be elicitated behaviorally.

3.3.1 Full-Dimensional Uncertainty and Full Identification

To make the underlying intuition for our identifying condition clear, we first derive it informally from the idea that the decision maker's uncertainty about the proper states is full-dimensional. Consider two acts f and g that have trade-offs between proper states. That is the act f is better than g in some states and g is better than fin some other states. Under full-dimensional uncertainty, there is uncertainty about the likelihood ratio between any proper states.³ Then especially there is uncertainty about the trade-offs between f and g. This means that in the mixture $\alpha f + (1 - \alpha)h$ with some act h, trading-off f for g can hedge ambiguity and the uncertainty can be observed. This is formalized in the next identification axiom by assuming that if two acts are unambiguously indifferent and so trading f for g does not hedge ambiguity, then they do not have trade-offs across states and are statewise indifferent.

Axiom 1 If $f, g \in H$ and $f \sim^* g$, then for all $s \in S$, $f_s \sim_s g_s$.

The main result of this paper is the following uniqueness result for the state dependent dual-self variational expected utility. The result states that Axiom 1 characterizes the separation and identification of the probabilities and intensities of preferences.

Theorem 1 (Full Identification) Assume that (u, \mathbb{C}) is a state dependent tight dual-self variational expected utility for \succeq . The following four conditions are equivalent:

- (1) \succeq satisfies Axiom 1.
- (2) $\operatorname{pr}_{S^{\mathcal{P}}} \overline{\operatorname{co}} \bigcup_{c \in \mathbb{C}} \operatorname{dom} c$ has a non-empty interior in $\Delta(S^{\mathcal{P}})$.
- (3) If $(\tilde{u}, \tilde{\mathbb{C}})$ is a state dependent tight dual-self variational expected utility for \succeq , then

$$\overline{\operatorname{co}}\bigcup_{\widetilde{c}\in\widetilde{\mathbb{C}}}\operatorname{dom}\widetilde{c}=\overline{\operatorname{co}}\bigcup_{c\in\mathbb{C}}\operatorname{dom}c.$$

(4) If $(\tilde{u}, \tilde{\mathbb{C}})$ is a state dependent tight dual-self variational expected utility for \succeq , then there are $\alpha \in \mathbb{R}_+$ and $B \in \mathbb{R}^S$ such that for all $s \in S^{\mathcal{P}}$,

$$\widetilde{u}_s = \alpha u_s + B_s.$$

The second condition shows that Axiom 1's uncertainty about every trade-off between proper states is the behavioral characterization of full-dimensional uncertainty for the proper states.

³Formally, full-dimensional uncertainty states that the set of probabilities has a non-empty interior. So especially there is uncertainty about the likelihood ratio between any states.

The third and fourth conditions show that the full-dimensional uncertainty characterizes the identification of the convex closure of the probabilities and the intensities of the utilities. That is we have partially recovered the state independent identification up to the levels of utilities. The lack of identification for the levels of utilities follows from the interchangeability of additive constants between the ambiguity index c and the utility function as illustrated in Section 2.

Before discussing the result, we show that with the state dependent dual-self expected utility, we additionally identify the utility levels.

Theorem 2 (Full Identification, Dual-Self) Assume that (u, \mathbb{P}) is a state dependent tight dual-self expected utility for \succeq . The following four conditions are equivalent:

- (1) \succeq satisfies Axiom 1.
- (2) $\operatorname{pr}_{S^{\mathcal{P}}} \overline{\operatorname{co}} \bigcup_{P \in \mathbb{P}} P$ has a non-empty interior in $\Delta(S^{\mathcal{P}})$.
- (3) If $(\tilde{u}, \tilde{\mathbb{P}})$ is a state dependent tight dual-self expected utility for \succeq , then

$$\overline{\operatorname{co}}\bigcup_{\widetilde{P}\in\widetilde{\mathbb{P}}}\widetilde{P}=\overline{\operatorname{co}}\bigcup_{P\in\mathbb{P}}P$$

(4) If $(\tilde{u}, \tilde{\mathbb{P}})$ is a state dependent tight dual-self expected utility for \succeq , then there are $\alpha \in \mathbb{R}_+$ and $\beta \in \mathbb{R}$ such that for all $s \in S^{\mathcal{P}}$,

$$\widetilde{u}_s = \alpha u_s + \beta.$$

Next, we discuss these results. First, these identification results provide the identification for all the special cases of dual-self variational expected utility and dual-self expected utility.

Second, the full-dimensionality of the set of probabilities for the proper states is a generic property, but it rules out important special cases: If \succeq satisfies the independence axiom, then Axiom 1 will not be satisfied. Especially, as is well-known, the state dependent expected utility is not identified. More generally, if there are unambiguous events, that is all of the probabilities agree on the probability of an event, then Axiom 1 will not be satisfied for these unambiguous events and the utilities and probabilities are not identified across these unambiguous events and their complements.

Third, under Axiom 1, the state dependent utilities are identified for all the proper states and the tight set of probabilities is identified. However, the set of ambiguity indices is not identified. This is symmetrical to the multiplicity of state independent dual-self variational expected utility (Chandrasekher et al., 2022). Additionally, the utilities for non proper states are not identifiable since these states do not affect the preferences.

Lastly, Section 2 highlighted the intuition of these identifications. This intuition generalizes to any finite number of states and is formalized in the Appendix.

3.3.2 Equally Crisp Acts and Full Identification

To get an insight into the identification, we can behaviorally capture the identification of intensities of utilities by acts that hedge ambiguity similarly as illustrated in our example in Section 2. These acts are such that trading one of the acts for the other does not change the hedged ambiguity or the preferences. This is formalized in the following property from Maccheroni et al.'s (2006) weak certainty independence axiom.

Definition Acts f and g are equally crisp if for all $h, h' \in H, \alpha \in (0, 1)$,

 $\alpha h + (1-\alpha)f \succeq \alpha h' + (1-\alpha)f \Longleftrightarrow \alpha h + (1-\alpha)g \succeq \alpha h' + (1-\alpha)g.$

We show later on in Section 4 that the existence of equally crisp acts, essentially, characterizes state dependent dual-self variational expected utility under standard axioms. The next result connects the full identification from Theorem 1 to equally crisp acts.

Proposition 3 Assume that (u, \mathbb{C}) is a state dependent dual-self variational expected utility for \succeq . The following two conditions are equivalent.

(1) \succeq satisfies Axiom 1.

(2) f and g are equally crisp acts iff there is $\alpha \in \mathbb{R}$ such that for all $s \in S^{\mathcal{P}}$,

$$u_s(f_s) - u_s(g_s) = \alpha.$$

This result shows that under Axiom 1, the utility differences across different states can be identified by equally crisp acts. This shows that intensities of utilities across states can be elicitated by identifying equally crisp acts that allows for the elicitation of beliefs. These equally crisp acts can be identified from the choice data. Additionally, this result shows that the identification of utility differences across states using equally crisp acts characterizes our identification axiom.

3.3.3 Crisp Acts and Full Identification

In the case of state dependent dual-self expected utility, we have a stronger behavioral identification for the utilities across states using acts that are unambiguous and do not hedge ambiguity as defined in Ghirardato et al. (2004). This identification was illustrated in our example in Section 2.

Definition
$$h \in H$$
 is a crisp act^4 if for all $f, g \in H, \alpha \in (0, 1)$,
 $f \succeq g \iff \alpha f + (1 - \alpha)h \succeq \alpha g + (1 - \alpha)h$.

We show later on in Section 4 that the existence of crisp acts, essentially, characterizes state dependent dual-self expected utility with a utility overlap across states under standard axioms. The next result connects the full identification from Theorem 2 to crisp acts.

Proposition 4 Assume that (u, \mathbb{P}) is a state dependent dual-self expected utility for \succeq . Then for the following two conditions, $(1) \Rightarrow (2)$.

- (1) \succeq satisfies Axiom 1.
- (2) f is a crisp act iff there is $\alpha \in \mathbb{R}$ such that for all $s \in S^{\mathcal{P}}$,

$$u_s(f_s) = \alpha.$$

 $^{^4\}mathrm{Cerreia-Vioglio}$ et al. (2011) use a different definition for crisp acts that does not extend to state dependent setting.

Additionally, if int $\bigcap_{s \in S^{\mathcal{P}}} u_s(\Delta(X_s)) \neq \emptyset$, then (1) and (2) are equivalent.

The first part of this result is symmetric to the previous Proposition 3 except that in here we can elicit the utility levels across the states by identifying the crisp acts. The second part shows, that if there is utility overlap with an interior across the states, then the identification of utilities with crisp acts characterizes our identification axiom.

3.4 Partial Identification

The previous results showed characterizations for the full identification. However, some states of the world might not be uncertain and in this case, the full identification is not possible. In this section, we offer behavioral conditions and conditions for the set of probabilities that characterize the identification of probabilities for a single state and relative likelihoods between two states. Additionally, we characterize the partial identification of the probabilities and utilities.

3.4.1 Probability Identification

The next result shows the identification of probabilities for a single state and how fulldimensional uncertainty can be extended to a single state by applying Axiom 1 only to a single state. Additionally, it provides a formalization for a notion of independent uncertainty for a single state.

Proposition 5 (Probability Identification) Assume that (u, \mathbb{C}) is a state dependent tight dual-self variational expected utility for \succeq . Fix any $s \in S^{\mathcal{P}}$. The following three conditions are equivalent:

- (1) If $f, g \in H$ and $f \sim^* g$, then $f_s \sim_s g_s$.
- (2) $S^{\mathcal{P}} = \{s\}$ or there are $p, q \in \overline{\operatorname{co}} \bigcup_{c \in \mathbb{C}} \operatorname{dom} c$ such that $p_s \neq q_s$ and for all $\tilde{s} \in S \setminus \{s\}$,

$$\frac{p_{\tilde{s}}}{1-p_s} = \frac{q_{\tilde{s}}}{1-q_s}.$$

(3) If $(\tilde{u}, \tilde{\mathbb{C}})$ is a state dependent tight dual-self variational expected utility for \succeq , then

$$\Big\{\widetilde{p}_s\Big|\widetilde{p}\in\overline{\mathrm{co}}\bigcup_{\widetilde{c}\in\widetilde{\mathbb{C}}}\mathrm{dom}\,\widetilde{c}\Big\}=\Big\{p_s\Big|p\in\overline{\mathrm{co}}\bigcup_{c\in\mathbb{C}}\mathrm{dom}\,c\Big\}.$$

The first condition shows that independent uncertainty about a state s means behaviorally that there is uncertainty about any trade-offs involving the state s^5 . The second condition shows that independent uncertainty about a state s is formalized as having uncertainty about the probability of the state s while keeping all the probabilities conditional on the event $S \setminus \{s\}$ the same. The last condition shows that this independent uncertainty about a state characterizes the identification of the probabilities for that state.⁶

3.4.2 Relative Likelihood Identification

Next, we move on to the identification of the relative likelihood of two states. When considering a gamble between two states the identification of the probabilities is not important since the gamble depends only on the relative likelihood of the states. The next result provides a behavioral condition for full-dimensional uncertainty about a relative likelihood and shows how this characterizes the identification of utilities and the relative likelihoods between two states. Additionally, it provides a formalization for a notion of independent uncertainty for a relative likelihood.

Proposition 6 (Relative Likelihood Identification) Assume that (u, \mathbb{C}) is a state dependent tight dual-self variational expected utility for \succeq . Fix any $s, s' \in S^{\mathcal{P}}, s \neq s'$. The following four conditions are equivalent:

- (1) If $f, g \in H$ are equally crisp and $f_s \succ_s g_s$, then $f_{s'} \succeq_{s'} g_{s'}$.
- (2) There are $p, q \in \overline{\text{co}} \bigcup_{c \in \mathbb{C}} \text{dom } c$ such that $\frac{p_s}{p_{s'}} \neq \frac{q_s}{q_{s'}}$ and for all $\tilde{s} \in S \setminus \{s, s'\}, p_{\tilde{s}} = q_{\tilde{s}}$.

⁵This follows from the observation that this condition is equivalent to if $f_s \not\sim_s g_s$ and $f \sim g$, then $f \not\sim^* g$.

 $^{^{6}}$ Identification of utilities for a single state is not possible since utilities are always identified only up to a common positive affine transformation.

(3) If $(\tilde{u}, \tilde{\mathbb{C}})$ is a state dependent tight dual-self variational expected utility for \succeq , then

$$\Big\{\frac{\widetilde{p}_s}{\widetilde{p}_{s'}}\Big|\widetilde{p}\in\overline{\mathrm{co}}\bigcup_{\widetilde{c}\in\widetilde{\mathbb{C}}}\mathrm{dom}\,\widetilde{c}\Big\}=\Big\{\frac{p_s}{p_{s'}}\Big|p\in\overline{\mathrm{co}}\bigcup_{c\in\mathbb{C}}\mathrm{dom}\,c\Big\}.$$

(4) If $(\tilde{u}, \tilde{\mathbb{C}})$ is a state dependent tight dual-self variational expected utility for \succeq , then there are $\alpha \in \mathbb{R}_+$ and $B \in \mathbb{R}^S$ such that

$$\widetilde{u}_s = \alpha u_s + B_s$$
 and $\widetilde{u}_{s'} = \alpha u_{s'} + B_{s'}$.

The first condition is a strengthening of Axiom 1 when applied to only two states. The idea of the axiom is the same as before: There is uncertainty about the likelihood of the state s compared to the state s'. However, when f and g are not indifferent, this uncertainty is observed in equally crisp acts.

The second condition shows that independent uncertainty about the relative likelihood of two states is formalized as having uncertainty about the relative likelihood of the two states while keeping all the other probabilities the same. The last two conditions show that this independent uncertainty about the relative likelihood characterizes the identification of the relative likelihoods and relative intensities of utilities for these two states.⁷

3.4.3 Partial Identification Characterization

Finally, we move on to the general partial identification characterization of the state dependent dual-self variational expected utility.

Theorem 7 (Partial Identification) Assume that (u, \mathbb{C}) is a state dependent tight dual-self variational expected utility for \succeq , \tilde{u} is a state dependent utility function, and $P \subseteq \Delta(S)$. The following two conditions are equivalent

⁷Additionally, with state dependent dual-self expected utility the relative utility levels are identified.

(1) There is $\widetilde{\mathbb{C}}$ such that $(\widetilde{u}, \widetilde{\mathbb{C}})$ is a state dependent tight dual-self variational expected utility for \succeq and

$$P = \overline{\operatorname{co}} \bigcup_{\widetilde{c} \in \widetilde{\mathbb{C}}} \operatorname{dom} \widetilde{c}.$$

(2) There are $x \in \mathbb{R}^{S}_{++}, B \in \mathbb{R}^{S}$, and $\alpha \in \mathbb{R}_{++}$ such that for all $p \in \bigcup_{c \in \mathbb{C}} \operatorname{dom} c$, $\sum_{s \in S} x_{s} p_{s} = 1$, for all $s \in S^{\mathcal{P}}$, $\widetilde{u}_{s} = \frac{\alpha}{x_{s}} u_{s} + B_{s}$, and $P = \left\{ (x_{s} p_{s})_{s \in S} \middle| p \in \overline{\operatorname{co}} \bigcup_{c \in \mathbb{C}} \operatorname{dom} c \right\}$.

