TAIL ESTIMATES AND OFF-DIAGONAL UPPER BOUNDS OF THE HEAT KERNEL

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ABSTRACT. We study the upper bounds of heat kernels of regular Dirichlet forms (with a jump part) on a doubling metric measure space. We prove an equivalent characterization of a certainL^q-estimate of the tail of the heat kernel outside balls in terms of the Faber-Krahn inequality, the generalized capacity condition, and the L^q-estimate of the tail of the jump kernel. As a consequence, we obtain a pointwise upper bound of the heat kernel with a polynomial decay in distance depending on the parameter q. In the case of Ahlfors regular measure, these results are valid for all $q \in [1, \infty]$, while in the general case of doubling measure we have to assume that $q \in [2, \infty]$. Thanks to the presence of the parameter q, our results cover much more general class of jump kernels than was previously possible. The proofs use new methods as well as the results of the previous works [24, 23] of the authors.

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1. Introduction

In this paper, we are concerned with the heat kernel upper bounds for a regular Dirichlet form $(\mathcal{E},\mathcal{F})$ with jump part on a metric space equipped with a doubling measure.

Let (M,d) be a locally compact separable metric space and let μ be a Radon measure on M with full support. The triple (M,d,μ) is referred to as a *metric measure space*. Let $(\mathcal{E},\mathcal{F})$ be a regular jump type Dirichlet form in $L^2 := L^2(M,\mu)$ associated with a Radon measure j defined on $M\times M\backslash\text{diag}$:

$$\mathcal{E}(u,v)=\iint_{M\times M\backslash\text{diag}}(u(x)-u(y))(v(x)-v(y))dj(x,y). \tag{1.1}$$

The Dirichlet form determines the heat semigroup $P_t=\exp(t\mathcal{L})$ in L^2, where $\mathcal{L}$ is the (negative definite) generator of the Dirichlet form. The *heat kernel* of $(\mathcal{E},\mathcal{F})$, denoted by $p_t(x,y)$, is by definition the integral kernel of the heat semigroup $\{P_t\}_{t>0}$. Besides, the heat kernel coincides with the transition density of the jump process associated with $(\mathcal{E},\mathcal{F})$.

If j is absolutely continuous with respect to $\mu\times\mu$, then the density $J(x,y):=\frac{dj}{d(\mu\times\mu)}$ is called *the jump kernel*. For example, if $M=\mathbb{R}^n$ and

$$J(x,y)=\frac{c}{|x-y|^{n+\beta}},\quad x,y\in\mathbb{R}^n$$

where $0<\beta<2$ and $c=c(n)$, then $\mathcal{L}=-(-\Delta)^{\beta/2}$. In this case the heat kernel is the transition density of a symmetric stable process of index β, and it admits the estimate

$$p_t(x,y)\simeq\frac{1}{t^{n/\beta}}\left(1+\frac{|x-y|}{t^{1/\beta}}\right)^{-(n+\beta)}. \tag{1.2}$$

Here the symbol $\simeq$ means that the ratio of both sides are bounded from above and below by two positive constants.

We aim at similar estimates of the heat kernel in a general metric measure space (M,d,μ). Denote by $B(x,r)$ open metric balls in M. Suppose for the moment that μ is α-regular for some $\alpha>0$, that is, for all $x\in M$ and $r>0$,

$$\mu(B(x,r))\simeq r^\alpha. \tag{V}$$

By a result of Grigor'yan and Kumagai (cf. [30]), if the heat kernel is stochastically complete and satisfies a self-similar estimate

$$p_t(x,y)\simeq t^{-\gamma}\Phi\left(\frac{d(x,y)}{t^{1/\beta}}\right)$$

for some $\beta,\gamma>0$ and some function Φ then it is necessarily the following estimate:

$$p_t(x,y)\simeq\frac{1}{t^{\alpha/\beta}}\left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}. \tag{1.3}$$

We refer to (1.3) as a *stable-like* estimate of the heat kernel because of its similarity to (1.2). A natural question aries: what conditions on the jump kernel J ensure (1.3)?

Chen and Kumagai proved in [11] that if $\beta<2$ then (1.3) is equivalent to the following condition:

$$J(x,y)\simeq d(x,y)^{-(\alpha+\beta)}\quad x,y\in M. \tag{J}$$

However, on most of fractal sets there exist regular Dirichlet forms with the jump kernel satisfying (J) with $\beta\geq 2$. In this case one needs one more condition: the *generalized capacity condition* denoted shortly (Gcap) that will be explained below.

Condition (Gcap) is closely related to the *cutoff Sobolev inequality* introduced by Barlow and Bass in [4], and to the *energy inequality* of Andres and Barlow in [1]. With help of this condition, the following result was proved by Grigor'yan, E.Hu and J.Hu in [21] and in a more general setting by Chen, Kumagai and Wang in [15].

Theorem 1.1. *Under the standing assumption (V) we have, for any $\beta>0$,*

$$(\text{Gcap})+(\text{J})\Leftrightarrow(1.3). \tag{1.4}$$

The above results deal with Dirichlet forms when the jump kernel admits comparable upper and lower bounds. However, there are many interesting jump measures when this is not the case. For instance, the jump kernel can vanish somewhere or may not exist at all. For such jump measures, only very limited results on heat kernel estimates are available. In paper [9], the authors considered on *ultrametric* spaces a class of jump kernels satisfying the following rather weak tail estimate: for all $x \in M$ and $r > 0$,

$$\int_{B(x,r)^c} J(x,y)d\mu(y) < \frac{C}{r^\beta}. \tag{TJ}$$

In [9, Theorem 2.8], we proved that, under the standing assumption (TJ), a certain Poincaré inequality (denoted there by (PI)) is equivalent to two sided estimates of the heat kernel that include the following upper bound

$$p_t(x,y) \le \frac{C}{t^{\alpha/\beta}}\left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-\beta} \tag{1.5}$$

and a certain weak lower bound. Let us emphasize that the exponent β here is smaller that the optimal exponent $\alpha+\beta$ in (1.3). However, the exponent β cannot be improved in this setting.

In the proof of the above result, the following *tail estimate* of the heat semigroup plays an important role: for any ball B of radius $r > 0$ and for any $t > 0$,

$$P_t \mathbf{1}_{B^c} \le \frac{Ct}{r^\beta} \quad \text{in } \frac{1}{4}B. \tag{TP}$$

(Here λB for $\lambda > 0$ means a ball of radius λr concentric to B.) Indeed, the most difficult part in [9] was to prove that (PI) + (TJ) $\Rightarrow$ (TP). Then the upper bound (1.5) follows easily from (TP) and other conditions.

It is clear that, under the hypothesis (V), the upper bound of the jump kernel in (J) implies (TJ). Similarly, the upper estimate of the heat kernel in (1.3) implies (1.5) as well as (TP).

One may ask whether there are other shapes of the heat kernel (and jump kernel) estimates between these two cases (1.3) and (1.5) (reps. between (J) and (TJ)).

In this paper we give a positive answer to this question by introducing one-parameter families of heat kernel and jump kernel estimates and by proving their equivalence (under certain standing hypotheses).

Assuming for simplicity that (V) holds, fix a parameter $q \in [1,\infty]$ and define the following L^q *tail estimate* of the jump kernel (see also Definition 2.5 below for a more general case): for all $x \in M$ and $r > 0$,

$$\|J(x,\cdot)\|_{L^q(B(x,r)^c)} \le \frac{C}{r^{\alpha/q'+\beta}}, \tag{TJ$_q$}$$

where $q' = \frac{q}{q-1}$ is the Hölder conjugate of q. Similarly, we introduce the L^q *tail estimate* of the heat kernel (see also Definition 2.10 for a more general case): for all $x \in M$, $r > 0$ and $t > 0$,

$$\|p_t(x,\cdot)\|_{L^q(B(x,r)^c)} \le C\left(t^{-\frac{\alpha}{\beta q'}} \wedge \frac{t}{r^{\alpha/q'+\beta}}\right) \simeq \frac{C}{t^{\alpha/(\beta q')}}\left(1 + \frac{r}{t^{1/\beta}}\right)^{-(\alpha/q'+\beta)}. \tag{TP$_q$}$$

as well as the following pointwise upper bound of the heat kernel (see also Definition 2.13 for a more general case): for all $x, y \in M$ and $t > 0$,

$$p_t(x,y) \le Ct^{-\frac{\alpha}{\beta q}}\left(t^{-\frac{\alpha}{\beta q'}} \wedge \frac{t}{d(x,y)^{\alpha/q'+\beta}}\right) \simeq \frac{1}{t^{\alpha/\beta}}\left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha/q'+\beta)}. \tag{UE$_q$}$$

Our main result for Ahlfors-regular spaces (Theorem 3.4) says the following: if (V) holds true, then

$$\text{(FK)} + \text{(Gcap)} + (\text{TJ}_q) \Leftrightarrow (\text{TP}_q) + \text{(C)} \Rightarrow (\text{UE}_q). \tag{1.6}$$

Here (FK) is a certain *Faber-Krahn inequality* (see Definitions 2.3, 3.3 for details), and condition (C) means that the Dirichlet form $(\mathcal{E},\mathcal{F})$ is conservative, that is, $P_t 1 = 1$ for all $t > 0$.

In fact, we prove this result in a more general setting of doubling measure (see Theorem 2.15) but in this case we have to assume that $q \in [2,\infty]$. Let us also mention that the most interesting and difficult part of the proof of (1.6) is the implication:

$$\text{(FK)} + \text{(Gcap)} + (\text{TJ}_q) \Rightarrow (\text{TP}_q).$$

We remark that the result (1.6) for the case when $q = \infty$ was partly proved by Grigor'yan, J. Hu and Lau in [28] and by Chen, Kumagai and Wang in [15], while (1.6) is entirely new when $q \in [2,\infty)$.

Note that our approach is purely analytical, and the metric space may be bounded or unbounded. Let us also emphasize the following novelties of this paper:

- Our starting point is the L^q-tail estimate of the jump kernel, rather than a more conventional pointwise estimate of J.
- Our main result in Theorem 2.15 is stated and proved for a general volume function $V(x,r)$ satisfying the volume doubling condition as well as for a general scaling function $W(x,r)$ (that replaces r^β) that may depend on point x, which covers many examples of metric measure spaces.

There are many works devoted to the study of heat kernels on metric measure spaces including fractals. For example, see [7] for the Sierpiński gasket, [17] for affine nested fractals, [32] for post-critically finite self-similar sets, [2], [3] for the Sierpiński carpets, and [34, 35] for a certain class of self-similar sets. Equivalent conditions for two-sided estimates of heat kernels for local Dirichlet forms on metric measure spaces were investigated in [29] and [31], whilst for non-local Dirichlet forms in [8], [11], [12], [13], [14], [15], [21]. Equivalent conditions only for upper estimates of heat kernels for local Dirichlet forms were studied in [1], [25], [26], [27, Section 6], [29, Section 9], [37], whilst for non-local Dirichlet forms, in [10], [28].

The structure of the paper is as follows.

In Section 2 we give all necessary definitions and state our main result – Theorem 2.15, for an arbitrary doubling measure μ and a general scaling function $W(x,r)$ that may depends on x. Here we assume that $q \in [2,\infty]$.

In Section 3 we state our second main result – Theorem 3.4, in the setting of an Ahlfors-regular measure μ and a specific scaling function $W(x,r) := r^\beta$ for some $\beta > 0$. Here we assume that $q \in [1,\infty]$.

In Section 4 we investigate the properties of condition (TP_q) and, in particular, prove its monotonicity with respect to q.

In Section 5 we investigate the properties of so-called ρ-local Dirichlet forms, for example, the ρ-locality, and prove the inequalities related to the associated resolvent. We also study the relation between the semigroups associated with the original and truncated Dirichlet forms.

In Section 6 we introduce a new metric equivalence to the original metric in some sense, and prove that the doubling (and reverse doubling) properties of μ are preserved by this change of metric. The purpose of this change of metric is to simplify the scaling function.

In Section 7 we rephrase all the conditions in question (for example, (TJ_q), (TP_q), (UE_q) etc) in terms of the new metric.

In Section 8 we apply the results of Sections 5-7 to obtain the upper estimates of heat kernels as well as the tail estimates of semigroups for truncated Dirichlet from under the new metric. This section is crucial in deriving the main results of this paper.

In Section 9 we prove the main implication

$$(\mathrm{FK}) + (\mathrm{Gcap}) + (\mathrm{TJ}_q) \Rightarrow (\mathrm{TP}_q)$$

for the new metric, and then come back to the original metric. We also prove that (TP_q) $\Rightarrow$ (UE_q).

In Section 10 we first investigate the consequences of condition (TP_q), in particular, we prove that

$$(\mathrm{TP}_q) \Rightarrow (\mathrm{FK}) + (\mathrm{TJ}_q)$$

(Lemma 10.3 and Proposition 10.4). Then we finally conclude the proof of our main Theorem 2.15.

In Section 11 we collect some known results needed in this paper.

Let us describe the main steps of the proof of the implications (1.6) in the general setting.

Step 0. We recall our previous results that will be used below. It was proved in [24, Theorem 14.1] that conditions (FK), (TJ_q) and (Gcap) imply the *survival estimates*, denoted by (S_+) and (S) (see Definitions 7.2 and 7.1) respectively, for all $q \in [1,\infty]$. Survival estimates play an important role in obtaining the exponential decay rate of heat kernels for truncated Dirichlet forms. Moreover, it was proved in [23, Corollary 2.14] that the same set of conditions also implied the existence and *on-diagonal upper estimate* (DUE) of the heat kernel for all $q \in [2,\infty]$ (see also Proposition 2.8). So, the rest of the proof mainly focuses on off-diagonal upper estimate of heat kernels and the tail estimate of semigroups.

Step 1. We consider a general Dirichlet form with truncated jump part:

$$\mathcal{E}^{(\rho)}(u,v) := \mathcal{E}^{(L)}(u,v) + \iint_{M\times B(x,\rho)} (u(x)-u(y))\,(v(x)-v(y))\,dj, \quad u,v\in\mathcal{F},$$

where $\rho > 0$ is fixed. We show that the resolvent of $(\mathcal{E}^{(\rho)}, \mathcal{F}(\Omega))$ satisfies various comparison inequalities (Propositions 5.3-5.5 and Lemma 5.6) which together with other conditions will be further used to prove that the heat kernel of $(\mathcal{E}^{(\rho)}, \mathcal{F})$ decays exponentially in distance. Besides, we investigate the relation between the semigroups of $(\mathcal{E}^{(\rho)}, \mathcal{F})$ and $(\mathcal{E}, \mathcal{F})$ (Proposition 5.11 and Lemma 5.12).

Step 2. We introduce a new metric d_* (Proposition 6.1) under which the scaling function becomes much simpler: $W_*(r) = r^\beta$ for some $\beta > 0$. This metric d_* is comparable to the original metric d in some sense (Propositions 6.2). Moreover, the doubling (and reverse doubling) property of μ are also satisfied with respect to d_* (Proposition 6.4).

Step 3. We rephrase all the condition (DUE), (S), (S_+), (TJ_q), (TJ) in terms of the new metric d_* as (DUE_*), (S_*), (S^*_+), (TJ^*_q), (TJ_*) respectively (Proposition 7.4). Then we set our main task: to obtain (TP^*_q) (see Definition 9.2), that is, for any ball $B_*(x,r)$ and any t in a finite inteval

$$\|p_t(x,\cdot)\|_{L^q(B_*^c)} \le C\left(\frac{1}{V_*(x,t^{1/\beta})^{1/q'}} \wedge \frac{t}{V_*(x,r)^{1/q'} r^\beta}\right). \tag{1.7}$$

Step 4. We study the following truncated bilinear form

$$\mathcal{E}^{(\rho)}(u,v) := \mathcal{E}^{(L)}(u,v) + \iint_{M\times B_*(x,\rho)} (u(x)-u(y))\,(v(x)-v(y))\,dj, \quad u,v\in\mathcal{F},$$

where $\rho > 0$ and $B_*(x,\rho)$ is a metric ball in (M, d_*). Under (TJ_*), we verify that $(\mathcal{E}^{(\rho)}, \mathcal{F})$ is also a regular Dirichlet form. In Proposition 8.2 and Lemma 8.3, we prove that the heat kernel $q_t^{(\rho)}(x,y)$ of $(\mathcal{E}^{(\rho)}, \mathcal{F})$ exists and satisfies on-diagonal upper estimate under conditions (DUE_*), (TJ_*). In particular, there is a common regular nest $\{F_k\}$ such that $q_t^{(\rho)}(x,\cdot) \in C(\{F_k\})$ for all $t > 0$, $x \in M$ and $\rho \in \mathbb{Q}_+$ (see Remark 8.4).

Step 5. Let $\{Q_t\}$ be the heat semigroup associated with truncated Dirichlet form and $\{Q^{B_*}\}$ be the Dirichlet semigroup for any ball B_*. We show that the tail of Q_t decays exponentially as shown in Lemmas 8.8 and 8.9: for any $k \ge 1$ and any ball B_* of radius $r \ge 4k\rho$,

$$Q_t \mathbf{1}_{B_*^c} \le 1 - Q_t^{B_*}\mathbf{1}_{B_*} \le C(\theta,k)\left(\frac{t}{\rho^\beta}\right)^{\frac{\theta k}{\theta+\beta}} \quad \text{in } \frac{1}{4}B_*, \tag{1.8}$$

where $\theta > 0$ is arbitrary.

Moreover, we give the relation between the tails of heat kernels $p_t(x,y)$ and $q_t(x,y)$ outside balls (see Lemmas 8.10 and 8.11): for any $t > 0$ and any ball B_* with $r > 0$,

$$\|p_t(x,\cdot)\|_{L^q(B_*^c)} \le \|q_t(x,\cdot)\|_{L^q(B_*^c)} + \frac{Ct}{V_*(x,\rho)^{1/q'}\rho^\beta}\exp\left(\frac{c't}{\rho^\beta}\right), \tag{1.9}$$

where $q' = \frac{q}{q-1} \in [1,\infty]$. In particular, in the case when $q = \infty$ and $q' = 1$, the above inequality gives the pointwise relation between $p_t(x,y)$ and $q_t(x,y)$.

Step 6. By (1.9), in order to prove (TP^*_q), we need firstly to obtain the following off-diagonal upper estimate of $q_t(x,y)$: for any $t > 0$ and $x, y \in M$,

$$q_t(x,y) \le \frac{C}{V_*(x,t^{1/\beta})}\exp\left(\frac{c't}{\rho^\beta}\right)\left(1 + \frac{\rho}{t^{1/\beta}}\right)^{\alpha_*}\exp\left(-c\frac{d_*(x,y)}{\rho}\right). \tag{1.10}$$

We prove this estimate by using (1.8) and on-diagonal upper estimate of $q_t(x,y)$ as well as other conditions (see Lemma 8.12).

Step 7. Using the semigroup property of $q_t(x,y)$, the fact that $\int_M q_t(x,y)d\mu(y) \le 1$ and (1.10), we show that

$$\|q_t(x,\cdot)\|_{L^q(B_*^c)} \le \left(\frac{C}{V_*(x,t^{1/\beta})}\exp\left(\frac{Ct}{\rho^\beta}\right)\left(1 + \frac{\rho}{t^{1/\beta}}\right)^{\alpha_*}\right)^{\frac{q-1}{q}}.$$

Consequently, using (1.9) we obtain

$$\|p_t(x,\cdot)\|_{L^q(B_*^c)} \le \frac{C}{V_*(x,t^{1/\beta})^{1/q'}}\exp\left(\frac{Ct}{\rho^\beta}\right)\left(1+\frac{\rho}{t^{1/\beta}}\right)^{\alpha_*/q'} + \frac{Ct}{V_*(x,\rho)^{1/q'}\rho^\beta}\exp\left(\frac{C't}{\rho^\beta}\right).$$

Since $\rho \in \mathbb{Q}_+$ in the above inequality is arbitrary, one can pass the limit as $\mathbb{Q}_+ \ni \rho \to t^{1/\beta}$ and obtain the first term on the right hand side of the inequality (1.7) (see the first part in the proof of Lemmas 9.4 and 9.6).

Step 8. To obtain the second term on the right hand side of (1.7), it suffices to consider the case when

$$r^\beta > t.$$

By (1.8), (1.10) and the semigroup property of $q_t(x,y)$, we show that the term $\|q_t(x,\cdot)\|_{L^q(B_*^c)}$ on the right hand side of (1.9) is controlled as follows: for any $x \in M$, $r > 0$ and $k \ge 1$ so that $r \ge 4k\rho$,

$$\|q_t(x,\cdot)\|_{L^q(B_*^c)} \le \frac{C(\theta,k)}{V_*(x,r)^{1/q'}}\exp\left(\frac{c't}{q'\rho^\beta}\right)\left(\frac{r}{\rho}\right)^{2\alpha_*/q'}\left(\frac{t}{\rho^\beta}\right)^{\frac{\theta k}{(\theta+\beta)q}-\frac{2\alpha_*}{\beta q'}}.$$

Consequently, by (1.9), we obtain for any $x \in M$, $r > 0$ and $k \ge 1$ so that $r \ge 4k\rho$,

$$\|p_t(x,\cdot)\|_{L^q(B_*^c)} \le \frac{C(\theta,k)}{V_*(x,r)^{1/q'}}\exp\left(\frac{c't}{q'\rho^\beta}\right)\left(\frac{r}{\rho}\right)^{2\alpha_*/q'}\left(\frac{t}{\rho^\beta}\right)^{\frac{\theta k}{(\theta+\beta)q}-\frac{2\alpha_*}{\beta q'}} + \frac{Ct}{V_*(x,\rho)^{1/q'}\rho^\beta}\exp\left(\frac{c't}{\rho^\beta}\right).$$

In the above inequality we set $\theta = \beta$ first and then choose k large enough such that $\frac{\theta k}{(\theta+\beta)q} - \frac{2\alpha_*}{\beta q'} > 1$. Moreover since the left hand side does not depend on ρ we pass the limit as $\mathbb{Q}_+ \ni \rho \to \frac{r}{4k}$ and see that the second term dominates the first term since $\frac{t}{r^\beta} < 1$. In particular, this yields the second term in (1.7), and hence, obtain (TP_q^*) (see the second part in the proof of Lemmas 9.4 and 9.6).

Step 9. In Lemma 9.7, we show that (TP_q^*) $\Leftrightarrow$ (TP_q); hence, we obtain the tail estimates of heat semigroup under the original metric d. On the other hand, the conservativeness follows from condition (S) (see Step 0) by using [20, Lemma 4.6, p. 3327]. Therefore, we obtain the implication "$\Rightarrow$" in the equivalence (1.6).

Step 10. We prove the consequences of (TP_q). It is easy to see that (TP_∞) $\Leftrightarrow$ (UE_∞). For $q \in [2,\infty)$, using the semigroup property and the Hölder inequality, we have for $t > 0$ and $x, y \in M$ with $R := \frac{1}{2}d_*(x,y) > 0$,

$$\begin{aligned}
p_t(x,y) &= \int_M p_{t/2}(x,z)p_{t/2}(z,y)d\mu(z)\\
&\le \int_{B(x,R)^c} p_{t/2}(x,z)p_{t/2}(z,y)d\mu(z) + \int_{B(y,R)^c} p_{t/2}(x,z)p_{t/2}(z,y)d\mu(z)\\
&\le \|p_{t/2}(x,\cdot)\|_{L^q(B(x,R)^c)}\|p_{t/2}(\cdot,y)\|_{L^{q'}} + \|p_{t/2}(x,\cdot)\|_{L^{q'}}\|p_{t/2}(\cdot,y)\|_{L^q(B(y,R)^c)}.
\end{aligned}$$

Since $q \ge 2$, and hence, $q' = \frac{q}{q-1} \le q$, we have not only (TP_q) but also ($\mathrm{TP}_{q'}$) (see Proposition 4.1). Therefore, by (TP_q), we have

$$\|p_{t/2}(x,\cdot)\|_{L^q(B(x,R)^c)} \le \frac{Ct}{V(x,R)^{1/q'}W(x,R)},$$

and by ($\mathrm{TP}_{q'}$),

$$\|p_{t/2}(\cdot,y)\|_{L^{q'}} \le \frac{C}{V(y,W^{-1}(y,t))^{1/q}}.$$

The terms $\|p_{t/2}(x,\cdot)\|_{L^{q'}}$ and $\|p_{t/2}(\cdot,y)\|_{L^q(B(y,R)^c)}$ can be similarly estimated by conditions (TP_q) and ($\mathrm{TP}_{q'}$). Combining all the above inequalities, one can obtain (UE_q) and the second implication "$\Rightarrow$" in (1.6) (see Lemma 9.8 for the details).

Step 11. For $q \in [2,\infty]$, the implication (TP_q) $\Rightarrow$ (DUE) follows from semigroup property and Proposition 4.1 (see Lemma 10.3(i)). Then, we use the idea in [26, p. 551-553] to prove (DUE) $\Rightarrow$ (FK) (see Proposition 10.4). The implication (TP_q) $\Rightarrow$ (TJ_q) + (S) is proved in Lemma 10.3, and (S) $\Rightarrow$ (Gcap) was proved in [24, Theorem 14.1]. This completes the proofs of the implication "$\Leftarrow$" in (1.6) and hence our main result - Theorem 2.15.

Step 12. The main reason that the parameter q in Theorem 2.15 has to be at least 2 is because $q \ge 2$ is used in the implication (FK) + (Gcap) + (TJ_q) $\Rightarrow$ (DUE) (see Proposition 2.8). However, when μ is

Ahlfors-regular, (DUE) follows directly from the Nash inequality, which itself follows from the Faber-Krahn inequality ($FK'_{\beta/\alpha}$) (see Lemma 10.11). Hence, the parameter q in this setting can take all the values from $[1, \infty]$ as stated in our second main result - Theorem 3.4.

Notation. Letters c, C, C', C_1, C_2, etc. are used to denote universal positive numbers, whose values may change at any occurrence. The letter $\overline{R} = \mathrm{diam} M \in (0, \infty]$ denotes the diameter of the metric space (M, d) throughout this paper. The usage of other letters depends on the context. The integral sign "$\int$" means the integration is taken over the whole space M. For two open sets $U, V \subset M$ and a measurable function F on $M \times M$, in the double integral $\iint_{U\times V} F(x, y)dj(x, y)$, the variable x is taken in U and y in V. Moreover, we may write $\iint_{U\times V} F(x, y)dj(x, y)$ as $\iint_{U\times V} F(x, y)dj$ for short. For a function u on M, the notation $\mathrm{supp}(u)$ means the support of u. For an open set U, the notation $A \Subset U$ means that A is a precompact open subset of U with $\overline{A} \subset U$. The notation $f \simeq g$ means that the ratio of the functions f and g is bounded from above and below by two positive constants for a specified range of the arguments. For a measurable function u on M, a set $U \subset M$ and $p \in [1, \infty)$, we use the notations $\|u\|_{L^p(U)} := \left(\int_U |u|^p d\mu\right)^{1/p}$ and $\|u\|_{L^\infty(U)} := \mathrm{esup}_{x\in U} |u(x)|$. Also we write $\|u\|_p := \|u\|_{L^p(M)}$ for simplicity for $p \in [1, \infty]$.

2. Main results for doubling measures

In this section we state our main results in a more general setting. As above, denote by $B(x, r)$ metric balls in the metric measure space (M, d, μ) that is

$$B(x, r) := \{y \in M : d(y, x) < r\}.$$

Since in general a ball as a subset of M does not determine x and r uniquely, we always require balls to have fixed centers and radii, even if they are not given explicitly. For any ball $B = B(x, r)$ and a positive number λ, denote by

$$\lambda B := B(x, \lambda r).$$

Set $V(x, r) := \mu(B(x, r))$. We say that (M, d, μ) satisfied the *volume doubling* condition, denoted by (VD), if there exists a constant $C \geq 1$ such that, for all $x \in M$ and all $r > 0$,

$$V(x, 2r) \leq CV(x, r). \tag{2.1}$$

In this case we also say that measure μ is *doubling*. Condition (VD) implies that $0 < V(x, r) < \infty$ for all $r > 0$. It is known that condition (VD) is equivalent to the following: there exist $\alpha, C > 0$ such that, for all $x, y \in M$ and all $0 < r \leq R < \infty$,

$$\frac{V(x, R)}{V(y, r)} \leq C\left(\frac{d(x, y) + R}{r}\right)^\alpha. \tag{2.2}$$

In particular, for all $x \in M$ and all $0 < r \leq R < \infty$,

$$\frac{V(x, R)}{V(x, r)} \leq C\left(\frac{R}{r}\right)^\alpha. \tag{2.3}$$

Throughout the paper, we fix a parameter $\overline{R} = \mathrm{diam}\, M$, that is, $\overline{R}$ is the diameter of M. We say that (M, d, μ) satisfies the *reverse volume doubling* condition, denoted by condition (RVD), if there exist two positive numbers C, α' such that, for all $x \in M$ and all $0 < r \leq R < \overline{R}$,

$$C^{-1}\left(\frac{R}{r}\right)^{\alpha'} \leq \frac{V(x, R)}{V(x, r)}. \tag{2.4}$$

Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in $L^2 := L^2(M, \mu)$ (see [19] for definition). In particular, the bilinear form $\mathcal{E}(u, v)$ is defined for all $u, v \in \mathcal{F}$, where $\mathcal{F}$ is a dense subspace of L^2, and $\mathcal{F}$ is complete with respect to the norm $\sqrt{\mathcal{E}_1(u)}$, where

$$\mathcal{E}_1(u) = \mathcal{E}(u) + \|u\|_{L^2}^2 \quad \text{and} \quad \mathcal{E}(u) := \mathcal{E}(u, u).$$

We assume throughout that $(\mathcal{E}, \mathcal{F})$ has no killing part (unless otherwise stated), that is, it admits the following unique *Beurling-Deny decomposition*:(cf. [19, Theorem 3.2.1 and Theorem 4.5.2]):

$$\mathcal{E}(u, v) = \mathcal{E}^{(L)}(u, v) + \mathcal{E}^{(J)}(u, v), \tag{2.5}$$

for all $u, v \in \mathcal{F}$, where Here $\mathcal{E}^{(L)}$ is the *local part* (or *diffusion part*), associated with a unique Radon measure $d\Gamma^{(L)}$ (the notions $\mathcal{E}^{(L)}(u,v), d\Gamma^{(L)}(u,v)$ are instead denoted by $\mathcal{E}^{(c)}(u,v), \frac{1}{2}d\mu^c_{\langle u,v\rangle}$ respectively in [19, see formula (3.2.22) on p.126]) as follows:

$$\mathcal{E}^{(L)}(u,v) = \int_M d\Gamma^{(L)}(u,v),$$

and $\mathcal{E}^{(J)}$ is the jump part associated with a unique Radon measure j defined on $M \times M \backslash \text{diag}$:

$$\mathcal{E}^{(J)}(u,v) = \iint_{M\times M\backslash\text{diag}} (u(x) - u(y))(v(x) - v(y))dj(x,y). \tag{2.6}$$

In this paper, we always assume that the measure j has the following shape:

$$dj(x,y) = J(x,dy)d\mu(x) \ \text{ in } M \times M.$$

Here $J(\cdot,\cdot)$ is kernel on $M \times \mathcal{B}(M)$ (where $\mathcal{B}(M)$ be the sigma-algebra of Borel sets of M), that is,

- for every fixed x in M, the map $E \mapsto J(x,E)$ is a measure on $\mathcal{B}(M)$;
- for every fixed E in $\mathcal{B}(M)$, the map $x \mapsto J(x,E)$ is a non-negative measurable function on M.

By the general theory of Dirichlet forms, $(\mathcal{E},\mathcal{F})$ has a *generator*, denoted by $\mathcal{L}$, that is a non-positive definite self-adjoint operator in L^2 that determines the *heat semigroup* $\{P_t\}_{t\geq 0}$ in L^2, given by $P_t = e^{t\mathcal{L}}$. The integral kernel of $\{P_t\}$ (should it exist) is denoted by $p_t(x,y)$ and is called the *heat kernel* of $(\mathcal{E},\mathcal{F})$. The heat kernel coincides with the transition density of the Hunt process associated with $(\mathcal{E},\mathcal{F})$.

Let $U \subset M$ be an open set, A be a Borel subset of U and $\kappa \geq 1$ be a real number. A *κ-cutoff function* of the pair (A,U) is any function ϕ in $\mathcal{F}$ such that

- $0 \leq \phi \leq \kappa$ μ-a.e. in M;
- $\phi \geq 1$ μ-a.e. in A;
- $\phi = 0$ μ-a.e. in U^c.

We denote by κ-cutoff(A,U) the collection of all κ-cutoff functions of the pair (A,U). Any 1-cutoff function will be simply referred to as a *cutoff function*. Clearly, $\phi \in \mathcal{F}$ is a cutoff function of (A,U) if and only if $0 \leq \phi \leq 1$, $\phi|_A = 1$ and $\phi|_{U^c} = 0$. Denote also

$$\text{cutoff}(A,U) := 1\text{-}\,\text{cutoff}(A,U).$$

Note that for every $\kappa \geq 1$,

$$\text{cutoff}(A,U) \subset \kappa\text{-}\,\text{cutoff}(A,U),$$

and that, if $\phi \in \kappa$-cutoff(A,U), then $1 \wedge \phi \in$ cutoff(A,U). It is known that if $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form in L^2, then cutoff(A,U) is not empty for any non-empty precompact set A with $\overline{A} \subset U$.

Let $\mathcal{F}'$ be a *vector space* defined by

$$\mathcal{F}' := \{v + a : v \in \mathcal{F},\ a \in \mathbb{R}\},$$

which, in particular, contains constant functions that may not be in L^2.

Our next purpose is to introduce condition (Gcap), that is called the *generalized capacity condition*. For that we need the notion of a *scaling function*. A function $W : M \times [0,\infty] \to [0,\infty]$ is called a scaling function if it satisfies the following conditions:

(i) for each $x \in M$, the function $W(x,\cdot)$ is continuous, strictly increasing, and $W(x,0) = 0$, $W(x,\infty) = \infty$;

(ii) there exist three positive constants C, β_1, β_2 (where $\beta_1 \leq \beta_2$) such that, for all $0 < r \leq R < \infty$ and for all $x, y \in M$ with $d(x,y) \leq R$,

$$C^{-1}\left(\frac{R}{r}\right)^{\beta_1} \leq \frac{W(x,R)}{W(y,r)} \leq C\left(\frac{R}{r}\right)^{\beta_2}. \tag{2.7}$$

Denote by $W^{-1}(x,\cdot)$ the inverse function of $r \mapsto W(x,r)$ for every $x \in M$. Clearly, (2.7) implies that, for all $x \in M$ and all $0 < r \leq R < \infty$

$$C^{-1}\left(\frac{R}{r}\right)^{1/\beta_2} \leq \frac{W^{-1}(x,R)}{W^{-1}(x,r)} \leq C\left(\frac{R}{r}\right)^{1/\beta_1}. \tag{2.8}$$

Scaling functions are commonly used, in particular, to describe the space/time scaling for the Hunt process associated with the Dirichlet form. For example, it is known that diffusions/jump processes on many fractal sets have the scaling function

$$W(x,r) := r^\beta$$

for some $\beta > 0$. For instance, for the diffusion on the Sierpiński gasket in $\mathbb{R}^2$, we have $\beta = \frac{\log 5}{\log 2}$. The value of β is called the *walk dimension* of the process. It characterizes how fast the process moves away from its starting point.

Definition 2.1 (Generalized capacity condition)**.** We say that condition (Gcap) is satisfied if there exist two numbers $\kappa \geq 1, C > 0$ such that, for any $u \in \mathcal{F}' \cap L^\infty$ and any pair of concentric balls $B_0 := B(x_0, R)$, $B := B(x_0, R+r)$ with $x_0 \in M$ and $0 < R < R + r < \overline{R}$, there exists $\phi \in \kappa$-cutoff(B_0, B) such that

$$\mathcal{E}(u^2\phi, \phi) \leq \sup_{x\in B} \frac{C}{W(x,r)} \int_B u^2 d\mu. \tag{2.9}$$

We remark that the function ϕ in (Gcap) may depend on u, but the constants κ, C are independent of u, B_0, B. Usually it is very difficult to verify (Gcap). However, there are some cases when (Gcap) is trivially satisfied for certain jump kernels (see conditions (TJ) and ($J_\leq$) below).

For a Borel measurable subset $U \subset M$ and $u \in \mathcal{F}'$, define the *energy measure* $d\Gamma_U(u)$ by

$$d\Gamma_U(u)(x) := d\Gamma^{(L)}(u)(x) + \int_M \mathbf{1}_U(y)(u(x) - u(y))^2 dj(x,y). \tag{2.10}$$

Here we use $\Gamma^{(L)}(u) := \Gamma^{(L)}(u,u)$ for short.

The following condition (ABB) (which is named after Andres, Barlow and Bass [1], [4]) is closely related to (Gcap) (see Lemma 10.6).

Definition 2.2. We say that condition (ABB) is satisfied if there exist $C_1 \geq 0$, $C_2 > 0$ such that, for any $u \in \mathcal{F}' \cap L^\infty$ and for any three concentric balls $B_0 := B(x_0, R)$, $B := B(x_0, R+r)$ and $\Omega := B(x_0, R')$ with $0 < R < R + r < R' < \overline{R}$, there exists $\phi \in$ cutoff(B_0, B) such that

$$\int_\Omega u^2 d\Gamma_\Omega(\phi) \leq C_1 \int_B \phi^2 d\Gamma_B(u) + \sup_{x\in\Omega} \frac{C_2}{W(x,r)} \int_\Omega u^2 d\mu,$$

where $\Gamma_B(u)$ is defined as in (2.10).

For a non-empty open subset U of M, denote by $C_0(U)$ the space of all continuous functions with compact supports contained in U. Let $\mathcal{F}(U)$ be a vector space defined by

$$\mathcal{F}(U) = \text{ the closure of } \mathcal{F} \cap C_0(U) \text{ in the norm } \sqrt{\mathcal{E}_1}. \tag{2.11}$$

By the theory of Dirichlet form, $(\mathcal{E}, \mathcal{F}(U))$ is a regular Dirichlet form on $L^2(U,\mu)$ if $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(M,\mu)$ (see, for example, [19, Theorem 4.4.3]). Denote by $\mathcal{L}^U$ the generator of the Dirichlet form $(\mathcal{E}, \mathcal{F}(U))$ and by $\lambda_1(U)$ the *bottom* of the spectrum of $\mathcal{L}^U$ in $L^2(U,\mu)$. It is known that

$$\lambda_1(U) = \inf_{u\in\mathcal{F}(U)\setminus\{0\}} \frac{\mathcal{E}(u)}{\|u\|_2^2}. \tag{2.12}$$

For any metric ball $B := B(x,r)$, set

$$W(B) := W(x,r).$$

Definition 2.3 (Faber-Krahn inequality)**.** We say that condition (FK) holds if there exist real numbers $\sigma \in (0,1]$ and $C, \nu > 0$ such that, for all balls B with radii $< \sigma\overline{R}$ and all non-empty open subsets U of B,

$$\lambda_1(U) \geq \frac{C^{-1}}{W(B)} \left(\frac{\mu(B)}{\mu(U)}\right)^\nu. \tag{2.13}$$

Sometimes, we label condition (FK) by (FK_ν) to emphasize the role of the exponent ν.

We introduce the condition (TJ) that provides estimates of *tails* of jump measures.

Definition 2.4 (Tail estimate of jump measure)**.** We say that condition (TJ) is satisfied if, for any ball B in M,

$$J(x,B^c) := \int_{B^c} J(x,dy) \le \frac{C}{W(B)}, \tag{2.14}$$

where $C \in [0,\infty)$ is a constant independent of B.

For a given number $1 \le q \le \infty$, let q' be the *Hölder conjugate* of q, that is,

$$q' := \frac{q}{q-1}$$

so that $q' = 1$ if $q = \infty$, and $q' = \infty$ if $q = 1$.

Let us introduce the condition (TJ_q) that provides a *tail estimate* of the jump kernel outside balls in L^q-norm.

Definition 2.5 (L^q-tail estimate of jump kernel)**.** For a given number $1 \le q \le \infty$, we say that condition (TJ_q) is satisfied if there exists a non-negative measurable function J (called the *jump kernel*) on $M \times M$ such that

$$dj(x,y) = J(x,y)d\mu(y)d\mu(x) \text{ in } M \times M,$$

and, for any $x \in M$ and any $R > 0$,

$$\|J(x,\cdot)\|_{L^q(B(x,R)^c)} \le \frac{C}{V(x,R)^{1/q'}\, W(x,R)}, \tag{2.15}$$

where $C \in [0,\infty)$ is a constant independent of x, R.

For example, if $q = 1$ then (2.15) coincides with (2.14). However, let us emphasize that the jump kernel $J(x,y)$ may not exist in condition (TJ), whereas it does in condition (TJ_q), in particular, in (TJ_1); hence,

$$(TJ_1) \Rightarrow (TJ). \tag{2.16}$$

For any x, y in M, denote by

$$V(x,y) := V(x,d(x,y)) \text{ and } W(x,y) := W(x,d(x,y)).$$

(note that $V(x,y)$ and $W(x,y)$ are not symmetric in x,y in general). If $q = \infty$ (and $q' = 1$) then (2.15) clearly becomes

$$J(x,y) \le \frac{C}{V(x,y)\,W(x,y)}, \tag{2.17}$$

for all $x \in M$ and μ-almost all $y \in M$. If (2.17) is satisfied for all $x,y \in M$ then we refer to this condition as ($J_\le$) so that

$$(J_\le) \Rightarrow (TJ_\infty).$$

Assume that $W(x,R) = R^\beta$ for any x in M and $R > 0$. Then the inequality (2.14) becomes

$$J(x,B(x,R)^c) \le \frac{C}{R^\beta} \quad \text{for all } x \in M \text{ and } R > 0.$$

This condition was introduced and studied in [9] on the ultrametric space. If in addition $V(x,R) \simeq R^\alpha$, then (2.17) becomes

$$J(x,y) \le \frac{C}{d(x,y)^{\alpha+\beta}} \quad \text{for all } x,y \in M.$$

This *pointwise* upper bound of the jump kernel is the starting point in most of literature, see for example [15], [21] and the references therein.

