Upper bounds for solutions of Leibenson's equation on Riemannian manifolds

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Abstract

We consider on Riemannian manifolds the Leibenson equation

 $\partial_t u = \Delta_p u^q$

that is also known as a doubly nonlinear evolution equation. We prove upper estimates of weak subsolutions to this equation on Riemannian manifolds with non-negative Ricci curvature in the case when p and q satisfy the conditions

$$1 and $1 \le q < \frac{1}{p-1}$.$$

We show that these estimates are optimal in terms of long time behaviour and nearoptimal in terms of long distance behavior.

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1 Introduction

Let M be an arbitrary Riemannian manifold. We consider solutions of the non-linear evolution equation

$$\partial_t u = \Delta_p u^q, \tag{1.1}$$

where p > 1, q > 0, u = u(x, t) is an unknown non-negative function of $x \in M$, $t \ge 0$, and Δ_p is the Riemannian *p*-Laplacian:

$$\Delta_p v = \operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right).$$

The equation (1.1) is frequently referred to as a *doubly non-linear parabolic equation*. For the physical meaning of this equation see [13, 20, 21].

When $M = \mathbb{R}^n$, G. I. Barenblatt [1] constructed for all p > 1, q > 0 spherically symmetric self-similar solutions of (1.1), that are nowadays called *Barenblatt solutions*.

Let us denote

$$D = 1 - q(p - 1).$$

If D < 0, then the Barenblatt solution has a finite propagation speed, and the same phenomenon occurs on arbitrary Riemannian manifolds (see [2, 6, 7, 13, 14, 15, 26]).

In the borderline case D = 0, the Barenblatt solutions is positive but decays exponentially in distance. Similar sub-Gaussian upper bounds of solutions of (1.1) on Riemannian manifolds were proved in the case D = 0 in [25].

In the present paper we are concerned with the case D > 0, that is, when

$$q(p-1) < 1. (1.2)$$

In this case, equation (1.1) is also called *singular*. If in addition

$$\beta := p - nD > 0,$$

then the Barenblatt solution satisfies the estimate

$$u(x,t) \simeq \frac{1}{t^{n/\beta}} \left(1 + \frac{|x|}{t^{1/\beta}} \right)^{-\frac{p}{D}}$$
(1.3)

(cf. Section 7.1), where the symbol " \simeq " means that the ratio of the terms is bounded from above and below by a positive constant.

In the present paper we prove upper bounds for solutions of the Leibenson equation (1.1) on geodesically complete Riemannian manifolds in the restricted singular case

$$1 (1.4)$$

(note that (1.4) implies (1.2)). We understand solutions of (1.1) in $M \times \mathbb{R}_+$ in a certain weak sense (see Section 2 for the definition).

Assume that the Riemannian manifold M is geodesically complete. Denote by μ the Riemannian measure on M, by d the geodesic distance and by B(x,r) the geodesic ball of radius r centered at x.

The main result of the present paper is as follows (cf. **Theorem 6.1**).

Theorem 1.1. Let M satisfy a relative Faber-Krahn inequality (see Section 3 for definition) and assume that, for all $x \in M$ and all $R \ge 1$,

$$\mu(B(x,R)) \ge cR^{\alpha},\tag{1.5}$$

for some $c, \alpha > 0$. Assume that (1.4) holds and that

$$\beta := p - \alpha D > 0. \tag{1.6}$$

Let u be a bounded non-negative solution of (1.1) in $M \times [0, \infty)$ with initial function $u_0 = u(\cdot, 0) \in L^1(M) \cap L^\infty(M)$. Set $A = \operatorname{supp} u_0$ and denote |x| = d(x, A). Then, for all t > 0 and all $x \in M$, we have

$$\|u\left(\cdot,t\right)\|_{L^{\infty}(B(x,\frac{1}{2}|x|))} \leq \frac{C}{t^{\frac{\alpha}{\beta}}} \Phi\left(1 + \frac{|x|}{t^{1/\beta}}\right),\tag{1.7}$$

where

$$\Phi(s) = s^{-\frac{p}{D}} \log^{\gamma}(1+s),$$

where the positive constants C and γ depend on $c, \alpha, p, q, ||u_0||_{L^{\infty}(M)}, ||u_0||_{L^1(M)}$ and on the constants in the relative Faber-Krahn inequality.

In particular, if the solution u is continuous then the left hand side of (1.7) can be replaced by u(x,t).

Note that the relative Faber-Krahn inequality is satisfied if, for example, M has non-negative Ricci curvature (see [4, 11, 23]).

Comparing the upper bound (1.7) from Theorem 1.1 with the estimate (1.3) of the Barenblatt solution, we see that the estimate (1.7) is sharp in \mathbb{R}^n up to a logarithmic term. A similar comparison takes place for some class of spherically symmetric manifolds (model manifolds) satisfying the relative Faber-Krahn inequality (cf. Section 7.1).

The restrictions (1.4) seem to be technical and we think, that the result should be true for a general range of p, q given by (1.2), as it is stated in the next conjecture.

Conjecture 1.2. Let M satisfy the relative Faber-Krahn inequality and (1.5). Assume that (1.2) and (1.6) hold, that is, D > 0 and $\beta > 0$. Then the estimate (1.7) holds with

$$\Phi(s) = s^{-\frac{p}{D}}$$

If p = 2 then (1.2) becomes q < 1. In this case, assuming that the *n*-dimensional Riemannian manifold satisfies a uniform Sobolev inequality, the long time decay of order $t^{-n/\beta}$ for solutions of (1.1) (thus matching (1.3)) was proved in [3].

If q = 1 then (1.2) amounts to p < 2. In this case qualitative properties of weak solutions of (1.1) in \mathbb{R}^n were proved in [8, 10].

The structure of the present paper is as follows.

In Section 2 we define the notion of a weak solution of the equation (1.1).

In Section 3 the aforementioned relative Faber-Krahn inequality is discussed.

In Section 4 we prove the main technical lemma (Lemma 4.2) about the long distance decay of solutions of (1.1). It says the following. Let u be a bounded non-negative solution of (1.1) in $M \times [0, \infty)$. Let $B_0 = B(x_0, R)$ be a ball such that the initial function $u(\cdot, 0) = u_0$ satisfies

$$u_0 = 0$$
 in B_0 .

Then, for all t > 0,

$$\|u\|_{L^{\infty}\left(\frac{1}{2}B_{0}\times[0,t]\right)} \leq C_{B_{0}}\left(\frac{t}{R^{p}}\right)^{\frac{1}{D}}\log^{\gamma}\left(2+\left(\frac{R^{p}}{t}\right)^{\frac{1}{D}}\frac{||u_{0}||_{L^{1}(M)}}{\mu\left(B_{0}\right)}\right),\tag{1.8}$$

where the positive constant C_{B_0} depends on the intrinsic geometry of B_0 and γ depends on p, q and on the constants in the relative Faber-Krahn inequality.

Let us emphasize that this Lemma 4.2 is valid for an arbitrary complete Riemannian manifold, and the estimate (1.8) depends on the *local* geometry of the manifold inside the ball B_0 .

In the proof of Lemma 4.2 we use a certain mean value inequality that is stated in Lemma 4.1 that we borrowed from our previous paper [14]. If one could improve the upper bound (1.8) to get rid of the log term then this would then imply the validity of Conjecture 1.2.

In Section 5 we prove the main lemma (Lemma 5.4) about the long time decay of solutions of (1.1). The main technical ingredient is a non-linear mean value inequality (Lemma 5.2) for solutions of (1.1), which says the following. Let u be a non-negative bounded solution in $Q = B \times [0, T]$, $B = B(x_0, R)$, T > 0. Then, for the cylinder

$$Q' = \frac{1}{2}B \times [\frac{1}{2}T, T],$$

we have

$$\|u\|_{L^{\infty}(Q')} \leq \left(\frac{C_B S}{\mu(B)} \int_Q u^{\sigma}\right)^{1/(\sigma+D)},$$

where

$$S = \frac{\|u\|_{L^{\infty}(Q)}^{D}}{T} + \frac{1}{R^{p}},$$

 $\sigma > 0$ is any and the constant C_B depends on p, q, σ and the intrinsic geometry of the ball B (in fact, on the Faber-Krahn inequality in B).

The proof of the mean value inequality of Lemma 5.2 in the present paper is inspired by the proof of a mean value inequality in our previous paper [14] and uses a modification of the classical De Giorgi iteration argument [5].

In Section 7 (Appendix) we mention the exact solutions of (1.1) on the model manifolds (generalizing the Barenblatt solutions).

We denote by c, c', C, C' positive constants whose value might change at each occurance.

