L^p-PARABOLICITY OF RIEMANNIAN MANIFOLDS

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Abstract. We describe the L^p-parabolicity of a Riemannian manifold in terms of a unified theory of nonlinear L^p-capacities, for the whole range $1 \le p \le \infty$. For example, the L^1-parabolicity is equivalent to the usual notion of parabolicity/recurrence, while L^2-parabolicity is equivalent to biparabolicity. For any $1 \le p \le \infty$, the L^p-parabolicity turns out to be equivalent to the L^q-Liouville property for positive superharmonic functions, where p and q are Hölder conjugate exponents. We also provide a new capacitary characterization of the L^1-Liouville property. Finally we obtain an almost optimal volume growth conditions for L^p-parabolicity for $1 < p \le 2$ as well as a sharp volume condition for all $1 < p < \infty$ for manifolds of non-negative Ricci curvature and for model manifolds.

Contents

1. Introduction

The properties of diffusion processes on Riemannian manifolds have been extensively investigated and have proven to be useful in the study of the geometric structure of the manifolds and the analytic properties of differential operators such as the Laplace-Beltrami operator (see, e.g., [3, 12, 17, 21, 25] and references therein).

One of such properties is recurrence: Brownian motion X_t on a manifold M is called *recurrent,* and the manifold M is called *parabolic*, if the process X_t starting at any point $x \in M$ visits infinitely often every neighborhood of x almost surely. The notion of parabolicity plays a crucial role in the classification of Riemann surfaces (cf. [2]), and it reveals deep connections between Stochastic Analysis, Potential Theory, Analysis and Geometry.

Parabolicity admits several equivalent characterizations, such as the non-existence of a positive Green function of the Laplace-Beltrami operator, the validity of the Liouville property for positive superharmonic functions, and also the vanishing of the harmonic capacity of some/every pre-compact set, to quote a few (see, e.g., [12]), which makes this notion an indispensable tool in Potential Theory and PDEs on Riemannian manifolds.

We say that a manifold M admits L^q-*Liouville property* for superharmonic functions if any positive superharmonic function from $L^q(M)$ is constant. For example, the L^∞-Liouville property is equivalent to the parabolicity of M (since the minimum of a superharmonic function with a constant is again superharmonic).

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At the other end of the scale, the L^1-Liouville property for superharmonic functions is equivalent to the fact that the expected lifetime of Brownian motion on M is infinite. This is closely related (although not equivalent) to the *stochastic completeness* of the manifold, which means that the lifetime of Brownian motion is infinite with probability one (cf. [12] and [5, 11, 20]).

It is natural to extend the investigation of the L^q-Liouville property for positive superharmonic functions to the entire scale of exponents $1 \le q \le \infty$, which is the main subject of this paper. In fact, the main purpose of this paper is to introduce a unified theory of nonlinear capacities in the setting of Riemannian manifolds and to describe the L^q-Liouville property in terms of L^p-capacities, where p and q are Hölder conjugate exponents.

In [1] D.R. Adams and L.I. Hedberg gave a systematic treatment of a general theory of L^p-capacities on the Euclidean space as a generalization of the so called (α, p)-capacities related to nonlinear potentials developed in [10, 14, 15, 19, 22]. For the problems we want to deal with we will restrict our attention to L^p-capacities defined in terms of the positive Green kernel of the manifold or of open subsets thereof, typically with Dirichlet boundary conditions.

Since on parabolic manifolds L^q-Liouville theorem holds for all $q \in [1, \infty]$, we assume in what follows that M is non-parabolic, in particular, M possesses a positive Green function of the Laplace-Beltrami operator. Let $1 \le p < \infty$. Denote by $g(x, y)$ the minimal positive Green function on M and define the L^p-capacity of a compact subset $K \subset M$ by

$$C_p(K)^{\frac{1}{p}} = \sup \left\{ \nu(K) \colon \nu \in \mathcal{M}^+(K), ||G\nu||_{L^q(M)} \le 1 \right\}, \tag{1.1}$$

where $p^{-1} + q^{-1} = 1$ and the potential $G\nu$ is given by

$$G\nu(x) = \int_K g(x, y) d\nu(y).$$

In the limiting case $p = 1$, the L^1-capacity coincides with the usual harmonic capacity $\mathrm{Cap}(K)$. The equivalence of these two notions of capacity is valid also in the case when M is parabolic as in this case $g \equiv \infty$ and, hence, the capacity of any compact set is zero.

We say that a manifold M is L^p-parabolic if the L^p-capacity of every compact subset of M has zero C_p-capacity. One of the main results of this paper, Theorem 3.11, says that, for any $1 \le p < \infty$, the L^p-parabolicity is equivalent to the L^q-Liouville property. This result extends the aforementioned equivalence between the parabolicity and the L^∞-Liouville property.

Although the other limiting case, namely $p = \infty$, is not covered by the definition given in (1.1), we obtain a characterization of the L^1-Liouville property in terms of a suitable L^∞-capacity (see Theorem 4.2). This in turn allows us to define a notion of L^∞-parabolicity equivalent to the L^1-Liouville property.

An important tool in establishing these equivalences, which is useful in its own right, is the characterization of the L^q-Liouville property, $1 \le q < \infty$, by means of the non-integrability of $g(x, \cdot)^q$ outside a pre-compact open set containing x, for some/every $x \in M$ (see Theorem 3.6). As a consequence of this characterization we prove that the L^2-parabolicity coincides with the *biparabolicity* of M, where the latter notion was introduced in [9] (see Subsection 3). Besides, this allows us to show that L^p-parabolicity implies L^s-parabolicity for every $1 \le p \le s < \infty$ (cf. Corollary 3.14).

As a further application of our results we also show that, on a complete Riemannian manifold M, if, for some $o \in M$ and sufficiently large $r > 1$, the volume $V(r)$ of the ball $B_r(o)$ centered at o of radius r satisfies

$$V(r) \leq C\frac{r^{2p}}{\log r}, \tag{1.2}$$

for some constant $C > 0$, then M is L^p-parabolic provided $1 < p \leq 2$ (cf. Theorem 5.1).

This extends the nearly optimal sufficient volume condition established in [9] for biparabolicity ($p = 2$), and is compatible with the known volume conditions for the usual parabolicity in the limiting case $p = 1$. For manifolds with non-negative Ricci curvature we obtain a sufficient integral condition (5.6) in the entire range $1 < p < \infty$ that implies the sharp volume condition

$$V(r) \leq Cr^{2p}(\log r)^{p-1}.$$

Moreover, the integral condition (5.6) turns out to be always valid for general model manifolds (see Section 5).

The paper is organized as follows. In Section 2 we describe the general theory of L^p-capacities on domains of a Riemannian manifold. In particular we show that they satisfy natural monotonicity and limit properties with respect to exhaustions (cf. Proposition 2.5), and we establish some useful results valid in this setting, for instance, Remark 2.9 dealing with the case $p = 1$, and Theorem 2.15, where we establish the equivalence between the C_p-capacity and a suitable Laplacian capacity.

Our main results are stated in Section 3. We establish there the connection between the L^p-parabolicity and the non-integrability of $g(x, \cdot)^q$, which yields the equivalence between the L^p-parabolicity and the L^q-Liouville property. Besides, it also shows that a pre-compact open set has zero C_p-capacity if and only if this holds for every such set (cf. Corollary 3.13). We end this section showing the equivalence between biparabolicity and L^2-parabolicity.

Section 4 is devoted to the limiting case $q = 1$. There, we introduce a C_∞-capacity and, by describing the extremal capacitary measure of compact sets in terms of the mean exit time of the manifold, we obtain a capacitary characterization of the L^1-Liouville property (see Theorem 4.2). We end the section collecting several properties of the C_∞-capacity which in particular imply that it is a Choquet capacity (cf. Proposition 4.3). The volume growth conditions for L^p-parabolicity are discussed in Section 5.

2. L^p-CAPACITY

Let (M, μ) be a Riemannian manifold without boundary endowed with the usual Riemannian measure μ and, for $1 < p < \infty$, denote by $L^p_+(M)$ the set of non-negative functions in $L^p(M)$. The general theory of nonlinear capacities presented in [1, Sect.2.3] is based on considering generic kernels, that is, non-negative functions $g \colon M \times M \to \mathbb{R}$, such that $g(x, \cdot)$ is lower semicontinuous on M and $g(\cdot, y)$ is measurable on M.

In what follows we will restrict our attention to subsets $U \subset M$ that admit a finite positive Dirichlet Green kernel $g^U(x, y)$ with zero boundary data, which is obtained by the usual exhaustion procedure (see e.g. [8, 20]), and satisfies $g^U(x, y) \leq g^V(x, y)$ for every $x, y \in U \subset V$. In particular, when $U = M$, we denote by $g(x, y)$ the positive minimal Green kernel of M.

Let $\mathcal{M}^+(U)$ be the set of non-negative Radon measure supported in U. For $\nu \in \mathcal{M}^+(U)$ define the potential $G^U\nu$ by

$$G^U\nu(y) \;=\; \int_U g^U(x, y)d\nu(x), \quad y \in U.$$

If $\nu = f\mu$ for some measurable non-negative function f, we will write $G^U f = G^U(f\mu)$. The potential $G^U\nu$ is well defined provided we allow it to take the value ∞.

As in [1, Sect.2.3], we define the relative variational C_p-capacity of the capacitor (E, U) as follows.

