The Annals of Applied Probability
2024, Vol. 34, No. 5, 4387–4407
https://doi.org/10.1214/24-AAP2069

HARMONIC MEASURE IN A MULTIDIMENSIONAL GAMBLER'S PROBLEM

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We consider a random walk in a truncated cone K_N, which is obtained by slicing cone K by a hyperplane at a growing level of order N. We study the behaviour of the Green function in this truncated cone as N increases. Using these results we also obtain the asymptotic behaviour of the harmonic measure.

The obtained results are applied to a multidimensional gambler's problem studied by Diaconis and Ethier (*Staist. Sci.* **37** (2022) 289–305). In particular we confirm their conjecture that the probability of eliminating players in a particular order has the same exact asymptotic behaviour as for the Brownian motion approximation. We also provide a rate of convergence of this probability towards this approximation.

1. Introduction. Consider a random walk $\{S(n), n \ge 1\}$ on a lattice R which is a linear transformation of the standard d-dimensional lattice $\mathbb{Z}^d$, $d \ge 1$, where

$$S(n) = X(1) + \cdots + X(n)$$

and $\{X(n), n \ge 1\}$ is a family of independent copies of a random vector $X = (X_1, \dots, X_d)$. Denote by $\mathbb{S}^{d-1}$ the unit sphere of $\mathbb{R}^d$ and Σ an open and connected subset of $\mathbb{S}^{d-1}$. Let K be the cone generated by the rays emanating from the origin and passing through Σ, that is, $\Sigma = K \cap \mathbb{S}^{d-1}$. Let τ be the exit time from K of the random walk, that is,

$$\tau := \inf\{n \ge 1 : S(n) \notin K\}.$$

In the series of papers [3–6] we have studied the tail behaviour of τ and have proved various conditional limit theorems for $S(n)$ conditioned on $\{\tau > n\}$.

In the present paper we are going to study some properties of walks conditioned to stay in K sliced by a hyperplane at a high level. Our attention to this problem was drawn by Persi Diaconis. In his joint paper [7] with Stewart Ethier, the following gambler's ruin problem with three players has been considered. The players have starting capitals A, B and C. At each step, a pair of players is chosen uniformly at random, the chosen players play a fair heads-or-tails game resulting in the transfer of one unit of capital between these players. The game is played until one of the players wins all $A + B + C$ units of capital. Since the total capital of all players remains constant, it suffices to keep track of capitals of two players only. As a result, one can model this game by the following 2-dimensional random walk. Let $Y(n)$ be independent random vectors with the uniform distribution on the set

$$\{(1,0), (-1,0), (0,1), (0,-1), (1,-1), (-1,1)\},$$

see Figure 1. One of the players is eliminated when the random walk

$$Z(n) = Y(1) + Y(2) + \cdots + Y(n)$$

Received February 2023; revised December 2023.
MSC2020 subject classifications. Primary 60G50; secondary 60F17, 60G40, 60J45.
Key words and phrases. Random walk, Brownian motion, first-passage time, overshoot, moving boundary.

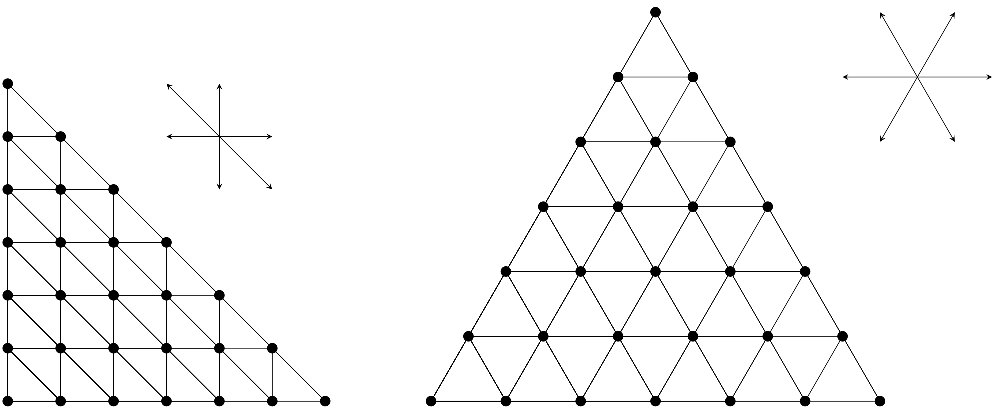

FIG. 1. *Random walks $Z(n)$ and $S(n) = TZ(n)$.*

starting at (A, B, C) hits the boundary of the triangle $\{(x, y) : x, y \geq 0, x + y \leq N\}$, where $N = A + B + C$. Using the results of [8] Diaconis and Ethier have shown that if $A = B = 1$ then there exist constants c_1 and c_2 such that

$$\frac{c_1}{N^3} \leq \mathbf{P}(\text{third player goes broke first}) \leq \frac{c_2}{N^3}. \tag{1}$$

(This means that $Z(n)$ exits the triangle via $\{(x, y) : x + y = N\}$.) The techniques of [8] give similar estimates for very general (inner uniform) domains. Diaconis and Ethier [7] also obtained a number of numerical results which showed a good fit of this probability with the corresponding asymptotics for the Brownian motion and strongly suggest there exists a positive constant c such that

$$\mathbf{P}(\text{third player goes broke first}) \sim \frac{c}{N^3} \quad \text{as } N \to \infty. \tag{2}$$

O'Connor and Saloff-Coste [13] have proven an analogue of (1) for the gambler's problem with four players. In this paper we will confirm that the lower and upper bounds can be strengthened and exact asymptotics hold, and, in particular, (2) and its analogues are true.

The purpose of the present note is to consider similar exit problems for a rather wide class of cones and walks and to prove that the Brownian approximation is valid under mild moment conditions on walks.

Let $\mathbb{H}$ be a hyperplane satisfying the following conditions:

(i) $0 \notin \mathbb{H}$;
(ii) $\mathbb{H}$ cuts the cone K into two parts, one of these parts is bounded;
(iii) there exists $\varepsilon > 0$ such that the set $\{x \in K : |x| < \varepsilon\}$ belongs to the bounded part.

Let $x_0 \in \mathbb{H}$ be such that $\text{dist}(0, \mathbb{H}) = |x_0|$. For every $t > 0$ we denote by $\mathbb{H}_t$ the hyperplane which is parallel to $\mathbb{H}$ and contains the point tx_0.

Define $\Gamma := K \cap \mathbb{H}$. For every $A \subseteq \Gamma$ we denote by $K^{(A)}$ the cone consisting of all rays starting at zero and going through points of the set A. Now we may define

$$A_{s,t} := K^{(A)} \cap \left(\bigcup_{u:s \leq u < t} \mathbb{H}_u \right), \quad 0 \leq s < t \leq \infty.$$

Clearly, $A_{s,t}$ is the part of $K^{(A)}$ between hyperplanes $\mathbb{H}_s$ and $\mathbb{H}_t$. If we take $A = \Gamma$, $s = N$ and $t = \infty$ then we get

$$\Gamma_{N,\infty} = K \cap \left(\bigcup_{u:u \geq N} \mathbb{H}_u \right).$$

We shall also use the notation

$$K_{0,N} = K \cap \Big(\bigcup_{u:u<N} \mathbb{H}_u \Big), \quad N > 0.$$

Now we can define the main objects of interest for us. Let

$$\sigma_N := \inf\{n \ge 1 : S(n) \notin K_{0,N}\}.$$

Relating to the gambler's problem described above, we see that the event "third player goes broke first" can be expressed in the following way: $Z(\sigma_N) \in \Gamma_{N,\infty}$. We will be interested in the behaviour of the probabilities

$$\mathbf{P}\big(S(\sigma_N) \in A_{N,\infty}\big), A \subset \Gamma \quad \text{and} \quad \mathbf{P}\big(S(\sigma_N) = y\big), y \in \Gamma_{N,\infty}.$$

There is a rather simple and known connection between the distribution of the random walk at an exit time and the Green function of the corresponding killed process. To describe this relation we denote by $G_N(x, y)$ the Green function corresponding to the exit time σ_N, that is,

$$G_N(x, y) := \sum_{n=0}^{\infty} \mathbf{P}_x(S_n = y, \sigma_N > n).$$

For any set A we also let

$$G_N(x, A) := \sum_{y \in A} G_N(x, y).$$

Then, by the total probability formula,

$$\begin{aligned}
\mathbf{P}_x\big(S(\sigma_N) = y\big) &= \sum_{n=0}^{\infty} \mathbf{P}_x\big(S(\sigma_N) = y, \sigma_N = n+1\big) \\
&= \sum_{n=0}^{\infty} \sum_{z \in K_N} \mathbf{P}_x\big(S(z) = y, \sigma_N > n\big)\mathbf{P}(X_{n+1} = y - z) \\
&= \sum_{z \in K_N} G_N(x, z)\mathbf{P}(X = y - z).
\end{aligned}$$

Thus, one needs a rather good control over the Green function G_N.

In order to formulate our results we recall some known properties of random walks conditioned to stay in K. First, let $V(x)$ denote the positive harmonic function for $S(n)$ killed at leaving the cone K, that is,

$$V(x) = \mathbf{E}_x\big[V\big(S(1)\big); \tau > 1\big], \quad x \in K.$$

This function has been constructed in our recent paper [5] under the following conditions:

(i) the cone K is either convex or star-like and C^2,

(ii) the random vector X has zero mean, unit covariance matrix and moments $\mathbf{E}|X|^p$, $\mathbf{E}|X|^{2+\varepsilon}$ are finite.