Let us recall the notion of a regular $\mathcal{E}$*-nest* (cf. [19, Section 2.1, p.66-69]). For an open set $U \subset M$, let

$$\mathrm{Cap}_1(U) := \inf\{\mathcal{E}(u) + \|u\|_2^2 : u \in \mathcal{F} \text{ and } u \ge 1\ \mu\text{-almost everywhere on } U\} \tag{2.18}$$

(note that $\mathrm{Cap}_1(U) = \infty$ if the set of functions u in (2.18) is empty). An increasing sequence of closed subsets $\{F_k\}_{k=1}^\infty$ of M is called an $\mathcal{E}$*-nest* of M if

$$\lim_{k\to\infty} \mathrm{Cap}_1(M \setminus F_k) = 0.$$

An $\mathcal{E}$-nest $\{F_k\}$ is said to be *regular* with respect to μ if, for each k,

$$\mu(U(x) \cap F_k) > 0 \text{ for any } x \in F_k \text{ and any open neighborhood } U(x) \text{ of } x.$$

For an $\mathcal{E}$-nest $\{F_k\}_{k=1}^{\infty}$, denote by

$$C(\{F_k\}) := \{u \text{ is a function on } M : u|_{F_k} \text{ is continuous for each } k\}. \tag{2.19}$$

A function $u : M \mapsto \mathbb{R} \cup \{\infty\}$ is said to be *quasi-continuous* if $u \in C(\{F_k\})$ for some $\mathcal{E}$-nest $\{F_k\}_{k=1}^{\infty}$.

Definition 2.6. A function $p_t(x, y)$ of three variables $(t, x, y) \in (0, \infty) \times M \times M$ is referred to as a *pointwise heat kernel* if it satisfies the following conditions, for all $t, s > 0$ and x, y in M.

(1) The measurability: $p_t(\cdot, \cdot)$ is jointly measurable on $M \times M$.
(2) The Markov property: $p_t(x, y) \geq 0$ and
$$\int_M p_t(x, y) d\mu(y) \leq 1.$$
(3) The symmetry: $p_t(x, y) = p_t(y, x)$.
(4) The semigroup property:
$$p_{s+t}(x, y) = \int_M p_s(x, z) p_t(z, y) d\mu(z).$$
(5) Approximation of identity: for any $f \in L^2$,
$$\int_M p_t(\cdot, y) f(y) d\mu(y) \to f$$
in L^2-norm as $t \to 0+$.

We say that $p_t(x, y)$ is the pointwise heat kernel of the Dirichlet form $(\mathcal{E}, \mathcal{F})$ if it satisfies in addition the following properties, for all x, y, t.

(1) There exists a regular $\mathcal{E}$-nest $\{F_k\}_{k=1}^{\infty}$ such that
$$p_t(x, \cdot) \in C(\{F_k\})$$
where $C(\{F_k\})$ is defined in (2.19).
(2) If one of points x, y lies outside $\cup_{k=1}^{\infty} F_k$, then
$$p_t(x, y) = 0. \tag{2.20}$$
(3) For any $f \in L^2$,
$$\int_M p_t(\cdot, y) f(y) d\mu(y) \in C(\{F_k\})$$
and
$$\int_M p_t(\cdot, y) f(y) d\mu(y) = P_t f,$$
where $P_t = \exp(t\mathcal{L})$.

The pointwise heat kernel $p_t(x, y)$ allows to extend the definition of the heat semigroup as follows: for any $1 \leq p \leq \infty$, define a *pointwise heat semigroup* in L^p still denoted by $\{P_t\}_{t>0}$, as follows:

$$P_t f(x) := \int_M p_t(x, y) f(y) d\mu(y), \quad f \in L^p$$

for every $t > 0$ and every $x \in M$.

We define the *on-diagonal upper estimate* (DUE) of the heat kernel.

Definition 2.7 (On-diagonal upper estimate)**.** We say that condition (DUE) is satisfied if the pointwise heat kernel $p_t(x, y)$ of $(\mathcal{E}, \mathcal{F})$ exists and, for any $C_0 \geq 1$, there exists a constant $C > 0$ such that, for all $x \in M$ and all $t < C_0 W(x, \overline{R})$,

$$p_t(x, x) \leq \frac{C}{V(x, W^{-1}(x, t))}. \tag{2.21}$$

The following on-diagonal upper estimate of heat kernel was proved in [23, Corollary 2.14].

Proposition 2.8. *Assume that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form in L^2 without killing part. Then, for any $q \in [2, \infty]$,*

$$(\text{VD}) + (\text{FK}) + (\text{Gcap}) + (\text{TJ}_q) \Rightarrow (\text{DUE}).$$

We will prove that, under the hypothesis of Proposition 2.8, certain type of off-diagonal upper estimate of heat kernel is also true (see condition (UE_q) below). Before that, let us introduce condition (TP), the *tail* estimate of the heat semigroup $\{P_t\}$ outside balls.

Definition 2.9 (Tail estimate of heat semigroup outside balls)**.** We say that condition (TP) holds if, for any ball $B = B(x,R)$ with $R \in (0,\overline{R})$ and any $t > 0$,

$$P_t \mathbf{1}_{B^c} \le \frac{Ct}{W(B)} \quad \text{in} \quad \frac{1}{4}B \tag{2.22}$$

for a positive constant C independent of B, t.

Let us define condition (TP_q) for $1 \le q \le \infty$, that is an L^q-estimate of the tail of the heat kernel outside balls.

Definition 2.10 (L^q-tail estimate of the heat kernel)**.** We say that condition (TP_q) is satisfied if the pointwise heat kernel $p_t(x,y)$ of the Dirichlet form (in the sense of Definition 2.6) exists and, for any ball $B := B(x,R)$ with $R \in (0,\overline{R})$ and any $t < W(x,\overline{R})$,

$$\|p_t(x,\cdot)\|_{L^q(B^c)} \le C\left(\frac{1}{V(x,W^{-1}(x,t))^{1/q'}} \wedge \frac{t}{V(x,R)^{1/q'}W(x,R)}\right), \tag{2.23}$$

where C is a positive constant independent of B, t.

Note that condition (TP) does not require the existence of the heat kernel, while condition (TP_q) does. Moreover, the inequality (2.23) in the case $q = \infty$ is equivalent to the following:

$$p_t(x,y) \le C\left(\frac{1}{V(x,W^{-1}(x,t))} \wedge \frac{t}{V(x,y)W(x,y)}\right).$$

For example, if $W(x,R) = R^\beta$ then condition (TP) becomes

$$P_t \mathbf{1}_{B(x,R)^c} \le \frac{Ct}{R^\beta} \quad \text{in } \frac{1}{4}B$$

for any ball $B = B(x,R)$ with $R > 0$ and any $t > 0$. If in addition $V(x,R) \simeq R^\alpha$ then (TP_q) becomes

$$\|p_t(x,\cdot)\|_{L^q(B^c)} \le C\left(\frac{1}{t^{\alpha/(\beta q')}} \wedge \frac{t}{R^{\alpha/q'+\beta}}\right).$$

Remark 2.11. If $\overline{R} < \infty$ and if (2.22) holds for $t < W(x,\overline{R})$, then (2.22) automatically holds also for any $t \ge W(x,\overline{R})$ by adjusting the value of constant C, since $P_t \mathbf{1}_{B^c} \le 1$ in M whilst

$$\frac{t}{W(x,R)} \ge \frac{W(x,\overline{R})}{W(x,R)} \ge 1 \quad \text{for any } 0 < R \le \overline{R}.$$

Therefore, in order to verify (2.22), it suffices to consider only the case when $t < W(x,\overline{R})$.

Remark 2.12. Note that (2.23) is equivalent to the following inequality

$$\|p_t(x,\cdot)\|_{L^q(B^c)} \le C \begin{cases} \frac{1}{V(x,W^{-1}(x,t))^{1/q'}} & \text{if } W(x,R) \le t, \\ \frac{t}{V(x,R)^{1/q'}W(x,R)} & \text{if } W(x,R) \ge t, \end{cases} \tag{2.24}$$

since we have

$$\frac{V(x,R)^{1/q'}}{V(x,W^{-1}(x,t))^{1/q'}} \le 1 \le \frac{t}{W(x,R)} \quad \text{if } W(x,R) \le t,$$
$$\frac{V(x,R)^{1/q'}}{V(x,W^{-1}(x,t))^{1/q'}} \ge 1 \ge \frac{t}{W(x,R)} \quad \text{if } W(x,R) \ge t.$$

The equivalence between (2.23) and (2.24) will be used later on.

Let us introduce condition (UE_q) that is called the *off-diagonal upper estimate* of the heat kernel.

Definition 2.13 (L^q-upper estimate of heat kernel)**.** For a given $1 \le q \le \infty$, we say that condition (UE$_q$) is satisfied if there exists a pointwise heat kernel $p_t(x,y)$ in the sense of Definition 2.6 such that, for all $x,y \in M$ and all $t < W(x,\overline{R}) \wedge W(y,\overline{R})$,

$$p_t(x,y) \le C\left(\frac{1}{V(x,W^{-1}(x,t))^{1/q'}} \wedge \frac{t}{V(x,y)^{1/q'}W(x,y)}\right)\left(\frac{1}{V(x,W^{-1}(x,t))^{1/q}} + \frac{1}{V(y,W^{-1}(y,t))^{1/q}}\right) \quad (2.25)$$

for some positive constant C independent of t,x,y.

For $q = \infty$, we simply write (UE) for (UE$_\infty$).

Remark 2.14. Consider the case when

$$V(x,r) \simeq r^\alpha, \quad W(x,r) = r^\beta,$$

where $\alpha,\ \beta \in (0,\infty)$. Then $W^{-1}(x,t) = t^{1/\beta}$ and

$$V(x,y) \simeq d(x,y)^\alpha,\ W(x,y) = d(x,y)^\beta.$$

The term on the right-hand side of (2.25) is equivalent to the following:

$$\begin{aligned}\left(\frac{1}{t^{\alpha/(q'\beta)}} \wedge \frac{t}{d(x,y)^{\alpha/q'}d(x,y)^\beta}\right)\left(\frac{1}{t^{\alpha/(q\beta)}} + \frac{1}{t^{\alpha/(q\beta)}}\right) &\simeq \frac{1}{t^{\alpha/(q'\beta)}}\left(1 \wedge \left(\frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha/q'+\beta)}\right)\cdot\frac{1}{t^{\alpha/(q\beta)}} \\ &\simeq \frac{1}{t^{\alpha/\beta}}\left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha/q'+\beta)}.\end{aligned}$$

In this case, condition (UE$_q$) is equivalent to

$$p_t(x,y) \le \frac{C}{t^{\alpha/\beta}}\left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha/q'+\beta)}. \quad (2.26)$$

In particular, for $q = 1$, (2.26) becomes

$$p_t(x,y) \le \frac{C}{t^{\alpha/\beta}}\left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-\beta},$$

which is the best heat kernel upper estimate in some cases on ultrametric spaces (cf. [9]).

For $q = \infty$, (2.26) becomes

$$p_t(x,y) \le \frac{C}{t^{\alpha/\beta}}\left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)},$$

which is the best possible heat kernel upper estimate on the fractal metric space, known also as a *stable-like* estimate (see for example [15] and [21]).

Condition (TP$_q$) implies condition (UE$_q$) when $2 \le q \le \infty$. However, the inverse implication may not be true. We will give below an example where (UE$_q$) holds but (TP$_q$) fails when $1 < q \le 2$, see Example 4.2.

We say that condition (C) is satisfied if the Dirichlet form $(\mathcal{E},\mathcal{F})$ is *conservative*, that is

$$P_t 1 \equiv 1 \text{ for each } t > 0.$$

The following theorem is one of the main results in this paper.

Theorem 2.15. *Assume that $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form in L^2 without killing part. If conditions* (VD), (RVD) *hold, then for any $2 \le q \le \infty$*

$$\begin{aligned}\text{(FK)} + \text{(Gcap)} + (\text{TJ}_q) &\Leftrightarrow \text{(FK)} + \text{(ABB)} + (\text{TJ}_q) \\ &\Leftrightarrow (\text{TP}_q) + \text{(C)} \\ &\Rightarrow (\text{UE}_q) + \text{(C)}.\end{aligned}$$

Remark 2.16. Note that condition (ABB) is stable under bounded perturbation of the Dirichlet form. Consequently, Theorem 2.15 shows that (TP$_q$) is stable under such perturbation.

The proof of Theorem 2.15 is highly non-trivial and long. Basically, the proof takes the entire paper and will be completed at the end of Section 10, after a series of propositions and lemmas. The most interesting and difficult part of Theorem 2.15 is the following key implication:

$$(\mathrm{FK}) + (\mathrm{Gcap}) + (\mathrm{TJ}_q) \Rightarrow (\mathrm{TP}_q). \tag{2.27}$$

whose proof is especially involved and consists of many steps. In the course of proof, we introduce a new metric d_* (see Section 6) in order to deal with the difficulties arising from possible dependence of $W(x,\cdot)$ on x. Under this new metric d_*, the measure μ is still doubling, but the scaling function has a simple form, and various conditions can be rephrased in a much simpler way (see Section 7). The idea of introducing the new metric was borrowed from [34] and [6].

The reverse volume doubling (RVD) is used only in the proof of the implication (DUE) ⇒ (FK) (which does not hold in general without (RVD)). Note also that (RVD) follows from (VD) if M is connected and unbounded (cf. [26, Corollary 5.3]); in this case, condition (RVD) can be dropped from the hypotheses of Theorem 2.15.

3. Main results for Ahlfors-regular measures

In our main result Theorem 2.15, the parameter q is always greater than or equal to 2 because we can only obtain the on-diagonal upper estimate (DUE) of heat kernel provided $q \geq 2$, and (DUE) plays an important role in the proof of Theorem 2.15.

In this section, we assume that the measure μ is Ahlfors-regular, which will allow us to state and prove the main results for the entire range $q \in [1, \infty]$.

Let us fix two numbers $\alpha > 0$ and $\beta > 0$. Recall that $\overline{R} = \operatorname{diam} M$ is the diameter of the metric space (M, d).

Definition 3.1. We say that measure μ is α-regular, or μ satisfies condition (V), if for all $x \in M$ and $r < \overline{R}$,

$$V(x,r) \simeq r^{\alpha}. \tag{3.1}$$

In this section we always assume that the condition (V) holds, and that the scaling function W is as follows:

$$W(x,r) := r^{\beta}, \tag{3.2}$$

for all $x \in M$ and $r > 0$.

Definition 3.2. We say that condition (FK′) holds if there exist two numbers $C, \nu > 0$ such that, for any non-empty open sets U,

$$\lambda_1(U) \geq C^{-1}\mu(U)^{-\nu} - \overline{R}^{-\beta}. \tag{3.3}$$

If necessary, we label condition (FK′) by (FK'_ν) to emphasize the role of the exponent ν.

Remark 3.3. It is always true that ($\mathrm{FK}'_{\beta/\alpha}$) ⇒ ($\mathrm{FK}_{\beta/\alpha}$).

Indeed, assume first that $\overline{R} = \infty$. Then, by ($\mathrm{FK}'_{\beta/\alpha}$) and (V), for any ball B of radius r and any open set $U \subset B$, we have

$$\lambda_1(U) \geq C^{-1}\mu(U)^{-\beta/\alpha} = C^{-1}\frac{1}{r^{\beta}}\left(\frac{r^{\alpha}}{\mu(U)}\right)^{\beta/\alpha} \simeq \frac{1}{W(B)}\left(\frac{\mu(B)}{\mu(U)}\right)^{\beta/\alpha}, \tag{3.4}$$

which gives ($\mathrm{FK}_{\beta/\alpha}$). Let now $\overline{R} < \infty$. Let B be a ball of radius $r \leq \sigma\overline{R}$ where $\sigma > 0$ is to be determined later. Then, for any open set $U \subset B$ we have $\mu(U) \leq \mu(B) \leq c(\sigma\overline{R})^{\alpha}$. Choosing $\sigma = \sigma(\alpha, \beta, c, C) > 0$ small enough we obtain that $C^{-1}\mu(U)^{-\beta/\alpha} \geq 2\overline{R}^{-\beta}$. Hence, ($\mathrm{FK}'_{\beta/\alpha}$) yields

$$\lambda_1(U) \geq \frac{1}{2}C^{-1}\mu(U)^{-\beta/\alpha},$$

which implies ($\mathrm{FK}_{\beta/\alpha}$) as in (3.4).

The following theorem states our main result when (V) is satisfied and $q \in [1, \infty]$.

Theorem 3.4. *Assume that $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form in L^2 without killing part. Assume that condition (V) is satisfied and the scaling function is given by (3.2). Then the following equivalences are satisfied:*

$$\begin{aligned}(\mathrm{FK}'_{\beta/\alpha})+(\mathrm{Gcap})+(\mathrm{TJ}) &\Leftrightarrow (\mathrm{FK}'_{\beta/\alpha})+(\mathrm{ABB})+(\mathrm{TJ})\\ &\Leftrightarrow (\mathrm{TP})+(\mathrm{DUE})+(\mathrm{C})\\ &\Leftrightarrow (\mathrm{TP}_1)+(\mathrm{UE}_1)+(\mathrm{C}).\end{aligned}\tag{3.5}$$

Moreover, we have for any $q\in(1,\infty]$,

$$\begin{aligned}(\mathrm{FK}'_{\beta/\alpha})+(\mathrm{Gcap})+(\mathrm{TJ}_q) &\Leftrightarrow (\mathrm{FK}'_{\beta/\alpha})+(\mathrm{ABB})+(\mathrm{TJ}_q)\\ &\Leftrightarrow (\mathrm{TP}_q)+(\mathrm{DUE})+(\mathrm{C})\\ &\Leftrightarrow (\mathrm{TP}_q)+(\mathrm{UE}_q)+(\mathrm{C}).\end{aligned}\tag{3.6}$$

Remark 3.5. The equivalences in (3.5) can be viewed as a version of (3.6) for $q=1$, where (TJ_1) is replaced by a weaker hypothesis (TJ) (cf. (2.16)).

The proof of Theorem 3.4 goes concurrently with that of Theorem 2.15 and will be completed in Section 10.

4. Condition (TP_q)

In this section, we show that condition (TP_q) is monotone in q. Thus, among all the conditions (TP), (TP_1), $\cdots$, (TP_∞), condition (TP) is the weakest, whilst condition (TP_∞) is the strongest one.

Proposition 4.1. *Assume that (VD) holds. Then, for all $1\le q_1<q_2\le\infty$,*

$$(\mathrm{TP}_{q_2}) \;\Rightarrow\; (\mathrm{TP}_{q_1})\Rightarrow\; (\mathrm{TP}_1)\Rightarrow\; (\mathrm{TP}).\tag{4.1}$$

Proof. Assume that condition (TP_{q_2}) holds. Fix a ball $B:=B(x,R)$ with $R>0$ and some $t<W(x,\overline{R})$. We distinguish two cases.

Case 1. Let $W(x,R)\le t$. By Remark 2.12, it suffices to prove that

$$\|p_t(x,\cdot)\|_{L^{q_1}(B^c)}\le\frac{C}{V(x,W^{-1}(x,t))^{1/q_1'}}.$$

If $q_1=1$ then this is trivially satisfied by $\|p_t(x,\cdot)\|_{L^1(B^c)}\le 1$ and $q_1'=\infty$. Let now $q_1>1$. Using (TP_{q_2}), the Hölder inequality with measure $p_t(x,y)d\mu(y)$ and the fact that $\|p_t(x,\cdot)\|_{L^1}\le 1$, we obtain

$$\begin{aligned}\|p_t(x,\cdot)\|_{L^{q_1}(B^c)} &= \left(\int_{B^c}p_t(x,y)^{q_1-1}\cdot p_t(x,y)d\mu(y)\right)^{1/q_1}\\ &\le\left(\left(\int_{B^c}p_t(x,y)^{(q_1-1)\frac{q_2-1}{q_1-1}}\cdot p_t(x,y)d\mu(y)\right)^{\frac{q_1-1}{q_2-1}}\left(\int_{B^c}p_t(x,y)d\mu(y)\right)^{1-\frac{q_1-1}{q_2-1}}\right)^{1/q_1}\\ &\le\left(\int_{B^c}p_t(x,y)^{q_2}d\mu(y)\right)^{\frac{1-1/q_1}{q_2-1}}=\left(\|p_t(x,\cdot)\|_{L^{q_2}(B^c)}\right)^{q_2'/q_1'}\\ &\le\left(\frac{C}{V(x,W^{-1}(x,t))^{1/q_2'}}\right)^{q_2'/q_1'}=\frac{C(q_1,q_2)}{V(x,W^{-1}(x,t))^{1/q_1'}},\end{aligned}$$

which was to be proved.

Case 2. Let $W(x,R)>t$. For any integer $n\ge0$, set $B_n:=B(x,2^nR)$ so that

$$W(B_n)=W(x,2^nR)\ge W(x,R)>t.$$

By condition (TP_{q_2}) we have, for any $n\ge0$,

$$\|p_t(x,\cdot)\|_{L^{q_2}(B_n^c)}\le\frac{Ct}{V(x,R_n)^{1/q_2'}W(x,R_n)}=\frac{Ct}{\mu(B_n)^{1/q_2'}W(B_n)}.$$

Using this and the Hölder inequality, we obtain, for any $n \geq 0$,

$$\begin{aligned}\|p_t(x,\cdot)\|_{L^{q_1}(B_{n+1}\setminus B_n)} &= \left(\int_{B_{n+1}\setminus B_n} p_t(x,y)^{q_1} d\mu(y)\right)^{1/q_1}\\ &\leq \left(\int_{B_{n+1}\setminus B_n} p_t(x,y)^{q_2} d\mu(y)\right)^{1/q_2} \mu(B_{n+1}\setminus B_n)^{1/q_1-1/q_2}\\ &\leq \|p_t(x,\cdot)\|_{L^{q_2}(B_n^c)}\, \mu(B_{n+1})^{1/q_1-1/q_2} \leq \frac{C't}{\mu(B_n)^{1/q_2'} W(B_n)} \mu(B_n)^{1/q_1-1/q_2}\\ &= \frac{C't}{\mu(B_n)^{1/q_1'} W(B_n)} \leq \frac{C't}{\mu(B)^{1/q_1'} W(B_n)}.\end{aligned}$$

Note that, by (2.7), $W(B_n) \geq c2^{n\beta_1} W(B)$, so that

$$\begin{aligned}\|p_t(x,\cdot)\|_{L^{q_1}(B^c)} &= \left(\sum_{n=0}^{\infty} \|p_t(x,\cdot)\|^{q_1}_{L^{q_1}(B_{n+1}\setminus B_n)}\right)^{1/q_1} \leq \sum_{n=0}^{\infty} \|p_t(x,\cdot)\|_{L^{q_1}(B_{n+1}\setminus B_n)}\\ &\leq \sum_{n=0}^{\infty} \frac{C't}{\mu(B)^{1/q_1'} W(B_n)} \leq \frac{Ct}{\mu(B)^{1/q_1'} W(B)},\end{aligned}$$

which proves (TP_{q_1}) by Remark 2.12. □

One of the claims of Theorem 2.15 is that

$$(TP_q) \Rightarrow (UE_q)$$

provided $q \geq 2$. Let us give an example showing that the opposite implication is not satisfied if $q = 2$, that is condition (TP_q) is strictly stronger than (UE_q) when $q = 2$. Probably, this is true for all $q \geq 2$.

Example 4.2. Let us fix $1 < q \leq 2$ and give an example when condition (UE_q) holds but condition (TP_q) fails, that is,

$$(UE_q) \not\Rightarrow (TP_q).$$

Let $\beta, \alpha_1, \alpha_2$ be three positive numbers. Let (M_i, d_i, μ_i) for $i = 1, 2$ be two ultrametric spaces, where each measure μ_i is α_i-regular. Let $J^{(i)}$ be a function on $M_i \times M_i$ for $i = 1, 2$ such that for all $x_i, y_i \in M_i$,

$$J_i(x_i, y_i) \simeq d_i(x_i, y_i)^{-(\alpha_i+\beta)}.$$

Let $(\mathcal{E}^{(i)}, \mathcal{F}^{(i)})$ for $i = 1, 2$ be two Dirichlet forms on $L^2(M_i, \mu_i)$ defined, respectively, by

$$\mathcal{E}^{(i)}(u,v) = \iint_{M_i\times M_i} (u(x_i) - u(y_i))(v(x_i) - v(y_i)) J_i(x_i, y_i) d\mu(x_i) d\mu(y_i), \quad u, v \in \mathcal{F}^{(i)},$$

where the space $\mathcal{F}^{(i)}$ is the closure of the set

$$\Big\{ \sum_{j=0}^{n} c_j \mathbf{1}_{B_j} : n \in \mathbb{N},\ c_j \in \mathbb{R},\ B_j \text{ is a compact ball} \Big\}$$

under the inner product

$$\sqrt{\mathcal{E}^{(i)}(\cdot,\cdot) + \|\cdot,\cdot\|^2_{L^2(M_i,\mu_i)}}.$$

The Dirichlet form $(\mathcal{E}^{(i)}, \mathcal{F}^{(i)})$ is regular and non-local (cf. [9, Theorem 2.2]). It turns out that the heat kernel $p_t^{(i)}(x_i, y_i)$ of the form $(\mathcal{E}^{(i)}, \mathcal{F}^{(i)})$ exists and satisfies the following two-sided estimates:

$$p_t^{(i)}(x_i, y_i) \simeq t^{-\frac{\alpha_i}{\beta}} \left(1 + \frac{d_i(x_i, y_i)}{t^{1/\beta}}\right)^{-(\alpha_i+\beta)} \tag{4.2}$$

for all $t > 0$ and all x_i, y_i in M_i, see for example [14], [21].

Let us construct a new ultrametric space (M, d, μ) by letting $M := M_1 \times M_2$, $\mu := \mu_1 \times \mu_2$, and

$$d(x,y) := \max\{d_1(x_1,y_1), d_2(x_2,y_2)\} \text{ for } x = (x_1,x_2),\ y = (y_1,y_2) \text{ in } M.$$

Clearly, for any point $x = (x_1, x_2)$ in M, any metric ball $B(x, r)$ in M is a direct product of balls $B(x_1, r)$ in M_1 and $B(x_2, r)$ in M_2, that is,

$$B(x, r) = B(x_1, r) \times B(x_2, r).$$

It follows that

$$V(x, r) = \mu(B(x, r)) = \mu_1(B(x_1, r))\mu_2(B(x_2, r)) \simeq r^{\alpha_1+\alpha_2} \simeq r^{\alpha}, \tag{4.3}$$

where $\alpha := \alpha_1 + \alpha_2$. For any point x in M and any $r > 0$, let

$$W(x, r) = r^{\beta}.$$

Define the measure j on $\mathcal{B}(M \times M)$ by $dj(x, y) = J(x, dy)d\mu(x)$, where $J(x, dy)$ is a kernel on $M \times \mathcal{B}(M)$ given by

$$J(x, dy) = J^{(1)}(x_1, y_1)d\mu_1(y_1)d\delta_{x_2}(y_2) + J^{(2)}(x_2, y_2)d\mu_2(y_2)d\delta_{x_1}(y_1) \tag{4.4}$$

for any points $x = (x_1, x_2)$, $y = (y_1, y_2)$ in M, where $d\delta_b(x)$ is the Dirac measure concentrated at point $b \in \mathbb{R}$.

By (4.3) and the definition (4.4), we have for any $r > 0$,

$$\begin{aligned}
&\sup_{x=(x_1,x_2)\in M} \int_{B(x,r)^c} J(x, dy)\\
&= \sup_{x=(x_1,x_2)\in M} \int_{B(x,r)^c} \left(J^{(1)}(x_1, y_1)d\mu_1(y_1)d\delta_{x_2}(y_2) + J^{(2)}(x_2, y_2)d\mu_2(y_2)d\delta_{x_1}(y_1)\right)\\
&\le \sup_{x_1\in M_1} \int_{B(x_1,r)^c} J^{(1)}(x_1, y_1)d\mu_1(y_1) + \sup_{x_2\in M_2} \int_{B(x_2,r)^c} J^{(2)}(x_2, y_2)d\mu_2(y_2)\\
&\le \frac{C}{r^\beta} + \frac{C}{r^\beta} = \frac{2C}{r^\beta}.
\end{aligned}$$

Hence, by [9, Theorem 2.2], the measure j determines a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(M, \mu)$, and the heat kernel $p_t(x, y)$ of $(\mathcal{E}, \mathcal{F})$ exists. It is known from the general theory that $p_t(x, y)$ satisfies

$$p_t(x, y) = p_t^{(1)}(x_1, y_1)p_t^{(2)}(x_2, y_2), \quad x = (x_1, x_2),\ y = (y_1, y_2) \in M. \tag{4.5}$$

Let $q \in (1, 2]$ be a given number. We choose α_1, α_2 so that

$$q' = \frac{q}{q-1} = \frac{\alpha}{\alpha_0} \in [2, \infty),$$

where $\alpha_0 := \min\{\alpha_1, \alpha_2\}$. Let us verify that condition (UE_q) is satisfied on M.

Indeed, we have by (4.5), (4.2) that, for any points x, y in M and any $t > 0$,

$$\begin{aligned}
p_t(x, y) &= p_t^{(1)}(x_1, y_1)p_t^{(2)}(x_2, y_2)\\
&\simeq \frac{1}{t^{\alpha_1/\beta}}\left(1 + \frac{d_1(x_1, y_1)}{t^{1/\beta}}\right)^{-(\alpha_1+\beta)} \frac{1}{t^{\alpha_2/\beta}}\left(1 + \frac{d_2(x_2, y_2)}{t^{1/\beta}}\right)^{-(\alpha_2+\beta)}\\
&\le \frac{C}{t^{(\alpha_1+\alpha_2)/\beta}}\left(1 + \frac{d(x, y)}{t^{1/\beta}}\right)^{-(\alpha_0+\beta)}\\
&= \frac{C}{t^{\alpha/\beta}}\left(1 + \frac{d(x, y)}{t^{1/\beta}}\right)^{-(\alpha/q'+\beta)},
\end{aligned}$$

thus showing that (UE_q) holds with $W(x, r) = r^\beta$.

Let us now disprove (TP_q). Fix

$$t > 0,\ R > t^{1/\beta} \text{ and } x = (x_1, x_2) \in M.$$

We need to estimate the lower bound of $\|p_t(x, \cdot)\|_{L^q(B(x,R)^c)}$.

Since $\mu_i(B(x_i, r)) \simeq r^{\alpha_i}$, $i = 1, 2$ for any $r > 0$, we can choose $a \ge 1$ large enough such that, for $i = 1, 2$,

$$\mu_i(B(x_i, ar) \setminus B(x_i, r)) = \mu_i(B(x_i, ar)) - \mu(B(x_i, r)) \ge cr^{\alpha_i}, \quad r > 0. \tag{4.6}$$

Using (4.5) and the fact that

$$\{y = (y_1, y_2) \in M : d_1(x_1, y_1) \ge R\} \subset B(x, R)^c,$$

we obtain

$$\begin{aligned}\int_{B(x,R)^c} p_t(x,y)^q d\mu(y) &\geq \int_{\{d_1(x_1,y_1)\geq R\}} p_t^{(1)}(x_1,y_1)^q d\mu_1(y_1) \int_{M_2} p_t^{(2)}(x_2,y_2)^q d\mu_2(y_2) \\ &\geq \int_{\{aR\geq d_1(x_1,y_1)\geq R\}} p_t^{(1)}(x_1,y_1)^q d\mu_1(y_1) \\ &\quad \times \int_{\{at^{1/\beta}\geq d_2(x_2,y_2)\geq t^{1/\beta}\}} p_t^{(2)}(x_2,y_2)^q d\mu_2(y_2).\end{aligned}$$

Using (4.2) and (4.6), we have

$$\begin{aligned}\int_{\{aR\geq d_1(x_1,y_1)\geq R\}} p_t^{(1)}(x_1,y_1)^q d\mu_1(y_1) &\geq ct^{-\frac{q\alpha_1}{\beta}} \left(\frac{aR}{t^{1/\beta}}\right)^{-q(\alpha_1+\beta)} \mu_1(B(x_1,aR)\setminus B(x_1,R)) \\ &\geq ct^{-\frac{q\alpha_1}{\beta}+\frac{q(\alpha_1+\beta)}{\beta}}(aR)^{-q(\alpha_1+\beta)}R^{\alpha_1} \\ &\geq c' t^q R^{-(q-1)\alpha_1 - q\beta},\end{aligned}$$

and

$$\begin{aligned}\int_{\{at^{1/\beta}\geq d_2(x_2,y_2)\geq t^{1/\beta}\}} p_t^{(2)}(x_2,y_2)^q d\mu_2(y_2) &\geq ct^{-\frac{q\alpha_2}{\beta}} \mu_2(B(x_2,at^{1/\beta})\setminus B(x_2,t^{1/\beta})) \\ &\geq c't^{-\frac{q\alpha_2}{\beta}+\frac{\alpha_2}{\beta}} = c't^{-\frac{(q-1)\alpha_2}{\beta}}.\end{aligned}$$

Combining the above three inequalities, we obtain

$$\|p_t(x,\cdot)\|_{L^q(B(x,R)^c)} \geq ct^{1-\frac{\alpha_2}{q'\beta}} R^{-\frac{\alpha_1}{q'}-\beta}.$$

If condition (TP_q) were satisfied, then we would have

$$\|p_t(x,\cdot)\|_{L^q(B(x,R)^c)} \leq CtR^{-\frac{\alpha}{q'}-\beta}.$$

Combining the above two inequalities and using that $\alpha = \alpha_1 + \alpha_2$, we obtain

$$t^{1-\frac{\alpha_2}{q'\beta}} R^{-\frac{\alpha_1}{q'}-\beta} \leq CtR^{-\frac{\alpha}{q'}-\beta},$$

which is equivalent to $R \leq Ct^{1/\beta}$. Hence, we obtain a contradiction for large enough $\frac{R}{t^{1/\beta}}$.

5. Truncated Dirichlet forms

In order to obtain the tail estimate of the heat semigroup $\{P_t\}_{t\geq 0}$ of $(\mathcal{E},\mathcal{F})$, we need to truncate the jump part $\mathcal{E}^{(J)}$. In this section, we study the truncation of a general Dirichlet form $(\mathcal{E},\mathcal{F})$ (not necessarily without killing part). Recall that any regular Dirichlet form $(\mathcal{E},\mathcal{F})$ can be decomposed into three parts as follows:

$$\mathcal{E}(u,v) = \mathcal{E}^{(L)}(u,v) + \mathcal{E}^{(J)}(u,v) + \mathcal{E}^{(K)}(u,v),$$

where $\mathcal{E}^{(L)}$ is the local part, $\mathcal{E}^{(J)}$ is the jump part associated with a unique Radon measure j on $M\times M\setminus\text{diag}$, and $\mathcal{E}^{(K)}$ is the *killing part*.

Fix a real number $\rho > 0$ and set

$$\mathcal{E}^{(\rho)}(u,v) := \mathcal{E}^{(L)}(u,v) + \mathcal{E}^{(j)}(u,v) + \mathcal{E}^{(K)}(u,v), \quad u,v\in\mathcal{F}, \tag{5.1}$$

where

$$\mathcal{E}^{(j)}(u,v) := \iint_{\{(x,y)\in M\times M: d(x,y)<\rho\}} (u(x)-u(y))(v(x)-v(y))\, dj.$$

The symmetric form $(\mathcal{E}^{(\rho)},\mathcal{F})$ may not be in general a regular Dirichlet form. In Subsection 5.3 we will prove that it is a regular Dirichlet form under an additional mild assumption. Currently we assume that $\mathcal{E}^{(\rho)}$ is a regular Dirichlet form on $L^2(M)$ with the domain $\mathcal{F}^{(\rho)} := \mathcal{F}$. We refer to the Dirichlet form $(\mathcal{E}^{(\rho)},\mathcal{F}^{(\rho)})$ as in (5.1) as a *ρ-local Dirichlet form*. If in addition $\mathcal{E}^{(K)} \equiv 0$, then the Dirichlet form $(\mathcal{E}^{(\rho)},\mathcal{F}^{(\rho)})$ is said to be *strongly* ρ-local.

In this section we always assume that the domain $\mathcal{F}$ of the Dirichlet form satisfies the following property:

$$\text{cutoff}(A,\Omega) \neq \emptyset \ \text{ for any non-empty open set } \Omega\subset M \text{ and any bounded set } A \text{ with } \overline{A}\subset\Omega. \tag{5.2}$$

Note that this property is always true under one of condition (S_+), (S_+^*), (S) and (S_*) introduced in Section 7. Moreover, it is also true for all compact sets A by the regularity of $(\mathcal{E}, \mathcal{F})$.

5.1. **Some properties of ρ-locality.** In this subsection, we study some properties of ρ-local Dirichlet forms.

Recall that the locality property of $\mathcal{E}^{(L)}$ means that if the functions $u, v \in \mathcal{F}$ have disjoint compact supports, then $\mathcal{E}^{(L)}(u, v) = 0$. The following proposition relaxes this assumption to bounded supports.

For any $r > 0$ and set $U \subset M$, denote by U_r the r-neighborhood of U:

$$U_r := \bigcup_{z \in U} B(z, r).$$

Proposition 5.1. Let $(\mathcal{E}, \mathcal{F})$ be a regular symmetric Dirichlet form on L^2, and $\mathcal{E}^{(L)}$ be its strongly local part. Assume that (5.2) is satisfied.

(i) If functions $u, v \in \mathcal{F}$ have disjoint bounded supports, then $\mathcal{E}^{(L)}(u, v) = 0$.

(ii) If functions $u, v \in \mathcal{F}$ have bounded supports and u is constant on a neighbourhood of $\operatorname{supp}(v)$, then $\mathcal{E}^{(L)}(u, v) = 0$.

Proof. **(i)**. Let $u, v \in \mathcal{F}$ have disjoint bounded supports.

We can choose two open sets U, V such that $\operatorname{supp}(u) \subset U$, $\operatorname{supp}(v) \subset V$ and $\operatorname{dist}(U, V) > 0$. Moreover, since $\operatorname{supp}(u)$, $\operatorname{supp}(v)$ are bounded, we have by hypothesis (5.2) that

$$\operatorname{cutoff}(\operatorname{supp}(u), U) \neq \emptyset \quad \text{and} \quad \operatorname{cutoff}(\operatorname{supp}(v), V) \neq \emptyset.$$

Consider three cases.

Case 1. Assume first that $0 \le u \le 1$ and $0 \le v \le 1$.

Choose some functions

$$\phi_1 \in \operatorname{cutoff}(\operatorname{supp}(u), U) \quad \text{and} \quad \phi_2 \in \operatorname{cutoff}(\operatorname{supp}(v), V).$$

There exist sequences $\{u_n\}$, $\{v_n\} \subset \mathcal{F} \cap C_0(M)$ such that

$$\lim_{n\to\infty} \mathcal{E}_1(u_n - u) = 0 \quad \text{and} \quad \lim_{n\to\infty} \mathcal{E}_1(v_n - v) = 0.$$

Without loss of generality, we can assume that $0 \le u_n \le 1$ and $0 \le v_n \le 1$ for all $n \ge 1$ by [19, Theorem 1.4.2(v), p. 28]. Note that by [19, Theorem 1.4.2(ii), p. 28], we have $\phi_1 u_n \in \mathcal{F}$ for any n, and

$$\begin{aligned}\sup_{n\ge1} \sqrt{\mathcal{E}_1(\phi_1 u_n)} &\le \|\phi_1\|_\infty \sup_{n\ge1} \sqrt{\mathcal{E}_1(u_n)} + \sup_{n\ge1} \|u_n\|_\infty \sqrt{\mathcal{E}_1(\phi_1)} \\ &\le \sup_{n\ge1} \sqrt{\mathcal{E}_1(u_n)} + \sqrt{\mathcal{E}_1(\phi_1)} < \infty.\end{aligned}$$

Moreover, $\phi_1 u_n$ converges to $\phi_1 u = u$ in L^2-norm as $n \to \infty$. Hence, by Lemma 11.2 in Appendix, a subsequence of $\phi_1 u_n$ (that we denote again by $\phi_1 u_n$) converges to $\phi_1 u = u$ weakly in $\mathcal{E}_1$-norm as $n \to \infty$. Similarly, $\phi_2 v_n$ converges to $\phi_2 v = v$ weakly in $\mathcal{E}_1$-norm as $n \to \infty$.