Definition 2.1. *For every $1 < p < \infty$ and $E \subset U$, let*

$$\Omega_E^U = \{f \in L_+^p(U) \colon G^U f(x) \geq 1 \ \ \textit{for all} \ \ x \in E\},$$

and define

$$C_p(E, U) = \inf_{\Omega_E^U} \int_U f^p d\mu. \tag{2.1}$$

We set $C_p(E, U) = \infty$ whenever $\Omega_E^U = \emptyset$, and write $C_p(E)$ instead of $C_p(E, M)$ if $U = M$.

Remark 2.2. *As we will see in Theorem 2.8 below, the quantity defined in (1.1) agrees with that defined in (2.1) for all Borel subsets.*

This set function is countably subadditive (see [1, Sec.2.3]), and it is easily seen to be monotone in E, that is, if $E_1 \subset E_2 \subset U$ then

$$C_p(E_1, U) \leq C_p(E_2, U).$$

Furthermore, $C_p(E, U)$ is a Choquet capacity for every $1 < p < \infty$ (cf. [1, Prop.2.3.12]). The usual monotonicity property with respect to U follows from the fact that if $f \in L_+^p(U_1)$ is such that $G^{U_1} f \geq 1$ on E, extending f by zero on $U_2 \backslash U_1$, by monotonicity, $G^{U_2} f \geq 1$ and therefore $\Omega_E^{U_1} \subseteq \Omega_E^{U_2}$.

We recall the following characterization of subsets with null capacity.

Proposition 2.3 (Prop. 2.3.7 in [1])**.** *Let $E \subset U$ and $1 < p < \infty$. Then $C_p(E, U) = 0$ if and only if there is $f \in L_+^p(U)$ such that*

$$E \subset \{x \in U \colon G^U f(x) \equiv \infty\}.$$

We shall say that a property holds C_p-quasieverywhere, C_p-q.e. for short, if it holds except on a subset E with $C_p(E, U) = 0$.

Remark 2.4. One can extend the definition of $G^U f$ to arbitrary measurable functions f by setting

$$G^U f(x) = G^U f^+(x) - G^U f^-(x),$$

whenever at least one of the summands on the right hand side is finite. Then, $G^U f(x)$ is well defined and finite C_p-q.e. by Proposition 2.3. It is easy to see that the restriction on the sign of the test functions in Ω_E^U can be dropped, that is

$$C_p(E, U) = \inf \left\{ ||f||_{L^p(U)}^p \colon f \in L^p(U), G^U f \geq 1 \ \ C_p\text{-q.e} \ \ \text{on} \ \ E \right\}.$$

The main advantage in using L^p spaces, for $1 < p < \infty$, is to have the existence of a capacitary function. Indeed, since Ω_E^U is a convex subset of the uniformly convex Banach space $L^p(M)$, by standard Functional Analysis results there exists an extremal function f^E belonging to the closure of Ω_E^U in $L^p(M)$ that minimizes the above capacity, i.e.,

$$C_p(E, U) = \int_U (f^E)^p d\mu.$$

The function f^E is called the capacitary function of E, and $G^U f^E$ is called the capacitary potential of E. Furthermore, $\overline{\Omega_E^U}$ can be explicitly described as (cf. [1, Prop.2.3.9])

$$\overline{\Omega_E^U} = \{f \in L_+^p(U) \colon G^U f(x) \geq 1 \ \ C_p\text{-q.e. on} \ \ E\}.$$

In the setting of Riemannian manifolds we have the following density result which will be instrumental in establishing connections between different nonlinear capacities.

Proposition 2.5. *Let $U \subset M$ be an open set and let $E \subset U$ be a compact subset. Then, for every $1 < p < \infty$, it holds*

$$C_p(E,U) = \inf\left\{\int_U \phi^p\, d\mu \,:\, 0 \le \phi \in C_c^\infty(U),\ G^U\phi \ge 1 \text{ on } E\right\}. \tag{2.2}$$

Proof. By definition, for every $\epsilon > 0$, there exists $f \in L^p_+(U)$ such that

$$G^U f(x) \ge 1 \text{ on } E \ \text{ and } \ \int_U f^p d\mu < C_p(E,U) + \epsilon/2.$$

Since

$$\int_U f^p d\mu = \inf_{c>1} \int_U (cf)^p d\mu$$

and, by homogeneity, $G^U(cf)(x) = cG^U f(x)$, choosing $c > 1$ close enough to 1 and replacing f with cf show that for every $\epsilon > 0$ there exists $f \in L^p_+(U)$ such that

$$\int_U f^p d\mu < C_p(E,U) + \epsilon \qquad \text{and} \qquad G^U f(x) \ge c > 1 \text{ on } E. \tag{2.3}$$

Finally, let U_n be an exhaustion of U by an increasing sequence of pre-compact open sets and set

$$f_n = \min\{n, f\}\mathbf{1}_{U_n},$$

where f is a function as in (2.3). Then for every n, $f_n \in L^p_+(U) \cap L^\infty(U)$ with compact support in U. Since $f_n \nearrow f$, by monotone convergence,

$$\int_U f_n^p d\mu \nearrow \int_U f^p d\mu$$

and

$$G^U f_n(x) \nearrow G^U f(x) \ \text{ pointwise on } U.$$

We claim there exists n_o such that $G^U f_n(x) > (1+c)/2$ for every $x \in E$ and $n \ge n_o$. Indeed, for every n let

$$V_n = \{x \in U \,:\, G^U f_n(x) > (1+c)/2\}.$$

Then, V_n is open, since $G^U f_n$ is lower semi-continuous, $V_n \subseteq V_{n+1}$ and $E \subset \cup_n V_n$, since $G^U f_n(x) \to G^U f(x) \ge c > (1+c)/2$ for every $x \in E$. By compactness there exists n_0 such that $E \subset V_n$ for every $n \ge n_0$.

By taking n large enough, and replacing f with f_n we conclude that for every $\epsilon > 0$ there exists $f \in L^p_+(U) \cap L^\infty_c(U)$ satisfying

$$\int_U f^p d\mu < C_p(E,U) + \epsilon \ \text{ and } \ G^U f \ge 1 + \delta > 1 \text{ on } E, \tag{2.4}$$

for some $\delta > 0$.

Suppose now that U is bounded with smooth boundary. Fix $1 < q < m/(m-2)$ and let $m/2 < q' < \infty$ be its conjugate exponent. Let f be as in (2.4) and let $0 \le \varphi_k \in C_c^\infty(U)$ be such that

$$\varphi_k \to f \quad \text{in } \ L^{q'}(U),$$

and therefore in every $L^s(U)$ with $s \le q'$. Thus

$$\int_U \varphi_k^p d\mu \to \int_U f^p d\mu < C_p(E,U) + \epsilon.$$

On the other hand, we claim that $h \to G^U h$ defines a bounded operator from $L^{q'}(U)$ into $L^\infty(E)$. Assuming the claim, we conclude that

$$||G^U f - G^U \varphi_k||_{L^\infty(E)} \le C||f - \varphi_k||_{L^{q'}} \to 0, \ \text{ as } \ k \to \infty,$$

so that, for k sufficiently large

$$G^U\varphi_k(x) > (1+c)/2 > 1 \quad \forall x \in E,$$

and (2.2) holds for U bounded with smooth boundary.

It remains to prove the claim. Indeed, according to [4, Thm. 4.17], there exists a constant C which depends only on the distance of x from ∂U such that

$$g^U(x,y) \le Cd(x,y)^{2-m} \quad \forall y \in U,$$

and it follows that for every $x \in E$ and fixed $1 < s < m/(m-2)$

$$\begin{aligned}\int_U g^U(x,y)^s d\mu(y) &\le C^s \int_{B(x,R)} d(x,y)^{s(2-m)} d\mu(y)\\ &\le C^s \int_0^R r^{s(2-m)} \left(B^{-1}\sinh(Br)\right)^{m-1} dr = C_1(s)^s,\end{aligned}$$

where $R = \operatorname{diam} U$ and $B > 0$ is such that

$$\mathrm{Ric}_M \ge -(m-1)B^2 \quad \text{on} \quad \{x : d(x,U) < R+1\}.$$

Thus, if s' is the exponent conjugate to s and $h \in L^{s'}$, for every $x \in E$,

$$\begin{aligned}|G^U h(x)| &= |\int_U g^U(x,y)h(y)d\mu(y)|\\ &\le \left(\int_U g^U(x,y)^s d\mu(y)\right)^{1/s} ||h||_{L^{s'}(U)} = C_1(s)||h||_{L^{s'}(U)},\end{aligned}$$

and the claim follows.

To conclude, let U be a general open set which admits a positive Dirichlet Green kernel $g^U = g$, and let U_n be an increasing exhaustion of U by pre-compact open sets with smooth boundary with $U_1 \supset E$. Recall that, denoting by g_n the Dirichlet Green kernel of U_n, we have (essentially by definition)

$$g(x,y) = \lim_n g_n(x,y).$$

Let f be a function as in (2.4) and let n_1 be such that $\operatorname{supp} f \subset U_{n_1}$. Since $g_n(x,y) \nearrow g(x,y)$ in U_{n_1}, for every $n \ge n_1$ and $x \in E$,

$$G^{U_n}f(x) = \int_{U_{n_1}} g_n(x,y)f(y)d\mu \nearrow \int_{U_{n_1}} g(x,y)f(y)d\mu = G^U f(x),$$

and arguing as above we deduce that for sufficiently large n, $G^{U_n}f(x) > 1 + \delta/2 > 1$ on E and therefore

$$C_p(E,U) \le C_p(E,U_n) \le \int_{U_{n_1}} f^p d\mu \le C_p(E,U) + \epsilon.$$

Since we have proved that

$$C_p(E,U_n) = \inf\left\{\int_{U_n} \varphi^p d\mu \,:\, 0 \le \varphi \in C_c^\infty(U_n) \text{ and } G^{U_n}f(x) \ge 1\right\}$$

the required conclusion follows. □

We note that in the last part of the above proof we have also shown the following.