2 Weak subsolutions

We consider in what follows the following evolution equation on a Riemannian manifold M:

$$\partial_t u = \Delta_p u^q. \tag{2.9}$$

By a subsolution of (2.9) we mean a non-negative function u satisfying

$$\partial_t u \le \Delta_p u^q \tag{2.10}$$

in a certain weak sense as explained below.

We assume throughout that

$$p > 1$$
 and $q > 0$.

Set

$$D = 1 - q(p - 1)$$

Let μ denote the Riemannian measure on M. For simplicity of notation, we frequently omit in integrations the notation of measure. All integration in M is done with respect to $d\mu$, and in $M \times \mathbb{R}$ – with respect to $d\mu dt$, unless otherwise specified.

Let Ω be an open subset of M and I be an interval in $[0, \infty)$.

Definition 2.1. We say that a non-negative function u = u(x,t) is a *weak subsolution* of (2.9) in $\Omega \times I$, if

$$u \in C(I; L^{1}(\Omega))$$
 and $u^{q} \in L^{p}_{loc}(I; W^{1,p}(\Omega))$ (2.11)

and (2.10) holds weakly in $\Omega \times I$, which means that for all $t_1, t_2 \in I$ with $t_1 < t_2$, and all non-negative test functions

$$\psi \in W_{loc}^{1,\infty}\left(I; L^{\infty}(\Omega)\right) \cap L_{loc}^{p}\left(I; W_{0}^{1,p}(\Omega)\right), \qquad (2.12)$$

we have

$$\left[\int_{\Omega} u\psi\right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\Omega} -u\partial_t\psi + |\nabla u^q|^{p-2} \langle \nabla u^q, \nabla \psi \rangle \le 0.$$
(2.13)

For different notions of weak solutions see also [9, 24]. Existence and uniqueness results for the Cauchy problem with the above notion of weak solutions of (2.9) were obtained in the euclidean case for example in [17, 18, 19, 22] and on manifolds in [16].

If u is of the class (2.11) then $\nabla(u^q)$ is defined as an element of $L^p(\Omega)$. Then we define ∇u as follows:

$$\nabla u := \begin{cases} q^{-1}u^{1-q}\nabla(u^q), & u > 0, \\ 0, & u = 0. \end{cases}$$

Remark 2.2. Note that it follows from (2.11) and (2.12) that the integrals in (2.13) are finite. Indeed, we have by Hölder's inequality

$$\begin{split} \int_{t_1}^{t_2} \int_{\Omega} |\nabla u^q|^{p-2} \left| \langle \nabla u^q, \nabla \psi \rangle \right| &\leq \int_{t_1}^{t_2} \int_{\Omega} |\nabla u^q|^{p-1} |\nabla \psi| \\ &\leq \left(\int_{t_1}^{t_2} \int_{\Omega} (|\nabla u^q|)^p \right)^{\frac{p-1}{p}} \left(\int_{t_1}^{t_2} \int_{\Omega} |\nabla \psi|^p \right)^{\frac{1}{p}}. \end{split}$$

Lemma 2.3. Let u(x,t) be a non-negative weak subsolution of (2.9) in $\Omega \times [0,\infty)$. Then $au(x, a^{-D}t)$ is a weak subsolution of (2.9) in $\Omega \times [0,\infty)$ for any a > 0.

Proof. Let us apply (2.13) with $a\psi(x, \tilde{t}) = a\psi(x, a^D t)$ noticing that $a\psi(x, a^D t)$ lies again in the class (2.12) if $\psi(x, t)$ does. Hence, for all $t_1, t_2 \in [0, \infty)$ with $t_1 < t_2$,

$$\begin{split} & \left[\int_{\Omega} u(x,t)a\psi(x,a^{D}t)\right]_{t_{1}}^{t_{2}} \\ & +\int_{t_{1}}^{t_{2}}\int_{\Omega} -au(x,t)\partial_{t}\psi(x,a^{D}t) + a|\nabla u(x,t)^{q}|^{p-2}\langle \nabla u(x,t)^{q},\nabla\psi(x,a^{D}t)\rangle dtd\mu \leq 0. \end{split}$$

We have

$$|\nabla (au)^q|^{p-2}\nabla (au)^q = a^{-D+1}|\nabla u^q|^{p-2}\nabla u^q$$

Thus, using that $dt = a^{-D} d\tilde{t}$, we obtain

$$\begin{split} &\int_{t_1}^{t_2} \int_{\Omega} a |\nabla u(x,t)^q|^{p-2} \langle \nabla u(x,t)^q, \nabla \psi(x,a^D t) \rangle dt d\mu \\ &= \int_{\widetilde{t_1}}^{\widetilde{t_2}} \int_{\Omega} |\nabla (au(x,a^{-D}\widetilde{t}))^q|^{p-2} \langle \nabla (au(x,a^{-D}\widetilde{t}))^q, \nabla \psi(x,\widetilde{t}) \rangle d\widetilde{t} d\mu, \end{split}$$

where $\tilde{t_1} = a^D t_1$ and $\tilde{t_2} = a^D t_2$. It follows that

$$\left[\int_{\Omega}au(x,a^{\delta}\widetilde{t})\psi(x,\widetilde{t})\right]_{\widetilde{t_{1}}}^{\widetilde{t_{2}}}$$

$$+\int_{\widetilde{t_1}}^{\widetilde{t_2}}\int_{\Omega}-u(x,a^{-D}\widetilde{t})\partial_{\widetilde{t}}\psi(x,\widetilde{t})+|\nabla u(x,a^{-D}\widetilde{t})^q|^{p-2}\langle\nabla u(x,a^{-D}\widetilde{t})^q,\nabla\psi(x,\widetilde{t})\rangle d\widetilde{t}d\mu\leq 0,$$

which proves the claim. \blacksquare

We cite from [13] and [14] the next three lemma.

Lemma 2.4. [14] Let u be a non-negative bounded weak subsolution of (2.9) in $\Omega \times [0,T)$. Assume that

$$1 and $1 \le q < \frac{1}{p-1}$. (2.14)$$

For any $\theta \geq 0$, define

$$f(s) = (s^a - \theta)^{1/a}_+,$$

where

$$a = \frac{1 - q(p-1)}{2 - p} = \frac{D}{2 - p}.$$
(2.15)

Then f(u) is also a weak subsolution of (2.9).



Figure 1: Function f(s)

Lemma 2.5. [13] (Caccioppoli type inequality) Let v = v(x,t) be a non-negative bounded subsolution to (2.9) in a cylinder $\Omega \times [0,T)$. Let $\eta(x,t)$ be a locally Lipschitz non-negative bounded function in $\Omega \times [0,T)$ such that $\eta(\cdot,t)$ has compact support in Ω for all $t \in [0,T)$. Fix some real σ such that

$$\sigma \ge \max\left(p, pq\right) \tag{2.16}$$

and set

$$\lambda = \sigma + D$$
 and $\alpha = \frac{\sigma}{p}$. (2.17)

Choose $0 \leq t_1 < t_2 < T$ and set $Q = \Omega \times [t_1, t_2]$. Then

$$\left[\int_{\Omega} v^{\lambda} \eta^{p}\right]_{t_{1}}^{t_{2}} + c_{1} \int_{Q} \left|\nabla \left(v^{\alpha} \eta\right)\right|^{p} \leq \int_{Q} \left[pv^{\lambda} \eta^{p-1} \partial_{t} \eta + c_{2} v^{\sigma} \left|\nabla \eta\right|^{p}\right], \qquad (2.18)$$

where c_1, c_2 are positive constants depending on p, q, λ .

In particular, if η does not depend on t, then

$$\left[\int_{\Omega} v^{\lambda} \eta^{p}\right]_{t_{1}}^{t_{2}} + c_{1} \int_{Q} |\nabla (v^{\alpha} \eta)|^{p} \leq c_{2} \int_{Q} v^{\sigma} |\nabla \eta|^{p}.$$

$$(2.19)$$

Let us observe for a later usage that

$$v^{\alpha}\eta \in L^p_{loc}\left([0,T]; W^{1,p}_0(\Omega)\right).$$
(2.20)

Indeed, using $\alpha \geq q$, we get that the function $\Phi(s) = s^{\frac{\alpha}{q}}$ is Lipschitz on any bounded interval in $[0, \infty)$. Thus, $v^{\alpha} = \Phi(v^q) \in W^{1,p}(\Omega)$ and

$$\left|\nabla v^{\alpha}\right| = \left|\Phi'(v^q)\nabla v^q\right| \le C\left|\nabla v^q\right|$$

whence

$$\int_{Q} |\nabla (v^{\alpha} \eta)|^{p} \leq C' \int_{Q} |\nabla v^{\alpha}|^{p} \eta^{p} + v^{\alpha p} |\nabla \eta|^{p} = C' \int_{Q} |\nabla v^{q}|^{p} \eta^{p} + v^{\sigma} |\nabla \eta|^{p},$$

which is finite since

$$\int_{Q} v^{\sigma} |\nabla \eta|^{p} \le \text{const } ||v||_{L^{\infty}}^{\sigma - pq} \int_{Q} v^{pq}$$

and proves (2.20).