This result shows that the set of beliefs is identified up to statewise multiplicative transformations for which the probabilities remain as probabilities when we do the statewise reciprocal transformation for the utilities. Additionally, common scalings and state dependent additive transformations for the utilities are possible. Especially, the theorem shows how the size of the set of probabilities restricts the possible transformations by the requirement that the multiplicative transformation must keep all the probabilities as probabilities.⁸⁹

4 Axioms for the Representations

4.1 State Dependent Dual-Self Variational Expected Utility

This section introduces five axioms that characterize the existence of a state dependent dual-self variational expected utility representation and an additional axiom that gives a state dependent dual-self expected utility representation. This axiomatization highlights the generality of the representation by showing that essentially

⁸Formally, these transformations are restricted by linearly independent probabilities in $\bigcup_{P \in \mathbb{P}} P$: If $x \in \mathbb{R}^S$ is such that for all probabilities p, $\sum_{s \in S} x_s p_s = 1$, then we can write x = 1 + (x - 1) where for all probabilities p, $\sum_{s \in S} (x_s - 1)p_s = 0$. Thus we can decompose any transformation to a sum of 1 and a vector orthogonal to the set of probabilities. Since \mathbb{R}^S can be decomposed to the sum of the linear span of $\bigcup_{P \in \mathbb{P}} P$ and its orthogonal complement, this shows the close connection between the size of the set of probabilities and the set of possible transformations.

⁹With state dependent dual-self expected utility, we have an additional restriction for utility levels that for all $p, q \in \overline{\text{co}} \bigcup_{P \in \mathbb{P}} P$, $\sum_{s \in S} p_s B_s = \sum_{s \in S} q_s B_s$.

the only standard ambiguity preferences that do not have a state dependent dual-self variational expected utility representation are such that every act contains different ambiguity.

The first four axioms define standard preferences under ambiguity following Cerreia-Vioglio et al. (2011) in the state dependent context. The first two axioms are standard assumptions that the preferences are a nontrivial weak order that satisfy continuity.

Axiom 2 \succeq is complete and transitive and there are $f, g \in H$ such that $f \succ g$.

Axiom 3 For all $f, g, h \in H$, the sets $\{\alpha \in [0, 1] | \alpha f + (1 - \alpha)g \succeq h\}$ and $\{\alpha \in [0, 1] | h \succeq \alpha f + (1 - \alpha)g\}$ are closed.

The next assumption for standard preferences under ambiguity is monotonicity.¹⁰ This is the main axiom ruling out state dependent utilities by assuming that the consequences in each state of the world are ranked similarly. We assume monotonicity only within each state which allows for a fully separate ranking of consequences in each state.

Axiom 4 For all $s \in S, x_s, y_s \in \Delta(X_s), f, g \in H$,

$$(x_s, f_{-s}) \succ (y_s, f_{-s}) \Longrightarrow (x_s, g_{-s}) \succeq (y_s, g_{-s}).$$

The next assumption for standard preferences under ambiguity is independence on lotteries.¹¹ We relax the independence axiom on lotteries within each state by allowing for weak preference reversals that could occur if the state might be impossible.¹²

Axiom 5 For all $s \in S, x_s, y_s, z_s \in \Delta(X_s), \alpha \in (0, 1)$,

$$x_s \succ_s y_s \Longrightarrow \alpha x_s + (1 - \alpha) z_s \succeq_s \alpha y_s + (1 - \alpha) z_s.$$

The last axiom for dual-self variational expected utility assumes that all the constant acts are equally crisp and share the same ambiguity. We relax this by assuming

¹⁰The state independent monotonicity assumes that if $f, g \in H$ are such that for all $s \in S$, $f_s \succeq g_s$, then $f \succeq g$ where f_s and g_s are acts that give the consequences f_s and g_s , respectively, in every state.

¹¹This is usually assumed implicitly through the certainty independence axiom or the weak certainty independence axiom.

 $^{^{12}}$ This weak independence was introduced in Einy (1989).

that there exist some equally crisp acts that are statewise ordered.

Axiom 6 There are $f^*, g^* \in H$ such that f^* and g^* are equally crisp and for all $s \in S^{\mathcal{P}}, f_s^* \succ_s g_s^*$.

The intuition for this axiom is that it captures the dispersional or relative nature of ambiguity. Ambiguity of an act comes from the uncertainty if the realized outcome is good or bad. However, these bad outcomes are only bad in a relative sense that the realized outcome could have been better. Now, this axiom is assuming that in any mixture with g^* trading g^* for f^* improves all the outcomes without affecting the relative comparisons or the ambiguity of the act. Finally, this axiom does not restrict how the outcomes are changed by trading g^* for f^* as long as the changes are strict improvements in all the proper states to rule out additional ambiguity from trade-offs across states.

These five axioms characterize state dependent dual-self variational expected utility representation.

Theorem 8 (Existence) The following two conditions are equivalent:

- (1) \succeq satisfies Axioms 2-6.
- (2) There is (u, \mathbb{C}) that is a state dependent tight dual-self variational expected utility for \succeq .

This result highlights the generality of the dual-self variational expected utility. Under the standard ambiguity assumptions, Axioms 2-5, the only requirement for the state dependent dual-self variational expected utility is that there are two acts that share the same ambiguity and are statewise ordered. In other words, essentially, the only standard preferences that do not have a state dependent dual-self variational representation are such that all the acts contain different ambiguity.

This characterization shows that the main axioms that the state dependent representation relaxes from the state independent representation are monotonicity and the weak certainty independence axiom. First, monotonicity is relaxed to hold only within each state. Second, for the weak certainty independence axiom, we infer from the decision maker's behavior which of the acts are equally crisp instead of assuming that constant acts are equally crisp.

This representation has multiple special cases as discussed above and this theorem provides the state dependent foundations for these representations.

4.2 State Dependent Dual-Self Expected Utility

Next, we show that adding an axiom stating that there exists an unambiguous act that cannot be used for ambiguity hedging characterizes state dependent dual-self expected utility with a utility overlap across the states.

Axiom 7 There exists a crisp act $c \in H$.

Theorem 9 (Existence, Dual-Self) The following two conditions are equivalent:

- (1) \succeq satisfies Axioms 2-7.
- (2) There is (u, \mathbb{P}) that is a state dependent tight dual-self expected utility for \succeq such that

$$\bigcap_{s \in S} u_s \Big(\Delta(X_s) \Big) \neq \emptyset$$

This result shows the generality of the state dependent dual-self expected utility. Essentially, the only standard ambiguity preferences that do not have a state dependent dual-self representation are such that every act is ambiguous or every act contains different ambiguity.

Remark Combining Axioms 6 and 7 to a stronger axiom that there exist crisp acts f^*, g^* such that for all $s \in S^{\mathcal{P}}, f_s^* \succ_s g_s^*$, gives a state dependent dual-self expected utility with int $\bigcap_{s \in S} u_s(\Delta(X_s)) \neq \emptyset$.

5 Related Literature

State independent dual-self and dual-self variational expected utilities were introduced in Chandrasekher et al. (2022) building on Ghirardato et al.'s (2004) approach of using Clarke derivatives to capture the beliefs of the decision maker. These models are non-convex generalizations of multiple prior preferences (Gilboa and Schmeidler, 1989) and variational preferences (Maccheroni et al., 2006).

The previous literature on state dependent utility has mainly focused on axiomatizing the state dependent expected utility using some additional information: Karni (2007) assumes preferences on conditional acts which are acts conditional on a given event happening. Karni (2011a; 2011b) and Tsakas (2023) assume preferences conditional on signals and uses updating of probabilities for identification. Karni et al. (1983) and Karni and Schmeidler (2016) assume preferences on hypothetical lotteries that are lotteries on state-consequence pairs. Lu (2019) uses two random choice data sets before and after updating beliefs for the identification of intensities of utilities.

Chew and Wang (2020) exemplifies that state and rank dependent expected utility can be identified under two states of the world. State and rank dependent expected utility is a special case of state dependent dual-self expected utility. Karni (2020) shows that in state and rank dependent expected utility with rank-dependent probabilities the utilities and probabilities are not identified.

Drèze (1958; 1961; 1987; 2004) studies state dependent maxmax expected utility in a rich setting when the acts are lotteries of Anscombe-Aumann acts in the context of moral hazard. However, in the Appendix, we show that the identification theorems are incorrect since Drèze does not focus on tight representations. Baccelli (2019) discusses Drèze's contribution extensively.

Baccelli (2019) studies the identification of state dependent subjective expected utility with act-dependent probabilities. However, in the Appendix, we show that the identification result is incorrect due to the non-uniqueness of the act-dependent probabilities.

Hill (2019) studies state dependent maxmin expected utility such that the best and the worst acts have a constant utility in the state independent Anscombe-Aumann setting without Risk Independence. He assumes that the best and the worst acts are crisp or unambiguous acts. Hill shows that when the best and the worst acts have a constant utility and the representation is linear between the best and the worst act, we recover the standard state independent identification. In contrast under Risk Independence, we consider the identification of representations for any state dependent utilities.

Additionally, our paper contributes to the literature on modeling choice under uncertainty that was recently discussed in Gilboa et al. (2020). Since Savage (1954), it has been the prevalent assumption in decision making under uncertainty that states of the world and consequences are separated and any consequence is achievable from any state of the world. With state independent utility, this leads to a difficulty in defining what are the consequences which are not affected by the realized state of the world as illustrated by the exchange between Aumann (1971) and Savage (1971). To resolve this problem, Savage (1954; 1967) used extended consequences that are pure experiences of receiving the original consequences in each state of the world. However, acts over these pure experiences are not observable and in Savage (1954) the identification of probabilities for the states relies crucially on unobservable imaginary acts (Shafer, 1986). In contrast, we allow for state dependent consequences and for state dependent utility. This relaxes the assumption that any consequence is achievable from any state of the world and we do not require the use of pure experiences as consequences. Our state dependent setting allows for more tractable and observable modeling of choice under uncertainty. Especially, we show that the beliefs of the decision maker can be observed even without simple bets on events.

6 Conclusion

The assumption of state independent utilities has been the simplifying, but nonideal, assumption to separate the subjective probabilities from the utilities. This paper provided a novel foundation for state dependent utility by studying models of non-expected utilities. We showed that with a state dependent version of dual-self expected utility (Chandrasekher et al., 2022) state dependent utilities and probabilities are fully identified under full-dimensional uncertainty. Additionally, with a state dependent version of more general dual-self variational expected utility (Chandrasekher et al., 2022) the intensities of preferences and probabilities are fully identified under full-dimensional uncertainty; however, the levels of utilities are not identified.

These identifications encompass those for state dependent versions of all the special cases and alternative representations of dual-self and dual-self variational expected utilities: maxmin expected utility (Gilboa and Schmeidler, 1989), Choquet expected utility (Schmeidler, 1989), α -maxmin expected utility (Ghirardato et al., 2004), invariant biseparable preferences (Ghirardato et al., 2004; Amarante, 2009), monotone mean-variance preferences (Maccheroni et al., 2009), multiplier preferences (Hansen and Sargent, 2001; Strzalecki, 2011), variational preferences (Maccheroni et al., 2006), monotonic mean-dispersion preferences (Grant and Polak, 2013), and vector expected utility (Siniscalchi, 2009).

Additionally, we offered an elicitation method for the state dependent utilities under full-dimensional uncertainty. First, we showed that with dual-self expected utility, constant utilities across states are identified by unambiguous acts. Second, with dual-self variational expected utility, constant utility differences between two acts are identified by equally ambiguous acts. This allows the elicitation of state dependent utilities and beliefs based on standard choice data in contrast to the previous approaches in the literature.

Finally, we considered the existence of the representations. First, we showed that, essentially, the only standard preferences under ambiguity that do not have a state dependent dual-self variational expected utility representation are such that every act contains different ambiguity. Second, we showed that, essentially, the only standard preferences without a state dependent dual-self expected utility representation are such that every act contains different ambiguity or every act is ambiguous. Our general framework allows for applications beyond just uncertainty. It is applied in Mononen (2024b) to the context of social choice theory. In there, our identification results show the identification of interpersonal utility comparisons and the fairness of the society. Second, in the context of intertemporal choice, our results provide foundations and identification for evolving tastes over time when there is uncertainty about the discount factor following Wakai (2008) and Mononen (2024a).

Appendix to "State Dependent Utility and Ambiguity"

This appendix clarifies the previous literature and proves the main identification result, Theorem 2, and the general partial identification result, Theorem 7. This appendix is organized as follows. First, Section A provides counterexamples for Baccelli's Proposition 1 (2019), Drèze's Theorem 8.2 (1987) and Drèze and Rustichini's Theorem 6.11 (2004). Second, Section B.1 introduces the notation used in the appendix. Section B.2 shows that for state dependent dual-self variational expected utility equally crisp acts are characterized by additivity in exchanging one crisp act for the other. Section B.3 shows how these identified directional derivatives give the uniqueness of the state dependent utilities and beliefs. Finally, Section B.4 shows that Axiom 1 captures full-dimensional uncertainty that gives the full-identification results, Theorems 1 and 2.

Next, we show Propositions 3 and 4 and characterize the identification with crisp and equally crisp acts. First, the identification with equally crisp acts, Proposition 3, follows directly from Corollary 15 and Lemma 19.

A Counterexamples for Previous Literature

A.1 A Counterexample for Baccelli's Proposition 1 (2019)

Example 1 State space $S = \{1, 2\}$. Preferences on \mathbb{R}^2 that are represented by $0.5x_1 + 0.5x_2$ if $(x_1, x_2) \neq (0, 0)$ and by x_1 if $(x_1, x_2) = (0, 0)$. Here, the act-dependent probabilities (0.5, 0.5) and (1, 0) are linearly independent. Proposition 1 claims that they are unique. However, the preferences have an alternative representation with state dependent utility for any $p \in (0, 1)$ by the utilities $u_1(x) = \frac{1}{p}x$ and $u_2(x) = \frac{1}{1-p}x$ and the representation $pu_1(x_1) + (1-p)u_2(x_2)$ if $(x_1, x_2) \neq (0, 0)$ and by $u_1(x_1)$ if

 $(x_1, x_2) = (0, 0)$. This contradicts Proposition 1.

A.2 A Counterexample for Drèze's Theorem 8.2 (1987) and Drèze and Rustichini's Theorem 6.11 (2004)

Example 2 State space $S = \{1, 2\}$. Preferences on $[0, 1]^2$ that have a maxmax representation with state dependent utility $u_1(x) = x$ and $u_2(x) = x - 1$ and the set of probabilities $\{(p, 1 - p) | p \in [\frac{1}{2}, 0]\}$. Since this set of probabilities contains two linearly-independent probabilities, Theorems 6.11 and 8.2 claim that the state dependent utility is unique up to a common positive affine transformation and the set of probabilities is unique. However, this set of probabilities is not tight since only the probability $(\frac{1}{2}, \frac{1}{2})$ is used in the maxmax representation. So, the preferences have an alternative maxmax representation with the same utility and a singleton set of probabilities of $(\frac{1}{2}, \frac{1}{2})$. By the example in Section 2, the state-dependent utility is not identified contradicting Theorems 6.11 and 8.2.