Proposition 2.6. *Let U be an open set of M and let $K \subset U$ be a compact set. Let also U_n be an increasing exhaustion of U by open sets with $K \subset U_1$ and denote by $g(x,y)$, $g_n(x,y)$ the Dirichlet Green kernels of U, U_n, respectively. Then*

$$C_p(K,U) = \lim_{n\to\infty} C_p(K,U_n).$$

Remark 2.7. For $1<p<\infty$, $C_p(\cdot,U)$ is a Choquet capacity, thus Borel subsets are capacitable, and the conclusions of Propositions 2.5 and 2.6 hold for every Borel subset $E\subset U$.

The capacity we have introduced admits a dual representation by means of non-negative Radon measures. This duality is possible due to a general version of the von Neumann minimax theorem (cf. [1, Thm.2.4.1]). For any $1<p<\infty$, let $1<q<\infty$ be its Hölder conjugate exponent, that is, $pq=p+q$.

Theorem 2.8 (Thm.2.5.1 in [1])**.** *Let $K\subset M$ be a compact subset, and let $1<p<\infty$. Then*

$$C_p(K,U)^{\frac{1}{p}}=\sup\{\nu(K)\colon \nu\in\mathcal{M}^+(K), ||G^U\nu||_{L^q(U)}\leq 1\}. \tag{2.5}$$

Remark 2.9. The limit case $p=1$ corresponds to the standard capacity $\mathrm{Cap}(K,U)$ since

$$\mathrm{Cap}(K,U)=\sup\{\nu(K)\colon \nu\in\mathcal{M}^+(K), ||G^U\nu||_{L^\infty(U)}\leq 1\}.$$

Indeed, $\mathrm{Cap}(K,U)$ is minimized by a function $w\in W_0^{1,2}(U)$ that satisfies

$$\begin{cases} \triangle w=0 \ \text{ in } \ U\backslash K,\\ w=0 \ \text{ on } \ \partial U, w=1 \ \text{ on } \ \partial K.\end{cases}$$

Moreover, w can be extended on K such that $\triangle w\leq 0$ weakly in U, and $w=1$ in K. Now, let $h=G^U\nu$ be the potential of a measure $\nu\in\mathcal{M}^+(K)$ such that $||G^U\nu||_{L^\infty(U)}\leq 1$. Since $\triangle h=-\nu$ weakly, we have that

$$\nu(K)=-\int_{\partial K}\partial_\eta h\,d\mu',$$

where μ' denotes the induced measure on ∂K, and η is the outward unit normal. Thus

$$\begin{aligned}\nu(K)^2&=\left(-\int_{\partial K}\partial_\eta h\,d\mu'\right)^2=\left(-\int_{\partial(U\backslash K)}w\partial_\eta h\,d\mu'\right)^2\\ &=\left(\int_{U\backslash K}\langle\nabla w,\nabla h\rangle d\mu\right)^2\leq\left(\int_{U\backslash K}|\nabla w|^2d\mu\right)\left(\int_{U\backslash K}|\nabla h|^2d\mu\right).\end{aligned}\tag{2.6}$$

On the other hand,

$$\begin{aligned}\int_{U\backslash K}|\nabla h|^2d\mu &= -\int_{U\backslash K}h\triangle h d\mu+\int_{\partial(U\backslash K)}h\partial_\eta h\,d\mu'\\ &\leq ||h||_{L^\infty(K)}\int_{\partial K}-\partial_\eta h\,d\mu'\leq\nu(K),\end{aligned}\tag{2.7}$$

and, inserting into (2.6), we obtain

$$\nu(K)\leq\int_{U\backslash K}|\nabla w|^2d\mu=\mathrm{Cap}(K,U).$$

By Riesz decomposition theorem [16, Thm.4.5.11] there is a unique measure $\nu^K\in\mathcal{M}^+(K)$ with $w=G^U\nu^K$. Choosing $h=w$, the inequalities in (2.6) and (2.7) become equalities, and the conclusion follows.

The set function defined by the right hand side of (2.5), replacing compact subsets K by arbitrary Borel subsets E, is easily seen to be monotone in E, and as a consequence of Theorem 2.8 and Remark 2.9, it satisfies the conditions of the capacitability given in [1, Prop.2.3.12]. It follows that the equality in (2.5) holds for every Borel subsets (see also [1, Cor.2.5.2]).

Definition 2.10. *For a Borel subset $E \subset U$ and $p \in [1, \infty)$ we define the L^p-capacity of the capacitor (E, U) by*

$$C_p(E,U) = \sup\{\nu(E)^p \colon \nu \in \mathcal{M}^+(E), ||G^U \nu||_{L^q(U)} \leq 1\},$$

where $q \in (1, \infty]$ is the Hölder conjugate exponent of p.

As in the case $p = 1$, for $1 < p < \infty$ there exists an extremal measure $\nu^K \in \mathcal{M}^+(K)$ realizing the C_p-capacity of a compact set $K \subset U$, whose potential is the extremal function $f^K \in L^p_+(U)$.

Corollary 2.11 (Thm.2.5.3 in [1]). *Let $K \subset U$ be a compact set, and $1 < p < \infty$. Then there is $\nu^K \in \mathcal{M}^+(K)$, called the capacitary measure of K, and a capacitary function $f^K = (G^U \nu^K)^{q-1}$, such that $G^U f^K(x) \geq 1$ C_p-q.e on K, and*

$$\nu^K(K) = \int_U \left(G^U \nu^K\right)^q d\mu = \int_U G^U f^K d\nu^K = C_p(K,U).$$

We end this section providing alternatives characterizations of the L^p-capacity. First of all, guided by Corollary 2.11, the capacitary potential of a measure $\nu \in \mathcal{M}^+(U)$ is defined by

$$\begin{aligned} V_p^\nu(x) &= G^U(G^U \nu)^{q-1}(x) \\ &= \int_U g^U(x,y) \left(\int_U g^U(z,y) d\nu(z) \right)^{q-1} d\mu(y). \end{aligned} \tag{2.8}$$

By analogy with the classical potential theory, the quantity

$$\int_U V_p^\nu d\nu = \int_U (G^U \nu)^q d\mu$$

is called the generalized energy (see [1, Def.2.5.4]).

Proposition 2.12 (Thm.2.5.5 in [1]). *Let $K \subset U$ be a compact set of M, and $1 < p < \infty$. Then*

$$\begin{aligned} G^U f^K(x) &= V_p^{\nu^K} \leq 1 \quad \textit{for all } \ x \in \operatorname{supp} \nu^K, \\ G^U f^K(x) &= V_p^{\nu^K} \geq 1 \quad C_p\textit{-q.e on } K. \end{aligned}$$

Moreover,

$$C_p(K,U) = \max\{\nu(K) : \nu \in \mathcal{M}^+(K), V_p^\nu(x) \leq 1 \quad \textit{for all } \ x \in \operatorname{supp} \nu\}.$$

Inspired by the variational definition of capacity by means of the generalized energy functional in the classical potential theory, we introduce the following set function which will provide a first alternative definition of C_p.

Definition 2.13. *Let $K \subset M$ be a compact set, and $1 < p < \infty$. We define*

$$Cap_{V_p}(K,U) = \inf \left\{ \int_U V_p^\nu d\nu \colon \nu \in \mathcal{M}^+(K), V_p^\nu \geq 1 \ C_p\textit{-q.e on } K \right\},$$

where V_p^ν is a capacitary potential for ν.

In the next result we are going to show the announced equivalence on compact sets.

Theorem 2.14. *Let $K \subset U$ be a compact subset of an open set $U \subset M$, and let $1 < p < \infty$. Then*

$$C_p(K,U) = Cap_{V_p}(K,U).$$

Proof. By Corollary 2.11, there is $\nu^K \in \mathcal{M}^+(K)$ such that

$$C_p(K,U) = \int_U \left(G^U \nu^K\right)^q d\mu = \int_U V^{\nu^K} d\nu,$$

where V^{ν^K} is the capacitary potential for K. Hence, taking infimum in the right hand side we obtain $\mathrm{Cap}_{V_p}(K,U) \leq C_p(K,U)$. For the reverse inequality we may assume $\mathrm{Cap}_{V_p}(K,U) < \infty$, for otherwise there is nothing to prove. So, let $\nu \in \mathcal{M}^+(K)$ such that $V_p^\nu \in L^1(U,d\nu)$ and $V_p^\nu(x) \geq 1$ C_p-q.e on K. Define $h(x) = (G^U\nu)^{q-1}(x)$. Since $pq - p = q$, $V_p^\nu = G^U(G^U\nu)^{q-1}$ and $\operatorname{supp}\nu \subset K$, we can conclude that $h \in L^p_+(U)$. Furthermore,

$$G^U h(x) = G^U(G^U\nu)^{q-1}(x) = V_p^\nu(x) \geq 1, \quad C_p\text{-q.e on } K.$$

Therefore,

$$C_p(K,U) \leq \int_U h^p d\mu = \int_U \left(G^U\nu\right)^q d\mu = \int_U V_p^\nu d\nu$$

and taking infimum we conclude the proof. □

The last result in this section establishes the equivalence between the C_p-capacity and the $(2,p)$-capacity (Laplacian capacity) with $1 < p < \infty$.