Lemma 2.6. [14] Let M be geodesically complete and v = v(x,t) be a bounded non-negative subsolution to (2.9) in $M \times [0,T)$. For any $\lambda \in [1,\infty]$, the function

$$t \mapsto \|v(\cdot, t)\|_{L^{\lambda}(M)}$$

is monotone decreasing in [0, T).

3 Faber-Krahn inequality

Let M be a connected Riemannian manifold of dimension n and d be the geodesic distance on M. For any $x \in M$ and r > 0, denote by B(x, r) the geodesic ball of radius r centered at x, that is,

$$B(x, r) = \{ y \in M : d(x, y) < r \}.$$

Let the ball B be precompact. Then the following Faber-Krahn inequality in B of order $p \ge 1$ holds: if $w \in W_0^{1,p}(B)$ is non-negative,

$$E = \{w > 0\}$$

and r(B) denotes the radius of the ball B, then

$$\int_{B} |\nabla w|^{p} \ge \frac{1}{r(B)^{p}} \left(\iota(B)\frac{\mu(B)}{\mu(E)}\right)^{\nu} \int_{B} w^{p}, \qquad (3.21)$$

where $\nu > 0$ and $\iota(B)$ is a positive constant that depends on the geometry of B. The value of ν is independent of B and can be chosen as follows:

$$\nu = \begin{cases} \frac{p}{n}, & \text{if } n > p, \\ \text{any number} \in (0, 1), & \text{if } n \le p. \end{cases}$$
(3.22)

Choosing $\iota(B)$ to be an optimal constant in (3.21) we obtain that the function

$$B \mapsto \frac{(\iota(B)\mu(B))^{\nu}}{r(B)^{p}} \tag{3.23}$$

is monotone decreasing with respect to the partial order \subset on balls.

We say that M satisfies a relative Faber-Krahn inequality of order p if (3.21) holds with $\iota(B) \ge \text{const} > 0$ for all geodesic balls B. For example, this holds if M is complete, non-compact and satisfies $Ricci_M \ge 0$ (see [4, 11, 23]).

4 Long distance decay

From now on we always assume that

$$1 and $1 \le q < \frac{1}{p-1}$. (4.24)$$

Let us denote

$$D = 1 - q(p - 1)$$

and note that, under condition (4.24), we have $D \in (0,1)$. We start with the following mean-value type inequality from [14].

Lemma 4.1. [14] Let the ball $B = B(x_0, r)$ be precompact. Let u be a non-negative bounded subsolution in $Q = B \times [0, t]$

such that

we have

$$u\left(\cdot,0
ight)=0 \ in \ B$$

 $O' \stackrel{1}{\longrightarrow} D \times [0, t]$

Let σ and λ be reals such that

$$\sigma > 0 \quad \text{and} \quad \lambda = \sigma + D.$$

$$(4.25)$$

Then, for the cylinder

$$Q = \frac{1}{2}D \times [0, t],$$

$$\|u\|_{L^{\infty}(Q')} \le \left(\frac{C}{\iota(B)\mu(B)r^p} \int_Q u^{\sigma}\right)^{1/\lambda},\tag{4.26}$$

where $\iota(B)$ is the Faber-Krahn constant in B, and the constant C depends on p, q, λ and the Faber-Krahn exponent ν .



Figure 2: Cylinders Q and Q'

The next lemma is the main result of this section.

Lemma 4.2. Assume that M is geodesically complete and let u be a bounded non-negative subsolution in $M \times [0,T]$. Let $B_0 = B(x_0, R)$ be a ball such that

$$u_0 := u(\cdot, 0) = 0$$
 in B_0 .

Then, for all $t \in [0, T]$,

$$\|u\|_{L^{\infty}\left(\frac{1}{2}B_{0}\times[0,t]\right)} \leq C\left(\frac{t}{\iota(B_{0})R^{p}}\right)^{\frac{1}{D}}\ln^{\gamma}\left(2+\left(\frac{\iota(B_{0})R^{p}}{t}\right)^{\frac{1}{D}}\frac{||u_{0}||_{L^{1}(M)}}{\mu(B_{0})}\right),$$

where the positive constants C and γ depend on p,q and the Faber-Krahn exponent ν .

Proof. Fix a point $x \in \frac{1}{2}B_0$ and $r \leq \frac{1}{2}R$ so that $B := B(x, r) \subset B_0$. Fix also some $t \leq T$ and set, for all $0 \leq k \leq l$,

$$Q_k = 2^k B \times [0, t]$$
 and $J_k = \int_{Q_k} u_k$

where l is the maximal non-negative integer such that

$$2^l r \le \frac{1}{2} R,\tag{4.27}$$

which implies $2^k B \subset B_0$ for all $0 \le k \le l$.

By Lemma 4.1 with $\sigma = 1$ and $\lambda = 1 + D$, we obtain for all $1 \le k \le l$,

$$||u||_{L^{\infty}(Q_{k-1})} \leq \left(\frac{C}{\iota(2^{k}B)\mu(2^{k}B)(2^{k}r)^{p}}\int_{Q_{k}}u\right)^{1/\lambda}.$$

It follows that

$$J_{k-1} = \int_{Q_{k-1}} u \le \mu \left(2^{k-1}B\right) t \|u\|_{L^{\infty}(Q_{k-1})}$$
$$\le \mu(B_0) t \left(\frac{C}{\iota(2^k B)\mu \left(2^k B\right) \left(2^k r\right)^p} J_k\right)^{1/\lambda}$$

whence

$$J_k \ge C^{-1}\iota(2^k B)\mu\left(2^k B\right)\left(2^k r\right)^p \left(\frac{J_{k-1}}{\mu\left(B_0\right)t}\right)^{\lambda}$$

Since by the monotonicity of the function (3.23),

$$\frac{\iota(2^k B)\mu\left(2^k B\right)}{\left(2^k r\right)^{p/\nu}} \ge \frac{\iota(B_0)\mu(B_0)}{R^{p/\nu}},$$

it follows that

$$\iota(2^k B) \mu\left(2^k B\right) \left(2^k r\right)^p \ge \iota(B_0) \mu(B_0) R^p \left(\frac{2^k r}{R}\right)^{p+p/\nu}$$

and

$$J_{k} \ge C^{-1}\iota(B_{0})\mu(B_{0})R^{p}\left(\frac{2^{k}r}{R}\right)^{p+p/\nu}\left(\frac{J_{k-1}}{\mu(B_{0})t}\right)^{\lambda} = \frac{A^{k}}{\Theta}J_{k-1}^{1+D},$$

where $A = 2^{p+p/\nu}$ and $\Theta = C \frac{\mu(B_0)^D t^{\lambda}}{\iota(B_0)R^p} \left(\frac{R}{r}\right)^{p+p/\nu}$. By Lemma 7.2 with $\omega = D$ we obtain

$$J_k \ge \left(\frac{J_0}{\left(A^{-1/D}\Theta\right)^{1/D}}\right)^{\lambda^{\kappa}} \left(A^{-k-1/D}\Theta\right)^{1/D},$$

that is,

$$J_0 \le \left(A^{k/D} \frac{J_k}{\xi}\right)^{\frac{1}{\lambda^k}} \xi = A^{\frac{k}{D\lambda^k}} \xi^{1-\frac{1}{\lambda^k}} J_k^{\frac{1}{\lambda^k}},$$

where $\xi = (A^{-1/D}\Theta)^{1/D}$. We have by Lemma 2.6

$$J_k \le \int_{B_0 \times [0,t]} u \le t \int_M u(\cdot, 0).$$

On the other hand, by Lemma 4.1,

$$\|u\|_{L^{\infty}\left(\frac{1}{2}B\times[0,t]\right)}^{\lambda} \leq \frac{C}{\iota(B)\mu(B)r^{p}}J_{0}$$

It follows that

$$\|u\|_{L^{\infty}\left(\frac{1}{2}B\times[0,t]\right)}^{\lambda} \leq \frac{CA^{\frac{k}{D\lambda^{k}}}}{\iota(B)\mu(B)r^{p}}\xi^{1-\frac{1}{\lambda^{k}}}\left(t||u_{0}||_{L^{1}(M)}\right)^{\frac{1}{\lambda^{k}}}.$$