The constant p in the second assumption depends on the cone K only and is equal to the degree of homogeneity of the positive solution to the classical Dirichlet problem in K. It can be found by solving an eigenvalue problem for a domain on a sphere, see [4] for further details.

Let $e_{\mathbb{H}} := -\frac{x_0}{|x_0|}$ be the vector perpendicular to $\mathbb{H}$ and directed to 0. We will require a one-dimensional projection of the random walk

$$S_{\mathbb{H}}(n) = \big(S(n), e_{\mathbb{H}}\big), \quad n \ge 0.$$

Let $v_{\mathbb{H}}(z)$ be the renewal function of the descending ladder height process of $(S_{\mathbb{H}}(n))_{n\ge 0}$.

THEOREM 1. *Assume that K is either convex or starlike and C^2. Assume also that X has zero mean, unit covariance matrix and $\mathbf{E}|X|^{(p\vee 2)+\varepsilon}<\infty$. Then, there exists C such that for any $A\subset int(\Gamma)$*

$$G_N(x,A_{N-j,N-j+1})\le C\frac{V(x)(1+\mathrm{dist}(y,\Gamma_N))}{N^p}j,\quad 1\le j\le\frac{N}{2}. \tag{3}$$

If $\mathbf{E}|X|^{p+d}<\infty$ then there exists C such that

$$G_N(x,y)\le C\frac{V(x)}{N^{p+d-1}}j,\quad y\in\Gamma_{N-j,N-j+1}, j\le\frac{N}{2}. \tag{4}$$

Furthermore, there exists a function h, depending on the cone K only, such that, uniformly in $j=o(N)$,

$$G_N(x,A_{N-j,N-j+1})\sim\frac{V(x)}{N^{p+d-1}}\sum_{y\in A_{N-j,N-j+1}}v_{\mathbb{H}}(\mathrm{dist}(y,\Gamma_N))h(y/N), \tag{5}$$

a closed form for the function h is given in (19).

Moreover, under the condition $\mathbf{E}|X|^{p+d}<\infty$,

$$G_N(x,y)=\frac{V(x)v_{\mathbb{H}}(\mathrm{dist}(y,\Gamma_N))}{N^{p+d-1}}h(y/N)+o\left(\frac{\mathrm{dist}(y,\Gamma_N)}{N^{p+d-1}}\right), \tag{6}$$

uniformly in $y\in K_{0,N}$ with $\mathrm{dist}(y,\Gamma_N)=o(N)$.

REMARK 1. Unfortunately we have no closed form expressions for the function h and we do not know how one can characterise it by analytical means. Such characterisations would be very interesting and, probably, they would give additional information on the behaviour of the Green function of the corresponding harmonic measure, see Theorem 2.

This result can be used to answer some questions on the gambler's problem mentioned above. It is easy to see that the random walk $Z(n)$ does not satisfy the conditions of Theorem 1: $\mathbf{E}Y_1^2=\mathbf{E}Y_2^2=\frac{2}{3}$ and $\mathbf{E}Y_1Y_2=-\frac{1}{3}$. To adjust this walk we apply the linear transformation given by the matrix

$$T=\begin{pmatrix}\sqrt{2} & \frac{\sqrt{2}}{2}\\ 0 & \frac{\sqrt{6}}{2}\end{pmatrix}.$$

The random walk $S(n)=TZ(n)$ satisfies the conditions of our theorem. The cone $\mathbb{R}^2_+$ transforms under this mapping into the wedge K with the opening angle $\frac{\pi}{3}$. Furthermore, the original cutting line transfers under T to the line $\{(x_1,x_2):\sqrt{3}x_1+x_2=\sqrt{6}\}$, see Figure 1. It is easy to see that the harmonic function for the Brownian motion in the wedge K is given by $u(x)=3x_1^2x_2-x_2^3$. In particular, $p=3$ in this case. Furthermore, simple calculations show that the function u is harmonic also for the discrete time walk $S(n)$ killed at leaving K. Thus, $V(x)=u(x)$ for all x in the set $K\cap(T\mathbb{Z}^2)$. Since the random walk $S(n)$ can not jump over the boundary $\Gamma_N=\{x:x_2+\sqrt{3}x_1=\sqrt{6}N\}$, every point $y\in\Gamma_N$ can be reached from $y^{(1)}:=y-(\sqrt{2},0)$ and from $y^{(2)}:=y-(\frac{\sqrt{2}}{2},\frac{\sqrt{6}}{2})$. This implies that

$$\mathbf{P}_x(S(\sigma_N)=y)=\frac{1}{6}G_n(x,y^{(1)})+\frac{1}{6}G_n(x,y^{(2)}).$$

In order to apply (6), it remains to calculate the renewal function $v_{\mathbb{H}}$. It is rather obvious that the increments of $S_{\mathbb{H}}(n)$ take values $-\frac{\sqrt{6}}{2}$, 0, $\frac{\sqrt{6}}{2}$ with equal probabilities. Therefore, $v_{\mathbb{H}}(r) = r$ for every $r \in \frac{\sqrt{6}}{2}\mathbb{N}$. In particular,

$$v_{\mathbb{H}}(\operatorname{dist}(y^{(1)}, \Gamma_N)) = v_{\mathbb{H}}(\operatorname{dist}(y^{(2)}, \Gamma_N)) = \frac{\sqrt{6}}{2}.$$

Plugging this into (6), we obtain

$$\mathbf{P}_x(S(\sigma_N) = y) = \frac{1}{\sqrt{6}} h\left(\frac{y}{N}\right) \frac{V(x)}{N^4} + o\left(\frac{1}{N^4}\right). \tag{7}$$

If the game starts with capitals A, B and $N - A - B$ then the random walk $S(n)$ starts at the point $x = (\sqrt{2}A + \frac{\sqrt{2}}{2}B, \frac{\sqrt{6}}{2}B)$. Recalling that $V(x) = 3x_1^2 x_2 - x_2^3$, one infers easily that

$$V(x) = 3\sqrt{6}AB(A + B) \quad \text{for } x = \left(\sqrt{2}A + \frac{\sqrt{2}}{2}B, \frac{\sqrt{6}}{2}B\right).$$

Summing over y the asymptotics in (7), we conclude that the probability, that the third player (the player with the starting capital $N - A - B$) goes bankrupt first, is asymptotically equivalent to

$$3\frac{AB(A + B)}{N^3} \int_\Gamma h(z)\, dz.$$

Furthermore, the probability that the first player wins the whole game is asymptotically equivalent to

$$3\frac{AB(A + B)}{N^3} \int_\Gamma (\sqrt{2}z_1 - 1) h(z)\, dz. \tag{8}$$

As the same arguments apply to the standard Brownian motion we obtain that the same asymptotics with the same constant should hold, see Corollary 4 below.

Let us now comment on the global asymptotics in (5). The right hand side is just the sum of local asymptotics from (6), the only advantage is the fact that for (5) we need a weaker moment assumption. Such a cumbersome expression is caused by the fact that we know only asymptotics for the renewal function $v_{\mathbb{H}}$. By the renewal theorem, $v_{\mathbb{H}}(r) \sim c_{\mathbb{H}} r$ as $r \to \infty$. Using this information, one easily infers from (5) that

$$G_N(x, A_{N-j,N-j+1}) \sim c_{\mathbb{H}} V(x) \frac{j}{N^p} \int_A h(z)\, dz$$

if $j \to \infty$ and $j = o(N)$.

We now formulate our result on the asymptotic behaviour of the harmonic measure.

THEOREM 2. *Assume that the random walk $S(n)$ and the cone K satisfy the conditions of Theorem 1. If, additionally, $\mathbf{E}|X|^{p+d}$ is finite then, uniformly in $y \in \Gamma_{N,\infty}$,*

$$\begin{aligned} &\mathbf{P}_x(S(\sigma_N) = y) \\ &\quad = \frac{V(x)}{N^{p+d-1}} h(y_N/N) \mathbf{E}[v_{\mathbb{H}}(\operatorname{dist}(y - X, \Gamma_N)); y - X \in K_{0,N}] + o\left(\frac{V(x)}{N^{p+d-1}}\right), \end{aligned} \tag{9}$$

where $y_N \in \Gamma_N$ is such that y and y_N belong to the same ray in the cone K.

The existence of a positive harmonic function V allows one to perform the Doob h-transform:

$$\mathbf{P}_x^{(V)}(S(1) \in dy) = \frac{V(y)}{V(x)} \mathbf{P}_x(S(1) \in dy, \tau > 1), \quad x, y \in K.$$

We can restate (9) under this new measure in the following way

$$
\begin{aligned}
&\mathbf{P}_x^{(V)}(S(\sigma_N)=y)\\
&\quad=\frac{V(y)}{N^{p+d-1}}h(y_N/N)\mathbf{E}[v_{\mathbb{H}}(\mathrm{dist}(y-X,\Gamma_N));\,y-X\in K_{0,N}]+o\left(\frac{V(y)}{N^{p+d-1}}\right).
\end{aligned}
\tag{10}
$$

Under some additional conditions on the cone, $V(y)=o(|y|^p)$ for y satisfying $\mathrm{dist}(y,\partial K)=o(|y|)$. This implies that

$$\mathbf{P}_x^{(V)}(S(\sigma_N)=y)=o(N^{-d+1})$$

for y satisfying $\mathrm{dist}(y,\partial K)=o(N)$. This shows that the behaviour of the harmonic measure of conditioned random walk differs from that of reflected walks. Dubedat [9] has found some examples of reflected walks in two-dimensional wedges with uniform harmonic measure. This difference can be explained by the fact that the entropic repulsion has a stronger effect than the reflection: the reflection is felt only at the boundary and the entropic repulsion is felt everywhere.