Let $U \subset M$ be a bounded open subset. Recall that $W_0^{2,p}(U)$ is the closure of $C_0^\infty(U)$ in the norm of $W^{2,p}(M)$. For a compact subset $K \subset U$ we define

$$\begin{aligned}\mathrm{Cap}_{2,p}(K,U) &= \inf\left\{\int_U |\Delta u|^p d\mu \colon u \in W_0^{2,p}(U), u \geq 1 \text{ on } K\right\}\\ &= \inf\left\{\int_U |\Delta u|^p d\mu \colon u \in C_c^\infty(U), u \geq 1 \text{ on } K\right\}.\end{aligned}$$

The following result is a consequence of Proposition 2.5.

Theorem 2.15. *Let $K \subset U$ be a compact subset of an open set $U \subset M$. Then for every $1 < p < \infty$ we have*

$$\mathrm{Cap}_{2,p}(K,U) = C_p(K,U).$$

Proof. Let us assume, without loss of generality, that $C_p(K,U) < \infty$. Due to Proposition 2.5 we can pick $0 \leq \phi \in C_c^\infty(U)$ such that $G^U\phi(x) \geq 1$ on K as a test function for $C_p(K,U)$. Setting $u(x) = G^U\phi(x) \in C_c^\infty(U)$, and observing that $u(x) \geq 1$ on K, by definition we have

$$\mathrm{Cap}_{2,p}(K,U) \leq \int_U |\Delta u|^p d\mu = \int_U \phi^p d\mu.$$

Hence, taking infimum on ϕ we conclude that

$$\mathrm{Cap}_{2,p}(K,U) \leq C_p(K,U).$$

For the reverse inequality, we take $u \in C_c^\infty(U)$ such that $u \geq 1$ on K. By Green representation we have $u = G^U(-\Delta u)$. By Remark 2.4 the function $-\Delta u \in L^p(U)$ is a test for $C_p(K,U)$, and thus

$$C_p(K,U) \leq \int_U |\Delta u|^p d\mu.$$

Again the result follows by taking infimum in the right hand side. □

Remark 2.16. I*t is worth to mention that in the limiting case $p = 1$ the C_1-capacity does not coincide with the* $\mathrm{Cap}_{2,1}$*-capacity. In fact, in* [6, Thm.E.1] *the authors proved the equality $C_1(K) = 2\mathrm{Cap}_{2,1}(K)$ for every compact set $K \subset \mathbb{R}^n$. It would be interesting to establish this equality on a general manifold.*

3. L^p-parabolicity and the L^q-Liouville property

Let $1 \leq q \leq \infty$. A manifold M satisfies the L^q-Liouville property, briefly M is L^q-Liouville, if every superharmonic function $u \in L^q_+(M)$ is constant. Since parabolic manifolds are trivially L^q-Liouville, we will restrict ourselves to the non-parabolic case, and let $g(x,y)$ be the positive finite Green kernel of M. Our main purpose in this section is to show that, for $1 < q < \infty$, the L^q-Liouville property holds on M if and only if $C_p(\overline{E}, M) = 0$ for some/every pre-compact open set $E \subset M$, with $pq = p + q$. The L^1-Liouville property and its capacitary description will be the topic of Section 4.

Extending the usual definition of parabolicity we set the following

Definition 3.1. *Let $1 \leq p < \infty$. A manifold M is said to be L^p-parabolic if $C_p(\overline{E}) = 0$ for every pre-compact open set $E \subset M$.*

With this definition we are going to prove that the L^q-Liouville property is equivalent to the L^p-parabolicity. As a first step in this direction, generalizing the characterization for the L^1-Liouville property, we show that the L^q-Liouville property amounts to the non-integrability of $g(x,\cdot)^q$ in the complement of a compact neighborhood of $x \in M$.

Proposition 3.2. *The L^q-Liouville property holds on M for $1 \leq q < \infty$ if and only if $g(x,\cdot)^q$ is non-integrable outside a ball $B_r(x)$ centered at $x \in M$ with radius $r > 0$.*

Proof. In one direction, if $g(x,\cdot)^q$ is integrable on $M \backslash B_r(x)$, then $u(y) = \min\{g(x,y), 1\}$ is a superharmonic function in $L^q_+(M)$ and M is not L^q-Liouville. Reciprocally, suppose that $f \in L^q_+(M)$ is a non-identically zero superharmonic function. By the minimum principle, f is positive on M. Let E be a pre-compact open set and $x_0 \in M$ such that $B_r(x_0) \subset \overline{E}$. The usual exhaustion argument applied to the Green function centered at x_0 gives the existence of a constant $C > 0$ such that $g(x_0, y) \leq Cf(y)$ for every $y \notin B_r(x_0)$. Then,

$$\int_{M \backslash B_r(x_0)} g(x_0,y)^q d\mu(y) \leq \int_{M \backslash B_r(x_0)} f(y)^q d\mu(y) < \infty.$$

$\square$

As a corollary of the above proposition we have the following hierarchy for the L^q-Liouville property.

Corollary 3.3. *If M is L^q-Liouville, then M is also L^r-Liouville for every $1 \leq r \leq q \leq \infty$.*

Proof. Since the Green kernel $g(x,\cdot)$ is harmonic in $M \backslash \{x\}$, and it is obtained as limit of the Dirichlet Green kernel of an exhaustion, we have that

$$\sup_{M \backslash B_r(x)} g(x,\cdot) \leq \sup_{\partial B_r(x)} g(x,\cdot),$$

for any $B_r(x) \subset M$. The conclusion follows immediately from Proposition 3.2. $\square$

Specializing (2.8) to measures with densities, we define the nonlinear Green operator G^q acting on all non-negative measurable functions f by the formula

$$G^q f(x) = G(Gf)^{q-1}(x) = \int_M g(x,z) \left(\int_M g(z,y) f(y) d\mu(y) \right)^{q-1} d\mu(z). \tag{3.1}$$

We begin with the following simple comparison result.

Lemma 3.4. *Given $x_0 \in M$ and two concentric balls $B_0 \subset B_1$ centered at x_0, there exists a constant $C > 0$, depending only on B_1, such that*

$$g(x,z) \leq Cg(x',z), \quad \textit{for every} \;\; x,x' \in B_0 \;\; \textit{and} \;\; z \in B_1^c.$$

Proof. It is a simple application of Harnack's inequality once it is noticed that for every $z \in B_1^c$, $x \mapsto g(x,z)$ is a positive harmonic function on B_1. $\square$

Remark 3.5. As a direct consequence of Lemma 3.4 we have that, for every function $f \in C_c^\infty(M)$, if $G^q f(x_0) = \infty$ for some point $x_0 \in M$, then $G^q f(x) = \infty$ for every point $x \in M$. Indeed, assume the contrary and suppose that there exists $x_1 \in M$ satisfying $G^q f(x_1) < \infty$. Let $B_0 \subset B_1$ be concentric balls with $x_0, x_1 \in B_0$. By the lemma there exists a constant $C > 0$ such that

$$g(x_0,z) \leq C g(x_1,z), \quad \text{for any} \;\; z \in B_1^c.$$

Since

$$Gf(z) = \int_M g(z,y) f(y) d\mu(y) \in C^\infty(M)$$

is bounded on $\overline{B}_1$ and since the Green kernel is locally integrable, we have

$$\begin{aligned} G^q f(x_0) &= \int_M g(x_0,z) \left(\int_M g(z,y) f(y) d\mu(y) \right)^{q-1} d\mu(z) \\ &= \int_{\overline{B}_1} g(x_0,z) (Gf(z))^{q-1} d\mu(z) + \int_{B_1^c} g(x_0,z)(Gf(z))^{q-1} d\mu(z) \\ &\leq \sup_{\overline{B}_1} (Gf)^{q-1} ||g(x_0,\cdot)||_{L^1(\overline{B}_1)} + c \int_{B_1^c} g(x_1,z)(Gf(z))^{q-1} d\mu(z) \\ &< \infty, \end{aligned}$$

which proves the claim. Indeed, the above argument also shows that $G^q f(x_0)$ is uniformly bounded in B_0 with a bound which depends only on $G^q f(x_1)$, on $\sup_{\overline{B}_1} (Gf)^{q-1}$ and on

$$||g(x_0,\cdot)||_{L^1(\overline{B}_1)} \leq \sup_{\overline{B}_0} G\varphi < \infty$$

where $\varphi \in C_c^\infty(M)$ is such that $\mathbf{1}_{\overline{B}_1} \leq \varphi$.

In the next result we establish the relation between the non-integrability of $g(x,\cdot)^q$ outside a ball and the explosion of the nonlinear Green operator G^q.