Using that

$$\iota(B)\mu(B) r^{p} \ge \frac{\iota(B_{0})\mu(B_{0})}{R^{p/\nu}} r^{p+p/\nu} = \iota(B_{0})\mu(B_{0})R^{p} \left(\frac{r}{R}\right)^{p+p/\nu}$$

we obtain

$$\begin{aligned} \|u\|_{L^{\infty}\left(\frac{1}{2}B\times[0,t]\right)}^{\lambda} &\leq \frac{CA^{\frac{k}{D\lambda^{k}}}}{\iota(B_{0})\mu(B_{0})R^{p}} \left(\frac{r}{R}\right)^{p+p/\nu} \xi^{1-\frac{1}{\lambda^{k}}} \left(t\||u_{0}||_{L^{1}(M)}\right)^{\frac{1}{\lambda^{k}}} \\ &= \frac{CA^{\frac{k}{D\lambda^{k}}}}{\iota(B_{0})\mu(B_{0})R^{p}} \left(\frac{r}{R}\right)^{p+p/\nu} \\ &\times \left(A^{-1/D}\frac{\mu\left(B_{0}\right)^{D}t^{\lambda}}{\iota(B_{0})R^{p}} \left(\frac{R}{r}\right)^{p+p/\nu}\right)^{\frac{1}{D}\left(1-\frac{1}{\lambda^{k}}\right)} \left(t||u_{0}||_{L^{1}(M)}\right)^{\frac{1}{\lambda^{k}}}.\end{aligned}$$

Let us choose k maximal possible, that is, k = l. Recall that $\lambda = 1 + D$, so that

$$\Gamma := \lambda \frac{1}{D} \left(1 - \frac{1}{\lambda^l} \right) + \frac{1}{\lambda^l} = 1 + \frac{1}{D} - \frac{1}{D\lambda^l}$$

and we obtain

$$\|u\|_{L^{\infty}\left(\frac{1}{2}B\times[0,t]\right)}^{\lambda} \leq CA^{\frac{l}{D\lambda^{l}}-\frac{1}{D^{2}}\left(1-\frac{1}{\lambda^{l}}\right)} \left(\frac{R}{r}\right)^{\left(p+\frac{p}{\nu}\right)\left(\frac{1}{D}\left(1-\frac{1}{\lambda^{l}}\right)-1\right)} \left(\frac{\|u_{0}\|_{L^{1}(M)}}{\mu(B_{0})}\right)^{\frac{1}{\lambda^{l}}} \left(\frac{t}{\iota(B_{0})R^{p}}\right)^{\Gamma}.$$

It follows from (4.27) that $\frac{R}{r} \simeq 2^l$, which yields

$$\|u\|_{L^{\infty}\left(\frac{1}{2}B\times[0,t]\right)}^{\lambda} \leq C2^{l\left(p+\frac{p}{\nu}\right)\left(\frac{1}{D}\left(1-\frac{1}{\lambda^{l}}\right)-1\right)} \left(\frac{\|u_{0}\|_{L^{1}(M)}}{\mu(B_{0})}\right)^{\frac{1}{\lambda^{l}}} \left(\frac{t}{\iota(B_{0})R^{p}}\right)^{\Gamma}.$$
(4.28)

Also, since $\frac{\Gamma}{\lambda} = \frac{1}{D} - \frac{1}{D\lambda^{l+1}}$, we deduce from (4.28),

$$\|u\|_{L^{\infty}\left(\frac{1}{2}B\times[0,t]\right)} \leq C2^{l\frac{1}{\lambda}\left(p+\frac{p}{\nu}\right)\left(\frac{1}{D}-1\right)} \left(\frac{t}{\iota(B_{0})R^{p}}\right)^{\frac{1}{D}} \left[\left(\frac{t}{\iota(B_{0})R^{p}}\right)^{-\frac{1}{D}} \frac{||u_{0}||_{L^{1}(M)}}{\mu(B_{0})}\right]^{\frac{1}{\lambda^{l+1}}} = C\left(\frac{t}{\iota(B_{0})R^{p}}\right)^{\frac{1}{D}} a^{l}b^{\frac{1}{\lambda^{l+1}}},$$

$$(4.29)$$

where

$$a = 2^{\frac{1}{\lambda} \left(p + \frac{p}{\nu} \right) \left(\frac{1}{D} - 1 \right)}$$
 and $b = \left(\frac{\iota(B_0) R^p}{t} \right)^{\frac{1}{D}} \frac{||u_0||_{L^1(M)}}{\mu(B_0)}.$

Let us now choose $l \in \mathbb{N}$ so that the right hand side of (4.29) is minimal. For that, consider first the case when $R^p \geq \frac{1}{a^{\frac{\lambda}{\log \lambda}}} \left(\frac{\mu(B_0)}{||u_0||_{L^1(M)}}\right)^D \frac{t}{\iota(B_0)}$. Then the right hand side of (4.29) is minimal if $\lambda^{l+1} = \frac{\ln \lambda}{\ln a} \ln b$, that is, we choose

$$l = \left\lfloor \frac{1}{\ln \lambda} \ln \left(\frac{\ln \lambda \ln b}{\lambda \ln a} \right) \right\rfloor.$$

Hence, we obtain from (4.29),

$$\|u\|_{L^{\infty}\left(\frac{1}{2}B\times[0,t]\right)} \leq C\left(\frac{t}{\iota(B_0)R^p}\right)^{\frac{1}{D}} \left(\frac{\ln\lambda\ln b}{\lambda\ln a}\right)^{\frac{\ln a}{\ln\lambda}} \exp\left(\frac{\ln a}{\ln\lambda}\right) = C\left(\frac{t}{\iota(B_0)R^p}\right)^{\frac{1}{D}} \ln^{\gamma} b,$$

where $\gamma = \frac{\ln a}{\ln \lambda}$. On the other hand, if $R^p \leq \frac{1}{a^{\frac{\lambda}{\log \lambda}}} \left(\frac{\mu(B_0)}{||u_0||_{L^1(M)}}\right)^D \frac{t}{\iota(B_0)}$, we have $b \leq a^{\frac{\lambda}{\log \lambda}}$ and whence,

$$||u||_{L^{\infty}(\frac{1}{2}B\times[0,t])} \le C\left(\frac{t}{\iota(B_0)R^p}\right)^{\frac{1}{D}}\ln^{\gamma}(2+b).$$

Covering $\frac{1}{2}B_0$ with balls $B(x_i, r)$ with small enough radius r we obtain

$$||u||_{L^{\infty}(\frac{1}{2}B_0 \times [0,t])} \le C\left(\frac{t}{\iota(B_0)R^p}\right)^{\frac{1}{D}} \ln^{\gamma}(2+b),$$

which proves the claim. \blacksquare

5 Long time decay

The main result of this section is Lemma 5.4.

5.1 Comparison in two cylinders

Let a be defined by (2.15), that is,

$$a = \frac{1 - q(p-1)}{2 - p} = \frac{D}{2 - p}.$$
(5.30)

Note that under condition (4.24) we have $a \in (0, 1]$.

Lemma 5.1. Consider two balls $B_0 = B(x_0, r_0)$ and $B_1 = B(x_0, r_1)$ with $0 < r_1 < r_0$ where B_0 is precompact. Assuming $0 < t_0 < t_1 < T$, consider two cylinders $Q_i = B_i \times [t_i, T]$, i = 0, 1. Let v_0 be non-negative bounded subsolution in Q_0 . Set for some $\theta > 0$

$$v_1 = (v_0^a - \theta)_+^{1/a}$$
,

where a is as in (5.30). Let σ and λ be reals satisfying (2.16) and (2.17). Set

$$J_i = \int_{Q_i} v_i^\sigma d\mu dt$$

Then

$$J_1 \le \frac{Cr_0^p S^{\nu}}{(\iota(B_0)\mu(B_0))^{\nu} \,\theta^{\frac{\lambda\nu}{a}} \,(r_0 - r_1)^p} J_0^{1+\nu},\tag{5.31}$$

where

$$S = \frac{\|v_0\|_{L^{\infty}(Q_0)}^D}{t_1 - t_0} + \frac{1}{(r_0 - r_1)^p},$$

 ν is the Faber-Krahn exponent, $\iota(B_0)$ is the Faber-Krahn constant in B_0 , and C depends on p, q and λ .



