Cash-constrained R&D investment*

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December 4, 2024

Abstract

We study endogenous, credit-financed innovation under uncertainty in dynamic contexts. In our model, a firm with limited cash reserves decides how much to invest in an R&D project, potentially using external financing. Investing more increases the probability of a sooner innovation, but higher repayment obligations also increase bankruptcy risk if the innovation takes longer. We show that the firm reduces its investment discontinuously if the financing cost is not favorable enough, in order to avoid the need for external financing. This insight implies that policies reducing financing costs can have discontinuous positive effects on investment, innovation rate and welfare. However, policy measures increasing the effectiveness of R&D might reduce the innovation rate and welfare due to a discontinuous reduction of R&D investment. Furthermore, we find that low financing costs can lead to over-investment. The welfare loss from cash constraints is more severe for radical innovations compared to incremental ones.

Keywords: Innovation, R&D investment, Cash constraints, Bankruptcy risk

1 Introduction

Firms' research and development (R&D) activities generate new growth opportunities and potentially greater returns. However, these activities typically require substantial investments that often exceed available cash reserves. The success of R&D is uncertain, as both the timing of successful innovations and the associated costs are unknown. Hence, determining optimal expenditures together with the amount of external financing is a crucial strategic decision for firms engaged in research and development. We study endogenous, credit-financed innovation under uncertainty in dynamic contexts.

Accessible financing resources are important for firms' innovation activities (Ayyagari et al., 2011; Gorodnichenko and Schnitzer, 2013; Lee et al., 2015; Santos et al., 2024). Large, established firms often use internal funds to finance their R&D investments (e.g., Hall and Lerner, 2010). However, small and medium sized enterprises (SMEs) are key players in the EU economy in terms of added value, accounting for 99.8% of all nonfinancial enterprises (Kraemer-Eis et al., 2023). These firms have much lower internal financial resources. Particularly start-ups and young firms often have only small sales volumes and revenue reserves (Fryges et al., 2015). Such firms may turn to external sources to ease the financial

*Financial support by the German Research Foundation (DFG) [CRC 1283/2 2021 – 317210226] is gratefully acknowledged.

constraints on their R&D activities. Among all possible external financing channels, credit financing (bank overdrafts and bank loans) accounts for 39% of the volume used by SMEs (Kraemer-Eis et al., 2023).[1] However, because of the inherent risks of innovation, access to external financing remains limited and costly for many firms (Brown et al., 2009; Eckel and Unger, 2023; Giudici and Paleari, 2000). In particular, firms may exhaust their cash reserves before achieving successful innovation, potentially leading to bankruptcy.

An illustrative example is Aradigm Corporation, a California-based biotech company that developed the inhaled antibiotic formulation Apulmiq. The company sought Food and Drug Administration (FDA) approval for Apulmiq but was rejected in 2018. According to the FDA's response letter, Aradigm would be required to conduct a new and costly two-year trial before reapplying for approval. The company had been attempting to raise funds to revive the program, as it "would not have enough funds to meet all of their future financial obligations." At that time, Aradigm had only USD 7.1 million in cash and equivalents but faced future obligations of USD 10–50 million. Consequently, the company filed for bankruptcy one year later.[2]

The aim of this paper is to analyze how limited cash reserves influence a firm's external financing decisions under technological uncertainty, specifically whether and how much it should borrow. We focus on the scenario of a monopoly firm with initial cash reserves that decides to invest in R&D, which can create new value upon success, although it is uncertain how long it will take to achieve the success. The firm faces a tradeoff: it can take out loans to invest heavily, thereby increasing the probability of sooner successful innovation, but also introducing default risks. A larger loan results in higher coupon payments and an earlier depletion of the firm's liquidity.

Within this framework, we address the following research questions: Under what conditions is it optimal for a firm with initial cash reserves to partially finance innovation projects externally? How does external financing affect the innovation rate and what are the associated distortions relative to the social optimum? How do the firm's financing decisions and resulting distortions vary based on the type of innovation project, whether incremental or radical? Finally, what policy implications can enhance the effectiveness of firms' R&D investments?

Our main results are as follows. First, we find that the firm's optimal investment and financing choice has a threshold structure with respect to the coupon rate, i.e., the price of a loan. Specifically, for coupon rates below this threshold, the firm takes a loan in order to invest strictly more than its cash reserve in R&D, with the optimal volumes of investment and loan depending on the exact value of the coupon rate. If the coupon rate is equal to the threshold or greater, however, the firm abstains from using any loan and constrains the investment by its internal financing capability, such that the optimal volume does not depend on the exact value of the coupon rate anymore.

In this context, a further remarkable insight with important implications is that the transition of the optimal investment size is not continuous at the threshold. The optimal investments for all coupon rates below the threshold exceed a lower bound strictly greater than the firm's cash reserve. This means the optimal investment experiences a downward jump if the coupon rate reaches the threshold, and this jump is even more pronounced if the

[1] Similarly, innovating small firms in the UK rely overly on debt financing (Freel, 1999). See also the references in Geelen et al. (2021) and Ning and Babich (2018) for further evidence that debt is a significant component of financing firms' innovation activities.

[2] https://seekingalpha.com/news/3434163-aradigm-files-for-bankruptcy-in-california

optimal investment constrained by internal financing is less than the full cash reserve.

Relatedly, we find a nonmonotone relationship between the coupon rate and the probability that the innovation is eventually attained. We then analyze the implications for social welfare and show that the welfare loss that is due to the firm's financial constraints jumps upward as the coupon rate crosses the critical threshold and the firm abstains from external financing.

But we establish also an over-investment result. When the coupon rate is relatively low, close to the risk-free interest rate, the firm tends to invest more than the social optimum. This occurs because the firm does not bear the full cost of bankruptcy, which incentivizes increased investment, and this effect may dominate the higher direct cost of external financing embodied by the coupon rate. For a given coupon rate, the tradeoff favors exploiting the possibility of default and investing more whenever the innovation rate per unit of investment is sufficiently small. That is, low investment effectiveness encourages the firm to invest more in R&D despite the implied higher bankruptcy risk. Considering also the effect of increasing the innovation rate, we show that this will decrease investment.

Finally, we study how the firm's optimal investment and the implied welfare loss depend on varying the type of innovation, from radical to incremental. Although the firm's investment approaches the socially optimal volume in both directions, the welfare loss is greatest for radical innovations because then the firm will take a substantial loan, which leads to an increased probability of default before achieving the innovation, whereas the firm will not invest more than its cash reserves for an incremental innovation, which then prevents default and ensures that the innovation is eventually attained.

Our paper relates to different streams of literature. On the one hand, starting with Jensen and Meckling (1976) and Myers (1977), there is a broad literature that studies the effects of debt financing on firms' investments into risky projects. The central question is how strong the incentives of borrowing firms are to shift risk to their lenders and whether the implied agency costs restrain debt financing or if there are ways to mitigate the resulting distortions (see, e.g., Chava and Roberts, 2008).

On the other hand, there is also a large body of literature on R&D investment and innovation uncertainty. A typical theme is to study the relation between market competition and firms' innovation (see, for example, Kamien and Schwartz, 1975; Loury, 1979; Lee and Wilde, 1980; Dasgupta and Stiglitz, 1980; Reinganum, 1982; Denicolò, 2000; Doraszelski, 2003). However, it is commonly assumed that firms can finance their innovation investments internally.

Only a smaller strand of literature considers the role of debt financing for innovation investments. Tanrısever et al. (2012) study a debt-financed startup firm's tradeoff between investing in process innovation to generate future growth opportunities and reducing short-term bankruptcy risk. Using a model where bankruptcy risk is caused only by the possibility of low cash flow from running operations, they show that optimal investment is either "conservative" to aim at high survival probability or "aggressive" to aim at high profit conditional on survival. Ning and Babich (2018) model an R&D investment game between two fully debt-financed firms and show that the under-investment incentive that arises from knowledge spillover can completely offset the over-investment incentive from debt financing, which then yields first-best investments. Zhang and Lee (2022) consider a manufacturer's problem whether to fund a supplier who has the potential to develop a new technology but no own resources. The manufacturer chooses both the volume and the mode of financing. Compared to equity, a loan implies a cost-shifting effect because of the supplier's limited liability, but it depends on the further circumstances which financing mode is optimal for the manufacturer.

These are two- or three-period models in which only the sequence of decisions but not time as such matters. In our setting, uncertainty about how long it takes to achieve an R&D success plays a central role. Such uncertainty is also taken into account by Geelen et al. (2021), who study innovation at the industry level with an optimal leveraging perspective on debt. Using a Schumpeterian growth model, they show that debt can foster innovation in aggregate although incumbents are hampered by the debt overhang effect that Myers (1977) identified, because it stimulates innovation by entrants instead. In their model, however, a firm's default decision is driven by maximizing the levered firm value rather than by liquidity depletion. We, in contrast, focus on the role of limited cash reserves.

Endogenizing the decision on the amount of debt and the corresponding volume of investment, we analyze the firm's tradeoff between technological risk and bankruptcy risk depending on key parameters (notably the cost of external financing), and we also take a welfare perspective.

The paper is organized as follows. Section 2 sets up the model and derives an instrumental representation of the firm's expected payoff. In Section 3 we characterize the firm's optimal credit and investment plans. Section 4 investigates the effect of the coupon rate on the firm's optimal credit and investment plans. In Section 5 we study how changes in the economic environment affect the firm's decisions and the corresponding innovation rate and welfare. Section 6 adds some further discussion of our results. Finally, the Appendix collects some supplementary results and the formal proofs.