Theorem 3.6. *For $q \in [1,\infty)$ the following assertions are equivalent.*

i) $G^q \equiv \infty$.
ii) $G^q f(x_0) = \infty$ *for some* $x_0 \in M$, *and some* $0 \leq f \in C_0^\infty(M)$.
iii) *There exist* $x \in M$ *and* $r > 0$ *such that* $g(x,\cdot) \notin L^q(M \backslash B_r(x))$.
iv) *For every* $x \in M$ *and every* $r > 0$, $g(x,\cdot) \notin L^q(M \backslash B_r(x))$.
v) M *is* L^q*-Liouville.*

Proof. The equivalence **iv**) $\Leftrightarrow$ **v**) is the content of Proposition 3.2.
i) $\Rightarrow$ **ii**) : Obvious.
ii) $\Rightarrow$ **iii**) : Let $f \in C_0^\infty(M)$ be a non-negative function and let $x_0 \in M$ be such that $G^q f(x_0) = \infty$. Take $r > 0$ sufficiently large to have $\operatorname{supp} f \subset B_r(x_0)$. By the local integrability of $g(x_0,\cdot)$,

$$\begin{aligned} (3.2) \quad & \int_{B_{2r}(x_0)} g(x_0,z) \left(\int_M g(z,y) f(y) d\mu(y) \right)^{q-1} d\mu(z) \\ &= \int_{B_{2r}(x_0)} g(x_0,z) \left(\int_{B_r(x_0)} g(z,y) f(y) d\mu(y) \right)^{q-1} d\mu(z) < \infty. \end{aligned}$$

On the other hand, by Lemma 3.4, there exists $C > 0$, which depends only on $B_{2r}(x_0)$, such that

$$g(z,y) \le Cg(z,x_0) \quad \forall z \notin B_{2r}(x_0) \text{ and } y \in B_r(x_0).$$

Therefore,

$$\int_{B_{2r}(x_0)^c} g(x_0,z)\left(\int_M g(z,y)f(y)d\mu(y)\right)^{q-1} d\mu(z) \tag{3.3}$$
$$\le (C||f||_{L^1(M)})^{q-1}\int_{B_{2r}(x_0)^c} g(x_0,z)g(z,x_0)^{q-1}d\mu(z)$$
$$\le (C||f||_{L^1(M)})^{q-1}\int_{B_r(x_0)^c} g(x_0,z)^q d\mu(z).$$

The assumption and (3.2) imply that the left hand side in (3.3) is infinite and we conclude that $g(x_0,\cdot) \notin L^q(M\backslash B_r(x_0))$.
iii) $\Rightarrow$ **iv)** : Assume that $g(x_0,\cdot) \notin L^q(M\backslash B_r(x_0))$ for given $x_0 \in M$ and $r_0 > 0$. Since $g(x_0,\cdot)$ is bounded in every annulus centered at x_0 it follows easily that **iv)** holds for $x_0 \in M$ and for every $r > 0$. Now, given $x \in M$, Lemma 3.4 applied to the ball $B_R(x_0)$ with $R = d(x,x_0)+1$ shows that there exists $C > 0$ such that $g(x_0,z) \le Cg(x,z)$ for every $z \in M\backslash B_R(x_0)$. Thus,

$$\infty = C^{-q}\int_{M\backslash B_R(x_0)} g(x_0,z)^q\, d\mu(z) \le \int_{M\backslash B_R(x_0)} g(x,z)^q\, d\mu(z)$$
$$\le \int_{M\backslash B_1(x)} g(x,z)^q\, d\mu(z).$$

iv) $\Rightarrow$ **i)** : Given a non-identically zero function $0 \le f \in C_c^\infty(M)$ let $B_r(x_0) \subset \operatorname{supp} f$ such that $\min_{B_r(x_0)} f = C_0 > 0$. Using Lemma 3.4 we can compute

$$\int_{M\backslash B_r(x_0)} g(x_0,z)^q d\mu(z) = \int_{M\backslash B_r(x_0)} g(x_0,z)[g(x_0,z)]^{q-1}d\mu(z) \le$$

$$\int_{M\backslash B_r(x_0)} g(x_0,z)\left[\frac{c}{C_0\mu(B_r(x_0))}\int_{B_r(x_0)} g(z,y)f(y)d\mu(y)\right]^{q-1} d\mu(z),$$

which yields

$$\int_{M\backslash B_r(x_0)} g(x_0,z)^q d\mu(z) \le \left(\frac{c}{C_0\mu(B_r(x_0))}\right)^{q-1} G^q f(x_0).$$

Thus, by our assumption we have $G^q f(x_0) = \infty$ and the conclusion follows by Remark 3.5. □

Remark 3.7. We observe for future use that the above proof actually shows that $G^q \equiv \infty$ if and only if, for some/every $x \neq y$ in M and some/every pre-compact open set U containing x and y,

$$\int_{M\backslash U} g(x,z)g(z,y)^{q-1}d\mu(z) = \infty.$$

Remark 3.8. If $G^q < \infty$, then Theorem 3.6 can be used to prove that for every $x \in M$, and every $r > 0$, there exists a constant $C > 0$, which depends only upon $B_{2r}(x)$, such that

$$\int_{M\backslash B_{2r}(x)} g(y,z)^q d\mu(z) < C \quad \text{for all} \quad y \in B_r(x).$$

Indeed, by assumption and Theorem 3.6,

$$\int_{M\setminus B_{2r}(x)} g(x,z)^q d\mu(z) < \infty,$$

and the conclusion follows again from Lemma 3.4.

Proposition 3.9. *M is L^q-Liouville, for $q \in [1,\infty)$, if and only if for every positive superharmonic function $f\colon M \to \mathbb{R}$, we have $G[f^{q-1}](x) \equiv \infty$ for some/every $x \in M$.*

Proof. Assume that M is L^q-Liouville and let f be a positive superharmonic function on M. Fix $y \in M$ and let U be a pre-compact neighborhood of y. As in Proposition 3.2 there is a constant $c > 0$ such that

$$g(z,y) \leq cf(z), \quad \text{for any} \quad z \notin U.$$

So that, raising to power $q-1$ and integrating we obtain

$$G[f^{q-1}](x) \geq c^{-1} \int_{M\setminus U} g(x,z) g(z,y)^{q-1} d\mu(z) = \infty.$$

The conclusion then follows by Remark 3.7. Conversely, let $\delta > 0$ be a small constant such that $m = \inf_{B_\delta(y)} g(y,\cdot) > \sup_{\partial B_1(y)} g(y,\cdot)$ and define $f = \min\{m, g(y,\cdot)\}$. Then f is a positive superharmonic function in M and

$$\int_M g(x,z) g(z,y)^{q-1} d\mu(z) \geq G[f^{q-1}](x) = \infty,$$

and the conclusion follows by Theorem 3.6. □

Remark 3.10. *In Proposition 3.9 we can use positive functions defined and harmonic in a complement of a compact set. In this case, if we fix a compact set $E \subset M$, we then have*

$$G^q \equiv \infty \quad \textit{if and only if} \quad \int_{M\setminus E} g(x,y) h(y)^{q-1} d\mu(y) = \infty.$$

As an application of Theorem 3.6 we now extend, from the classical case $q = \infty$, the characterization of the L^q-Liouville property in terms of the existence of pre-compact open sets with zero C_p-capacity to the range $1 < q \leq \infty$, with p and q Hölder conjugate exponents.

Theorem 3.11. *Let M be a manifold and $p \in [1,\infty)$. Then $C_p(\overline{E}) = 0$ for some/every pre-compact open subset $E \subset M$ if and only if M is L^q-Liouville. Therefore, M is L^p-parabolic if and only it is L^q-Liouville.*

Proof. By Theorem 3.6 we assume $G^q \equiv \infty$ and let $E \subset M$ be a pre-compact open set. Since the C_p-capacity is monotone and subadditive and E is pre-compact, it suffices to prove that $C(B_1(y_0)) = 0$ for any $y_0 \in M$. As in the proof of Theorem 3.6, by Lemma 3.4 there exists $C > 0$, depending only on $B_2(y_0)$, such that $g(y_0,z) \leq Cg(y,z)$ for all $y \in B_1(y_0)$ and for all $z \notin B_2(y_0)$. Given any measure $0 \neq \nu \in \mathcal{M}^+(B_1(y_0))$ we estimate

$$\begin{aligned}
||G\nu||_{L^q(M)} &\geq \left(\int_{M\setminus B_2(y_0)} \left(\int_{B_1(y_0)} g(y,z) d\nu(y) \right)^q d\mu(z) \right)^{\frac{1}{q}} \\
&\geq C\nu(B_1(y_0)) \left(\int_{M\setminus B_2(y_0)} g(y_0,z)^q d\mu(z) \right)^{\frac{1}{q}} \\
&= \infty,
\end{aligned}$$

where in the last equality we have applied Theorem 3.6 item **iv**). Thus, if $||G\nu||_{L^q(M)} \leq 1$, $\nu \equiv 0$ and by Theorem 2.8 we have $C_p(B_1(y_0)) = 0$.

For the converse, we are going to show that if $G^q < \infty$ then $C_p(\overline{E}) > 0$ for every pre-compact open subset E with $\mu(\overline{E}) > 0$. So, assume $G^q < \infty$ and that there exists some pre-compact open subset E such that $C_p(\overline{E}) = 0$. For every x, let $F_x = B_1(x) \cap \overline{E}$ whenever the right hand side is not empty.

Notice first of all that since $G^q < \infty$, then $G\chi_{F_x} \in L^q(M)$. Indeed, if $\psi \in C_c^\infty(M)$ is non-negative and $\psi = 1$ on $B_1(x)$, then

$$\begin{aligned}\int_{B_2(x)} (G\chi_{F_x})^q(y)d\mu(y) &\le \int_{B_2(x)} \left(\int_M g(y,z)\psi(z)d\mu(z)\right)^q d\mu(y)\\ &\le \mu(B_2(x)) \sup_{\overline{B}_2(x)} |G\psi|^q < \infty.\end{aligned}$$

On the other hand, by Remark 3.8, there exists a constant C depending only on $B_2(x)$ such that

$$\int_{M\setminus B_2(x)} g(y,z)^q d\mu(y) \le C, \quad \forall z \in B_1(x),$$

and therefore, by Minkowski's integral inequality,

$$\begin{aligned}\left[\int_{M\setminus B_2(x)} (G\chi_{F_x}(y))^q d\mu(y)\right]^{\frac{1}{q}} &= \left[\int_{M\setminus B_2(x)} \left(\int_{F_x} g(y,z)d\mu(z)\right)^q d\mu(y)\right]^{\frac{1}{q}}\\ &\le \int_{F_x}\left(\int_{M\setminus B_2(x)} g(y,z)^q d\mu(y)\right)^{\frac{1}{q}} d\mu(z)\\ &\le C^{\frac{1}{q}}\mu(B_1(x)),\end{aligned}$$

and our claim follows.