Figure 3: Cylinders Q_0 and Q_1

Proof. From Lemma 2.4 we know that v_1 is also a subsolution. Let $\eta(x,t) = \eta(x)$ be a bump function of B_1 in $B_{1/2} := B\left(x_0, \frac{r_0+r_1}{2}\right)$. Recall that by (2.20), $v_1^{\alpha}\eta \in L^p\left([t_0,T]; W_0^{1,p}(B)\right)$, where α is defined by (2.17), that is $\alpha = \frac{\sigma}{p}$. Hence, applying the Faber-Krahn inequality (3.21) in ball B_0 for any $t \in [t_0,T]$ we get that

$$\int_{B_1} v_1^{\sigma} \le \int_{B_0} \left(v_1^{\alpha} \eta \right)^p \le r_0^p \left(\frac{\mu(D_t)}{\iota(B_0)\mu(B_0)} \right)^{\nu} \int_{B_0} |\nabla(v_1^{\alpha} \eta)|^p,$$
(5.32)

where we used that $\alpha p = \sigma$ and $\eta = 1$ in B_1 and

$$D_t = \{v_1^{\alpha}\eta(\cdot, t) > 0\} = \{v_1 > 0\} \cap \{\eta > 0\} = \{v_0(\cdot, t) > \theta^{1/a}\} \cap B_{1/2}.$$

Also, note that $\eta_t = 0$ and $|\nabla \eta| \le \frac{2}{r_0 - r_1}$. From (2.19) we therefore obtain

$$c_1 \int_{t_1}^T \int_{B_0} |\nabla (v_1^{\alpha} \eta)|^p \le \int_{t_1}^T \int_{B_0} v_1^{\sigma} |\nabla \eta|^p \le \frac{c_3}{(r_0 - r_1)^p} J_0,$$
(5.33)

where $c_3 = c_2 2^p$ and we used that $v_1 \leq v_0$.

Let us now apply Lemma 2.5 to function v_0 in $B_0 \times [t_0, T]$. Take

$$\eta\left(x,t\right) = \eta_{1}\left(x\right)\eta_{2}\left(t\right),$$

where η_1 is a bump function of $B_{1/2}$ in B_0 so that

$$|\nabla \eta_1| \le \frac{2}{r_0 - r_1}$$

and η_2 is a bump function of $[t_1, T]$ in $[t_0, T]$, that is,

$$\eta_{2}(t) = \begin{cases} 1, t \ge t_{1} \\ \frac{t-t_{0}}{t_{1}-t_{0}}, t_{0} \le t \le t_{1} \end{cases}$$

so that

$$|\partial_t \eta_2| \le \frac{1}{t_1 - t_0}$$

From (2.18) we obtain

$$\left[\int_{B_0} v_0^{\lambda} \eta^p\right]_{t_0}^T \le \int_{t_0}^T \int_{B_0} \left[p\eta^{p-1} \partial_t \eta v_0^{\lambda} + c_2 \, |\nabla \eta|^p \, v_0^{\sigma} \right] = \int_{t_0}^T \int_{B_0} \left[p\eta^{p-1} \partial_t \eta v_0^D + c_2 \, |\nabla \eta|^p \right] v_0^{\sigma}.$$

Hence, for any $t \in [t_1, T]$, using that $\eta_2(t_0) = 0$ and $\eta(x, t) = 1$ for $x \in B_{1/2}$ and $t \ge t_1$,

$$\int_{B_{1/2}} v_0^{\lambda}(\cdot, t) \le c_4 \int_{t_0}^T \int_{B_0} \left[\frac{||v_0||_{L^{\infty}}^D}{t_1 - t_0} + \frac{1}{(r_0 - r_1)^p} \right] v_0^{\sigma} \le c_4 S J_{0,0}$$

where $c_4 = \max(p, c_3)$. Thus, we deduce

$$\mu\left(D_{t}\right) \leq \frac{1}{\theta^{\lambda/a}} \int_{B_{1/2}} v_{0}^{\lambda}\left(\cdot, t\right) \leq \frac{c_{4}SJ_{0}}{\theta^{\lambda/a}}$$

Combining this with (5.32) and (5.33) we obtain

$$J_1 = \int_{t_1}^T \int_{B_1} v_1^{\sigma} \le r_0^p \left(\frac{c_4 S J_0}{\iota(B_0)\mu(B_0)\theta^{\lambda/a}}\right)^{\nu} \frac{c_3}{c_1 (r_0 - r_1)^p} J_0,$$

which implies (5.31) and finishes the proof.

5.2 Iterations and the mean value theorem

Lemma 5.2. Let the ball $B = B(x_0, R)$ be precompact. Let u be a non-negative bounded subsolution in $Q = B \times [0, T]$. Let σ and λ be reals such that

$$\sigma > 0 \quad \text{and} \quad \lambda = \sigma + D.$$
 (5.34)

Then, for the cylinder

$$Q' = \frac{1}{2}B \times [\frac{1}{2}T, T]$$

we have

$$\|u\|_{L^{\infty}(Q')} \le \left(\frac{CS}{\iota(B)\mu(B)} \int_{Q} u^{\sigma}\right)^{1/\lambda},\tag{5.35}$$

where

$$S = \frac{\|u\|_{L^{\infty}(Q)}^{D}}{T} + \frac{1}{R^{p}},$$
(5.36)

 $\iota(B)$ is the Faber-Krahn constant in B, and the constant C depends on p, q, λ and ν .



Figure 4: Cylinders Q' and Q

Proof. Let us first prove (5.35) for σ large enough as in Lemma 2.5. Consider sequences

$$r_k = \left(\frac{1}{2} + 2^{-k-1}\right) R, \quad t_k = \left(1 - 2^{-k}\right) \frac{T}{2}$$

where k = 0, 1, 2, ..., so that $r_0 = R$ and $r_k \searrow \frac{1}{2}R$ as $k \to \infty$ $t_0 = 0$ and $t_k \nearrow \frac{1}{2}T$ as $k \to \infty$. Set $B_k = B(x_0, r_k)$, $Q_k = B_k \times [t_k, T]$ so that $B_0 = B$, $Q_0 = Q$ and $Q_\infty := \lim_{k \to \infty} Q_k = Q'$. Choose some $\theta > 0$ to be specified later and define a sequence of functions $\{u_k\}$ by

$$u_0 = u, \quad u_k = \left(u_{k-1}^a - 2^{-k}\theta\right)_+^{1/a} \text{ for } k \ge 1$$

where a is given by (5.30). The function $f_{\theta}(s) = (s^a - \theta)^{1/a}_+$ has the property that $f_{\theta_1} \circ f_{\theta_2} = f_{\theta_1+\theta_2}$. Hence, we obtain

$$u_k = \left(u^a - \frac{1}{2}\theta - \dots - \frac{1}{2^k}\theta\right)_+^{1/a} = \left(u^a - \left(1 - 2^{-k}\right)\theta\right)_+^{1/a}.$$

Set

$$J_k = \int_{Q_k} u_k^{\sigma}.$$

Since u_k is a subsolution, we obtain by Lemma 5.1 that

$$J_{k+1} \le \frac{Cr_k^p S_k^{\nu}}{\left(\iota(B_k)\mu(B_k)\right)^{\nu} \left(2^{-(k+1)}\theta\right)^{\frac{\lambda\nu}{a}} \left(r_k - r_{k+1}\right)^p} J_k^{1+\nu},$$

where

$$S_k = \frac{\|u\|_{L^{\infty}(Q_k)}^D}{t_{k+1} - t_k} + \frac{1}{(r_k - r_{k+1})^p}$$

By monotonicity of the function (3.23), we have

$$\frac{r_k^p}{(\iota(B_k)\mu(B_k))^{\nu}} \le \frac{R^p}{(\iota(B)\mu(B))^{\nu}}.$$

Since $r_k - r_{k+1} = 2^{-k-2}R$ and $t_{k+1} - t_k = 2^{-k-2}T$, it follows that

$$S_k \le 2^{(k+2)p} \left(\frac{\|u\|_{L^{\infty}(Q)}^D}{T} + \frac{1}{R^p} \right) = 2^{(k+2)p} S.$$

Hence,

$$J_{k+1} \leq \frac{C2^{(k+1)\frac{\lambda\nu}{a}}2^{(k+2)p(1+\nu)}S^{\nu}}{(\iota(B)\mu(B))^{\nu}\theta^{\frac{\lambda\nu}{a}}}J_{k}^{1+\nu} = \frac{A^{k}J_{k}^{1+\nu}}{\Theta}$$

where

$$A = 2^{\frac{\lambda\nu}{a} + (1+\nu)p}$$
 and $\Theta = c \left(\frac{\iota(B)\mu(B)\theta^{\frac{\lambda}{a}}}{S}\right)^{\nu}$.