We will now move to the rate of convergence for the three gamblers problems. First we discuss a Brownian motion analogue. Let $B(t)=(B_1(t),B_2(t))$ be a standard Brownian motion. Let $\mathcal{T}_N$ be an equilateral triangle with vertices $(0,0)$, $(\sqrt{2}N,0)$, $(\frac{1}{\sqrt{2}}N,\sqrt{\frac{3}{2}}N)$. This model is the limiting case corresponding to the linear transformation $TZ(n)$ of the original game. Then, starting from a position (x_1,x_2) we run this two-dimensional Brownian motion until it hits one of the edges. When this happens one of the players gets eliminated and we run a one-dimensional Brownian motion on this edge until it hits one of the vertices. We will discuss the probability $P^{bm,(321)}_{x_1,x_2}(N)$ of the event that the third player gets eliminated first, then the second.

It can be found as follows. Let $\mathcal{T}_N^{(1)}$, $\mathcal{T}_N^{(2)}$ and $\mathcal{T}_N^{(3)}$ be the edges of this triangle, between corresponding vertices $(\sqrt{2}N,0)$, $(0,0)$ and $(\frac{1}{\sqrt{2}}N,\sqrt{\frac{3}{2}}N)$. Consider the stopping times

$$
\begin{aligned}
\tau_N^{bm}&:=\inf\{t>0:B(t)\notin\mathcal{T}_N\},\\
\sigma_N^{bm}&:=\inf\{t>0:B(t)\in\mathcal{T}_N^{(3)}\}.
\end{aligned}
$$

Then,

$$P^{bm,(321)}_{x_1,x_2}(N)=\mathbf{E}_{x_1,x_2}\left[\frac{\sqrt{\frac{3}{2}}N-B_2(\tau_N^{bm})}{\sqrt{\frac{3}{2}}N};\,\tau_N^{bm}=\sigma_N^{bm}\right]\quad\text{as } N\to\infty.$$

A solution to this problem was obtained earlier in [2] via a conformal mapping of the above triangle to the unit disk. Here we present a solution via a conformal mapping of the triangle $\mathcal{T}_N$ to the upper half plane. The form of the explicit solution, see (27) below, allows us to find the asymptotics and the corresponding harmonic function.

PROPOSITION 3. *For fixed x_1,x_2 the following asymptotics hold, as $N\to\infty$,*

$$P^{bm,(321)}_{x_1,x_2}(N)\sim\frac{\Gamma(1/3)^9}{96\sqrt{6}\pi^4}\frac{u(x)}{N^3},\tag{11}$$

$$\mathbf{P}_{x_1,x_2}(\textit{third player gets eliminated first})\sim\frac{\Gamma(1/3)^9}{48\sqrt{6}\pi^4}\frac{u(x)}{N^3},\tag{12}$$

where $u(x)=3x_1^2x_2-x_2^3$ is the positive harmonic function in the wedge K with the opening angle $\pi/3$.

Let $P^{(321)}_{x_1,x_2}(N)$ be the corresponding probability for the random walks in $\mathcal{T}_N$, that is, that the third player gets eliminated first and the second player gets eliminated second.

COROLLARY 4. *Consider the random walk $Z(n)$. For fixed y_1, y_2 the following asymptotics hold, as $N \to \infty$,*

$$P^{(321)}_{y_1,y_2}(N) \sim \frac{\Gamma(1/3)^9}{32\pi^4}\frac{y_1y_2(y_1+y_2)}{N^3},$$

$$\mathbf{P}_{y_1,y_2}(\textit{third player gets eliminated first}) \sim \frac{\Gamma(1/3)^9}{16\pi^4}\frac{y_1y_2(y_1+y_2)}{N^3}.$$

PROOF. By arguments leading to (8) the asymptotics is the same as the asymptotics for the Brownian motion given in Proposition 3. We just need to recalculate the harmonic function in the original coordinates. Recall that $x_1 = \frac{1}{\sqrt{2}}y + \sqrt{2}y_2$, $x_2 = \sqrt{3/2}y_2$. This will result in

$$u(x_1,x_2) = 3\sqrt{6}\frac{y_1y_2(y_1+y_2)}{N^3}. \qquad \square$$

We can now compare the theoretical result with computations in Table 4 in [7]. The results from exact computation for $y_1 = y_2 = 1$ and $N = 50, 100, 150, 200, 250, 300$ (rounded to 15 significant figures) rapidly converge to a limit with $N^3 P^{(321)}_{1,1}(N) = 4.55979450208$. The asymptotics in Corollary 4 is as follows,

$$P^{(321)}_{1,1}(N) \sim \frac{\Gamma(1/3)^9}{16\pi^4}\frac{1}{N^3}.$$

Since

$$\frac{\Gamma(1/3)^9}{16\pi^4} \approx 4.5597944999598458,$$

the answer is in excellent agreement with these exact computations.

Now we will move to the rate of convergence. It was conjectured in [7], Conjecture 4.2, that for starting points x_1, x_2 sufficiently away from the edges

$$\left|P^{(321)}_{x_1,x_2}(N) - P^{bm,(321)}_{x_1,x_2}(N)\right| = O\left(\frac{1}{N^4}\right).$$

Using the estimates for the Green function we have established the following rate of convergence.

PROPOSITION 5. *There exists a constant C such that*

$$\left|P^{(321)}_{x_1,x_2}(N) - P^{bm,(321)}_{x_1,x_2}(N)\right| \le \frac{C}{N^3}.$$

This rate convergence is quite fast, but not the same as conjectured in [7]. It is not clear to us which rate of convergence is the right one. However it is likely that some symmetries of the problem can be used to obtain the conjectured rate of convergence.

REMARK 2. The problem with three gamblers can also be reformulated as the problem of three noncolliding (ordered [3]) random walks. For that consider for $i = 1, 2, 3$, the random walk

$$S^{(i)}_n = S^{(i)}_0 + X^{(i)}_1 + \cdots + X^{(i)}_n, \quad n = 1, 2, \ldots,$$

where $\{X_j^{(i)}, j = 1, 2, \dots, i = 1, 2, 3\}$ are i.i.d. random variables distributed as X,

$$\mathbf{P}(X = 1) = \mathbf{P}(X = 0) = \frac{1}{2}.$$

Let $W = \{x = (x_1, x_2, x_3), x_1 < x_2 < x_3\}$ be the Weyl chamber. Consider the stopping time

$$\tau := \min\{n \ge 1 : (S_n^{(1)}, S_n^{(2)}, S_n^{(3)}) \notin W\}.$$

It is known that for these random walks the positive harmonic function on W is given by the Vandermonde determinant $(x_3 - x_2)(x_2 - x_1)(x_3 - x_1)$. Now let

$$\tau_N := \min\{n \ge 1 : S_n^{(3)} - S_n^{(1)} = N\}.$$

Then, the walk (A_n, B_n) defined as

$$A_n = S_n^{(3)} - S_n^{(2)}, \qquad B_n = S_n^{(2)} - S_n^{(1)},$$

for $n = 0, 1, 2, 3, \dots$ will have the following transition probabilities $p(x, y) = \mathbf{P}(A_n - A_{n-1} = x, B_n - B_{n-1} = y)$:

$$p(0, 1) = p(0, -1) = p(1, 0) = p(-1, 0) = p(1, -1) = p(-1, 1) = \frac{1}{8}, \qquad p(0, 0) = \frac{1}{4}.$$

Thus, for $n < \tau_N \wedge \tau$ the Markov chain (A_n, B_n) will follow the same path as the Markov chain $(Z(n))$. The only difference is that it will move slower, as it stays at the same place with probability $\frac{1}{4}$. This means that the distribution of the exit point is the same for (A_n, B_n) and thus one can study the harmonic measure for noncolliding random walks to obtain the harmonic measure for $(Z(n))$. For noncolliding simple random walks there is an exact distribution for the exit time. Namely one can make use of a discrete version of the Karlin–McGregor formula from [12], see the last paragraph there. However, this exact formula involves an infinite number of reflections and seems to be difficult for the asymptotic analysis or numerical computations.

In the rest of the paper we give proofs of the above statements. Proof of each statement is given in a separate section.

2. Proof of Theorem 1.

Fix $\varepsilon > 0$ and split the Green function G_N into two parts:

$$G_N(x, y) := G_N^{(1)}(x, y) + G_N^{(2)}(x, y),$$

where

$$G_N^{(1)}(x, y) := \sum_{n \le \varepsilon N^2} \mathbf{P}_x(S(n) = y, \sigma_N > n),$$

$$G_N^{(2)}(x, y) := \sum_{n > \varepsilon N^2} \mathbf{P}_x(S(n) = y, \sigma_N > n).$$

We will start with analysis of $G_N^{(1)}(x, y)$. This part corresponds to large deviations of the random walk.