2 Model

A firm considers how much to invest in an R&D project in order to generate an innovation that has a net present value of $V > 0$ when it occurs. The investment size, denoted as $x \geq 0$, determines the innovation arrival rate, represented by $h(x)$, where h is a strictly increasing and differentiable function with $h(0) = 0$. The variable x represents the present value of the cost associated with making a contractual commitment to research and development, as described by Loury (1979) and the subsequent literature (Weeds, 2002; Clark and Konrad, 2008; Galasso and Simcoe, 2011; Bloom et al., 2013). Alternatively, it can also capture the situation where the firm invests in a technological development with no direct market impact until a possible later time (Cohen and Levinthal, 1994).

Let $\tau \geq 0$ denote the arrival time of the innovation, which is, thus, exponentially distributed with mean $1/h(x)$. To mark the dependence on x, we write F_x for the cumulative distribution function of τ, and the corresponding density function and expectation operator are f_x and E_x, respectively.

To finance the investment, the firm can use its given initial liquidity $w > 0$ and additionally take a loan; other funding is not available. Thus, if the firm decides to take a loan of size $l \geq 0$, it can invest at most

$$x \leq w + l. \tag{1}$$

In particular, any investment $x > w$ requires a loan. Whenever the firm takes a loan, it commits to making the coupon payment cl as along as its liquidity is positive, where $c > 0$ is the fixed coupon rate. Also the proceeds from a successful innovation must be used for continuing the coupon payments. Once the firm's liquidity is exhausted, however, it has to default on the loan and abandon all operations—including the ongoing R&D project if the innovation has not occurred, yet.

Liquidity is kept in a cash account where it earns the risk-free interest rate $r > 0$, which we assume to be less than c.[3] The objective of the firm is to choose l and x satisfying the feasibility constraint (1) in order to maximize the expected net present value of all cash inflows and outflows, i.e.,

$$U(l,x) := E_x \left[\mathbf{1}_{\{\tau \le T\}} e^{-r\tau} V - \int_0^T e^{-rs} cl \, ds \right] + w + l - x,$$

where T is the (possibly infinite) time of default. Since T depends on τ, it is stochastic, too. We shall next characterize T and then derive a more instrumental expression for $U(l,x)$ in Proposition 1.

2.1 Characterizing bankruptcy

The balance of the firm's cash account immediately after investment is $w + l - x$. If the firm decides to invest without a loan, which implies $x \le w$ by (1), it has no obligations to make any further payments and, thus, will never default. Once it takes a loan, however, then, depending on (l,x) and the arrival time of the innovation, it might not always be able to perpetually pay the coupon cl. We can distinguish the following four scenarios.

1. The first scenario is that the perpetual coupon cl can be fully financed from the reserved cash $w + l - x$, which grows at the rate r. Then the firm does not default and we let $T = \infty$. This subsumes the case when $l = 0$.

2. In the second scenario, the reserved cash alone is not enough to cover the coupon payments forever. However, the innovation arrives before the cash account turns negative, which would happen at time

$$T_0(l,x) := \frac{1}{r} \ln \left(\frac{cl}{(c-r)l + rx - rw} \right). \tag{2}$$

   The coupon then can be financed forever after adding the proceeds from the innovation to the remaining cash. Hence, also in this case the firm does not default and $T = \infty$.

3. In the third scenario, the innovation does not arrive before the firm runs out of the reserved cash, which causes default at $T = T_0(l,x) < \infty$.

4. In the last scenario, the innovation arrives in time again to prevent immediate default, but its proceeds and the remaining cash are not sufficient to finance the coupon perpetually. In this case default is only postponed to a finite time $T > T_0(l,x)$. In Proposition 1, we show that whether this or the second scenario occurs is determined by the arrival time of the innovation. Specifically, default will happen if and only if the innovation arrives after the critical time

$$T_1(l,x) := \frac{1}{r} \ln \left(\frac{rV}{(c-r)l + rx - rw} \right). \tag{3}$$

[3] Since r is the risk-free rate whereas the lender faces some risk of reduced coupon payments due to firm bankruptcy, the coupon rate c should be larger than r even under a perfectly competitive lender market. Nevertheless, we analyze also the case $c = r$ (in Appendix B) and find that it has some degenerate effects.

Figure 1 shows a timeline for the arrival of the innovation that illustrates the different scenarios if the coupon cannot be fully financed from the reserved cash, i.e., cases 2–4. The ordering of $T_0(l,x)$ and $T_1(l,x)$ depends on the loan size l, and scenario 4 disappears if $T_0(l,x) \leq T_1(l,x)$. We will come back to this figure when we discuss optimal loan sizes in Section 3.

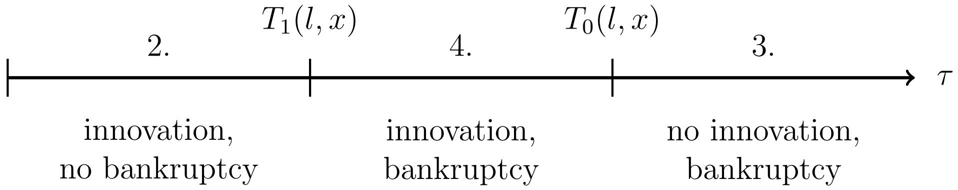


Figure 1: Scenarios depending on the innovation arrival time τ.

Proposition 1. *Whenever* (l,x) *satisfies*

$$w \geq \left(\frac{c}{r} - 1\right) l + x, \tag{4}$$

then $T = \infty$ *and*

$$U(l,x) = E_x e^{-r\tau} V + w - \left(\frac{c}{r} - 1\right) l - x. \tag{5}$$

Otherwise, i.e., whenever

$$w < \left(\frac{c}{r} - 1\right) l + x, \tag{6}$$

we have $T = \infty$ *if and only if* $\tau \leq \min(T_0(l,x), T_1(l,x))$*, and further*

$$U(l,x) = E_x \left[\mathbf{1}_{\{\tau \leq \min(T_0(l,x),T_1(l,x))\}} \underbrace{\left(e^{-r\tau} V + w - \left(\frac{c}{r} - 1\right) l - x \right)}_{=:\pi} \right]. \tag{7}$$

Condition (4) corresponds to the first scenario described above, whereas condition (6) subsumes the other three scenarios, which then depend on when the innovation occurs. The expression for the expected firm value in (7) reflects that, if the firm needs the proceeds from the innovation for covering the coupon payments, it only makes a gain if the innovation arrives sufficiently fast (scenario 2), and then the net present value of the gain is the term denoted by π, which is indeed positive whenever $\tau < T_1(l,x)$. In all other cases, when the firm defaults at a finite time T, its liquidity is eventually used up for the coupon payments.

Figure 2 illustrates for which combinations of investment x and loan size l the firm faces the risk of bankruptcy. Increasing the size of the loan reduces the range of investments that can be done without generating bankruptcy risk, because more liquidity is needed after investment to guarantee full coverage of the coupon payment without relying on innovation proceeds. The figure also highlights that any investment above the initial liquidity w necessarily induces some bankruptcy risk. Investments below w can be done without bankruptcy risk since no loan is needed. Whether it is optimal to abstain from taking a loan in this case will be analyzed in the next section.

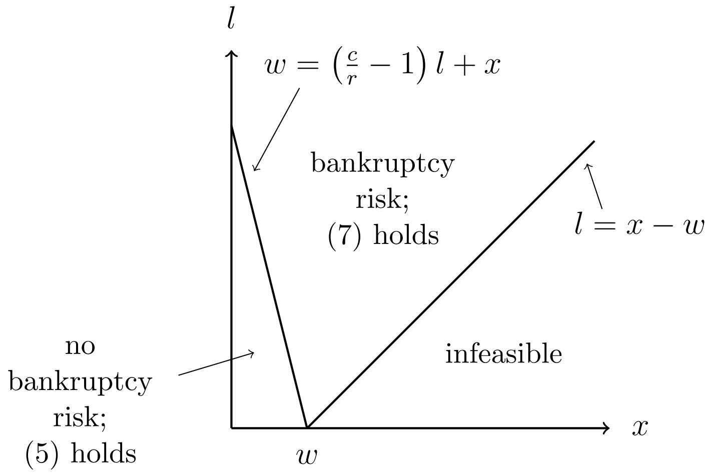


Figure 2: Credit-and-investment plans (l, x) as distinguished in Proposition 1.

3 Optimal Investment and Financing Decisions

First, as a benchmark, consider the case in which the firm has sufficient liquidity to finance the investment internally; it just chooses the investment $x \geq 0$, thereby determining the innovation rate $h(x)$. In this case the expected profit is

$$U_0(x) := E_x e^{-r\tau} V - x = \frac{h(x)}{r + h(x)} V - x,$$

so the first-order condition for optimal investment is given by

$$\frac{h'(x) r V}{(r + h(x))^2} = 1.$$

In Lemma 1 in the Appendix we show that the profit function $U_0(x)$ is strictly concave if the innovation rate $h(x)$ is concave. Then there is a unique optimal investment size, denoted by x_0^*, which can be identified by the first order condition.

With limited cash reserves, however, such a simple characterization is not possible. Instead, we now distinguish two candidate classes for the optimal investment size and show that for each candidate there is a unique optimal loan size to finance it. Afterward we will address the main question which of the two classes optimal investment belongs to.