Now let us apply Lemma 7.1 with $\omega = \nu$: if

$$\Theta \ge A^{1/\nu} J_0^{\nu},$$
 (5.37)

then, for all $k \ge 0$,

 $J_k \le A^{-k/\nu} J_0.$

In terms of θ the condition (5.37) is equivalent

$$c\left(\frac{\iota(B)\mu(B)\theta^{\frac{\lambda}{a}}}{S}\right)^{\nu} \ge A^{1/\nu}J_0^{\nu}$$

that is,

$$\theta^{\frac{\lambda}{a}} \ge \frac{CSJ_0}{\iota(B)\mu(B)}.$$

Hence, we choose θ to have equality here. For this θ we obtain $J_k \to 0$ as $k \to \infty$, which implies that $u^a \leq \theta$ in Q_{∞} . Hence,

$$\|u\|_{L^{\infty}(Q')} \le \left(\frac{CSJ_0}{\iota(B)\mu(B)}\right)^{1/\lambda}$$

which proves (5.35).

Now we prove (5.35) for any $\sigma > 0$. Let σ_0 be such that (5.35) is already known for $\sigma = \sigma_0$, and let $\sigma < \sigma_0$. Denote

$$\lambda_0 = \sigma_0 + D$$
 and $\lambda = \sigma + D$

so that $\lambda < \lambda_0$.

Consider, for $k \ge 0$, sequences $r_k = \left(1 - \frac{1}{2^{k+1}}\right)R$ and $t_k = 2^{-(k+1)}T$ so that $r_0 = \frac{1}{2}R$, $t_0 = \frac{1}{2}T$ and $r_k \uparrow R$, $t_k \downarrow 0$ as $r \to \infty$. Set $B_k = B(x_0, r_k)$ and $\tilde{Q}_k = B_k \times [t_k, T]$. Denoting also $B = B(x_0, R)$, we see that

$$\frac{1}{2}B \subset B_k \subset B \text{ and } B_k \uparrow B \text{ as } k \to \infty$$

and thus $\widetilde{Q}_0 = Q'$ and $\widetilde{Q}_k \uparrow Q$. Set also $\rho_k = r_{k+1} - r_k = \frac{1}{2^{k+2}}R$. Let us also use the notation $\chi(B) = \iota(B)\mu(B)$. For any point $(x, \tau) \in \widetilde{Q}_k$, let s be such that

$$\tau < s < \min\left(\tau + \frac{1}{2}t_k, T\right).$$

Then applying (5.35) from the first part of the proof in $B(x, \rho_k) \times [s - t_k, s)$, we obtain

$$\begin{aligned} \|u\|_{L^{\infty}(B(x,\frac{1}{2}\rho_k)\times[s-\frac{1}{2}t_k,s))}^{\lambda_0} &\leq \frac{CS_k}{\chi\left(B(x,\rho_k)\right)} \int_{B(x,\rho_k)\times[s-t_k,s)} u^{\sigma_0} \\ &\leq \frac{CS_k}{\chi\left(B(x,\rho_k)\right)} \|u\|_{L^{\infty}(B(x,\rho_k)\times[s-t_k,s))}^{\sigma_0-\sigma} \int_{B(x,\rho_k)\times[s-t_k,s)} u^{\sigma}, \end{aligned}$$

where

$$S_k = \frac{\|u\|_{L^{\infty}(B(x,\rho_k) \times [s-t_k,s))}^D}{t_k} + \frac{1}{\rho_k^p}.$$

Since $B(x, \rho_k) \subset B_{k+1} \subset B$, we have by the monotonicity of (3.23), $\frac{\chi(B(x, \rho_k))}{\rho_k^{p/\nu}} \ge \frac{\chi(B)}{R^{p/\nu}}$ whence

$$\frac{1}{\chi(B(x,\rho_k))} \le \frac{(R/\rho_k)^{p/\nu}}{\chi(B)} = \frac{2^{(k+2)p/\nu}}{\chi(B)}$$

Hence, we obtain

$$\|u\|_{L^{\infty}(B(x,\frac{1}{2}\rho_{k})\times[s-\frac{1}{2}t_{k},s))}^{\lambda_{0}} \leq \frac{C2^{kp(1+1/\nu)}S}{\chi(B)} \|u\|_{L^{\infty}(\widetilde{Q}_{k+1})}^{\lambda_{0}-\lambda} \int_{Q} u^{\sigma}.$$

Covering \widetilde{Q}_k by a sequence of sets $B(x, \frac{1}{2}\rho_k) \times [s - \frac{1}{2}t_k, s))$ with $(x, \tau) \in \widetilde{Q}_k$, we obtain

$$\|u\|_{L^{\infty}(\tilde{Q}_{k})}^{\lambda_{0}} \leq \frac{C2^{kp(1+1/\nu)}S}{\chi(B)} \|u\|_{L^{\infty}(\tilde{Q}_{k+1})}^{\lambda_{0}-\lambda} \int_{Q} u^{\sigma}.$$
(5.38)

Setting $J_k = \|u\|_{L^{\infty}(\widetilde{Q}_k)}^{-(\lambda_0 - \lambda)}$, we rewrite (5.38) as follows:

$$J_{k+1} \le \frac{A^k}{\Theta} J_k^{\frac{\lambda_0}{\lambda_0 - \lambda}} = \frac{A^k}{\Theta} J_k^{1+\omega},$$

where

$$A = 2^{p(\nu^{-1}+1)}, \ \Theta^{-1} = \frac{CS}{\chi(B)} \int_Q u^{\sigma} \text{ and } \omega = \frac{\lambda_0}{\lambda_0 - \lambda} - 1 = \frac{\lambda}{\lambda_0 - \lambda}.$$

Applying Lemma 7.1, we obtain

$$J_k \le \left(\frac{J_0}{\left(A^{-1/\omega}\Theta\right)^{1/\omega}}\right)^{(1+\omega)^k} \left(A^{-1/\omega}\Theta\right)^{1/\omega},$$

that is,

$$J_0 \ge \left(A^{-1/\omega}\Theta\right)^{1/\omega} \left(\left(A^{1/\omega}\Theta^{-1}\right)^{1/\omega}J_k\right)^{\frac{1}{(1+\omega)^k}}$$

Since $J_k \ge ||u||_{L^{\infty}(\tilde{Q}_k)}^{-(\lambda_0 - \lambda)} =: \text{const} > 0$, we see that

$$\liminf_{k \to \infty} \left(\left(A^{1/\omega} \Theta^{-1} \right)^{1/\omega} J_k \right)^{\frac{1}{(1+\omega)^k}} \ge 1,$$

whence

$$J_0 \ge \left(A^{-1/\omega}\Theta\right)^{1/\omega}$$

It follows that

$$\|u\|_{L^{\infty}(\widetilde{Q}_{0})}^{\lambda_{0}-\lambda} \leq A^{1/\omega^{2}} \left(\frac{CS}{\chi(B)} \int_{Q} u^{\sigma}\right)^{1/\omega}$$

and finally

$$\|u\|_{L^{\infty}(Q')} \leq \left(\frac{CS}{\chi(B)} \int_{Q} u^{\sigma}\right)^{1/\lambda}$$

where A^{1/ω^2} is absorbed into C and finishes the proof.

5.3 Initial estimate of the long time decay

Lemma 5.3. Assume that M is geodesically complete and satisfies the relative Faber-Krahn inequality. Let u be a non-negative bounded subsolution in $M \times [0, \infty)$ with initial function $u_0 = u(\cdot, 0)$. Set

$$\tau = \|u_0\|_{L^{\infty}(M)}^D$$

Let $\sigma \geq 1$ and $\lambda = \sigma + D$. Then, for all T > 0 and all $x \in M$,

$$||u(\cdot,T)||_{L^{\infty}\left(B(x,\frac{1}{2}(T/\tau)^{1/p})\right)} \leq C\left(\frac{\tau}{\mu\left(B(x,(T/\tau)^{1/p})\right)} \int_{M} u_{0}^{\sigma}\right)^{1/\lambda},$$

where C depends on p, q and the constants in the relative Faber-Krahn inequality.