LEMMA 1. *Assume that* $\mathbf{E}|X|^p < \infty$. *There exists a function* $f : \mathbb{R} \to [0, 1]$ *satisfying* $\lim_{\varepsilon \to 0} f(\varepsilon) = 0$ *such that*

$$G_N^{(1)}(x, A_{N-j,N-j+1}) \le C_A f(\varepsilon) \frac{V(x)}{N^p} j.$$

PROOF. Let

$$\theta := \inf\{k \ge 1 : S(k) \in A_{N-j,N-j+1}\}.$$

Then, by the strong Markov property,

$$\begin{aligned} G_N^{(1)}(x, A_{N-j,N-j+1}) &= \mathbf{E}_x\left[\sum_{n=\theta}^{\varepsilon N^2} \mathbf{1}\{S(n) \in A_{N-j,N-j+1}, \sigma_N > n\}; \theta \le \varepsilon N^2\right] \\ &\le \mathbf{E}_x\left[\mathbf{E}_{S(\theta)}\left[\sum_{n=0}^{\infty} \mathbf{1}\{S(n) \in A_{N-j,N-j+1}, \sigma_N > n\}\right]; \theta \le \varepsilon N^2, \tau > \theta\right]. \end{aligned}$$

For every $y \in \Gamma_{N-j,N-j+1}$,

$$\begin{aligned} &\mathbf{E}_y\left[\sum_{n=0}^{\infty} \mathbf{1}\{S(n) \in A_{N-j,N-j+1}, \sigma_N > n\}\right] \\ &\quad \le \mathbf{E}_{\mathrm{dist}(y,\mathbb{H})}\left[\sum_{n=0}^{\infty} \mathbf{1}\{S_{\mathbb{H}}(n) \in [j-1, j]\}; \tau_{\mathbb{H}} > n\right] \le Cj, \end{aligned}$$

see, for example, [14], Chapter 19, for the latter inequality for the Green function of one-dimensional random walk on a half-line.

Therefore,

$$G_N^{(1)}(x, A_{N-j,N-j+1}) \le Cj\mathbf{P}_x(\theta \le \varepsilon N^2, \tau > \theta).$$

Applying the Doob h-transform with the positive harmonic function V, we have

$$\mathbf{P}_x(\theta \le \varepsilon N^2, \tau > \theta) = V(x)\mathbf{E}_x^{(V)}\left[\frac{1}{V(S(\theta))}; \theta \le \varepsilon N^2\right].$$

Since $A \subset \mathrm{int}(\Gamma)$ the distance $\mathrm{dist}(A_{N-j,N-j+1}) \ge c_A N$ for all $j \le N/2$. Therefore, $\inf_{y \in A_{N-j,N-j+1}} V(y) \ge \widehat{c}_A N^p$. This implies that

$$\mathbf{P}_x(\theta \le \varepsilon N^2, \tau > \theta) \le C_A \frac{V(x)}{N^p}\mathbf{P}_x^{(V)}(\theta \le \varepsilon N^2).$$

By the functional limit theorem for the random walk conditioned to stay in a cone, see Theorem 3 in [11], $\mathbf{P}_x^{(V)}(\theta \le \varepsilon N^2) \le f(\varepsilon)$ for some f with required properties. □

LEMMA 2. *Assume that* $\mathbf{E}|X|^{p+d} < \infty$. *There exists a function* $f : \mathbb{R} \to [0, 1]$ *satisfying* $\lim_{\varepsilon \to 0} f(\varepsilon) = 0$ *such that*

$$G_N^{(1)}(x, y) \le Cf(\varepsilon)\frac{V(x)}{N^{p+d-1}} j$$

uniformly in $y \in A_{N-j,N-j+1}$ *and uniformly in* $A \subset \Gamma$.

PROOF. Fix some $\delta > 0$ and let

$$\theta_y := \inf\{k \ge 1 : |S(k) - y| \le \delta N\}.$$

Repeating the same arguments as in Lemma 1, we obtain

$$G_N^{(1)}(x, y) \le \mathbf{E}_x\left[\mathbf{E}_{S(\theta_y)}\left[\sum_{n=0}^{\infty} \mathbf{1}\{S(n) = y\}, \tau_{\mathbb{H}} > n\right]; \tau > \theta_y, \theta_y \le \varepsilon N^2\right].$$

Since $\mathbf{E}|X|^{d+1} < \infty$ we can apply Theorem 1.2 from [10], which implies that

$$\mathbf{E}_{S(\theta_y)}\left[\sum_{n=0}^{\infty} \mathbf{1}\{S(n) = y\}, \tau_{\mathbb{H}} > n\right] \le C\frac{1 + \mathrm{dist}(S(\theta_y), N\Gamma)}{1 + |S(\theta_y) - y|^d} j.$$

Therefore,

$$\begin{aligned}
G_N^{(1)}(x, y) &\le \frac{CNj}{N^d}\mathbf{P}_x\Big(\big|y - S(\theta_y)\big| > \frac{\delta}{2}N, \tau > \theta_y, \theta_y \le \varepsilon N^2\Big) \\
&\quad + CNj\mathbf{P}_x\Big(\big|y - S(\theta_y)\big| \le \frac{\delta}{2}N, \tau > \theta_y, \theta_y \le \varepsilon N^2\Big) \\
&\le C\frac{Cj}{N^{d-1}}\mathbf{P}_x\Big(\big|y - S(\theta_y)\big| > \frac{\delta}{2}N, \tau > \theta_y, \theta_y \le \varepsilon N^2\Big) \\
&\quad + CNj\mathbf{P}\Big(\big|X(\theta_y)\big| > \frac{\delta}{2}N\Big) \\
&\le \frac{Cj}{N^{d-1}} \frac{V(x)}{\inf_{z \in K:\, \frac{\delta}{2}N \le |z-y| \le \delta N} V(z)} \widehat{\mathbf{P}}_x\big(\theta_y \le \varepsilon N^2\big) \\
&\quad + CNj\mathbf{P}\Big(\big|X(\theta_y)\big| > \frac{\delta}{2}N\Big).
\end{aligned}$$

Making use of the lower bound for the harmonic function

$$\inf_{z \in K:\, \frac{\delta}{2}N \le |z-y| \le \delta N} V(z) \ge c_\delta N^p$$

and the moment assumption $\mathbf{E}|X|^{p+d} < \infty$ we arrive at the conclusion. □

In order to determine the behaviour of $G_N^{(2)}$ we study each probability separately.

Let f_r be the density of the measure $\mathbf{P}(M_K(1) \in \cdot, M_K(t) \in K_{0,1/r}$ for all $t \le 1)$. This meander M_K is defined and studied in [11], where the functional central limit theorem that we are using in the next lemma is proved.

LEMMA 3. *Let n be such that $\frac{n}{N^2} \to r^2$ as $N \to \infty$. Then,*

$$\sup_{y \in K_{0,N}} \left| n^{p/2+d/2}\mathbf{P}_x\big(S(n) = y, \sigma_N > n\big) - \varkappa V(x) f_r\Big(\frac{y}{\sqrt{n}}\Big)\right| \to 0. \tag{13}$$

PROOF. Applying Theorem 1 in [11], one concludes easily that

$$\begin{aligned}
&\mathbf{P}_x\big(S(n) \in \sqrt{n}D, S(k) \in K_{0,N} \text{ for all } k \le n | \tau > n\big) \\
&\quad \to \mathbf{P}\big(M_K(1) \in D, M_K(t) \in K_{0,1/r} \text{ for all } t \le 1\big).
\end{aligned}$$

Arguing exactly in the same way as in [4], Theorem 5, we obtain the desired local limit theorem from the above global limit theorem. □

LEMMA 4. *There exists an independent of the random walk function $g(z, r)$ such that*

$$\begin{aligned}
&\frac{n^{p/2+d/2+1/2}}{V(x) v_{\mathbb{H}}(\mathrm{dist}(y, \Gamma_N))}\mathbf{P}_x\big(S(n) = y, \sigma_N > n\big) \\
&\quad = \varkappa\sqrt{\frac{2}{\pi}} 2^{p/2+d/2} g\Big(\frac{y}{\sqrt{n}}, \frac{N}{\sqrt{n/2}}\Big) + o(1),
\end{aligned}$$

uniformly in $\frac{n}{N^2} \in [\varepsilon, \varepsilon^{-1}]$ and in $\mathrm{dist}(y, N\Gamma) = o(N)$.