Theorem 1. *An optimal credit-investment plan (l^*, x^*) exists, and optimal investment has to belong to one of the following two classes.*

1. *$x \leq w$. For any such x, the optimal loan size is $l^*(x) = 0$, and then the payoff is*

$$U(0, x) = U_0(x) + w = \frac{h(x)}{r + h(x)} V - x + w. \tag{8}$$

2. *$w < x \leq w + (r/c)(V - w)$. For any such x, the optimal loan size $l^*(x)$ is the unique l in $(w - x, (r/c)V)$ that solves the first-order condition*

$$h(x) e^{-h(x) T_0(l,x)} \left(V - \frac{cl}{r}\right) \frac{x - w}{cl^2} - \left(1 - e^{-h(x) T_0(l,x)}\right) \left(\frac{c}{r} - 1\right) = 0. \tag{9}$$

For any l in $(w-x,(r/c)V)$, we have $T_0(l,x) \leq T_1(l,x)$, and then the payoff is

$$U(l,x) = \frac{h(x)}{r+h(x)} V \left(1 - e^{-(r+h(x))T_0(l,x)}\right) + \left(w - \left(\frac{c}{r} - 1\right) l - x\right) \left(1 - e^{-h(x)T_0(l,x)}\right). \tag{10}$$

The first class are all investments that can be fully financed from the internal funds; then one should not take any loan. Any bigger investment cannot be financed without a loan. The upper bound on x in the second class shows, however, that it is not worthwhile to undertake excessive investments financed by loans. The intuition for this bound is that once the investment exceeds it, the total financing cost (including that for the loan) is so high that even an immediate innovation occurring with certainty at time $t = 0$ would not be worth it. We remark that the second class is empty when $w \geq V$, which implies that if the firm's initial funds are at least equal to what it can gain from the innovation, it should not use any loan.

It is possible to provide also some intuition for determining the optimal loan size when it is worthwhile to consider investments $x > w$, i.e., when $w < V$, which is the case we are primarily interested in. Then $U(l,x)$ satisfies (7), where increasing l obviously decreases the potential gain denoted by π. A bigger loan also decreases the critical time $T_1(l,x)$, because when the innovation arrives the cash reserve still needs to be big enough for the firm to be able to perpetually pay the coupon then. This means that whenever $T_1(l,x) < T_0(l,x)$, also the probability of realizing the gain π is decreasing in l, so one should then decrease l until $T_1(l,x) \geq T_0(l,x)$. Thus, an optimal loan size prevents that scenario 4 happens—if the innovation arrives in time to prevent immediate default, the firm will not default at all (in contrast to the case shown in Figure 1). The corresponding condition for the loan size is $l \leq (r/c)V$. Once this holds, however, a tradeoff arises: if l is further decreased, π continues to increase, but $T_0(l,x)$ then *decreases*, because the cash reserve from a smaller loan is depleted *earlier* (for fixed investment size x). Thus, for $l < (r/c)V$, one needs to balance the size of the potential gain and the probability of realizing it—which does the first-order condition (9).

Now, in order to determine the optimal investment size x^*, we need to study the two functions given by (8) and (10) on their respective domains.[4] Although a full closed form representation of the optimal values of x^* and l^* in both cases is not possible, we can shed some light on the behavior of the optimal loan size and the resulting payoff at the transition between the two classes $x \leq w$ and $x > w$, before we then proceed to further characterize the optimal investment size x^*.

Proposition 2. *Suppose $w < V$. Then the optimal loan size $l^*(x)$ is continuous at $x = w$. Nevertheless, the implied payoff $U(l^*(x),x)$ is discontinuous at $x = w$.*

4 Impact of the Coupon Rate

The economic question we are interested in is when it is optimal to take a loan in order to be able to invest more than w. This depends of course strongly on the cost represented by the

[4]A graphical illustration of these relevant domains is provided in the Appendix; see Figure 6 in Appendix D. Concerning this illustration, note that the bound $l \leq (r/c)V$ matters only if $w < (r/c)V$. Otherwise, for any $x > w \geq (r/c)V$, it is implied by $(c/r-1)l + x \leq V$.

coupon rate c. In the following, we study the impact of c on the optimal investment decision and show in particular that it is discontinuous.

Therefore, to stress the dependence of $U(l, x)$ on the parameter c, we will write $U(l, x; c)$, which, thus, still satisfies (5) or (7) depending on whether (l, x) satisfies (4) or (6).

This also allows us to introduce the notation

$$U^*(c) = \sup\{U(l, x; c) \mid w + l - x \geq 0\}$$

for the value of the firm's investment problem depending on the parameter c. Then the question when optimal investment requires a loan can be rephrased as when $U^*(c)$ exceeds the value that can be attained without loan. The latter is the maximum of $U(0, x)$ subject to $x \leq w$ and does not depend on c. Fix an arbitrary constrained maximizer

$$\bar{x}^* \in \operatorname{argmax}\{U(0, x) \mid x \leq w\},$$

which indeed exists because $U(0, x)$ inherits continuity in x from $h(x)$; see (8).

Proposition 3. *There is a critical threshold $c^* < \infty$ such that for any $c \geq c^*$ it is optimal to invest only $\bar{x}^* \leq w$, whereas for any $c < c^*$ optimal investment must exceed w and, thus, requires to take a loan.*

In particular, if h is concave, then for any $c \geq c^*$ it is optimal to invest

$$\bar{x}^* = \min\{x_0^*, w\}.$$

Proposition 3 shows that the firm has an incentive to invest more than its initial liquidity w if and only if the cost of taking a loan, the coupon rate c, is low enough. Otherwise the firm's investment will be (self-)constrained by its ability to finance internally. It is possible that the threshold c^* equals r so that it is optimal to invest $\bar{x}^* \leq w$ regardless of c—for instance when the firm has a very high initial liquidity $w \geq V$. In fact, since the proof of Proposition 3 covers also the limiting case $c = r$, we have $c^* > r$ if and only if it is optimal to take a loan under the lowest conceivable cost, i.e., if and only if $U^*(r) > U(0, \bar{x}^*)$ (see Appendix B for an analysis of the special case $c = r$).

It is of course intuitive that the firm refrains from taking a loan if the cost is too high. But the attractiveness to invest more than the internal funds is not slowly fading away. Instead, we obtain the striking result that the coupon rate has a discontinuous effect on investment at the critical threshold c^*.

Proposition 4. *Assume $c^* > r$. Then there is a threshold $\hat{x} > w$ such that for any $c < c^*$ optimal investment is at least $\hat{x}$. Hence, x^* does not approach w as c increases towards c^*, rather it exhibits a downward jump when c reaches c^*.*

Since investment exhibits a downward jump as the coupon rate crosses the threshold c^*, this implies that also the optimal size of the loan jumps at $c = c^*$. In light of Proposition 4 this follows directly from $l^* \geq x^* - w$ for $c < c^*$ and $l^* = 0$ for $c \geq c^*$.

5 Economic Analysis

In this section we analyze the implications of variation of key parameters on the optimal investment and financing strategy as well as on the resulting innovation rate and welfare.

To carry out a welfare analysis we assume that the social value of the innovation is given by $V^S = V + S$, such that $S \geq 0$ denotes the difference between the social value and the value for the firm. Expected discounted welfare is then given by

$$W(l, x) = E_x\left[e^{-r\tau}(V+S)\right] - x = \frac{h(x)}{r+h(x)}(V+S)\left(1 - e^{-(r+h(x))T_0}\right) - x \tag{11}$$

Accordingly, the welfare maximum under internal financing is given by

$$W^* = W(0, x_0^s) = \frac{h(x_0^s)}{r+h(x_0^s)}(V+S) - x_0^s,$$

with x_0^s denoting the socially optimal level of investment. We denote by $\Delta W = W^* - W(l^*, x^*)$ the welfare loss arising under optimal firm behavior in the presence of cash constraints.

Since a full analytic characterization of the dependence of optimal investment and loan size on key parameters of the model as well as the associated welfare effects is not possible, we rely on a numerical approach to analyze these issues. To this end, we specify the innovation rate as a linear function of investment, i.e., we have $h(x) = \alpha x$, $\alpha > 0$. In this case the unconstrained profit function (for unlimited internal funds) is given by

$$U_0(x) = \frac{\alpha x}{r + \alpha x}V - x,$$

so that the uniquely optimal investment is

$$x_0^* = \frac{\sqrt{\alpha r V} - r}{\alpha}$$

if $\alpha V > r$, whereas it is not worthwhile to invest into research if $\alpha V \leq r$. Furthermore, under the linear innovation rate the socially optimal investment (under internal financing) is given by

$$x_0^s = \frac{\sqrt{\alpha r(V+S)} - r}{\alpha}.$$

Figure 3 illustrates Proposition 3 by demonstrating the downward jump of optimal firm investment x^* as the coupon rate c crosses the threshold c^*.[5] The shape of the optimal loan size $l^*(x)$ and of the value function underlying the jump are illustrated in Figure 7 in Appendix D. Panels (c) and (d) of Figure 3 show that the downward jump induces a nonmonotone relationship between the innovation probability and the coupon rate and that the welfare loss exhibits an upward jump as c crosses c^*. Panel (a) also highlights that, for values of the coupon rate close to the interest free rate, investment, and therefore also the innovation rate, are higher under financial constraints compared to the scenario with internal financing. Intuitively, this is due to the fact that in the presence of cash constraints, and hence a positive default probability, the expected costs covered by the firm are smaller than x, which increases the incentives to choose a larger investment. This effect is countervailed by the fact that, if the firm does not go bankrupt, marginal costs of investment are larger under external than under internal financing, but this second effect is dominated for small c. It should be noted that, if the private and the social value of the innovation coincide,

[5] In our numerical illustration we use the default parameter setting: $\alpha = 0.02, V = 14, S = 0, w = 2.1, c = 0.0525, r = 0.05,$.