Passing again to subsequences we can assume that the Cesaro means

$$\tilde{u}_n := \frac{1}{n} \sum_{k=1}^{n} \phi_1 u_k \quad \text{and} \quad \tilde{v}_n := \frac{1}{n} \sum_{k=1}^{n} \phi_2 v_k$$

converge to u and v in $\mathcal{E}_1$-norm, respectively. In particular, we have

$$\lim_{n\to\infty} \mathcal{E}^{(L)}(\tilde{u}_n - u) = 0 \quad \text{and} \quad \lim_{n\to\infty} \mathcal{E}^{(L)}(\tilde{v}_n - v) = 0.$$

On the other hand,

$$\operatorname{supp}(\phi_1 u_n) \subset \overline{U} \cap \operatorname{supp}(u_n) \quad \text{and} \quad \operatorname{supp}(\phi_2 v_n) \subset \overline{V} \cap \operatorname{supp}(v_n)$$

for each n. Hence, for any $m, n \ge 1$ $\operatorname{supp}(\tilde{u}_n)$ and $\operatorname{supp}(\tilde{v}_n)$ are compact and

$$\operatorname{dist}(\operatorname{supp}(\tilde{u}_n), \operatorname{supp}(\tilde{v}_n)) \ge \operatorname{dist}(\overline{U}, \overline{V}) > 0.$$

Therefore, it follows from the locality of $\mathcal{E}^{(L)}$ that, for all $n \ge 1$,

$$\mathcal{E}^{(L)}(\tilde{u}_n, \tilde{v}_n) = 0.$$

Pass to the limit as $n \to \infty$, and obtain $\mathcal{E}^{(L)}(u, v) = 0$.

Case 2. Assume now that $u,\ v \in L^\infty$.

Set $c := (\|u\|_\infty \vee \|v\|_\infty)^{-1}$. Then all functions $cu_+,\ cu_-,\ cv_+,\ cv_-$ take values in $[0, 1]$, and by the result of Case 1, we have

$$\mathcal{E}^{(L)}(cu_+, cv_+) = \mathcal{E}^{(L)}(cu_+, cv_-) = \mathcal{E}^{(L)}(cu_-, cv_+) = \mathcal{E}^{(L)}(cu_-, cv_-) = 0.$$

Consequently, $\mathcal{E}^{(L)}(u, v) = 0$.

Case 3. Consider now the general case of u and v.

For any $n \geq 1$, define

$$u_n := (-n) \vee u \wedge n \quad \text{and} \quad v_n := (-n) \vee v \wedge n.$$

Then the supports of u_n and v_n are disjoint, and, hence, by Case 2, we have $\mathcal{E}^{(L)}(u_n, v_n) = 0$ for all n. Since $\{u_n\}$ and $\{v_n\}$ converge in $\mathcal{E}_1$-norm to u and v, respectively, we conclude that $\mathcal{E}^{(L)}(u, v) = 0$.

(ii). Suppose that the functions $u, v \in \mathcal{F}$ have bounded supports and u is constant on an open set U with $\mathrm{supp}(v) \subset U$.

Case 1. Consider first the case when $\mathrm{supp}(v)$ is compact.

Choose a precompact open set V such that $\mathrm{supp}(v) \subset V \subset \overline{V} \subset U$ and choose $\phi \in \mathrm{cutoff}(\overline{V}, U) \cap C_0(M)$. Let c be the constant such that $u|_U = c$. Since $\phi|_V = 1$ and $\mathrm{supp}(\phi)$ is compact, it follows from the strong locality of $\mathcal{E}^{(L)}$ that $\mathcal{E}^{(L)}(\phi, v) = 0$. On the other hand, since $u - c\phi = 0$ on V, we have that $\mathrm{supp}(u - c\phi) \subset V^c$, so that $\mathrm{supp}(u - c\phi)$ is bounded and disjoint with $\mathrm{supp}(v)$. Hence, by the result in (i), we obtain that $\mathcal{E}^{(L)}(u - c\phi, v) = 0$. It follows that

$$\mathcal{E}^{(L)}(u, v) = \mathcal{E}^{(L)}(u - c\phi, v) + \mathcal{E}^{(L)}(c\phi, v) = 0.$$

Case 2. Consider the general case when $\mathrm{supp}(v)$ is just bounded.

Let $\varepsilon := \frac{1}{2}\mathrm{dist}(\mathrm{supp}(v), U^c) > 0$ and $V := (\mathrm{supp}(v))_\varepsilon$ be the ε-neighborhood of $\mathrm{supp}(v)$ so that $\overline{V} \subset U$. Choose $\psi \in \mathrm{cutoff}(\mathrm{supp}(v), V)$ (by (5.2)). Then, by the argument in Case 1 of the proof of (i), we can take a sequence $\{\tilde{v}_n\} \subset \mathcal{F}$ of functions with compact supports such that $\mathrm{supp}(\tilde{v}_n) \subset \overline{V} \subset U$ for all n and

$$\lim_{n\to\infty} \mathcal{E}^{(L)}(\tilde{v}_n - v) = 0.$$

Since u is constant on U and, hence, on a neighbourhood of $\tilde{v}_n$, it follows from the result in Case 1 that $\mathcal{E}^{(L)}(u, \tilde{v}_n) = 0$. Passing to the limit as $n \to \infty$, and using the above formula, we obtain $\mathcal{E}^{(L)}(u, v) = 0$. □

The following corollary shows that the (strongly) ρ-local Dirichlet form $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)})$ (as in (5.1)) possesses some properties analogous to those of $\mathcal{E}^{(L)}$.

Corollary 5.2. Let $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)})$ be the regular ρ-local Dirichlet form on L^2 as in (5.1). Assume that (5.2) is satisfied.

(i) If functions $u, v \in \mathcal{F}^{(\rho)}$ have bounded supports and $\mathrm{dist}(\mathrm{supp}(u), \mathrm{supp}(v)) > \rho$, then $\mathcal{E}^{(\rho)}(u, v) = 0$.

(ii) Suppose that in addition $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)})$ is strongly ρ-local. If functions $u, v \in \mathcal{F}^{(\rho)}$ have bounded supports and v is constant on a neighbourhood of $\overline{(\mathrm{supp}(u))_\rho}$, then $\mathcal{E}^{(\rho)}(u, v) = 0$.

Proof. It suffices to prove (ii). Since $\mathcal{F}^{(\rho)} = \mathcal{F}$, any cutoff function for $\mathcal{E}$ is also a cutoff function for $\mathcal{E}^{(\rho)}$.

Suppose that $u, v \in \mathcal{F}^{(\rho)}$ have bounded supports and u is constant on a neighbourhood of $\overline{(\mathrm{supp}(v))_\rho}$. It follows from Proposition 5.1(ii) that $\mathcal{E}^{(L)}(u, v) = 0$.

It remains to prove that $\mathcal{E}^{(j)}(u, v) = 0$. Let $A := \mathrm{supp}(u)$. Using the facts that $v = \text{const}$ on A_ρ so that $v(x) - v(y) = 0$ on $A_\rho \times A_\rho$ as well as $u = 0$ on $A^c \supset A_\rho^c$ so that $u(x) - u(y) = 0$ on $A_\rho^c \times A_\rho^c$, we obtain

$$\begin{aligned}
\mathcal{E}^{(j)}(u, v) &= \iint_{M\times M} (u(x) - u(y))(v(x) - v(y))\mathbf{1}_{\{d(x,y)<\rho\}} dj \\
&= \left(\iint_{A_\rho\times A_\rho} + \iint_{A_\rho\times A_\rho^c} + \iint_{A_\rho^c\times A_\rho} + \iint_{A_\rho^c\times A_\rho^c} \right) (u(x) - u(y))(v(x) - v(y))\mathbf{1}_{\{d(x,y)<\rho\}} dj \\
&= 2 \iint_{A_\rho\times A_\rho^c} (u(x) - u(y))(v(x) - v(y))\mathbf{1}_{\{d(x,y)<\rho\}} dj
\end{aligned}$$

$$= 2\left(\iint_{A\times A_\rho^c} + \iint_{(A_\rho\setminus A)\times A_\rho^c}\right)(u(x)-u(y))(v(x)-v(y))\mathbf{1}_{\{d(x,y)<\rho\}}dj$$
$$= 0,$$

where in the last line we have used that $\mathbf{1}_{\{d(x,y)<\rho\}} = 0$ on $A \times A_\rho^c$ so that the first integral vanishes, and $u(x) - u(y) = 0$ on $(A_\rho \setminus A) \times A_\rho^c$ (as $u = 0$ outside A), so that the second integral is also zero. Finally, it follows from (5.1) that $\mathcal{E}^{(\rho)}(u, v) = 0$. □

5.2. **Resolvents.** In this subsection, we give some general facts on the resolvent associated with the ρ-local Dirichlet form.

For $\rho > 0$ and for any non-empty subset Ω of M, let $\mathcal{F}^{(\rho)}(\Omega)$ be a vector space defined by

$$\mathcal{F}^{(\rho)}(\Omega) = \text{ the closure of } \mathcal{F}^{(\rho)} \cap C_0(\Omega) \text{ in the norm } \sqrt{\mathcal{E}_1^{(\rho)}}. \tag{5.3}$$

Then $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)}(\Omega))$ is a regular Dirichlet form in $L^2(\Omega, \mu)$. Let

$$Q_t^\Omega := Q_t^{(\rho),\Omega}$$

be the heat semigroup in L^2 associated with $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)}(\Omega))$. For any $\lambda > 0$, let $R_\lambda^\Omega := R_\lambda^{(\rho),\Omega}$ be the resolvent associated with $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)}(\Omega))$ that is defined by

$$R_\lambda^\Omega f = \int_0^\infty e^{-\lambda s} Q_s^\Omega f\, ds, \quad f \in L^2. \tag{5.4}$$

When $\Omega = M$, we drop the superscript Ω by writing

$$Q_t := Q_t^\Omega \text{ and } R_\lambda := R_\lambda^\Omega.$$

For simplicity, denote by

$$\mathcal{E}_\lambda^{(\rho)}(u, v) := \mathcal{E}^{(\rho)}(u, v) + \lambda(u, v) \text{ for any } u, v \in \mathcal{F}^{(\rho)}. \tag{5.5}$$

It is known (see for example [19, formula (1.3.7), p. 20]) that, for any open subset Ω,

$$\mathcal{E}_\lambda^{(\rho)}(R_\lambda^\Omega f, g) = (f, g) \quad \text{for all } f \in L^2(\Omega) \text{ and } g \in \mathcal{F}^{(\rho)}(\Omega). \tag{5.6}$$

The following statement gives a relation between the functions $1 - Q_t^\Omega \mathbf{1}_\Omega$ and $1 - \lambda R_\lambda^\Omega \mathbf{1}_\Omega$.

Proposition 5.3. *For any open subset $\Omega \subset M$ and all $t, \lambda > 0$,*

$$1 - Q_t^\Omega \mathbf{1}_\Omega \le e^{\lambda t}\left(1 - \lambda R_\lambda^\Omega \mathbf{1}_\Omega\right) \text{ in } M. \tag{5.7}$$

Proof. Note that the function $s \mapsto Q_s^\Omega \mathbf{1}_\Omega$ is non-increasing. Hence, for any $t > 0$,

$$1 - \lambda R_\lambda^\Omega \mathbf{1}_\Omega = \int_0^\infty (1 - Q_s^\Omega \mathbf{1}_\Omega)\lambda e^{-\lambda s} ds \ge \int_t^\infty (1 - Q_s^\Omega \mathbf{1}_\Omega)\lambda e^{-\lambda s} ds$$
$$\ge (1 - Q_t^\Omega \mathbf{1}_\Omega) \int_t^\infty \lambda e^{-\lambda s} ds = (1 - Q_t^\Omega \mathbf{1}_\Omega) e^{-\lambda t},$$

which is equivalent to (5.7). □

Note that the above the inequality (5.7) is true for general Markovian semigroups and their resolvents.

Proposition 5.4. *Let $\rho \ge 0$ and $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)})$ be a strongly ρ-local Dirichlet form in L^2. Assume that (5.2) is satisfied. Let $\lambda > 0$ and U be a non-empty bounded open subset of M. If a function $u \in \mathcal{F}^{(\rho)} \cap L^\infty(M)$ is such that $0 \le u \le 1$ in U_ρ and*

$$\mathcal{E}_\lambda^{(\rho)}(u, \psi) \le 0, \quad \forall\, 0 \le \psi \in \mathcal{F}^{(\rho)}(U), \tag{5.8}$$

where $\mathcal{E}_\lambda^{(\rho)}$ is defined by (5.5), then

$$u \le 1 - \lambda R_\lambda^U \mathbf{1}_U \quad \text{in } U_\rho. \tag{5.9}$$

Proof. We will apply elliptic maximum principle (Proposition 11.6 in Appendix). It suffices to prove (5.9) in U, since (5.9) is automatically true in $U_\rho \setminus U$ as $u \le 1$ in U_ρ and $R_\lambda^U \mathbf{1}_U = 0$ in $U_\rho \setminus U$. We show that for any open set $V \Subset U$,

$$u \le 1 - \lambda R_\lambda^V \mathbf{1}_V \ \text{ in } V. \tag{5.10}$$

hen (5.9) will follow by taking an exhaustion $V \uparrow U$ in (5.10) and using [25, Lemma 4.13, p. 119].

Choose some $\phi \in \text{cutoff}(V_\rho, U_\rho)$ (by (5.2)), and consider the function

$$v := \phi u - (\phi - \lambda R_\lambda^V \mathbf{1}_V).$$

Since $u, \phi \in \mathcal{F}^{(\rho)} \cap L^\infty(M)$, we conclude by [19, Theorem 1.4.2, p. 28],

$$\phi u \in \mathcal{F}^{(\rho)} \cap L^\infty(M).$$

Consequently, $v \in \mathcal{F}^{(\rho)}$. On the other hand, since $0 \le u \le 1$ in U_ρ, we have

$$v = \phi u - \phi + \lambda R_\lambda^V \mathbf{1}_V \le \lambda R_\lambda^V \mathbf{1}_V \in \mathcal{F}^{(\rho)}(V).$$

It follows from [25, Lemma 4.4, p. 114] that

$$v_+ \in \mathcal{F}^{(\rho)}(V). \tag{5.11}$$

Let $0 \le \psi \in \mathcal{F}^{(\rho)} \cap C_0(V)$. Let us prove that v satisfies

$$\mathcal{E}_\lambda^{(\rho)}(v, \psi) \le 0. \tag{5.12}$$

Indeed, note that $\text{supp}((\phi - 1)u) \subset (V_\rho)^c$ and $\text{supp}(\psi) \subset V$, so that the distance between $\text{supp}((\phi - 1)u)$ and $\text{supp}(\psi)$ is strictly greater than ρ. By Corollary 5.2(i), we have

$$\mathcal{E}_\lambda^{(\rho)}((\phi - 1)u, \psi) = 0.$$

Combining this and (5.8), we obtain,

$$\mathcal{E}_\lambda^{(\rho)}(\phi u, \psi) = \mathcal{E}_\lambda^{(\rho)}(u, \psi) + \mathcal{E}_\lambda^{(\rho)}((\phi - 1)u, \psi) \le \mathcal{E}_\lambda^{(\rho)}((\phi - 1)u, \psi) = 0.$$

On the other hand, since $\text{supp}(\psi) \subset V$ and $\phi = 1$ in V_ρ so that ϕ is constant in the ρ-neighborhood of the support of ψ, it follows from Corollary 5.2(ii) that $\mathcal{E}^{(\rho)}(\phi, \psi) = 0$. Hence,

$$\mathcal{E}_\lambda^{(\rho)}(\phi, \psi) = \mathcal{E}^{(\rho)}(\phi, \psi) + \lambda(\phi, \psi) = \lambda(\phi, \psi) = \lambda \|\psi\|_1.$$

By (5.6), we have

$$\mathcal{E}_\lambda^{(\rho)}(R_\lambda^V \mathbf{1}_V, \psi) = (\mathbf{1}_V, \psi) = \|\psi\|_1.$$

Therefore, combining the above three formulas, we obtain

$$\begin{aligned}
\mathcal{E}_\lambda^{(\rho)}(v, \psi) &= \mathcal{E}_\lambda^{(\rho)}(\phi u - \phi + \lambda R_\lambda^V \mathbf{1}_V, \psi) \\
&= \mathcal{E}_\lambda^{(\rho)}(\phi u, \psi) - \mathcal{E}_\lambda^{(\rho)}(\phi, \psi) + \lambda \mathcal{E}_\lambda^{(\rho)}(R_\lambda^V \mathbf{1}_V, \psi) \\
&\le 0 - \lambda \|\psi\|_1 + \lambda \|\psi\|_1 = 0,
\end{aligned}$$

thus showing (5.12). For a general function $0 \le \psi \in \mathcal{F}^{(\rho)}(V)$, we can apply (5.12) for a sequence of functions $\{\psi_n\} \subset \mathcal{F}^{(\rho)} \cap C_0(V)$ converging to ψ, and obtain (5.12) also for this ψ.

Consequently, it follows from (5.11) and (5.12) that the function v satisfies all the assumptions in the elliptic maximum principle in V (see Proposition 11.6 in Appendix). We conclude that

$$v = \phi u - (\phi - \lambda R_\lambda^V \mathbf{1}_V) \le 0 \ \text{ in } V,$$

which yields (5.10) as $\phi = 1$ on V. □

We remark that Proposition 5.4 can be viewed as an extension to the ρ-local case of [25, Corollary 4.15] that was proved for strongly local Dirichlet forms.

Proposition 5.5. *Assume that (5.2) is satisfied. Fix $\lambda > 0$, $\rho > 0$. Let Ω be a non-empty bounded open subset of M, and $U \subset \Omega$ be an open subset such that $U_\rho \subset \Omega$. Then, for μ-almost all $z \in U_\rho$*

$$1 - \lambda R_\lambda^\Omega \mathbf{1}_\Omega(z) \le \left(1 - \lambda R_\lambda^U \mathbf{1}_U(z)\right) \left\|1 - \lambda R_\lambda^\Omega \mathbf{1}_\Omega\right\|_{L^\infty(U_\rho)}. \tag{5.13}$$

Observe that the inclusion $U \subset \Omega$ implies that

$$1 - \lambda R^{\Omega}_{\lambda}\mathbf{1}_{\Omega} = \int_0^{\infty} \lambda e^{-\lambda s}(1 - Q^{\Omega}_s \mathbf{1}_{\Omega})\, ds \le \int_0^{\infty} \lambda e^{-\lambda s}(1 - Q^{U}_s \mathbf{1}_{U}) ds = 1 - \lambda R^{U}_{\lambda}\mathbf{1}_{U} \quad \text{in } M \tag{5.14}$$

because $Q^U_t \mathbf{1}_U \le Q^{\Omega}_t \mathbf{1}_{\Omega}$ in M for any $t > 0$. The inequality (5.13) gives a sharper upper bound of the function $1 - \lambda R^{\Omega}_{\lambda}\mathbf{1}_{\Omega}$ in terms of $1 - \lambda R^{U}_{\lambda}\mathbf{1}_{U}$. In the next lemma, the inequality (5.13) will be used to get the tail estimate $Q_t \mathbf{1}_{B^c}$ for any metric ball B.

Proof of Proposition 5.5. Let $V \subset U$ be an arbitrary precompact open subset; then $\overline{V_\rho} \subset U_\rho \subset \Omega$. Choose some $\phi \in \text{cutoff}(V_\rho, U_\rho)$ (by (5.2)) and consider the function

$$u := c_0(\phi - \lambda R^{\Omega}_{\lambda}\mathbf{1}_{\Omega}),$$

where c_0 is a constant given by

$$c_0^{-1} = \left\| \phi - \lambda R^{\Omega}_{\lambda}\mathbf{1}_{\Omega} \right\|_{L^\infty(U_\rho)}.$$

We will apply Proposition 5.4 to show

$$u \le 1 - \lambda R^{V}_{\lambda}\mathbf{1}_{V} \quad \text{in } V_\rho \text{ (also in } U_\rho). \tag{5.15}$$

Indeed, note that $u \in \mathcal{F}^{(\rho)} \cap L^\infty(M)$ and $0 \le u \le 1$ in U_ρ. We need to verify that u satisfies (5.8) in V that is, for all $0 \le \psi \in \mathcal{F}^{(\rho)}(V)$. By Corollary 5.2(ii) and using the fact that $\phi = 1$ on V_ρ, we have

$$\mathcal{E}^{(\rho)}(\phi, \psi) = 0.$$

It follows that

$$\mathcal{E}^{(\rho)}_{\lambda}(\phi, \psi) = \mathcal{E}^{(\rho)}(\phi, \psi) + \lambda(\phi, \psi) = \lambda(\phi, \psi) = \lambda \|\psi\|_1.$$

On the other hand, by (5.6) we have

$$\mathcal{E}^{(\rho)}_{\lambda}(R^{\Omega}_{\lambda}\mathbf{1}_{\Omega}, \psi) = (\mathbf{1}_{\Omega}, \psi) = \|\psi\|_1.$$

Combining the above two formulas, we obtain that

$$\begin{aligned}\mathcal{E}^{(\rho)}_{\lambda}(u, \psi) &= c_0 \mathcal{E}^{(\rho)}_{\lambda}(\phi - \lambda R^{\Omega}_{\lambda}\mathbf{1}_{\Omega}, \psi) = c_0 \left(\mathcal{E}^{(\rho)}_{\lambda}(\phi, \psi) - \lambda \mathcal{E}^{(\rho)}_{\lambda}(R^{\Omega}_{\lambda}\mathbf{1}_{\Omega}, \psi) \right) \\ &= c_0 \left(\lambda \|\psi\|_1 - \lambda \|\psi\|_1 \right) = 0,\end{aligned}$$

thus proving (5.8).

By inequality (5.9) of Proposition 5.4, we obtain (5.15). Combining (5.15) and the fact that $\phi = 1$ in V_ρ, we obtain that

$$\begin{aligned}1 - \lambda R^{\Omega}_{\lambda}\mathbf{1}_{\Omega} &= \phi - \lambda R^{\Omega}_{\lambda}\mathbf{1}_{\Omega} \le c_0^{-1} \left(1 - \lambda R^{V}_{\lambda}\mathbf{1}_{V} \right) \\ &= \left(1 - \lambda R^{V}_{\lambda}\mathbf{1}_{V} \right) \left\| 1 - \lambda R^{\Omega}_{\lambda}\mathbf{1}_{\Omega} \right\|_{L^\infty(U_\rho)} \quad \text{in } V_\rho \text{ (also in } U_\rho).\end{aligned}$$

Passing to the limit as $V \uparrow U$, we obtain (5.13). □

Lemma 5.6. *Assume that (5.2) is satisfied. Fix $\lambda > 0$, $\rho > 0$. Let B be a ball in M of radius $R > 0$, and $k \ge 1$ be an integer satisfying*

$$4k\rho < R.$$

Assume also that, for any $z \in B$,

$$1 - \lambda R^{B(z,\rho)}_{\lambda}\mathbf{1}_{B(z,\rho)} \le a \quad \text{in } \frac{1}{4}B(z, \rho) \tag{5.16}$$

for some positive constant a. Then

$$1 - \lambda R^{B}_{\lambda}\mathbf{1}_{B} \le a^k \quad \text{in } \frac{1}{4}B. \tag{5.17}$$

Proof. We divide the proof into two steps.

Step 1. Fix $y \in \frac{1}{4}B$ and prove that

$$1 - \lambda R_\lambda^{B(y,\frac{3}{4}R)} \mathbf{1}_{B(y,\frac{3}{4}R)} \le a^k \quad \text{in } B(y,\rho). \tag{5.18}$$

Indeed, for any $0 \le n \le k$, set $B_n := B(y,(2n+1)\rho) \subset B(y, \frac{3}{4}R) \subset B$ and

$$m_n := \left\| 1 - \lambda R_\lambda^{B(y,\frac{3}{4}R)} \mathbf{1}_{B(y,\frac{3}{4}R)} \right\|_{L^\infty(B_n)}.$$

For any $1 \le n \le k$ and for any $z \in B_{n-1}$ we have, $B(z,\rho) \subset B(z,2\rho) \subset B_n$. Applying (5.13) with $\Omega = B(y, \frac{3}{4}R)$, $U = B(z,\rho)$ for any $z \in B_{n-1} \subset B$ and using (5.16), we obtain

$$\begin{aligned}
1 - \lambda R_\lambda^{B(y,\frac{3}{4}R)} \mathbf{1}_{B(y,\frac{3}{4}R)} &\le \left(1 - \lambda R_\lambda^{B(z,\rho)} \mathbf{1}_{B(z,\rho)}\right) \|1 - \lambda R_\lambda^{B(y,\frac{3}{4}R)} \mathbf{1}_{B(y,\frac{3}{4}R)}\|_{L^\infty(B(z,2\rho))} \\
&\le \left(1 - \lambda R_\lambda^{B(z,\rho)} \mathbf{1}_{B(z,\rho)}\right) \|1 - \lambda R_\lambda^{B(y,\frac{3}{4}R)} \mathbf{1}_{B(y,\frac{3}{4}R)}\|_{L^\infty(B_n)} \\
&= \left(1 - \lambda R_\lambda^{B(z,\rho)} \mathbf{1}_{B(z,\rho)}\right) m_n \\
&\le a m_n \quad \text{in } \frac{1}{4} B(z,\rho).
\end{aligned}$$

Covering B_{n-1} by at most countable balls like $\frac{1}{4}B(z,\rho)$, we obtain from the above inequality that

$$m_{n-1} = \left\| 1 - \lambda R_\lambda^{B(y,\frac{3}{4}R)} \mathbf{1}_{B(y,\frac{3}{4}R)} \right\|_{L^\infty(B_{n-1})} \le a m_n.$$

Iterating this inequality and using the fact that $m_k \le 1$, we obtain

$$\left\| 1 - \lambda R_\lambda^{B(y,\frac{3}{4}R)} \mathbf{1}_{B(y,\frac{3}{4}R)} \right\|_{L^\infty(B_0)} = m_0 \le a^k m_k \le a^k,$$

which is exactly (5.18).

Step 2. Since $B(y, \frac{3}{4}R) \subset B$ for any $y \in \frac{1}{4}B$, we have by (5.18),

$$1 - \lambda R_\lambda^B \mathbf{1}_B \le 1 - \lambda R_\lambda^{B(y,\frac{3}{4}R)} \mathbf{1}_{B(y,\frac{3}{4}R)} \le a^k \quad \text{in } B(y,\rho).$$

Covering $\frac{1}{4}B$ by at most countable family of balls like $B(y,\rho)$, we obtain (5.17). □

5.3. **Relation between two semigroups.** In this subsection, we always assume that the following condition holds:

$$\omega(\rho) := \sup_{x \in M} \int_{B(x,\rho)^c} J(x,dy) < \infty \tag{5.19}$$

and investigate the relationship between the original heat semigroup $\{P_t\}$ and the ρ-truncated heat semigroup $\{Q_t\}$.

Lemma 5.7. *Under the hypothesis (5.19) the bilinear form $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)})$ is a regular Dirichlet form.*

Proof. By the symmetry of j, we have, for all $u \in \mathcal{F}$,

$$\begin{aligned}
\mathcal{E}(u,u) &= \mathcal{E}^{(\rho)}(u,u) + \iint_{M \times B(x,\rho)^c} (u(x) - u(y))^2 \, dj \\
&\le \mathcal{E}^{(\rho)}(u,u) + 2 \iint_{M \times B(x,\rho)^c} \left(u(x)^2 + u(y)^2\right) dj \\
&\le \mathcal{E}^{(\rho)}(u,u) + 4 \iint_{M \times B(x,\rho)^c} u(x)^2 J(x,dy) d\mu(x).
\end{aligned}$$

Using (5.19), we obtain

$$\begin{aligned}
\mathcal{E}(u,u) &\le \mathcal{E}^{(\rho)}(u,u) + 4 \int_M u(x)^2 J(x, B(x,\rho)^c) d\mu(x) \\
&\le \mathcal{E}^{(\rho)}(u,u) + 4\omega(\rho) \|u\|_2^2
\end{aligned}$$

$$\leq (4\omega(\rho) \vee 1)\mathcal{E}_1^{(\rho)}(u,u), \quad u \in \mathcal{F},$$

where $\mathcal{E}_1^{(\rho)}(u,v) = \mathcal{E}^{(\rho)}(u,v) + (u,v),\ u,v \in \mathcal{F}$. Hence, it follows that

$$\mathcal{E}_1^{(\rho)}(u,u) \leq \mathcal{E}_1(u,u) \leq 2(4\omega(\rho) \vee 1)\mathcal{E}_1^{(\rho)}(u,u), \quad u \in \mathcal{F}.$$

Therefore, the quadratic forms $\mathcal{E}_1^{(\rho)}$ and $\mathcal{E}_1$ are equivalent, which implies that $(\mathcal{E}^{(\rho)}, \mathcal{F})$ is also a regular Dirichlet form. □

For any $\rho > 0$, we define the operator $A^{(\rho)}$ by

$$A^{(\rho)} f(x) = 2\int_M (f(y) - f(x))\mathbf{1}_{B(x,\rho)^c}(y)J(x,dy). \tag{5.20}$$

assuming that $f \in \mathcal{F}$ and that the integral in the right hand side is well defined. Note that we always use here a quasi-continuous version of f, since the measure $dj(x,y) = J(x,dy)d\mu(x)$ charges no set of zero capacity.

Proposition 5.8. *Fix some $\rho > 0$ and $q \in [1,\infty]$. Assume that (5.19) is true. Then, for any $f \in \mathcal{F} \cap L^q$,*

$$\|A^{(\rho)} f\|_q \leq 4\omega(\rho)\|f\|_q < \infty. \tag{5.21}$$

Proof. For the case when $q = \infty$, the inequality (5.21) follows directly from Proposition 11.9 in Appendix. Let $q \in [1,\infty)$. By the Hölder inequality, we have, for any $f \in \mathcal{F} \cap L^q$,

$$\begin{aligned}
\|A^{(\rho)} f\|_q^q &= 2^q \int_M \left| \int_{B(x,\rho)^c} (f(y) - f(x))J(x,dy) \right|^q d\mu(x) \\
&\leq 2^q \int_M \int_{B(x,\rho)^c} |f(y) - f(x)|^q J(x,dy) \cdot \left(\int_{B(x,\rho)^c} J(x,dy) \right)^{q-1} d\mu(x) \\
&\leq 2^q \omega(\rho)^{q-1} \int_M \int_M 2^{q-1}\mathbf{1}_{\{d(x,y)\geq\rho\}}(|f(x)|^q + |f(y)|^q)J(x,dy)d\mu(x) \\
&= 2^{2q}\omega(\rho)^{q-1} \int_M |f(x)|^q \int_{B(x,\rho)^c} J(x,dy)d\mu(x) \\
&= 4^q\omega(\rho)^q \int_M |f(x)|^q d\mu(x),
\end{aligned}$$

thus showing (5.21). □

Remark 5.9. Let $q = 2$. If (5.19) is satisfied then by (5.21) the operator $A^{(\rho)}$ is bounded in L^2-norm and, hence, can be extended to a bounded operator on the entire space L^2.

Next, we compare the semigroups $\{P_t\}$ and $\{Q_t\}$ by means of the following abstract Phillips theorem.

Proposition 5.10 ([38, Theorem 3.5 and eq. (13)])**.** *Let Δ be the (non-positive definite) infinitesimal generator of a strongly continuous semigroup $\{Q_t\}_{t\geq 0}$ on a Banach space $\mathcal{H}$, and let A be a bounded linear operator from $\mathcal{H}$ to $\mathcal{H}$. Then the semigroup $\{P_t\}_{t\geq 0}$ generated by $\Delta + A$ can be expressed by*

$$P_t = \sum_{n=0}^{\infty} Q_t^{(n)},$$

where $Q_t^{(0)} = Q_t$, and

$$Q_t^{(n)} = \int_0^t Q_{t-s}AQ_s^{(n-1)}ds \ \text{ for each } n \geq 1$$

is well-defined, strongly continuous in t on $\mathcal{H}$. If in addition $\{Q_t\}_{t\geq 0}$ is contractive on $\mathcal{H}$, that is, $\|Q_t\| \leq 1$, then

$$\|Q_t^{(n)}\| \leq \frac{(t\|A\|)^n}{n!} \ \text{ for each } n \geq 0. \tag{5.22}$$

The following statement gives a relationship between two heat semigroups $\{P_t\}$ and $\{Q_t\}$.

Proposition 5.11. *Assume that (5.19) is satisfied. Then, for all $\rho, t > 0$ and $f \in L^2$,*

$$P_t f = Q_t f + \int_0^t Q_s A^{(\rho)} P_{t-s} f ds, \tag{5.23}$$

where operator $A^{(\rho)}$ is defined by (5.20).

Proof. Observe that, for all $f, g \in \mathcal{F}$,

$$\begin{aligned}\mathcal{E}(f,g) = \mathcal{E}^{(\rho)}(f,g) + \iint_{M\times B(x,\rho)^c} (f(x)-f(y))(g(x)-g(y))dj \\ = \mathcal{E}^{(\rho)}(f,g) - 2\int_M g(x)\left(\int_{B(x,\rho)^c}(f(y)-f(x))J(x,dy)\right)d\mu(x) \\ = \mathcal{E}^{(\rho)}(f,g) - (A^{(\rho)}f, g).\end{aligned}$$

Since the operator $A^{(\rho)}$ is bounded, the Dirichlet form $\mathcal{E}^{(\rho)}(f,g)$ is a bounded perturbation of $\mathcal{E}(f,g)$, which implies that the generators $\mathcal{L}^{(\rho)}$ and $\mathcal{L}$ of these Dirichlet forms have the same domains and

$$\mathcal{L} = \mathcal{L}^{(\rho)} + A^{(\rho)}. \tag{5.24}$$

Therefore, applying Proposition 5.10 with $\Delta = \mathcal{L}^{(\rho)}$, $A = A^{(\rho)}$ we obtain that

$$P_t = \sum_{n=0}^{\infty} Q_{t,n}, \tag{5.25}$$

where $Q_{t,0} = Q_t$, and

$$Q_{t,n} = \int_0^t Q_{t-s}A^{(\rho)}Q_{s,n-1}ds, \quad n \geq 1.$$

It remains to show (5.23).

Indeed, the series $\sum_{n=0}^{\infty} Q_{t,n}$ is absolutely convergent in the operator norm of $\|\cdot\|$ in L^2 since, for any $t > 0$,

$$\begin{aligned}\int_0^t \left\|Q_{t-s}A^{(\rho)}Q_{s,n}\right\| ds &\leq \int_0^t \left\|A^{(\rho)}Q_{s,n}\right\| ds \quad (\text{since } Q_t \text{ is contractive in } L^2) \\ &\leq \int_0^t \|A^{(\rho)}\|\, \|Q_{s,n}\| ds \quad (\text{by (5.21) and Remark 5.9}) \\ &\leq \int_0^t \|A^{(\rho)}\| \frac{(s\|A^{(\rho)}\|)^n}{n!} ds \quad (\text{by (5.22)}) \\ &= \frac{\|A^{(\rho)}\|^{n+1}}{n!}\int_0^t s^n ds \leq \frac{1}{(n+1)!}(4\omega(\rho)t)^{n+1} \quad (\text{by (5.21)}),\end{aligned}$$

which yields that

$$\sum_{n=0}^{\infty}\int_0^t \left\|Q_{t-s}A^{(\rho)}Q_{s,n}\right\| ds \leq \sum_{n=0}^{\infty}\frac{(4\omega(\rho)t)^{n+1}}{(n+1)!} = \exp(4\omega(\rho)t) - 1.$$

Exchanging the order of summation and integration, we obtain from (5.25) that, for any $f \in L^2$ and any $t > 0$,

$$\begin{aligned}P_t f = \sum_{n=0}^{\infty} Q_{t,n} f &= Q_t f + \sum_{n=1}^{\infty}\int_0^t Q_{t-s}A^{(\rho)}Q_{s,n-1} f ds \\ &= Q_t f + \int_0^t Q_{t-s}A^{(\rho)}\left(\sum_{n=1}^{\infty} Q_{s,n-1} f\right) ds \\ &= Q_t f + \int_0^t Q_{t-s}A^{(\rho)}\left(\sum_{k=0}^{\infty} Q_{s,k} f\right) ds \\ &= Q_t f + \int_0^t Q_{t-s}A^{(\rho)}P_s f ds \quad (\text{by (5.25) again}),\end{aligned}$$

which yields (5.23) by changing variables $t-s$ to s. □

The next lemma was proved in [28, Proposition 4.6, p. 6412] under the assumption that the jump kernel exists, but the same proof works also in the present setting.

Lemma 5.12. *Assume that (5.19) is satisfied. Let $\Omega \subset M$ be a non-empty open set. Let $\{Q_t^\Omega\}$ be the heat semigroup associated with the part Dirichlet from $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)}(\Omega))$ of the truncated ρ-local Dirichlet form $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)})$ (cf. (5.1)). Then, for any $t>0$ and any $f \in L^\infty$,*

$$\|P_t^\Omega f - Q_t^\Omega f\|_\infty \leq 2\omega(\rho) t \|f\|_\infty. \tag{5.26}$$

6. A new metric

In this section, we will introduce a new metric d_* on M, which is topologically equivalent to the original metric d. Under this new metric d_*, the scaling function $W(x,R)$ becomes independent of point x, while the measure μ is still doubling (resp., reverse doubling). The new metric d_* will be used to construct a truncated Dirichlet form.

Recall that $W(x,y) := W(x, d(x,y))$, where $x, y \in M$, and set

$$D(x,y) := W(x,y) + W(y,x). \tag{6.1}$$

By the right inequality in (2.7), we see that, for all $x, y \in M$,

$$\frac{W(x,d(x,y))}{W(y,d(x,y))} \leq C\left(\frac{d(x,y)}{d(x,y)}\right)^{\beta_2} = C,$$

that is, $W(x,y) \leq CW(y,x)$, which implies by interchanging x, y that

$$W(x,y) \simeq W(y,x).$$

It follows from (6.1) that, for all $x, y \in M$,

$$W(x,y) \leq D(x,y) \leq C' W(x,y) \tag{6.2}$$

for some constant $C' > 0$.

Clearly, the function D is symmetric, that is $D(x,y) = D(y,x)$, and it vanishes if and only if $x = y$. Let us show that $D(x,y)$ is a quasi-metric on M.

Proposition 6.1. *There exists a constant $C_1 \geq 1$ such that for all $x, y, z \in M$,*

$$D(x,y) \leq C_1(D(x,z) + D(z,y)). \tag{6.3}$$

Consequently, there exist two constants β, $C_2 > 0$ and a metric d_* on M such that

$$C_2^{-1} d_*(x,y)^\beta \leq D(x,y) \leq C_2 d_*(x,y)^\beta \tag{6.4}$$

for all $x, y \in M$.

Let us observe that if $W(x,r) = r^{\widetilde{\beta}}$ for some $\widetilde{\beta} > 0$, then $\beta = \widetilde{\beta}$ and

$$d(x,y) \simeq d_*(x,y), \quad x,\ y \in M.$$

Proof. By the triangle inequality, we have

$$d(x,y) \leq d(x,z) + d(z,y) \leq 2\max\{d(x,z), d(z,y)\}.$$

Assume without loss of generality that

$$d(x,y) \leq 2d(x,z).$$

It follows from (6.2), the monotonicity of $W(x,\cdot)$ and the right inequality in (2.7) that

$$D(x,y) \leq C' W(x,y) \leq C' W(x, 2d(x,z)) \leq C_1 W(x, d(x,z)) \leq C_1 D(x,z)$$

for some $C_1 \geq 1$, thus proving (6.3). Hence, $D(x,y)$ is a quasi-metric on M.

The second claim follows from (6.3) by [33, Proposition 14.5]. □

In the rest of the paper, β will always denote the constant from Proposition 6.1.

Define the function F by

$$F(x,R) := W(x,R)^{1/\beta}, \quad x \in M, \ R > 0. \tag{6.5}$$

Clearly, the function $F(x,\cdot)$ is strictly increasing on $[0,\infty]$ for any $x \in M$ and, by (6.4),

$$L^{-1}d_*(x,y) \le F(x,d(x,y)) = W(x,y)^{1/\beta} \le Ld_*(x,y), \tag{6.6}$$

for some constant $L > 1$ and all $x,y \in M$. For any $x \in M$, let $F^{-1}(x,\cdot)$ be the inverse of the function $t \mapsto F(x,t)$, so that

$$F^{-1}(x,t) = W^{-1}(x,t^{\beta}), \quad t > 0. \tag{6.7}$$

Denote by $B_*(x,r)$ balls with respect to metric d_*, that is

$$B_*(x,r) := \{y \in M : d_*(y,x) < r\}. \tag{6.8}$$

Proposition 6.2. *There exists a number $L_0 \ge L^2 > 1$ such that the following properties are true.*

(i) *For all $x \in M$ and all $r > 0$,*

$$B_*(x,L_0^{-1}r) \subset B(x,F^{-1}(x,L^{-1}r)) \subset B_*(x,r). \tag{6.9}$$

(ii) *For all $x \in M$ and all $R > 0$,*

$$B(x,L_0^{-1}R) \subset B_*(x,L^{-1}F(x,R)) \subset B(x,R). \tag{6.10}$$

Consequently, the metrics d_* and d are topologically equivalent.

Proof. Let $L_0 > 1$ be a constant to be determined later.