Proof. We apply Lemma 5.2 with $\sigma \ge 1$ and $\lambda = \sigma + D$. Fix some T > 0 and choose R from the equation

$$\frac{\tau}{T} = \frac{1}{R^p}.$$

Fix also some $x \in M$ and set B = B(x, R),

$$Q = B \times [0, T]$$
 and $Q' = \frac{1}{2}B \times [\frac{1}{2}T, T].$

Observe that by Lemma 2.6

$$\int_{Q} u^{\sigma} = \int_{0}^{T} \int_{B} u^{\sigma} \le T \int_{M} u_{0}^{\sigma}$$

and $||u||_{L^{\infty}(Q)}^{D} \leq ||u_{0}||_{L^{\infty}(M)}^{D} = \tau$. By the choice of R we have

$$S = \frac{\|u\|_{L^{\infty}(Q)}^{D}}{T} + \frac{1}{R^{p}} \le \frac{\tau}{T} + \frac{1}{R^{p}} = \frac{2}{R^{p}}$$

Using also that $\iota(B) \ge \text{const} > 0$ by assumption and applying Lemma 5.2, we obtain that

$$\|u\|_{L^{\infty}(Q')} \leq \left(\frac{C}{\iota(B)\mu(B)R^p} \int_Q u^{\sigma}\right)^{1/\lambda} \leq \left(\frac{C\tau}{\mu\left(B(x, (T/\tau)^{1/p})\right)} \int_M u_0^{\sigma}\right)^{1/\lambda}$$

whence the claim follows. \blacksquare

5.4 Optimal long time decay

The next lemma is the main result about long time decay.

Lemma 5.4. Assume that M is geodesically complete and satisfies the relative Faber-Krahn inequality. Assume that, for all $x \in M$ and $R \ge 1$,

$$\mu(B(x,R)) \ge cR^{\alpha},\tag{5.39}$$

for some $c, \alpha > 0$. Assume also that

$$\beta := p - D\alpha > 0.$$

Let u be a non-negative bounded subsolution in $M \times [0, \infty)$ with initial function $u_0 = u(\cdot, 0)$. Then, for all t > 0, we have

$$\|u(\cdot,t)\|_{L^{\infty}(M)} \le \frac{C}{t^{\alpha/\beta}} \left(\|u_0\|_{L^1(M)} + \|u_0\|_{L^{\infty}(M)} \right)^{p/\beta},$$
(5.40)

where C depends on c, α, p, q and on the constants in the relative Faber-Krahn inequality.

Proof. Denote $t_0 := ||u_0||_{L^{\infty}(M)}^D$ and observe first that, for $t < t_0$, the right hand side of (5.40) is bounded below by

$$\frac{C}{t_0^{\alpha/\beta}} \left(\|u_0\|_{L^{\infty}(M)} \right)^{p/\beta} = C \|u_0\|_{L^{\infty}(M)}^{p/\beta - D\alpha/\beta} = C \|u_0\|_{L^{\infty}(M)}$$

so that (5.40) is trivially satisfied by Lemma 2.6.

Hence, we assume in what follows that $t \ge t_0$. Let us first consider the case when $||u_0||_{L^{\infty}} = 1$, that is, $t_0 = 1$. Denote

$$F(t) = \left\| u(\cdot, t) \right\|_{L^{\infty}(M)}$$

and note that $F(t) \leq 1$. The function $u(\cdot, t + \cdot)$ is a subsolution in $M \times [0, \infty)$ with the initial function $u(\cdot, t)$. Hence, applying Lemma 5.3 to subsolution $u(\cdot, t + \cdot)$ and with $\sigma = 1$ and $\tau = \|u(\cdot, t)\|_{L^{\infty}(M)}^{D} \leq t_{0}$, we obtain that

$$||u(\cdot,2t)||_{L^{\infty}\left(B(x,\frac{1}{2}(t/\tau)^{1/p})\right)} \leq C\left(\frac{\tau}{\mu\left(B(x,(t/\tau)^{1/p})\right)}||u(\cdot,t)||_{L^{1}(M)}\right)^{\frac{1}{1+D}}$$

Setting $\lambda = 1 + D$ and using (5.39) with $R = (t/\tau)^{1/p} \ge 1$, we obtain

$$\begin{aligned} ||u(\cdot,2t)||_{L^{\infty}\left(B(x,\frac{1}{2}(t/\tau)^{1/p})\right)} &\leq C\left(\frac{\tau}{(t/\tau)^{\alpha/p}}||u_{0}||_{L^{1}(M)}\right)^{\frac{1}{\lambda}} \\ &= C\left(\frac{1}{t^{\alpha/p}}F(t)^{D(1+\frac{\alpha}{p})}||u_{0}||_{L^{1}(M)}\right)^{\frac{1}{\lambda}}. \end{aligned}$$

Covering M with a countable sequence of balls $B(x_i, \frac{1}{2}(t/\tau)^{1/p})$ with $x_i \in M$, it follows that

$$F(2t) \le C \left(\frac{1}{t^{\alpha/p}} F(t)^{D(1+\frac{\alpha}{p})} ||u_0||_{L^1(M)}\right)^{\frac{1}{\lambda}}.$$
(5.41)

By the monotonicity of function F(t) (Lemma 2.6), it suffices to prove (5.40) when $t = 2^k$, $k \ge 0$. By (5.41), we obtain for all $k \ge 0$,

$$F(2^{k+1}) \le C\left(\frac{1}{(2^{k+1})^{\alpha/p}}F\left(2^k\right)^{D(1+\frac{\alpha}{p})}||u_0||_{L^1(M)}\right)^{\frac{1}{\lambda}}.$$
(5.42)

Note that

$$\frac{1}{\lambda}D\left(1+\frac{\alpha}{p}\right) - 1 = \frac{D\left(p+\alpha\right) - p\left(1+D\right)}{\left(1+D\right)p} = \frac{D\alpha - p}{\left(1+D\right)p} = -\frac{\beta}{\lambda p}$$

Denoting $F_k = F(2^k)$ we obtain from (5.42) for $G_k = \log_2 F_k$, that

$$G_{k+1} \le \left(1 - \frac{\beta}{\lambda p}\right)G_k - \frac{\alpha}{\lambda p}k + c,$$

where

$$c = \frac{1}{\lambda} \left(\log_2 \|u_0\|_{L^1} \right)_+ + \log_2 C.$$

Note that

$$1 - \frac{\beta}{\lambda p} > 0.$$

The equation

$$g_{k+1} = \left(1 - \frac{\beta}{\lambda p}\right)g_k - \frac{\alpha}{\lambda p}k + c$$

has a general solution in the form

$$g_k = K \left(1 - \frac{\beta}{\lambda p}\right)^k + Ak + B,$$

where K, A, B are some constants. The constants A and B are determined from the condition that Ak + B is a solution, that is,

$$A(k+1) + B = \left(1 - \frac{\beta}{\lambda p}\right)(Ak + B) - \frac{\alpha}{\lambda p}k + c$$

whence

$$A = -\frac{\alpha}{\beta}.$$

Further, we have

$$A + B = \left(1 - \frac{\beta}{\lambda p}\right)B + c$$

so that

$$B = \frac{c - A}{\beta / \lambda p} = \frac{\lambda p}{\beta} c + \frac{\alpha \lambda p}{\beta^2} = \frac{p}{\beta} \left(\log_2 \|u_0\|_{L^1} \right)_+ + \widetilde{c},$$

where

$$\widetilde{c} = \frac{\lambda p}{\beta} \log_2 C + \frac{\alpha \lambda p}{\beta^2}$$

Let us choose the initial condition for g_k as follows:

$$g_0 = G_0 = G(1),$$

which allows to determine K as follows:

$$K = G(1) - B \le 0,$$

because $G(1) = \log_2 F(1) \le 0$ and B > 0. Hence, we obtain, for all $k \ge 0$,

$$G_k \le g_k \le Ak + B = -\frac{\alpha}{\beta}k + \frac{p}{\beta}\log_2\left(\|u_0\|_{L^1}\right)_+ + \widetilde{c},$$

whence, for $t = 2^k$,

$$F(t) \le \widetilde{C}t^{-\alpha/\beta} (1 + ||u_0||_{L^1})^{p/\beta}$$

which finishes the proof of (5.40) in the case $||u_0||_{L^{\infty}} = 1$.

In the general case when $||u_0||_{L^{\infty}}$ and hence, t_0 is arbitrary, consider the function

$$u': (x,\widetilde{t}) \mapsto \|u_0\|_{L^{\infty}}^{-1} u(x,t),$$

where $\tilde{t} = \|u_0\|_{L^{\infty}}^{-D} t$, which satisfies $\|u'_0\|_{L^{\infty}} = 1$ and is a subsolution of (2.9) by Lemma 2.3. Hence, we obtain by the previous part of the proof, for $\tilde{t} \ge 1$,

$$\left\| u'\left(\cdot,\widetilde{t}\right) \right\|_{L^{\infty}(M)} \leq \frac{C}{\left(\widetilde{t}\right)^{\alpha/\beta}} \left(1 + ||u'_{0}||_{L^{1}(M)} \right)^{p/\beta}.$$

Noticing that $\tilde{t} \ge 1 \Leftrightarrow t \ge t_0$, we conclude, for $t \ge t_0$,

$$\|u(\cdot,t)\|_{L^{\infty}(M)} \leq \frac{C}{t^{\alpha/\beta}} \|u_0\|_{L^{\infty}(M)}^{D\alpha/\beta+1} \left(1 + \|u_0\|_{L^{\infty}(M)}^{-1} \|u_0\|_{L^1(M)}\right)^{p/\beta},$$

which finishes the proof of (5.40), because $D\alpha/\beta + 1 = p/\beta$.