B Identification Characterizations

B.1 Preliminaries

For clarity, we assume, without loss of generality, that $S = S^{\mathcal{P}}$ since the probability for not proper states is always zero and identified and the utilities are not identified. Additionally, for clarity, we focus on state dependent niveloid representation that is an alternative representation for state dependent dual-self variational representation. Before defining this, we introduce some notation. Let u be an affine state dependent utility and $I: u(H) \to \mathbb{R}$ be a function. Denote by $\overline{1} \in \mathbb{R}^S$ a constant vector of 1. We say that

• I is monotonic if for all $\varphi, \psi \in u(H)$ such that for all $s \in S, \varphi_s \ge \psi_s, I(\varphi) \ge I(\psi)$.

- I is C-additive if for all $\varphi \in u(H), \alpha \ge 0$ such that $\varphi + \alpha \overline{1} \in u(H), I(\varphi + \alpha \overline{1}) = I(\varphi) + \alpha$.
- I is positive homogeneous if for all $\varphi \in u(H), \alpha > 0$ such that $\alpha \varphi \in u(H),$ $I(\alpha \varphi) = \alpha I(\varphi).$

Definition (u, I) is a state dependent niveloid representation for \succeq if for each $s \in S$, $u_s : \Delta(X_s) \to \mathbb{R}$ is affine, $I : u(H) \to \mathbb{R}$ is C-additive and monotonic, and for all $f, g \in H$,

$$f \succeq g \iff I(u(f)) \ge I(u(g)).$$

Especially, a state dependent niveloid representation such that I is positive homogeneous is an alternative representation for state dependent dual-self expected utility.

We use the following notation. Let $A \subseteq \mathbb{R}^S$ be a convex set. For every $\varphi \in \operatorname{int} A, \xi \in \mathbb{R}^S$, the *Clarke upper derivative* of I at φ in the direction ξ is

$$I^{\circ}(\varphi;\xi) = \limsup_{\psi \to \varphi, t \searrow 0} \frac{I(\psi + t\xi) - I(\psi)}{t}.$$

The Clarke subdifferential of I at φ is the set

$$\partial I(\varphi) = \{ \chi \in \mathbb{R}^S | \, \forall \, \xi \in \mathbb{R}^S, \chi \cdot \xi \le I^{\circ}(\varphi; \xi) \}.$$

Ghirardato et al. (2004) has shown that the union of all Clarke subdifferentials, $\bigcup_{\varphi \in \operatorname{int} u(H)} \partial I(\varphi)$, captures the decision maker's beliefs.

Example 3 Assume that (u, \mathbb{C}) is a state dependent tight dual-self variational expected utility for \succeq . Define $I : u(H) \to \mathbb{R}$ by for all $f \in H$,

$$I(u(f)) = \max_{c \in \mathbb{C}} \min_{p \in \Delta(S)} \sum_{s \in S} p_s u_s(f_s) + c(p).$$

Then I is a state dependent niveloid representation for \succsim and

$$\overline{\operatorname{co}} \bigcup_{c \in \mathbb{C}} \operatorname{dom} c = \bigcup_{\varphi \in \operatorname{int} u(H)} \partial I(\varphi).$$

Example 4 Assume that (u, \mathbb{P}) is a state dependent tight dual-self expected utility for \succeq . Define $I : u(H) \to \mathbb{R}$ by for all $f \in H$,

$$I(u(f)) = \max_{P \in \mathbb{P}} \min_{p \in P} \sum_{s \in S} p_s u_s(f_s).$$

Then I is a state dependent niveloid representation for \succeq such that I is positive homogeneous and

$$\overline{\operatorname{co}}\bigcup_{P\in\mathbb{P}}P=\bigcup_{\varphi\in\operatorname{int}u(H)}\partial I(\varphi)$$

We start with a lemma showing that the state dependent utilities are identified up to statewise positive affine transformations.

Lemma 10 Assume that (u, I) and (\tilde{u}, \tilde{I}) are state dependent niveloid representations for \succeq . Then, for each $s \in S^{\mathcal{P}}$, there are $A_s > 0$ and $B_s \in \mathbb{R}$ such that $\tilde{u}_s = A_s u_s + B_s$.

Proof. Let $s \in S^{\mathcal{P}}$. By the monotonicity of I, we have for all $x_s, y_s \in \Delta(X_s)$ if $x_s \succ y_s$ then $u_s(x_s) > u_s(y_s)$ and $\tilde{u}_s(x_s) > \tilde{u}_s(y_s)$. Additionally, clearly, Axioms 2-5 are necessary and satisfied by \succeq . Thus by Lemma 26, \succeq_s is complete, transitive, non-trivial, and continuous. So by Mononen (2022), there are $A_s > 0$ and $B_s \in \mathbb{R}$ such that $\tilde{u}_s = A_s u_s + B_s$

B.2 State Dependent Niveloid Representation Additive Between Equally Crisp Acts

We first show that any state dependent niveloid representation is additive between equally crisp acts. The next lemma shows that the set of equally crisp acts is convex. This simple proof is omitted.

Lemma 11 Assume that f^*, g^* and \tilde{f}^*, \tilde{g}^* are equally crisp acts and $\alpha \in [0, 1]$. Then $\alpha f^* + (1 - \alpha)\tilde{f}^*, \alpha g^* + (1 - \alpha)\tilde{g}^*$ are equally crisp acts.

Lemma 12 Assume that (u, I) is a state dependent niveloid representation for \succeq . If f^* and g^* are equally crisp acts then for all $\alpha \in (0, 1)$

$$I(\alpha u(f^*) + (1-\alpha)u(g^*)) = \alpha I(u(f^*)) + (1-\alpha)I(u(g^*)).$$

Proof. First, assume that there exist $m, n \in \mathbb{N}$ such that $\alpha = \frac{m}{n}$. Let $h \in H, \beta \in (0, 1)$ be such that $u(h) \in \operatorname{int} u(H)$. We show that

$$I\left(\beta u(h) + (1-\beta)\left(\alpha u(f^*) + (1-\alpha)u(g^*)\right)\right)$$

$$= \alpha I\left(\beta u(h) + (1-\beta)u(f^*)\right) + (1-\alpha)I\left(\beta u(h) + (1-\beta)u(g^*)\right).$$
(1)

Now there exist $\tilde{h} \in H, c \in \mathbb{R}$ and $\tilde{n} \in \mathbb{N}$ such that $\beta u(h) + (1 - \beta)u(f^*) + c1, \beta u(h) + (1 - \beta)u(g^*) + c1, \in \text{int } u(H)$ and

$$\beta u(\tilde{h}) + (1 - \beta)u(g^*) = \beta u(h) + (1 - \beta)u(g^*) + c1$$

and

$$\beta \tilde{h} + (1-\beta)g^* \sim \beta h + (1-\beta) \left(\frac{1}{n\tilde{n}}f^* + \frac{n\tilde{n}-1}{n\tilde{n}}g^*\right).$$

Since f^* and g^* are equally crisp, we have for all $m \in \{1, \ldots, n\tilde{n}\}$,

$$\beta \tilde{h} + (1-\beta) \left(\frac{m-1}{n\tilde{n}} f^* + \frac{n\tilde{n}-m+1}{n\tilde{n}} g^* \right) \sim \beta h + (1-\beta) \left(\frac{m}{n\tilde{n}} f^* + \frac{n\tilde{n}-m}{n\tilde{n}} \right) g^*.$$

By C-additivity of I, for all $m \in \{1, \ldots, n\tilde{n}\},\$

$$I\left(\beta u(h) + (1-\beta)\left(\frac{m-1}{n\tilde{n}}u(f^*) + \frac{n\tilde{n}-m+1}{n\tilde{n}}u(g^*)\right)\right) + c$$
$$= I\left(\beta u(h) + (1-\beta)\left(\frac{m}{n\tilde{n}}u(f^*) + \frac{n\tilde{n}-m}{n\tilde{n}}u(g^*)\right)\right).$$

Thus for each $m \in \{0, \ldots, n\tilde{n}\},\$

$$I\left(\beta u(h) + (1-\beta)\left(\frac{m}{n\tilde{n}}u(f^*) + \frac{n\tilde{n}-m}{n\tilde{n}}u(g^*)\right)\right) = I\left(\beta u(h) + (1-\beta)u(g^*)\right) + mc.$$
(2)

Hence, by (2) for $m = n\tilde{n}$

$$c = \frac{1}{n\tilde{n}} \left(I \Big(\beta u(h) + (1-\beta)u(f^*) \Big) - I \Big(\beta u(h) + (1-\beta)u(g^*) \Big) \right).$$

Thus, (2) for $m = m\tilde{n}$ shows (1). By the continuity of I, by taking $\beta \to 0$ shows the claim for all $\alpha \in [0, 1]$.

The next lemma shows that the constant directional derivative to the direction $u(f^*) - u(g^*)$ gives the additivity of the representation to the direction $u(f^*) - u(g^*)$ that characterizes equally crisp acts.

Lemma 13 Assume that (u, I) is a state dependent niveloid representation for \succeq and f^* and g^* are equally crisp acts. Then for all $h \in H, \beta \in (0, 1)$

$$I(\beta u(h) + (1 - \beta)u(f^*)) - I(\beta u(h) + (1 - \beta)u(g^*)) = (1 - \beta)(I(u(f^*)) - I(u(g^*))).$$

Proof. Assume first that $f^* \succ g^*$. Denote

$$A = \left\{ \alpha \in [0,\beta] \middle| I \bigl(\alpha u(h) + (1-\alpha)u(f^*) \bigr) - I \bigl(\alpha u(h) + (1-\alpha)u(g^*) \bigr) \\ = (1-\alpha) \Bigl(I \bigl(u(f^*) \bigr) - I \bigl(u(g^*) \bigr) \Bigr) \right\}.$$

Since I is continuous, A is closed. Since A is non-empty, max A exists. We show that max $A = \beta$. Let $\alpha^* \in A$ and $\alpha^* < \beta$. Since $\alpha^* h + (1 - \alpha^*) f^* \succ \alpha^* h + (1 - \alpha^*) g^*$, there exist $\tilde{\alpha} \in (\alpha^*, \beta)$ and $\gamma \in (0, 1)$ such that

$$\tilde{\alpha}h + (1 - \tilde{\alpha})(0.5f^* + 0.5g^*) \sim \alpha^*h + (1 - \alpha^*) \Big(\gamma f^* + (1 - \gamma)g^*\Big).$$
(3)

Let $\varepsilon < \min\{(1-\gamma)(1-\alpha^*), 0.5(1-\tilde{\alpha})\}$. By writing $\tilde{\alpha}h + (1-\tilde{\alpha})(0.5f^*+0.5g^*)$ and $\alpha^*h + (1-\alpha^*)(\gamma f^* + (1-\gamma)g^*)$ as probability ε mixtures with g^* , we have since f^* and g^* are equally crisp.

$$\tilde{\alpha}h + (1 - \tilde{\alpha})\left(\left(0.5 + \frac{1}{1 - \tilde{\alpha}}\varepsilon\right)f^* + \left(0.5 - \frac{1}{1 - \tilde{\alpha}}\varepsilon\right)g^*\right)$$

$$\sim \alpha^*h + (1 - \alpha^*)\left(\left(\gamma + \frac{1}{1 - \alpha^*}\varepsilon\right)f^* + \left(1 - \gamma - \frac{1}{1 - \alpha^*}\varepsilon\right)g^*\right).$$
(4)

Thus by Lemmas 11 and 12 and (3,4), $\tilde{\alpha} \in A$. Thus max $A = \beta$.

Assume next that $f^* \sim g^*$. We show that for all $\alpha \in (0,1)$, $\alpha h + (1-\alpha)f^* \sim \alpha h + (1-\alpha)g^*$. Assume, per contra, that there exist $\alpha^* \in (0,1)$ with $\alpha^* h + (1-\alpha^*)f^* \succ \alpha^* h + (1-\alpha^*)g^*$. Denote

$$A = \Big\{ \alpha \in [0, \alpha^*] \Big| \alpha h + (1 - \alpha) f^* \not\sim \alpha h + (1 - \alpha) g^* \Big\}.$$

By the first part, for all $\alpha \in A$,

$$I(\alpha u(h) + (1 - \alpha)u(f^*)) - I(\alpha u(h) + (1 - \alpha)u(g^*))$$

= $\frac{1 - \alpha}{1 - \alpha^*} \Big(I(\alpha^* u(h) + (1 - \alpha^*)u(f^*)) - I(\alpha^* u(h) + (1 - \alpha^*)u(g^*)) \Big).$

Thus by the continuity of I, A is closed. But also the complement of A in $[0, \alpha^*]$ is nonempty and closed which is a contradiction.

Finally, equally crisp acts can be characterized by having certainty about their expected utility difference. First, we relate Clarke derivatives to standard derivatives.

Lemma 14 Assume that (u, I) is a state dependent niveloid representation for \succeq . Then for all $f \in H$ with $u(f) \in \operatorname{int} u(H)$,

$$\partial I(u(f)) = \operatorname{convex} \left\{ \lim_{i \to \infty} \nabla I(\varphi_i) \Big| (\varphi_i)_{i=1}^{\infty} \subseteq u(H), \lim_{i \to \infty} \varphi_i = u(f), \forall i \in \mathbb{N}, I \text{ differentiable at } \varphi_i \right\}$$

Proof. Since I is Lipschitz on u(H), this follows directly from Clarke's (1983) Theorem 2.5.1.

Corollary 15 Assume that (u, I) is a state dependent niveloid representation for \succeq . f and g are equally crisp acts iff there exist $\alpha \in \mathbb{R}$ such that for all $p \in \bigcup_{\varphi \in \operatorname{int} u(H)} \partial I(\varphi)$

$$p \cdot (u(f) - u(g)) = \alpha.$$

Especially, then $\alpha = I(u(f)) - I(u(g))$.

Proof. \implies : Follows directly from Lemmas 13 and 14.

 $\iff: \text{By Chandrasekher et al.'s (2022) proof of Theorem 1, there exists } \mathbb{C} \subseteq \{c : \Delta(S) \to \mathbb{R} \cup \{\infty\} | c \text{ is convex} \}$ such that for all $\varphi \in u(H)$, $I(\varphi) = \max_{c \in \mathbb{C}} \min_{p \in \text{dom } c} p \cdot \varphi + c(p)$ and for each $c \in \mathbb{C}$, dom $c \subseteq \bigcup_{\varphi \in \text{int } u(H)} \partial I(\varphi)$. By the assumption, we have for all $h \in H$ and $\beta \in [0, 1]$

$$I\left(u(\beta h + (1-\beta)f)\right) = \max_{c \in \mathbb{C}} \min_{p \in \text{dom}\,c} \beta p \cdot u(h) + (1-\beta)p \cdot u(f) + c(p)$$
$$= \max_{c \in \mathbb{C}} \min_{p \in \text{dom}\,c} \beta p \cdot u(h) + (1-\beta)p \cdot u(g) + c(p) + (1-\beta)\alpha = I\left(u(\beta h + (1-\beta)g)\right) + (1-\beta)\alpha.$$

This shows that f and g are equally crisp acts.