(i). For some fixed $x \in M$ and $r > 0$, let

$$R' := F^{-1}(x,L^{-1}r).$$

We show the left inclusion in (6.9). Indeed, for any $y \in B_*(x,L_0^{-1}r)$, we have

$$d_*(x,y) < L_0^{-1}r,$$

and, hence, by (6.6),

$$F(x,d(x,y)) \le Ld_*(x,y) < LL_0^{-1}r.$$

It follows that

$$d(x,y) < F^{-1}(x,LL_0^{-1}r) \le F^{-1}(x,L^{-1}r) = R',$$

provided $LL_0^{-1} \le L^{-1}$ that is,

$$L_0 \ge L^2. \tag{6.11}$$

Thus, the left inclusion of (6.9) holds provided L_0 satisfies (6.11).

Let us show the right inclusion in (6.9). Indeed, for any $y \in B(x,R')$, we have by (6.6) and the definition of R',

$$d_*(x,y) \le LF(x,d(x,y)) \le LF(x,R') = L(L^{-1})r = r,$$

whence the right inclusion in (6.9) follows.

(ii). For some fixed point $x \in M$ and $R > 0$, let

$$r' := L^{-1}F(x,R).$$

Let us verify the left inclusion in (6.10). Indeed, for any $y \in B(x,L_0^{-1}R)$, we have

$$d(x,y) < L_0^{-1}R,$$

and then, by (6.6)

$$d_*(x,y) \le LF(x,d(x,y)) < LF(x,L_0^{-1}R) \le L^{-1}F(x,R) = r',$$

provided that

$$LF(x,L_0^{-1}R) \le L^{-1}F(x,R) \text{ for all } x \in M, \tag{6.12}$$

which proves the left inclusion in (6.10).

Let us now prove the right inclusion in (6.10). Indeed, for any $y \in B_*(x,r')$, we have by (6.6) and the definition of r' that

$$F(x,d(x,y)) \le Ld_*(x,y) < Lr' = F(x,R),$$

showing that

$$d(x,y) < R.$$

This proves the right inclusion in (6.10) provided L_0 satisfies (6.12).

It remains to pick up $L_0 \geq 1$ so that both (6.11) and (6.12) are satisfied. Indeed, by (6.5) and the left inequality in (2.7), we obtain for all $x \in M$,

$$\frac{L^{-1}F(x,R)}{LF(x,L_0^{-1}R)} = \frac{1}{L^2}\left(\frac{W(x,R)}{W(x,L_0^{-1}R)}\right)^{1/\beta} \geq \frac{1}{L^2}\left(C^{-1}\left(\frac{R}{L_0^{-1}R}\right)^{\beta_1}\right)^{1/\beta} = \frac{(CL_0^{\beta_1})^{1/\beta}}{L^2},$$

from which, we see that (6.12) is satisfied if

$$\frac{(C^{-1}L_0^{\beta_1})^{1/\beta}}{L^2} \geq 1 \quad \Leftrightarrow \quad L_0 \geq (CL^{2\beta})^{1/\beta_1}.$$

Therefore, if we choose

$$L_0 := L^2 \vee (CL^{2\beta})^{1/\beta_1}$$

then both (6.11) and (6.12) are satisfied, which completes the proof. □

Denote the diameter of M under the metric d_* by

$$\overline{R}_* := \sup\{d_*(x,y) | \; x,y \in M\}.$$

Recall that $\overline{R} = \operatorname{diam} M$ denotes the diameter of M under the metric d.

Proposition 6.3. *Let C_W denote the constant in (2.7) and let $C := LC_W^{1/\beta}$. Then, for any $x \in M$,*

$$C^{-1}W(x,\overline{R})^{1/\beta} \leq \overline{R}_* \leq CW(x,\overline{R})^{1/\beta}. \tag{6.13}$$

Proof. Fix $x \in M$. By (6.6), we have that $W(x,\overline{R}) = \infty$ if and only if $\overline{R}_* = \infty$. Hence, it suffices to consider the case when $\overline{R} < \infty$.

By the left inequality in (6.6), we have for all $z,y \in M$,

$$d_*(z,y) \leq LF(z,d(z,y)) \leq LF(z,\overline{R}) = LW(z,\overline{R})^{1/\beta}.$$

On the other hand, we have by (2.7)

$$\frac{W(z,\overline{R})^{1/\beta}}{W(x,\overline{R})^{1/\beta}} \leq \left(C_W\left(\frac{\overline{R}}{\overline{R}}\right)^{\beta_2}\right)^{1/\beta} = C_W^{1/\beta}. \tag{6.14}$$

Combining the above two inequalities and using the arbitrariness of z,y, we obtain the right inequality in (6.13) with $C = LC_W^{1/\beta}$.

Let us prove the left inequality in (6.13). Indeed, by the right inequality in (6.6), we have, for all $z,y \in M$,

$$F(z,d(z,y)) \leq Ld_*(z,y) \leq L\overline{R}_*.$$

On the other hand, we have by (2.7)

$$W(x,\overline{R})^{1/\beta} = \frac{F(x,\overline{R})}{F(z,d(z,y))}F(z,d(z,y)) \leq \left(C_W\left(\frac{\overline{R}}{d(z,y)}\right)^{\beta_2}\right)^{1/\beta} F(z,d(z,y)).$$

Combining the above two inequalities, we obtain

$$W(x,\overline{R})^{1/\beta} \leq \left(C_W\left(\frac{\overline{R}}{d(z,y)}\right)^{\beta_2}\right)^{1/\beta} L\overline{R}_*.$$

Passing to the limit in the above inequality as $d(z,y) \uparrow \overline{R}$, we obtain the left inequality in (6.13) with the same constant $C = LC_W^{1/\beta}$. □

For any $x \in M$ and $r > 0$, let $V_*(x,r)$ be the volume of the ball $B_*(x,r)$, that is,

$$V_*(x,r) := \mu(B_*(x,r)).$$

Proposition 6.4. *Assume that (VD) is satisfied. Then the following statements are true.*

(i) *Condition* (VD$_*$) *is satisfied, that is, there exists a constant* $C > 0$ *such that, for all* $x \in M$ *and* $r > 0$,

$$V_*(x, 2r) \le CV_*(x, r). \tag{6.15}$$

Consequently, there exists $\alpha_* > 0$ *such that for all* $x, y \in M$ *and* $0 < s \le r$ *with* $d_*(x, y) \le r$,

$$\frac{V_*(x, r)}{V_*(y, s)} \le C \left(\frac{r}{s}\right)^{\alpha_*}.$$

(ii) *Assume in addition that* (RVD) *is satisfied. Then condition* (RVD$_*$) *is also satisfied, that is, there exists* $\alpha'_* > 0$ *such that for all* $x \in M$ *and all* $0 < s \le r < \overline{R}_*$,

$$\frac{V_*(x, r)}{V_*(x, s)} \ge C^{-1} \left(\frac{r}{s}\right)^{\alpha'_*}. \tag{6.16}$$

Proof. Let $L, L_0 \ge 1$ be the same constants as in Proposition 6.2. Fix a point $x \in M$.

(i). Fix $r > 0$ and set

$$R_1 = F^{-1}(x, 2L_0L^{-1}r) \quad \text{and} \quad R_2 = F^{-1}(x, L^{-1}r).$$

Since $F(x, \cdot)$ is strictly increasing and $L_0 \ge 1$, we have $0 < R_2 < R_1 < \infty$. Using the left inclusion in (6.9) with r replaced by $2L_0r$, we obtain

$$B_*(x, 2r) \subset B(x, R_1).$$

Similarly, we have by the right inclusion in (6.9)

$$B(x, R_2) \subset B_*(x, r).$$

Hence, using the definition of V_*, (6.7) and the right inequality in (2.8), we obtain

$$\begin{aligned}
\frac{V_*(x, 2r)}{V_*(x, r)} &\le \frac{V(x, R_1)}{V(x, R_2)} \le C \left(\frac{R_1}{R_2}\right)^{\alpha} = C \left(\frac{F^{-1}(x, 2L_0L^{-1}r)}{F^{-1}(x, L^{-1}r)}\right)^{\alpha} \\
&= C \left(\frac{W^{-1}(x, (2L_0L^{-1}r)^{\beta})}{W^{-1}(x, (L^{-1}r)^{\beta})}\right)^{\alpha} \le C \left(C \left(\frac{(2L_0L^{-1}r)^{\beta}}{(L^{-1}r)^{\beta}}\right)^{1/\beta_1}\right)^{\alpha} = C \left(C(2L_0)^{\beta/\beta_1}\right)^{\alpha},
\end{aligned}$$

thus proving (VD$_*$).

(ii). For $0 < s \le r < \overline{R}_*$, let

$$R_s := F^{-1}(x, \delta L^{-1}s) \quad \text{and} \quad R_r := F^{-1}(x, \delta L^{-1}r),$$

where the constant $\delta > 0$ is small enough such that, by (6.13),

$$R_s \le R_r = W^{-1}(x, (\delta L^{-1}r)^{\beta}) < W^{-1}(x, (\delta L^{-1})^{\beta} CW(x, \overline{R})) \le W^{-1}(x, W(x, \overline{R})) = \overline{R}.$$

By (6.9), we have

$$B_*(x, \delta L_0^{-1}s) \subset B(x, R_s) \quad \text{and} \quad B(x, R_r) \subset B_*(x, r).$$

Hence, using the definition of V_*, (VD$_*$), (6.7), and the left inequality in (2.8), we obtain

$$\begin{aligned}
\frac{V_*(x, r)}{V_*(x, s)} &\ge C^{-1} \frac{V_*(x, r)}{V_*(x, \delta L_0^{-1}s)} \quad \text{(by (VD}_*\text{) and } L_0 \ge 1) \\
&\ge C^{-1} \frac{V(x, R_r)}{V(x, R_s)} \ge C' \left(\frac{R_r}{R_s}\right)^{\alpha'} \quad \text{(by (RVD))} \\
&= C' \left(\frac{F^{-1}(x, L^{-1}r)}{F^{-1}(x, L^{-1}s)}\right)^{\alpha'} = C' \left(\frac{W^{-1}(x, (L^{-1}r)^{\beta})}{W^{-1}(x, (L^{-1}s)^{\beta})}\right)^{\alpha'} \\
&\ge C' \left(C^{-1} \left(\frac{(L^{-1}r)^{\beta}}{(L^{-1}s)^{\beta}}\right)^{1/\beta_2}\right)^{\alpha'} = C \left(\frac{r}{s}\right)^{\alpha'\beta/\beta_2},
\end{aligned}$$

thus proving (RVD$_*$) with $\alpha'_* = \alpha'\beta/\beta_2$. □

7. Heat semigroup and jump measure under the new metric

In this section, we shall reformulate some properties of the heat semigroup and jump measure under the new metric d_*. The advantage of the change of metric is that the scaling function under the metric d_* becomes

$$W_*(x,r) := W_*(r) := r^\beta \quad x \in M, \ r > 0,$$

where $\beta > 0$ is given by (6.4). The new scaling function $W_*(x,r) = r^\beta$ is independent of point x and, hence, is much simpler to deal with.

Let us first introduce conditions (S) and (S_+). For any open set $\Omega \subset M$, let $\{P_t^\Omega\}$ be the heat semigroup of the Dirichlet form $(\mathcal{E}, \mathcal{F}(\Omega))$.

Definition 7.1. We say that condition (S) (*survival estimate*) holds if there exist two constants $\varepsilon, \delta \in (0,1)$ such that, for any ball B in M of radius $< \overline{R}$ and any $t \le \delta W(B)$,

$$P_t^B \mathbf{1}_B \ge \varepsilon \ \text{ in } \frac{1}{4}B.$$

Definition 7.2. We say that condition (S_+) holds if there exist two constants $\varepsilon \in (0,1)$ and $c > 0$ such that, for any ball B of radius $< \overline{R}$ and all $t > 0$,

$$P_t^B \mathbf{1}_B \ge \varepsilon - \frac{ct}{W(B)} \ \text{ in } \frac{1}{4}B.$$

Let us emphasize that, in contrast to condition (S), there is no restriction on the range of time t in condition (S_+). In fact, we have

$$(S) \Leftrightarrow (S_+). \tag{7.1}$$

Indeed, it is clear that (S_+) $\Rightarrow$ (S) by choosing the constant δ in (S) small enough. To show the opposite implication (S) $\Rightarrow$ (S_+), it suffices to consider the case when $t > \delta W(B)$. In this case, this implication follows by setting the constant c in (S_+) to be δ^{-1} so that $P_t^B \mathbf{1}_B \ge 0 > \varepsilon - 1 > \varepsilon - \frac{ct}{W(B)}$.

It is proved in [24, Theorem 14.1] that, under the condition (VD),

$$(FK) + (Gcap) + (TJ) \Rightarrow (S_+) \Rightarrow (S) \Rightarrow (Gcap). \tag{7.2}$$

Remark 7.3. We remark that the constant c in the condition (S_+) in [24, Theorem 14.1] is required to be in $(0,1)$, which is different from that in this paper. However, condition (S_+) in [24] can be replaced by (S_+) in this paper, and all the results in [24] are also true.

It is proved in [23, Proposition 3.1] that, under the condition (VD), for any $1 \le q_1 \le q_2 \le \infty$,

$$(TJ_{q_2}) \Rightarrow (TJ_{q_1}) \Rightarrow (TJ_1) \Rightarrow (TJ). \tag{7.3}$$

In this section, we look at conditions (DUE), (S), (S_+), (TJ_q), (TJ) under the new metric d_*. For that, let us introduce conditions (DUE_*), (S_*), (S_+^*), (TJ_q^*), (TJ_*) as follows.

- Condition (DUE_*): The heat kernel $p_t(x,y)$ of $(\mathcal{E},\mathcal{F})$ exists pointwise on $(0,\infty) \times M \times M$, and there exists a regular $\mathcal{E}$-nest $\{F_k\}$ such that the following properties are true.
  (*a*) For any $x \in M$ and $t > 0$,
  $$p_t(x,\cdot) \in C(\{F_k\}).$$
  (*b*) For any $C_0 \ge 1$, there exists a constant $C > 0$ such that for all $x, y \in M$ and all $t < C_0(\overline{R}_*)^\beta$,
  $$p_t(x,x) \le \frac{C}{V_*(x, t^{1/\beta})}. \tag{7.4}$$
- Condition (S_*): There exist $\varepsilon, \delta_* \in (0,1)$ such that, for any metric ball $B_* = B_*(x,r)$ of radius $r < 2\overline{R}_*$,
  $$P_t^{B_*} \mathbf{1}_{B_*} \ge \varepsilon \ \text{ in } \frac{1}{4}B_*, \tag{7.5}$$
  provided $t^{1/\beta} \le \delta_* r$.
- Condition (S_+^*): There exist $\varepsilon \in (0,1)$ and $c > 0$ such that, for any metric ball $B_* = B_*(x,r)$ with $r < 2\overline{R}_*$ and any $t > 0$,
  $$P_t^{B_*} \mathbf{1}_{B_*} \ge \varepsilon - \frac{ct}{r^\beta} \ \text{ in } \frac{1}{4}B_*. \tag{7.6}$$

- Condition (TJ$_*$): There exists $C \in [0, \infty)$ such that, for any $x \in M$ and any $r > 0$,
$$J(x, B_*(x, r)^c) \le \frac{C}{r^\beta}. \tag{7.7}$$
- Condition (TJ$_q^*$) for some $1 \le q \le \infty$: There exists a non-negative function J such that
$$dj(x, y) = J(x, y)d\mu(y)d\mu(x) \ \text{ in } M \times M,$$
and, for any $x \in M$ and any $r > 0$,
$$\|J(x, \cdot)\|_{L^q(B_*(x,r)^c)} \le \frac{C}{V_*(x, r)^{1/q'} r^\beta}, \tag{7.8}$$
where $q' = \frac{q}{q-1}$ and $C \in [0, \infty)$ is independent of x, r.

Proposition 7.4. *The following statements are true.*

(i) (VD) + (DUE) ⇒ (DUE$_*$).

(ii) (S) ⇒ (S$_*$). *Moreover,*
$$(S_+) \Leftrightarrow (S_+^*). \tag{7.9}$$

(iii) (TJ) ⇔ (TJ$_*$), *and, for any* $1 \le q \le \infty$,
$$(VD) + (TJ_q) \Rightarrow (TJ_q^*).$$
Consequently, under (VD), *for any* $1 \le q_1 \le q_2 \le \infty$,
$$(TJ_{q_2}) \Rightarrow (TJ_{q_2}^*) \Rightarrow (TJ_{q_1}^*) \Rightarrow (TJ_*). \tag{7.10}$$

Proof. Since (VD) ⇒ (VD$_*$), we can assume throughout the proof that (VD$_*$) is satisfied.

(i). Fix $x \in M$ and $t < C_0(\overline{R}_*)^\beta$. The existence and continuity property of the heat kernel are satisfied by (DUE), so we need only to verify the inequality (7.4). Indeed, we have by (DUE)
$$p_t(x, x) \le \frac{C}{V(x, W^{-1}(x, t))}. \tag{7.11}$$
By (6.13), there exists a small enough constant $c \in (0, 1)$ such that
$$ct < cC_0(\overline{R}_*)^\beta \le cC_0C^\beta W(x, \overline{R}) < W(x, \overline{R}),$$
whence
$$R := W^{-1}(x, ct) < \overline{R} \ \left(\Leftrightarrow \ \ ct = W(x, R) < W(x, \overline{R})\right).$$
By (6.10), the ball $B(x, R)$ contains a ball $B_*(x, r)$, where
$$r := L^{-1}F(x, R) = L^{-1}W(x, R)^{1/\beta} = L^{-1}(ct)^{1/\beta},$$
which implies that
$$V(x, W^{-1}(x, ct)) = \mu(B(x, R)) \ge \mu(B_*(x, r)) = V_*(x, L^{-1}(ct)^{1/\beta}).$$
Hence, it follows from (7.11), the above inequality and (VD$_*$), that
$$p_t(x, x) \le \frac{C}{V(x, W^{-1}(x, t))} \le \frac{C}{V(x, W^{-1}(x, ct))} \le \frac{C}{V_*(x, L^{-1}(ct)^{1/\beta})} \le \frac{C'}{V_*(x, t^{1/\beta})},$$
which was to be proved.

(ii). Fix $x \in M$ and $r < 2\overline{R}_*$. By (6.13), we have
$$r < 2\overline{R}_* \le 2CW(x, \overline{R})^{1/\beta} = 2CF(x, \overline{R}),$$
where $C = LC_W^{1/\beta} \ge L$. Set
$$R := F^{-1}(x, (2C)^{-1}r) < \overline{R} \quad \text{ and } \quad B := B(x, R).$$
Using (6.9) with r replaced by $(2C)^{-1}Lr$, we obtain
$$\begin{aligned} B_*(x, L_0^{-1}(2C)^{-1}Lr) &\subset B(x, F^{-1}(x, L^{-1}(2C)^{-1}Lr)) \\ &= B(x, R) \subset B_*(x, (2C)^{-1}Lr) \\ &\subset B_*(x, r) =: B_*. \end{aligned}$$

It follows that
$$\frac{1}{4}B \supset \frac{1}{4}B_*(x, L_0^{-1}(2C)^{-1}Lr) = (4^{-1}L_0^{-1}(2C)^{-1}L)B_* =: \eta B_*$$
which together with condition (S) yields
$$P_t^{B_*}\mathbf{1}_{B_*} \geq P_t^B \mathbf{1}_B \geq \varepsilon \quad \text{in } \frac{1}{4}B \supset \eta B_* \tag{7.12}$$
provided that
$$t \leq \delta W(x, R) = \delta(2C)^{-\beta} r^\beta.$$
Let us show that (7.12) holds also in $\frac{1}{4}B_*$ (not only in ηB_*). This can be done by using the standard covering arguments. Indeed, for any $z \in \frac{1}{4}B_*$, since $U := B_*(z, \frac{1}{4}r) \subset B_*$, we see by (7.12) that
$$P_t^{B_*}\mathbf{1}_{B_*} \geq P_t^U \mathbf{1}_U \geq \varepsilon \quad \text{in } \eta B_*(z, \frac{1}{4}r)$$
provided $t \leq \delta(2C)^{-\beta}(\frac{1}{4}r)^\beta$. Covering $\frac{1}{4}B_*$ by a countable family of balls like $B_*(z, \frac{\eta}{4}r)$, we conclude that
$$P_t^{B_*}\mathbf{1}_{B_*} \geq \varepsilon \quad \text{in } \frac{1}{4}B_*$$
provided $t \leq \delta(2C)^{-\beta}(\frac{1}{4}r)^\beta$, thus showing that condition (S_*) holds with $\delta_* := \delta^{1/\beta}(8C)^{-1}$.

For the equivalence (7.9), let us first prove the implication (S_+) $\Rightarrow$ (S_+^*). Fix some ball $B_* := B_*(x, r)$ with $r < 2\overline{R}_*$ and set
$$R := F^{-1}(x, L^{-1}c_0 r) \quad \text{and} \quad B := B(x, R),$$
where $c_0 > 0$ is a small constant such that $R < \overline{R}$ (which can be done thanks to (6.13) and (2.8)). Then, by (6.9) with r replaced by $c_0 r$, we see that
$$B_*(x, L_0^{-1}c_0 r) \subset B \subset B_*(x, c_0 r) \subset B_*.$$
Thus, by condition (S_+), we obtain, for any $t > 0$,
$$P_t^{B_*}\mathbf{1}_{B_*} \geq P_t^B \mathbf{1}_B \geq \varepsilon - \frac{ct}{W(x,R)} = \varepsilon - \frac{ct}{(L^{-1}r)^\beta} = \varepsilon - \frac{cL^\beta t}{r^\beta} \quad \text{in } \frac{1}{4}B \supset \frac{1}{4}B_*(x, L_0^{-1}c_0 r) = \frac{c_0}{4L_0}B_*,$$
where the constants $\varepsilon, c > 0$ come from condition (S_+). Moreover, by standard covering arguments, we have
$$P_t^{B_*}\mathbf{1}_{B_*} \geq \varepsilon - \frac{c' t}{r^\beta} \quad \text{in } \frac{1}{4}B_* \quad \text{(not only in } \frac{c_0}{4L_0}B_*)$$
for any $r < 2\overline{R}_*$ and any $t > 0$, which proves (S_+^*).

It remains to prove the converse implication (S_+^*) $\Rightarrow$ (S_+). Fix a ball $B := B(x, R)$ with $R < \overline{R}$, and
$$r := L^{-1}F(x, R).$$
By (6.10), we have
$$B(x, L_0^{-1}R) \subset B_* := B_*(x, r) \subset B(x, R) = B.$$
Hence, by (S_+^*), we obtain
$$P_t^B \mathbf{1}_B \geq P_t^{B_*}\mathbf{1}_{B_*} \geq \varepsilon - \frac{ct}{r^\beta} = \varepsilon - \frac{ct}{L^{-\beta}W(x,R)} \quad \text{in } \frac{1}{4}B_* \supset \frac{1}{4}B(x, L_0^{-1}R) = \frac{1}{4L_0}B.$$
where the constants $\varepsilon, c > 0$ come from condition (S_+^*). Moreover, by standard covering arguments, we can prove the above inequality also holds in $\frac{1}{4}B$, which proves (S_+).

(iii). Fix some $x \in M$ and $r > 0$. We have by (6.9)
$$B_*(x, L_0^{-1}r) \subset B(x, R) \subset B_*(x, r),$$
where
$$R := F^{-1}(x, L^{-1}r) \;\Leftrightarrow\; F(x, R) = L^{-1}r \;\Leftrightarrow\; W(x, R) = (L^{-1}r)^\beta.$$
It follows that
$$V(x, R) = \mu(B(x, R)) \geq \mu(B_*(x, L_0^{-1}r)) = V_*(x, L_0^{-1}r).$$

Hence, if (TJ) holds, then

$$J(x, B_*(x,r)^c) \le J(x, B(x,R)^c) \le \frac{C}{W(x,R)} = \frac{C}{(L^{-1}r)^\beta},$$

so that condition (TJ$_*$) holds as well.

Similarly, we can use the right inclusion in (6.10) to prove (TJ$_*$) $\Rightarrow$ (TJ). Here we omit the details.

If (TJ$_q$) holds for some $1 \le q \le \infty$, then we obtain similarly

$$\begin{aligned}\|J(x,\cdot)\|_{L^q(B_*(x,r)^c)} \le \|J(x,\cdot)\|_{L^q(B(x,R)^c)} &\le \frac{C}{V(x,R)^{1/q'}W(x,R)} \\ &\le \frac{C}{V_*(x,L_0^{-1}r)^{1/q'}W(x,R)} = \frac{C}{V_*(x,L_0^{-1}r)^{1/q'}(L^{-1}r)^\beta} \\ &\le \frac{C'}{V_*(x,r)^{1/q'}r^\beta} \quad \text{(by (VD}_*\text{))},\end{aligned}$$

thus proving (TJ$_q^*$).

Finally, the implication (7.10) follows from the similar arguments that lead to (7.3). □

Remark 7.5. Proposition 7.4 says that if conditions (DUE), (S), (S$_+$), (TJ$_q$), (TJ) are satisfied for a scaling function $W(x,r)$, that may depend on x, then the parallel conditions (DUE$_*$), (S$_*$), (S$_+^*$), (TJ$_q^*$), (TJ$_*$) are also satisfied for a new scaling function $W_*(x,r) = r^\beta$, that is independent of x, under the metric d_*. This property is crucial for the study of a truncated Dirichlet form in the next section.

8. Truncated Dirichlet form under new metric

In this section, we will consider the ρ-truncated Dirichlet form $(\mathcal{E}^{(\rho)},\mathcal{F}^{(\rho)})$ defined in Section 5 for any number $\rho > 0$ but under the new metric d_*, and obtain the heat kernel estimates for the truncated Dirichlet form. Unless otherwise stated, all balls in this section are defined under the new metric d_*.

Recall that $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form without killing part, and the jump part is as in (2.6). For any $\rho > 0$, set

$$\mathcal{E}^{(\rho)}(u,v) := \mathcal{E}^{(L)}(u,v) + \iint_{M\times B_*(x,\rho)} (u(x)-u(y))(v(x)-v(y))\,dj, \quad u,v\in\mathcal{F}, \tag{8.1}$$

where $B_*(x,\rho)$ is an open ball under the new metric d_* as defined in (6.8).

Clearly, if condition (TJ$_*$) or (TJ) (which implies (TJ$_*$) by Proposition 7.4(iii)) holds, then

$$\omega(\rho) := \operatorname*{esup}_{x\in M} J(x, B_*(x,\rho)^c) < \infty, \tag{8.2}$$

and $(\mathcal{E}^{(\rho)},\mathcal{F}^{(\rho)})$ is a regular ρ-local Dirichlet form by Lemma 5.7. Besides, all the results in Subsection 5.3 can be applied in the present setting.

Denote by $\{Q_t^\Omega\}$ the heat semigroup of the Dirichlet form $(\mathcal{E}^{(\rho)},\mathcal{F}^{(\rho)}(\Omega))$ restricted to a non-empty open set $\Omega\subset M$ (the superscript ρ in Q_t^Ω is omitted). If $\Omega = M$, then $\{Q_t\} := \{Q_t^\Omega\}$ is the heat semigroup of $(\mathcal{E}^{(\rho)},\mathcal{F}^{(\rho)})$.

Remark 8.1. Since $\mathcal{F} = \mathcal{F}^{(\rho)}$, all the cutoff functions defined for the Dirichlet form $(\mathcal{E},\mathcal{F})$ are also cutoff functions for $(\mathcal{E}^{(\rho)},\mathcal{F}^{(\rho)})$.

8.1. On-diagonal upper estimate of truncated heat kernel.

We need the notions of the subcaloric and caloric functions. Let I be an interval in $\mathbb{R}$. A function $u: I\to L^2$ is said to be *weakly differentiable* at $t\in I$, if for any $\varphi\in L^2$, the function $(u(\cdot),\varphi)$ is differentiable at t, that is, the limit

$$\lim_{\varepsilon\to 0}\left(\frac{u(t+\varepsilon)-u(t)}{\varepsilon},\varphi\right)$$

exists. In this case, by the principle of uniform boundedness, there is some $w\in L^2$ such that

$$\lim_{\varepsilon\to 0}\left(\frac{u(t+\varepsilon)-u(t)}{\varepsilon},\varphi\right) = (w,\varphi)$$

for any $\varphi \in L^2$. The function w is called the *weak derivative* of u at t, and we write $w = \partial_t u$. Weak derivative satisfies the following *product rule*. Let $g : I \mapsto \mathbb{R}$ be differentiable at $t \in I$ in the classical sense, and $u : I \to L^2$ be weakly differentiable at $t \in I$. Then the function $v := gu : I \to L^2$ is also weakly differentiable at t.

For an open subset $\Omega \subset M$, a function $u : I \to \mathcal{F}$ is called *subcaloric* in $I \times \Omega$ with respect to $(\mathcal{E}, \mathcal{F})$ if u is weakly differentiable in L^2 at any $t \in I$ and, for any $t \in I$ and any non-negative $\varphi \in \mathcal{F}(\Omega)$,

$$(\partial_t u, \varphi) + \mathcal{E}(u(t, \cdot), \varphi) \le 0.$$

A function u is called *caloric* if the above inequality is replaced by equality, that is, if

$$(\partial_t u, \varphi) + \mathcal{E}(u(t, \cdot), \varphi) = 0.$$

For example, for any $f \in L^2(\Omega)$, the function $u(t, \cdot) = P_t^\Omega f$ is caloric in $(0, \infty) \times \Omega$.

Proposition 8.2. *Let Ω be an open subset of M. Under condition (8.2), for any $t > 0$ and any $0 \le f \in L^2$, we have*

$$Q_t^\Omega f \le e^{2\omega(\rho)t} P_t^\Omega f \quad \text{in } \Omega \text{ (also in } M). \tag{8.3}$$

Consequently, if condition (TJ$_*$) hold, then

$$Q_t^\Omega f \le \exp\Big(\frac{2ct}{\rho^\beta}\Big) P_t^\Omega f \quad \text{in } \Omega \text{ (also in } M), \tag{8.4}$$

where $c > 0$ is the same constant as in condition (TJ$_*$) (independent of ρ, t, f and Ω).

Proof. Let $f \in L^2$ be nonnegative in M and

$$u(t, x) := Q_t^\Omega f(x) \quad t > 0,\ x \in M.$$

Clearly, the function u is caloric in $(0, \infty) \times \Omega$ with respect to $\mathcal{E}^{(\rho)}$, that is, for any $t > 0$ and any $0 \le \varphi \in \mathcal{F}$

$$(\partial_t u(t, \cdot), \varphi) + \mathcal{E}^{(\rho)}(u(t, \cdot), \varphi) = 0. \tag{8.5}$$

Consider the following function defined for all $t > 0$ and $x \in M$:

$$v(t, x) := \exp(-2\omega(\rho)t) u(t, x) - P_t^\Omega f(x)$$

Clearly, the function $v(t, \cdot)$ satisfies the boundary and initial conditions:

$$v_+(t, \cdot) \le \exp(-2\omega(\rho)t) u(t, x) \in \mathcal{F}^{(\rho)}(\Omega) = \mathcal{F}(\Omega),$$
$$v_+(t, \cdot) \to 0 \ \text{ in the norm of } L^2(\Omega) \text{ as } t \to 0.$$

Note that the function $P_t^\Omega f$ is caloric in $(0, \infty) \times \Omega$ with respect to $\mathcal{E}$, that is, for any $t > 0$ and any $\varphi \in \mathcal{F}$

$$(\partial_t P_t^\Omega f, \varphi) + \mathcal{E}(P_t^\Omega f, \varphi) = 0. \tag{8.6}$$

Moreover, the function v is subcaloric with respect to $\mathcal{E}$ in $(0, \infty) \times \Omega$, since for any $0 \le \varphi \in \mathcal{F}(\Omega)$

$$\begin{aligned}
(\partial_t v(t, \cdot), \varphi) + \mathcal{E}(v(t, \cdot), \varphi) = &\ \exp(-2\omega(\rho)t)\Big(- 2\omega(\rho)(u(t, \cdot), \varphi) + (\partial_t u(t, \cdot), \varphi) + \mathcal{E}^{(\rho)}(u(t, \cdot), \varphi) \\
&+ \int_M \int_{B_*(x,\rho)^c} (u(t, x) - u(t, y))(\varphi(x) - \varphi(y)) dj\Big) \\
&+ (\partial_t P_t^\Omega f, \varphi) + \mathcal{E}(P_t^\Omega f, \varphi) \\
\le &\ \exp(-2\omega(\rho)t)\Big(- 2\omega(\rho)(u(t, \cdot), \varphi) \quad \text{(by (8.5) and (8.6))} \\
&+ \int_M \int_{B_*(x,\rho)^c} [u(t, x)\varphi(x) + u(t, y)\varphi(y)] dj\Big) \\
\le &\ \exp(-2\omega(\rho)t)(- 2\omega(\rho)(u(t, \cdot), \varphi) + 2\omega(\rho)(u(t, \cdot), \varphi)) \\
= &\ 0.
\end{aligned}$$

Therefore, by the parabolic maximum principle (Proposition 11.7 in Appendix), we obtain

$$v(t, x) = \exp(-2\omega(\rho)t) Q_t^\Omega f(x) \le P_t^\Omega f(x) \ \text{ for } (t, x) \in (0, \infty) \times \Omega,$$

thus showing (8.3).

The inequality (8.4) follows from (8.3) and (TJ$_*$). □

Next we show the existence and on-diagonal upper bound of the pointwise heat kernel for the truncated Dirichlet form $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)})$ for any $\rho > 0$.

Lemma 8.3. *Assume that conditions* (VD$_*$), (TJ$_*$) *and* (DUE$_*$) *hold. Then, for any* $\rho > 0$, *the Dirichlet form* $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)})$ *possesses a pointwise heat kernel* $q_t^{(\rho)}(x,y)$ *on* $(0,\infty)\times M\times M$ *(see Definition 2.6). Moreover, for any* $C_0 \geq 1$, *there exist constant* $C,\ c>0$ *such that, for any* $x\in M$ *and any* $t < C_0(\overline{R}_*)^\beta$,

$$q_t^{(\rho)}(x,x) \leq \frac{C}{V_*(x,t^{1/\beta})}\exp\left(\frac{ct}{\rho^\beta}\right). \tag{8.7}$$

Proof. Fix a ball $B_* := B(x,r)$ for some $x\in M$ and $r>0$. Using conditions (VD$_*$) and (DUE$_*$), and following the arguments in the proofs of [23, Lemma 6.3 and Corollary 6.4], one can obtain that

$$\|P_t f\|_{L^\infty(B_*)} \leq \frac{C'}{\sqrt{\mu(B_*)}}\left(\frac{r}{\overline{R}_*}\vee 1\right)^{\frac{\alpha_*}{2}}\left(\frac{r^\beta}{t}+1\right)^{\frac{\alpha_*}{2\beta}}\|f\|_2.$$

Using (8.4) with $\Omega = M$ and (TJ$_*$), we obtain for any $t>0,\ r>0$ and $f\in L^2$,

$$\|Q_t f\|_{L^\infty(B_*)} \leq \exp\left(\frac{ct}{\rho^\beta}\right)\|P_t f\|_{L^\infty(B_*)} \leq \exp\left(\frac{ct}{\rho^\beta}\right)\frac{C'}{\sqrt{\mu(B_*)}}\left(\frac{r}{\overline{R}_*}\vee 1\right)^{\frac{\alpha_*}{2}}\left(\frac{r^\beta}{t}+1\right)^{\frac{\alpha_*}{2\beta}}\|f\|_2.$$

Therefore, it follows from Theorem 11.8 (with $p=2$) in Appendix that the heat kernel $q_t^{(\rho)}(x,y)$ of $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)})$ exists on $(0,\infty)\times M\times M$, and, moreover, for any $z\in B_* = B_*(x,r)$ and any $t>0$,

$$\|q_t^{(\rho)}(z,\cdot)\|_{L^2} \leq \exp\left(\frac{ct}{\rho^\beta}\right)\frac{C'}{\sqrt{\mu(B_*)}}\left(\frac{r}{\overline{R}_*}\vee 1\right)^{\frac{\alpha_*}{2}}\left(\frac{r^\beta}{t}+1\right)^{\frac{\alpha_*}{2\beta}}.$$

(See also [23, Eq. (6.13) and Remark 6.8].)

Let us verify (8.7). Indeed, for given $C_0\geq 1$ and $t< C_0(\overline{R}_*)^\beta$, setting $r := t^{1/\beta} < C_0^{1/\beta}\overline{R}_*$ in the above inequality and using (VD$_*$) and (2.7), we obtain

$$q_{2t}^{(\rho)}(x,x) \leq \exp\left(\frac{ct}{\rho^\beta}\right)\frac{C'}{\sqrt{V_*(x,t^{1/\beta})}}\left(\frac{t^{1/\beta}}{\overline{R}_*}\vee 1\right)^{\frac{\alpha_*}{2}}\left(\frac{t}{t}+1\right)^{\frac{\alpha_*}{2\beta}} \leq \frac{C''}{\sqrt{V_*(x,(2t)^{1/\beta})}}\exp\left(\frac{ct}{\rho^\beta}\right),$$

which proves (8.7).

It remains to observe that the $\mathcal{E}^{(\rho)}$-nest $\{F_k^{(\rho)}\}_{k=1}^\infty$ is also $\mathcal{E}$-nest, which follows directly from the fact that $\mathcal{E}_1$ and $\mathcal{E}_1^{(\rho)}$ are equivalent. □

Remarks 8.4. (i). Note that the $\mathcal{E}$-nest $\{F_k^{(\rho)}\}_{k=1}^\infty$ in Lemma 8.3 may depend on ρ. Let $\{F_k\}$ be the $\mathcal{E}$-nest in condition (DUE) (see also [23, Lemma 6.6]). By [19, Theorem 2.1.2(i)] and its proof, there exists a common regular $\mathcal{E}$-nest $\{\widetilde{F}_k\}_{k=1}^\infty$ such that, for any positive rational number $\rho\in\mathbb{Q}_+$ and for each k,

$$\widetilde{F}_k \subset F_k^{(\rho)} \quad\text{and}\quad \widetilde{F}_k\subset F_k.$$

Consequently, for any $\rho\in\mathbb{Q}_+$,

$$C(\{F_k^{(\rho)}\})\subset C(\{\widetilde{F}_k\}) \quad\text{and}\quad C(\{F_k\})\subset C(\{\widetilde{F}_k\}),$$

Hence, we can modify the heat kernels $p_t(x,y)$ and $q_t^{(\rho)}(x,y)$ by letting

$$p_t(x,y) = q_t^{(\rho)}(x,y) = 0$$

for any $t>0$, whenever x,y lie outside the union set $\cup_{k=1}^\infty \widetilde{F}_k$.

In the rest of this paper, we rename $\{\widetilde{F}_k\}_{k=1}^\infty$ by $\{F_k\}_{k=1}^\infty$ so that (2.20) holds for both heat kernels $p_t(x,y)$ and $q_t^{(\rho)}(x,y)$ simultaneously for all $t>0$ and $\rho\in\mathbb{Q}_+$.

(ii). Under the hypothesis of Lemma 8.3, for any $f \in L^2$ and $t > 0$, the function $P_t f$ has a quasi-continuous version that belongs to $C(\{F_k\})$, for example, $\int_M p_t(\cdot, y) f(y) d\mu(y)$. We always this version of $P_t f$, that is, for any $f \in L^2$,

$$P_t f(x) = \int_M p_t(x, y) f(y) d\mu(y) \text{ for any } x \in M \text{ and } t > 0 \tag{8.8}$$

so that

$$P_t f \in C(\{F_k\}).$$

Similarly, by Lemma 8.3, we can replace the truncated heat semigroup $\{Q_t f\}_{t>0}$ by its pointwise realization, by setting

$$Q_t f(x) = \int_M q_t^{(\rho)}(x, y) f(y) d\mu(y) \text{ for any } x \in M \text{ and } t > 0 \tag{8.9}$$

so that $Q_t f \in C(\{F_k^{(\rho)}\})$. In particular, for any $\rho \in \mathbb{Q}_+$, $f \in L^2$ and any $t > 0$,

$$Q_t f \in C(\{F_k\}).$$

By the standard arguments, we can first extend (8.8) and (8.9) to all positive functions $f \in \mathcal{B}_+(M)$, and then to all $f \in \mathcal{B}(M)$ whenever the integrals in (8.8) and (8.9) make sense.

(iii). If in addition $P_t f$ is continuous for all $t > 0$ and $f \in L^2$, then, by the proof of Theorem 11.8, the $\mathcal{E}$-nest $\{F_k\}$ can be taken as $F_k := M$ for all $k \geq 1$. Similarly, if $Q_t f$ is continuous for all $t > 0$ and $f \in L^2$, then $\mathcal{E}^{(\rho)}$-nest $\{F_k^{(\rho)}\}$ can also be take as $F_k^{(\rho)} := M$ for all $k \geq 1$.

In the remainder of this subsection, we prove the following statements that will be used later on.

Proposition 8.5. *Under the hypothesis of Lemma 8.3, the following statements are true.*

(i) *Let g be a continuous function in an open subset U of M and f be a non-negative Borel function in M. If the following inequality*

$$P_t f(x) \leq g(x) \tag{8.10}$$

holds for some $t > 0$ and for μ-almost all $x \in U$, then it also holds for all $x \in U$.

(ii) *Let h be a continuous function on $U \times U$. If the following inequality*

$$p_t(x, y) \leq h(x, y) \tag{8.11}$$

holds for some $t > 0$ and $(\mu \times \mu)$-almost all $(x, y) \in U \times U$, then it also holds for all $(x, y) \in U \times U$.

The above results are valid for Q_t and $q_t^{(\rho)}$ (when $\rho \in \mathbb{Q}_+$) under similar assumptions.