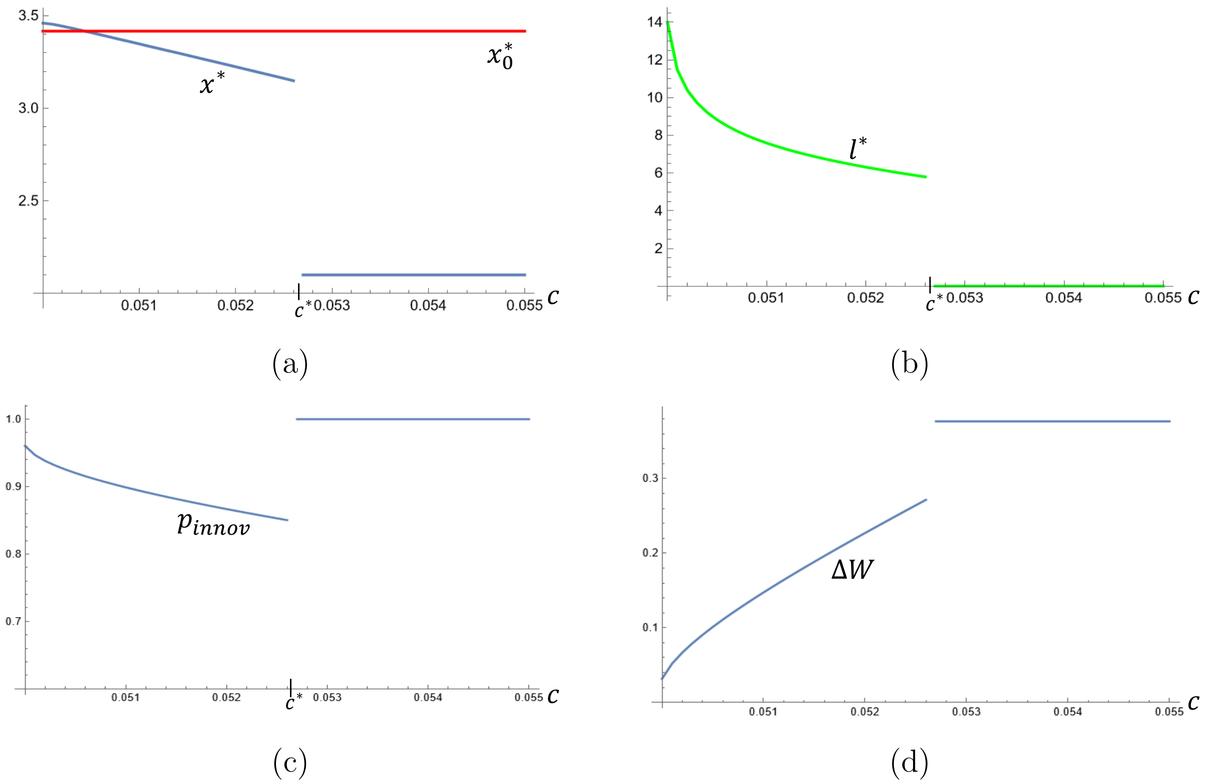


Figure 3: (a) Optimal investment x^*, (b) optimal size of the loan l^*, (c) innovation probability and (d) welfare loss for varying coupon rates c.

i.e., $S = 0$, the optimal investment of the firm is also higher than the socially optimal level (which coincides with x_0^* in this case). For larger values of S, however, the socially optimal investment always exceeds the optimal firm level (see Figure 8 in Appendix D). Finally, panel (b) shows that the optimal size of the loan, if a loan is used, exceeds investment by a factor of at least 2. Such a large loan size is optimal because it gives the firm a substantial liquidity buffer, which the firm needs to reduce the probability of becoming insolvent before the innovation has arrived.

Figure 4 shows the effect of the cash constraints for varying effectiveness of innovation investment. The deviation of investment from the unconstrained optimum is smaller for projects with low effectiveness of innovation investment, and for projects with very small values of α investment is even above the level with internal financing. The intuition for this effect is similar to that for the effect of increasing c, which we discussed above. External financing has two countervailing effects on expected investment costs, a direct positive one, since $c > r$, and an indirect negative one, since in the case of bankruptcy the firm no longer pays the coupon rate. The importance of the second effect decreases with α, because the bankruptcy probability goes down as the effectiveness of innovation expenditures becomes larger. Hence, an increase in α has a positive effect on the expected investment cost. Although an increase in α also increases the marginal return on investment, Figure 4 illustrates that the cost effect dominates and x^* decreases with α in the range of α values where a loan is taken. If α is sufficiently large such that it is optimal for the firm to not use external financing, under our parametrization the value of α has no impact on the size of investment, since it is determined by the level of available liquidity w.

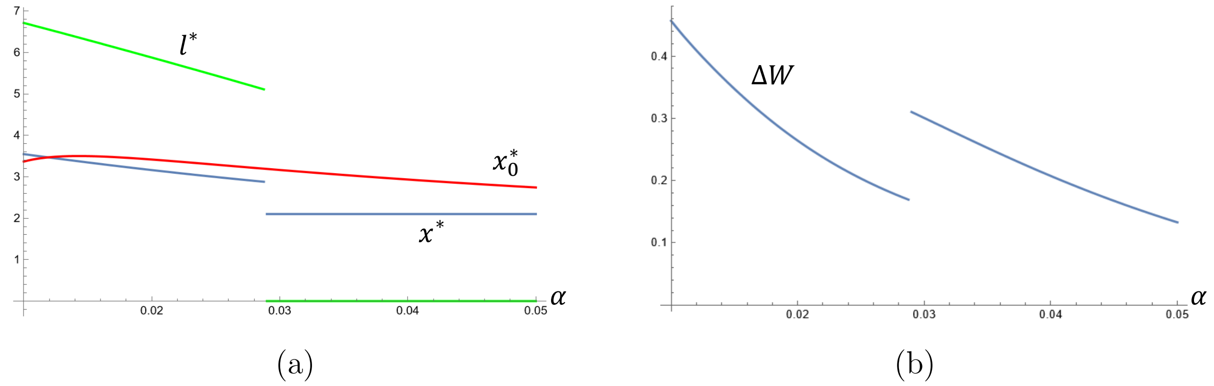


Figure 4: (a) Optimal investment x^* and optimal loan size $l^*(x)$ as well as (b) induced welfare loss for varying effectiveness of innovation investment (α).

The welfare loss induced by cash constraints is the highest for high-risk innovations, i.e., small values of α. This is due to the large probability that the firm has to default before the innovation is reached.

Keeping V constant while α is increased implies that the expected value of the innovation project increases with α. To analyze how the type of innovation, radical versus incremental, as such influences the effect of financial constraints, we now consider a simultaneous variation of α and V with the property that the optimal expected welfare with internal financing, W^* is constant across this variation. We refer to innovations with a small success rate (α), but high value (V) as radical, whereas innovations with large α and small V are labeled as incremental.

Figure 5 shows how optimal investment and welfare loss are affected if innovations become more incremental (i.e., α increases). The deviation of the optimal investment from the socially optimal level is small for very radical and for highly incremental innovations. The largest (downward) distortion of investment due to cash constraints occurs for intermediate types of innovations. The welfare loss induced by cash constraints is smaller the more incremental the innovation is, with the exception of values of α where making the innovation slightly more incremental triggers a transition from external to internal financing. In this region of the parameter space a more incremental innovation induces a larger welfare loss. Considering actual welfare under cash constraints (rather than welfare loss) shows that in this case the transition to a more incremental innovation actually reduces welfare (see Figure 9 in Appendix D).

6 Discussion and Conclusions

This study explores credit-financed innovation under uncertainty. A firm with limited cash reserve must decide on the volume and the mode of R&D investment. Choosing a larger volume increases the probability of success until any given point in time, but relying on external financing introduces bankruptcy risk. We show that optimal R&D investment exhibits a jump when the mode of financing switches from internal to (partly) external financing, which implies that lowering financing costs can sharply boost investment and innovation. If the firm faces very low financing costs, it might over-invest relative to the socially optimal level. The firm's optimal investment turns out to be the lower, however, the higher R&D effectiveness is. Distinguishing different types of innovations, we show that cash constraints reduce welfare

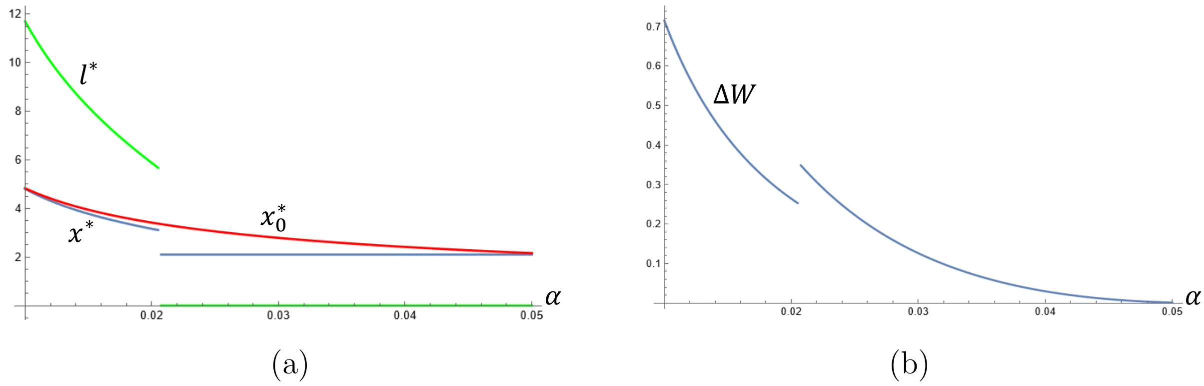


Figure 5: (a) Optimal investment x^* and optimal loan size $l^*(x)$ as well as (b) induced welfare loss for varying effectiveness of innovation investment (α) if V is adjusted such that the expected welfare under internal financing is fixed.

more strongly in case of radical than incremental innovations.