B.3 Partial Identification

Define the unambiguous preference \succeq^* by for all $f, g \in H$,

$$f \succeq^* g \iff \alpha f + (1 - \alpha)h \succeq \alpha g + (1 - \alpha)h$$
 for all $\alpha \in (0, 1]$ and $h \in H$.

Definition (u, P) is a state dependent Bewley expected utility for \succeq^* if $u = (u_s)_{s \in S}$ and for all $s \in S$, $u_s : \Delta(X_s) \to \mathbb{R}$ is affine and $P \subseteq \Delta(S)$ is closed, convex, and nonempty such that for each $p \in P, s \notin S^{\mathcal{P}}$, $p_s = 0$ and for all $f, g \in H$,

$$f \succeq^* g \iff \sum_{s \in S} p_s u_s(f_s) \ge \sum_{s \in S} p_s u_s(g_s) \text{ for all } p \in P.$$

We show in the online appendix Proposition S.1 the uniqueness of state dependent Bewley expected utility representation.

We omit the proof the next standard result.

Lemma 16 Assume that (u, I) is a state dependent niveloid representations for \succeq . Then $\left(u, \overline{\operatorname{co}} \bigcup_{\varphi \in \operatorname{int} u(H)} \partial I(\varphi)\right)$ is a state dependent Bewley expected utility for \succeq^*

The next lemma shows the general identification for dual-self variational representation's utilities and probabilities. For all $x \in \mathbb{R}^S$ and $p \in \Delta(S)$, define multiplications statewise

$$xp = (x_s p_s)_{s \in S}.$$

This induces multiplications of $x \in \mathbb{R}^S$ with sets $P \in \Delta(S)$.

Lemma 17 Assume that (u, I) and (\tilde{u}, \tilde{I}) are state dependent niveloid representations for \succeq . Then there exist $x \in \mathbb{R}^{S}_{++}, B \in \mathbb{R}^{S}$, and $\alpha \in \mathbb{R}_{++}$ such that for all $p \in \bigcup_{\varphi \in \operatorname{int} u(H)} \partial I(\varphi), \sum_{s \in S} x_{s} p_{s} = 1$,

$$\overline{\operatorname{co}} \bigcup_{\varphi \in \operatorname{int} \widetilde{u}(H)} \partial \widetilde{I}(\varphi) = x \overline{\operatorname{co}} \bigcup_{\varphi \in \operatorname{int} u(H)} \partial I(\varphi).$$

and

$$(\tilde{u}_s)_{s\in S^{\mathcal{P}}} = \left(\frac{\alpha}{x_s}u_s + B_s\right)_{s\in S^{\mathcal{P}}}$$

Proof. By Lemma 10, there exist $A \in \mathbb{R}^{S}_{++}, B \in \mathbb{R}^{S}_{++}$ such that for all $s \in S^{\mathcal{P}}$,

$$\widetilde{u}_s = A_s u_s + B_s.$$

Let $f^*, g^* \in H$ with $\tilde{u}(f^*), \tilde{u}(g^*) \in \operatorname{int} \tilde{u}(H)$ and $c \in \mathbb{R}_{++}$ be such that for all $s \in S^{\mathcal{P}}$, $\tilde{u}_s(f_c^*) = \tilde{u}_s(g_c^*) + c.$ By Corollary 15, f^* and g^* are equally crisp. By Corollary 15, for all $p \in \bigcup_{\varphi \in int u(H)} \partial I(\varphi)$

$$p \cdot A^{-1} = p \cdot \left(u(f^*) - u(g^*) \right) = I(u(f^*)) - I(u(g^*)).$$

Denote $\alpha = I(u(f)) - I(u(g))$ and $x = A^{-1}\alpha^{-1}$.

By Lemma 16, $(u, \overline{\operatorname{co}} \bigcup_{\varphi \in \operatorname{int} u(H)} \partial I(\varphi))$ and $(\tilde{u}, \overline{\operatorname{co}} \bigcup_{\varphi \in \operatorname{int} \tilde{u}(H)} \partial \tilde{I}(\varphi))$ give state dependent Bewley expected utilities for \succeq^* . By the uniqueness of state dependent Bewley expected utility, Proposition S.1, we have

$$\overline{\operatorname{co}} \bigcup_{\varphi \in \operatorname{int} \widetilde{u}(H)} \partial \widetilde{I}(\varphi) = x \overline{\operatorname{co}} \bigcup_{\varphi \in \operatorname{int} u(H)} \partial I(\varphi).$$

Theorem 7 follows as a direct corollary of Lemma 17 by Example 3.

B.4 Full Identification

Next, we move on to full-identification and show how Axiom 1 characterizes the full identification. We start by showing that any utility difference between two acts can be realized with strict preference for all the strict utility differences.

Lemma 18 Assume that (u, I) is a state dependent niveloid representation for \succeq . Then there exist $f \in H$ with $u(f) \in \operatorname{int} u(H)$ such that for all $g \in H$ and $s \in S^{\mathcal{P}}$, $f_s \succ_s g_s$ iff $u_s(f_s) > u_s(g_s)$ and $f_s \succeq_s g_s$ iff $u_s(f_s) \ge u_s(g_s)$.

Proof. By Lemma 14 and Chandrasekher et al.'s (2022) Supplementary Appendix's Remark 3, for each $s \in S^{\mathcal{P}}$ there exist $f^s \in H$ such that $u(f^s) \in \operatorname{int} u(H)$ and I is differentiable at $u(f^s)$ with $\nabla I(u(f^s))_s > 0$. Then $(f^s_s)_{s \in S}$ satisfies the claim. \Box

The next result shows that Axiom 1 characterizes the full-dimensional uncertainty that gives the identification by previous partial identification.

Lemma 19 Assume that (u, I) is a state dependent niveloid representation for \succeq . The following two are equivalent: (I) \succeq satisfies Axiom 1.

(II) $\operatorname{pr}_{S^{\mathcal{P}}} \overline{\operatorname{co}} \bigcup_{\varphi \in \operatorname{int} u(H)} \partial I(\varphi)$ has a non-empty interior in $\Delta(S^{\mathcal{P}})$.

Proof. Assume that (I) holds. Let $f \in H$ be as in Lemma 18 and $g \in H$ be such that for all $p \in \bigcup_{\varphi \in \operatorname{int} u(H)} \partial I(\varphi)$, $p \cdot (u(f) - u(g))$. By Lemma 13 and Corollary 15, $f \sim^* g$. By Axiom 1 and the choice of f, for all $s \in S^{\mathcal{P}}$, $u_s(f_s) = u_s(g_s)$. Since $u(f) \in \operatorname{int} u(H)$, this shows that $(\operatorname{pr}_{S^{\mathcal{P}}} \overline{\operatorname{co}} \bigcup_{\varphi \in \operatorname{int} u(H)} \partial I(\varphi))^{\perp} = \{\overline{0}\}$. Hence, following Boyd and Vandenberghe's (2004) Section 2.5.2 it has a non-empty interior in $\Delta(S^{\mathcal{P}})$.

Assume that (II) holds. Assume that $f \sim^* g$. Especially, f and g are equally crisp. By Corollary 15,

$$\operatorname{pr}_{S^{\mathcal{P}}} u(f) - u(g) \in \left(\operatorname{pr}_{S^{\mathcal{P}}} \overline{\operatorname{co}} \bigcup_{\varphi \in \operatorname{int} u(H)} \partial I(\varphi) \right)^{\perp} = \{ \overline{0} \}$$

that shows (I) by the monotonicity of I.

Finally, Theorems 1 and 2 follow directly from Lemmas 17, 19, and 23.

B.5 Partial Identification for State Dependent Dual-Self Representation

The next lemma shows that the values of the state dependent niveloid representation are unique up to a positive affine transformation.

Lemma 20 Assume that (u, I) and (\tilde{u}, \tilde{I}) are state dependent niveloid representations for \succeq . Let f^* and g^* be equally crisp acts such that $f^* \succ g^*$. If $f, g \in H$ with $u(f), u(g) \in \operatorname{int} u(H)$, then

$$\widetilde{I}(u(f)) - \widetilde{I}(\widetilde{u}(g)) = \frac{\widetilde{I}(\widetilde{u}(f^*)) - \widetilde{I}(\widetilde{u}(g^*))}{I(u(f^*)) - I(u(g^*))} \Big(I(u(f)) - I(u(g))\Big)$$

Proof. If $f \sim g$, then the claims follows by the representations. So assume that $f \succ g$. Now there exist $\beta \in (0, 1), h, h' \in H$ with $u(h), u(h') \in \operatorname{int} u(H)$ such that

$$u(f) = \beta u(h) + (1 - \beta)u(g^*) \text{ and } u(g) = \beta u(h') + (1 - \beta)u(g^*).$$
 (5)

Let $n \in \mathbb{N}$ and $\gamma \in (0, 1)$ be such that

$$\frac{1}{n}\Big(I\Big(u(f)\Big) - I\Big(u(g)\Big)\Big) = (1-\beta)\gamma\Big(I\Big(u(f^*)\Big) - I\Big(u(g^*)\Big)\Big).$$
(6)

For each $i \in \{0, \ldots n\}$, there exist $\alpha^i \in [0, 1]$ such that

$$I\left(\alpha^{i}u(f) + (1 - \alpha^{i})u(g)\right) = \frac{i}{n}I\left(u(f)\right) + \frac{n - i}{n}I\left(u(g)\right)$$

By (5), for each $i \in \{0, ..., n-1\}$,

$$\alpha^{i}f + (1 - \alpha^{i})g \sim \beta(\alpha^{i}h + (1 - \alpha^{i})h') + (1 - \beta)g^{*}$$

and by Lemma 13,

$$\alpha^{i+1}f + (1 - \alpha^{i+1})g \sim \beta(\alpha^{i}h + (1 - \alpha^{i})h') + (1 - \beta)(\gamma f^* + (1 - \gamma)g^*).$$

Hence, by Lemma 13,

$$\widetilde{I}\left(\alpha^{i+1}\widetilde{u}(f) + (1-\alpha^{i+1})\widetilde{u}(g)\right) - \widetilde{I}\left(\alpha^{i}\widetilde{u}(f) + (1-\alpha^{i})\widetilde{u}(g)\right) = (1-\beta)\gamma\left(\widetilde{I}\left(\widetilde{u}(f^{*})\right) - \widetilde{I}\left(\widetilde{u}(g^{*})\right)\right).$$

Thus,

$$\begin{split} \widetilde{I}\big(\widetilde{u}(f)\big) - \widetilde{I}\big(\widetilde{u}(g)\big) &= \sum_{i=0}^{n-1} \widetilde{I}\big(\alpha^{i+1}\widetilde{u}(f) + (1-\alpha^{i+1})\widetilde{u}(g)\big) - \widetilde{I}\big(\alpha^{i}\widetilde{u}(f) + (1-\alpha^{i})\widetilde{u}(g)\big) \\ &= n(1-\beta)\gamma\Big(\widetilde{I}\big(\widetilde{u}(f^*)\big) - \widetilde{I}\big(\widetilde{u}(g^*)\big)\Big) \end{split}$$

that shows the claim by (6).

As a direct corollary the values of the state dependent niveloid representation are unique up to a positive affine transformation.

Lemma 21 Assume that (u, I) and (\tilde{u}, \tilde{I}) are state dependent niveloid representations for \succeq . Let f^* and g^* be equally crisp acts such that $f^* \succ g^*$. Denote

$$a = \frac{I(\widetilde{u}(f^*)) - I(\widetilde{u}(g^*))}{I(u(f^*)) - I(u(g^*))} \text{ and } b = \widetilde{I}(\widetilde{u}(g^*)) - \alpha I(u(g^*)).$$

Then for all $f \in H$,

$$\widetilde{I}(\widetilde{u}(f)) = aI(u(f)) + b.$$

Next, we move on to partial identification of dual-self representation. The next lemma shows how for dual-self representation the derivative gives the value of the representation.

Lemma 22 Assume that (u, I) is a state dependent niveloid representation for \succeq and I is positive homogeneous. Let $f \in H$ be such that $u(f) \in \operatorname{int} u(H)$. If I is differentiable at u(f) with derivative p, then $I(u(f)) = p \cdot u(f)$.

Proof. Since $u(f) \in \operatorname{int} u(H)$, there exists $\alpha^0 > 0$ such that $(1 + \alpha^0)u(f) \in u(H)$. Since u(H) is convex and I is positive homogeneous, we have for all $\alpha \in (0, \alpha^0)$,

$$I((1+\alpha)u(f)) = (1+\alpha)I(u(f)) \Longrightarrow \frac{I(\alpha u(f) + u(f)) - I(u(f))}{\alpha} = I(u(f)).$$

Thus by taking the limit $\alpha \to 0$, we have by the differentiability of I at u(f),

$$I(u(f)) = \lim_{\alpha \to 0} \frac{I(\alpha u(f) + u(f)) - I(u(f))}{\alpha} = p \cdot u(f).$$

The next lemma shows the identification for probabilities and utilities.

Lemma 23 Assume that (u, I) and (\tilde{u}, \tilde{I}) are state dependent niveloid representations for \succeq that are positive homogeneous. Then there exist $x \in \mathbb{R}^{S}_{++}, y \in \mathbb{R}^{S}, \alpha \in \mathbb{R}_{++}$, and $\beta \in \mathbb{R}$ such that for all $p \in \bigcup_{\varphi \in \operatorname{int} u(H)} \partial I(\varphi), \sum_{s \in S} x_s p_s = 1, \sum_{s \in S} y_s p_s = 0$,

$$\bigcup_{\varphi \in \operatorname{int} \widetilde{u}(H)} \partial \widetilde{I}(\varphi) = x \bigcup_{\varphi \in \operatorname{int} u(H)} \partial I(\varphi).$$

and for all $S^{\mathcal{P}}$,

$$(\widetilde{u}_s)_{s\in S^{\mathcal{P}}} = \left(\frac{\alpha}{x_s}(u_s+y_s)+\beta\right)_{s\in S^{\mathcal{P}}}.$$

Proof. By Lemma 17, there exist $x \in \mathbb{R}_{++}^S$, $B \in \mathbb{R}^S$, and $\alpha \in \mathbb{R}_{++}$ such that for all $p \in \bigcup_{\varphi \in \operatorname{int} u(H)} \partial I(\varphi), \sum_{s \in S} x_s p_s = 1$,

$$(\widetilde{u}_s)_{s\in S^{\mathcal{P}}} = \left(\frac{\alpha}{x_s}u_s + B_s\right)_{s\in S^{\mathcal{P}}}.$$

From Lemma 21, it follows that I is differentiable at $\varphi \in \operatorname{int} u(H)$ with derivative $\nabla I(\varphi)$ iff \tilde{I} is differentiable at $\frac{\alpha}{x}\varphi + B \in \operatorname{int} \tilde{u}(H)$ with derivative $x\nabla I(\varphi)$.