Furthermore, uniformly in $y \in K_{0,N}$,

$$\mathbf{P}_x\big(S(n) = y, \sigma_N > n\big) \le C \frac{V(x) v_{\mathbb{H}}(\mathrm{dist}(y, \Gamma_N))}{n^{p/2+d/2+1/2}}.$$

PROOF. Set $m = [n/2]$. We first make use of the time inversion as follows,

$$\begin{aligned} &\mathbf{P}_x\big(S(n) = y, \sigma_N > n\big) \\ &\quad = \sum_z \mathbf{P}_x\big(S(m) = z, \sigma_N > m\big)\mathbf{P}_z\big(S(n-m) = y, \sigma_N > n-m\big) \\ &\quad = \sum_z \mathbf{P}_x\big(S(m) = z, \sigma_N > m\big)\mathbf{P}_y\big(\widehat{S}(n-m) = z, \widehat{\sigma}_N > n-m\big), \end{aligned} \tag{14}$$

where $\widehat{S}(k) = -X(1) - X(2) - \cdots - X(k)$, $k \ge 1$ is a time-reversed random walk and $\widehat{\sigma}_N$ is the corresponding stopping time for $(\widehat{S}(n))_{n\ge 0}$, that is,

$$\widehat{\sigma}_N = \inf\{n \ge 1 : \widehat{S}(n) \notin K_{0,N}\}.$$

Define also the exit time of $\widehat{S}(n)$ from the half space with the boundary $\mathbb{H}_N$:

$$\widehat{\tau}_{\mathbb{H}} := \inf\{n \ge 1 : (\widehat{S}(n), x_0) \ge N\}.$$

It is clear that

$$\mathbf{P}_y(\widehat{\tau}_{\mathbb{H}} > n) = \mathbf{P}_{\mathrm{dist}(y,\Gamma_N)}(\tau_{\mathbb{H}} > n).$$

Using the known results for one-dimensional walks, we have

$$\mathbf{P}_y(\sigma_N > n-m) \le \mathbf{P}_y(\widehat{\tau}_H > n) \le C \frac{1 + \mathrm{dist}(y, \Gamma_N)}{\sqrt{n}} \quad \text{uniformly in } y \tag{15}$$

and, uniformly in y with $\mathrm{dist}(y, \Gamma_N) = o(\sqrt{n})$,

$$\mathbf{P}_y(\widehat{\tau}_H > n) \sim \sqrt{\frac{2}{\pi}} \frac{v_{\mathbb{H}}(\mathrm{dist}(y, N\Gamma))}{n^{1/2}}. \tag{16}$$

Combining Lemma 3 with (15), we obtain

$$\begin{aligned} &\frac{n^{p/2+d/2}}{V(x)} \mathbf{P}_x\big(S(n) = y, \sigma_N > n\big) \\ &\quad = \varkappa 2^{p/2+d/2} \mathbf{E}_y\left[f_{n/\sqrt{2}N}\left(\frac{\widehat{S}(n-m)}{\sqrt{m}}\right); \widehat{\sigma}_N > n-m\right] + o\big(\mathrm{dist}(y, \Gamma_N)\big). \end{aligned}$$

By the functional CLT for random walks conditioned to stay in a half-space,

$$\mathbf{E}_y\left[f_{n/\sqrt{2}N}\left(\frac{\widehat{S}(n-m)}{\sqrt{m}}\right)\mathbf{1}\{\widehat{\sigma}_N > n-m\} \big| \widehat{\tau}_H > n\right] - g\left(\frac{y}{\sqrt{n}}, \frac{n}{\sqrt{2}N}\right) = o(1),$$

where

$$g(z, r) = \mathbf{E}_{z_r}\big[f_{r/\sqrt{2}}(M_{\mathbb{H}}(1)); M_{\mathbb{H}}(t) \in K_{0,1/r}\big],$$

where $z_r \in \Gamma_{1/r}$ is such that z and z_r belong to the same ray. Clearly, this function is continuous in both coordinates.

Combining this with (16), we have

$$\begin{aligned} &\frac{n^{p/2+d/2+1/2}}{V(x) v_{\mathbb{H}}(\mathrm{dist}(y, \Gamma_N))} \mathbf{P}_x\big(S(n) = y, \sigma_N > n\big) \\ &\quad = \varkappa \sqrt{\frac{2}{\pi}} 2^{p/2+d/2} g\left(\frac{y}{\sqrt{n}}, \frac{N}{\sqrt{n/2}}\right) + o(1). \end{aligned}$$

To prove the upper bound we notice that, by Lemma 27 from [4],

$$\mathbf{P}_x\big(S(m)=z,\sigma_N>m\big)\le \mathbf{P}_x\big(S(m)=z,\tau>m\big)\le C\frac{V(x)}{m^{p/2+d/2}}.$$

Plugging this into (14), we conclude that

$$\mathbf{P}_x\big(S(n)=y,\sigma_N>n\big)\le C\frac{V(x)}{m^{p/2+d/2}}\mathbf{P}_y(\sigma_N>n-m).$$

Applying now (15) completes the proof. □

COMPLETION OF THE PROOF OF THEOREM 1. Making use of Lemma 1 and Lemma 2 with $\varepsilon=\infty$ we obtain the upper bound (3) and its local version.

We will find now asymptotics for $G_N^{(2)}(x,y)$. According to the first part of Lemma 4,

$$\begin{aligned}&\frac{1}{V(x)v_{\mathbb{H}}(\mathrm{dist}(y,N\Gamma))}\sum_{n=\varepsilon N^2}^{N^2/\varepsilon}\mathbf{P}_x\big(S(n)=y,\sigma_N>n\big)\\ &\quad = c_0\sum_{n=\varepsilon N^2}^{N^2/\varepsilon} n^{-p/2-d/2-1/2}g\Big(\frac{y}{\sqrt{n}},\frac{N}{\sqrt{n/2}}\Big)+o\Big(\frac{1}{N^{p+d-1}}\Big).\end{aligned}\tag{17}$$

Note next that, as $N\to\infty$,

$$\begin{aligned}&\sum_{n=\varepsilon N^2}^{N^2/\varepsilon} n^{-p/2-d/2-1/2}g\Big(\frac{y}{\sqrt{n}},\frac{N}{\sqrt{n/2}}\Big)\\ &\quad=\frac{1}{N^{p+d-1}}\sum_{n=\varepsilon N^2}^{N^2/\varepsilon}\Big(\frac{N}{\sqrt{n}}\Big)^{p+d+1}g\Big(\frac{y}{N}\frac{N}{\sqrt{n}},\frac{N}{\sqrt{n/2}}\Big)\frac{1}{N^2}\\ &\quad=\frac{2}{N^{p+d-1}}\sum_{n=\varepsilon N^2}^{N^2/\varepsilon}\Big(\frac{N}{\sqrt{n}}\Big)^{p+d}g\Big(\frac{y}{N}\frac{N}{\sqrt{n}},\frac{N}{\sqrt{n/2}}\Big)\frac{1}{N^2}\frac{N}{2\sqrt{n}}\\ &\quad=\frac{2+o(1)}{N^{p+d-1}}\int_{\varepsilon^{1/2}}^{\varepsilon^{-1/2}} r^{-p-d}g\Big(\frac{y}{N}r^{-1},\sqrt{2}r^{-1}\Big)\,dr.\end{aligned}\tag{18}$$

Finally, using the uniform upper bound from Lemma 3, we have

$$\begin{aligned}\sum_{n=N^2/\varepsilon}^{\infty}\mathbf{P}_x\big(S(n)=y,\sigma_N>n\big)&\le CV(x)v_{\mathbb{H}}\big(\mathrm{dist}(y,N\Gamma)\big)\sum_{n=N^2/\varepsilon}^{\infty}n^{-p/2-d/2-1/2}\\ &\le C\varepsilon^{p/2+d/2-1/2}\frac{V(x)v_{\mathbb{H}}(\mathrm{dist}(y,N\Gamma))}{N^{p+d-1}}.\end{aligned}$$

Combining this with (17) and (18), we obtain

$$\begin{aligned}\limsup_{N\to\infty}\bigg|&\frac{N^{p+d-1}}{V(x)v_{\mathbb{H}}(\mathrm{dist}(y,N\Gamma))}G_N^{(2)}(x,y)\\ &-2c_0\int_{\varepsilon^{1/2}}^{\infty}r^{-p-d}g\Big(\frac{y}{N}r^{-1},\sqrt{2}r^{-1}\Big)\,dr\bigg|\le C\varepsilon^{p/2+d/2-1/2}.\end{aligned}$$

Combining this estimate with the bound from Lemma 2 and letting $\varepsilon\to 0$, we infer that (6) holds with

$$h(z)=2c_0\int_0^{\infty}r^{-p-d}g\big(zr^{-1},\sqrt{2}r^{-1}\big)\,dr.\tag{19}$$

□

3. Proof of Theorem 2. As we have already mentioned in the Introduction,

$$\mathbf{P}_x\big(S(\sigma_N)=y\big)=\sum_{z\in K_{0,N}} G_N(x,z)\mathbf{P}(y-X=z). \tag{20}$$

We first notice that

$$\sum_{z\in K_{0,N/2}} G_N(x,z)\mathbf{P}(y-X=z)\le \mathbf{P}\big(|X|>N/2\big)\max_{z\in K} G_N(x,z).$$

Applying Lemma 27 from [4], we have

$$\begin{aligned}G_N(x,z)&\le 1+\sum_{n=1}^{\infty}\mathbf{P}_x\big(S(n)=z;\tau>n\big)\\&\le 1+C(x)\sum_{n=1}^{\infty}n^{-p/2-d/2}\le C'(x)\quad \text{uniformly in } z\in K.\end{aligned}$$

Due to the assumption $\mathbf{E}|X|^{p+d}<\infty$, $\mathbf{P}(|X|>N/2)=o(N^{-p-d})$. Therefore,

$$\sum_{z\in K_{0,N/2}} G_N(x,z)\mathbf{P}(y-X=z)=o\big(N^{-p-d}\big). \tag{21}$$

Fix now a sequence R_N such that $R_N\to\infty$ and $R_N=o(N)$. According to the upper bound (4),

$$\begin{aligned}&\sum_{z\in\Gamma_{N/2,N-R_N}} G_N(x,z)\mathbf{P}(y-X=z)\\&\qquad\le C\frac{V(x)}{N^{p+d-1}}\sum_{z\in\Gamma_{N/2,N-R_N}}\operatorname{dist}(z,\Gamma_N)\mathbf{P}(y-X=z).\end{aligned}$$