Proof. (i). Let $K \subset U$ be compact. Assume first that $0 \leq f \in L^2$. Let $\phi \in C_0(U)$ be such that $\mathbf{1}_K \leq \phi \leq \mathbf{1}_U$. By (8.10), we have for μ-almost all $x \in M$,

$$\phi(x) P_t f(x) \leq \phi(x) g(x). \tag{8.12}$$

Let $\{F_k\}$ be the $\mathcal{E}$-nest as in Remark 8.4(i). Since $\phi g \in C_0(M)$ and $\phi P_t f \in C(\{F_k\})$, by [19, Theorem 2.1.2(ii), p. 69], we see that (8.12) holds true for all $x \in M$. In particular, we have (8.10) for all $x \in K$, as $\phi|_K = 1$. Since $K \subset U$ is arbitrary, we obtain (8.10) for all $x \in U$.

For a general non-negative Borel function f, let

$$f_n := (f \wedge n) \mathbf{1}_K \in L^2(M), \quad n \geq 1.$$

It follows from above that (8.10) is true for each f_n and for every $x \in U$, since $0 \leq f_n \leq f$. Passing to the limit as $n \to \infty$ and $K \uparrow U$, we obtain, for every $x \in K$,

$$P_t f(x) = \lim_{n\to\infty,\ K\uparrow U} P_t f_n(x) \leq g(x), \quad \forall\, x \in U.$$

(ii). Let $0 \leq f \in L^\infty(M)$ with compact support in U. Multiplying by f both sides of (8.11), we obtain, for μ-almost all $x \in U$

$$P_t f(x) = \int_M p_t(x, y) f(y) d\mu(y) \leq \int_M h(x, y) f(y) d\mu(y). \tag{8.13}$$

Since $f \in L^\infty$ has compact support and $h \in C(U \times U)$, it follows that the function

$$x \mapsto \int_M h(x,y) f(y) d\mu(y)$$

is continuous in U. Hence, using (8.10), we obtain (8.13) for all $x \in U$. From this, we see that (8.11) holds for all $x \in U$ and μ-almost all $y \in U$. Moreover, by condition (DUE$_*$) (or (DUE)), we see that $p_t(x,\cdot) \in C(\{F_k\})$. Then, similar arguments in (i) will lead to the inequality (8.11) for all $(x,y) \in U \times U$. □

Remark 8.6. Recall that there is the relation (5.23) between P_t and Q_t that holds almost everywhere in M. Under the hypotheses (VD$_*$), (TJ$_*$) and (DUE$_*$), we conclude by Proposition 8.5 that (5.23) holds pointwise for all $\rho \in \mathbb{Q}_+$ and $f \in L^2 \cap L^\infty$, that is,

$$P_t f(x) = Q_t f(x) + \int_0^t Q_s A^{(\rho)} P_{t-s} f(x) ds \quad \text{for all } x \in M. \tag{8.14}$$

The identity (8.14) plays an important role in deriving the upper bounds of heat kernels.

Indeed, by Remark 8.4(ii), we know that $P_t f,\ Q_t f \in C(\{F_k\})$ for $t > 0$ and $f \in L^2$ when $\rho \in \mathbb{Q}_+$. Fix $\rho \in \mathbb{Q}_+$ and $f \in L^2 \cap L^\infty$. By (5.21), we have, for any $0 < s < t$,

$$Q_s A^{(\rho)} P_{t-s} f \in C(\{F_k\}).$$

Then it follows from [19, Theorem 2.1.2, p. 69] and (5.21) that, for any $x \in M$,

$$\begin{aligned} |Q_s A^{(\rho)} P_{t-s} f(x)| &\le \|Q_s A^{(\rho)} P_{t-s} f\|_\infty \\ &\le \|Q_s\|_{L^\infty \to L^\infty} \cdot 4\omega(\rho) \cdot \|P_{t-s}\|_{L^\infty \to L^\infty} \|f\|_\infty \\ &\le 4\omega(\rho) \|f\|_\infty < \infty. \end{aligned}$$

Hence, by the dominated convergence theorem, we obtain

$$\int_0^t Q_s A^{(\rho)} P_{t-s} f \in C(\{F_k\}).$$

By [19, Theorem 2.1.2, p. 69], we conclude that (8.14) holds for any $f \in L^2 \cap L^\infty$ and $t > 0$.

8.2. **Tail estimate for truncated semigroup.** Recall that, for any open set $\Omega \subset M$, $\{Q_t^\Omega\}$ denotes the heat semigroup associated with the part Dirichlet from $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)}(\Omega))$ of the truncated ρ-local Dirichlet form defined by (8.1) for $\rho > 0$. In this subsection, we give pointwise tail estimate of the heat semigroup $\{Q_t^{B_*}\}$ of any ρ-local Dirichlet form $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)}(B_*))$ for any ball B_*.

Proposition 8.7. *If every ball in M has finite measure and conditions (S$_+^*$), (TJ$_*$) hold, then, for any ball $B_* := B_*(x,r)$ with $r > 0$ and any $t > 0$,*

$$1 - Q_t^{B_*} \mathbf{1}_{B_*} \le 1 - \varepsilon + C\left(r^{-\beta} + \rho^{-\beta}\right) t \quad \text{in } \frac{1}{4} B_* \tag{8.15}$$

where $\varepsilon \in (0,1)$ and $C > 0$ are two constants independent of ρ, t, B_*.

Proof. By condition (S$_+^*$), for any ball $B_* := B_*(x,r)$ with $r < 2\overline{R}_*$ and any $t > 0$,

$$1 - P_t^{B_*} \mathbf{1}_{B_*} \le 1 - \varepsilon + C r^{-\beta} t \ \text{ in } \frac{1}{4} B_*, \tag{8.16}$$

where $\varepsilon \in (0,1)$ and $C > 0$ are two constants independent of t, B_*. Let us prove that (8.16) holds also when $r \ge 2\overline{R}_*$ (and $\overline{R}_* < \infty$). Since every ball has finite measure, it follows from [20, Lemma 4.6, p. 3327] that condition (S$_*$) (and, hence, (S$_+^*$)) implies that $(\mathcal{E}, \mathcal{F})$ is conservative. Hence, when $r \ge 2\overline{R}_*$, we have $B_* = M$ whence $1 - P_t^{B_*} \mathbf{1}_{B_*} = 1 - P_t 1 = 0$, for all $t > 0$, which implies (8.16) for all $r > 0$ and $t > 0$.

Consequently, it follows from (5.26) with $\Omega = B_*$, $f = \mathbf{1}_{B_*}$ that

$$\begin{aligned} 1 - Q_t^{B_*} \mathbf{1}_{B_*} &\le 1 - P_t^{B_*} \mathbf{1}_{B_*} + \frac{Ct}{\rho^\beta} \left\| \mathbf{1}_{B_*} \right\|_\infty \\ &\le 1 - \varepsilon + C r^{-\beta} t + C \rho^{-\beta} t \quad \text{in } \frac{1}{4} B_*, \end{aligned}$$

which proves (8.15). □

In the next two lemma we obtain two different estimates of $Q_t\mathbf{1}_{B_*^c}$.

Lemma 8.8. *If every ball in M has finite measure and conditions (S_+^*), (TJ_*) hold, then there exist positive constants C, c, c' such that, for any ball $B_* := B_*(x_0, r)$ of radius $r > 0$ and any $t > 0$,*

$$Q_t\mathbf{1}_{B_*^c} \le 1 - Q_t^{B_*}\mathbf{1}_{B_*} \le C\exp\left(-c\frac{r}{\rho} + c'\frac{t}{\rho^\beta}\right) \quad in\ \frac{1}{4}B_*. \tag{8.17}$$

Proof. Fix a ball $B_* := B_*(x_0, r)$ with $r > 0$ and $t > 0$. The inequality (8.17) is trivially satisfied if $r \le 4\rho$, since $1 - Q_t^{B_*}\mathbf{1}_{B_*} \le 1$ in M. In the sequel, assume that $r > 4\rho$.

Since (S_+^*) and (TJ_*) are satisfied, we obtain by Proposition 8.7, that, for any $z \in M$,

$$1 - Q_t^{B_*(z,\rho)}\mathbf{1}_{B_*(z,\rho)} \le 1 - \varepsilon + c_0\left(\rho^{-\beta} + \rho^{-\beta}\right)t = 1 - \varepsilon + 2c_0\rho^{-\beta}t \quad \text{in } \frac{1}{4}B_*(z,\rho). \tag{8.18}$$

Recall that, for any $\lambda > 0$ and a ball $B_*(y, r')$, the resolvent $R_\lambda^{B_*(y,r')}$ of the heat semigroup $\{Q_t^{B_*(y,r')}\}$ is given by (5.4), that is, by

$$R_\lambda^{B_*(y,r')}f = \int_0^\infty e^{-\lambda s}Q_s^{B_*(y,r')}f\,ds \quad \text{for } f \in L^2.$$

Then by (8.18), for any $\lambda > 0$ and $z \in B_*$

$$\begin{aligned} 1 - \lambda R_\lambda^{B_*(z,\rho)}\mathbf{1}_{B_*(z,\rho)} &= \int_0^\infty \lambda e^{-\lambda s}(1 - Q_s^{B_*(z,\rho)}\mathbf{1}_{B_*(z,\rho)})\,ds \\ &\le \int_0^\infty \lambda e^{-\lambda s}\left(1 - \varepsilon + 2c_0\rho^{-\beta}s\right)ds \\ &= 1 - \varepsilon + 2c_0\lambda^{-1}\rho^{-\beta} =: c(\rho,\lambda) \quad \text{in } \frac{1}{4}B_*(z,\rho). \end{aligned}$$

Next, let $k \ge 1$ be an integer such that

$$k < \frac{r}{4\rho} \le k + 1,$$

in particular, $4k\rho < r$. Since (S_+^*) $\Rightarrow$ (S_*) and every ball has finite measure, by [20, Lemma 4.5, p. 3326], we have that $\mathrm{cutoff}(A, U) \neq \emptyset$ for any bounded measurable set A and for any open set U with $\overline{A} \subset U$ (that is, (5.2) is satisfied). Hence, by Lemma 5.6 and (5.17), we obtain, for any $\lambda > 0$,

$$1 - \lambda R_\lambda^{B_*}\mathbf{1}_{B_*} \le c(\rho,\lambda)^k = (1 - \varepsilon + 2c_0\lambda^{-1}\rho^{-\beta})^k \quad \text{in } \frac{1}{4}B_*.$$

Setting $\lambda = \frac{4c_0}{\varepsilon\rho^\beta}$ in the above inequality, we obtain

$$1 - \lambda R_\lambda^{B_*}\mathbf{1}_{B_*} \le (1 - \varepsilon/2)^k = \exp\left(-k\ln\frac{2}{2-\varepsilon}\right) \le \exp\left(-\left(\frac{r}{4\rho} - 1\right)\ln\frac{2}{2-\varepsilon}\right) \quad \text{in } \frac{1}{4}B_*.$$

Moreover, with the above choice of λ, using (5.7) with $\Omega = B_*$, we obtain from the above inequality

$$\begin{aligned} 1 - Q_t^{B_*}\mathbf{1}_{B_*} &\le e^{\lambda t}\left(1 - \lambda R_\lambda^{B_*}\mathbf{1}_{B_*}\right) \le e^{\lambda t}\exp\left(-\ln\frac{2}{2-\varepsilon}\left(\frac{r}{4\rho} - 1\right)\right) \\ &= \exp\left(-\ln\frac{2}{2-\varepsilon}\left(\frac{r}{4\rho} - 1\right) + \frac{4c_0t}{\varepsilon\rho^\beta}\right) \\ &= C\exp\left(-c\frac{r}{\rho} + c'\frac{t}{\rho^\beta}\right) \quad \text{in } \frac{1}{4}B_*. \end{aligned}$$

which is exactly (8.17) where $C = \frac{2}{2-\varepsilon}$, $c = \frac{1}{4}\ln\frac{2}{2-\varepsilon}$ and $c' = \frac{4c_0}{\varepsilon}$. □

Lemma 8.9. *If conditions (S_+^*) and (TJ_*) hold, then, for any $t > 0$, $\theta > 0$, any integer $k \ge 1$, and any ball $B_* := B_*(x_0, r)$ with $r > 4k\rho$,*

$$Q_t\mathbf{1}_{B_*^c} \le 1 - Q_t^{B_*}\mathbf{1}_{B_*} \le C(\theta,k)\left(\frac{t}{\rho^\beta}\right)^{\frac{\theta k}{\theta+\beta}} \quad in\ \frac{1}{4}B_*. \tag{8.19}$$

Here the constant $C(\theta, k) > 0$ is independent of t, B_*, ρ.

Proof. Let $\{Q_t^{(\overline{\rho})}\}$ be the heat semigroup associated with a regular, strongly $\overline{\rho}$-local truncated Dirichlet form $(\mathcal{E}^{(\overline{\rho})}, \mathcal{F}^{(\overline{\rho})})$ defined by (8.1) for $\overline{\rho} > 0$. Let $\{Q_t^{(\overline{\rho}),U}\}$ be the heat semigroup associated with Dirichlet form $(\mathcal{E}^{(\overline{\rho})}, \mathcal{F}^{(\overline{\rho})}(U))$ for any open set U.

Step 1. Let us prove that for any $\theta > 0$, $s > 0$, any ball $B_*(z, \overline{r})$ with $z \in M$ and $\overline{r} > 0$,

$$1 - P_s^{B_*(z,\overline{r})} \mathbf{1}_{B_*(z,\overline{r})} \le C(\theta) \left(\frac{s}{\overline{r}^\beta}\right)^{\frac{\theta}{\theta+\beta}} \quad \text{in } \frac{1}{4} B_*(z, \overline{r}). \tag{8.20}$$

If $\frac{s}{\overline{r}^\beta} \ge 1$, then (8.20) is trivial, since $1 - P_s^{B_*(z,\overline{r})} \mathbf{1}_{B_*(z,\overline{r})} \le 1$ in M. Hence, let us assume that

$$s < \overline{r}^\beta. \tag{8.21}$$

Let $\overline{\rho} > 0$ be a number to determined later. Applying (5.26) with $\Omega = B_*(z, \overline{r})$, $f = \mathbf{1}_{B_*(z,\overline{r})}$ and using (TJ_*), we obtain, for any $s > 0$,

$$1 - P_s^{B_*(z,\overline{r})} \mathbf{1}_{B_*(z,\overline{r})} \le 1 - Q_s^{(\overline{\rho}),B_*(z,\overline{r})} \mathbf{1}_{B_*(z,\overline{r})} + \frac{Cs}{\overline{\rho}^\beta} \left\| \mathbf{1}_{B_*(z,\overline{r})} \right\|_\infty . \tag{8.22}$$

Since (S_+^*) and (TJ_*) are satisfied, we obtain by combining (8.22) and (8.17), that

$$1 - P_s^{B_*(z,\overline{r})} \mathbf{1}_{B_*(z,\overline{r})} \le C \exp\left(-c\frac{\overline{r}}{\overline{\rho}} + c' \frac{s}{\overline{\rho}^\beta}\right) + \frac{Cs}{\overline{\rho}^\beta} \quad \text{in } \frac{1}{4} B_*(z, \overline{r}). \tag{8.23}$$

We will minimize the right hand side of the above inequality by choosing $\overline{\rho}$ that satisfies

$$s \le \overline{\rho}^\beta \quad \text{and} \quad \overline{\rho} \le \overline{r}. \tag{8.24}$$

Assuming that $\overline{\rho}$ satisfies (8.24) for the moment, applying (8.23) and using the elementary inequality

$$e^{-ca} \le c_2(\theta) a^{-\theta} \quad \text{for all } a > 0,$$

we obtain that, for any ball $B_*(z, \overline{r})$ with $z \in B_*$ and $\overline{r} < r$,

$$\begin{aligned} 1 - P_s^{B_*(z,\overline{r})} \mathbf{1}_{B_*(z,\overline{r})} &\le C \exp\left(-c\frac{\overline{r}}{\overline{\rho}} + c' \frac{s}{\overline{\rho}^\beta}\right) + \frac{Cs}{\overline{\rho}^\beta} \\ &\le Ce^{c'} \exp\left(-c\frac{\overline{r}}{\overline{\rho}}\right) + \frac{Cs}{\overline{\rho}^\beta} \\ &\le C(\theta)\left(\left(\frac{\overline{\rho}}{\overline{r}}\right)^\theta + \frac{s}{\overline{\rho}^\beta}\right) \quad \text{in } \frac{1}{4} B_*(z, \overline{r}). \end{aligned} \tag{8.25}$$

Now choose $\overline{\rho}$ such that $\left(\frac{\overline{\rho}}{\overline{r}}\right)^\theta = \frac{s}{\overline{\rho}^\beta}$, that is,

$$\overline{\rho} = \left(\overline{r}^\theta s\right)^{\frac{1}{\theta+\beta}} .$$

Note that the number $\overline{\rho}$ satisfies (8.24), since

$$\frac{\overline{\rho}^\beta}{s} = \left(\frac{\overline{r}^\beta}{s}\right)^{\frac{\theta}{\theta+\beta}} > 1 \quad \text{and} \quad \frac{\overline{\rho}}{\overline{r}} = \left(\frac{s}{\overline{r}^\beta}\right)^{\frac{1}{\theta+\beta}} < 1.$$

Therefore, substituting the above value of $\overline{\rho}$ into (8.25), we obtain that, for any $\theta > 0$, any ball $B_*(z, \overline{r})$ with $z \in M$ and $\overline{r} > 0$ and for any $s > 0$,

$$1 - P_s^{B_*(z,\overline{r})} \mathbf{1}_{B_*(z,\overline{r})} \le 2C(\theta) \left(\frac{\overline{\rho}}{\overline{r}}\right)^\theta = 2C(\theta) \left(\frac{s}{\overline{r}^\beta}\right)^{\frac{\theta}{\theta+\beta}} \quad \text{in } \frac{1}{4} B_*(z, \overline{r}),$$

thus proving (8.20).

Step 2. We turn to prove (8.19). It suffices to consider the case that

$$\frac{t}{\rho^\beta} < 1.$$

Using (TJ$_*$) and (5.26) with $\Omega = B_*(z,\rho)$ (where $z \in B_*$) and $f = \mathbf{1}_{B_*(z,\rho)}$, we obtain by (8.20) that, for any $\theta > 0$ and any $s > 0$,

$$1 - Q_s^{B_*(z,\rho)}\mathbf{1}_{B_*(z,\rho)} \le 1 - P_s^{B_*(z,\rho)}\mathbf{1}_{B_*(z,\rho)} + \frac{Cs}{\rho^\beta}\left\|\mathbf{1}_{B_*(z,\rho)}\right\|_\infty \le C(\theta)\left(\frac{s}{\rho^\beta}\right)^{\frac{\theta}{\theta+\beta}} + \frac{Cs}{\rho^\beta} \quad \text{in } \frac{1}{4}B_*(z,\rho).$$

Then, for any $\lambda > 0$ and $z \in M$,

$$\begin{aligned} 1 - \lambda R_\lambda^{B_*(z,\rho)}\mathbf{1}_{B_*(z,\rho)} &= \int_0^\infty \lambda e^{-\lambda s}(1 - Q_s^{B_*(z,\rho)}\mathbf{1}_{B_*(z,\rho)})\,ds \\ &\le \int_0^\infty \lambda e^{-\lambda s}\left(C(\theta)\left(\frac{s}{\rho^\beta}\right)^{\frac{\theta}{\theta+\beta}} + \frac{Cs}{\rho^\beta}\right)ds \\ &= \int_0^\infty e^{-s}\left(C(\theta)\left(\frac{s}{\lambda\rho^\beta}\right)^{\frac{\theta}{\theta+\beta}} + \frac{Cs}{\lambda\rho^\beta}\right)ds \\ &\le C'(\theta)\left(\lambda\rho^\beta\right)^{-\frac{\theta}{\theta+\beta}} + C\left(\lambda\rho^\beta\right)^{-1} =: c(\rho,\lambda) \quad \text{in } \frac{1}{4}B_*(z,\rho). \end{aligned}$$

Since $4k\rho < r$, it follows from Lemma 5.6 that

$$1 - \lambda R_\lambda^{B_*}\mathbf{1}_{B_*} \le c(\rho,\lambda)^k = \left(C'(\theta)\left(\lambda\rho^\beta\right)^{-\frac{\theta}{\theta+\beta}} + C\left(\lambda\rho^\beta\right)^{-1}\right)^k \quad \text{in } \frac{1}{4}B_*.$$

Moreover, setting $\lambda = t^{-1}$ in the above inequality and using (5.7) with $\Omega = B_*$, we obtain by the above inequality

$$\begin{aligned} 1 - Q_t^{B_*}\mathbf{1}_{B_*} &\le e^{\lambda t}\left(1 - \lambda R_\lambda^{B_*}\mathbf{1}_{B_*}\right) \le e^{\lambda t}\left(C'(\theta)\left(\lambda\rho^\beta\right)^{-\frac{\theta}{\theta+\beta}} + C\left(\lambda\rho^\beta\right)^{-1}\right)^k \\ &= e\left(C'(\theta)\left(\frac{t}{\rho^\beta}\right)^{\frac{\theta}{\theta+\beta}} + C\frac{t}{\rho^\beta}\right)^k \\ &= C(\theta,k)\left(\frac{t}{\rho^\beta}\right)^{\frac{\theta k}{\theta+\beta}} \quad \text{in } \frac{1}{4}B_*. \end{aligned}$$

where we also use the assumption that $t < \rho^\beta$ and the fact that $\frac{\theta}{\theta+\beta} < 1$. □

In the remainder of this subsection, we will obtain the relation of two heat kernels $p_t(x,y)$ and $q_t(x,y)$ in the norm of L^q outside ball B_* for any $1 < q \le \infty$.

Lemma 8.10. *Assume that* (VD$_*$), (DUE$_*$), (S$_+^*$), (TJ$_q^*$) *hold for some* $1 < q < \infty$. *Let* $q_t(x,y)$ *be the heat kernel of the* ρ*-local truncated Dirichlet form* $(\mathcal{E}^{(\rho)},\mathcal{F}^{(\rho)})$ *defined by* (8.1) *for any* $\rho \in \mathbb{Q}_+$. *Then, for any* $t > 0$ *and any ball* $B_* := B_*(x,r)$ *with* $r > 0$,

$$\|p_t(x,\cdot)\|_{L^q(B_*^c)} \le \|q_t(x,\cdot)\|_{L^q(B_*^c)} + \frac{Ct}{V_*(x,\rho)^{1/q'}\rho^\beta}\exp\left(\frac{c't}{\rho^\beta}\right), \tag{8.26}$$

where C, C' *are two positive constants independent of* t, x, B_*, ρ, *and* $q' = \frac{q}{q-1}$ *as before.*

Proof. Since conditions (TJ$_*$) (which follows from (TJ$_q^*$) by (7.10)) and (DUE$_*$) hold, we see by Lemma 8.3 and Remark 8.4 that, for any $\rho \in \mathbb{Q}_+$, the truncated Dirichlet form $(\mathcal{E}^{(\rho)},\mathcal{F}^{(\rho)})$ possesses a quasi-continuous heat kernel $q_t(x,y)$ on $(0,\infty)\times M \times M$.

Fix a ball $B_* := B_*(x,r)$ with $r > 0$ and fix $t > 0$. Without loss of generality, assume that

$$\|p_t(x,\cdot)\|_{L^q(B_*^c)} > 0,$$

otherwise, nothing is needed to prove. It suffices to consider the case $r < \overline{R}_*$, as otherwise, $B_*^c = \emptyset$ and $\|p_t(x,\cdot)\|_{L^q(B_*^c)} = 0$. The equality (8.14) yields that, for any $f \in L^2 \cap L^\infty$,

$$\int_M p_t(x,y)f(y)d\mu(y) = \int_M q_t(x,y)f(y)d\mu(y) + \int_0^t Q_sA^{(\rho)}P_{t-s}f(x)ds. \tag{8.27}$$

Let us use the operator $A^{(\rho)}$ defined in (5.20), that is, (under hypothesis (TJ_q^*))

$$A^{(\rho)}f(y) = 2\int_M (f(z) - f(y))J(y,z)\mathbf{1}_{\{d_*(y,z)\geq\rho\}}d\mu(z).$$

We need to estimate the term $Q_sA^{(\rho)}P_{t-s}f(x)$. To do this, let us introduce the function $h_{s,x} : M \mapsto \mathbb{R}_+$ by

$$h_{s,x}(z) := \int_M q_s(x,w)J_\rho(w,z)d\mu(w), \quad z \in M,$$

where $J_\rho(w,z) := J(w,z)\mathbf{1}_{\{d_*(w,z)\geq\rho\}}$. Then, for any $s \in (0,t)$ and any $0 \leq f \in L^2 \cap L^\infty$,

$$\begin{aligned}
Q_sA^{(\rho)}P_{t-s}f(x) &= \int_M q_s(x,y)\cdot A^{(\rho)}P_{t-s}f(y)d\mu(y)\\
&\leq 2\int_M q_s(x,y)\left(\int_M P_{t-s}f(z)J(y,z)\mathbf{1}_{\{d_*(y,z)\geq\rho\}}d\mu(z)\right)d\mu(y) \quad \text{(by definition (5.20))}\\
&= 2\int_M P_{t-s}f(z)\left(\int_M q_s(x,y)J(y,z)\mathbf{1}_{\{d_*(y,z)\geq\rho\}}d\mu(y)\right)d\mu(z)\\
&= 2(P_{t-s}f, h_{s,x}) = 2(f, P_{t-s}h_{s,x})\\
&\leq 2\|f\|_{q'}\|P_{t-s}h_{s,x}\|_q \quad \text{(by Hölder inequality)}\\
&\leq 2\|f\|_{q'}\|h_{s,x}\|_q \quad \text{(by contractivity of } P_t \text{ in } L^q\text{)},
\end{aligned}$$

where $q' := \frac{q}{q-1}$. Combining this and (8.27), we obtain that

$$\begin{aligned}
\int_M p_t(x,y)f(y)d\mu(y) &\leq \int_M q_t(x,y)f(y)d\mu(y) + \int_0^t Q_sA^{(\rho)}P_{t-s}f(x)ds\\
&\leq \int_M q_t(x,y)f(y)d\mu(y) + 2\|f\|_{q'}\int_0^t \|h_{s,x}\|_q ds. \qquad (8.28)
\end{aligned}$$

Let K be a bounded set under the metric d_*. Consider the function

$$f(\cdot) := p_t(x,\cdot)^{q-1}\mathbf{1}_{B_*^c\cap K}(\cdot).$$

Observe that $f \in L^\infty(M)$ because by (DUE_*) we have, for any $y \in M$,

$$\begin{aligned}
p_t(x,y) &= \int_M p_{t/2}(x,z)p_{t/2}(z,y)d\mu(z) \leq \|p_{t/2}(x,\cdot)\|_2\|p_{t/2}(\cdot,y)\|_2\\
&= \sqrt{p_{t/2}(x,x)p_{t/2}(y,y)} \leq \frac{C}{\sqrt{V_*(x,t^{1/\beta})V_*(y,t^{1/\beta})}},
\end{aligned}$$

which together with (VD_*) and the fact that f is supported in a bounded set K, yields that f is bounded. It follows that also $f \in L^1(M)$ and, hence, $f \in L^{q'}(M)$. Note that

$$\|f\|_{q'} = \left(\int_{B_*^c\cap K} p_t(x,y)^q d\mu(y)\right)^{1/q'} = \|p_t(x,\cdot)\|_{L^q(B_*^c\cap K)}^{q-1}. \qquad (8.29)$$

Applying (8.28) with the above function f, we obtain

$$\int_{B_*^c\cap K} p_t(x,y)^q d\mu(y) \leq \|f\|_{q'}\|q_t(x,\cdot)\|_{L^q(B_*^c)} + 2\|f\|_{q'}\int_0^t \|h_{s,x}\|_q ds.$$

Dividing by $\|f\|_{q'}$ on the both sides of the above inequality and using (8.29), we obtain

$$\|p_t(x,\cdot)\|_{L^q(B_*^c\cap K)} \leq \|q_t(x,\cdot)\|_{L^q(B_*^c)} + 2\int_0^t \|h_{s,x}\|_q ds.$$

Since the bounded set K is arbitrary, we conclude that

$$\|p_t(x,\cdot)\|_{L^q(B_*^c)} \leq \|q_t(x,\cdot)\|_{L^q(B_*^c)} + 2\int_0^t \|h_{s,x}\|_q ds. \qquad (8.30)$$

It remains to estimate the term $\|h_{s,x}\|_q$. By condition (TJ_q^*), we have

$$\|J_\rho(w,\cdot)\|_q \leq \frac{C}{V_*(w,\rho)^{1/q'}\rho^\beta}, \quad w \in M. \tag{8.31}$$

Defining measure ν by

$$d\nu(w) := q_s(x,w)d\mu(w),$$

and using Minkowski's inequality for integrals (cf. [18, on p.194]), we obtain that

$$\begin{aligned}
\|h_{s,x}\|_q &= \left(\int_M \left(\int_M J_\rho(w,z)\cdot q_s(x,w)d\mu(w)\right)^q d\mu(z)\right)^{1/q} \\
&= \left(\int_M \left(\int_M J_\rho(w,z)d\nu(w)\right)^q d\mu(z)\right)^{1/q} \leq \int_M \|J_\rho(w,\cdot)\|_q d\nu(w) \\
&\leq \int_M \frac{C}{V_*(w,\rho)^{1/q'}\rho^\beta} d\nu(w) \quad \text{(by (8.31))} \\
&= \frac{C}{\rho^\beta}\int_M \frac{q_s(x,w)}{V_*(w,\rho)^{1/q'}} d\mu(w).
\end{aligned} \tag{8.32}$$

Let us estimate the last integral. Let $B_0 := \emptyset$ and

$$B_k := B_*(x,k\rho) \ \text{ for } k \geq 1.$$

Then

$$\int_M \frac{q_s(x,w)}{V_*(w,\rho)^{1/q'}} d\mu(w) = \sum_{k=1}^\infty \int_{B_k\setminus B_{k-1}} \frac{q_s(x,w)}{V_*(w,\rho)^{1/q'}} d\mu(w) =: \sum_{k=1}^\infty I_k.$$

By (VD_*), we have, for any $k \geq 1$ and any $w \in B_k$,

$$\frac{1}{V_*(w,\rho)} = \frac{V_*(x,\rho)}{V_*(w,\rho)}\frac{1}{V_*(x,\rho)} \leq \frac{C(k+1)^{\alpha_*}}{V_*(x,\rho)}, \tag{8.33}$$

and then

$$I_1 \leq \frac{2^{\alpha_*}C}{V(x,\rho)}\int_M q_s(x,w)d\mu(w) \leq \frac{2^{\alpha_*}C}{V(x,\rho)}.$$

On the other hand, by Proposition 8.5 and (8.17) with t, r replaced by s, $(k-1)\rho$ respectively, we have, that for any $k \geq 2$

$$Q_s\mathbf{1}_{B_{k-1}^c}(x) \leq C\exp\left(-c\frac{(k-1)\rho}{\rho} + c'\frac{s}{\rho^\beta}\right) = C\exp\left(-c(k-1) + c'\frac{s}{\rho^\beta}\right).$$

Combining this, (8.33) and (VD_*), we obtain, for any $k \geq 2$,

$$\begin{aligned}
I_k &= \int_{B_k\setminus B_{k-1}} \frac{q_s(x,w)}{V_*(w,\rho)^{1/q'}} d\mu(w) \leq \left(\frac{C(k+1)^{\alpha_*}}{V_*(x,\rho)}\right)^{1/q'} \int_{B_k\setminus B_{k-1}} q_s(x,w)d\mu(w) \\
&\leq \left(\frac{C(k+1)^{\alpha_*}}{V_*(x,\rho)}\right)^{1/q'} Q_s\mathbf{1}_{B_{k-1}^c}(x) \leq \left(\frac{C(k+1)^{\alpha_*}}{V_*(x,\rho)}\right)^{1/q'} \cdot C\exp\left(-c(k-1) + c'\frac{s}{\rho^\beta}\right) \\
&\leq \frac{C'}{V_*(x,\rho)^{1/q'}} \exp\left(c'\frac{s}{\rho^\beta}\right)\cdot (k+1)^{\alpha_*/q'}\exp(-ck).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_M \frac{q_s(x,w)}{V_*(w,\rho)^{1/q'}} d\mu(w) &= \sum_{k=1}^\infty I_k \leq \frac{C'}{V_*(x,\rho)^{1/q'}}\exp\left(c'\frac{s}{\rho^\beta}\right)\sum_{k=1}^\infty (k+1)^{\alpha_*/q'}\exp(-ck) \\
&\leq \frac{C}{V_*(x,\rho)^{1/q'}}\exp\left(\frac{c's}{\rho^\beta}\right).
\end{aligned} \tag{8.34}$$

Combining this, (8.32) and (8.30), we obtain (8.26). □

The following lemma is an analogue of the above lemma for the case when $q = \infty$.

Lemma 8.11. *Assume that* (VD$_*$), (DUE$_*$), (S$^*_+$), (J$^*_\le$) = (TJ$^*_\infty$) *hold true. Let* $q_t(x,y)$ *be the heat kernel of the* ρ*-local truncated Dirichlet form* $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)})$ *defined by* (8.1) *for any* $\rho \in \mathbb{Q}_+$. *Then, for any* $t > 0$ *and any* $x, y, \in M$ *with* $x \neq y$,

$$p_t(x,y) \le q_t(x,y) + \frac{Ct}{V_*(x,\rho)\rho^\beta} \exp\left(\frac{c't}{\rho^\beta}\right). \tag{8.35}$$

Proof. Let $B_*(x,r)$ with $r > 0$. By (8.28), we see that for any $0 \le f \in L^1(B_*^c) \cap L^\infty$

$$\int_{B_*^c} p_t(x,y) f(y) d\mu(y) \le \int_{B_*^c} q_t(x,y) f(y) d\mu(y) + 2\|f\|_1 \int_0^t \|h_{s,x}\|_\infty ds,$$

from which, it follows that, for μ-almost all y in B_*^c,

$$p_t(x,y) \le q_t(x,y) + 2\int_0^t \|h_{s,x}\|_\infty ds. \tag{8.36}$$

Since $p_t(x,\cdot) \in C(\{F_k\})$ and $q_t(x,\cdot) \in C(\{F_k\})$ by Remark 8.4, we see that (8.36) holds true for all $y \in M$ with $d_*(x,y) > r$. It remains to estimate the term $\|h_{s,x}\|_\infty$. Indeed, by the similar arguments in the proof of Lemma 8.10, we can prove that (8.34) also holds true for $q' = 1$ but using (J$^*_\le$) = (TJ$^*_\infty$) instead of (TJ$^*_q$). Hence, we obtain

$$\begin{aligned}
\|h_{s,x}\|_\infty &\le \int_M q_s(x,w)\left(\operatorname*{esup}_{z\in M} \mathbf{1}_{\{d_*(w,z)>\rho\}} J(w,z)\right) d\mu(w) \\
&\le \int_M q_s(x,w) \frac{C}{V_*(w,\rho)\rho^\beta} d\mu(w) = \frac{C}{\rho^\beta}\int_M \frac{q_s(x,w)}{V_*(w,\rho)} d\mu(w) \\
&\le \frac{C}{V_*(x,\rho)\rho^\beta} \exp\left(\frac{c's}{\rho^\beta}\right).
\end{aligned}$$

Combining this and (8.36), we finish the proof. □

8.3. **Off-diagonal upper estimate of truncated heat kernel.** In this subsection, we derive off-diagonal upper bound of the heat kernel $q_t(x,y)$, for the truncated Dirichlet form $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)})$ where $\rho \in \mathbb{Q}_+$.

Lemma 8.12. *Let* $(\mathcal{E}^{(\rho)}, \mathcal{F}^{(\rho)})$ *be a regular Dirichlet form in* L^2 *with* $\mathcal{E}^{(\rho)}$ *defined by* (8.1) *for* $\rho \in \mathbb{Q}_+$. *If conditions* (VD$_*$), (DUE$_*$), (S$^*_+$), (TJ$_*$) *hold, then for any* $C_0 \ge 1$, $t < C_0(\overline{R}_*)^\beta$ *and any* $x, y \in M$,

$$q_t(x,y) \le \frac{C}{V_*(x,t^{1/\beta})} \exp\left(\frac{c't}{\rho^\beta}\right)\left(1 + \frac{\rho}{t^{1/\beta}}\right)^{\alpha_*} \exp\left(-c\frac{d_*(x,y)}{\rho}\right), \tag{8.37}$$

where the constants $C, c, c' > 0$ *are independent of* t, x, y, ρ. *Consequently,*

$$q_t(x,y) \le \frac{C}{V_*(x,t^{1/\beta})} \exp\left(\frac{c't}{\rho^\beta}\right)\left(1 + \frac{\rho}{t^{1/\beta}}\right)^{\alpha_*}. \tag{8.38}$$

Proof. Fix $C_0 \ge 1$, $t < C_0(\overline{R}_*)^\beta$ and $x, y \in M$. If $x = y$, then (8.37) follows directly from Lemma 8.3. In the sequel, assume $x \neq y$ and set $r := \frac{1}{3} d_*(x,y)$. We consider two cases.

Case 1: $\rho \ge r$. In this case, since $t/2 < 2^{-1}C_0(\overline{R}_*)^\beta$, we have by Lemma 8.3 that

$$q_t(x,y) = \sqrt{q_{t/2}(x,x) q_{t/2}(y,y)} \le \frac{C}{\sqrt{V_*(x,(t/2)^{1/\beta}) V_*(y,(t/2)^{1/\beta})}} \exp\left(\frac{c't}{\rho^\beta}\right).$$

Moreover, by using (VD$_*$) and the fact that $\rho \ge r$, we have

$$\frac{V_*(x,t^{1/\beta})}{V_*(y,(t/2)^{1/\beta})} \le C\left(\frac{d_*(x,y) + t^{1/\beta}}{(t/2)^{1/\beta}}\right)^{\alpha_*} \le C\left(\frac{3\rho + t^{1/\beta}}{(t/2)^{1/\beta}}\right)^{\alpha_*} \le C'\left(1 + \frac{\rho}{t^{1/\beta}}\right)^{\alpha_*}.$$

Combining the above two inequalities and using the fact that $\frac{d_*(x,y)}{\rho} \le 3$, we obtain (8.37).