By Lemma 22, we have for all $\varphi \in \operatorname{int} u(H)$ such that φ is a differentiability point of I.

$$\alpha \nabla I(\varphi) \cdot \varphi + \nabla I(\varphi) \cdot xB = x \nabla I(\varphi) \cdot \left(\frac{\alpha}{x}\varphi + B\right) = \tilde{I}(\varphi)$$
$$= \alpha I(\varphi) + \beta = \alpha \nabla I(\varphi) \cdot \varphi + \beta$$

and so $\nabla I(\varphi) \cdot xB = \beta$. Hence, by Lemma 14, for all $p \in \bigcup_{\varphi \in \operatorname{int} u(H)} \partial I(\varphi)$,

 $p \cdot xB = \beta.$

Defining $\alpha y = xB - x\beta$ shows the claim.

B.6 Identification with Crisp Acts

Next, we show Proposition 4 and characterize the identification with crisp acts. First, we show that state dependent niveloid representation is affine between any act and a crisp act.

Lemma 24 Assume that (u, I) is a state dependent niveloid representation for \succeq . If f^* is crisp act, then for all $h \in H, \alpha \in (0, 1)$,

$$I(\alpha u(h) + (1 - \alpha)u(f^*)) = \alpha I(u(h)) + (1 - \alpha)I(u(f^*)).$$

Proof. First, assume that $u(h) \in \operatorname{int} u(H)$. Now there exist $\tilde{h} \in H, c \in \mathbb{R}$ and $n \in \mathbb{N}$ $\alpha \in (0, 1)$ such that $u(h) + c1 \in \operatorname{int} u(H), \ \alpha^n = \beta$

$$u(\tilde{h}) = u(h) + c1$$
 and $\tilde{h} \sim \alpha h + (1 - \alpha)f^*$.

Since f^* is crisp, we have for all $m \in \{1, \ldots, n\}$,

$$\alpha^{m-1}\tilde{h} + (1 - \alpha^{m-1})f^* \sim \alpha^m h + (1 - \alpha^m)f^*$$

and by the representation

$$I(\alpha^{m-1}u(h) + (1 - \alpha^{m-1})f^*) + \alpha^{m-1}c = I(\alpha^m u(h) + (1 - \alpha^m)f^*).$$

Thus for each $m \in \{0, \ldots, n\}$,

$$I\left(\alpha^m u(h) + (1 - \alpha^m)f^*\right) = I\left(u(h)\right) + \frac{1 - \alpha^m}{1 - \alpha}c$$

By taking the limit $m \to \infty$, we have by the continuity of I

$$\frac{c}{1-\alpha} = I(u(f^*)) - I(u(h))$$

and so

$$I(\alpha^n u(h) + (1 - \alpha^n)f^*) = I(u(h)) + (1 - \alpha^n)\left(I(u(f^*)) - I(u(h))\right)$$
$$= (1 - \alpha^n)I(u(f^*)) + \alpha^n I(u(h)).$$

Finally, by the continuity of I, this shows the results for all $h \in H$.

Next, we show that the expected utility of crisp acts is the same for all probabilities.

Lemma 25 Assume that (u, I) is a state dependent niveloid representation for \succeq that is positive homogeneous. Then $c \in H$ is crisp iff for all $p \in \bigcup_{\varphi \in \operatorname{int} u(H)} \partial I(\varphi)$,

$$p \cdot u(c) = I(u(c)).$$

Proof. \Leftarrow : By Chandrasekher et al.'s (2022) proof of Theorem 1, there exists $\mathbb{P} \in \mathcal{K}(\Delta(S))$ such that for all $\varphi \in u(H)$, $I(\varphi) = \max_{P \in \mathbb{P}} \min_{p \in P} p \cdot \varphi$ and for each $P \in \mathbb{P}$, $P \subseteq \bigcup_{\varphi \in \operatorname{int} u(H)} \partial I(\varphi)$. By the assumption, we have for all $f \in H$ and $\alpha \in [0, 1]$

$$I\left(u(\alpha f + (1 - \alpha)c)\right) = \max_{P \in \mathbb{P}} \min_{p \in P} \alpha p \cdot u(f) + (1 - \alpha)p \cdot u(c)$$
$$= \max_{P \in \mathbb{P}} \min_{p \in P} \alpha p \cdot u(f) + (1 - \alpha)I\left(u(c)\right) = \alpha I\left(u(f)\right) + (1 - \alpha)I\left(u(c)\right)$$

This shows that c is a crisp act.

 \implies : Let $h \in H$ be such that $u(h) \in \operatorname{int} u(H)$ and I is differentiable at u(h). Then since I is affine between u(h) and u(c) by Lemma 24,

$$I(u(c)) = \nabla I(u(h)) \cdot u(h) + \nabla I(u(h)) \cdot (u(c) - u(h)) = \nabla I(u(h)) \cdot u(c).$$

Finally, Proposition 4 follows directly from Lemmas 19 and 25.

C Existence of State Dependent Dual-Self Expected Utility and Dual-Self Variational Expected Utility

C.1 Statewise Weak Affine Representations

Next, we show that each \succeq_s has a weak affine representation as defined next.

Definition $u_s : \Delta(X_s) \to \mathbb{R}$ is a weak affine representation for \succeq_s if u_s is affine and for all $x, y \in \Delta(X_s)$

$$x \succ_s y \Longrightarrow u_s(x) > u_s(y).$$

We show first that \succeq_s is a weak order that is mixture continuous. This follows directly from Axioms 2-4 and we omit the standard proof.

Lemma 26 Let $s \in S$. If \succeq satisfies Axioms 2-4, then \succeq_s is complete, transitive, and for all $x_s, y_s, z_s \in \Delta(X_s)$, the sets $\{\alpha \in [0,1] | \alpha x_s + (1-\alpha)y_s \succeq_s z_s\}$ and $\{\alpha \in [0,1] | z_s \succeq_s \alpha x_s + (1-\alpha)y_s\}$ are closed.

Next, we show the existence of weak affine representation for \succeq .

Lemma 27 If \succeq satisfies Axioms 2-5 and $s \in S$, then there exists an affine u_s : $\Delta(X_s) \to \mathbb{R}$ such that u_s is a weak affine representation for \succeq_s .

Proof. Follows directly from, Lemma 26, Axiom 5, and Mononen (2022, Theorem 1).

The next result shows statewise monotonicity results with the weak affine representation.

Lemma 28 Assume that for each $s \in S^{\mathcal{P}}$, \succeq_s has a weak affine representation with u_s . Then

- (1) For all $s \in S$, $x_s, y_s \in \Delta(X_s)$, if $u_s(x_s) \ge u_s(y_s)$, then $x_s \succeq y_s$.
- (2) For all $f, g \in H$, if for all $s \in S^{\mathcal{P}}$, $u_s(f_s) \ge u_s(g_s)$, then $f \succeq g$.

Proof. The first one follows from the negation of the definition of weak representation. The second one follows from the first one and the transitivity of \succeq .

C.2 Preferences Monotonic Between Equally Crisp Acts

Next we show that \succeq is monotonic when moving from one equally crisp act to another one.

Lemma 29 Assume that \succeq satisfies Axioms 2-5, for each $s \in S^{\mathcal{P}}$, $u_s : \Delta(X_s) \to \mathbb{R}$ is a weak affine representation for \succeq_s and f^* and g^* are equally crisp acts such that for all $s \in S^{\mathcal{P}}$,

$$u_s(f_s^*) > u_s(g_s^*).$$
 (7)

Then for all $\alpha, \alpha' \in [0, 1]$,

$$\alpha \ge \alpha' \iff \alpha f^* + (1 - \alpha)g^* \succeq \alpha' f^* + (1 - \alpha')g^*$$

Proof. For all $\alpha \in [0, 1]$, denote

$$f^* \alpha g^* = \alpha f^* + (1 - \alpha)g^*.$$

Assume, per contra, that there exist $1 > \alpha^* > \alpha_* > 0$ such that

$$f^* \alpha^* g^* \sim f^* \alpha_* g^* \tag{8}$$

By the nontriviality and continuity, there exists $f^0 \not\sim f^* \alpha^* g^*$ with $u(f^0) \in \operatorname{int} u(H)$. W.l.o.g. assume that $f^* \alpha^* g^* \succ f^0$. Denote

$$\alpha^f = \sup\{\alpha \in [0,1] | \alpha f^0 + (1-\alpha) f^* \alpha^* g^* \succeq f^* \alpha^* g^* \}.$$
(9)

By the continuity, $\alpha^f < 1$ and $\alpha^f f^0 + (1 - \alpha^f) f^* \alpha^* g^* \sim f^* \alpha^* g^*$. Since by Lemma 11, $f^* \alpha^* g^*$ and $f^* \alpha_* g^*$ are equally crisp,

$$\alpha^{f} f^{0} + (1 - \alpha^{f}) f^{*} \alpha_{*} g^{*} \sim \alpha^{f} f^{*} \alpha^{*} g^{*} + (1 - \alpha^{f}) f^{*} \alpha_{*} g^{*} = f^{*} (\alpha^{f} \alpha^{*} + (1 - \alpha^{f}) \alpha_{*}) g^{*}.$$
(10)

By (7) and since for each $s \in S$, u_s is affine, there exist $0 < \varepsilon < 1 - \alpha^f$ such that for each $s \in S^{\mathcal{P}}$, Then we have for all $s \in S^{\mathcal{P}}$,

$$u_s\Big((\alpha^f + \varepsilon)f_s^0 + (1 - \alpha^f - \varepsilon)f_s^*\alpha^*g_s^*\Big) \ge u_s\Big(\alpha^f f_s^0 + (1 - \alpha^f)f_s^*\alpha_*g_s^*\Big).$$
(11)

By Lemma 28,

$$(\alpha^{f} + \varepsilon)f^{0} + (1 - \alpha^{f} - \varepsilon)f^{*}\alpha^{*}g^{*} \stackrel{(11)}{\succeq} \alpha^{f}f^{0} + (1 - \alpha^{f})f^{*}\alpha_{*}g^{*} \stackrel{(10)}{\succeq} f^{*}(\alpha^{f}\alpha^{*} + (1 - \alpha^{f})\alpha_{*})g^{*} \stackrel{(7)}{\succeq} f^{*}\alpha_{*}g^{*} \stackrel{(8)}{\sim} f^{*}\alpha^{*}g^{*}.$$

However, this contradicts the definition of α^f in (9). Thus if $1 > \alpha^* > \alpha_* > 0$, then by (7) and Lemma 28, $f^*\alpha^*g^* \succ f^*\alpha_*g^*$. Finally, if $1 \ge \alpha^* > \alpha_* \ge 0$, then by above, (7), and Lemma 28,

$$f^*\alpha^*g^* \succeq f^*(2/3\alpha^* + 1/3\alpha_*)g^* \succ f^*(1/3\alpha^* + 2/3\alpha_*)g^* \succeq f^*\alpha_*g^*.$$

The next result generalizes the previous monotonicity when the equally crisp acts are mixed with another act.

Lemma 30 Assume that \succeq satisfies Axioms 2-5, for each $s \in S^{\mathcal{P}}$, $u_s : \Delta(X_s) \to \mathbb{R}$ is a weak affine representation for \succeq_s and f^* and g^* are equally crisp acts such that for all $s \in S^{\mathcal{P}}$, $u_s(f_s^*) > u_s(g_s^*)$. Then for all $f \in H, \beta \in [0, 1)$,

$$\alpha \ge \alpha' \iff \beta f + (1-\beta) \left(\alpha f^* + (1-\alpha)g^* \right) \succeq \beta f + (1-\beta) \left(\alpha' f^* + (1-\alpha')g^* \right).$$

Proof. Follows directly from Lemmas 11 and 29 since $\beta f + (1 - \beta)f^*, \beta f + (1 - \beta)g^*$ are equally crisp acts and for each $s \in S^{\mathcal{P}} u_s$ is affine.

C.3 Existence of State Dependent Dual-Self Variational Expected Utility

The next step show the existence of C-additive and monotonic representation.

Proposition 31 Assume \succeq satisfies Axioms 2-6 and for all $s \in S^{\mathcal{P}}$, $u_s : \Delta(X_s) \to \mathbb{R}$ is a weak affine representation for \succeq_s such that $u_s(f_s^*) - u_s(g_s^*) = 1$. Then there exists $I : u(H) \to \mathbb{R}$ such that (u, I) is a state dependent niveloid representation for \succeq .

Proof. For all $\alpha \in [0, 1]$, denote

$$f^* \alpha g^* = \alpha f^* + (1 - \alpha)g^*$$

Denote

$$H^{\circ} = \left\{ f \in H \middle| u(f) \in \operatorname{int} u(H) \right\}.$$

We endow H° with the order topology.

Define the partition of H° ,

$$\mathcal{B} = \Big\{ \{g \in H^{\circ} | \exists \alpha \in \mathbb{R}, u(g) = u(f) + \bar{1}\alpha \} \Big| f \in H^{\circ} \Big\}.$$

Define the mapping $B: H^{\circ} \to \mathcal{B}$ by for each $f \in H^{\circ}, f \in B(f) \in \mathcal{B}$.

For each $B \in \mathcal{B}$, choose $f^B \in B$ and define the function $\varphi^B : B \to R$ by the following. For $f \in B$, there exists $\alpha^f \in \mathbb{R}$ such that $u(f) = u(f^B) + \bar{1}\alpha^f$ and define $\psi^B(f) = \alpha^f$.

First, H° and each $B \in \mathcal{B}$ are connected in the order topology by Axiom 3. Second, for each $f \in B \in \mathcal{B}$, there exist $f^1, f^2 \in B$ such that $f^1 \succ f \succ f^2$ by Lemma 30. Third, ψ^B is continuous in the order topology by Axiom 3.

Finally, let $f, g \in H^{\circ}$. Since $u(f), u(g) \in \operatorname{int} u(H)$, there exist $\tilde{f}, \tilde{g} \in H$ and $\hat{\beta} \in (0, 1)$ such that $u(\tilde{f}) = u(f) - \hat{\beta}(u(f^{*1/2}g^*) - u(f)), u(\tilde{g}) = u(g) - \hat{\beta}(u(f^{*1/2}g^*) - u(g)) \in \operatorname{int} u(H)$. Denoting $\beta = \frac{1}{1+\hat{\beta}}$, we have

$$u(\beta \tilde{f} + (1 - \beta)(f^{*1/2}g^*)) = u(f) \text{ and } u(\beta \tilde{g} + (1 - \beta)(f^{*1/2}g^*)) = u(g).$$

Let $f' \in B(f)$ and $g' \in B(g)$ be such that $\beta \tilde{f} + (1 - \beta)f^* \succ f', g' \succ \beta \tilde{f} + (1 - \beta)g^*$. Let $\alpha^f, \alpha^g \in (0, 1)$ be such that

$$\beta \widetilde{f} + (1-\beta)(f^*\alpha^{f'}g^*) \sim f' \text{ and } \beta \widetilde{f} + (1-\beta)(f^*\alpha^{g'}g^*) \sim g'.$$

By Lemma 30, we have $u(f) + (\frac{1}{2} - \alpha^{f'})\bar{1} = u(f')$ and $u(g) + (\frac{1}{2} - \alpha^{g'})\bar{1} = u(g')$. Especially $\varphi^{B(f)}(f') - \varphi^{B(f)}(f) = \alpha^{f'} - \frac{1}{2}$ and $\varphi^{B(g)}(g') - \varphi^{B(g)}(g) = \alpha^{g'} - \frac{1}{2}$.