Our analysis provides guidance for the effectiveness of different policy instruments available to a social planer interested in maximizing welfare. Our results about the impact of a variation of the parameter c show that the size of the welfare gain induced by measures which reduce the cost of external financing of R&D depends crucially on the size of the coupon rate in the absence of a policy. Examples of measures reducing c are subsidies for external R&D financing, loan guarantees, or public financing schemes. Such policies are especially effective if the coupon rate without policy is marginally above the threshold that prevents the firm from using a loan to finance its investment.

For policy measures that induce an increase in the parameter α, such as R&D subsidies or tax shields for innovation investments as well as policies strengthening public institutions carrying out research that is relevant for the desired innovation, our analysis shows that in the presence of financial constraints such policies might have a detrimental effect. If due to such policy measures the value of α crosses the threshold above which the firm ceases to finance its innovation expenditures externally, then this results not only in a larger downward distortion of innovation expenditures, but also in a lower innovation rate, higher welfare loss and a decrease in welfare (see Appendix D). These insights call for a combination of policies increasing α with such reducing c, since a reduction in c pushes the threshold above which the firm finances internally upwards and therefore avoids the potential negative implications of an increase of the effectiveness of innovation investment α.

Appendix

A Auxiliary Results

Lemma 1. *Suppose the innovation rate $h(x)$ is concave. Then the profit function $U_0(x) = \frac{h(x)}{r+h(x)}V - x$ is strictly concave and has a unique maximizer x_0^*. If*

$$h'(0) \le \frac{r}{V},$$

then $x_0^* = 0$; otherwise, x_0^* is in $(0, V)$ and the unique solution of the first-order condition

$$\frac{h'(x)rV}{(r+h(x))^2} = 1.$$

Proof. Note that the derivative of the profit function is

$$U_0'(x) = \frac{h'(x)rV}{(r+h(x))^2} - 1.$$

By assumption, h is strictly increasing. This implies that the denominator on the right-hand side is strictly increasing and that the numerator is strictly positive. If h is concave, however, the numerator is nonincreasing, so that $U_0'(x)$ is strictly decreasing. It follows that $U_0(x)$ is strictly concave and that its unique maximizer is $x_0^* = 0$ if $U_0'(0) \le 0$, which is equivalent to $h'(0) \le r/V$ by $h(0) = 0$. Thus, suppose $U_0'(0) > 0$. Since $h(x)$ is strictly increasing from $h(0) = 0$, we have $U_0(0) = 0 > U_0(x)$ for any $x > V$. By strict concavity and $U_0'(0) > 0$ it follows that $U_0(x)$ must have a unique maximizer x_0^* in $(0, V)$, which necessarily solves the first-order condition $U_0'(x) = 0$. □

B The limit case $c = r$

In the main text, we assumed that the coupon rate c is higher than the risk-free interest rate r, because the firm must default with positive probability if it cannot finance the investment fully internally. In this section, we consider the limit case $c = r$, because it represents a further benchmark and because it is useful for proving and applying our main qualitative results, Propositions 3 and 4.

Without surprise, $c = r$ is a degenerate case. Then the firm can take excessive loans at no cost. Thus, it turns out that the only requirement for an optimal loan size is to be big enough.

Proposition 5. *Suppose $c = r$. Then Proposition 1 holds unchanged and simplifies as follows. Whenever $x \le w$,*

$$U(l, x) = E_x e^{-r\tau} V + w - x. \tag{12}$$

Otherwise, whenever $x > w$,

$$U(l, x) = E_x \left[\mathbf{1}_{\{\tau \le \min(T_0(l,x), T_1(l,x))\}} \underbrace{\left(e^{-r\tau} V + w - x \right)}_{=:\pi} \right], \tag{13}$$

where

$$T_0(l,x) = \frac{1}{r}\ln\left(\frac{l}{x-w}\right) \quad \textit{and} \quad T_1(l,x) = \frac{1}{r}\ln\left(\frac{V}{x-w}\right). \tag{14}$$

Further, as in Theorem 1, an optimal plan (l^*, x^*) exists, and optimal investment still has to belong to one of the following two classes, but the optimal loan size is not unique anymore.

1. $x \le w$. *Then $U(l,x)$ is constant in l by* (12).

2. $w < x \le V$. *For any such x, the optimal loan sizes are all $l \ge V$. Then $T_0(l,x) \ge T_1(l,x)$ and*

$$\begin{aligned} U(l,x) &= E_x\left[\mathbf{1}_{\{\tau \le T_1(l,x)\}}\left(e^{-r\tau}V + w - x\right)\right] \\ &= E_x\left[\max\left\{e^{-r\tau}V + w - x, 0\right\}\right] \\ &= \frac{h(x)}{r+h(x)}V - (x-w) + \frac{r}{r+h(x)}V\left(\frac{x-w}{V}\right)^{\frac{r+h(x)}{r}}. \end{aligned} \tag{15}$$

The right-hand side of (15) is the sum of the value of financially unconstrained investment and the value of defaulting on the loan (of any size $l \ge V$). It can be used for checking whether the critical threshold c^* from Proposition 3 exceeds r, which is the case if and only if the maximum of the whole sum for $w \le x \le V$ is greater than the maximum of the first summand—which is equal to the right-hand side of (12)—for $x \le w$.

Proof. The proof of Proposition 1 applies identically for $c = r$, and the simplified representation resulting from $c = r$ is immediate. From the proof of Theorem 1, also the proofs of Lemmas 2 and 3 apply identically for $c = r$. This yields the bound $x \le V$ for optimal investment and implies existence of an optimal plan (l^*, x^*). The two candidate classes for optimal investment still follow from the upper bound, but the characterization of the optimal loan sizes now obtains from (12)–(14). This is immediate for $x \le w$. For $w < x \le V$, it is also clear from (13) and (14) that $U(l,x)$ is constant in all $l \ge V$, because then $T_0(l,x) \ge T_1(l,x)$ and the latter is constant in l. Feasibility requires $l \ge x - w$. On the interval $[x-w, V]$, $T_0(l,x)$ is strictly increasing in l from zero to $T_1(l,x)$, which implies by (13) that also $U(l,x)$ is strictly increasing in l, because the term denoted by π is strictly positive for all $\tau < T_1(l,x)$ and the distribution of τ has full support on $[0,\infty)$. It follows that the optimal loan size is any $l \ge V$. Then (13) and (14) immediately imply (15). □

C Proofs

Proof of Proposition 1. The firm's liquidity left in the cash account after taking a loan of size $l \ge 0$ and making an investment of size $x \ge 0$ is $w_0 = w + l - x$, which is required to be nonnegative by assumption; cf. (1). Paying the coupon cl from the cash account and capitalizing the interest earned at rate r, the liquidity w_t evolves according to $dw_t = (rw_t - cl)\,dt$. Thus, absent the innovation, we have

$$w_t = e^{rt}(w+l-x) - \int_0^t e^{r(t-s)}cl\,ds = e^{rt}\left(w - \left(\frac{c}{r}-1\right)l - x\right) + \frac{cl}{r}. \tag{16}$$

The coupon can be fully financed from the cash reserve if w_t given by (16) never becomes negative, which is if and only if $w \geq \left(\frac{c}{r} - 1\right) l + x$, which is condition (4). Then we have $T = \infty$ independently of τ. Plugging this into (2), the definition of $U(l, x)$, immediately yields identity (5).

Now suppose instead $w < \left(\frac{c}{r} - 1\right) l + x$, which is condition (6). Then w_t given by (16) turns negative at some $t \in [0, \infty)$, precisely at $t = T_0(l, x)$ defined in (2). This means the firm defaults at time $T = T_0(l, x) < \infty$ if $\tau > T_0(l, x)$.

Thus, suppose $\tau \leq T_0(l, x)$. Then the innovation generates additional liquidity with net present value V at time τ. Together with the cash left in the savings account given by (16) for $t = \tau$, this is enough to finance the remaining coupon payments if and only if

$$e^{r\tau} \left(w - \left(\frac{c}{r} - 1\right) l - x\right) + \frac{cl}{r} + V \geq \frac{cl}{r}, \tag{17}$$

which is if and only if $\tau \leq T_1(l, x)$ as defined in (3). Thus, if $\tau \leq \min(T_0(l, x), T_1(l, x))$, we have $T = \infty$, but if $T_1(l, x) < \tau \leq T_0(l, x)$, the firm still runs out of cash at a finite time t. Note that in the latter case, the precise time of default is

$$T = \frac{1}{r} \ln \left(\frac{cl}{(c - r)l + rx - rw - re^{-r\tau} V} \right), \tag{18}$$

which is strictly greater than $T_0(l, x)$, because the latter is finite and adding $V > 0$ at time $\tau \leq T_0(l, x)$ strictly increases the cash balance. This completes the characterization of T.