6 Combined estimate

The following theorem is our main result (equivalent to Theorem 1.1 from the Introduction).

Theorem 6.1. Assume that M is geodesically complete and satisfies the relative Faber-Krahn inequality. Assume that, for all $x \in M$ and $R \ge 1$,

$$\mu(B(x,R)) \ge cR^{\alpha},$$

for some $c, \alpha > 0$. Assume that (4.24) holds and that

$$\beta := p - D\alpha > 0.$$

Let u be a bounded non-negative subsolution in $M \times [0, \infty)$ with initial function $u_0 = u(\cdot, 0) \in L^1(M) \cap L^\infty(M)$ and set $A = \text{supp } u_0$. Denote |x| = d(x, A). Then, for all t > 0 and all $x \in M$, we have

$$\|u(\cdot,t)\|_{L^{\infty}(B(x,\frac{1}{2}|x|))} \le \frac{C}{t^{\frac{\alpha}{\beta}}} \Phi\left(1 + \frac{|x|}{t^{1/\beta}}\right), \tag{6.43}$$

where

$$\Phi(s) = s^{-\frac{p}{D}} \log^{\gamma}(1+s),$$

where the positive constants C and γ depend on $c, \alpha, p, q, ||u_0||_{L^1(M)}, ||u_0||_{L^{\infty}(M)}$ and on the constants in the relative Faber-Krahn inequality.

Proof. Let us first prove that for all t > 0 and all $x \in M \setminus A$, we have

$$\|u\left(\cdot,t\right)\|_{L^{\infty}(B(x,\frac{1}{2}|x|))} \leq \frac{C_1}{t^{\alpha/\beta}} \wedge C_2\left(\frac{t}{|x|^p}\right)^{\frac{1}{D}} \log^{\gamma}\left(2 + \left(\frac{|x|^{\beta}}{t}\right)^{\frac{1}{D}}\right),\tag{6.44}$$

where the positive constants C_1, C_2, γ depend on $c, \alpha, p, q, ||u_0||_{L^1(M)}, ||u_0||_{L^{\infty}(M)}$ and on the constants in the relative Faber-Krahn inequality.

By Lemma 5.4 we have

$$\|u(\cdot,t)\|_{L^{\infty}(M)} \leq \frac{C}{t^{\alpha/\beta}} \left(\|u_0\|_{L^1(M)} + \|u_0\|_{L^{\infty}(M)} \right)^{p/\beta}$$

which gives the first term in (6.44). In order to obtain the second term in (6.44), we apply Lemma 4.2 in the ball $B_x = B(x, |x|)$ that is disjoint with supp u_0 and deduce

$$\begin{aligned} \|u(\cdot,t)\|_{L^{\infty}(\frac{1}{2}B_x)} &\leq C\left(\frac{t}{\iota(B_x)|x|^p}\right)^{\frac{1}{D}}\log^{\gamma}\left(2+\left(\frac{\iota(B_x)|x|^p}{t}\right)^{\frac{1}{D}}\frac{||u_0||_{L^1(M)}}{|x|^{\alpha}}\right) \\ &\leq C\left(\frac{t}{|x|^p}\right)^{\frac{1}{D}}\log^{\gamma}\left(2+\frac{|x|^{\frac{\beta}{D}}}{t^{\frac{1}{D}}}\right).\end{aligned}$$

Now let us show how (6.44) implies (6.43). In the case when $\frac{|x|}{t^{1/\beta}} \leq C'$ for some constant C' > 1, we have $\Phi\left(1 + \frac{|x|}{t^{1/\beta}}\right) \geq \text{const} > 0$, which yields (6.43). On the other hand, if $\frac{|x|}{t^{1/\beta}} \geq C'$, we see that

$$\frac{1}{t^{\frac{\alpha}{\beta}}}\Phi\left(1+\frac{|x|}{t^{1/\beta}}\right) = \frac{1}{t^{\frac{\alpha}{\beta}}}\left(1+\frac{|x|}{t^{1/\beta}}\right)^{-p/D}\log^{\gamma}\left(2+\frac{|x|}{t^{1/\beta}}\right) \simeq \frac{t^{1/D}}{|x|^{p/D}}\log^{\gamma}\left(2+\frac{|x|}{t^{1/\beta}}\right),$$

because $\frac{p}{\beta D} - \frac{\alpha}{\beta} = \frac{1}{D}$, which finishes the proof of (6.43) also in this case.

Remark 6.2. The model manifold mentioned in Section 7.1 satisfies the volume doubling property, the Poincaré inequality, and consequently, also the relative Faber-Krahn inequality (see Proposition 4.10 in [12]). From (7.2), we have on that manifold the estimate

$$\|u\left(\cdot,t\right)\|_{L^{\infty}\left(B\left(x,\frac{1}{2}|x|\right)\right)} \simeq \frac{1}{t^{\frac{\alpha}{\beta}}} \left(1 + \frac{|x|}{t^{1/\beta}}\right)^{-p/D}$$

which shows that our estimate (6.43) is only logarithmically off the sharp result.

7 Appendix

7.1 Radial solutions on polynomial models

Let M be a model manifold, that is $M = (0, +\infty) \times \mathbb{S}^{n-1}$ as topological spaces and M is equipped with the Riemannian metric ds^2 given by

$$ds^2 = dr^2 + \psi^2(r)d\theta^2,$$

where $\psi(r)$ is a smooth positive function on $(0, +\infty)$ and $d\theta^2$ is the standard Riemannian metric on \mathbb{S}^{n-1} . We define $S(r) = \psi^{n-1}(r)$, which is called the *profile* of the model manifold.

In the following, we assume that, for some $\alpha \in (0, n]$ and all $r \geq r_0$,

$$S\left(r\right) = Cr^{\alpha - 1}$$

Let us denote D = 1 - q (p - 1). Similarly to Proposition 5.1 in [14] one can show that if D > 0 and $p > \alpha D$, then the following function is a non-negative solution of (1.1) in $M \setminus B_{r_0} \times \mathbb{R}_+$:

$$u(r,t) = \frac{1}{t^{\alpha/\beta}} \left(C + \kappa \left(\frac{r}{t^{1/\beta}} \right)^{\frac{p}{p-1}} \right)^{-1/\tau},$$
(7.1)

where C > 0 and

$$\beta=p-\alpha D,\quad \tau=\frac{D}{p-1},\quad \kappa=\tau\frac{p-1}{pq\beta^{\frac{1}{p-1}}}$$

It follows from (7.1) that

$$u(r,t) \simeq \frac{1}{t^{\frac{\alpha}{\beta}}} \left(1 + \frac{r}{t^{1/\beta}}\right)^{-p/D}.$$
(7.2)

7.2 Auxiliary lemmas

Lemma 7.1. [13] Let a sequence $\{J_k\}_{k=0}^{\infty}$ of non-negative reals satisfy

$$J_{k+1} \le \frac{A^k}{\Theta} J_k^{1+\omega} \text{ for all } k \ge 0.$$

where $A, \Theta, \omega > 0$. Then, for all $k \ge 0$,

$$J_k \le \left(\left(A^{1/\omega} \Theta^{-1} \right)^{1/\omega} J_0 \right)^{(1+\omega)^k} \left(A^{-k-1/\omega} \Theta \right)^{1/\omega}.$$

In particular, if $\Theta \ge A^{1/\omega}J_0^\omega$, then $J_k \le A^{-k/\omega}J_0$ for all $k \ge 0$.

The next lemma is a version of Lemma 7.1 with the opposite inequality sign, and the proof is analogous to that of Lemma 7.1.

Lemma 7.2. Let a sequence $\{J_k\}_{k=0}^{\infty}$ of non-negative reals satisfy

$$J_k \ge \frac{A^k}{\Theta} J_{k-1}^{1+\omega} \quad for \ all \ k \ge 1.$$

where $A, \Theta, \omega > 0$. Then, for all $k \ge 0$,

$$J_k \ge \left(\left(A^{1/\omega} \Theta^{-1} \right)^{1/\omega} J_0 \right)^{(1+\omega)^k} \left(A^{-k-1/\omega} \Theta \right)^{1/\omega}.$$

In particular, if $\Theta \leq A^{1/\omega}J_0^\omega$, then $J_k \geq A^{-k/\omega}J_0$ for all $k \geq 0$.

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