So we have

$$\begin{split} f' \succeq g' \iff \beta \tilde{f} + (1-\beta)(f^* \alpha^{f'} g^*) \succeq \beta \tilde{f} + (1-\beta)(f^* \alpha^{g'} g^*) \iff \alpha^{f'} \ge \alpha^{g'} \\ \iff \alpha^{f'} - \frac{1}{2} \ge \alpha^{g'} - \frac{1}{2} \iff \varphi^{B(f)}(f') - \varphi^{B(f)}(f) \ge \varphi^{B(g)}(g') - \varphi^{B(g)}(g). \end{split}$$

Thus by Mononen (2021, Theorem 1), there exist $\tau: \mathcal{B} \to \mathbb{R}$ such that for all $f, g \in H^{\circ}$

$$f \succeq g \iff \psi^{B(f)}(f) + \tau(B(f)) \ge \psi^{B(g)}(g) + \tau(B(g))$$

By defining $I : \operatorname{int} u(H) \to \mathbb{R}$ by for all $f \in \operatorname{int} u(H)$, $I(u(f)) = \varphi^{B(f)}(f) + \tau(B(f))$, (u, I) is a state dependent niveloid representation for \succeq in H° .

Finally, since I is 1-lipschitz continuous, it can be extended continuously to u(H)with $\tilde{I}: u(H) \to \mathbb{R}$ that inherits C-additivity and monotonicity.

Proposition 32 If \succeq satisfies Axioms 2-6, then there exists (u, \mathbb{C}) that is a dual-self variational representation for \succeq .

Proof. By Lemma 27 and Proposition 31, for all $s \in S$, there exist affine $u_s : \Delta(X_s) \to \mathbb{R}$ and $I : u(H) \to \mathbb{R}$ such that (u, I) is a state dependent niveloid representation for \succeq and $u(f^*) = \overline{1}, u(g^*) = \overline{0}$. By Chandrasekher et al.'s (2022) Lemmas A.5 and S.3.2, there exists $\mathbb{C} \subseteq \{c : \Delta(S) \to \mathbb{R} \cup \{\infty\} | c \text{ is convex}\}$ such that for all $\varphi \in \operatorname{int} u(H),$ $I(\varphi) = \max_{c \in \mathbb{C}} \min_{p \in \Delta(S)} p \cdot \varphi + c(p).$

C.4 Existence of State Dependent Dual-Self Expected Utility

Next, we show that if there exist a crisp act with a constant utility, then a state dependent dual-self variational representation is a state dependent dual-self representation.

Proposition 33 If \succeq satisfies Axioms 2-7, then there exists (u, \mathbb{P}) that is a dual-self representation for \succeq .

Proof. Let f^*, g^* be equally crisp acts as in Axiom 6 and c a crisp act as in Axiom 7. By Lemma 27, there exist and for each $s \in S^{\mathcal{P}}$, $u_s : \Delta(X_s) \to \mathbb{R}$ is a weak affine representation for \succeq_s such that $u_s(f_s^*) - u_s(g_s^*) = 1$ and $u_s(c_s) = 0$. By Proposition 31, there exists $I: u(H) \to \mathbb{R}$ such that (u, I) is a state dependent niveloid representation for \succeq . By adding a constant, we can assume that $I(\bar{0}) = 0$. By Lemma 24, for all $h \in H, \alpha \in (0, 1)$

$$I(\alpha u(h)) = \alpha I(u(h))$$

and so I is positively homogeneous. By Chandrasekher et al.'s (2022) proof of Theorem 1, there exists $\mathbb{P} \in \mathcal{K}(\Delta(S))$ such that for all $\varphi \in u(H)$, $I(\varphi) = \max_{P \in \mathbb{P}} \min_{p \in P} p$.

References

 φ .

- Amarante, Massimiliano (2009). Foundations of neo-Bayesian statistics. Journal of Economic Theory 144(5), pp. 2146–2173.
- Anscombe, Francis J. and Aumann, Robert J. (1963). A Definition of Subjective Probability. *The annals of mathematical statistics* 34(1), pp. 199–205.
- Arrow, Kenneth J. (1971). Essays in the Theory of Risk-Bearing. Chicago: Markham Publishing Company.
- Aumann, Robert J. (1971). Letter from Robert Aumann to Leonard Savage, 8 January 1971. Published in Drèze (1987).
- Baccelli, Jean (2019). The Problem of State-Dependent Utility: A Reappraisal. The British Journal for the Philosophy of Science.
- Boyd, Stephen and Vandenberghe, Lieven (2004). *Convex Optimization*. Cambridge University Press.
- Cerreia-Vioglio, Simone; Ghirardato, Paolo; Maccheroni, Fabio; Marinacci, Massimo, and Siniscalchi, Marciano (2011). Rational preferences under ambiguity. *Economic Theory* 48(2-3), pp. 341–375.
- Chandrasekher, Madhav; Frick, Mira; Iijima, Ryota, and Le Yaouanq, Yves (2022). Dual-Self Representations of Ambiguity Preferences. *Econometrica* 90(3), pp. 1029–1061.
- Chew, Soo Hong and Wang, Wenqian (2020). On the robustness of indeterminacy in subjective probability. *Economics Letters* 188, p. 108939.
- Clarke, Frank H. (1983). Optimization and Nonsmooth Analysis. Canadian Mathematical Society series of monographs and advanced texts. Wiley.
- Drèze, Jacques (1958). Individual Decision Making under Partially Controllable Uncertainty. PhD thesis. Columbia University. Republished in 1971.

- Drèze, Jacques (1961). Les fondements logiques de l'utilité cardinale et de la probabilité subjective. La décision, pp. 73–97. Translated and published as Chapter 3 in Drèze (1987).
- (1987). Essays on economic decisions under uncertainty. Cambridge University Press.
- Drèze, Jacques and Rustichini, Aldo (2004). State-Dependent Utility and Decision Theory. Handbook of Utility Theory: Volume 2 Extensions. Ed. by Salvador Barberà; Peter Hammond, and Christian Seidl. Springer Science & Business Media, pp. 839–892.
- Einy, Ezra (1989). On preference relations which satisfy weak independence property. Journal of Mathematical Economics 18(3), pp. 291–300.
- Ghirardato, Paolo; Maccheroni, Fabio, and Marinacci, Massimo (2004). Differentiating ambiguity and ambiguity attitude. Journal of Economic Theory 118(2), pp. 133–173.
- Gilboa, Itzhak; Minardi, Stefania; Samuelson, Larry, and Schmeidler, David (2020). States and Contingencies: How to Understand Savage without Anyone Being Hanged. *Revue économique* 71(2), pp. 365–386.
- Gilboa, Itzhak and Schmeidler, David (1989). Maxmin expected utility with nonunique prior. *Journal of mathematical economics* 18(2), pp. 141–153.
- Grant, Simon and Polak, Ben (2013). Mean-dispersion preferences and constant absolute uncertainty aversion. Journal of Economic Theory 148(4), pp. 1361–1398.
- Hansen, Lars Peter and Sargent, Thomas J. (2001). Robust Control and Model Uncertainty. American Economic Review 91(2), pp. 60–66.
- Hill, Brian (2019). A Non-Bayesian Theory of State-Dependent Utility. *Econometrica* 87(4), pp. 1341–1366.
- Karni, Edi (2007). Foundations of Bayesian theory. Journal of Economic Theory 132(1), pp. 167–188.

- Karni, Edi (2011a). A theory of Bayesian decision making with action-dependent subjective probabilities. *Economic Theory* 48(1), pp. 125–146.
- (2011b). Subjective Probabilities on a State Space. American Economic Journal: Microeconomics 3(4), pp. 172–85.
- (2020). On the indeterminacy of the representation of beliefs by probabilities.
 Economics Letters 196, p. 109535.
- Karni, Edi and Schmeidler, David (2016). An expected utility theory for statedependent preferences. *Theory and Decision* 81(4), pp. 467–478.
- Karni, Edi; Schmeidler, David, and Vind, Karl (1983). On State Dependent Preferences and Subjective Probabilities. *Econometrica* 51(4), pp. 1021–1031.
- Lu, Jay (2019). Bayesian Identification: A Theory for State-Dependent Utilities. American Economic Review 109(9), pp. 3192–3228.
- Maccheroni, Fabio; Marinacci, Massimo, and Rustichini, Aldo (2006). Ambiguity Aversion, Robustness, and the Variational Representation of Preferences. *Econometrica* 74(6), pp. 1447–1498.
- Maccheroni, Fabio; Marinacci, Massimo; Rustichini, Aldo, and Taboga, Marco (2009). Portfolio Selection with Monotone Mean-Variance Preferences. *Mathematical Finance* 19(3), pp. 487–521.
- Mononen, Lasse (2021). From Local to Global Variational Representation.
- (2022). Latent Expected Utility.
- (2024a). Dynamically Consistent Intertemporal Dual-Self Expected Utility.
- (2024b). Observable Interpersonal Utility Comparisons.
- Savage, Leonard J. (1954). The Foundations of Statistics. Wiley publications in statistics. Wiley.
- (1967). Difficulties in the Theory of Personal Probability. *Philosophy of Science* 34(4), pp. 305–310.
- (1971). Letter from Leonard Savage to Robert Aumann, 27 January 1971. Published in Drèze (1987).

Schmeidler, David (1989). Subjective Probability and Expected Utility without Additivity. *Econometrica* 57(3), pp. 571–587.

Shafer, Glenn (1986). Savage Revisited. Statistical science 1(4), pp. 463–485.

- Siniscalchi, Marciano (2009). Vector Expected Utility and Attitudes Toward Variation. *Econometrica* 77(3), pp. 801–855.
- Strzalecki, Tomasz (2011). Axiomatic Foundations of Multiplier Preferences. Econometrica 79(1), pp. 47–73.
- Tsakas, Elias (2023). Belief identification by proxy.
- Wakai, Katsutoshi (2008). A Model of Utility Smoothing. *Econometrica* 76(1), pp. 137–153.

Online Appendix to "State Dependent Utility and Ambiguity"

Not intended for publication

This online appendix is organized as follows. Section B.5 showes the partial identification for state dependent dual-self expected utility. This shows Theorem 2 as a corollary. Section C proves the existence of state dependent dual-self expected utility and state dependent dual-self variational expected utilities, Theorems 8 and 9. Section B.6 characterizes the identification with crisp and equally crisp acts and shows Propositions 3 and 4. Finally, Appendix S.2 characterizes the full identification of probabilities for a single state and for relative likelihoods between two states and shows Proposition 5 and Proposition 6.

S.1 State Dependent Bewley Uniqueness

Assume that $S^{\mathcal{P}} = S$. We show the uniqueness of the state dependent Bewley representation.

Proposition S.1 If (u, C) and (\tilde{u}, \tilde{C}) are state dependent Bewley representations for \succeq , then there exist $a \in \mathbb{R}^{S}_{++}, b \in \mathbb{R}^{S}$ such that

$$\widetilde{C} = \left\{ \widetilde{p} \in \Delta(S) \middle| \exists p \in C, \forall s \in S, \widetilde{p}_s = \frac{a_s^{-1} p_s}{\sum_{s \in S} a_s^{-1} p_s} \right\}$$

and for all $s \in S^{\mathcal{P}}$,

$$\widetilde{u}_s = a_s u_s + b_s.$$

Proof. By the representation, we have for all $x_s, y_s \in \Delta(X_s)$, $u_s(x_s) \geq u_s(y_s)$ iff $\tilde{u}_s(x_s) \geq \tilde{u}_s(y_s)$. By the vNM uniqueness theorem, there exist $a \in \mathbb{R}^{S}_{++}$, $b \in \mathbb{R}^{S}$ such that for each $s \in S$,

$$\widetilde{u}_s = a_s u_s + b.$$

Denote $a^{-1} = (a_s^{-1})_{s \in S}$. Define multiplications with a and a^{-1} statewise. Define $\widehat{C} = \left\{ \left(\frac{a_s^{-1} p_s}{\sum_{s' \in S} a_{s'}^{-1} p_{s'}} \right)_{s \in S} \middle| p \in C \right\}.$

Now $\hat{C} \subseteq \Delta(S)$ since for all $s \in S$, $a_s > 0$.

Define $\succeq^{\widehat{C}}$ and $\succeq^{\widetilde{C}}$ on \mathbb{R}^S by for all $\varphi, \psi \in \mathbb{R}^S$,

$$\varphi \succsim^{\widetilde{C}} \psi \iff \forall \, \widehat{p} \in \widehat{C}, \, \widehat{p} \cdot \varphi \ge \widehat{p} \cdot \psi \text{ and}$$
$$\varphi \succsim^{\widetilde{C}} \psi \iff \forall \, \widetilde{p} \in \widetilde{C}, \, \widetilde{p} \cdot \varphi \ge \widetilde{p} \cdot \psi.$$

Let $\varphi, \psi \in \mathbb{R}^S$. We show that $\varphi \succeq^{\widehat{C}} \psi \iff \varphi \succeq^{\widetilde{C}} \psi$. Since $S = S^{\mathcal{P}}$, there exist $\theta \in \operatorname{int} u(H)$ and $\alpha^* \in (0, 1]$ such that $\alpha^* a^{-1} \varphi + (1 - \alpha^*) \theta, \alpha^* a^{-1} \psi + (1 - \alpha^*) \theta \in u(H)$. Let $f, g \in H$ be such that

$$\alpha^* a^{-1} \varphi + (1 - \alpha^*) \theta = u(f) \text{ and } \alpha^* a^{-1} \psi + (1 - \alpha^*) \theta = u(g).$$

Now we have

$$\begin{split} \varphi \succeq^{\widehat{C}} \psi \iff \forall \widehat{p} \in \widehat{C}, \widehat{p} \cdot \varphi \ge \widehat{p} \cdot \psi \\ \Leftrightarrow \qquad \forall \widehat{p} \in \widehat{C}, \alpha^{\star} \widehat{p} \cdot \varphi + (1 - \alpha^{\star}) \widehat{p} \cdot a\theta \ge \alpha^{\star} \widehat{p} \cdot \psi + (1 - \alpha^{\star}) \widehat{p} \cdot a\theta \\ \Leftrightarrow \qquad \forall \widehat{p} \in \widehat{C}, \widehat{p} \cdot a \left(\alpha^{\star} a^{-1} \varphi + (1 - \alpha^{\star}) \theta \right) \ge \widehat{p} \cdot a \left(\alpha^{\star} a^{-1} \psi + (1 - \alpha^{\star}) \theta \right) \\ \Leftrightarrow \qquad \forall p \in C, \frac{a^{-1} p}{\sum_{s \in S} a_s^{-1} p_s} \cdot au(f) \ge \frac{a^{-1} p}{\sum_{s \in S} a_s^{-1} p_s} \cdot au(g) \\ \Leftrightarrow \qquad \forall p \in C, p \cdot u(f) \ge p \cdot u(g) \iff f \succeq g \iff \forall \widetilde{p} \in \widetilde{C}, \widetilde{p} \cdot \widetilde{u}(f) \ge \widetilde{p} \cdot \widetilde{u}(g) \\ \Leftrightarrow \qquad \forall \widetilde{p} \in \widetilde{C}, \widetilde{p} \cdot a \left(\alpha^{\star} a^{-1} \varphi + (1 - \alpha^{\star}) \theta \right) + \widetilde{p} \cdot b \ge \widetilde{p} \cdot a \left(\alpha^{\star} a^{-1} \psi + (1 - \alpha^{\star}) \theta \right) + \widetilde{p} \cdot b \\ \Leftrightarrow \qquad \forall \widetilde{p} \in \widetilde{C}, \widetilde{p} \cdot \varphi \ge \widetilde{p} \cdot \psi \iff \varphi \succeq^{\widetilde{C}} \psi. \end{split}$$

Thus $\succeq^{\widetilde{C}}$ has state independent Bewley representations with $(\mathrm{Id}, \widetilde{C})$ and $(\mathrm{Id}, \widehat{C})$. By its uniqueness (Ghirardato et al., 2004), $\widetilde{C} = \widehat{C}$.