Next, since $C_p(\overline{E}) = 0$, by definition of capacity, for every $\epsilon > 0$ there exists a function $f \in L_+^p(M)$ such that $Gf \ge 1$ on $\overline{E}$ and

$$\int_M f^p d\mu < \epsilon^p.$$

We can therefore estimate

$$\begin{aligned}\mu(F_x) &\le \int_{F_x} Gf(y)d\mu(y)\\ &= \int_M \chi_{F_x}(y)Gf(y)d\mu(y)\\ &= \int_{M\times M} g(y,z)f(z)\chi_{F_x}(y)d\mu(z)d\mu(y)\\ &= \int_M f(z)G\chi_{F_x}(z)d\mu(z)\\ &\le ||f||_{L^p(M)}||G\chi_{F_x}||_{L^q(M)} < \epsilon||G\chi_{F_x}||_{L^q(M)},\end{aligned}$$

and, letting $\epsilon \to 0$, we conclude that

$$\mu(F_x) = \mu(B_1(x) \cap E) = 0$$

for every x such that $B_1(x) \cap \overline{E} \neq \emptyset$. Since $\overline{E}$ is compact, it can be covered by a countable family of balls $B_1(x_k)$. Hence, $\overline{E} = \cup_k F_{x_k}$ and by subadditivity we conclude that $\mu(\overline{E}) = 0$, as required. □

Remark 3.12. In the above proof, since M is second countable, we have proved that if $G^q < \infty$ then $C_p(K) = 0$ implies $\mu(K) = 0$ for every compact set K. It means that, in this case, the capacity C_p gives a more refined measure than the Lebesgue one. Furthermore, if $G^q = \infty$ then $C_p(E) = 0$ for all subsets $E \subset M$.

Corollary 3.13. *There is a pre-compact open subset $E \subset M$ with $C_p(E) > 0$ if and only if $C_p(F) > 0$ for every pre-compact open set $F \subset M$.*

As a notable consequence of Theorem 3.11 and Corollary 3.3 we can derive the hierarchy of the L^p-parabolicity.

Corollary 3.14. *If M is L^p-parabolic, then M is also L^s-parabolic for every $1 \le p \le s < \infty$.*

In the recent paper [9] the authors introduced the concept of biparabolicity for a Riemannian manifold in terms of a Liouville type condition for the bilaplacian operator. A function $u \in C^4(M)$ is said to be bi-superharmonic if $\Delta u \le 0$ and $\Delta^2 u \ge 0$.

Definition 3.15. *A manifold M is biparabolic if any positive bi-superharmonic function $u \in C^4(M)$ is harmonic, $\Delta u = 0$.*

One of the main results in [9], Theorem 3.1, is a characterization of the biparabolicity of M by the explosion of the Green operator G^2 defined in (3.1). Thus, Theorems 3.6 and 3.11 give rise to the following application.

Corollary 3.16. *A manifold M is biparabolic if and only if M is L^2-parabolic, equivalently, L^2-Liouville.*

In this connection, we note that Proposition 3.9 and Remark 3.10 extend [9, Lem.3.2].

4. A capacitary approach to the L^1-Liouville property

Since the positive minimal Green kernel is locally integrable, according to Proposition 3.2, the L^1-Liouville property on a manifold M is equivalent to the non-integrability of $g(x,\cdot)$ (see also [11, 12]). From this one can easily deduce that every stochastically complete manifold satisfies the L^1-Liouville property and, on model manifolds, the two properties are equivalent. In general this equivalence is not true in any dimension (see [5, 20]). Note that, if M is not L^1-Liouville, the function

$$E(x) = \int_M g(x,y)d\mu(y),$$

which is usually referred to as the mean exit time of M, is a positive solution to the equation

$$\Delta E + 1 = 0.$$

As in the previous subsection we are going to restrict ourselves to the non-parabolic case. Since there is no duality when $q = 1$, we will define the relevant capacity via positive measures.

Definition 4.1. *Let $K \subset U$ be a compact subset of an open set $U \subseteq M$, we define*

$$C_\infty(K,U) = \sup\{\nu(K) \colon \nu \in \mathcal{M}^+(K), ||G^U \nu||_{L^1(U)} \le 1\}.$$

When $U = M$, we simply write $C_\infty(K)$ in place of $C_\infty(K,M)$. A manifold M is then said to be L^∞-parabolic if $C_\infty(K) = 0$ for every compact subset $K \subset M$.

The set function C_∞ is monotone increasing in K, and monotone decreasing in U. In the next theorem we provide a description of the extremal measures for the C_∞-capacity which in turn will easily yield the required capacitary characterization of the L^1-Liouville property and its equivalence with the L^∞-parabolicity defined above.

Theorem 4.2. *Let K be a compact subset of an open set $U \subseteq M$. There exists an extremal measure $\nu^K \in \mathcal{M}^+(K)$ such that*

$$C_\infty(K,U) = \nu^K(K).$$

Moreover, $\nu^K = \left(\min_K E^U\right)^{-1}\delta_{x_0}$ *where* δ_{x_0} *is the Dirac measure centered at* $x_0 \in \partial K$*, and* $E^U(x_0) = \min_K E^U$*. In particular, a manifold* M *is* L^1*-Liouville if and only if* $C_\infty(K) = 0$ *for some/every compact set* K*.*

Proof. We first consider the case where, for some/every point $x \in U$, we have

$$E^U(x) = \int_U g^U(x,y)d\mu(y) < \infty.$$

In this case, E^U is a smooth function, and for any test measure $\nu \in \mathcal{M}^+(K)$ there holds

$$1 \geq \int_U G^U\nu(y)d\mu(y) \geq \nu(K)\min_K E^U.$$

Hence,

$$C_\infty(K,U) \leq \frac{1}{\min_K E^U}.$$

On the other hand, if $E^U(x_0) = \min_K E^U$ for some $x_0 \in \partial K$ (here we use that E^U is superharmonic), then the Dirac measure $\nu_0 = (E^U(x_0))^{-1}\delta_{x_0} \in \mathcal{M}^+(K)$ satisfies

$$\int_U G^U\nu_0(y)d\mu(y) = \frac{1}{E^U(x_0)}\int_U g^U(x_0,y)d\mu(y) = 1.$$

Moreover,

$$\nu_0(K) = \frac{1}{E^U(x_0)} \geq C_\infty(K,U),$$

that is, ν_0 is the extremal measure for the capacity. Now, assume that

$$E^U(x) = \int_M g^U(x,y)d\mu(y) = \infty$$

for some/every $x \in U$, and let $\nu \in \mathcal{M}^+(K)$ be such that $\nu(K) > 0$. Then

$$\begin{aligned}\int_U G^U\nu(y)d\mu(y) &= \int_U d\mu(y)\int_U g^U(x,y)d\nu(x)\\ &= \int_K d\nu(x)\int_U g^U(x,y)d\mu(y) = \infty,\end{aligned}$$

showing that

$$C_\infty(K,U) = 0.$$

The second statement is now a consequence of the already noted fact that M is L^1-Liouville if and only if $E \equiv \infty$. □

As in Proposition 2.6 the monotonicity of $C_\infty(K,U)$ with respect to U gives rise to the following identity

$$C_\infty(K,U) = \lim_{n\to\infty} C_\infty(K,U_n),$$

where $\{U_n\}$ is a compact exhaustion of any open set $U \subset M$, such that $K \subset U_1$. It is easy to see that such a limit does not depend on the exhaustion and satisfies

$$C_\infty(K,U) \leq \lim_{n\to\infty} C_\infty(K,U_n).$$

For the other inequality we consider $g_n(x,y)$ the Dirichlet Green function associated to U_n, and extend $g_n(x,y)$ by zero on $U\backslash U_n$. For every $y \in U$, and $x \in U$ fixed, we know that $g_n(x,y) \nearrow g^U(x,y)$. It follows that

$$E_n(x) \doteq \int_U g_n(x,y)d\mu(y) \nearrow \int_U g^U(x,y)d\mu(y) \doteq E^U(x)$$

for any $x \in U$. As before, if $E^U(x) < \infty$ then E^U is a smooth function, and by Dini's theorem we have $E_n \nearrow E^U$ uniformly on compact sets, in particular on K. Now, by uniform convergence $\min_K E_n \to \min_K E^U$ and the conclusion follows from Theorem 4.2. If $E^U(x) = \infty$, since $E_n \to E^U$ locally uniformly it follows that $\min_K E_n \to \infty$.

We may extend the definition of the set function $C_\infty(\,\cdot\,, U)$ to arbitrary sets via the standard min-max procedure. First, if $G \subset U$ is an open subset we let

$$C_\infty(G, U) \doteq \sup\{C_\infty(K, U) \colon K \subset G \ \text{ is compact}\},$$

and, for every $F \subset U$, we define

$$C_\infty(F, U) \doteq \inf\{C_\infty(G, U) \colon G \supset F \ \text{ is open}\}.$$

For any open subset $F \subset U$ we can show that

$$C_\infty(F, U) = \left(\inf_F E^U\right)^{-1}. \tag{4.1}$$

Indeed, if $G \subset U$ is open, it follows easily from the properties of inf/sup that, for every $G \subset U$ open,

$$\inf_G E^U = \inf_{K \subset G} \inf_K E^U,$$

where $K \subset G$ is compact, whence

$$C_\infty(G, U) = \sup_{K \subset G} C_\infty(K, U) = \left(\inf_{K \subset G} \inf_K E^U\right)^{-1} = \left(\inf_G E^U\right)^{-1}.$$

Similarly, using the fact that, for every $F \subset U$,

$$\inf_F E^U = \sup_{G \supset F} \inf_G E^U,$$

where $G \subset U$ is open, we deduce that

$$C_\infty(F, U) = \inf_{G \supset F} C_\infty(G, U) = \left(\inf_F E^U\right)^{-1}.$$

We now collect the properties of the set function C_∞ in the following proposition.