It remains to verify the identity (7), given that condition (6) holds. Therefore, consider the definition of $U(l, x)$ given in (2). We now apply the already obtained characterization of T. First, note that we have $\tau \leq T$ if and only if $\tau \leq T_0(l, x)$, because $\tau > T_0(l, x)$ implies $T = T_0(l, x)$, whereas $\tau \leq T_0(l, x)$ implies $T > T_0(l, x)$. Next, to consider the different cases for T also in the integral in (2), let T_2 denote the right-hand side of (18). Then

$$\begin{aligned} U(l, x) = E_x \Bigg[& \mathbf{1}_{\{\tau \leq T_0(l,x)\}} e^{-r\tau} V - \mathbf{1}_{\{\tau > T_0(l,x)\}} \int_0^{T_0(l,x)} e^{-rs} cl \, ds \\ & - \mathbf{1}_{\{T_1(l,x) < \tau \leq T_0(l,x)\}} \int_0^{T_2} e^{-rs} cl \, ds \\ & - \mathbf{1}_{\{\tau \leq \min(T_0(l,x), T_1(l,x))\}} \int_0^{\infty} e^{-rs} cl \, ds \Bigg] + w + l - x. \end{aligned}$$

By definition of $T_0(l, x)$, the value of the first integral is $w + l - x$; similarly, the second integral is equal to $w + l - x + e^{-r\tau} V$; and the third integral is simply cl/r. This yields (7). □

C.1 Proofs of Theorem 1 and Proposition 2

Proof of Theorem 1. The proof is split up into several lemmas. First, we establish some upper bounds for the optimal loan and investment sizes, which we can then impose.

Lemma 2. *Optimal credit-investment plans have to satisfy*

$$\left(\frac{c}{r} - 1\right) l + x \leq V, \tag{19}$$

which, by (1)*, further implies*

$$x \le w + \frac{r}{c}(V - w).$$

Proof. Note that refraining from any investment yields $U(0,0) = w$ by $h(0) = 0$. Now consider any pair (l, x) violating (19), so $(c/r-1)l + x > V$. In case (4) holds, (5) and $V > 0$ together imply

$$U(l, x) \le V + w - (c/r - 1)l - x < w,$$

so (l, x) is suboptimal in this case. In case (6) holds, (7) together with $r > 0$, $V > 0$, and $w > 0$ implies

$$U(l, x) \le E_x[\mathbf{1}_{\{\tau \le \min(T_0(l,x), T_1(l,x))\}}(V - (c/r - 1)l - x)] + w \le w,$$

where equality would hold throughout only if $\tau = 0$ with probability one, which we cannot have for any x. Thus, also in this case (l, x) is suboptimal. This proves (19). Combining it with (1), which is equivalent to $l \ge x - w$, yields the second claimed inequality. $\square$

Now existence of an optimal plan (l^*, x^*) follows from the next lemma.

Lemma 3. *$U(l, x)$ is upper semicontinuous (usc).*

Proof. Using the density function $f_x(t) = h(x)e^{-h(x)t}$ of τ, (5) implies

$$U(l, x) = \frac{h(x)}{r + h(x)}V + w - \left(\frac{c}{r} - 1\right) l - x \tag{20}$$

in case $w \ge (c/r - 1)l + x$ and (7) implies

$$\begin{aligned} U(l, x) = \quad & \frac{h(x)}{r + h(x)}V \left(1 - e^{-(r+h(x))\min(T_0(l,x),T_1(l,x))}\right) \\ & + \left(w - \left(\frac{c}{r} - 1\right) l - x\right) \left(1 - e^{-h(x)\min(T_0(l,x),T_1(l,x))}\right) \end{aligned} \tag{21}$$

in case $w < (c/r - 1)l + x$. Both functions on the right-hand sides of (20) and (21) are continuous on the (sub-)domain given by the respective case. Therefore, to show that $U(l, x)$ is usc, it is enough to verify that $\limsup U(l_n, x_n) \le U(l, x)$ for any sequence (l_n, x_n) with limit (l, x) such that $w < (c/r - 1)l_n + x_n$ but, in the limit, $w = (c/r - 1)l + x$. By (7) and $V > 0$ then

$$U(l_n, x_n) \le E\left[e^{-r\tau}V + \mathbf{1}_{\{\tau \le \min(T_0(l,x),T_1(l,x))\}}\left(w - \left(\frac{c}{r} - 1\right) l_n - x_n\right)\right].$$

Here, the right-hand side converges to $E[e^{-r\tau}V] = h(x)V/(r + h(x))$, because the term $w - (c/r - 1)l_n - x_n$ is not random and vanishes in the limit. By (20), we also have $U(l, x) = h(x)V/(r + h(x))$ at the limit (l, x), which now implies upper semicontinuity. $\square$

Next, to characterize the optimal loan size $l^*(x)$ for any relevant x, consider first the case $x \le w$.

Lemma 4. *For any $x \le w$, the optimal loan size is $l^*(x) = 0$.*

Proof. If $x \leq w$, then (1) holds for any $l \geq 0$. In particular, $l = 0$ is feasible. From (5), it is also clear that $l = 0$ is uniquely optimal among all $l \geq 0$ satisfying (4), because the distribution of τ is fixed by x alone, and the cost of the loan is linear with slope $c/r - 1 > 0$.

Now consider any $l > 0$ such that (6) holds. The cost for such a loan, as implied by (7), is less than linear, because the firm defaults on the coupon payment if the innovation does not occur early enough. Nevertheless, we now have

$$U(l, x) \leq E_x \left[\mathbf{1}_{\{\tau \leq \min(T_0(l,x), T_1(l,x))\}} \left(e^{-r\tau} V + w - x\right)\right] < E_x \left[e^{-r\tau} V + w - x\right] = U(0, x),$$

where the first inequality is implied by (7) and $c > r$, and the second one follows from $V > 0$, $x \leq w$, and finiteness of $T_0(l, x)$ given that (6) holds. □

Equation (8) simply follows from Proposition 1 and applying the density function $f_x(t) = h(x)e^{-h(x)t}$ of τ.

By Lemma 2, it only remains to consider the case $w < x \leq w + (r/c)(V - w)$.

Lemma 5. *For any x such that $w < x \leq w + (r/c)(V - w)$, the optimal loan size $l^*(x)$ is the unique l in $(w - x, (r/c)V)$ that solves the first-order condition*

$$h(x)e^{-h(x)T_0(l,x)} \left(V - \frac{cl}{r}\right) \frac{x - w}{cl^2} - \left(1 - e^{-h(x)T_0(l,x)}\right) \left(\frac{c}{r} - 1\right) = 0. \tag{9}$$

Proof. First note that the given upper bound on x and $w > 0$ together imply

$$x - w < (r/c)V. \tag{22}$$

Further, $x > w$ and $c > r$ together imply that (6) holds for any $l \geq 0$, in particular for any feasible l satisfying (1). This means that $U(l, x)$ satisfies (7).

We first consider all $l \geq (r/c)V$; these are feasible by (22). We are going to argue that $l = (r/c)V$ is in fact better than any $l > (r/c)V$. Therefore, consider $l = (r/c)V$. Then $T_0(l, x) = T_1(l, x)$ by their definition; see (2) and (3). Moreover, (22) implies that $T_0(l, x) > 0$ for $l = (r/c)V$. Hence, we have $T_0(l, x) = T_1(l, x) > 0$ in (7), so $\min(T_0(l, x), T_1(l, x)) = T_1(l, x)$ and the event $\{\tau < T_1(l, x)\}$ has positive probability. Note that also the term denoted by π is strictly positive whenever $\tau < T_1(l, x)$ (and $\pi = 0$ if $\tau = T_1(l, x)$).

Now, if we increase l, then both $T_1(l, x)$ and π decrease by $c > r$. It follows that the argument of the expectation in (7), which is never negative, cannot increase, and it actually decreases with positive probability. Thus, also $U(l, x)$ decreases, which shows that $l = (r/c)V$ was better.

Next, we consider all feasible $l \leq (r/c)V$. Then $T_0(l, x) \leq T_1(l, x)$ by their definition and $c > 0$. Since $T_0(l, x)$ is increasing in l whenever $x > w$, we cannot repeat the previous argument. Instead, we analyze the derivative of $U(l, x)$ with respect to l. Therefore, note that applying the inequality $T_0(l, x) \leq T_1(l, x)$ and the density function $f_x(t) = h(x)e^{-h(x)t}$ of τ in (7) yields

$$U(l, x) = \frac{h(x)}{r + h(x)} V \left(1 - e^{-(r+h(x))T_0(l,x)}\right) + \left(w - \left(\frac{c}{r} - 1\right) l - x\right) \left(1 - e^{-h(x)T_0(l,x)}\right). \tag{10}$$

Then, using also the definition of $T_0(l, x)$, we obtain[6]

$$\frac{\partial U(l, x)}{\partial l} = h(x)e^{-h(x)T_0(l,x)} \left(V - \frac{cl}{r}\right) \frac{x - w}{cl^2} - \left(1 - e^{-h(x)T_0(l,x)}\right) \left(\frac{c}{r} - 1\right). \tag{23}$$

[6] In particular, we have $(c/r - 1)l + x - w = e^{-rT_0(l,x)} cl/r$, which further implies $e^{-rT_0(l,x)} \partial T_0(l, x)/\partial l = (x - w)/(cl^2)$.