S.2 Relative Likelihood and Probability Characterizations

Next, we move on to proving Proposition 5 and Proposition 6. We show these results by characterizing the possible transformations for probabilities that keep the relative likelihoods between two states the same and that keep the probabilities for a single state the same behaviorally and in terms of the set of probabilities.

S.2.1 Relative Likelihood Identification

Our first lemma for relative likelihood identification shows that our behavioral assumption for relative likelihood identification characterizes their identification. This follows as a corollary of this lemma and our previous partial identification result.

Lemma S.2 Assume that (u, I) is a state dependent niveloid representation for \succeq . Fix any $s, s' \in S^{\mathcal{P}}, s \neq s'$. The following two are equivalent:

- (1) If $f, g \in H$ are equally crisp and $f_s \succ_s g_s$, then $f_{s'} \succeq_{s'} g_{s'}$.
- (2) For all $x \in \left(\bigcup_{\varphi \in \operatorname{int} u(H)} \partial I(\varphi)\right)^{\perp}, x_s = x_{s'}.$

Proof. We will first show that $(1) \Rightarrow (2)$. Assume, per contra, there exists $x \in \left(\bigcup_{\varphi \in \operatorname{int} u(H)} \partial I(\varphi)\right)^{\perp}$, $x_s \neq x_{s'}$. Assume w.l.o.g. $x_s > x_{s'}$. Let $c = -\frac{1}{2}x_s - \frac{1}{2}x_{s'}$. By Lemma 18, there exist $f, g \in H, a > 0$ such that $f_s \succ_s g_s$ and $g_{s'} \succ_{s'} f_{s'}$ and for all $\tilde{s} \in S^{\mathcal{P}} u_{\tilde{s}}(f_{\tilde{s}}) - u_{\tilde{s}}(g_{\tilde{s}}) = a(x_{\tilde{s}} + c)$. By Corollary 15, f and g are equally crisp acts which contradicts (1).

Second, we show that $(2) \Rightarrow (1)$. Assume, per contra, that there exist equally crisp acts f and g such that $f_s \succ_s g_s$ and $g_{s'} \succ_{s'} f_{s'}$. Denote $x = u(f) - u(g) - [I(u(f)) - I(u(g))]\overline{1}$. By Corollary 15, $x \in \left(\bigcup_{\varphi \in \operatorname{int} u(H)} \partial I(\varphi)\right)^{\perp}$. Now, $u_s(f_s) > u_s(g_s)$ and $u_{s'}(g_{s'}) > u_{s'}(f_{s'})$, so $x_s \neq x_{s'}$ that contradicts (2).

The next lemma characterizes in terms of probabilities when the relative likelihood for states is identified.

Lemma S.3 Assume that $P \subseteq \Delta(S)$, P is convex, $P \neq \emptyset$, $s, s' \in S$, and $s \neq s'$. The following two are equivalent:

- (1) For all $x \in P^{\perp}$, $x_s = x_{s'}$.
- (2) There exist $p, q \in P$ such that for all $\tilde{s} \in S \setminus \{s, s'\}$, $p_{\tilde{s}} = q_{\tilde{s}}$ and $\frac{p_s}{p_{s'}} \neq \frac{q_s}{q_{s'}}$.

Proof. We will first show that $(2) \Rightarrow (1)$. Assume, per contra, that there exists $x \in P^{\perp}$ such that $x_s \neq x_{s'}$. Let p, q be as in (2). Then

$$\sum_{\tilde{s}\in S} x_{\tilde{s}} p_{\tilde{s}} = 0 = \sum_{\tilde{s}\in S} x_{\tilde{s}} q_{\tilde{s}}$$

Since for all $\tilde{s} \in S \setminus \{s, s'\}$, $p_{\tilde{s}} = q_{\tilde{s}}$ and so $p_s + p_{s'} = q_s + q_{s'}$,

 $(x_s - x_{s'})p_s + x_s(p_s + p_{s'}) = x_sp_s + x_{s'}p_{s'} = x_sq_s + x_{s'}q_{s'} = (x_s - x_{s'})q_s + x_s(q_s + q_{s'}).$

Hence,

$$(x_s - x_{s'})(p_s - q_s) = 0.$$

Since $x_s \neq x_{s'}$ by the counter assumption, we have $p_s = q_s$ which is a contradiction.

Next, we show that $(1) \Rightarrow (2)$. Denote $n^{\perp} = \dim P^{\perp}$ and $n = \dim P$. Now $n^{\perp} + n = |S|$. There exist $(c^i)_{i=1}^{n^{\perp}} \subseteq P^{\perp}$ linearly independent vectors and $(p^i)_{i=1}^n \subseteq P$ linearly independent vectors. First, we show that $n \ge 2$. Since P is non-empty, $n \ge 1$. Assume, per contra, n = 1. Thus P is a singleton $P = \{p\}$. If $p_s = 0$, then $(1_s, 0_{-s}) \in P^{\perp}$ which is a contradiction. Similarly, $p_{s'} \ne 0$. Now, $\left((\frac{1}{p_{s'}})_s, (-\frac{1}{p_s})_{s'}, 0_{-s,s'}\right) \in P^{\perp}$ that is a contradiction. So $n \ge 2$.

We consider two cases. First, assume that for all $i \in \{1, \ldots, n^{\perp}\}$,

$$c_s^i = 0. (12)$$

Now we can consider the matrix C formed by row vectors $(c^i)_{i=1}^{n^{\perp}}$ where the columns are S. By (12), we can reduce C into Smith normal form \tilde{C} formed by row vectors $(\tilde{c}^i)_{i=1}^{n^{\perp}}$ such that for each $i \in \{1, \ldots, n^{\perp}\}$, there exists $s^i \in S \setminus \{s, s'\}$ such that $\tilde{c}_{s^i}^i = -1$, for all $j \in \{1, \ldots, n^{\perp}\} \setminus \{i\}$, $\tilde{c}_{s^j}^i = 0$, and for all $k, l \in \{1, \ldots, n^{\perp}\}$, $k \neq l, s^k \neq s^l$. Now $(\tilde{c}^i)_{i=1}^{n^{\perp}} \subseteq P^{\perp}$ are linearly independent and for all $i \in \{1, \ldots, n^{\perp}\}$,

$$\tilde{c}_s^i = \tilde{c}_{s'}^i = 0. \tag{13}$$

Denote $S^{\perp} = \{s^1 \dots, s^{n^{\perp}}\}$. Now we have for all $i \in \{1, \dots, n^{\perp}\}$ and $p \in P$, $n_{i} = \sum_{i} \tilde{c}^i_i p^j_i.$ (14)

$$p_{s^i} = \sum_{\tilde{s} \in S \setminus S^\perp} \tilde{c}^i_{\tilde{s}} p^j_{\tilde{s}}.$$
(14)

Let $\pi : \{1, \ldots, n-2\} \to S \setminus (S^{\perp} \cup \{s, s'\})$ be a one-to-one function. We show by induction that for each $i \in \{0, \ldots, n-2\}$, there exists a linearly independent collection of probabilities $(p^{j,i})_{j=1}^{n-i} \subseteq P$ such that for all $j, k \in \{1, \ldots, n-i\}, 1 \leq l \leq i$,

$$p_{\pi(l)}^{j,i} = p_{\pi(l)}^{k,i}.$$
(15)

For the first step i = 0, define for all $j \in \{1, ..., n\}$, $\tilde{p}^{j,i} = p^j$. For the induction step, assume that for $0 \le i \le n-2$, there exists $(p^{j,i})_{j=1}^{n-i}$ that satisfy (15). First, we show that

$$\min\left\{p_{\pi(i+1)}^{j,i} \middle| j \in \{1,\dots,n-i\}\right\} \neq \max\left\{p_{\pi(i+1)}^{j,i} \middle| j \in \{1,\dots,n-i\}\right\}.$$
(16)

Assume, per contra, that for all $j, k \in \{1, \ldots, n-i\}, p_{\pi(i+1)}^{j,i} = p_{\pi(i+1)}^{k,i}$. Let P^i be a matrix formed by the column vectors $(p^{j,i})_{j=1}^{n-i}$. The column rank of P^i is n-isince columns are linearly independent. Thus the row rank of P^i is n-i. However, each row $\tilde{s} \in S^{\perp}$ is linearly dependent on rows $S \setminus S^{\perp}$ by (14) and each row $\tilde{s} \in$ $\{\pi(1), \ldots, \pi(i), \pi(i+1)\}$ is constant and so since the rows sum to 1, linearly dependent on the rows $S \setminus \{\pi(1), \ldots, \pi(i), \pi(i+1)\}$. Thus the maximum row rank for P^i is $|S| - n^{\perp} - i - 1 = n - i - 1$ which is a contradiction. This shows (16).

Let

$$j^* \in \arg \max \left\{ p_{\pi(i+1)}^{j,i} \middle| j \in \{1, \dots, n-i\} \right\} \text{ and } j_* \in \arg \min \left\{ p_{\pi(i+1)}^{j,i} \middle| j \in \{1, \dots, n-i\} \right\}.$$

By (16), there exists $\beta^i \in (0, 1)$ such that for all $j \in \{1, \dots, n-i\} \setminus j_*$, $\frac{1}{2} p_{\pi(i+1)}^{j,i} + \frac{1}{2} p_{\pi(i+1)}^{j^*,i} > \beta^i > p_{\pi(i+1)}^{j_*,i}.$

Thus for each $j \in \{1, \ldots, n-i\} \setminus j_*$, there exists $\alpha^j \in (0, 1)$ such that

$$\alpha^{j} \left(\frac{1}{2} p_{\pi(i+1)}^{j,i} + \frac{1}{2} p_{\pi(i+1)}^{j^{*},i} \right) + (1 - \alpha^{j}) p_{\pi(i+1)}^{j_{*},i} = \beta^{i}$$

Denote for $j \in \{1, \ldots, n-i-1\} \setminus j_*$,

$$\tilde{p}^{(j-1(j>j_*)),i+1} = \alpha^j \left(\frac{1}{2}p^{j,i} + \frac{1}{2}p^{j^*,i}\right) + (1-\alpha^j)p^{j_*,i}$$

where $\mathbb{1}(j > j_*)$ is an indicator function for $j > j_*$. Now $(\tilde{p}^{j,i+1})_{j=1}^{n-i-1}$ are linearly independent since they have been created using elementary column operations from P^i . Additionally, it satisfies (15). This shows the induction step and concludes the induction.

Since $n \ge 2$, by the induction, there exist $p^*, p^{\dagger} \in P$ that are linearly independent and for all $\tilde{s} \in S \setminus (S^{\perp} \cup \{s, s'\}), p^*_{\tilde{s}} = p^{\dagger}_{\tilde{s}}$.

By (13,14), for all $\tilde{s} \in S \setminus \{s, s'\}$,

$$p_{\tilde{s}}^* = p_{\tilde{s}}^\dagger.$$

This shows the claim since p^{\dagger} and p^{*} are linearly independent.

Second, we consider the case that there exists $i \in \{1, ..., n^{\perp}\}$ such that $c_s^i \neq 0$. First, if a vector $\tilde{c} \in \mathbb{R}^S$ is such that $\tilde{c}_s = \tilde{c}_{s'} \neq 0$ and for all $\tilde{s} \in S \setminus \{s, s'\}, \tilde{c}_{\tilde{s}} = 0$, then $\tilde{c} \notin P^{\perp}$ (17)

since each $p \in P$ is non-negative.

Next, consider the matrix C formed by row vectors $(c^i)_{i=1}^{n^{\perp}}$ where the columns are S. Denote $s^{n^{\perp}} = s, s^{n^{\perp}+1} = s'$. By (17), we can reduce C into Smith normal form \tilde{C} formed by row vectors $(\tilde{c}^i)_{i=1}^{n^{\perp}}$ such that for each $i \in \{1, \ldots, n^{\perp} - 1\}$, there exists $s^i \in S \setminus \{s, s'\}$ such that for each $\hat{i} \in \{1, \ldots, n^{\perp}\}$, $\tilde{c}_{s^i}^i = -1$, for all $j \in \{1, \ldots, n^{\perp}\} \setminus \{i\}$, $\tilde{c}_{s^j}^i = 0$, and for all $k, l \in \{1, \ldots, n^{\perp}\}, k \neq l, s^k \neq s^l$. Now $(\tilde{c}^i)_{i=1}^{n^{\perp}} \subseteq P^{\perp}$ are linearly independent and for all $i \in \{1, \ldots, n^{\perp} - 1\}$,

$$\tilde{c}_{s}^{i} = \tilde{c}_{s'}^{i} = 0 \text{ and } \tilde{c}_{s}^{n^{\perp}} = \tilde{c}_{s'}^{n^{\perp}} = -1.$$

Denote $S^{\perp} = \{s^1 \dots, s^{n^{\perp}}, s^{n^{\perp}+1}\}$. Now we have for all $i \in \{1, \dots, n^{\perp}-1\}$ and $p \in P$,

$$p_{s^{i}} = \sum_{\tilde{s} \in S \setminus S^{\perp}} \tilde{c}^{i}_{\tilde{s}} p_{\tilde{s}} \text{ and } p_{s^{n^{\perp}}} + p_{s^{n^{\perp}+1}} = \sum_{\tilde{s} \in S \setminus S^{\perp}} \tilde{c}^{n^{\perp}}_{\tilde{s}} p_{\tilde{s}}.$$
(18)

Next, we show that there exists $s^{\dagger} \in S \setminus S^{\perp}$ such that

$$\sum_{i=1}^{n^{\perp}} \tilde{c}_{s^{\dagger}}^i \neq -1.$$
(19)

Assume, per contra, for all $\tilde{s} \in S \setminus S^{\perp}$,

$$\sum_{i=1}^{n^{\perp}} \tilde{c}_{\tilde{s}}^{i} = -1.$$
 (20)

Then, we have for $p \in P$,

$$1 = \sum_{\tilde{s} \in S} p_{\tilde{s}} \stackrel{(18)}{=} \sum_{\tilde{s} \in S \setminus S^{\perp}} \left(1 + \sum_{i=1}^{n^{\perp}} \tilde{c}_{\tilde{s}}^{i} \right) p_{\tilde{s}} \stackrel{(20)}{=} 0$$

that is a contradiction which shows (19).