Proposition 4.3. *C_∞ enjoys the following properties.*

1) *Given $F_1 \subset F_2 \subset U$ subsets of an open set $U \subset M$, we have*
$$C_\infty(F_1, U) \le C_\infty(F_2, U).$$
2) *Given $U_1 \subset U_2 \subset M$ open sets such that $F \subset U_1$, it holds*
$$C_\infty(F, U_1) \ge C_\infty(F, U_2).$$
3) *Given F_1, F_2 arbitrary subsets of an open set $U \subset M$, there holds*
$$C_\infty(F_1 \cup F_2, U) \le C_\infty(F_1, U) + C_\infty(F_2, U) - C_\infty(F_1 \cap F_2, U).$$
4) *Let K_i be a decreasing sequence of compact sets contained in an open set $U \subset M$, then*
$$C_\infty(\cap_{i=1}^\infty K_i, U) = \lim_{i \to \infty} C_\infty(K_i, U).$$
5) *Given an increasing sequence of arbitrary sets F_i contained in an open set $U \subset M$ there holds*
$$C_\infty(\cup_i F_i, U) = \lim_{i \to \infty} C_\infty(F_i, U).$$

Proof. Items 1) and 2) are an easy consequence of (4.1). For item 3), let $F_1, F_2 \subset U$ be arbitrary sets. Equation (4.1) yields

$$\begin{aligned}
C_\infty(F_1 \cup F_2, U) &= \left(\inf_{F_1 \cup F_2} E^U\right)^{-1} \\
&= \left[\min\{\inf_{F_1} E^U, \inf_{F_2} E^U\}\right]^{-1} \\
&\leq \left(\inf_{F_1} E^U\right)^{-1} + \left(\inf_{F_2} E^U\right)^{-1} - \left(\inf_{F_1 \cap F_2} E^U\right)^{-1} \\
&= C_\infty(F_1, U) + C_\infty(F_2, U) - C_\infty(F_1 \cap F_2, U).
\end{aligned}$$

Item 4) will follow from the following identity

$$\min_{\cap_n K_i} E^U = \lim_{i \to \infty} \min_{K_i} E^U.$$

Indeed, by monotonicity it is clear that the limit exists, and the left hand side is no less than the right hand side. On the other hand, for every i let $x_i \in K_i$ be the point minimizing E^U on K_i. By passing to a subsequence we may assume that $x_i \to x \in \cap_i K_i$. Hence,

$$\min_{\cap_i K_i} E^U \leq E^U(x) = \lim_{i \to \infty} E^U(x_i) = \lim_{i \to \infty} \min_{K_i} E^U.$$

To prove item 5) let us set $F = \cup_i F_i$. The claim follows from the fact that $\inf_F E^U = \lim_i \inf_{F_i} E^U$ which in turn is easily seen as follows. On the one hand, monotonicity shows that the limit exists and the left hand side is less than or equal to the right hand side. On the other hand, for every $\epsilon > 0$, there exists $x_\epsilon \in F$ such that $E^U(x_\epsilon) < \inf_F E^U + \epsilon$. Since $x_\epsilon \in F_i$ for large enough i, for such i's $\inf_{F_i} E^U \leq E^U(x_\epsilon) \leq \inf_F E^U + \epsilon$, whence letting $i \to \infty$ and $\epsilon \to 0$ we conclude that $\lim_i \inf_{F_i} E^U \leq \inf_F E^U$ and our claim follows. □

The properties listed in Proposition 4.3 show that $C_\infty(\cdot, U)$ is a Choquet capacity, and thus, all Borel sets $F \subset U$ are capacitable (cf. [1, Thm.2.3.11]), that is,

$$\begin{aligned}
C_\infty(F, U) &= \inf\{C_\infty(G, U) \colon G \supset F, G \text{ open}\} \\
&= \sup\{C_\infty(K, U) \colon K \subset F, K \text{ compact}\}.
\end{aligned}$$

5. Volume conditions for the L^p-parabolicity

In this section we first provide a sufficient pointwise volume condition for the validity of the L^p-parabolicity of a complete manifold M in the range $1 < p \leq 2$, which extends the results obtained in [9] for $p = 2$, and is compatible with the known volume conditions for the usual parabolicity in the limit $p \to 1$. For manifolds with non-negative Ricci curvature we obtain a sufficient integral condition in the whole range $1 < p < \infty$ that implies an improved volume condition. Moreover, this integral condition turns out to be always valid, and essentially sharp, for general model manifolds.

In what follows we are going to denote by $V(o, r)$, or simply $V(r)$, the volume of the ball $B_r(o)$ and by $S(o, r)$, or simply $S(r)$, the area of the sphere $\partial B_r(o)$, where $o \in M$ is a given reference point.

Theorem 5.1. *Let M be a complete manifold and let $1 < p \leq 2$. Assume that, for some $o \in M$ and sufficiently large r, it holds*

$$V(r) \leq C \frac{r^{2p}}{\log r}, \tag{5.1}$$

for some constant $C > 0$. Then M is L^p-parabolic.

Proof. From Theorems 3.6 and 3.11 it is sufficient to prove that $G^q\varphi \equiv \infty$ for some $0 \le \varphi \in C_c^\infty(M)$, and $pq = p+q$. Recall that the Green kernel is given by

$$g(x,y) = \int_M p_t(x,y)d\mu(y),$$

where $p_t(x,y)$ is the heat kernel of M. By Jensen's inequality and the semi-group property of the heat kernel we obtain

$$\begin{aligned} G^q\varphi(o) &= \int_0^\infty dt \int_M \left(\int_M g(y,z)\varphi(z)d\mu(z)\right)^{q-1} p_t(o,y)d\mu(y) \\ &\ge \int_0^\infty \left(\int_M p_t(o,y)\int_M g(y,z)\varphi(z)d\mu(z)d\mu(y)\right)^{q-1} dt \\ &= \int_0^\infty \left(\int_0^\infty P_{t+s}\varphi(o)ds\right)^{q-1} dt \\ &= \int_0^\infty \left(\int_t^\infty P_\tau\varphi(o)d\tau\right)^{q-1} dt, \end{aligned}$$

where P_t is the heat operator acting on $C_c^\infty(M)$ by

$$P_t\varphi(x) = \int_M p_t(x,y)\varphi(y)d\mu(y).$$

To estimate $P_t\varphi(o)$ from below we argue as in [9]: let $\operatorname{supp}\varphi \subset B_R(o)$. It is proved in [7] (see also [13, Thm.16.5]) that a polynomial volume estimate of the form $V(r) \le Cr^\nu$, for $r \ge r_0$ implies that there exist constants $t_0 = t_0(r_0)$ and $K = K(o, r_0, C, \nu)$ such that the following heat kernel diagonal lower bound

$$p_t(o,o) \ge \frac{1}{4}[V(\sqrt{Kt\log t})]^{-1}$$

holds for every $t \ge t_0 = t_0(r_0)$. Together with the local parabolic Härnack inequality in [23, 24] we obtain

$$\begin{aligned} P_\tau\varphi(o) &= \int_{B_R(o)} p_\tau(o,y)\varphi(y) \ge c\int_{B_R(o)} p_t(o,o)\varphi(y) \\ &\ge c\|\varphi\|_{L^1}\tau^{-\frac{q}{q-1}}(\log\tau)^{-\frac{1}{q-1}} \quad \text{for} \quad \tau \ge \tau_0, \end{aligned} \tag{5.2}$$

where $c > 0$ depends on r_o, C, ν, R and on the geometry of $B_R(o) \subset M$. Inserting this into the above inequality we conclude that

$$\begin{aligned} G^q\varphi(o) &\ge c\int_{\tau_0}^\infty \left(\int_t^\infty \tau^{-\frac{q}{q-1}}(\log\tau)^{-\frac{1}{q-1}}d\tau\right)^{q-1} dt \\ &\asymp \int_{\tau_0}^\infty \frac{dt}{t\log t} = \infty. \end{aligned}$$

□

Remark 5.2. A sharp estimate in the range $1 < p \le 2$ can be obtained if we assume that M satisfies the following heat kernel diagonal lower bound

$$p_t(o,o) \ge \frac{c}{V(\sqrt{t})}, \tag{5.3}$$

for some $c > 0$ and for all $t > 0$. In this case, we can replace (5.1) by the sharp inequality

$$V(r) \le Cr^{2p}(\log r)^{p-1},$$

to obtain (5.2). Thus, the result follows as in the proof of Theorem 5.1. For instance, (5.3) holds if M satisfies the volume doubling condition and a comparable diagonal heat kernel upper estimate or, equivalently, the relative Faber-Krahn inequality (see [13, Thm.15.21 and Cor.16.7]).

Let us now consider the class of geodesically complete Riemannian manifolds with non-negative Ricci curvature. In this class, P. Li and S.-T. Yau [18] established the following Green function estimate

$$C^{-1}\int_r^\infty \frac{t\,dt}{V(x,t)} \le g(x,y) \le C\int_r^\infty \frac{t\,dt}{V(x,t)}, \tag{5.4}$$

where $r = d(x,y)$ and $C > 0$. A characterization of L^p-parabolicity in the whole range $1 < p < \infty$ can be deduced from Theorems 3.6 and 3.11.