Since the second moment of $|X|$ is finite,

$$\sum_{z\in\Gamma_{0,N-R_N}}\operatorname{dist}(z,\Gamma_N)\mathbf{P}(y-X=z)=o(1). \tag{22}$$

Consequently,

$$\sum_{z\in\Gamma_{N/2,N-R_N}} G_N(x,z)\mathbf{P}(y-X=z)=o\big(N^{-p-d+1}\big). \tag{23}$$

Since $R_N=o(N)$, for $z\in\Gamma_{N-R_N,N}$ we may apply (6):

$$\begin{aligned}&\sum_{z\in\Gamma_{N-R_N,N}} G_N(x,z)\mathbf{P}(y-X=z)\\&\quad=\frac{V(x)}{N^{p+d-1}}\sum_{z\in\Gamma_{N-R_N,N}} v_{\mathbb{H}}\big(\operatorname{dist}(z,\Gamma_N)\big)h(z/N)\mathbf{P}(y-X=z)\\&\qquad+o\bigg(\frac{V(x)}{N^{p+d-1}}\sum_{z\in\Gamma_{N-R_N,N}} v_{\mathbb{H}}\big(\operatorname{dist}(z,\Gamma_N)\big)\mathbf{P}(y-X=z)\bigg).\end{aligned}$$

The finiteness of the second moment of $|X|$ implies that

$$\sum_{z\in\Gamma_{N-R_N,N}} v_{\mathbb{H}}\big(\operatorname{dist}(z,\Gamma_N)\big)\mathbf{P}(y-X=z)=O(1)$$

and that

$$\sum_{z\in\Gamma_{N-R_N,N}} v_{\mathbb{H}}\big(\mathrm{dist}(z,\Gamma_N)\big)h(z/N)\mathbf{P}(y-X=z)$$
$$= h(y_N/N)\sum_{z\in\Gamma_{N-R_N,N}} v_{\mathbb{H}}\big(\mathrm{dist}(z,\Gamma_N)\big)\mathbf{P}(y-X=z)+o(1).$$

(Recall that y_N is the point on Γ_N which lies on the same ray as y.)

Therefore,

$$\sum_{z\in\Gamma_{N-R_N,N}} G_N(x,z)\mathbf{P}(y-X=z)$$
$$=\frac{V(x)}{N^{p+d-1}}h(y_N/N)\sum_{z\in\Gamma_{N-R_N,N}} v_{\mathbb{H}}\big(\mathrm{dist}(z,\Gamma_N)\big)\mathbf{P}(y-X=z)+o\biggl(\frac{V(x)}{N^{p+d-1}}\biggr).$$

Combining this with (22), we finally get

$$\sum_{z\in\Gamma_{N-R_N,N}} G_N(x,z)\mathbf{P}(y-X=z)$$
$$=\frac{V(x)}{N^{p+d-1}}h(y_N/N)\mathbf{E}\big[v_{\mathbb{H}}\big(\mathrm{dist}(y-X,\Gamma_N)\big);y-X\in K_{0,N}\big]+o\biggl(\frac{V(x)}{N^{p+d-1}}\biggr). \tag{24}$$

Pugging (21), (23) and (24) into (20), we get (9).

4. Three gambler's problem: Continuous case. In this section we give a proof of Proposition 3. Let $x=(x_1,x_2)$. To find the probabilities of interest we need to solve the following Dirichlet problem

$$\begin{cases}\Delta v_N(x)=0 & x\in\mathcal{T}_N,\\ v_N(x)=0 & x\in\mathcal{T}_N^{(2)}\cup\mathcal{T}_N^{(1)},\\ v_N(x)=N\phi(x/N) & x\in\mathcal{T}_N^{(3)}\end{cases} \tag{25}$$

with appropriately chosen boundary value functions ϕ.

Note first that by scaling

$$v_N(x)=Nv_1(x/N),\quad x\in\mathcal{T}_N\cup\partial\mathcal{T}_N. \tag{26}$$

Problem (25) can be solved using conformal mappings. In view of the scaling it is sufficient to consider the conformal mapping of the triangle $\mathcal{T}_{1/\sqrt{2}}$ to the upper half plane. This mapping is given by the formula

$$w(z)=\frac{1}{2}+\frac{27}{2B(\frac{1}{3},\frac{1}{3})^3}\wp'\biggl(\overline{z}e^{-\pi i/3};0,-\frac{1}{27^2}B\biggl(\frac{1}{3},\frac{1}{3}\biggr)^6\biggr),$$

where $\wp'$ is the first derivative of Weierstrass's elliptic function and $B(a,b)$ denotes the Euler beta function. This mapping transforms the edge from 0 to 1 into the half line from 1 to $+\infty$, the edge from 0 to $e^{\pi i/3}$ into the half line from $-\infty$ to 0 and the edge from 1 to $e^{\pi i/3}$ into the half line from 1 to 0. The inverse mapping from the upper half plane to the triangle is given by

$$z(w)=\overline{\frac{1}{B(\frac{1}{3},\frac{1}{3})}\int_0^w\frac{dt}{t^{2/3}(1-t)^{2/3}}e^{-\pi i/3}}+e^{\pi i/3}.$$

On the half-plane the solution to the Dirichlet problem is given by the Poisson kernel for the half-plane, see [1], Theorem 1.7.2. As a result the solution to (25) can be written down as follows

$$v_{1/\sqrt{2}}(z) = \frac{1}{\pi\sqrt{2}} \int_{-\infty}^{\infty} \frac{\operatorname{Im} w(z)}{|t - w(z)|^2} \phi\big(z(t)\big)\, dt = \frac{1}{\pi\sqrt{2}} \int_0^1 \frac{\operatorname{Im} w(z)}{|t - w(z)|^2} \phi\big(z(t)\big)\, dt. \tag{27}$$

This solution is harmonic in the triangle and continuous at its boundary points, where the function ϕ is continuous.

Plug in now the initial condition $z = \frac{1}{N}(x_1 + ix_2)$. Since $\wp'$ has a pole of order 3 at 0, we have

$$w(z) \sim -\frac{27}{B(\frac{1}{3}, \frac{1}{3})^3} \frac{N^3}{(x_1 + ix_2)^3}.$$

Then, uniformly in $t \in (0, 1)$,

$$\frac{\operatorname{Im} w(z)}{|t - w(z)|^2} \sim \frac{\operatorname{Im} w(z)}{|w(z)|^2} \sim -\frac{B(\frac{1}{3}, \frac{1}{3})^3}{27N^3} |x_1 + ix_2|^6 \operatorname{Im}\left(\frac{1}{(x_1 + ix_2)^3}\right)$$
$$= -\frac{B(\frac{1}{3}, \frac{1}{3})^3}{27N^3} \operatorname{Im}(x_1 - ix_2)^3 = \frac{B(\frac{1}{3}, \frac{1}{3})^3}{27N^3} \big(3x_1^2 x_2 - x_2^3\big).$$

Hence,

$$v_{1/\sqrt{2}}(z) \sim \frac{B(\frac{1}{3}, \frac{1}{3})^3}{27N^3} \big(3x_1^2 x_2 - x_2^3\big) \frac{1}{\pi\sqrt{2}} \int_0^1 \phi\big(z(t)\big)\, dt.$$

To analyse the probability in (12) we notice that $\mathbf{P}_{x_1,x_2}$(third player gets eliminated first) $= v_N(x)/N$, where v_N solves (25) with $\phi(x) \equiv 1$. Plugging in this boundary condition, we obtain

$$v_{1/\sqrt{2}}(z) \sim \frac{B(\frac{1}{3}, \frac{1}{3})^3}{27\sqrt{2}\pi N^3} \big(3x_1^2 x_2 - x_2^3\big).$$

Using the scaling we obtain,

$$\mathbf{P}_{x_1,x_2}(\text{third player gets eliminated first})$$
$$= v_1(x/N)$$
$$= \sqrt{2} v_{1/\sqrt{2}}\big(x/(\sqrt{2}N)\big) \sim \frac{B(\frac{1}{3}, \frac{1}{3})^3}{54\sqrt{2}\pi N^3} \big(3x_1^2 x_2 - x_2^3\big) = \frac{\Gamma(1/3)^9}{48\sqrt{6}\pi^4} \frac{u(x)}{N^3}.$$

Next we notice that the probability in (11) is given by $P^{bm,(321)}_{x_1,x_2}(N) = \frac{v_N(x)}{N}$, where v_N solves (25) with boundary value $\phi(x) = 1 - \frac{2x_2}{\sqrt{3}}$. For this function we have

$$\int_0^1 \phi\big(z(t)\big)\, dt = \int_0^1 \left(1 - \frac{2}{\sqrt{3}} \operatorname{Im} z(t)\right) dt$$
$$= \frac{1}{B(1/3, 1/3)} \int_0^1 \int_0^t \frac{du}{u^{2/3}(1 - u)^{2/3}}\, dt.$$

Integrating by parts we obtain

$$\int_0^1 \phi\big(z(t)\big)\, dt = \frac{1}{B(1/3, 1/3)} \int_0^1 (1 - u)^{1/3} u^{-2/3}\, du = \frac{B(4/3, 1/3)}{B(1/3, 1/3)}.$$

Then,

$$v_{1/\sqrt{2}}(z) \sim \frac{B(\frac{1}{3},\frac{1}{3})^2 B(\frac{4}{3},\frac{1}{3})}{27\sqrt{2}\pi N^3}(3x_1^2x_2 - x_2^3)$$
$$\sim \frac{B(\frac{1}{3},\frac{1}{3})^3}{54\sqrt{2}\pi N^3}(3x_1^2x_2 - x_2^3) = \frac{\Gamma(1/3)^9}{48\sqrt{3}\pi^4\sqrt{2}}\frac{3x_1^2x_2 - x_2^3}{N^3}.$$

Using the scaling we obtain

$$v_1(z) \sim \frac{\Gamma(1/3)^9}{96\sqrt{6}\pi^4}\frac{u(x)}{N^3},$$

which implies the statement.