Here, since $T_0(l,x)$ is strictly increasing in l given $x > w$, and $c > r$, the right-hand side is strictly decreasing in $l \leq (r/c)V$. It becomes strictly positive if l approaches $x - w$, because then $T_0(l,x)$ vanishes and $V > cl/r$ by (22). For $l = (r/c)V$, the right-hand side of (23) is strictly negative by $c > r$. In consequence, there exists a unique l in $(x - w, (r/c)V)$ such that the right-hand side of (23) is zero, i.e., (9) holds. This is the unique optimal l. $\square$

Since equation (10) has been established along the proof of Lemma 5, the proof of Theorem 1 is now complete. $\square$

Proof of Proposition 2. Since $l^*(x) = 0$ for any $x \leq w$, we need to show that $l^*(x)$ vanishes as $x \searrow w$. By $c > r$ and $V > w$, Theorem 1 implies that $l^*(x) < (r/c)V$ for any small enough $x > w$. Now fix any $l \in (0, (r/c)V)$. By (1), l is feasible for any $x \leq w + l$, so in particular for any $x \in (w, w + l]$. Then the partial derivative $\partial U(l,x)/\partial l$ satisfies (23). Keeping l fixed, the derivative becomes strictly negative if we let x approach w, because then $T_0(l,x)$ tends to $(1/r)\ln(c/c - r) > 0$. This means that l is too big to be optimal for any x close to w, because we showed that $\partial U/\partial l$ is strictly decreasing in $l \in [x - w, (r/c)V]$. Thus, since l can be chosen arbitrarily close to zero, $l^*(x)$ must vanish as $x \searrow w$.

To prove that $U(l^*(x), x)$ is discontinuous at $x = w$, consider any sequence (x_n) such that $x_n > w$ but $\lim x_n = x = w$. Combine this with any sequence (l_n) such that l_n is feasible, i.e., $l_n \geq x_n - w$; cf. (1). Then $U(l_n, x_n)$ satisfies (7), because $(c/r - 1)l_n + x_n > w$; cf. (6). It follows that

$$U(l_n, x_n) \leq E_x \left[\mathbf{1}_{\{\tau \leq T_0(l_n, x_n)\}} e^{-r\tau} V \right] = \frac{h(x_n)}{r + h(x_n)} V \left(1 - e^{-(r + h(x_n)) T_0(l_n, x_n)} \right).$$

However, $x_n > w$ ensures that $T_0(l_n, x_n)$ is bounded by $(1/r)\ln(c/(c - r))$, which is finite by $c > r$. This implies the discontinuity $\limsup U(l_n, x_n) < h(x)V/(r + h(x)) = U(0, x)$ (since $x = w$). $\square$

C.2 Proofs of Propositions 3 and 4

In the proofs of Propositions 3 and 4, we will appeal to the monotonicity and continuity of $U(l, x; c)$ in c, which is also (partly) inherited by $U^*(c)$. We are going to establish these properties first. Further, it will be convenient to explicitly include the limit case $c = r$, which is discussed in Section B. In particular, we may apply Proposition 1 also for $c = r$ (see Proposition 5).

Lemma 6. *For any fixed feasible pair (l, x), $U(l, x; c)$ is nonincreasing in $c \geq r$.*

Proof. Fix any feasible pair (l, x). For any sufficiently small c such that (4) holds, $U(l, x; c)$ satisfies (5), where the right-hand side is clearly nonincreasing in c. For all bigger c, $U(l, x; c)$ satisfies (7). To see that also there the right-hand side is nonincreasing in c, note that the term denoted by π is nonincreasing and nonnegative for all $\tau \leq T_1(l, x)$. Further, both $T_0(l, x)$ and $T_1(l, x)$ are nonincreasing in c for any feasible l, so the same applies to the indicator function. Since the distribution of τ does not depend on c, it follows that the expectation is indeed nonincreasing in c. Finally, note that the right-hand side of (5) is at least equal to $E_x[e^{-r\tau}V]$ for any c sufficiently small such that (4) holds, whereas the right-hand side of (7) is at most equal to $E_x[e^{-r\tau}V]$ for any bigger c such that (6) holds, because $V \geq 0$. The proof is now complete. $\square$

Lemma 7. *For any fixed feasible pair (l, x), $U(l, x; c)$ is continuous in $c \geq r$.*

Proof. Fix any feasible pair (l, x) and any $c_0 \geq r$. If $w > (c_0/r - 1)l + x$, then (4) holds for all c in a neighborhood of c_0 and, thus, $U(l, x; c)$ is continuous in $c = c_0$ by (5). The same argument applies if $l = 0$, because then $x \leq w$ by (1). Next, suppose $w < (c_0/r - 1)l + x$. Then (6) holds for all c in a neighborhood of c_0 and, thus, it is enough to show that the right-hand side of (7) is continuous in $c = c_0$ by dominated convergence. Therefore, note that the term denoted by π is nonnegative for all $\tau \leq T_1(l, x)$, bounded from above by $V + w - x$, and continuous in c. Further, both $T_0(l, x)$ and $T_1(l, x)$ are continuous in $c = c_0$ by $w < (c_0/r - 1)l + x$, $r > 0$, and (1). It follows that also the indicator function is continuous in $c = c_0$ except if $\tau = \min(T_0(l, x), T_1(l, x))$ for $c = c_0$. But the latter event has probability zero, so also the expectation on the right-hand side of (7) is continuous in $c = c_0$. Finally, suppose $w = (c_0/r - 1)l + x$ and $l > 0$. Then, by (5), $U(l, x; c)$ is left-continuous in $c = c_0$ and $U(l, x; c_0) = E_x e^{-r\tau} V$. The right-hand limit is determined by the right-hand side of (7). There, as $c \to c_0$, π converges boundedly again, now to $e^{-r\tau} V$, whereas both $T_0(l, x)$ and $T_1(l, x)$ diverge to ∞. Since τ is finite with probability one, this shows that $U(l, x; c)$ is also right-continuous in $c = c_0$. □

Proof of Proposition 3. By definition, $U^*(c) \geq U(0, \bar{x}^*)$. We will show that there is some finite c^* such that $U^*(c) > U(0, \bar{x}^*)$ if and only if $c < c^*$. Then it follows that optimal investment for any $c < c^*$ must exceed w, because by Lemma 4 (for $c > r$) and Proposition 5 (for $c = r$) no feasible plan (l, x) with $x \leq w$ yields a payoff greater than $U(0, \bar{x}^*)$.

As the first step, note that $U^*(c)$ is nonincreasing in c, because this is the case for each of the functions $U(l, x; c)$ by Lemma 6. If $U^*(r) \leq U(0, \bar{x}^*)$, it follows that $c^* = r$ is a suitable threshold.

Thus, suppose from now on $U^*(r) > U(0, \bar{x}^*)$ and let c^* be the supremum of all $c \geq r$ such that still $U^*(c) > U(0, \bar{x}^*)$. We show next that c^* is finite and $U^*(c^*) = U(0, \bar{x}^*)$.

Concerning finiteness of c^*, note that $U(0, \bar{x}^*) \geq U(0, 0) = w > 0$. Further, in the proof of Lemma 2 we showed that $U(l, x) < w$ for any plan violating (19), so in particular for any plan with $x > V$. Therefore, it is enough to show that there is an upper bound for $\{U(l, x; c) \mid V \geq x > w \text{ and } w + l - x \geq 0\}$ that converges to zero as $c \to \infty$. Hence, consider any feasible plan with $V \geq x > w$ and any $c > r$. Since $x > w$, $U(l, x; c)$ satisfies (7) and there the term denoted by π is less than V. Thus,

$$U(l, x; c) < E_x \left[\mathbf{1}_{\{\tau \leq \min(T_0(l,x), T_1(l,x))\}} \right] V \leq E_x \left[\mathbf{1}_{\{\tau \leq T_0(l,x)\}} \right] V.$$

The definition of $T_0(l, x)$ in (2), $x > w$, and $l \geq x - w > 0$ imply

$$T_0(l, x) < \frac{1}{r} \ln \left(\frac{cl}{(c - r)l} \right) = \frac{1}{r} \ln \left(\frac{c}{c - r} \right),$$

so we further have

$$U(l, x; c) < F_x \left(\frac{1}{r} \ln \left(\frac{c}{c - r} \right) \right) V \leq F_V \left(\frac{1}{r} \ln \left(\frac{c}{c - r} \right) \right) V,$$

where the last inequality follows from $V \geq x$ and the fact that the arrival rate $h(x)$ is increasing in x. The last bound on $U(l, x; c)$ does not depend on (l, x) anymore and indeed vanishes as $c \to \infty$, because $F_V(0) = 0$. This shows that c^* is finite.

To see that $U^*(c^*) = U(0, \bar{x}^*)$, note that $U^*(c) \leq U(0, \bar{x}^*)$ for all $c > c^*$ by construction of c^* and that $U^*(c)$ is lower semicontinuous, because it is the pointwise supremum of a family of functions $U(l, x; c)$ indexed by (l, x) that are themselves (lower semi-)continuous in c by Lemma 7.

Finally, assume h is concave. Then $U_0(x)$ is strictly concave by Lemma 1. Hence, since $U(0, x) = U_0(x) + w$, it follows that $\bar{x}^* = \min\{x_0^*, w\}$. □

Proof of Proposition 4. Recall that, for any given $c \geq r$, there exists an optimal plan (l^*, x^*) by Theorem 1 (see Proposition 5 for $c = r$). If $c < c^*$, then necessarily $x^* > w$ by Proposition 3.

To argue that x^* stays even bounded away from w for all $c \in [r, c^*)$, we divide the latter interval into two parts. Therefore, fix any $c' \in (r, c^*)$. First, consider any sequence (c_n) from $[r, c']$ as well as two sequences (x_n) and (l_n) such that (l_n, x_n) is an optimal plan for $c = c_n$. In consequence of Lemma 6, $U^*(c)$ is nonincreasing in c. Thus, by $c_n \leq c' < c^*$,

$$U(l_n, x_n; c_n) = U^*(c_n) \geq U^*(c') > U(0, \bar{x}^*). \tag{24}$$

In particular, we have $x_n > w$ for all n. By way of contradiction, however, assume $\liminf x_n = w$. Passing to a subsequence if necessary, it is no loss to assume in fact $\lim x_n = w$.