Case 2: $\rho < r$. In this case, consider disjoint balls

$$U := B_*(x,r), \quad V := B_*(y,r).$$

By Lemma 8.3, we have, for any $w', z' \in M$ and $s < C_0(\overline{R}_*)^\beta$,

$$q_{2s}(w',z') = \sqrt{q_s(w',w')q_s(z',z')} \leq \frac{C}{\sqrt{V_*(w',s^{1/\beta})V_*(z',s^{1/\beta})}} \exp\left(\frac{c's}{\rho^\beta}\right).$$

In particular, the function $q_s(w',z')$ is locally bounded in $\mathbb{R}_+ \times M \times M$. Hence, by Corollary 11.5 in Appendix, with $s = t$ and $\Omega = M$, we see that, for μ-almost all $w \in U$, $z \in V$,

$$q_{2t}(w,z) \leq \left(1 - Q_t^U \mathbf{1}_U(w)\right) \sup_{t<t'\leq 2t} \|q_{t'}(\cdot,z)\|_{L^\infty(U_\rho)} + \left(1 - Q_t^V \mathbf{1}_V(z)\right) \sup_{t<t'\leq 2t} \|q_{t'}(\cdot,w)\|_{L^\infty(V_\rho)}.$$

On the other hand, applying (8.17) with B_* replaced by U, we have

$$1 - Q_t^U \mathbf{1}_U(w) \leq C \exp\left(-c\frac{r}{\rho} + c'\frac{t}{\rho^\beta}\right) \quad \mu\text{-a.a. } w \in \frac{1}{4}U.$$

Similarly, applying (8.17) with B_* replaced by V, we have

$$1 - Q_t^V \mathbf{1}_V(z) \leq C \exp\left(-c\frac{r}{\rho} + c'\frac{t}{\rho^\beta}\right) \quad \mu\text{-a.a. } z \in \frac{1}{4}V.$$

Therefore, combining the above three inequalities, we obtain that for μ-almost all $w \in \frac{1}{4}U$, $z \in \frac{1}{4}V$,

$$q_{2t}(w,z) \leq 2C \exp\left(-c\frac{r}{\rho} + c'\frac{t}{\rho^\beta}\right) \sup_{t<t'\leq 2t} \operatorname*{esup}_{w'\in U_\rho, z'\in V_\rho} q_{t'}(w',z'). \tag{8.39}$$

Let us estimate the term $\sup_{t<t'\leq 2t} \operatorname{esup}_{w'\in U_\rho, z'\in V_\rho} q_{t'}(w',z')$. Indeed, since $\rho < r$, we have for any $w' \in U_\rho$ and $z' \in V_\rho$,

$$d_*(x,w') \leq r + \rho < 2r, \quad \text{and} \quad d_*(y,z') \leq r + \rho < 2r.$$

Moreover, since $t < C_0(\overline{R}_*)^\beta$, we have for any $t' \in (t, 2t]$,

$$t' \leq 2t < 2C_0(\overline{R}_*)^\beta.$$

Therefore, by Lemma 8.3, we have

$$\begin{aligned} \sup_{t<t'\leq 2t} \operatorname*{esup}_{w'\in U_\rho, z'\in V_\rho} q_{t'}(w',z') &\leq \operatorname*{esup}_{w'\in U_\rho, z'\in V_\rho} \frac{C}{\sqrt{V_*(w',t^{1/\beta})V_*(z',t^{1/\beta})}} \exp\left(\frac{c't}{\rho^\beta}\right) \\ &\leq \frac{C'}{V_*(x,t^{1/\beta})} \exp\left(\frac{c't}{\rho^\beta}\right)\left(1 + \frac{r}{t^{1/\beta}}\right)^{\alpha_*}, \end{aligned} \tag{8.40}$$

where we have used the fact that, using the doubling property (VD$_*$), for any points $w' \in U_\rho$, $z' \in V_\rho$

$$\begin{aligned} \frac{V_*(x,t^{1/\beta})}{V_*(w',t^{1/\beta})} &\leq C\left(\frac{d_*(x,w') + t^{1/\beta}}{t^{1/\beta}}\right)^{\alpha_*} \leq C'\left(1 + \frac{r+\rho}{t^{1/\beta}}\right)^{\alpha_*} \leq C'\left(1 + \frac{2r}{t^{1/\beta}}\right)^{\alpha_*}, \\ \frac{V_*(x,t^{1/\beta})}{V_*(z',t^{1/\beta})} &\leq C\left(\frac{d_*(x,z') + t^{1/\beta}}{t^{1/\beta}}\right)^{\alpha_*} \leq C\left(\frac{d_*(x,y) + d_*(y,z') + t^{1/\beta}}{t^{1/\beta}}\right)^{\alpha_*} \\ &\leq C'\left(1 + \frac{4r+\rho}{t^{1/\beta}}\right)^{\alpha_*} \leq C'\left(1 + \frac{5r}{t^{1/\beta}}\right)^{\alpha_*}. \end{aligned}$$

Now, combining (8.39) and (8.40), we have

$$\begin{aligned} q_{2t}(w,z) &\leq 2C \exp\left(-c\frac{r}{\rho} + c'\frac{t}{\rho^\beta}\right) \sup_{t<t'\leq 2t} \operatorname*{esup}_{w'\in U_\rho, z'\in V_\rho} q_{t'}(w',z') \\ &\leq \frac{C'}{V_*(x,t^{1/\beta})} \exp\left(-c\frac{r}{\rho} + 2c'\frac{t}{\rho^\beta}\right)\left(1 + \frac{r}{t^{1/\beta}}\right)^{\alpha_*}. \end{aligned}$$

Moreover, by Proposition 8.5, the above inequality holds true for all points $w \in \frac{1}{4}U$ and $z \in \frac{1}{4}V$. In particular, for $(w,z) = (x,y)$, we obtain

$$q_{2t}(x,y) \leq \frac{C}{V_*(x,t^{1/\beta})} \exp\left(\frac{2c't}{\rho^\beta}\right) \exp\left(-c\frac{r}{\rho}\right)\left(1 + \frac{r}{t^{1/\beta}}\right)^{\alpha_*}.$$

Finally, let us observe that

$$\exp\left(-c\frac{r}{\rho}\right)\left(1+\frac{r}{t^{1/\beta}}\right)^{\alpha_*} \leq C\exp\left(-\frac{c}{2}\frac{r}{\rho}\right)\left(1+\frac{\rho}{t^{1/\beta}}\right)^{\alpha_*},$$

which follows from the elementary inequality $\exp(\lambda) \geq c(1+\lambda)^{\alpha_*}$. Combining the above two inequalities, we conclude that

$$q_{2t}(x,y) \leq \frac{C'}{V_*(x,t^{1/\beta})}\exp\left(\frac{2c't}{\rho^\beta}\right)\exp\left(-\frac{c}{2}\frac{r}{\rho}\right)\left(1+\frac{\rho}{t^{1/\beta}}\right)^{\alpha_*},$$

thus proving (8.37) by renaming t by $\frac{t}{2}$ and substituting $r = \frac{d_*(x,y)}{3}$. □

9. Heat kernel upper bound

In this section, we will prove the upper estimate of $\|p_t(x,\cdot)\|_{L^q(B^c)}$ for any $1 \leq q \leq \infty$. To do this, we first obtain the upper estimate for $\|p_t(x,\cdot)\|_{L^q(B_*^c)}$ under metric d_*, and then translate to the original metric d.

9.1. Tail estimate in L^q under new metric.

Let us first introduce conditions (TP$_*$) and (TP$_q^*$).

Definition 9.1 (Condition (TP$_*$)). We say that condition (TP$_*$) is satisfied if

$$P_t\mathbf{1}_{B_*^c} \leq \frac{Ct}{r^\beta} \quad \text{in } \frac{1}{4}B_* \tag{9.1}$$

for any ball $B_* := B_*(x,r)$ of radius $r \in (0,\overline{R}_*)$ and any $t > 0$, where $C > 0$ is a constant independent of t, B_*.

Definition 9.2 (Condition (TP$_q^*$)). For a number $1 \leq q < \infty$, we say that condition (TP$_q^*$) is satisfied if the pointwise heat kernel $p_t(x,y)$ exists in the sense of Definition 2.6, and, for any $C_0 \geq 1$, there exists $C > 0$ such that, for any ball $B_* = B_*(x,r)$ of radius $r \in (0,\overline{R}_*)$ and any $t < C_0(\overline{R}_*)^\beta$,

$$\|p_t(x,\cdot)\|_{L^q(B_*^c)} \leq C\left(\frac{1}{V_*(x,t^{1/\beta})^{1/q'}} \wedge \frac{t}{V_*(x,r)^{1/q'}r^\beta}\right), \tag{9.2}$$

where $q' = \frac{q}{q-1}$ as before.

We start with the following lemma.

Lemma 9.3. *The following implication is true:*

$$(\text{VD}_*) + (\text{S}_+^*) + (\text{TJ}_*) \Rightarrow (\text{TP}_*).$$

Proof. Let $B_* := B_*(x,r)$ with $r \in (0,\overline{R}_*)$ and let $t > 0$. We need to show (9.1). We can assume that $t < r^\beta$, because otherwise (9.1) is trivial since $P_t\mathbf{1}_{B_*^c} \leq 1$.

Using (5.26) with $\Omega = M$, $f = \mathbf{1}_{B_*^c}$ and (8.19), we have, that for any integer $k \geq 1$, any $0 < \rho < \frac{r}{4k} < \frac{\overline{R}_*}{4k}$, and any $\theta, t > 0$,

$$P_t\mathbf{1}_{B_*^c} \leq Q_t\mathbf{1}_{B_*^c} + \frac{Ct}{\rho^\beta}\left\|\mathbf{1}_{B_*^c}\right\|_\infty \leq C(\theta,k)\left(\frac{t}{\rho^\beta}\right)^{\frac{\theta k}{\theta+\beta}} + \frac{Ct}{\rho^\beta} \quad \text{in } \frac{1}{4}B_*.$$

Setting here $\theta = \beta$, $k = 3$ and $\rho = \frac{r}{5k} = \frac{r}{15}$, we obtain

$$P_t\mathbf{1}_{B_*^c} \leq C(\beta)\left(\frac{15^\beta t}{r^\beta}\right)^{3/2} + \frac{15^\beta Ct}{r^\beta} \leq C(\beta)\left(\frac{15^\beta t}{r^\beta}\right)^{3/2} + \frac{15^\beta Ct}{r^\beta} \leq C'\frac{t}{r^\beta} + \frac{15^\beta Ct}{r^\beta} = C\frac{t}{r^\beta} \quad \text{in } \frac{1}{4}B_*,$$

which proves (9.1). □

Let us prove a similar implication for (TP$_q^*$) for $1 \leq q < \infty$.

Lemma 9.4. *For any* $1 \leq q < \infty$, *we have*

$$(\text{VD}_*) + (\text{DUE}_*) + (\text{S}_+^*) + (\text{TJ}_q^*) \Rightarrow (\text{TP}_q^*).$$

Proof. Let $B_* := B_*(x, r)$ with $r \in (0, \overline{R}_*)$ and let $t < C_0(\overline{R}_*)^\beta$ with $C_0 \geq 1$. By remark 8.4(i), the pointwise heat kernel $p_t(x, y)$ exists in the sense of Definition 2.6. So, we need only to show the inequality (9.2).

Let us first prove that

$$\|p_t(x,\cdot)\|_{L^q(B_*^c)} \leq \frac{C}{V_*(x, t^{1/\beta})^{1/q'}}, \tag{9.3}$$

for some positive constant C independent of t, x. It suffices to consider the case when $1 < q < \infty$ since $\|p_t(x,\cdot)\|_{L^1(B_*^c)} \leq 1$.

To do this, we need to estimate the term $\|q_t(x,\cdot)\|_{L^q(B_*^c)}$ in (8.26) for $\rho \in \mathbb{Q}_+$. Indeed, we have by (8.38) in Lemma 8.12 that

$$\begin{aligned}
\|q_t(x,\cdot)\|_{L^q(B_*^c)} &= \left(\int_{B_*^c} q_t(x,y)^{q-1} \cdot q_t(x,y) d\mu(y)\right)^{\frac{1}{q}} \\
&\leq \sup_{y\in M} q_t(x,y)^{\frac{q-1}{q}} \left(\int_M q_t(x,y) d\mu(y)\right)^{\frac{1}{q}} \leq \sup_{y\in M} q_t(x,y)^{\frac{q-1}{q}} \\
&\leq \left(\frac{C}{V_*(x, t^{1/\beta})} \exp\left(\frac{Ct}{\rho^\beta}\right)\left(1 + \frac{\rho}{t^{1/\beta}}\right)^{\alpha_*}\right)^{\frac{q-1}{q}}.
\end{aligned}$$

Therefore, it follows from (8.26) that

$$\begin{aligned}
\|p_t(x,\cdot)\|_{L^q(B_*^c)} &\leq \|q_t(x,\cdot)\|_{L^q(B_*^c)} + \frac{Ct}{V_*(x,\rho)^{1/q'}\rho^\beta} \exp\left(\frac{c't}{\rho^\beta}\right) \\
&\leq \frac{C}{V_*(x, t^{1/\beta})^{1/q'}} \exp\left(\frac{Ct}{\rho^\beta}\right)\left(1 + \frac{\rho}{t^{1/\beta}}\right)^{\alpha_*/q'} + \frac{Ct}{V_*(x,\rho)^{1/q'}\rho^\beta} \exp\left(\frac{C't}{\rho^\beta}\right).
\end{aligned}$$

Choose a rational ρ close to $t^{1/\beta}$, we obtain (9.3).

Let us next prove that

$$\|p_t(x,\cdot)\|_{L^q(B_*(x,r)^c)} \leq \frac{Ct}{V_*(x,r)^{1/q'} r^\beta} \tag{9.4}$$

for some C independent of t, r, x. By (9.3), it suffices to consider the case when

$$r^\beta > t. \tag{9.5}$$

We also assume that

$$\rho \in (0, r] \cap \mathbb{Q}_+. \tag{9.6}$$

By (8.37) and (VD_*), we have that, for any $t < C_0(\overline{R}_*)^\beta$ and any $x \in M$,

$$\begin{aligned}
\sup_{y\in B_*^c} q_t(x,y) &\leq \sup_{y\in B_*^c} \frac{C}{V_*(x, t^{1/\beta})} \exp\left(\frac{c't}{\rho^\beta}\right)\left(1 + \frac{\rho}{t^{1/\beta}}\right)^{\alpha_*} \exp\left(-c\frac{d_*(x,y)}{\rho}\right) \\
&\leq \frac{C}{V_*(x,r)}\left(\frac{r}{t^{1/\beta}}\right)^{\alpha_*} \exp\left(\frac{c't}{\rho^\beta}\right)\left(1 + \frac{\rho}{t^{1/\beta}}\right)^{\alpha_*} \exp\left(-c\frac{r}{\rho}\right) \\
&\leq \frac{C}{V_*(x,r)}\left(\frac{r}{t^{1/\beta}}\right)^{\alpha_*} \exp\left(\frac{c't}{\rho^\beta}\right) \exp\left(-c\frac{r}{\rho}\right) \\
&\quad + \frac{C}{V_*(x,r)}\left(\frac{r}{t^{1/\beta}}\right)^{\alpha_*} \exp\left(\frac{c't}{\rho^\beta}\right)\left(\frac{\rho}{t^{1/\beta}}\right)^{\alpha_*} \exp\left(-c\frac{r}{\rho}\right) \\
&\leq \frac{C'}{V_*(x,r)}\left(\frac{r}{t^{1/\beta}}\right)^{2\alpha_*} \exp\left(\frac{c't}{\rho^\beta}\right)
\end{aligned} \tag{9.7}$$

where we have used the fact in the last line that

$$\left(\frac{\rho}{t^{1/\beta}}\right)^{\alpha_*} \exp\left(-c\frac{r}{\rho}\right) = \left(\frac{r}{t^{1/\beta}}\right)^{\alpha_*}\left(\frac{r}{\rho}\right)^{-\alpha_*} \exp\left(-c\frac{r}{\rho}\right) \leq C\left(\frac{r}{t^{1/\beta}}\right)^{\alpha_*}.$$

On the other hand, we have by (8.19), Remark 8.4 and Proposition 8.5 that, for any $\theta > 0$ and for any integer $k \geq 1$ with $r > 4k\rho$ (noting that (9.6) is clearly satisfied since $\frac{r}{\rho} > 4k \geq 4$),

$$\int_{B_*^c} q_t(x,y)d\mu(y) = Q_t \mathbf{1}_{B_*^c}(x) \leq C(\theta)\left(\frac{t}{\rho^\beta}\right)^{\frac{\theta k}{\theta+\beta}}. \tag{9.8}$$

Consequently, combining (9.7) and (9.8), it follows that, for any $\theta > 0$ and for any integer $k \geq 1$ with $r > 4k\rho$,

$$\begin{aligned}
\|q_t(x,\cdot)\|_{L^q(B_*^c)} &= \left(\int_{B_*^c} q_t(x,y)^{q-1}\cdot q_t(x,y)d\mu(y)\right)^{1/q} \leq \sup_{y\in B_*^c} q_t(x,y)^{\frac{q-1}{q}}\left(\int_{B_*^c} q_t(x,y)d\mu(y)\right)^{1/q} \\
&\leq \left(\frac{C'}{V_*(x,r)}\left(\frac{r}{t^{1/\beta}}\right)^{2\alpha_*}\exp\left(\frac{c't}{\rho^\beta}\right)\right)^{1/q'}\cdot\left(C(\theta)\left(\frac{t}{\rho^\beta}\right)^{\frac{\theta k}{\theta+\beta}}\right)^{1/q} \\
&\leq \frac{C'(\theta)}{V_*(x,r)^{1/q'}}\exp\left(\frac{c't}{q'\rho^\beta}\right)\left(\frac{r}{t^{1/\beta}}\right)^{2\alpha_*/q'}\left(\frac{t}{\rho^\beta}\right)^{\frac{\theta k}{(\theta+\beta)q}} \\
&= \frac{C'(\theta)}{V_*(x,r)^{1/q'}}\exp\left(\frac{c't}{q'\rho^\beta}\right)\left(\frac{r}{\rho}\right)^{2\alpha_*/q'}\left(\frac{t}{\rho^\beta}\right)^{\frac{\theta k}{(\theta+\beta)q}-\frac{2\alpha_*}{\beta q'}}.
\end{aligned}$$

Therefore, substituting the above inequality into (8.26), we obtain

$$\begin{aligned}
\|p_t(x,\cdot)\|_{L^q(B_*^c)} &\leq \|q_t(x,\cdot)\|_{L^q(B_*^c)} + \frac{Ct}{V_*(x,\rho)^{1/q'}\rho^\beta}\exp\left(\frac{C't}{\rho^\beta}\right) \\
&\leq \frac{C'(\theta)}{V_*(x,r)^{1/q'}}\exp\left(\frac{c't}{q'\rho^\beta}\right)\left(\frac{r}{\rho}\right)^{2\alpha_*/q'}\left(\frac{t}{\rho^\beta}\right)^{\frac{\theta k}{(\theta+\beta)q}-\frac{2\alpha_*}{\beta q'}} \\
&\quad + \frac{Ct}{V_*(x,\rho)^{1/q'}\rho^\beta}\exp\left(\frac{c't}{\rho^\beta}\right).
\end{aligned} \tag{9.9}$$

Now let $\theta = \beta$ and choose the integer $k \geq 1$ such that

$$\frac{\theta k}{(\theta+\beta)q} - \frac{2\alpha_*}{\beta q'} = \frac{k}{2q} - \frac{2\alpha_*}{\beta q'} \geq 1,$$

for example, let

$$k = 1 + \left\lfloor 2q\left(1 + \frac{2\alpha_*}{\beta q'}\right)\right\rfloor.$$

Choosing the rational ρ close to $\frac{r}{4k}$ and using (VD$_*$) and (9.5), we obtain

$$\|p_t(x,\cdot)\|_{L^q(B_*^c)} \leq \frac{C}{V_*(x,r)^{1/q'}}\left(\left(\frac{t}{r^\beta}\right)^{\frac{\theta k}{(\theta+\beta)q}-\frac{2\alpha_*}{\beta q'}} + \frac{t}{r^\beta}\right) \leq \frac{C't}{V_*(x,r)^{1/q'}r^\beta},$$

thus proving (9.4).

Finally, condition (TP$_q^*$) follows directly from (9.3) and (9.4). □

Let us define condition (TP$_q^*$) for $q = \infty$.

Definition 9.5 (Condition (TP$_\infty^*$))**.** We say that condition (TP$_\infty^*$) is satisfied if the pointwise heat kernel $p_t(x,y)$ exists in the sense of Definition 2.6, and for any $C_0 \geq 1$, there exists $C > 0$ such that for any $x,y \in M$ and any $t < C_0(\overline{R}_*)^\beta$,

$$p_t(x,y) \leq C\left(\frac{1}{V_*(x,t^{1/\beta})} \wedge \frac{t}{V_*(x,d_*(x,y))d_*(x,y)^\beta}\right). \tag{9.10}$$

Lemma 9.6. *For $q = \infty$, we have*

$$(\mathrm{VD}_*) + (\mathrm{DUE}_*) + (\mathrm{S}_+^*) + (\mathrm{TJ}_\infty^*) \Rightarrow (\mathrm{TP}_\infty^*).$$

Proof. Fix $x, y \in M$ and $t < C_0(\overline{R}_*)^\beta$ with $C_0 \geq 1$.

Let us first prove that $p_t(x,y) \leq \frac{C}{V_*(x,t^{1/\beta})}$. Indeed, by (8.35) and (8.38), we have for any $\rho \in \mathbb{Q}_+$,

$$p_t(x,y) \leq \frac{C}{V_*(x,t^{1/\beta})} \exp\left(\frac{c't}{\rho^\beta}\right)\left(1+\frac{\rho}{t^{1/\beta}}\right)^{\alpha_*} + \frac{C}{V_*(x,t^{1/\beta})} \exp\left(\frac{c't}{\rho^\beta}\right).$$

Taking here ρ close to $t^{1/\beta}$, we obtain

$$\begin{aligned} p_t(x,y) &\leq \frac{C'}{V_*(x,t^{1/\beta})} \exp\left(\frac{c't}{t}\right)\left(1+\frac{t^{1/\beta}}{t^{1/\beta}}\right)^{\alpha_*} + \frac{Ct}{V_*(x,t^{1/\beta})t} \exp\left(\frac{c't}{t}\right) \\ &= \frac{C}{V_*(x,t^{1/\beta})}. \end{aligned}$$

It remains to show that

$$p_t(x,y) \leq \frac{Ct}{V_*(x,d_*(x,y))d_*(x,y)^\beta}. \tag{9.11}$$

It suffices to consider the case when

$$d_*(x,y)^\beta > t.$$

By (8.35), we need to estimate $q_t(x,y)$. Let

$$r := 2d_*(x,y) > 2t^{1/\beta},$$

so that $M \subset B_*(x,r)^c \cup B(y,r)^c$.

By semigroup property of $q_t(x,y)$, we have

$$\begin{aligned} q_t(x,y) &= \int_M q_{t/2}(x,z)q_{t/2}(z,y)d\mu(z) \\ &\leq \left(\int_{B_*(x,r)^c} + \int_{B_*(y,r)^c}\right) q_{t/2}(x,z)q_{t/2}(z,y)d\mu(z) \\ &\leq \sup_{z\in M} q_{t/2}(z,y) \int_{B_*(x,r)^c} q_{t/2}(x,z)d\mu(z) + \sup_{z\in M} q_{t/2}(x,z) \int_{B_*(y,r)^c} q_{t/2}(z,y)d\mu(z). \end{aligned}$$

We need to estimate the terms on the right hand side of the above inequality. By (8.38), (VD) and the assumption that $r > 2t^{1/\beta}$, we obtain

$$\begin{aligned} \sup_{z\in M} q_{t/2}(z,y) &\leq \frac{C}{V_*(y,(t/2)^{1/\beta})} \exp\left(\frac{c't}{2\rho^\beta}\right)\left(1+\frac{\rho}{(t/2)^{1/\beta}}\right)^{\alpha_*} \\ &\leq \frac{C}{V_*(x,r)} \frac{V_*(x,r)}{V_*(y,(t/2)^{1/\beta})} \exp\left(\frac{c't}{2\rho^\beta}\right)\left(1+\frac{\rho}{t^{1/\beta}}\right)^{\alpha_*} \\ &\leq \frac{C'}{V_*(x,r)} \left(\frac{r}{t^{1/\beta}}\right)^{\alpha_*} \exp\left(\frac{c't}{2\rho^\beta}\right)\left(1+\frac{\rho}{t^{1/\beta}}\right)^{\alpha_*}. \end{aligned}$$

Similarly, we obtain the same estimate of $\sup_{z\in M} q_{t/2}(x,z)$:

$$\sup_{z\in M} q_{t/2}(x,z) \leq \frac{C'}{V_*(x,r)} \left(\frac{r}{t^{1/\beta}}\right)^{\alpha_*} \exp\left(\frac{c't}{2\rho^\beta}\right)\left(1+\frac{\rho}{t^{1/\beta}}\right)^{\alpha_*}.$$

On the other hand, by (8.19), Remark 8.4 and Proposition 8.5, we obtain for any $\theta > 0$, any integer $k \geq 1$ and $\rho \in \mathbb{Q}_+$ with $4k\rho < r$,

$$\int_{B_*(x,r)^c} q_{t/2}(x,z)d\mu(z) = Q_{t/2}\mathbf{1}_{B_*(x,r)^c}(x) \leq C(\theta,k)\left(\frac{t}{2\rho^\beta}\right)^{\frac{\theta k}{\theta+\beta}},$$

and

$$\int_{B_*(y,r)^c} q_{t/2}(z,y)d\mu(z) = Q_{t/2}\mathbf{1}_{B_*(y,r)^c}(y) \leq C(\theta,k)\left(\frac{t}{2\rho^\beta}\right)^{\frac{\theta k}{\theta+\beta}},$$

where $C(\theta,k)$ is a constant independent of t, x, y, ρ.

Finally, combining the above five inequalities, we obtain that, for any $\theta > 0$, $k \geq 1$ and $\rho \in (0, \frac{r}{4k}) \cap \mathbb{Q}_+$,

$$q_t(x,y) \leq \frac{2C(\theta,k)}{V_*(x,r)} \left(\frac{r}{t^{1/\beta}}\right)^{\alpha_*} \exp\left(\frac{c't}{\rho^\beta}\right)\left(1+\frac{\rho}{t^{1/\beta}}\right)^{\alpha_*}\left(\frac{t}{\rho^\beta}\right)^{\frac{\theta k}{\theta+\beta}},$$

Substituting the above inequality into (8.35), we obtain that

$$\begin{aligned} p_t(x,y) &\leq q_t(x,y) + \frac{Ct}{V_*(x,\rho)\rho^\beta} \exp\left(\frac{c't}{\rho^\beta}\right) \\ &\leq \frac{2C(\theta,k)}{V_*(x,r)} \left(\frac{r}{t^{1/\beta}}\right)^{\alpha_*} \exp\left(\frac{c't}{\rho^\beta}\right)\left(1+\frac{\rho}{t^{1/\beta}}\right)^{\alpha_*}\left(\frac{t}{\rho^\beta}\right)^{\frac{\theta k}{\theta+\beta}} \\ &\quad + \frac{Ct}{V_*(x,\rho)\rho^\beta} \exp\left(\frac{c't}{\rho^\beta}\right). \end{aligned} \tag{9.12}$$

Set in the above inequality $\theta = \beta$ and take

$$k = 1 + \left\lfloor 2\left(1+\frac{2\alpha_*}{\beta}\right)\right\rfloor$$

so that

$$\frac{k}{2} - \frac{2\alpha_*}{\beta} \geq 1. \tag{9.13}$$

Passing to the limit in (9.12) as the rational ρ increases to $\frac{r}{5k}$, we obtain by (VD$_*$) that

$$\begin{aligned} p_t(x,y) &\leq \frac{C_1(\beta,k)}{V_*(x,r)} \left(\frac{r}{t^{1/\beta}}\right)^{\alpha_*} \exp\left(\frac{c't}{(r/5k)^\beta}\right)\left(1+\frac{r/5k}{t^{1/\beta}}\right)^{\alpha_*}\left(\frac{t}{(r/5k)^\beta}\right)^{\frac{k}{2}} \\ &\quad + \frac{Ct}{V_*(x,r/5k)(r/5k)^\beta} \exp\left(\frac{c't}{(r/5k)^\beta}\right) \\ &\leq \frac{C(k)}{V_*(x,r)}\left(\frac{t}{r^\beta}\right)^{\frac{k}{2}-\frac{2\alpha_*}{\beta}} + \frac{C'(k)}{V_*(x,r)}\frac{t}{r^\beta} \\ &\leq \frac{C(k)t}{V_*(x,r)r^\beta}, \end{aligned}$$

where we have used the facts that $\frac{t}{r^\beta} < 1$ and then

$$\left(\frac{t}{r^\beta}\right)^{\frac{k}{2}-\frac{2\alpha_*}{\beta}} \leq \frac{t}{r^\beta} \quad \text{(by (9.13))}.$$

Thus we have proved (9.11). □

9.2. **Tail estimate in L^q under the original metric.**

Lemma 9.7. *Assume that* (VD) *is satisfied. For any* $1 \leq q \leq \infty$, *we have*

$$(\mathrm{TP}^*_q) \Leftrightarrow (\mathrm{TP}_q), \tag{9.14}$$

$$(\mathrm{TP}_*) \Leftrightarrow (\mathrm{TP}). \tag{9.15}$$

Proof. For the equivalence (9.14), it suffices to prove the implication (TP$^*_q$) ⇒ (TP$_q$) since the other direction can be proved similarly.

Indeed, assume that condition (TP$^*_q$) is true. Fix $x \in M$, $R \in (0, \overline{R})$ and $t < W(x, \overline{R})$. It follows from (6.13) that $t < C_0(\overline{R}_*)^\beta$ for some $C_0 > 0$. Let

$$r := L^{-1}F(x,R) \text{ so that } W(x,R) = F(x,R)^\beta = (Lr)^\beta. \tag{9.16}$$

By (6.10), we have

$$B(x, L_0^{-1}R) \subset B_*(x,r) \subset B(x,R) \tag{9.17}$$

so that

$$V(x,R) = \mu(B(x,R)) \geq \mu(B_*(x,r)) = V_*(x,r) \geq V(x, L_0^{-1}R).$$

Let us assume that $1 \le q < \infty$ since the case when $q = \infty$ can be proved similarly. Using condition (TP_q^*), it follows from above that

$$\|p_t(x,\cdot)\|_{L^q(B(x,R)^c)} \le \|p_t(x,\cdot)\|_{L^q(B_*(x,r)^c)} \le C\left(\frac{1}{V_*(x,t^{1/\beta})^{1/q'}} \wedge \frac{t}{V_*(x,r)^{1/q'}r^\beta}\right)$$
$$\le C\left(\frac{1}{V_*(x,t^{1/\beta})^{1/q'}} \wedge \frac{t}{V(x,L_0^{-1}R)^{1/q'}L^{-\beta}W(x,R)}\right).$$

On the other hand, using the second inclusion in (6.9) with $r = t^{1/\beta}$, we have by (6.7)

$$V_*(x,t^{1/\beta}) \ge V(x,F^{-1}(x,L^{-1}t^{1/\beta})) = V(x,W^{-1}(x,L^{-\beta}t)).$$

Therefore, combining the above two inequalities and using (VD) and (2.8), we conclude that

$$\|p_t(x,\cdot)\|_{L^q(B(x,R)^c)} \le C'\left(\frac{1}{V(x,W^{-1}(x,t))^{1/q'}} \wedge \frac{t}{V(x,R)^{1/q'}W(x,R)}\right),$$

thus showing that condition (TP_q) is true.

For the equivalence (9.15), it suffices to prove the implication (TP_*) $\Rightarrow$ (TP) since the opposite direction can be handled similarly.

Assuming that condition (TP_*) is true and using (9.17), (9.16), we obtain

$$P_t\mathbf{1}_{B(x,R)^c} \le P_t\mathbf{1}_{B_*(x,r)^c} \le \frac{Ct}{r^\beta} = \frac{Ct}{L^{-\beta}W(x,R)} \quad \text{in } \frac{1}{4}B_* \supset B(x,\frac{1}{4}L_0^{-1}R).$$

By standard covering arguments, this inequality still holds in $\frac{1}{4}B(x,R)$, thus proving (TP). □

9.3. **Off-diagonal upper bound.** We show that condition (TP_q) will lead to condition (UE_q).

Lemma 9.8. *For $2 \le q \le \infty$, we have*

$$(VD) + (TP_q) \Rightarrow (UE_q).$$

Proof. Fix two points $x, y \in M$ and set

$$R = \frac{1}{2}d(x,y).$$

Let $t < W(x,\overline{R}) \wedge W(y,\overline{R})$.

We first assume that $q \in [2,\infty)$. In this case, we have $q' = \frac{q}{q-1} \le 2 \le q$. It follows from (4.1) that condition ($TP_{q'}$) is also true.

Using the semigroup property and the Hölder inequality, we have

$$\begin{aligned} p_t(x,y) &= \int_M p_{t/2}(x,z)p_{t/2}(z,y)d\mu(z) \\ &\le \int_{B(x,R)^c} p_{t/2}(x,z)p_{t/2}(z,y)d\mu(z) + \int_{B(y,R)^c} p_{t/2}(x,z)p_{t/2}(z,y)d\mu(z) \\ &\le \|p_{t/2}(x,\cdot)\|_{L^q(B(x,R)^c)}\|p_{t/2}(\cdot,y)\|_{q'} + \|p_{t/2}(x,\cdot)\|_{q'}\|p_{t/2}(\cdot,y)\|_{L^q(B(y,R)^c)}. \end{aligned} \tag{9.18}$$

We estimate the term $\|p_{t/2}(\cdot,y)\|_{q'}$.

Indeed, since $t < W(y,\overline{R})$, by condition ($TP_{q'}$), there exists a constant $C > 0$ such that for any $R' < \overline{R}$,

$$\|p_{t/2}(\cdot,y)\|_{L^{q'}(B(y,R')^c)} \le \frac{C}{V(y,W^{-1}(y,t))^{1/q}}.$$

Since $R' < \overline{R}$ is arbitrary, passing to the limit in the above inequality as $R' \downarrow 0$, we obtain

$$\|p_{t/2}(\cdot,y)\|_{q'} \le \frac{C}{V(y,W^{-1}(y,t))^{1/q}}.$$

Similarly, since $t < W(x,\overline{R})$, we have

$$\|p_{t/2}(x,\cdot)\|_{q'} \le \frac{C}{V(x,W^{-1}(x,t))^{1/q}}.$$

Substituting the above two inequalities and condition (TP$_q$) into (9.18), we obtain

$$p_t(x,y) \le C\left(\frac{1}{V(x,W^{-1}(x,t/2))^{1/q'}} \wedge \frac{t/2}{V(x,R)^{1/q'}W(x,R)}\right)\frac{1}{V(y,W^{-1}(y,t/2))^{1/q}}$$
$$+ C\left(\frac{1}{V(y,W^{-1}(y,t/2))^{1/q'}} \wedge \frac{t/2}{V(y,R)^{1/q'}W(y,R)}\right)\frac{1}{V(x,W^{-1}(x,t/2))^{1/q}}. \tag{9.19}$$

We claim that, for $2 \le q < \infty$,

$$\frac{1}{V(y,W^{-1}(y,t))^{1/q'}} \wedge \frac{t}{V(y,x)^{1/q'}W(y,x)} \le C\left(\frac{1}{V(x,W^{-1}(x,t))^{1/q'}} \wedge \frac{t}{V(x,y)^{1/q'}W(x,y)}\right) \tag{9.20}$$

for a positive constant C independent of t, x, y.

Indeed, by condition (VD) and (2.7),

$$\begin{aligned} C_1^{-1}V(y,x) \le V(x,y) \le C_1 V(y,x), \\ C_1^{-1}W(y,x) \le W(x,y) \le C_1 W(y,x) \end{aligned} \tag{9.21}$$

for a positive constant $C_1 \ge 1$ independent of x, y. Let us divide the proof into two cases.

Case 1: $W(y,x) > t$. In this case, we have $d(x,y) > W^{-1}(y,t)$, which gives by (9.21) that

$$W(x,y) = W(x,d(x,y)) \ge C_1^{-1}W(y,d(x,y)) > C_1^{-1}t.$$

From this and using (2.8), we see that

$$d(x,y) \ge W^{-1}(x, C_1^{-1}t) \ge C^{-1}W^{-1}(x,t).$$

Therefore, it follows from (9.21), (VD) that

$$\begin{aligned} \frac{1}{V(y,W^{-1}(y,t))^{1/q'}} \wedge \frac{t}{V(y,x)^{1/q'}W(y,x)} &= \frac{t}{V(y,x)^{1/q'}W(y,x)} \le C\frac{t}{V(x,y)^{1/q'}W(x,y)} \\ &= C\left(\frac{t}{V(x,y)^{1/q'}W(x,y)} \wedge \frac{t}{V(x,C^{-1}W^{-1}(x,t))^{1/q'}(C_1^{-1}t)}\right) \\ &\le C\left(\frac{t}{V(x,y)^{1/q'}W(x,y)} \wedge \frac{C'}{V(x,W^{-1}(x,t))^{1/q'}}\right), \end{aligned}$$

thus showing (9.20) in this case.

Case 2: $W(y,x) \le t$. In this case, we have $d(x,y) \le W^{-1}(y,t)$. By (11.2) in Appendix,

$$C^{-1} \le \frac{V(x,W^{-1}(x,t))}{V(y,W^{-1}(y,t))} \le C \tag{9.22}$$

for a positive constant C independent of x, y, t. From this and using (9.21), we obtain (9.20). This proves our claim.

By (9.20), the factor in front of the second term on the right-hand side of (9.19) is bounded by

$$\begin{aligned} \frac{1}{V(y,W^{-1}(y,t/2))^{1/q'}} \wedge \frac{t/2}{V(y,R)^{1/q'}W(y,R)} &\le C\left(\frac{1}{V(y,W^{-1}(y,t))^{1/q'}} \wedge \frac{t}{V(y,R)^{1/q'}W(y,R)}\right) \\ &\le C'\left(\frac{1}{V(x,W^{-1}(x,t))^{1/q'}} \wedge \frac{t}{V(x,y)^{1/q'}W(x,y)}\right). \end{aligned}$$

Therefore, combining this and (9.19) and substituting $R = \frac{1}{2}d(x,y)$, we obtain

$$p_t(x,y) \le C'\left(\frac{1}{V(x,W^{-1}(x,t))^{1/q'}} \wedge \frac{t}{V(x,y)^{1/q'}W(x,y)}\right)\left(\frac{1}{V(y,W^{-1}(y,t))^{1/q}} + \frac{1}{V(x,W^{-1}(x,t))^{1/q}}\right),$$

which is the inequality (2.25) in (UE$_q$) in the case when $2 \le q < \infty$.

Consider now the case $q = \infty$. In this case $q' = 1$. Fix $y \ne x$ and let $R = \frac{1}{2}d(x,y)$. Using (2.20) and the fact that $p_t(x,\cdot) \in C(\{F_k\})$, we have by condition (TP$_\infty$) that for every point $z \in M$ with $d(x,z) > R$,

$$p_t(x,z) \le \|p_t(x,\cdot)\|_{L^\infty(B(x,R)^c)} \le C\left(\frac{1}{V(x,W^{-1}(x,t))} \wedge \frac{t}{V(x,R)W(x,R)}\right).$$

In particular, the above inequality holds true for $z = y$ since $d(x, y) > R$, thus showing that (2.25) holds in the case when $q = \infty$.

Therefore, we always have that (2.25) holds for $t < W(x, \overline{R}) \wedge W(y, \overline{R})$, thus showing condition (UE_q). □

10. Proofs of main results

In this section, we first give some consequences of the tail estimate (TP_q). And then, we prove main theorems.

Proposition 10.1. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2. Let $U, V \subset \mathcal{B}(M)$ with $U \cap V = \emptyset$, and $f, g \in \mathcal{F}$ be non-negative Borel functions such that* $\mathrm{supp}(f) \subset U$ *and* $\mathrm{supp}(g) \subset V$*. Then,*

$$\int_U f(x) \int_V g(y) J(x, dy) d\mu(x) \leq \liminf_{t \to 0} \frac{1}{2t} \int_U f(x) P_t g(x) d\mu(x).$$

Proof. Since $\mathrm{supp}(f) \cap \mathrm{supp}(g) \subset U \cap V = \emptyset$, we obtain for any $t > 0$,

$$-(f, g - P_t g) = (f, P_t g) = \int_U f(x) P_t g(x) d\mu(x),$$

whence by [19, Lemma 1.3.4(i)],

$$|\mathcal{E}(f, g)| = \lim_{t \to 0} \left| \frac{1}{t} |(f, g - P_t g) \right| \leq \liminf_{t \to 0} \frac{1}{t} \int_U f(x) P_t g(x) d\mu(x).$$

On the other hand, using (2.6), we obtain

$$\begin{aligned} \mathcal{E}(f, g) = \mathcal{E}^{(J)}(f, g) &= \iint_{M \times M \setminus \mathrm{diag}} (f(x) - f(y))(g(x) - g(y)) dj(x, y) \\ &= -2 \iint_{U \times V} f(x) g(y) J(x, dy) d\mu(x). \end{aligned}$$

Combining the above two inequalities we finish the proof. □

Lemma 10.2. *We have*

$$(\mathrm{TP}) \Rightarrow (\mathrm{TJ}).$$

Proof. Fix some $x_0 \in M$ and $R > 0$, and set $B := B(x_0, R)$. Let $f, g \in \mathcal{F}$ be non-negative Borel functions such that $\mathrm{supp}(f) \subset B$ and $g \leq \mathbf{1}_{(4B)^c}$. By (TP) we have for any $t > 0$,

$$P_t g \leq P_t \mathbf{1}_{(4B)^c} \leq \frac{Ct}{W(x_0, R)} \quad \text{in } B.$$

Using the above inequality and applying Proposition 10.1 for $U = B$ and $V = (4B)^c$, we obtain for any $t > 0$,

$$\int_B f(x) \int_{(4B)^c} g(y) J(x, dy) d\mu(x) \leq \frac{C}{W(x_0, R)} \int_B f d\mu.$$

Passing to the limit as $g \uparrow \mathbf{1}_{(4B)^c}$, we obtain

$$\begin{aligned} \int_{B(x_0,R)} f(x) J(x, B(x, 5R)^c) d\mu(x) &\leq \int_{B(x_0,R)} f(x) J(x, B(x_0, 4R)^c) d\mu(x) \\ &\leq \frac{C}{W(x_0, R)} \int_{B(x_0,R)} f d\mu. \end{aligned}$$

Since f is arbitrary, there exists a Borel set $N_{x_0,R}$ of measure 0 depending on x_0 and R such that, for all $x \in B(x_0, R) \setminus N_{x_0,R}$,

$$J(x, B(x, 5R)^c) \leq \frac{C}{W(x_0, R)} \leq \frac{C'}{W(x, 5R)}, \tag{10.1}$$

where in the last inequality, we have used the fact that by the right inequality in (2.7),

$$W(x, 5R) \leq c W(x_0, R).$$

For a fixed $R > 0$, since M can be covered by at most countable balls like $B(x_0, R)$, there exists a measurable set N_R with $\mu(N_R) = 0$ such that (10.1) holds for all $x \in M \setminus N_R$.

Next, set

$$N := \cup_{R\in\mathbb{Q}_+} N_R.$$

Then $\mu(N) = 0$ and (10.1) also holds for all $x \in M \setminus N$ and all rationals $R \in \mathbb{Q}_+$.

For any real $R > 0$, taking a sequence $\{R_n\} \subset (0, R) \cap \mathbb{Q}$ such that $R_n \uparrow R$ as $n \uparrow \infty$, we obtain by (10.1) and the right inequality in (2.7),

$$\begin{aligned} J(x, B(x, 5R)^c) &\le \liminf_{n\to\infty} J(x, B(x, 5R_n)^c) \le \liminf_{n\to\infty} \frac{C}{W(x, 5R_n)} \\ &\le \liminf_{n\to\infty} \frac{C'}{W(x, 5R)} \left(\frac{R}{R_n}\right)^{\beta_2} = \frac{C'}{W(x, 5R)}, \end{aligned}$$

showing that (10.1) holds true for all $x \in M \setminus N$ and all $R > 0$. Let us set $J(x, B(x, 5R)^c) = 0$ for any $x \in N$ so that (10.1) is satisfied for all $x \in M$ and $R > 0$. Renaming R by $R/5$ in (10.1), we conclude that condition (TJ) is true. □

Recall that condition (C) means that the Dirichlet form is conservative.