Let $\pi : \{1, \ldots, n-2\} \to S \setminus (S^{\perp} \cup \{s^{\dagger}\})$ be a one-to-one function. By an induction as in the previous case for each $i \in \{0, \ldots, n-2\}$, there exists a collection of linearly independent probabilities $(\tilde{p}^{j,i})_{j=1}^{n-i}$ such that for all $j, k \in \{1, \ldots, n-i\}, 1 \leq l \leq i$,

$$p_{\pi(l)}^{j,i} = p_{\pi(l)}^{k,i}.$$

Since $n \ge 2$, by the induction, there exist $p^*, p^{\dagger} \in P$ that are linearly independent and for all $\tilde{s} \in S \setminus (S^{\perp} \cup \{s^{\dagger}\}), p^*_{\tilde{s}} = p^{\dagger}_{\tilde{s}}$.

Now we have for $p \in \{p^*, p^{\dagger}\}$,

$$1 = \sum_{\tilde{s}\in S} p_{\tilde{s}} \stackrel{(18)}{=} \sum_{\tilde{s}\in S\setminus S^{\perp}} \left(1 + \sum_{i=1}^{n^{\perp}} \tilde{c}_{\tilde{s}}^{i} \right) p_{\tilde{s}} \stackrel{(19)}{\Longrightarrow} p_{s^{\dagger}} = \frac{1 - \sum_{\tilde{s}\in S\setminus (S^{\perp}\cup\{s^{\dagger}\})} \left(1 + \sum_{i=1}^{n^{\perp}} \tilde{c}_{\tilde{s}}^{i} \right) p_{\tilde{s}}}{1 + \sum_{i=1}^{n^{\perp}} \tilde{c}_{s^{\dagger}}^{i}}.$$

Thus $p_{s^{\dagger}}^* = p_{s^{\dagger}}^{\dagger}$.

Now for all $\tilde{s} \in S \setminus S^{\perp}$, $p_{\tilde{s}}^* = p_{\tilde{s}}^{\dagger}$. By (18), for all $\tilde{s} \in S \setminus \{s, s'\}$, $p_{\tilde{s}}^* = p_{\tilde{s}}^{\dagger}$ and $p_s^* + p_{s'}^* = p_s^{\dagger} + p_{s'}^{\dagger}$.

This shows the claim since p^{\dagger} and p^{*} are linearly independent.

The relative likelihood identification characterization, Proposition 6, directly follows from Lemmas S.2 and S.3 and Theorem 7.

S.2.2 Probability Identification

The identification for a probability is similar to the identification for the relative likelihood of some states. First, we show that our behavioral assumption for probability identification characterizes its identification.

Lemma S.4 Assume that (u, I) is a state dependent niveloid representation for \succeq . Fix any $s \in S^{\mathcal{P}}$. The following two are equivalent:

- (1) If $f, g \in H$ and $f \sim^* g$, then $f_s \sim_s g_s$.
- (2) For all $f, g \in H$ such that $f \sim g$ and $f_s \succ_s g_s$, there exist $h \in H$ and $\alpha \in (0, 1)$ such that

(3) For all
$$x \in \left(\bigcup_{\varphi \in \operatorname{int} u(H)} \partial I(\varphi)\right)^{\perp}, x_s = 0.$$

Proof. We will first show that $(1) \Rightarrow (2)$. Assume, per contra, w.l.o.g. that there exists $\tilde{x} \in \left(\bigcup_{\varphi \in \operatorname{int} u(H)} \partial I(\varphi)\right)^{\perp}, \tilde{x}_s > 0$. Assume w.l.o.g. $\tilde{x}_s > 0$. Now $x = (\tilde{x}_{S^{\mathcal{P}}}, 0_{-S^{\mathcal{P}}}) \in \left(\bigcup_{\varphi \in \operatorname{int} u(H)} \partial I(\varphi)\right)^{\perp}$.

By Lemma 18, there exist $f, g \in H$, a > 0 such that $f_s \succ_s g_s$, $u(f), u(g) \in \operatorname{int} u(H)$, and u(f) - u(g) = ax. By Corollary 15, f and g are equally crisp and $f \sim g$. By Lemma 13, $f \sim^* g$ that contradicts (1).

Second, we show that $(2) \Rightarrow (1)$. Assume, per contra, w.l.o.g. that there exist $f, g \in H, s' \in S$ such that $f \sim^* g, f_s \succ_s g_s$. Especially, by the definition, f and g are equally crisp acts. By Corollary 15, $u(f) - u(g) \in \left(\bigcup_{\varphi \in \text{int } u(H)} \partial I(\varphi)\right)^{\perp}$. Additionally, by the monotonicity of $I, u_s(f_s) - u_s(g_s) > 0$ that contradicts (2).

The second lemma characterizes in terms of the set of probabilities when the probabilities for a single state are identified.

Lemma S.5 Assume that $P \subseteq \Delta(S)$, P is convex, $P \neq \emptyset$, and $s \in S$. The following two are equivalent:

- (1) For all $x \in P^{\perp}$, $x_s = 0$ and there exist $s' \in S$, $s \neq s'$ and $p \in P$ such that $p_{s'} > 0$.
- (2) There exist $p, q \in P$ such that $p_s \neq q_s$ and for all $\tilde{s} \in S \setminus \{s\}$,

$$\frac{p_{\tilde{s}}}{1-p_s} = \frac{q_{\tilde{s}}}{1-q_s}.$$
(21)

Proof. We show first that $(2) \Rightarrow (1)$. First, by (2), $|P| \ge 2$ and so there exist $s' \in S$, $s \ne s'$ and $p \in P$ such that $p_{s'} > 0$. Assume, per contra, that there exists $x \in P^{\perp}$ such that $x_s \ne 0$. Let p, q be as in (2). Since $p_s \ne 1$ or $q_s \ne 1$, we have by (21) since $|S| \ge 2$,

$$p_s \neq 1 \text{ and } q_s \neq 1.$$
 (22)

Now, we have

$$\sum_{\tilde{s}\in S} x_{\tilde{s}} p_{\tilde{s}} = 0 \text{ and } \sum_{\tilde{s}\in S} x_{\tilde{s}} q_{\tilde{s}} = 0$$

By (22), we have

$$\sum_{\tilde{s}\in S\setminus\{s\}} x_{\tilde{s}} \frac{p_{\tilde{s}}}{1-p_s} + x_s \frac{p_s}{1-p_s} = 0 = \sum_{\tilde{s}\in S\setminus\{s\}} x_{\tilde{s}} \frac{q_{\tilde{s}}}{1-q_s} + x_s \frac{q_s}{1-q_s}$$

By (21), we have

$$x_s \frac{p_s}{1 - p_s} = x_s \frac{q_s}{1 - q_s} \xrightarrow{x_s \neq 0} \frac{p_s}{1 - p_s} = \frac{q_s}{1 - q_s}$$

Now $x \mapsto \frac{x}{1-x}$ is strictly increasing function for $x \in [0,1)$ and so by (22),

$$\frac{p_s}{1-p_s} = \frac{q_s}{1-q_s} \Rightarrow p_s = q_s.$$

This contradicts (2).

Next, we show that (1) \Rightarrow (2). Denote $n^{\perp} = \dim P^{\perp}$ and $n = \dim P$. Now $n^{\perp} + n = |S|$. There exist $(c^i)_{i=1}^{n^{\perp}} \subseteq P^{\perp}$ linearly independent vectors and $(p^i)_{i=1}^n \subseteq P$ linearly independent vectors.

First, we show that $n \ge 2$. Since P is non-empty, $n \ge 1$. Assume, per contra, n = 1. Thus P is a singleton $P = \{p\}$. If $p_s = 0$, then $(1_s, 0_{-s}) \in P^{\perp}$ which is a contradiction. By assumption, there exists $s' \in S, s' \ne s$ such that $p_{s'} > 0$. Now $\left(\left(\frac{1}{p_{s'}}\right)_{s}, \left(-\frac{1}{p_s}\right)_{s'}, 0_{-s,s'}\right) \in P^{\perp}$ that is a contradiction. So $n \ge 2$.

Next, consider the matrix C formed by row vectors $(c^i)_{i=1}^{n^{\perp}}$ and columns S. We can reduce C into Smith normal form \tilde{C} formed by row vectors $(\tilde{c}^i)_{i=1}^{n^{\perp}}$ such that for each $i \in \{1, \ldots, n^{\perp}\}$, there exists $s^i \in S \setminus \{s\}$ such that $\tilde{c}_{s^i}^i = -1$, for all $j \in \{1, \ldots, n^{\perp}\} \setminus \{i\}$, $\tilde{c}_{s^j}^i = 0$, and for all $k, l \in \{1, \ldots, n^{\perp}\}, k \neq l, s^k \neq s^l$. Now $(\tilde{c}^i)_{i=1}^{n^{\perp}} \subseteq P^{\perp}$ are linearly independent and for all $i \in \{1, \ldots, n^{\perp}\}, \tilde{c}_s^i = 0$.

Denote $S^{\perp} = \{s^1 \dots, s^{n^{\perp}}\}$. Now we have for all $i \in \{1, \dots, n^{\perp}\}$ and $p \in P$,

$$p_{s^i} = \sum_{\tilde{s} \in S \setminus (S^\perp \cup \{s\})} \tilde{c}^i_{\tilde{s}} p_{\tilde{s}}.$$
(23)

Let $\pi : \{1, \ldots, n-1\} \to S \setminus (S^{\perp} \setminus \{s\})$ be a one-to-one function such that $\pi(n) = s$. We show by induction that for each $i \in \{0, \ldots, n-1\}$, there exists a collection of linearly independent probabilities $(p^{j,i})_{j=1}^n \subseteq P$ such that for all $1 \leq m \leq i$, and $j,k \in \{1,\ldots,n\} \setminus \{m\},\$

$$p_{\pi(m)}^{j,i} = p_{\pi(m)}^{k,i} > p_{\pi(m)}^{m,i}$$

This follows symmetrically to the proof in Lemma S.3 since at each step for i < n-1,

$$\min\left\{p_{\pi(i+1)}^{j,i} \middle| j \in \{i+1,\dots,n\}\right\} \neq \max\left\{p_{\pi(i+1)}^{j,i} \middle| j \in \{i+1,\dots,n\}\right\}$$

by linear independence and since for all $k \leq i$ and $j, l \in \{i+1, \ldots, n\}, p_{\pi(k)}^{j,i} = p_{\pi(k)}^{l,i}$.

Denote $p^* = p^{n,n-1}$. Since $n \ge 2$, by taking a convex combination of $(p^{j,n-1})_{j=1}^{n-1}$, there exist a > 1 and $p^{\dagger} \in P$ such that p^* and p^{\dagger} are linearly independent and for all $\tilde{s} \in S \setminus (S^{\perp} \cup \{s\}), p^*_{\tilde{s}} = ap^{\dagger}_{\tilde{s}} > 0$. Let $s^{\dagger} \in S \setminus (S^{\perp} \cup \{s\})$. Now we have for all $\tilde{s} \in S \setminus (S^{\perp} \cup \{s\}),$

$$\frac{p_{\tilde{s}}^*}{p_{s^{\dagger}}^*} = \frac{ap_{\tilde{s}}^{\dagger}}{ap_{s^{\dagger}}^{\dagger}} = \frac{p_{\tilde{s}}^{\dagger}}{p_{s^{\dagger}}^{\dagger}}$$

By (23), we have for all $i \in \{1, \ldots, n^{\perp}\}$ and $p \in P$,

$$\frac{p_{s^i}}{p_{s^\dagger}} = \sum_{\tilde{s} \in S \setminus (S^\perp \cup \{s\})} \tilde{c}^i_{\tilde{s}} \frac{p_{\tilde{s}}}{p_{s^\dagger}}.$$

Thus for all $\tilde{s} \in S \setminus \{s\}$,

$$\frac{p_{\tilde{s}}^*}{p_{s^{\dagger}}^*} = \frac{p_{\tilde{s}}^{\dagger}}{p_{s^{\dagger}}^{\dagger}}.$$
(24)

By taking the sum over \tilde{s} , we have

$$\frac{1-p_s^*}{p_{s^\dagger}^*} = \frac{1-p_s^\dagger}{p_{s^\dagger}^\dagger}.$$

Thus by multiplying (24) by $\frac{p_{s^{\dagger}}^*}{1-p_s^*}$, we have for all $\tilde{s} \in S \setminus \{s\}$,

$$\frac{p_{\tilde{s}}^*}{1-p_s^*} = \frac{p_{\tilde{s}}^\dagger}{1-p_s^\dagger}.$$
 (25)

Finally, we show that $p_s^* \neq p_s^{\dagger}$. Assume, per contra, $p_s^* = p_s^{\dagger}$. Then by (25) for all $\tilde{s} \in S$, $p_{\tilde{s}}^* = p_{\tilde{s}}^{\dagger}$, which contradicts that p^* and p^{\dagger} are linearly independent. Thus $p_s^* \neq p_s^{\dagger}$ which shows the claim.

The previous two lemmas give the probability identification characterization.

Proposition S.6 (Probability Identification) Assume that (u, \mathbb{C}) is a state dependent tight dual-self variational expected utility for \succeq and $s \in S^{\mathcal{P}}$. The following

three conditions are equivalent:

(1) If $f, g \in H$ are such that $f \sim g$ and $f_s \succ_s g_s$, then there exist $h \in H$ and $\alpha \in (0, 1)$ such that

$$\alpha h + (1 - \alpha)f \not\sim \alpha h + (1 - \alpha)g.$$

(2) $S^{\mathcal{P}} = \{s\}$ or there are $p, q \in \overline{\operatorname{co}} \bigcup_{c \in \mathbb{C}} \operatorname{dom} c$ such that $p_s \neq q_s$ and for all $\tilde{s} \in S \setminus \{s\}$, $p_{\tilde{s}} \qquad q_{\tilde{s}}$

$$\frac{p_s}{1-p_s} = \frac{q_s}{1-q_s}$$

(3) If $(\tilde{u}, \tilde{\mathbb{C}})$ is a state dependent tight dual-self variational expected utility for \succeq , then

$$\Big\{\widetilde{p}_s\Big|\widetilde{p}\in\overline{\mathrm{co}}\bigcup_{\widetilde{c}\in\widetilde{\mathbb{C}}}\mathrm{dom}\,\widetilde{c}\Big\}=\Big\{p_s\Big|p\in\overline{\mathrm{co}}\bigcup_{c\in\mathbb{C}}\mathrm{dom}\,c\Big\}.$$

Proof. First if $S^{\mathcal{P}} = \{s\}$, then (1) is always true since $\overline{\operatorname{co}} \bigcup_{c \in \mathbb{C}} \operatorname{dom} c = \{(1_s, 0_{-s})\}$ and there does not exist $f, g \in H$ such that $f_s \succ_s g_s$ and $f \sim g$ by the representation. Additionally, (2) and (3) are always true.

Second assume that $S^{\mathcal{P}} \neq \{s\}$. (1) \iff (2) follows from Lemmas S.4 and S.5. (2) \iff (3) follows from Theorem 7 and Lemma S.5.