By Lemma 6, $U(l_n, x_n; c)$ is nonincreasing in c. Further, for $c = r$ and each x_n, $l = V$ is optimal by Proposition 5, because Theorem 1 implies $x_n \leq w + (r/c)(V - w) \leq V$. Thus, we have

$$U(l_n, x_n; c_n) \leq U(l_n, x_n; r) \leq U(V, x_n; r)$$

for all n. Since $U(l, x)$ is upper semicontinuous by Lemma 3, and $x_n \to w$, it follows that

$$\limsup U(l_n, x_n; c_n) \leq U(V, w; r).$$

Finally, since $l = 0$ is optimal for any $x \leq w$ (by Proposition 5 for $c = r$), we have

$$U(V, w; r) \leq U(0, \bar{x}^*),$$

which together with the previous inequality contradicts (24). This shows that $\liminf x_n > w$.

Next, let (c_n) be any sequence from $[c', c^*)$. As before, assume (l_n, x_n) is an optimal plan for $c = c_n$. Then $c_n < c^*$ still implies $x_n > w$ and $U(l_n, x_n; c_n) > U(0, \bar{x}^*)$. By way of contradiction, assume again $\liminf x_n = w$ and hence without loss that $\lim x_n = w$.

By $x_n > w$ and the feasibility constraint (1), we have $l_n > 0$. Moreover, $U(l_n, x_n; c_n)$ must satisfy (7). It follows that

$$U(l_n, x_n; c_n) \leq E_{x_n}\left[\mathbf{1}_{\{\tau \leq T_0(l_n, x_n)\}} e^{-r\tau} V\right] = \frac{h(x_n)}{r + h(x_n)} V \left(1 - e^{-(r+h(x_n))T_0(l_n, x_n)}\right),$$

where $T_0(l_n, x_n)$ needs to be evaluated for $c = c_n$, see (2). Note that, for any $x > w$ and $l > 0$, $T_0(l, x)$ is bounded by $(1/r)\ln(c/(c - r))$, which is decreasing in c for any $r > 0$. Thus, since $c_n \geq c' > r > 0$, the lim sup of $T_0(l_n, x_n)$ for $c = c_n$ is bounded by $(1/r)\ln(c'/(c' - r)) < \infty$ as $n \to \infty$. Together with the continuity of h, this implies

$$\limsup U(l_n, x_n; c_n) < h(w)V/(r + h(w)) = U(0, w) \leq U(0, \bar{x}^*),$$

which contradicts that $U(l_n, x_n; c_n) > U(0, \bar{x}^*)$ for all n. This shows again that in fact $\liminf x_n > w$. □

D Additional Figures

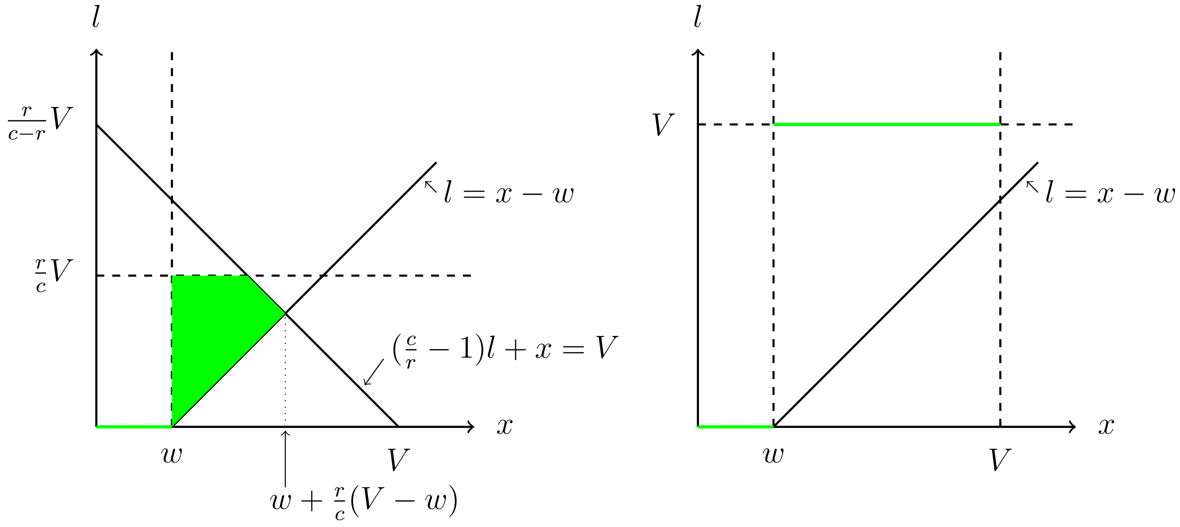


Figure 6: Relevant pairs (l, x) (colored) for $c > r$ (left) and $c = r$ (right).

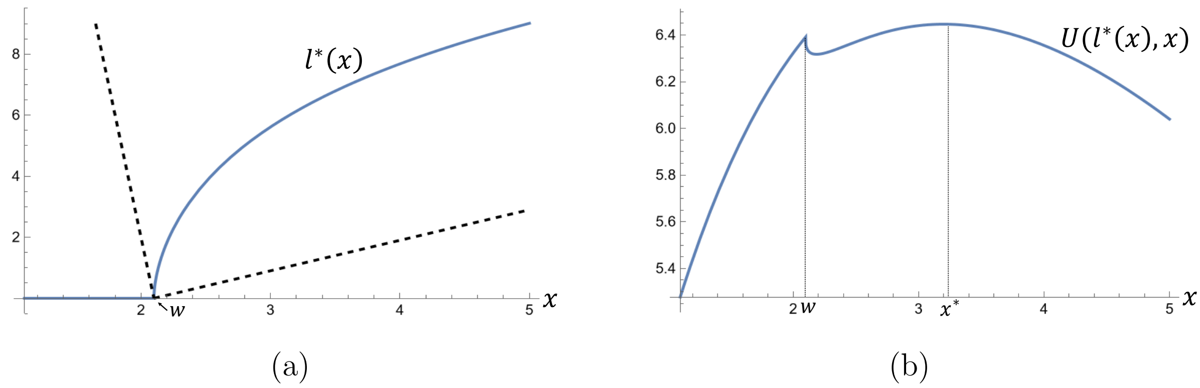


Figure 7: Illustration of (a) the optimal loan size $l^*(x)$ and (b) the firm profit under the optimal loan size $U(l^*(x), x)$ for a value of the coupon rates c just below the threshold c^* ($c = 0.0522 < c^* = 0.05265$). The dashed black lines indicate the boundaries between the regions of no bankruptcy risk, bankruptcy risk and infeasible investment (see Figure 2).

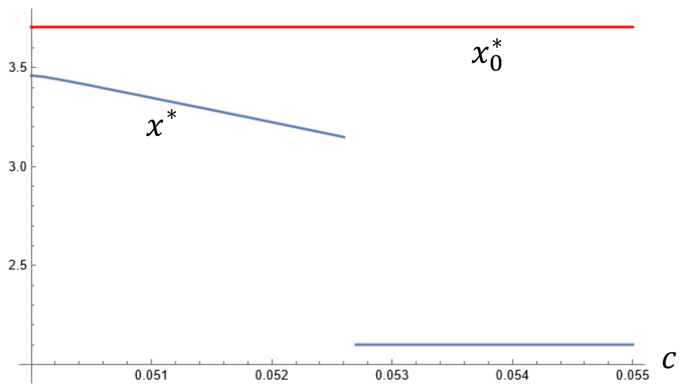
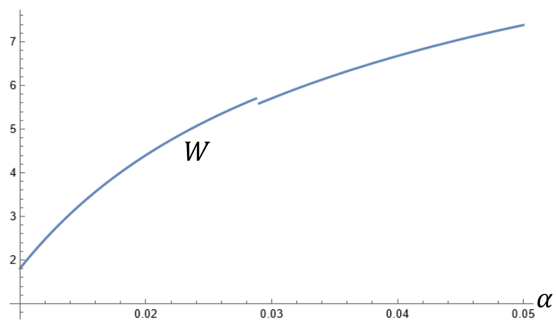
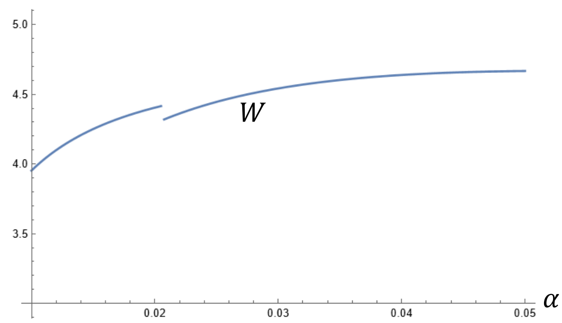


Figure 8: Optimal innovation investment of the firm (x^*) and socially optimal investment (x_0^*) if the social value of the innovation is 10% higher than the private value ($S = 0.1V$). The figure illustrates that if the social value exceeds the private value of innovation firm investments are below the socially optimal level for all values of c.



(a)



(b)

Figure 9: Welfare under optimal firm behavior for increasing effectiveness of innovation investment (α) and constant value of V (a) as well as V adjusted such that the expected welfare under internal financing is fixed (b).

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