REMARK 3. There are other conformal mappings which can be used to obtain some representations for solutions to (25). For example, Hajek [2] has used a mapping of the triangle onto the unit disk. In our opinion, the use of the Weierstrass function gives a handier representation. Since this is a meromorphic function, it allows for a rather standard asymptotic analysis. The following fact is not used in our paper, but the Weierstrass function is very convenient for numerical approximations and is contained in many computer packages.

5. Rate of convergence: Proof of Proposition 5.

5.1. *Extension of the harmonic function.* We will first extend the harmonic function $v_N(x)$ to obtain better estimates for its derivatives. Let $\mathcal{H}_N$ is the hexagon obtained by rotation of $\mathcal{T}_N$ about origin 5 times by $\frac{\pi}{3}$ each time. Let $\widetilde{\mathcal{T}}_N$ be the triangle obtained by the union of the reflection of $\mathcal{T}_N$ with respect the edge $\mathcal{T}_N^{(3)}$ and the edge $\mathcal{T}_N^{(3)}$. The resulting region $\mathcal{H}_N \cup \widetilde{\mathcal{T}}_N$ can be seen at Figure 2.

LEMMA 5. *Function v_N can be extended to $\mathcal{H}_N \cup \widetilde{\mathcal{T}}_N$ in such a way that it is harmonic on this region.*

PROOF. Note that using the standard Schwarz reflection principle (see [1], Theorem 1.3.6) we can construct a harmonic extension of the function $v_N(x)$ over $\mathcal{T}_N^{(1)}$ and $\mathcal{T}_N^{(2)}$. For that note that $v_N(x) = 0$ for $x \in \mathcal{T}_N^{(1)} \cup \mathcal{T}_N^{(2)}$ and is continuous at the boundary of $\mathcal{T}_N$ except vertices of $\mathcal{T}_N^{(3)}$. Then Theorem 1.7.5 implies continuity of $v_N(x)$ on the closure $\mathcal{T}_N$ except vertices of $\mathcal{T}_N^{(3)}$ and hence [1], Theorem 1.3.6, is applicable.

The construction of the reflection (and proof of harmonicity) is as follows. As $v_N(x) = 0$ for $x \in \mathcal{T}_N^{(1)} \cup \mathcal{T}_N^{(2)}$ we can extend it to the reflection of the triangle over the line $\{x_2 = 0\}$ by the usual formula

$$v_N(x_1, x_2) = -v_N(x_1, -x_2), \quad x_2 \in -\mathcal{T}_N.$$

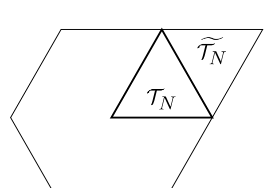

FIG. 2. *Region $\mathcal{H}_N \cup \widetilde{\mathcal{T}_N}$ to which v_N is extended.*

The resulting function is indeed harmonic as for $x \in \mathcal{T}_N$ or $x \in -\mathcal{T}_N$ it is equal to the average over all sufficiently small balls about x. For x at the boundary such that $x_2 = 0$ we note that the average over all sufficiently small balls is equal to 0 by cancellations in our construction and hence is equal to the value of the function. Since v_N is equal to the average over all small balls for all x in the region under consideration it is harmonic.

The reflection over other side of the triangle with 0 boundary conditions is analogous. Applying the reflection principle several times we obtain that function $v_N(x)$ has a harmonic continuation to the hexagon $\mathcal{H}_N$.

It is also possible to extend the side with nonzero boundary conditions. Indeed, rotating the triangle to simplify the notation we can assume that nonzero boundary conditions are on the side connecting $(0,0)$ and $(\sqrt{2}N, 0)$ and are given by

$$v_N(x_1, 0) = \frac{x_1}{\sqrt{2}N}.$$

Note that function $\frac{x_1}{\sqrt{2}N}$ is harmonic over the whole plane. Now put

$$\widetilde{v}_N(x_1, x_2) = v_N(x_1, x_2) - \frac{x_1}{\sqrt{2}N}.$$

Function $\widetilde{v}_N(x_1, x_2)$ is harmonic over $\mathcal{T}_N$ and is equal to 0 at the boundary. Hence we can extend it to $-\mathcal{T}_N$ by the same formula

$$\widetilde{v}_N(x_1, x_2) = -\widetilde{v}_N(x_1, -x_2), \quad x_2 \in -\mathcal{T}_N.$$

Then,

$$v_N(x_1, x_2) = \widetilde{v}_N(x_1, x_2) + \frac{x_1}{\sqrt{2}N}$$

is a harmonic extension of the original function with the required boundary conditions. Thus, we have shown that the extension exists to the above hexagon and the adjacent equilateral triangle with the side given by $\mathcal{T}_N^{(3)}$. □

5.2. *Diffusion approximation.* For $y \in \mathcal{T}_N$ let $\delta(y)$ be the distance from y to the closest vertex of the triangle. Let $\mathcal{C}_n$ be the convex hull of

$$(\overline{\mathcal{T}}_N \cap T\mathbb{Z}^2) \setminus \{\text{vertices of } \mathcal{T}_n\}.$$

Using the above extension of the harmonic function we can bound the derivatives of the harmonic function v_N in exactly the same way as in [4], Lemma 7, to obtain.

LEMMA 6. *For any k there exists a constant c_k such that for $x \in \mathcal{C}_N$ and $\alpha : |\alpha| \le k$,*

$$\left|\frac{\partial^\alpha v_N(x)}{\partial x^\alpha}\right| \le c_k \frac{N}{\delta(x)^{|\alpha|}}. \tag{28}$$

PROOF. First note that there exists c_0 such that for each $x \in \mathcal{C}_N$ the ball $B(x, c_0\delta(x))$ lies inside $\mathcal{H}_N \cup \widetilde{\mathcal{T}}_N$. Hence, the extension of $v_n(x)$ together with its derivatives is harmonic over $B(x, c_0\delta(x))$.

Applying the mean-value formula for harmonic functions to function $\frac{\partial v_N(x)}{\partial x}$ we obtain

$$\begin{aligned}\left|\frac{\partial v_N(x)}{\partial x}\right| &= \left|\frac{1}{\mathrm{Vol}(B(x, c_0\delta(x)))} \int_{B(x,c_0\delta(x))} \frac{\partial v_N(y)}{\partial y}\, dy\right| \\ &= \left|\frac{1}{(c_0\delta(y))^d \alpha(d)} \int_{\partial B(x,c_0\delta(x))} v_N \nu_i\, ds\right|\end{aligned}$$

$$\le \frac{d\alpha(d)(c_0\delta(y))^{d-1}}{\alpha(d)(c_0\delta(y))^d}\max_{y\in\partial B(y,c_0\delta_0(y))} v_N(y)$$
$$\le N\frac{d}{\delta(x)},$$

where we used the fact that $v_N(x)\le N$. As the classical maximum principle is not directly applicable (v_N has a discontinuity at the vertex), we recall the representation of v_N via the Poisson kernel for the half plane and [1], Theorem 1.7.5, to establish that $v_N(x)\le N$. Here $\alpha(d)$ is the volume of the unit ball and we used the Gauss–Green theorem. In the second line of the display ν_i is the outer normal and integration takes place on the surface of the ball $B(x,\delta(y))$. The higher derivatives can be treated likewise. The claim of the lemma immediately follows. □

We will now return to the random walk. Let v_N be the solution of (25) in $\mathcal{T}_N$. Next we can estimate the error similar to Lemma 8 of [4]. Let $f_N(x):=\mathbf{E}v_N(x+X)-v_N(x)$.