Lemma 10.3. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2. Then the following statements are true.*

(i) *For $2 \le q \le \infty$,*

$$(\mathrm{VD}) + (\mathrm{TP}_q) \Rightarrow (\mathrm{DUE}).$$

(ii) *For $1 < q \le \infty$,*

$$(\mathrm{VD}) + (\mathrm{TP}_q) \Rightarrow (\mathrm{TJ}_q).$$

(iii) *For $1 \le q \le \infty$,*

$$(\mathrm{TP}) + (\mathrm{C}) \Rightarrow (\mathrm{S}_+).$$

Consequently,

$$(\mathrm{VD}) + (\mathrm{TP}_q) + (\mathrm{C}) \Rightarrow (\mathrm{S}_+).$$

Proof. (i). Since condition (TP_q) holds for $2 \le q \le \infty$, condition (TP_2) is also true by (4.1). Thus, for any ball $B := B(x, R)$ of radius $R \in (0, \overline{R})$ and any $t < W(x, \overline{R})$,

$$\|p_t(x, \cdot)\|_{L^2(B^c)} \le C\left(\frac{1}{V(x, W^{-1}(x, t))^{1/2}} \wedge \frac{t}{V(x, R)^{1/2} W(x, R)}\right) \le \frac{C}{V(x, W^{-1}(x, t))^{1/2}}.$$

Passing to the limit as $R \downarrow 0$ and using the semigroup property of p_t, we obtain that for any $x \in M$ and any $t < W(x, \overline{R})$,

$$p_t(x, x) = \|p_t(x, \cdot)\|_2^2 \le \frac{C^2}{V(x, W^{-1}(x, t))} \tag{10.2}$$

To prove condition (DUE), we need to extend the above inequality to any $t < C_0 W(x, \overline{R})$ with $C_0 \ge 1$ if $\overline{R} < \infty$. Indeed, for any $y \in B(x, \frac{1}{2}W^{-1}(x, t))$ and $f \in L^2$, applying (10.2) for the point y and $t < W(x, \overline{R})$, and using (11.2) in Appendix, we have that

$$\begin{aligned} |P_t f(y)| &= \left|\int_M p_t(y, z) f(z) d\mu(z)\right| \le \|p_t(y, \cdot)\|_2 \|f\|_2 \\ &\le \frac{C}{V(y, W^{-1}(y, t))^{1/2}} \|f\|_2 \le \frac{C'}{V(x, W^{-1}(x, t))^{1/2}} \|f\|_2, \end{aligned}$$

thus showing that [23, Eq. (6.2)] holds true. Therefore, by [23, Remark 6.8], the inequality (10.2) holds true for any $t < C_0 W(x, \overline{R})$ with $C_0 \ge 1$, that is, condition (DUE) holds true.

(ii). Let us first consider the case when $1 < q < \infty$. We first prove that the jump kernel J exists. Indeed, fix $x_0 \in M$ and $R > 0$. Applying Proposition 10.1 for $U := B(x_0, R)$, $V := B(x_0, 2R)^c$ and for $0 \le f, g \in \mathcal{F}$ with $\mathrm{supp}(f) \subset B(x_0, R)$ and $\mathrm{supp}(g) \subset B(x_0, 2R)^c$, we obtain by (VD), (2.7), (TP_q) and Hölder inequality that

$$\begin{aligned} &\int_U f(x) \int_V g(y) J(x, dy) d\mu(x) \\ &\le \liminf_{t\to 0} \frac{1}{2t} \int_U f(x) \int_V p_t(x, y) g(y) d\mu(y) d\mu(x) \end{aligned}$$

$$
\begin{aligned}
&\le \liminf_{t\to 0}\frac{1}{2t}\int_U f(x)\left(\int_V g^{q'}d\mu\right)^{1/q'}\left(\int_V p_t(x,y)^q d\mu(y)\right)^{1/q}d\mu(x)\\
&\le \liminf_{t\to 0}\frac{1}{2t}\int_U f(x)\cdot\|g\|_{L^{q'}(V)}\|p_t(x,\cdot)\|_{L^q(B(x,R)^c)}d\mu(x)\quad(\text{since } V\subset B(x,R)^c)\\
&\le \liminf_{t\to 0}\|g\|_{L^{q'}(V)}\frac{1}{2t}\int_U f(x)\cdot\frac{Ct}{V(x,R)^{1/q'}W(x,R)}d\mu(x)\quad(\text{by (TP}_q))\\
&= C\|g\|_{L^{q'}(V)}\int_U \frac{f(x)}{V(x,R)^{1/q'}W(x,R)}d\mu(x) \qquad (10.3)\\
&\le \frac{C'\|g\|_{L^{q'}(V)}}{V(x_0,R)^{1/q'}W(x_0,R)}\int_U f(x)d\mu(x)\quad(\text{by (VD) and (2.7)}).
\end{aligned}
$$

Since $\mathcal{F}\cap C_0(M)$ is dense in $C_0(M)$ and $L^{q'}(V)$ is separable, we can choose a sequence $\{g_n\}_{n=1}^{\infty}\subset\mathcal{F}\cap L^{q'}(B(x_0,2R)^c)$ with $\mathrm{supp}(g_n)\subset B(x_0,2R)^c$ for all $n\ge 1$ such that $\{g_n\}_{n=1}^{\infty}$ is dense in $L^{q'}(B(x_0,2R)^c)$. Since the function $0\le f\in\mathcal{F}$ with $\mathrm{supp}(f)\subset U$ is arbitrary in the above inequality, there exists $N_{x_0,R}\in\mathcal{B}(M)$ with $N_{x_0,R}\subset B(x_0,R)$ and $\mu(N_{x_0,R})=0$ such that for all $x\in B(x_0,R)\setminus N_{x_0,R}$ and all $n\ge 1$,

$$
\int_{B(x_0,2R)^c}|g_n(y)|J(x,dy)\le\frac{C'\|g_n\|_{L^{q'}(B(x_0,2R)^c)}}{V(x_0,R)^{1/q'}W(x_0,R)}.
$$

Since $\{g_n\}_{n=1}^{\infty}$ is dense in $L^{q'}(B(x_0,2R)^c)$, by Fatou's Lemma, (VD) and the right inequality in (2.7), we obtain that for all $x\in B(x_0,R)\setminus N_{x_0,R}$ and $g\in L^{q'}(B(x_0,2R)^c)$,

$$
\begin{aligned}
\int_{B(x,3R)^c}|g(y)|J(x,dy)&\le\int_{B(x_0,2R)^c}|g(y)|J(x,dy)\\
&\le\liminf_{n\to\infty}\frac{C\|g_n\|_{L^{q'}(B(x_0,2R)^c)}}{V(x_0,R)^{1/q'}W(x_0,R)}\\
&\le\frac{C\|g\|_{L^{q'}(B(x_0,2R)^c)}}{V(x_0,R)^{1/q'}W(x_0,R)}\\
&\le\frac{C\|g\|_{L^{q'}(M)}}{V(x,3R)^{1/q'}W(x,3R)}.\quad(\text{by (VD) and (2.7)}) \qquad (10.4)
\end{aligned}
$$

For a fixed $R>0$, since M can be covered by at most countable balls like $B(x_0,R)$, there exists a measurable set N_R with $\mu(N_R)=0$ such that (10.4) holds for all $x\in M\setminus N_R$ and $g\in L^{q'}(M)$.

Next, set

$$
N := \cup_{R\in\mathbb{Q}_+}N_R.
$$

Then $\mu(N)=0$ and (10.4) also holds for all $x\in M\setminus N$, all rationals $R\in\mathbb{Q}_+$ and $g\in L^{q'}(M)$.

For any real $R>0$, choosing a sequence $\{R_n\}\subset(0,R)\cap\mathbb{Q}$ such that $R_n\uparrow R$ as $n\uparrow\infty$, by (10.4), (VD) and the right inequality in (2.7), we obtain for all $x\in M\setminus N$, all $R>0$ and $g\in L^{q'}(M)$.

$$
\begin{aligned}
\int_{B(x,3R)^c}|g(y)|J(x,dy)&\le\liminf_{n\to\infty}\frac{C\|g\|_{L^{q'}(M)}}{V(x,3R_n)^{1/q'}W(x,3R_n)}\\
&\le\liminf_{n\to\infty}\frac{C\|g\|_{L^{q'}(M)}}{V(x,3R)^{1/q'}W(x,3R)}\cdot\left(\frac{R}{R_n}\right)^{\alpha+\beta_2}\quad(\text{by (VD) and (2.7)})\\
&=\frac{C\|g\|_{L^{q'}(M)}}{V(x,3R)^{1/q'}W(x,3R)}. \qquad (10.5)
\end{aligned}
$$

Therefore, for any $x\in M\setminus N$ and any $R>0$, $J(x,dy)$ is absolutely continuous with respect to $d\mu(y)$ on $B(x,3R)^c$, and hence, the derivative

$$
J_x(y) := \frac{J(x,dy)}{d\mu(y)}, \qquad (10.6)
$$

exists on $M \setminus \{x\}$ and satisfies

$$\|J_x\|_{L^q(B(x,3R)^c)} \leq \frac{C}{V(x,3R)^{1/q'}W(x,3R)}. \tag{10.7}$$

Now, let us prove that the function $J_x(y)$ has a jointly measurable version, say $J(x,y)$, in $(x,y) \in M \times M$. Indeed, we fix $R > 0$, and consider the function $(x,y) \mapsto J_x(y)\mathbf{1}_{B(x,R)^c}(y)$. First of all, by the fact that $J(x,dy)$ is a kernel on $M \times \mathcal{B}(M)$ and (10.7), we obtain that the map $x \mapsto J_x\mathbf{1}_{B(x,R)^c}$ from M to $L^q(M)$ is weakly measurable since for any $g \in L^{q'}(M)$, the function

$$x \mapsto \int_M \mathbf{1}_{B(x,R)^c}(y)g(y)J(x,dy) = \int_{B(x,R)^c} g(y)J(x,dy)$$

is measurable. Secondly, since $L^q(M)$ is separable, by Pettis' measurability theorem (see [39, Chapter V, Section 4]), the map $x \mapsto J_x\mathbf{1}_{B(x,R)^c}$ from M to $L^q(M)$ is strongly measurable. Thirdly, for any ball $B(o,k)$ with $o \in M$ and $k > R$, we have by (VD) and the right inequality in (2.7),

$$\frac{1}{V(x,R)^{1/q'}W(x,R)} \leq \frac{C}{V(o,k)^{1/q'}W(o,k)},$$

which together with (10.7) implies that the function $x \mapsto \|J_x\mathbf{1}_{B(x,R)^c}\|_{L^q(M)}$ belongs to $L^1(B(o,k))$. This shows that $x \mapsto J_x\mathbf{1}_{B(x,R)^c}$ is Bochner integrable on $B(o,k)$ by Bochner's theorem (see [39, Chapter V, Section 5]). Finally, by [16, Chapter III, Section 11, Theorem 17], any Bochner integrable mapping admits a jointly measurable version. This shows that $x \mapsto J_x\mathbf{1}_{B(x,R)^c}$ admits a jointly measurable version (depending on R) on $B(o,k)$. Since $k > R$ is arbitrary, there is a jointly measurable function $J^{(R)}(x,y)$ in $(x,y) \in M \times M$ such that for μ-a.a. $x \in M$,

$$J_x(y)\mathbf{1}_{B(x,R)^c}(y) = J^{(R)}(x,y), \quad \mu\text{-a.a } y \in M.$$

Moreover, for any $R > r$, we have for μ-a.a. $y \in B(x,R)^c$,

$$J^{(R)}(x,y) = J_x(y)\mathbf{1}_{B(x,R)^c}(y) = J_x(y)\mathbf{1}_{B(x,r)^c}(y) = J^{(r)}(x,y).$$

Hence, we can define the jointly measurable function $J(x,y)$:

$$J(x,y) = \lim_{\mathbb{Q}_+ \ni R \downarrow 0} J^{(R)}(x,y), \quad x,y \in M,$$

such that for any $R > 0$ and μ-a.a. $x \in M$,

$$J_x(y)\mathbf{1}_{B(x,R)^c}(y) = J(x,y), \quad \mu\text{-a.a } y \in M.$$

Therefore, by (10.6), we obtain that

$$dj(x,y) = J(x,dy)d\mu(x) = J(x,y)d\mu(y)d\mu(x). \tag{10.8}$$

Moreover, the function $J(x,y)$ can be symmetric since the measurable j is symmetric. That is, we have proved that the jump kernel $J(x,y)$ exists.

Using(10.8) and repeating the arguments that lead to (10.5), we obtain for all $x \in M \setminus N$, all $R > 0$ and $g \in L^{q'}(M)$.

$$\int_{B(x,3R)^c} |g(y)|J(x,y)d\mu(y) \leq \frac{C\|g\|_{L^{q'}(M)}}{V(x,3R)^{1/q'}W(x,3R)},$$

which, by the arbitrariness of g, implies that for all $x \in M \setminus N$ and $R > 0$,

$$\|J(x,\cdot)\|_{L^q(B(x,3R)^c)} \leq \frac{C}{V(x,3R)^{1/q'}W(x,3R)}.$$

Renaming R by $R/3$, we obtain the inequality in condition (TJ_q), hence, proving (TJ_q).

It remains to consider the case when $q = \infty$. Indeed, by Lemma 9.8, condition (UE_∞) is true, from which, one can obtain (TJ_∞). In fact, one can similarly obtain (10.3) directly from condition (UE_∞), that is, for any non-negative $f,g \in \mathcal{F}$ with $\mathrm{supp}(f) \subset U$, $\mathrm{supp}(g) \subset V$, and $U \cap V = \emptyset$,

$$\iint_{U\times V} f(x)g(y)dj(x,y) \leq C\iint_{U\times V} \frac{f(x)g(y)}{V(x,y)W(x,y)}d\mu(y)d\mu(x).$$

Hence, the measure j is absolutely continuous with respect to $\mu\times\mu$ on $M\times M\backslash\text{diag}$, and hence, there is some $\mathcal{B}(M\times M\setminus\text{diag})$-measurable function $J(x,y)$ such that $dj(x,y)=J(x,y)d\mu(x)d\mu(y)$, and

$$J(x,y)\le\frac{C}{V(x,y)W(x,y)}\text{ on }M\times M\setminus\text{diag},$$

showing that condition (TJ$_\infty$) is true. We remark that the similar result was also obtained in [5, Thoerem 1.2] and [15, Proposition 3.3], but the conservativeness of $(\mathcal{E},\mathcal{F})$ was used. Here we do not need the conservativeness.

(iii). We show (VD) + (TP$_q$) + (C) $\Rightarrow$ (TP) + (C) $\Rightarrow$ (S$_+$).

Under (VD), since condition (TP$_q$) for $1\le q\le\infty$ is true, condition (TP) is always satisfied by (4.1), and then (TP$_*$) is true by (9.15). That is, for any ball $B_*:=B_*(x,r)$ of radius $r<\overline{R}_*$ and any $t>0$, we have

$$P_t\mathbf{1}_{B_*^c}\le\frac{Ct}{r^\beta}\quad\text{ in }\frac{1}{4}B_*.$$

Hence, we obtain by using [27, Lemma 6.1, p. 2634] that

$$1-P_t^{B_*}\mathbf{1}_{B_*}\le\frac{2Ct}{(r/2)^\beta}\quad\text{ in }\frac{1}{4}B_*.$$

Moreover, by standard covering arguments, one can extend the above inequality from $r\in(0,\overline{R}_*)$ to $r\in(0,2\overline{R}_*)$. That is, we have proved (S$_+^*$), and then, (S$_+$) by (7.9).

□

We show that condition (FK) will follow from (VD), (RVD), (DUE).

Proposition 10.4. *Assume that $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form in L^2. Then*

$$\text{(VD)}+\text{(RVD)}+\text{(DUE)}\Rightarrow(\text{FK}_{\beta_1^2/(\alpha\beta_2)}),\tag{10.9}$$

where α is the constant from (2.2) and β_1,β_2 are the constants from (2.7).

Proof. Fix $B:=B(x_0,R)$ with $0<R<\sigma\overline{R}$, where $\sigma\in(0,1)$ will be determined later on. We divide the proof into four steps.

Step 1. We show that

$$\operatorname*{esup}_B p_t^B:=\operatorname*{esup}_{x,y\in B}p_t^B(x,y)\le\frac{K}{\mu(B)}h\left(\frac{R}{W^{-1}(x_0,t)}\right)\quad\text{for all }0<t\le T,\tag{10.10}$$

where T is defined by

$$T:=W(x_0,\lambda K^{1/\alpha}R),$$

$\lambda,K\ge1$ are two positive constants to be determined, and h is the function defined by

$$h(s)=\begin{cases}s^\alpha & \text{if }0<s\le1,\\ s^{\alpha\beta_2/\beta_1} & \text{if }s>1.\end{cases}\tag{10.11}$$

Indeed, by (DUE) and [23, Eq. (6.18) in Corollary 6.9], if $x,y\in B$ and

$$t\le T<W(x,\overline{R})\wedge W(y,\overline{R}),\tag{10.12}$$

then we have

$$\begin{aligned}p_t^B(x,y)\le p_t(x,y)&\le\frac{C_1}{\sqrt{V(x,W^{-1}(x,t))V(y,W^{-1}(y,t))}}\\&=\frac{C_1}{V(x_0,R)}\sqrt{\frac{V(x_0,R)}{V(x,W^{-1}(x,t))}}\sqrt{\frac{V(x_0,R)}{V(y,W^{-1}(y,t))}}.\end{aligned}\tag{10.13}$$

We will first choose large λ,K and then choose a small σ such that (10.10) is satisfied. To do this, we need to estimate the term $\frac{V(x_0,R)}{V(x,W^{-1}(x,t))}$ from above for any $x\in B$.

Indeed, let $t \in (0, T]$ and $x \in B$. Denote by

$$\begin{aligned} R_{x_0,t} &:= W^{-1}(x_0, t) \quad \text{and} \quad R_{x,t} := W^{-1}(x, t), \\ R_{x_0,T} &:= W^{-1}(x_0, T) = \lambda K^{1/\alpha} R, \end{aligned} \tag{10.14}$$

so that $R_{x_0,T} \geq R_{x_0,t}$, and

$$W(x_0, R_{x_0,t}) = t = W(x, R_{x,t}) \leq T = W(x_0, R_{x_0,T}). \tag{10.15}$$

Since the following argument is sensitive to constants, we denote the constants in (VD), (RVD) and (2.7) by C_V, C_R, C_W respectively.

Case 1 when $R \leq R_{x_0,t}$. In this case, if $R_{x_0,t} > R_{x,t}$, then

$$d(x_0, x) < R \leq R_{x_0,t},$$

from which, by the left inequality in (2.7) and using (10.15), we see

$$C_W^{-1} \left(\frac{R_{x_0,t}}{R_{x,t}} \right)^{\beta_1} \leq \frac{W(x_0, R_{x_0,t})}{W(x, R_{x,t})} = \frac{t}{t} = 1,$$

and so,

$$R_{x,t} \geq C_W^{-1/\beta_1} R_{x_0,t}.$$

If $R_{x_0,t} \leq R_{x,t}$, the above inequality is also true since $C_W \geq 1$. Therefore,

$$\begin{aligned} \frac{V(x_0,R)}{V(x,R_{x,t})} &= \frac{V(x_0,R_{x_0,T})}{V(x,R_{x,t})} \frac{V(x_0,R)}{V(x_0,R_{x_0,T})} \leq \frac{V(x_0,R_{x_0,T})}{V(x, C_W^{-1/\beta_1} R_{x_0,t})} \frac{V(x_0,R)}{V(x_0,R_{x_0,T})} \\ &\leq C_V \left(\frac{R_{x_0,T}}{C_W^{-1/\beta_1} R_{x_0,t}} \right)^{\alpha} \frac{V(x_0,R)}{V(x_0,R_{x_0,T})} \quad \text{(by (2.3) and the fact that } R_{x_0,T} \geq R_{x_0,t}) \\ &\leq C_V \left(\frac{R_{x_0,T}}{C_W^{-1/\beta_1} R_{x_0,t}} \right)^{\alpha} \cdot C_R \left(\frac{R}{R_{x_0,T}} \right)^{\alpha'} \quad \text{(by (RVD))} \\ &= C_V \left(\frac{\lambda K^{1/\alpha} R}{C_W^{-1/\beta_1} R_{x_0,t}} \right)^{\alpha} \cdot C_R \left(\frac{R}{\lambda K^{1/\alpha} R} \right)^{\alpha'} \quad \text{(by (10.14))} \\ &= C_V C_R C_W^{\alpha/\beta_1} \lambda^{\alpha-\alpha'} K^{-\alpha'/\alpha} \cdot K \left(\frac{R}{R_{x_0,t}} \right)^{\alpha} \\ &= C_V C_R C_W^{\alpha/\beta_1} \lambda^{\alpha-\alpha'} K^{-\alpha'/\alpha} \cdot K h \left(\frac{R}{R_{x_0,t}} \right) \leq \frac{K}{C_1} h \left(\frac{R}{R_{x_0,t}} \right), \end{aligned} \tag{10.16}$$

provided that

$$C_V C_R C_W^{\alpha/\beta_1} \lambda^{\alpha-\alpha'} K^{-\alpha'/\alpha} \leq C_1^{-1}. \tag{10.17}$$

Since $x \in B$ is arbitrary, we also have for $y \in B$

$$\frac{V(x_0,R)}{V(y,R_{y,t})} \leq \frac{K}{C_1} h \left(\frac{R}{R_{x_0,t}} \right) \tag{10.18}$$

provided that (10.17) holds.

Plugging (10.16), (10.18) into (10.13), we obtain for any x, y in B

$$p_t^B(x,y) \leq \frac{C_1}{V(x_0,R)} \frac{K}{C_1} h \left(\frac{R}{R_{x_0,t}} \right) = \frac{K}{\mu(B)} h \left(\frac{R}{W^{-1}(x_0,t)} \right),$$

thus showing (10.10), provided that (10.17) is satisfied.

Case 2 when $R > R_{x_0,t}$. In this case, if $R_{x_0,t} \leq R_{x,t}$, then by (VD)

$$\frac{V(x_0,R)}{V(x,R_{x,t})} \leq \frac{V(x_0,R)}{V(x,R_{x_0,t})} \leq C_V \left(\frac{R}{R_{x_0,t}} \right)^{\alpha} \leq C_V \left(\frac{R}{R_{x_0,t}} \right)^{\alpha\beta_2/\beta_1} = C_V h \left(\frac{R}{R_{x_0,t}} \right).$$

If $R_{x_0,t} > R_{x,t}$, then $R > R_{x,t}$ and so, by (VD),

$$\frac{V(x_0,R)}{V(x,R_{x,t})} \le C_V\left(\frac{R}{R_{x,t}}\right)^{\alpha}.$$

Moreover, by (2.7) and (10.15),

$$C_W^{-1}\left(\frac{R}{R_{x,t}}\right)^{\beta_1} \le \frac{W(x_0,R)}{W(x,R_{x,t})} = \frac{W(x_0,R)}{W(x_0,R_{x_0,t})} \le C_W\left(\frac{R}{R_{x_0,t}}\right)^{\beta_2}.$$

It follows from the above two inequalities that

$$\frac{V(x_0,R)}{V(x,R_{x,t})} \le C_V\left(\frac{R}{R_{x,t}}\right)^{\alpha} \le C_V C_W^{2\alpha/\beta_1}\left(\frac{R}{R_{x_0,t}}\right)^{\alpha\beta_2/\beta_1} = C_V C_W^{2\alpha/\beta_1} h\left(\frac{R}{R_{x_0,t}}\right), \tag{10.19}$$

no matter $R_{x_0,t} \le R_{x,t}$ or $R_{x_0,t} > R_{x,t}$. Since $x \in B$ is arbitrary, we also have $y \in B$

$$\frac{V(x_0,R)}{V(y,R_{y,t})} \le C_V C_W^{2\alpha/\beta_1} h\left(\frac{R}{R_{x_0,t}}\right). \tag{10.20}$$

Therefore, plugging (10.19), (10.20) into (10.13), we obtain

$$\begin{aligned} p_t^B(x,y) &\le \frac{C_1}{V(x_0,R)}\sqrt{\frac{V(x_0,R)}{V(x,W^{-1}(x,t))}}\sqrt{\frac{V(x_0,R)}{V(y,W^{-1}(y,t))}} \\ &\le \frac{C_1}{V(x_0,R)} C_V C_W^{2\alpha/\beta_1} h\left(\frac{R}{R_{x_0,t}}\right) \le \frac{K}{\mu(B)} h\left(\frac{R}{W^{-1}(x_0,t)}\right), \end{aligned}$$

provided that

$$C_1 C_V C_W^{2\alpha/\beta_1} \le K. \tag{10.21}$$

thus showing (10.10).

So far we have proven (10.10), provided that assumptions (10.12), (10.17) and (10.21) are all satisfied, which will be confirmed later on.

Step 2. We further show that there exists a constant $C > 0$ such that

$$\operatorname*{esup}_B p_t^B \le \frac{C}{\mu(B)} h\left(\frac{R}{W^{-1}(x_0,t)}\right) \quad \text{for all } t > 0. \tag{10.22}$$

Indeed, note that (10.10) holds for $t = T$. We claim that it also holds for $t = 2T$. As a matter of fact, let $R_{x_0,2T} := W^{-1}(x_0, 2T)$, that is,

$$W(x_0, R_{x_0,2T}) = 2T.$$

Note that

$$R_{x_0,2T} \ge R_{x_0,T} = \lambda K^{1/\alpha} R \ge R,$$

where $R_{x_0,T}$ is given in (10.14). By (10.11) and (10.10), we obtain for all x, y in B,

$$\begin{aligned} p_{2T}^B(x,y) &= \int_B p_T^B(x,z)p_T^B(z,y)d\mu(z) \le \mu(B)(\operatorname*{esup}_B p_T^B)^2 \\ &\le \mu(B)\left(\frac{K}{\mu(B)} h\left(\frac{R}{R_{x_0,T}}\right)\right)^2 = \frac{K^2}{\mu(B)}\left(\frac{R}{R_{x_0,T}}\right)^{2\alpha} \\ &= K\left(\frac{R_{x_0,2T}}{R_{x_0,T}}\right)^{\alpha}\left(\frac{R}{R_{x_0,T}}\right)^{\alpha} \cdot \frac{K}{\mu(B)}\left(\frac{R}{R_{x_0,2T}}\right)^{\alpha}. \end{aligned} \tag{10.23}$$

By the left inequality in (2.7), we have

$$C_W^{-1}\left(\frac{R_{x_0,2T}}{R_{x_0,T}}\right)^{\beta_1} \le \frac{W(x_0,R_{x_0,2T})}{W(x_0,R_{x_0,T})} = \frac{2T}{T} = 2.$$

Plugging this inequality into (10.23) and then using (10.14), we obtain

$$p_{2T}^B(x,y) \le K\left(\frac{R_{x_0,2T}}{R_{x_0,T}}\right)^{\alpha}\left(\frac{R}{R_{x_0,T}}\right)^{\alpha} \cdot \frac{K}{\mu(B)}\left(\frac{R}{R_{x_0,2T}}\right)^{\alpha}$$

$$\leq K \cdot (2C_W)^{\alpha/\beta_1} \cdot \left(\frac{R}{\lambda K^{1/\alpha} R}\right)^{\alpha} \cdot \frac{K}{\mu(B)} \left(\frac{R}{R_{x_0,2T}}\right)^{\alpha}$$
$$\leq (2C_W)^{\alpha/\beta_1} \lambda^{-\alpha} \cdot \frac{K}{\mu(B)} \left(\frac{R}{R_{x_0,2T}}\right)^{\alpha},$$

thus showing that (10.10) holds for $t = 2T$, provided that

$$(2C_W)^{\alpha/\beta_1} \lambda^{-\alpha} \leq 1. \tag{10.24}$$

We now first choose a large number λ such that (10.24) is satisfied, and then choose a large number K such that both (10.17) and (10.21) are satisfied. We will verify (10.12) by choosing small enough σ later. In the rest of the proof, we will fix these choices of λ and K.

We turn to show (10.22). Indeed, we see by induction that (10.10) holds at $t = 2^n T$ for any integer $n = 0, 1, 2, \cdots$. Since the function $t \mapsto \operatorname{esup}_B p_t^B$ is non-increasing (cf. [26, Lemma 3.9]), we obtain that, for $2^n T \leq t < 2^{n+1} T$,

$$\operatorname*{esup}_B p_t^B \leq \operatorname*{esup}_B p_{2^n T}^B \leq \frac{K}{\mu(B)} h\left(\frac{R}{W^{-1}(x_0, 2^n T)}\right). \tag{10.25}$$

Let us estimate the term $\frac{R}{W^{-1}(x_0,2^n T)}$. Indeed, by (2.7) and the monotonicity of $W^{-1}(x_0, \cdot)$, we have

$$\frac{R}{W^{-1}(x_0, 2^n T)} = \frac{R}{W^{-1}(x_0, 2^{n+1} T)} \frac{W^{-1}(x_0, 2^{n+1} T)}{W^{-1}(x_0, 2^n T)}$$
$$\leq \frac{R}{W^{-1}(x_0, 2^{n+1} T)} \left(C_W \frac{W(x_0, W^{-1}(x_0, 2^{n+1} T))}{W(x_0, W^{-1}(x_0, 2^n T))}\right)^{1/\beta_1}$$
$$\leq \frac{R}{W^{-1}(x_0, t)} \cdot \left(C_W \frac{2^{n+1} T}{2^n T}\right)^{1/\beta_1} = \frac{R}{W^{-1}(x_0, t)} \cdot (2C_W)^{1/\beta_1}.$$

Moreover, using the fact that $t \geq 2^n T = 2^n W(x_0, \lambda K^{1/\alpha} R) \geq W(x_0, R)$ and using (10.11), we have

$$h\left(\frac{R}{W^{-1}(x_0, 2^n T)}\right) = \left(\frac{R}{W^{-1}(x_0, 2^n T)}\right)^{\alpha} \leq \left(\frac{R}{W^{-1}(x_0, t)} (2C_W)^{1/\beta_1}\right)^{\alpha}$$
$$= (2C_W)^{\alpha/\beta_1} \left(\frac{R}{W^{-1}(x_0, t)}\right)^{\alpha} = (2C_W)^{\alpha/\beta_1} h\left(\frac{R}{W^{-1}(x_0, t)}\right).$$

Plugging this inequality into (10.25), we obtain

$$\operatorname*{esup}_B p_t^B \leq \frac{K}{\mu(B)} h\left(\frac{R}{W^{-1}(x_0, 2^n T)}\right) \leq \frac{K (2C_W)^{\alpha/\beta_1}}{\mu(B)} h\left(\frac{R}{W^{-1}(x_0, t)}\right),$$

This proves (10.22) by setting $C = (2C_W)^{\alpha/\beta_1} K$.

Step 3. We show (FK$_\nu$).

Indeed, let U be a non-empty open subset of B. Let $R_{x_0,t} = W^{-1}(x_0, t)$ be as in (10.14). Using the fact that $p_t^U \leq p_t^B$ and the Cauchy-Schwarz inequality, we have from (10.22) that, for any $f \in \mathcal{F}(U)$ and any $t > 0$,

$$\left(P_t^U f, f\right) = \int_U \int_U p_t^U(x, y) f(x) f(y) d\mu(x) d\mu(y) \leq \frac{C}{\mu(B)} h\left(\frac{R}{R_{x_0,t}}\right) \|f\|_1^2$$
$$\leq \frac{C\mu(U)}{\mu(B)} h\left(\frac{R}{R_{x_0,t}}\right) \|f\|_2^2.$$

Since the function $t^{-1}(f - P_t^U f, f)$ monotonously increases to $\mathcal{E}(f, f)$ as t goes to 0, it follows that

$$\mathcal{E}(f, f) \geq \frac{1}{t}\left(f - P_t^U f, f\right) = \frac{1}{t}\left(\|f\|_2^2 - (P_t^U f, f)\right),$$

from which, we see that for any non-zero $f \in \mathcal{F}(U)$ and any $t > 0$

$$\frac{\mathcal{E}(f, f)}{\|f\|_2^2} \geq \frac{1}{t}\left(1 - \frac{C\mu(U)}{\mu(B)} h\left(\frac{R}{R_{x_0,t}}\right)\right). \tag{10.26}$$

Since $t > 0$ in (10.26) is arbitrary, we will choose t to satisfy the identity

$$\frac{C\mu(U)}{\mu(B)} h\left(\frac{R}{R_{x_0,t}}\right) = \frac{1}{2},$$

that is,

$$h\left(\frac{R}{R_{x_0,t}}\right) = \frac{1}{2C}\frac{\mu(B)}{\mu(U)} := a.$$

If $a \le 1$, we have by definition of h in (10.11)

$$h\left(\frac{R}{R_{x_0,t}}\right) = \left(\frac{R}{R_{x_0,t}}\right)^{\alpha} = a,$$

that is, $R_{x_0,t} = W^{-1}(x_0, t) = a^{-1/\alpha}R$, and so by (2.7)

$$t = W(x_0, a^{-1/\alpha}R) \le C_W \left(a^{-1/\alpha}\right)^{\beta_2} W(x_0, R) = C_W \left(\frac{1}{2C}\frac{\mu(B)}{\mu(U)}\right)^{-\beta_2/\alpha} W(x_0, R).$$

Then, it follows from (10.26) that

$$\frac{\mathcal{E}(f,f)}{\|f\|_2^2} \ge \frac{1}{2t} \ge \frac{C'}{W(x_0, R)}\left(\frac{\mu(B)}{\mu(U)}\right)^{\beta_2/\alpha} \quad \text{if } a \le 1. \tag{10.27}$$

On the other hand, if $a > 1$, then by definition (10.11)

$$h\left(\frac{R}{R_{x_0,t}}\right) = \left(\frac{R}{R_{x_0,t}}\right)^{\alpha\beta_2/\beta_1} = a,$$

that is, $R_{x_0,t} = W^{-1}(x_0, t) = a^{-\beta_1/(\alpha\beta_2)}R$, and so by (2.7)

$$t = W(x_0, a^{-\beta_1/(\alpha\beta_2)}R) \le C_W \left(a^{-\beta_1/(\alpha\beta_2)}\right)^{\beta_1} W(x_0, R) = C_W \left(\frac{1}{2C}\frac{\mu(B)}{\mu(U)}\right)^{-\beta_1^2/(\alpha\beta_2)} W(x_0, R).$$

Then, it follows from (10.26) that

$$\frac{\mathcal{E}(f,f)}{\|f\|_2^2} \ge \frac{1}{2t} \ge \frac{C'}{W(x_0, R)}\left(\frac{\mu(B)}{\mu(U)}\right)^{\beta_1^2/(\alpha\beta_2)} \quad \text{if } a > 1. \tag{10.28}$$

In both cases (10.27) and (10.28), we always have that, using the fact that $\frac{\mu(B)}{\mu(U)} \ge 1$,

$$\lambda_{\min}(U) = \inf_{f \in \mathcal{F}(U)\setminus\{0\}} \frac{\mathcal{E}(f,f)}{\|f\|_2^2} \ge \frac{C'}{W(x_0, R)}\left(\frac{\mu(B)}{\mu(U)}\right)^{\nu},$$

where ν is given by

$$\nu = \min\{\beta_2/\alpha, \beta_1^2/(\alpha\beta_2)\} = \beta_1^2/(\alpha\beta_2)$$

since $\beta_1 \le \beta_2$, thus proving condition (FK) with $\nu = \beta_1^2/(\alpha\beta_2)$.

Step 4. Finally, it remains to verify (10.12). This can be achieved by choosing the value of σ. Without loss of generality, assume that $\overline{R} < \infty$; otherwise $W(x, \overline{R}) = \infty$, and (10.12) is trivially satisfied.

For any x in B, since $d(x, x_0) < R < \sigma\overline{R} < \overline{R}$, we see by (2.7)

$$\frac{W(x_0, \overline{R})}{W(x, \overline{R})} \le C_W \left(\frac{\overline{R}}{\overline{R}}\right)^{\beta_2} = C_W,$$

and so (10.12) will be secured if

$$T < C_W^{-1} W(x_0, \overline{R}). \tag{10.29}$$

On the other hand, if $R < \sigma\overline{R}$, then by (2.7),

$$T = W(x_0, \lambda K^{1/\alpha}R) \le W(x_0, \lambda K^{1/\alpha}(\sigma\overline{R})) \le C_W \left(\lambda K^{1/\alpha}\sigma\right)^{\beta_1} W(x_0, \overline{R}).$$

Now, we can choose σ to be sufficiently small such that

$$C_W \left(\lambda K^{1/\alpha}\sigma\right)^{\beta_1} \le C_W^{-1} < 1.$$

With the choice of the above σ, we conclude that (10.29) is true, which in turn implies that (10.12) is secured. □

Definition 10.5 (Capacity upper bound)**.** We say that the condition ($\mathrm{Cap}_{\le}$) is satisfied if there exists a constant $C > 0$ such that for all balls B of radii R less than $\overline{R}$

$$\mathrm{cap}(\frac{1}{2}B, B) \le C\frac{\mu(B)}{W(B)}. \tag{10.30}$$

The authors proved in [24, Theorem 2.11] that under mild assumptions, (Gcap) ⇔ (ABB) + ($\mathrm{Cap}_{\le}$). While, in the following lemma, we prove that under the same assumptions, (ABB) ⇒ ($\mathrm{Cap}_{\le}$), and consequently, (Gcap) ⇔ (ABB).

Lemma 10.6. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2 without killing part. Then, we have*

$$(\mathrm{VD}) + (\mathrm{TJ}) + (\mathrm{ABB}) \Rightarrow (\mathrm{Cap}_{\le}). \tag{10.31}$$

Consequently, under conditions (VD), (FK) *and* (TJ), *we have the following equivalence:*

$$(\mathrm{Gcap}) \Leftrightarrow (\mathrm{ABB}). \tag{10.32}$$

Proof. Let $B := B(x_0, R)$ with $x_0 \in M$ and $R < \overline{R}$. We divide the proof of (10.31) into two cases.

Case 1: $R < \frac{1}{2}\overline{R}$. Applying (ABB) for $B_0 := \frac{1}{2}B$, B, $\Omega := 2B$ and $u = 1$, we have that there exists $\phi \in \mathrm{cutoff}(B_0, B)$ such that

$$\int_{2B} d\Gamma_{2B}(\phi) \le \sup_{x\in 2B} \frac{c_1}{W(x, R/2)} \int_{2B} d\mu = \sup_{x\in 2B} \frac{c_1\mu(2B)}{W(x, R/2)}.$$

Then, by (2.5), (VD), (TJ) and (2.7), we have

$$\begin{aligned}
\mathcal{E}(\phi, \phi) &= \int_{2B} d\Gamma_{2B}(\phi) + 2\iint_{(2B)\times(2B)^c} (\phi(x) - \phi(y))^2 dj(x, y)\\
&= \int_{2B} d\Gamma_{2B}(\phi) + 2\iint_{B\times(2B)^c} \phi(x)^2 J(x, dy)d\mu(x)\\
&\le \sup_{x\in 2B} \frac{c_1\mu(2B)}{W(x, R/2)} + 2\int_B J(x, B(x, R)^c)d\mu(x)\\
&\le \sup_{x\in 2B} \frac{c_1\mu(2B)}{W(x, R/2)} + \int_B \frac{c_2}{W(x, R)} d\mu(x)\\
&\le \frac{c_1\mu(B)}{W(B)} \sup_{x\in 2B} \frac{W(x_0, R)}{W(x, R/2)} \frac{\mu(2B)}{\mu(B)} + \frac{c_2}{W(B)} \int_B \frac{W(x_0, R)}{W(x, R)} d\mu(x)\\
&\le \frac{C\mu(B)}{W(B)},
\end{aligned}$$

which is the inequality in ($\mathrm{Cap}_{\le}$).

Case 2: $\frac{1}{2}\overline{R} \le R < \overline{R}$ (when $\overline{R} < \infty$). By (VD), there exists an integer $N > 0$ depending only on the constant in (VD) and $\{x_i,\ 1 \le i \le N\} \subset \frac{1}{2}B$ such that $\frac{1}{2}B \subset \cup_{i=1}^N B(x_i, \frac{1}{4}R)$. Similar to Case 1, for each x_i, one can find $\phi_i \in \mathrm{cutoff}(B(x_i, \frac{1}{4}R), B(x_i, \frac{1}{2}R))$ such that

$$\mathcal{E}(\phi_i, \phi_i) \le \frac{C_i V(x_i, \frac{1}{2}R)}{W(x_i, \frac{1}{2}R)}.$$

Define

$$\phi := \bigvee_{i=1}^N \phi_i.$$

Clearly, $\phi \in \mathrm{cutoff}(\frac{1}{2}B, B)$. Moreover, by the subadditivity of capacity and (2.7), we have

$$\mathcal{E}(\phi, \phi) \le \sum_{i=1}^N \frac{C_i V(x_i, \frac{1}{2}R)}{W(x_i, \frac{1}{2}R)} \le \sum_{i=1}^N \frac{C_i V(x_0, R)}{W(x_0, R)} \frac{W(x_0, R)}{W(x_i, \frac{1}{2}R)} \le \frac{C\mu(B)}{W(B)},$$

which is the inequality in ($\mathrm{Cap}_{\leq}$). Hence we obtain ($\mathrm{Cap}_{\leq}$).