To state the next results, fix some $x \in M$ and set $V(r) = V(x,r)$.

Proposition 5.3. *Let M be a complete Riemannian manifold with Ric ≥ 0, and let $1 < p < \infty$. Then M is L^p-parabolic if and only if*

$$\int^\infty \left(\int_r^\infty \frac{t\,dt}{V(t)}\right)^{\frac{p}{p-1}} V'(r)dr = \infty. \tag{5.5}$$

Proof. By Theorem 3.6, M is L^q-Liouville with $q = \frac{p}{p-1}$ if and only if for some/all $\varepsilon > 0$

$$\int_{M\setminus B_\varepsilon(x)} g(x,y)^q d\mu(y) = \infty.$$

Integrating the Li-Yau Green function estimate (5.4) we obtain

$$\int_{M\setminus B_\varepsilon(x)} g(x,y)^q d\mu(y) \asymp \int_\varepsilon^\infty \left(\int_r^\infty \frac{t\,dt}{V(t)}\right)^q V'(r)dr.$$

Finally, by Theorem 3.11, M is L^p-parabolic if and only if M is L^q-Liouville. □

As a first consequence, we can obtain a sharp sufficient integral volume condition to the validity of the L^p-parabolicity.

Corollary 5.4. *Let M be a complete Riemannian manifold with Ric ≥ 0, and let $1 < p < \infty$. If*

$$\int^\infty r\left(\int_r^\infty \frac{tdt}{V(t)}\right)^{\frac{1}{p-1}} dr = \infty, \tag{5.6}$$

where $V(t) = V(x,t)$ for some $x \in M$, then M is L^p-parabolic.

Proof. Since every parabolic manifold is L^p-parabolic for every $1 < p < \infty$, let us assume that M is non-parabolic. By the Li-Yau estimate (5.4) the inner integral in (5.6) is finite for any $r > 0$. Integration by parts yields

$$\begin{aligned}\int_1^\infty \left(\int_r^\infty \frac{tdt}{V(t)}\right)^{\frac{p}{p-1}} V'(r)dr \ge & -\int_1^\infty V(r)\frac{d}{dr}\left(\int_r^\infty \frac{tdt}{V(t)}\right)^{\frac{p}{p-1}} dr \\ & - V(1)\left(\int_1^\infty \frac{tdt}{V(t)}\right)^{\frac{p}{p-1}} \\ = & \frac{p}{p-1}\int_1^\infty r\left(\int_r^\infty \frac{tdt}{V(t)}\right)^{\frac{1}{p-1}} dr \\ & - V(1)\left(\int_1^\infty \frac{tdt}{V(t)}\right)^{\frac{p}{p-1}},\end{aligned}$$

and the conclusion follows by Proposition 5.3. □

Remark 5.5. For example, if

$$V(r) \leq Cr^{2p} (\log r)^{p-1}$$

then (5.6) is satisfied. In particular, if M has dimension $n \geq 2$ and satisfies $\mathrm{Ric} \geq 0$, by the Bishop-Gromov volume comparison theorem $V(r) \leq Cr^n$, for some $C > 0$, and therefore M is L^p-parabolic for every $p \geq n/2$.

Motivated by this result we make the following

Conjecture 5.6. *Let M be a complete Riemannian manifold, and let $p \in (1, \infty)$. If*

$$\int^{\infty} r \left(\int_r^{\infty} \frac{tdt}{V(t)} \right)^{\frac{1}{p-1}} dr = \infty,$$

then M is L^p-parabolic.

We conclude this section by observing that Conjecture 5.6 holds also for the class of model manifolds (M_σ, ds^2), where $M_\sigma = \mathbb{R}^n$ and, in polar coordinates (r, θ),

$$ds^2 = dr^2 + \sigma^2(r) d\theta^2$$

for a smooth, positive function σ on $(0, \infty)$. In this case, the Green kernel with pole at $o = 0$ is radial and it is given by

$$g(o, x) = \int_r^{\infty} \frac{dt}{S(t)}, \quad \text{if } x = (r, \theta),$$

where $S(r) = c_m \sigma^{m-1}(r)$ is the Riemannian volume of the sphere $\partial B_r(o)$ centered at the pole o with radius $r > 0$.

Similarly to Proposition 5.3, Theorems 3.6 and 3.11 give the following characterization for the L^p-parabolicity of model manifolds.

Proposition 5.7. *Let M_σ be a model manifold as above and $p \in (1, \infty)$. Then M_σ is L^p-parabolic if and only if*

$$\int^{\infty} \left(\int_r^{\infty} \frac{dt}{S(t)} \right)^{\frac{p}{p-1}} S(r) dr = \infty. \tag{5.7}$$

To show that Conjecture 5.6 holds for the class of model manifolds we will need the next lemma.

Lemma 5.8. *Let f be a continuously differentiable function on an interval (a, b) such that $f > 0$ and $f' > 0$. Then*

$$\int_a^b \frac{dt}{f'(t)} \geq \frac{1}{2} \int_a^b \frac{(t-a)\, dt}{f(t)}. \tag{5.8}$$

Proof. Changing t to $t - a$, we reduce (5.8) to

$$\int_0^{b-a} \frac{dt}{f'(t+a)} \geq \frac{1}{2} \int_0^{b-a} \frac{tdt}{f(t+a)}.$$

Hence, renaming $b - a$ into b and $f(t + a)$ into f, we see that it suffices to prove that

$$\int_0^b \frac{dt}{f'(t)} \geq \frac{1}{2} \int_0^b \frac{tdt}{f(t)} \tag{5.9}$$

that is, (5.8) in the case $a = 0$.

Multiplying and dividing by $f'(t)^{1/2}$ and using the Cauchy-Schwarz inequality we get

$$\int_0^b \frac{t}{f(t)} dt \leq \left(\int_0^b \frac{t^2}{f(t)^2} f'(t) dt \right)^{1/2} \left(\int_0^b \frac{1}{f'(t)} dt \right)^{1/2}.$$

Integrating by parts in the first integral on the right hand side gives

$$\int_0^b \frac{t^2}{f(t)^2} f'(t)dt = -\frac{b^2}{f(b)} + 2\int_0^b \frac{t}{f(t)}dt \le 2\int_0^b \frac{t}{f(t)}dt,$$

whence inserting into the above inequality and simplifying yield (5.8). ☐

Corollary 5.9. *Let $p \in (1, \infty)$. If*

$$\int^\infty r\left(\int_r^\infty \frac{tdt}{V(t)}\right)^{\frac{1}{p-1}} dr = \infty \tag{5.10}$$

then M_σ is L^p-parabolic.

Proof. From a simple change of variables in the external integral we can rewrite (5.6) as

$$\int^\infty r\left(\int_{2r}^\infty \frac{tdt}{V(t)}\right)^{\frac{1}{p-1}} dr = \infty.$$

By Lemma 5.8, we have

$$\int_{2r}^\infty \frac{tdt}{V(t)} \le 2\int_r^\infty \frac{(t-r)\,dt}{V(t)} \le 4\int_r^\infty \frac{dt}{S(t)}.$$

Hence, (5.6) implies that

$$\int^\infty r\left(\int_r^\infty \frac{dt}{S(t)}\right)^{\frac{1}{p-1}} dr = \infty. \tag{5.11}$$

Let us show that (5.11) implies (5.7). For that, consider the function

$$f(r) = \left(\int_r^\infty \frac{dt}{S(t)}\right)^{-\frac{1}{p-1}}.$$

We may assume that M_σ is non-parabolic, and therefore $f(r) > 0$, for otherwise M_σ is automatically L^p-parabolic. It follows from (5.11) that

$$\int_1^\infty \frac{rdr}{f(r)} = \infty,$$

whence also

$$\int_1^\infty \frac{(r-1)\,dr}{f(r)} = \infty.$$

By Lemma 5.8 we have

$$\int_1^\infty \frac{(r-1)\,dr}{f(r)} \le 2\int_1^\infty \frac{dr}{f'(r)},$$

and, hence,

$$\int_1^\infty \frac{dr}{f'(r)} = \infty. \tag{5.12}$$

We clearly have

$$f'(r) = \frac{1}{p-1}\left(\int_r^\infty \frac{dt}{S(t)}\right)^{-\frac{p}{p-1}} \frac{1}{S(r)}.$$

Substituting this into (5.12) we obtain (5.7). Hence, M_σ is L^p-parabolic by Proposition 5.7. ☐

The next example shows the almost optimality of the pointwise volume condition described in Theorem 5.1 and the optimality of the integral volume condition given in Corollaries 5.4 and 5.9.

Example 5.10. Let M_σ be a model manifold with

$$S(r) = c_m \sigma(r)^{m-1} = c_m r^{\alpha-1} (\log r)^\beta$$

for $r \geq 2$, with $\alpha > 2$ and real β. Note that in this case

$$V(r) \asymp r^\alpha (\log r)^\beta.$$

Since $S(r)^{-1}$ is integrable at infinity, the manifold M_σ is non-parabolic and its Green kernel with pole at o satisfies for all $r \gg 1$

$$g(r) = \int_r^\infty \frac{dt}{S(t)} \asymp r^{2-\alpha} (\log r)^{-\beta}.$$

It follows that

$$\left(\int_r^\infty \frac{dt}{S(t)} \right)^{\frac{p}{p-1}} S(r) \asymp r^{\frac{p+1-\alpha}{p-1}} (\log r)^{-\frac{\beta}{p-1}}.$$

Therefore, the integral in (5.7) is divergent and, hence, M_σ is L^p-parabolic by Proposition 5.7, if and only if either $\alpha < 2p$, or $\alpha = 2p$ and $\beta \leq p - 1$.

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