LEMMA 7. *There exists a constant C such that for $x\in\mathcal{T}_N\cap T\mathbb{Z}^2$,*

$$|f_N(x)|\le\frac{CN}{\delta(x)^6}. \tag{29}$$

PROOF. Recall that $S(n)=TZ(n)$, that is,

$$S_1(n)=\sqrt{2}Y_1(n)+\frac{1}{\sqrt{2}}Y_2(n),\qquad S_2(n)=\sqrt{\frac{3}{2}}Y_2(n),\quad n=0,1,2,\dots$$

We will compute now the first five moments of (X_1,X_2). The details of computation are given in the Appendix. We have, for positive integers n and m,

$$\mathbf{E}[X_1^{2n+1}]=\mathbf{E}[X_2^{2n+1}]=0,$$
$$\mathbf{E}[X_2^2]=\mathbf{E}[X_1^2]=1,\qquad \mathbf{E}[X_1X_2]=0,$$
$$\mathbf{E}[X_1^{2n}X_2^{2m-1}]=\mathbf{E}[X_1^{2n-1}X_2^{2m}]=0,$$
$$\mathbf{E}[X_1^3X_2]=\mathbf{E}[X_1X_2^3]=0,$$
$$\mathbf{E}[X_1^2X_2^2]=\frac{1}{2},\qquad \mathbf{E}[X_2^4]=\mathbf{E}[X_1^4]=\frac{3}{2}.$$

Similar to Lemma 8 of [4] we first write down the Taylor expansion up to the sixth term. Now these moments allow us to write down the first five terms of the Taylor expansion,

$$\begin{aligned}\mathbf{E}[v_N(x_1+X_1,x_2+X_2)]&=u(x)+\frac{1}{2}\Delta v_N(x_1,x_2)\\&\quad+\frac{1}{24}\big((v_N)_{x_1^4}(x_1,x_2)\mathbf{E}[X_1^4]+6(v_N)_{x_1^2x_2^2}(x_1,x_2)\mathbf{E}[X_1^2X_2^2]\\&\quad+(v_N)_{x_2^4}(x_1,x_2)\mathbf{E}[X_2^4]\big)\\&\quad+R_6(x)=v_N(x)+\frac{1}{2}\Delta v_N(x_1,x_2)\\&\quad+\frac{1}{24}\Big(\frac{3}{2}(v_N)_{x_1^4}(x_1,x_2)+3(v_N)_{x_1^2x_2^2}(x_1,x_2)\frac{3}{2}(v_N)_{x_2^4}(x_1,x_2)\Big)\\&\quad+R_6(x).\end{aligned}$$

Now note that since v_N is harmonic

$$\Delta v_N = 0$$

and

$$\frac{3}{2}(v_N)_{x_1^4} + 3(v_N)_{x_1^2 x_2^2}\frac{3}{2}(v_N)_{x_2^4} = \frac{3}{2}\Delta(v_N)_{x_1x_1} + \frac{3}{2}\Delta(v_N)_{x_2x_2} = 0.$$

Hence, only the terms starting from the sixth matter and

$$\left|\mathbf{E}\left[(v_N)(x_1+X_1, x_2+X_2)\right] - u(x)\right| = \left|R_6(x)\right| \le C \max_{z\in C(x),|\alpha|=6}\left|\frac{\partial^\alpha}{\partial z^\alpha}(v_N)(z)\right|,$$

where $C(x)$ is a convex hull of points achievable from x in one jump. Applying the estimates for sixth partial derivatives proved in Lemma 6 we arrive at the conclusion. □

Recall that $P^{bm,(321)}_{x_1,x_2}(N) = \frac{v_N(x)}{N}$, where $v_N(x)$ solves the Dirichlet problem (25) with $\phi(x) = 1 - \sqrt{\frac{2}{3}\frac{x_2}{N}}$. Then,

$$N P^{(321)}_{x_1,x_2}(N) =: V_N(x) = \mathbf{E}_x\left[\phi(S_{\sigma_N})\right] = \mathbf{E}_x v_N(S_{\sigma_N}).$$

Next note that

$$\mathbf{E}_x v_N(S_{\sigma_N}) - v_N(x) = \mathbf{E}_x \sum_{n=0}^{\sigma_N - 1} f_N(S_k) = \sum_{y\in\mathcal{T}_N\cap T\mathbb{Z}^2} G_N(x,y) f_N(y),$$

and, therefore,

$$V_N(x) = \mathbf{E}_x v_N(S_{\sigma_N}) = v_N(x) + \sum_{y\in\mathcal{T}_N\cap T\mathbb{Z}^2} G_N(x,y) f_N(y). \tag{30}$$

PROOF OF PROPOSITION 5. We will bound the series in (30) to obtain the result. First note that by reversing time we obtain the following estimate from (4),

$$G_N(x,y) \le C\frac{V(y)}{N^3}.$$

By symmetry this bound holds near all three vertices and hence we can write,

$$G_N(x,y) \le C\frac{\delta(y)^3}{N^3}.$$

Then, using Lemma 7 we estimate

$$\sum_{y\in\mathcal{T}_N\cap T\mathbb{Z}^2} G_N(x,y)\left|f_N(y)\right| \le C_\delta \sum_{y\in\mathcal{T}_N\cap T\mathbb{Z}^2} \frac{\delta(y)^3}{N^3}\frac{N}{\delta(y)^6} \le \frac{C}{N^2},$$

as the series $\sum_{y\in\mathcal{T}_N\cap T\mathbb{Z}^2}\frac{1}{\delta(y)^3}$ converges. Therefore,

$$\left|P^{(321)}_{x_1,x_2}(N) - P^{bm,(321)}_{x_1,x_2}(N)\right| = \left|\frac{V_N(x) - v_N(x)}{N}\right| \le \frac{C}{N^3},$$

as required. □

APPENDIX: COMPUTATION OF MOMENTS

First we will write down the moments of (Y_1, Y_2). We have, for positive integer n

$$\mathbf{E}[(Y_1)^{2n}] = \mathbf{E}[(Y_2)^{2n}] = \frac{2}{3},$$
$$\mathbf{E}[(Y_1)^{2n-1}] = \mathbf{E}[(Y_2)^{2n-1}] = 0.$$

Also, for all positive integers n and m

(31)
$$\mathbf{E}[Y_1^{2n} Y_2^{2m-1}] = \mathbf{E}[Y_1^{2n-1} Y_2^{2m}] = 0,$$

(32)
$$\mathbf{E}[Y_1^{2n-1} Y_2^{2m-1}] = -\frac{1}{3},$$

(33)
$$\mathbf{E}[(Y_1)^{2n}(Y_2)^{2m}] = \frac{1}{3}.$$

Then, it follows from (31) that odd moments disappear,

$$\mathbf{E}[X_1^{2n+1}] = \mathbf{E}[X_2^{2n+1}] = 0.$$

Even moments are given by

$$\mathbf{E}[X_2^2] = \frac{3}{2}\mathbf{E}[Y_2^2] = 1, \qquad \mathbf{E}[X_2^4] = \frac{9}{4}\mathbf{E}[Y_2^4] = \frac{3}{2}$$

and

$$\mathbf{E}[X_1^2] = \frac{1}{2}\mathbf{E}[(2Y_1 + Y_2)^2] = \frac{1}{2}\left(4\frac{2}{3} - 4\frac{1}{3} + \frac{2}{3}\right) = 1,$$
$$\mathbf{E}[X_1^4] = \frac{1}{4}\mathbf{E}[(2Y_1 + Y_2)^4] = \frac{1}{4}\left(16 \cdot \frac{2}{3} - 4 \cdot 8\frac{1}{3} + 6 \cdot 4\frac{1}{3} - 4 \cdot 2\frac{1}{3} + \frac{2}{3}\right) = \frac{3}{2}.$$

Mixed moments are given by

$$\mathbf{E}[X_1 X_2] = \frac{\sqrt{3}}{2}\mathbf{E}[Y_2^2 + 2Y_1 Y_2] = 0,$$
$$\mathbf{E}[X_1^2 X_2] = \frac{\sqrt{3}}{2\sqrt{2}}\mathbf{E}[Y_2^3 + 4Y_2^2 Y_1 + 4Y_2 Y_1^2] = 0.$$

Similar to the latter expression one can use (31) to show for any positive integers n and m that

$$\mathbf{E}[X_1^{2n} X_2^{2m-1}] = \mathbf{E}[X_1^{2n-1} X_2^{2m}] = 0.$$

Next

$$\begin{aligned}
\mathbf{E}[X_1^3 X_2] &= \frac{\sqrt{3}}{4}\mathbf{E}[(Y_2 + 2Y_1)^3 Y_2] \\
&= \frac{\sqrt{3}}{4}(\mathbf{E}[Y_2^4 + 6\mathbf{E}[Y_2^3 Y_1] + 12\mathbf{E}[Y_2^2 Y_1^2] + 8\mathbf{E}[Y_2 Y_1^3]]) \\
&= \frac{\sqrt{3}}{4}\left(\frac{2}{3} - 6 \cdot \frac{1}{3} + 12 \cdot \frac{1}{3} - 8 \cdot \frac{1}{3}\right) = 0
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{E}[X_1 X_2^3] &= \frac{3\sqrt{3}}{4}\mathbf{E}[(Y_2 + 2Y_1)Y_2^3] \\
&= \frac{3\sqrt{3}}{4}(\mathbf{E}[Y_2^4] + 2\mathbf{E}[Y_2^3 Y_1]) = \frac{3\sqrt{3}}{4}(2/3 - 2/3) = 0.
\end{aligned}$$

Finally,

$$\begin{aligned}\mathbf{E}[X_1^2X_2^2] &= \frac{3}{4}\mathbf{E}[(Y_2+2Y_1)^2Y_2^2] \\ &= \frac{3}{4}(\mathbf{E}[Y_2^4]+4\mathbf{E}[Y_2^3Y_1]+4\mathbf{E}[Y_2^2Y_1^2]) \\ &= \frac{3}{4}\left(\frac{2}{3}-4\cdot\frac{1}{3}+4\cdot\frac{1}{3}\right)=\frac{1}{2}.\end{aligned}$$

Acknowledgments. We are grateful to Persi Diaconis for drawing attention to this problem and a number of encouraging discussions. We are very grateful to the reviewer for the detailed reading and helpful and constructive comments.

Funding. D. Denisov was supported by a Leverhulme Trust Research Project Grant RPG-2021-105.

V. Wachtel was supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation)—Project-ID 317210226—SFB 1283.

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