Finally, (10.32) follows directly from (10.31) and [24, Theorem 2.11]. □

By (7.1), (7.2) and (10.32), we obtain that under condition (VD),

$$(\mathrm{FK}) + (\mathrm{ABB}) + (\mathrm{TJ}) \Rightarrow (\mathrm{S}_+) \Leftrightarrow (\mathrm{S}) \Rightarrow (\mathrm{Gcap}). \tag{10.33}$$

Theorem 10.7. *Under condition (VD), we have*

$$\begin{aligned}(\mathrm{FK}) + (\mathrm{Gcap}) + (\mathrm{TJ}) &\Leftrightarrow (\mathrm{FK}) + (\mathrm{ABB}) + (\mathrm{TJ})\\ &\Rightarrow (\mathrm{S}) + (\mathrm{TJ})\\ &\Rightarrow (\mathrm{TP}).\end{aligned}$$

Proof. The first equivalence follows directly from (10.32). The rest conclusions follows from the following implications:

$$\begin{aligned}(\mathrm{FK}) + (\mathrm{ABB}) + (\mathrm{TJ}) &\Rightarrow (\mathrm{S}_+) \quad ((10.33))\\ (\mathrm{S}_+) &\Leftrightarrow (\mathrm{S}) \quad ((7.1))\\ (\mathrm{S}_+) &\Leftrightarrow (\mathrm{S}_+^*) \quad (\text{Proposition 7.4(ii)})\\ (\mathrm{TJ}) &\Leftrightarrow (\mathrm{TJ}_*) \quad (\text{Proposition 7.4(iii)})\\ (\mathrm{VD}) &\Rightarrow (\mathrm{VD}_*) \quad (\text{Proposition 6.4(i)})\\ (\mathrm{VD}_*) + (\mathrm{S}_+^*) + (\mathrm{TJ}_*) &\Rightarrow (\mathrm{TP}_*) \quad (\text{Lemma 9.3})\\ (\mathrm{TP}_*) &\Rightarrow (\mathrm{TP}) \quad (\text{Lemma 9.7}).\end{aligned}$$

□

The next theorem contains a number of equivalent conditions for (TP_q) + (C) that constitute a substantial part of the proof of the main Theorem 2.15 below.

Theorem 10.8. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2 without killing part.*

(i) *For $q \in (1, \infty]$, under conditions (VD) and (DUE), we have*

$$(\mathrm{TP}_q) + (\mathrm{C}) \Leftrightarrow (\mathrm{S}) + (\mathrm{TJ}_q) \Rightarrow (\mathrm{Gcap}) + (\mathrm{TJ}_q) \Rightarrow (\mathrm{ABB}) + (\mathrm{TJ}_q) \tag{10.34}$$

and

$$(\mathrm{VD}) + (\mathrm{FK}) + (\mathrm{ABB}) + (\mathrm{TJ}_q) \Rightarrow (\mathrm{S}). \tag{10.35}$$

(ii) *For $q \in [2, \infty]$, under condition (VD), we have*

$$\begin{aligned}(\mathrm{TP}_q) + (\mathrm{C}) &\Leftrightarrow (\mathrm{DUE}) + (\mathrm{S}) + (\mathrm{TJ}_q)\\ &\Rightarrow (\mathrm{DUE}) + (\mathrm{Gcap}) + (\mathrm{TJ}_q)\\ &\Rightarrow (\mathrm{DUE}) + (\mathrm{ABB}) + (\mathrm{TJ}_q)\end{aligned} \tag{10.36}$$

and

$$(\mathrm{VD}) + (\mathrm{FK}) + (\mathrm{ABB}) + (\mathrm{TJ}_q) \Rightarrow (\mathrm{S}) + (\mathrm{DUE}). \tag{10.37}$$

(iii) *For $q \in (1, \infty]$, under conditions (VD), (RVD) and (DUE), we have*

$$(\mathrm{TP}_q) + (\mathrm{C}) \Leftrightarrow (\mathrm{S}) + (\mathrm{TJ}_q) \Leftrightarrow (\mathrm{Gcap}) + (\mathrm{TJ}_q) \Leftrightarrow (\mathrm{ABB}) + (\mathrm{TJ}_q). \tag{10.38}$$

(iv) *For $q \in [2, \infty]$, under conditions (VD) and (RVD), we have*

$$(\mathrm{TP}_q) + (\mathrm{C}) \Leftrightarrow (\mathrm{DUE}) + (\mathrm{S}) + (\mathrm{TJ}_q) \Leftrightarrow (\mathrm{DUE}) + (\mathrm{Gcap}) + (\mathrm{TJ}_q) \tag{10.39}$$

$$\Leftrightarrow (\mathrm{FK}) + (\mathrm{Gcap}) + (\mathrm{TJ}_q) \Leftrightarrow (\mathrm{FK}) + (\mathrm{ABB}) + (\mathrm{TJ}_q) \tag{10.40}$$

Proof. (i). In the proof we use (S_+) instead of (S) as these two conditions are equivalent by (7.1). The implication

$$(\mathrm{TP}_q) + (\mathrm{C}) \Leftarrow (\mathrm{S}_+) + (\mathrm{TJ}_q)$$

in (10.34) follows from the following sequence of implications:

$$\begin{aligned}(\mathrm{VD}) + (\mathrm{DUE}) &\Rightarrow (\mathrm{DUE}_*) \quad (\text{Proposition 7.4(i)})\\ (\mathrm{VD}) + (\mathrm{TJ}_q) &\Rightarrow (\mathrm{TJ}_q^*) \quad (\text{Proposition 7.4(iii)})\end{aligned}$$

$$
\begin{aligned}
(\mathrm{VD}) + (\mathrm{TJ}_q^*) &\Rightarrow (\mathrm{TJ}_*) \quad ((7.10))\\
(\mathrm{S}_+) &\Rightarrow (\mathrm{S}_+^*) \quad (\text{Proposition 7.4(ii)})\\
(\mathrm{S}_+^*) + (\mathrm{TJ}_*) &\Rightarrow \text{Inequality (8.15)} \quad (\text{Proposition 8.7})\\
\text{Inequality (8.15)} + (\mathrm{S}_+^*) &\Rightarrow \text{Inequality (8.17)} \quad (\text{Lemma 8.8})\\
\text{Inequality (8.17)} + (\mathrm{TJ}_*) &\Rightarrow \text{Inequality (8.19)} \quad (\text{Lemma 8.9})\\
(\mathrm{VD}) &\Rightarrow (\mathrm{VD}_*) \quad (\text{Proposition 6.4})\\
(\mathrm{VD}_*) + (\mathrm{DUE}_*) + (\mathrm{S}_+^*) + (\mathrm{TJ}_*) &\Rightarrow \text{upper estimate of truncated heat kernel } q_t(x,y) \quad (\text{Lemma 8.12})\\
(\mathrm{VD}_*) + (\mathrm{DUE}_*) + (\mathrm{S}_+^*) + (\mathrm{TJ}_q^*) &\Rightarrow (\mathrm{TP}_q^*) \text{ for } 1 < q \le \infty \quad (\text{Lemmas 9.4 and 9.6})\\
(\mathrm{VD}) + (\mathrm{TP}_q^*) &\Rightarrow (\mathrm{TP}_q) \text{ for } 1 \le q \le \infty \quad (\text{Lemma 9.7})\\
(\mathrm{S}_+^*) &\Rightarrow (\mathrm{S}_*) \Rightarrow (\mathrm{C}) \quad ([20, \text{Lemma 4.6, p. 3327}]).
\end{aligned}
$$

The implication

$$(\mathrm{TP}_q) + (\mathrm{C}) \Rightarrow (\mathrm{S}_+) + (\mathrm{TJ}_q)$$

in (10.34) is proved as follows:

$$
\begin{aligned}
(\mathrm{VD}) + (\mathrm{TP}_q) + (\mathrm{C}) &\Rightarrow (\mathrm{S}_+) \quad (\text{Lemma 10.3(iii)})\\
(\mathrm{VD}) + (\mathrm{TP}_q) &\Rightarrow (\mathrm{TJ}_q) \quad (\text{Lemma 10.3(ii)}).
\end{aligned}
$$

The rest implications in (10.34) follow directly from (10.33) and the following implication:

$$
\begin{aligned}
(\mathrm{VD}) + (\mathrm{TJ}_q) &\Rightarrow (\mathrm{TJ}) \quad ((7.3))\\
(\mathrm{Gcap}) + (\mathrm{TJ}) &\Rightarrow (\mathrm{ABB}) \quad ([24, \text{Lemma 6.2}]).
\end{aligned}
$$

The implication (10.35) follows from the following implications.

$$
\begin{aligned}
(\mathrm{VD}) + (\mathrm{TJ}_q) &\Rightarrow (\mathrm{TJ}) \quad ((7.3))\\
(\mathrm{VD}) + (\mathrm{FK}) + (\mathrm{ABB}) + (\mathrm{TJ}) &\Rightarrow (\mathrm{S}_+) \quad ((10.33)).
\end{aligned}
$$

(ii). The formula (10.36) follows from (10.34) and the following implication:

$$(\mathrm{VD}) + (\mathrm{TP}_q) \Rightarrow (\mathrm{DUE}) \quad (\text{Lemma 10.3(i)}).$$

The implication (10.37) follows from (10.35) and the following implications:

$$
\begin{aligned}
(\mathrm{VD}) + (\mathrm{FK}) + (\mathrm{ABB}) + (\mathrm{TJ}) &\Rightarrow (\mathrm{Gcap}) \quad ((10.33))\\
(\mathrm{VD}) + (\mathrm{FK}) + (\mathrm{Gcap}) + (\mathrm{TJ}_q) &\Rightarrow (\mathrm{DUE}) \quad (\text{Proposition 2.8}).
\end{aligned}
$$

(iii). The formula (10.38) follows from (10.34), (10.35) and the following implication:

$$(\mathrm{VD}) + (\mathrm{RVD}) + (\mathrm{DUE}) \Rightarrow (\mathrm{FK}) \quad (\text{Proposition 10.9}).$$

(iv). The formula (10.39) follows from (10.36), (10.37) and the following implications:

$$
\begin{aligned}
(\mathrm{VD}) + (\mathrm{RVD}) + (\mathrm{DUE}) &\Rightarrow (\mathrm{FK}) \quad (\text{Proposition 10.9})\\
(\mathrm{VD}) + (\mathrm{FK}) + (\mathrm{Gcap}) + (\mathrm{TJ}) &\Rightarrow (\mathrm{S}_+) \quad ((7.2))\\
(\mathrm{VD}) + (\mathrm{FK}) + (\mathrm{Gcap}) + (\mathrm{TJ}_q) &\Rightarrow (\mathrm{DUE}). \quad (\text{Proposition 2.8})
\end{aligned}
$$

□

We now prove the main Theorem 2.15.

Proof of Theorem 2.15. The statement of this theorem is contained in the equivalences (10.39), (10.40) and the implication

$$(\mathrm{TP}_q) \Rightarrow (\mathrm{UE}_q). \quad (\text{Lemma 9.8})$$

□

In the rest of this section, we are to prove Theorem 3.4, which states the result on heat kernel estimates when condition (V) is satisfied. Before that we need to introduce the condition (Nash) and to prove some lemmas. Note that if (V) is satisfied, then both (VD) and (RVD) are satisfied with $\alpha = \alpha'$. Recall that $\overline{R} = \operatorname{diam} M$ is the diameter of M.

Definition 10.9 (Nash inequality)**.** We say that condition (Nash) holds if there exist two numbers $C, \nu > 0$ such that, for any $u \in \mathcal{F} \cap L^1(M)$,

$$\|u\|_2^{2(1+\nu)} \le C\left(\mathcal{E}(u,u) + \overline{R}^{-\beta}\|u\|_2^2\right)\|u\|_1^{2\nu}. \tag{10.41}$$

If necessary, we label condition (Nash) by (Nash_ν) to emphasize the role of the exponent ν.

Lemma 10.10. *For any $\nu > 0$,*

$$(\text{FK}'_\nu) \Leftrightarrow (\text{Nash}_\nu).$$

Proof. For the implication "⇒", we use the approach of [26, Lemma 5.4]. Fix a quasi-continuous function $u \in \mathcal{F} \cap L^1(M)$. Without loss of generality, we can assume that $u \ge 0$ since $\mathcal{E}(|u|,|u|) \le \mathcal{E}(u,u)$. If $\|u\|_1 = 0$ then there is nothing to prove. Hence, we assume that $\|u\|_1 > 0$. For any $s > 0$, set

$$E_s := \{x \in M : u(x) > s\},$$

and note that

$$\mu(E_s) \le \frac{1}{s}\int_{E_s} u\,d\mu \le \frac{\|u\|_1}{s}.$$

Fix $\varepsilon > 0$ and choose an open set U_s be an open set such that $E_s \subset U_s$ and $\mu(U_s \setminus E_s) < \varepsilon$. Since $(u-s)_+(x) = 0$ for all $x \in E_s^c$ and $E_s \subset U_s$, we have $(u-s)_+ \in \mathcal{F}(U_s)$. Then, by the Markov property of $(\mathcal{E}, \mathcal{F})$ and (2.12), we have for any $s > 0$,

$$\mathcal{E}(u,u) \ge \mathcal{E}((u-s)_+, (u-s)_+) \ge \lambda_1(U_s)\int_{U_s}(u-s)_+^2\,d\mu.$$

Since $u \ge 0$, we have

$$\int_{U_s}(u-s)_+^2\,d\mu = \int_M (u-s)_+^2\,d\mu \ge \int_M (u^2 - 2su)\,d\mu = \|u\|_2^2 - 2s\|u\|_1.$$

On the other hand, since

$$\mu(U_s) \le \mu(E_s) + \varepsilon \le \frac{\|u\|_1}{s} + \varepsilon,$$

we have by (FK'_ν) that

$$\lambda_1(U_s) \ge c\mu(U_s)^{-\nu} - \overline{R}^{-\beta} \ge c\left(\frac{\|u\|_1}{s} + \varepsilon\right)^{-\nu} - \overline{R}^{-\beta}.$$

Combining the above inequalities and letting $\varepsilon \to 0$, we obtain, for any $s > 0$,

$$\mathcal{E}(u,u) \ge \left(c\left(\frac{\|u\|_1}{s}\right)^{-\nu} - \overline{R}^{-\beta}\right)(\|u\|_2^2 - 2s).$$

Choosing $s = \frac{\|u\|_2^2}{4\|u\|_1}$ in the above inequality, we obtain (10.41).

Now let us prove the implication "⇐". Fix a non-empty open subset U. Let $u \in \mathcal{F}(U) \setminus \{0\}$. It follows from ($\text{Nash}_\nu$) and the inequality $\|u\|_1 \le \sqrt{\mu(U)}\|u\|_2$ that

$$\|u\|_2^{2(1+\nu)} \le C\left(\mathcal{E}(u,u) + \overline{R}^{-\beta}\|u\|_2^2\right)\left(\sqrt{\mu(U)}\|u\|_2\right)^{2\nu},$$

that is

$$\|u\|_2^2 \le C\left(\mathcal{E}(u,u) + \overline{R}^{-\beta}\|u\|_2^2\right)\mu(U)^\nu,$$

which together with (2.12) yields (FK'_ν). □

Note that under (3.1) and (3.2), the inequality (2.21) in condition (DUE) becomes

$$p_t(x,x) \le Ct^{-\alpha/\beta}, \tag{10.42}$$

for all $t < \overline{R}^\beta$, $x \in M$.

Lemma 10.11. *We have*

$$(\mathrm{FK}'_{\beta/\alpha}) \Leftrightarrow (\mathrm{Nash}_{\beta/\alpha}) \Leftrightarrow (\mathrm{DUE}).$$

Proof. By Lemma 10.10, it suffices to prove the second equivalence. Let us first prove the implication $(\mathrm{Nash}_{\beta/\alpha}) \Rightarrow (\mathrm{DUE})$.

Recall that by [10, Theorem 2.1], $(\mathrm{Nash}_{\beta/\alpha})$ is equivalent to the *ultracontractivity* of the heat semigroup $\{P_t\}_{t\ge 0}$, that is, to

$$\|P_t f\|_\infty \le Ce^{\overline{R}^{-\beta}t} t^{-\alpha/\beta}\|f\|_1, \quad t>0,\ f\in L^1(M). \tag{10.43}$$

On the other hand, by Theorem 11.8 (for $p=1$, $\mathcal{S}=\{M\}$ and $\phi(M,t)=Ce^{\overline{R}^{-\beta}t}t^{-\alpha/\beta}$) in Appendix, the heat semigroup is ultracontractive if and only if there exists a regular nest $\mathcal{E}$-nest $\{F_k\}$ such that, for any $t>0$ and $x\in M$, $p_t(x,\cdot)\in C(\{F_k\})$ and

$$p_t(x,y) \le Ce^{\overline{R}^{-\beta}t}t^{-\alpha/\beta}, \quad t>0,\ x,y\in M. \tag{10.44}$$

Clearly, (10.44) implies (10.42) and, hence, (DUE).

Let us prove the converse implication $(\mathrm{DUE}) \Rightarrow (\mathrm{Nash}_{\beta/\alpha})$. By (10.42) and the semigroup property, we have, for any $t<\overline{R}^\beta$ and $x,y\in M$,

$$\begin{aligned} p_t(x,y) &= \int_M p_{t/2}(x,z)p_{t/2}(z,y)d\mu(z) \le \|p_{t/2}(x,\cdot)\|_2\|p_{t/2}(\cdot,y)\|_2 \\ &= \sqrt{p_t(x,x)p_t(y,y)} \le Ct^{-\alpha/\beta}. \end{aligned}$$

If $\overline{R}=\infty$ then we have (10.44) and, hence, (10.43) and $(\mathrm{Nash}_{\beta/\alpha})$. Let $\overline{R}<\infty$. Then we only need to verify (10.44) for $t\ge \overline{R}^\beta$. Using the above inequality for $t=t_0 := \frac{1}{2}(\overline{R})^\beta$, we obtain

$$\begin{aligned} p_t(x,y) &= \int_M p_{t-t_0}(x,z)p_{t_0}(z,y)d\mu(z) \le Ct_0^{-\alpha/\beta}\int_M p_{t-t_0}(x,z)d\mu(z) \\ &\le Ct_0^{-\alpha/\beta} \le C'e^{\frac{t}{2t_0}}t^{-\alpha/\beta} = C'e^{\overline{R}^{-\beta}t}t^{-\alpha/\beta}. \end{aligned}$$

□

Now we are to prove Theorem 3.4. Note that under (3.1) and (3.2), conditions (S_+^*), (S_*), (TJ_*), (TP_*) are the same to (S_+), (S), (TJ), (TP) respectively.

Proof of Theorem 3.4. (i). We first prove the equivalences in (3.5). The first equivalence in (3.5) follows from (10.32) in Lemma 10.6.

The implication

$$(\mathrm{FK}'_{\beta/\alpha}) + (\mathrm{Gcap}) + (\mathrm{TJ}) \Rightarrow (\mathrm{TP}) + (\mathrm{DUE}) + (\mathrm{C})$$

is proved as follows:

$$\begin{aligned} (\mathrm{FK}'_{\beta/\alpha}) &\Rightarrow (\mathrm{DUE}) \quad (\text{Lemma } 10.11) \\ (\mathrm{FK}'_{\beta/\alpha}) + (\mathrm{Gcap}) + (\mathrm{TJ}) &\Rightarrow (\mathrm{S}_+) \Rightarrow (\mathrm{S}) \quad (\text{Remark } 3.3 \text{ and } (7.2)) \\ (\mathrm{S}_+) + (\mathrm{TJ}) &\Rightarrow (\mathrm{TP}) \quad (\text{Lemma } 9.3) \\ (\mathrm{S}) &\Rightarrow (\mathrm{C}) \quad ([20, \text{Lemma 4.6, p. 3327}]) \end{aligned}$$

The implication

$$(\mathrm{TP}) + (\mathrm{DUE}) + (\mathrm{C}) \Rightarrow (\mathrm{FK}'_{\beta/\alpha}) + (\mathrm{Gcap}) + (\mathrm{TJ})$$

is proved as follows:

$$\begin{aligned} (\mathrm{DUE}) &\Rightarrow (\mathrm{FK}'_{\beta/\alpha}) \quad (\text{Lemma } 10.11) \\ (\mathrm{TP}) + (\mathrm{C}) &\Rightarrow (\mathrm{S}_+) \quad (\text{Lemma } 10.3\text{(iii)}) \end{aligned}$$

$$\begin{aligned}(S_+) &\Rightarrow (S) \Rightarrow (\text{Gcap}) \quad ((7.2))\\ (\text{TP}) &\Rightarrow (\text{TJ}) \quad (\text{Lemma } 10.2).\end{aligned}$$

For the last equivalence in (3.5), it suffices to prove that

$$(\text{TP}) + (\text{DUE}) \Rightarrow (\text{TP}_1) + (\text{UE}_1).$$

Indeed, under condition (DUE), the heat kernel exists, and then it is trivial that (TP) = (TP_1). It remains to prove that

$$(\text{TP}) + (\text{DUE}) \Rightarrow (\text{UE}_1).$$

The argument here is similar to the proof of [9, Lemma 12.6]. Indeed, fix $x, y \in M$ with $x \neq y$ and $t < \overline{R}^{\beta}$. Let

$$R := d(x,y).$$

Note that the inequality (2.21) in condition (DUE) implies (10.44) (see the proof of Lemma 10.11). By semigroup property and (TP) + (DUE), we have

$$\begin{aligned}p_t(x,y) &= \int_M p_{t/2}(x,z)p_{t/2}(z,y)d\mu(z)\\ &\leq \left(\int_{B(x,R)^c} + \int_{B(y,R)^c}\right) p_{t/2}(x,z)p_{t/2}(z,y)d\mu(z)\\ &\leq Ct^{-\alpha/\beta}\int_{B(x,R)^c} p_{t/2}(x,z)d\mu(z) + Ct^{-\alpha/\beta}\int_{B(y,R)^c} p_{t/2}(z,y)d\mu(z) \quad (\text{by } (10.44))\\ &= Ct^{-\alpha/\beta}P_{t/2}\mathbf{1}_{B(x,R)^c}(x) + Ct^{-\alpha/\beta}P_{t/2}\mathbf{1}_{B(y,R)^c}(y)\\ &\leq Ct^{-\alpha/\beta}\left(1 \wedge \frac{t}{R^\beta}\right) \quad (\text{by (TP)}),\end{aligned}$$

which yields (UE_1) (see also Remark 2.14).

(ii). The first equivalence in (3.6) follows from the first equivalence in (3.5) and the implication (TJ_q) ⇒ (TJ) (by (7.3)). For the rest equivalence, the implication

$$(\text{FK}'_{\beta/\alpha}) + (\text{Gcap}) + (\text{TJ}_q) \Rightarrow (\text{TP}_q) + (\text{DUE}) + (\text{C})$$

follows from the following implications:

$$\begin{aligned}(\text{FK}'_{\beta/\alpha}) &\Rightarrow (\text{DUE}) \quad (\text{Lemma } 10.11)\\ (\text{TJ}_q) &\Rightarrow (\text{TJ}) \quad ((7.3))\\ (\text{FK}'_{\beta/\alpha}) + (\text{Gcap}) + (\text{TJ}) &\Rightarrow (S_+) \Rightarrow (S) \quad (\text{Remark } 3.3 \text{ and } (7.2))\\ (\text{DUE}) + (S_+) + (\text{TJ}_q) &\Rightarrow (\text{TP}_q) \quad (\text{Lemmas } 9.4 \text{ and } 9.6)\\ (S) &\Rightarrow (\text{C}) \quad ([20, \text{Lemma } 4.6, \text{p. } 3327])\end{aligned}$$

The implication

$$(\text{TP}_q) + (\text{DUE}) + (\text{C}) \Rightarrow (\text{FK}'_{\beta/\alpha}) + (\text{Gcap}) + (\text{TJ})$$

follows from the following implications:

$$\begin{aligned}(\text{DUE}) &\Rightarrow (\text{FK}'_{\beta/\alpha}) \quad (\text{Lemma } 10.11)\\ (\text{TP}_q) + (\text{C}) &\Rightarrow (S_+) \quad (\text{Lemma } 10.3(\text{iii}))\\ (S_+) &\Rightarrow (S) \Rightarrow (\text{Gcap}) \quad ((7.2))\\ (\text{TP}_q) &\Rightarrow (\text{TJ}_q) \quad (\text{Lemma } 10.3(\text{ii})).\end{aligned}$$

For the last equivalence in (3.6), it suffices to prove that

$$(\text{TP}_q) + (\text{DUE}) \Rightarrow (\text{UE}_q).$$

Similar to the above arguments, we fix $x, y \in M$ with $x \neq y$ and $t < \overline{R}^{\beta}$. Let

$$R := d(x,y) \quad \text{and} \quad q' := \frac{q}{q-1}$$

Note that the inequality (2.21) in condition (DUE) implies (10.44) (see the proof of Lemma 10.11). By semigroup property and (TP_q) + (DUE), we have

$$\begin{aligned} p_t(x,y) &= \int_M p_{t/2}(x,z)p_{t/2}(z,y)d\mu(z) \\ &\le \left(\int_{B(x,R)^c} + \int_{B(y,R)^c}\right) p_{t/2}(x,z)p_{t/2}(z,y)d\mu(z) \\ &\le \|p_{t/2}(x,\cdot)\|_{L^q(B(x,R)^c)}\|p_{t/2}(\cdot,y)\|_{L^{q'}(B(x,R)^c)} \\ &\quad + \|p_{t/2}(y,\cdot)\|_{L^q(B(y,R)^c)}\|p_{t/2}(x,\cdot)\|_{L^{q'}(B(y,R)^c)} \\ &\le C\left(t^{-\alpha/(\beta q')} \wedge \frac{t}{R^{\alpha/q'+\beta}}\right)\left(\|p_{t/2}(\cdot,y)\|_{L^{q'}(B(x,R)^c)} + \|p_{t/2}(x,\cdot)\|_{L^{q'}(B(y,R)^c)}\right) \end{aligned}$$

where we have used (TP_q). Next, using

$$\|f\|_{q'} \le \|f\|_\infty^{1/q} \|f\|_1^{1/q'}$$

and that by (10.44) $\|p_{t/2}(\cdot,y)\|_\infty \le Ct^{-\alpha/\beta}$ and $\|p_{t/2}(\cdot,y)\|_1 \le 1$, we obtain

$$p_t(x,y) \le C\left(t^{-\alpha/(\beta q')} \wedge \frac{t}{R^{\alpha/q'+\beta}}\right)t^{-\alpha/(\beta q)},$$

that is, (UE_q) (see also Remark 2.14). □

11. Appendix

In this appendix, we collect some facts that have been used in this paper.

11.1. **Miscellaneous issues.**

Proposition 11.1. *Assume that condition* (VD) *holds and W satisfies* (2.7)*. Then there exists a constant $C > 0$ such that, for all $t > 0$ and all points $x, y \in M$ with $d(x,y) \le W^{-1}(x,t) \vee W^{-1}(y,t)$,*

$$C^{-1} \le \frac{W^{-1}(x,t)}{W^{-1}(y,t)} \le C, \tag{11.1}$$

$$C^{-1} \le \frac{V(x,W^{-1}(x,t))}{V(y,W^{-1}(y,t))} \le C, \tag{11.2}$$

Proof. Let $t > 0$. Assume that

$$d(x,y) \le W^{-1}(x,t) \vee W^{-1}(y,t).$$

Without loss of generality, assume that

$$d(x,y) \le W^{-1}(x,t),$$

otherwise, both inequalities (11.1) and (11.2) are still true by exchanging the order of x, y. Denote the constants in (VD) and (2.7) by C_V, C_W respectively. Let us divided the proof into two cases.

Case 1. $W^{-1}(x,t) \ge W^{-1}(y,t)$. Since $d(x,y) \le W^{-1}(x,t)$, we have by the left inequality in (2.7)

$$C_W^{-1}\left(\frac{W^{-1}(x,t)}{W^{-1}(y,t)}\right)^{\beta_1} \le \frac{W(x,W^{-1}(x,t))}{W(y,W^{-1}(y,t))} = \frac{t}{t} = 1,$$

thus showing that (11.1) holds for $C := C_W^{1/\beta_1}$.

Let us prove (11.2). Indeed, since $d(x,y) \le W^{-1}(x,t)$, we have by (VD) and (11.1),

$$\frac{V(x,W^{-1}(x,t))}{V(y,W^{-1}(y,t))} \le C_V\left(\frac{W^{-1}(x,t)}{W^{-1}(y,t)}\right)^\alpha \le C_V C_W^{\alpha/\beta_1},$$

thus showing the right inequality in (11.2). On the other hand, Since $d(x,y) \le W^{-1}(x,t)$, we see by (VD)

$$\frac{V(y,W^{-1}(y,t))}{V(x,W^{-1}(x,t))} \le \frac{V(y,W^{-1}(x,t))}{V(x,W^{-1}(x,t))} \le C_V\left(\frac{W^{-1}(x,t)}{W^{-1}(x,t)}\right)^\alpha = C_V,$$

thus showing the left inequality in (11.2).

Case 2. $W^{-1}(x,t) < W^{-1}(y,t)$. Since $d(x,y) \le W^{-1}(x,t) < W^{-1}(y,t)$, we have by the left inequality in (2.7) that

$$C_W^{-1}\left(\frac{W^{-1}(y,t)}{W^{-1}(x,t)}\right)^{\beta_1} \le \frac{W(y, W^{-1}(y,t))}{W(x, W^{-1}(x,t))} = \frac{t}{t} = 1,$$

thus showing that (11.1) holds again for $C := C_W^{1/\beta_1}$.

Let us prove (11.2). Indeed, since $d(x,y) \le W^{-1}(x,t) < W^{-1}(y,t)$, we have by (VD)

$$\frac{V(x, W^{-1}(x,t))}{V(y, W^{-1}(y,t))} \le \frac{V(x, W^{-1}(y,t))}{V(y, W^{-1}(y,t))} \le C_V\left(\frac{W^{-1}(y,t)}{W^{-1}(y,t)}\right)^{\alpha} = C_V,$$

thus showing the right inequality in (11.2). On the other hand, Since $d(x,y) \le W^{-1}(x,t) < W^{-1}(y,t)$, we have by (VD) and (11.1),

$$\frac{V(y, W^{-1}(y,t))}{V(x, W^{-1}(x,t))} \le C_V\left(\frac{W^{-1}(y,t)}{W^{-1}(x,t)}\right)^{\alpha} \le C_V C_W^{\alpha/\beta_1},$$

thus showing the left inequality in (11.2). □

The following was proved in [36, Lemma 2.12].

Lemma 11.2. *Let $(\mathcal{E},\mathcal{F})$ be a Dirichlet form in L^2. If*

$$f_n \xrightarrow{L^2} f, \quad \sup_n \mathcal{E}(f_n) < \infty,$$

then $f \in \mathcal{F}$, and there exists a subsequence, still denoted by $\{f_n\}$, such that $f_n \overset{\mathcal{E}}{\rightharpoonup} f$ weakly, that is,

$$\mathcal{E}(f_n,\varphi) \to \mathcal{E}(f,\varphi)$$

as $n \to \infty$ for any $\varphi \in \mathcal{F}$. And there exists a subsequence $\{f_{n_k}\}$ such that its Cesaro mean $\frac{1}{n}\sum_{k=1}^n f_{n_k}$ converges to f in $\mathcal{E}_1$-norm. Moreover, we have

$$\mathcal{E}(f) \le \liminf_{n\to\infty} \mathcal{E}(f_n).$$

11.2. **Comparison inequalities.** Recall the notion of the ρ-local Dirichlet form in Subsection 5.

The following proposition is essentially the same as [27, Theorem 4.3, p. 2627]. Here we replace the compactness of U_ρ in [27, Theorem 4.3, p. 2627] by the assumption that $\mathrm{cutoff}(U_\rho, M) \neq \emptyset$.

Proposition 11.3. *Assume that $(\mathcal{E},\mathcal{F})$ is some ρ-local regular Dirichlet form for $\rho \ge 0$. Let U be an open set such that $\mathrm{cutoff}(U_\rho, M) \neq \emptyset$, and let u be subcaloric in $(0,T_0)\times U$ where $T_0 \in (0,+\infty]$. Assume that $u(t,\cdot) \in L^\infty(M)$ for each $t \in (0,T_0)$, and*

$$u_+(t,\cdot) \xrightarrow{L^2(U)} 0 \qquad \text{as } t \to 0. \tag{11.3}$$

Then for any compact subset K of U, any $t \in (0,T_0)$, and for almost all $x \in U_\rho$,

$$u(t,x) \le \left(1 - P_t^U \mathbf{1}_U(x)\right) \sup_{0<s\le t} \|u_+(s,\cdot)\|_{L^\infty(U_\rho\setminus K)},$$

provided that $\sup_{0<s\le t}\|u_+(s,\cdot)\|_{L^\infty(U_\rho\setminus K)} < \infty$.

Proof. Note that the set U_ρ in [27, Theorem 4.3] is required to be precomact, while we only assume that $\mathrm{cutoff}(U_\rho, M) \neq \emptyset$. However, the proof is this proposition is parallel to that in [27, Theorem 4.3].

Indeed, the compactness of U_ρ is used in three places in the proof of [27, Theorem 4.3]. Firstly, the compactness of U_ρ implies that $\mathrm{cutoff}(U_\rho, M) \neq \emptyset$, which is our assumption. Secondly, it is used in [27, Theorem 2.9] to make sure that the set $K \cap \overline{\Omega}$ is compact. However, this is true since K is compact in our assumption. Thirdly, the compactness of U_ρ implies that $\mu(U) < \infty$. To overcome this difficulty, we can take a sequence of precompact open sets $\{U_i\}_{i\ge1}$ such that $U_i \uparrow U$ as $i \to \infty$ and $K \subset U_1$. Applying [27, Theorem 4.3] for each U_i, we obtain for μ-a.a $x \in (U_i)_\rho$,

$$u(t,x) \le \left(1 - P_t^{U_i}\mathbf{1}_{U_i}(x)\right) \sup_{0<s\le t} \|u_+(s,\cdot)\|_{L^\infty((U_i)_\rho\setminus K)}$$

$$\leq \left(1 - P_t^{U_i}\mathbf{1}_{U_i}(x)\right) \sup_{0<s\leq t} \|u_+(s,\cdot)\|_{L^\infty(U_\rho\setminus K)},$$

thus showing this proposition by passing to the limit as $i \to \infty$. □

By using proposition 11.3 and repeating the arguments in [27, Corollary 4.8 and Remark 4.9], we have the following result.

Corollary 11.4. *Assume that $(\mathcal{E},\mathcal{F})$ is some ρ-local regular Dirichlet form for $\rho \geq 0$. Let U,Ω be open sets such that $U_\rho \subset \Omega$ and $\mathrm{cutoff}(U_\rho,\Omega) \neq \emptyset$. Then, for any $0 \leq f \in L^\infty(M)$, any $t > 0$ and for μ-a.a. $x \in U_\rho$,*

$$P_t^\Omega f(x) - P_t^U f(x) \leq \left(1 - P_t^U\mathbf{1}_U(x)\right) \sup_{s\in(0,t]\cap\mathbb{Q}} \|P_s^\Omega f\|_{L^\infty(U_\rho\setminus K)}, \tag{11.4}$$

where K is a compact subset of U.

Using (11.4) and repeating the proof of [27, Theorem 5.1], we have the following.

Corollary 11.5. *Assume that $(\mathcal{E},\mathcal{F})$ is some ρ-local regular Dirichlet form for $\rho \geq 0$. Let U,V,Ω be three open sets such that $U_\rho \cup V_\rho \subset \Omega$. Assume that the Dirichlet heat kernel p_t^Ω exists and is locally bounded in $\mathbb{R}_+ \times \Omega \times \Omega$. Then for any $s,t > 0$ and μ-a.a. $x \in U, y \in V$,*

$$\begin{aligned} p_{t+s}^\Omega(x,y) \leq & \int_M p_t^U(x,z)p_s^V(z,y)d\mu(z) \\ & + \left(1 - P_t^U\mathbf{1}_U(x)\right) \sup_{s<t'\leq t+s} \left\|p_{t'}^\Omega(\cdot,y)\right\|_{L^\infty(U_\rho)} + \left(1 - P_t^V\mathbf{1}_V(y)\right) \sup_{t<t'\leq t+s} \left\|p_{t'}^\Omega(\cdot,x)\right\|_{L^\infty(V_\rho)}. \end{aligned} \tag{11.5}$$

11.3. **Maximum principle.** The following is elliptic maximum principle.

Proposition 11.6 ([25, Proposition 4.6, p. 116])**.** *Suppose that $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form. Let $\lambda > 0$ and Ω be a non-empty open subset of M. If $u \in \mathcal{F}$ satisfies*

$$\begin{cases} \mathcal{E}(u,\phi) + \lambda(u,\phi) \leq 0, & \forall\, 0 \leq \phi \in \mathcal{F}(\Omega), \\ u_+ \in \mathcal{F}(\Omega), \end{cases}$$

then $u \leq 0$ a.e. in Ω.

The following is parabolic maximum principle.

Proposition 11.7 ([25, Proposition 4.11, p. 117])**.** *Suppose that $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form. Fix $T \in (0,+\infty]$ and an open subset $\Omega \subset M$. If $u : (0,T) \mapsto \mathcal{F}$ is a subcaloric function in $(0,T]\times\Omega$ and satisfies*

$$\begin{cases} u_+(t,\cdot) \in \mathcal{F}(\Omega) \ \text{ for any } t \in (0,T) \\ u_+(t,\cdot) \overset{L^2(\Omega)}{\longrightarrow} 0 \ \text{ as } t \to 0, \end{cases}$$

then $u \leq 0$ a.e. on $(0,T) \times \Omega$.

11.4. **The existence of heat kernel.** The following result shows that the existence of heat kernels follows from some generalized ultracontractivity of semigroups.

Theorem 11.8 ([22, Theorem 2.2])**.** *Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form on $L^2(M,\mu)$ for a metric measure space (M,d,μ), and let $\{P_t\}_{t>0}$ be the associated heat semigroup on L^2. Fix $T_0 \in (0,\infty]$ and $1 \leq p \leq 2$. Assume that there exist a countable family $\mathcal{S}$ of open sets with $M = \cup_{U\in\mathcal{S}}U$ and a function $\varphi : \mathcal{S}\times(0,T_0) \mapsto \mathbb{R}_+$ such that, for each $t \in (0,T_0)$, $U \in \mathcal{S}$ and each $f \in L^p \cap L^2$*

$$\|P_t f\|_{L^\infty(U)} \leq \varphi(U,t)\|f\|_p.$$

Then $\{P_t\}_{t>0}$ possesses a heat kernel $p_t(x,y)$ in $(0,\infty)\times M\times M$ that satisfies Definition 2.6 for some regular $\mathcal{E}$-nest $\{F_n\}_{n=1}^\infty$ of M, and

$$p_t(x,y) = 0 \ \text{ for any } t > 0$$

whenever one of points x,y lies outside $\cup_{n=1}^\infty F_n$. Moreover, for each $t \in (0,T_0)$ and $x \in U$

$$\|p_t(x,\cdot)\|_{p'} \leq \varphi(U,t),$$

where $p' = \frac{p}{p-1}$ is the Hölder conjugate of p, and for any $1 \leq q \leq p'$.

$$\|p_t(x,\cdot)\|_q \leq (\varphi(U,t))^{(q-1)(p-1)}.$$

11.5. **Essential supremum.** The notion of the μ-regular $\mathcal{E}$-*nest* $\{F_k\}$ is given in Section 2.

Proposition 11.9. *Let $B_2 \subset B_1$ be two metric balls such that $B_1 \setminus B_2 \neq \emptyset$. Then for any quasi-continuous $v \in \mathcal{F}$,*

$$\int_{B_1\setminus B_2} v(y)J(x,dy) \le \Big(\operatorname*{esup}_{B_1} v\Big)\int_{B_2^c} J(x,dy) \quad \textit{for q.e. } x \in B_2.$$

Proof. By [19, Lemma 4.5.4(i), p. 184], the measure j charges no part of $M \times M\backslash\text{diag}$ whose projection on the factor M is exceptional. Furthermore, by [19, Theorem 4.2.1(ii), p. 161], a set $N \subset M$ is exceptional if and only if $\mathrm{Cap}_1(N) = 0$. By [19, Theorem 2.1.2(i), p. 69], there is a μ-regular $\mathcal{E}$-nest $\{F_k\}$ such that $v \in C\{F_k\}$. Set $F := \bigcup_{k\ge1} F_k$, whose complement is exceptional.

Hence, it follows that

$$\int_{B_1\setminus B_2} v(y)J(x,dy) = \int_{(B_1\setminus B_2)\cap F} v(y)J(x,dy) \quad \text{for q.e. } x \in B_2.$$

Moreover, by [24, Proposition 15.3 in Appendix], we have that for any x,

$$\int_{(B_1\setminus B_2)\cap F} v(y)J(x,dy) \le \Big(\sup_{(B_1\setminus B_2)\cap F} v\Big)\int_{(B_1\setminus B_2)\cap F} J(x,dy) \le \Big(\sup_{B_1\cap F} v\Big)\int_{B_2^c} J(x,dy) = \Big(\operatorname*{esup}_{B_1} v\Big)\int_{B_2^c} J(x,dy).$$

□

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