IMPLICIT SCHEME FOR SEMILINEAR SPDEs WITH ADDITIVE NOISE UNDER NON-GLOBAL LIPSCHITZ AND IRREGULAR DRIFT

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ABSTRACT. We consider a mixed finite element method combined with the implicit Euler scheme to approximate second-order semilinear parabolic stochastic partial differential equations (SPDEs) with additive noise. The nonlinearity is of Nemytskii type, satisfies a one-sided Lipschitz condition, exhibits polynomial growth, and includes irregular components. Such SPDEs serve as suitable models for various phenomena, including advection-reaction-diffusion processes. We prove the strong convergence of the fully discrete scheme to the mild solution, achieving convergence rate in time approximatively 1. We obtain a convergence rate in space that depends on the spatial dimension and the order of the polynomial growth of the nonlinearity. The analysis is challenging due to the irregularities of the nonlinear drift function and the absence of a global Lipschitz condition. Numerical experiments are provided to illustrate the theoretical results.

Keywords Stochastic partial differential equations. Finite element method. Implicit Euler method. Strong convergence. One-sided Lipschitz condition. Polynomial growth condition.

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1. Introduction

We consider a mixed finite element approximation of the following SPDE

$$dX(t) + AX(t)dt = F(X(t))dt + dW(t), \quad X(0) = X_0, \quad t \in (0, T], \tag{1}$$

which is defined in the Hilbert space $L^2(\Lambda)$, where $\Lambda \subset \mathbb{R}^d$ ($d = 1, 2, 3$) is bounded and has smooth boundary or is a convex polygon. The final time $T > 0$ is fixed. The unbounded linear operator A is not necessarily self-adjoint. The noise $W(t) = W(x,t)$ is a Q-Wiener process defined in the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\in[0,T]})$, where $Q : \mathcal{H} \longrightarrow \mathcal{H}$ is a positive linear self-adjoint operator. The filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ is assumed to be normal (see [36, Definition 2.1.11]) . The noise can be represented as follows (see e.g., [9, 36])

$$W(x,t) = \sum_{i=0}^{\infty} \sqrt{q_i} e_i(x) \beta_i(t), \quad t \in [0, T], \quad x \in \Lambda, \tag{2}$$

where the q_i and the e_i, $i \in \mathbb{N}^d$ are respectively the eigenvalues and the eigenfunctions of the covariance operator Q. In (2), $(\beta_i)_i$ are independent identically distributed (i.i.d.) standard Brownian motions. Equations of type (1) are used to model different real world phenomena (such as phase separation in multi-component alloy) in different fields such as biology, chemistry and physics (see e.g., [38, 13, 39]). In many situations, explicit solutions of SPDEs are unknown, therefore numerical methods are powerful tools to provide realistic approximations. Numerical approximation of SPDEs is therefore an active research area and has attracted a lot of attentions since two decades (see e.g., [30, 43, 45, 25, 19, 20, 46] and the references therein). The convergence analysis of many numerical schemes in the literature are for SPDEs with global Lipschitz drift function F. However, for many realistic problems, the nonlinear function F does not satisfy the global Lipschitz condition. A typical example is the stochastic Allen-Cahn equation with cubic nonlinearity (i.e., Equation (1) with $F(u) = u - u^3$). It is well known that the standard Euler-Maruyama method for stochastic differential equations (SDEs) with non-global Lipschitz drift diverges (cf. [17]). For SPDEs driven by non-global Lipschitz nonlinearity, exponential integrators and linear implicit method are proved to be divergent (cf. [2]). For such SDEs and SPDEs, implicit schemes were proved to converge to the mild solution (cf. [31, 22]). Recently, explicit tamed methods were proved to be efficient for such SDEs (see e.g., [18, 40]). This taming strategy is being extended to the case of SPDEs with non-global Lipschitz drift (see e.g., [3, 44]). Numerical approximations of SPDEs with non-global Lipschitz drift is currently a hot topic, see e.g., [5, 4, 6] for explicit schemes and [23, 22] for implicit schemes. However, almost all the above-mentioned references on implicit scheme for SPDEs with non-global Lipschitz drift are restricted to the stochastic Allen-Cahn equation. In this paper, we extend the strong convergence of the implicit Euler scheme to SPDEs with more general nonlinearities, see Assumption 2.3. We also consider the unbounded linear operator not to be necessarily self-adjoint.

Another interesting feature of a numerical scheme is its convergence rate. In all the above mentioned references as well as in many Euler-type schemes for SPDEs, the optimal convergence rate in time is $\frac{1}{2}$. For instance the optimal convergence rate in [5, 4] is $\frac{1}{2}$, providing that $\|A^{\frac{1}{2}}Q^{\frac{1}{2}}\|_{\mathcal{L}_2(\mathcal{H})} < \infty$ (which corresponds to Assumption 2.2 with $\beta = 2$). Note that the restriction of convergence rate to $\frac{1}{2}$ is due to the presence of terms like $\|F(X(s))-F(X(t_j))\|$ in the error analysis, which by the global Lipschitz condition and temporal regularity of the mild solution can be bounded as: $\|F(X(s)) - F(X(t_j))\| \leq C|s - t_j|^{\min\left(\frac{1}{2},\frac{\beta}{2}\right)}$, with β as in Assumption 2.2. This therefore leads to the optimal convergence rate $\frac{1}{2}$. There are recent results on Euler-type methods overcoming the barrier rate $\frac{1}{2}$ (see e.g., [21, 20, 43, 44, 45, 15, 37] and the references therein). The main strategy there to overcome the barrier rate $\frac{1}{2}$ consists on applying Taylor's formula of order 2 to $F(X(s)) - F(X(t_j))$. Therefore, one needs to require the drift function F to be twice differentiable. This excludes many nonlinearities, such as $F(u) = \frac{u}{1+|u|} + u - u^5$, $u \in \mathcal{H}$, which is only once differentiable. Such approach is no longer applicable when the nonlinear drift function is only once differentiable. In this paper, we prove that when F satisfies the one-sided Lipschitz condition, is polynomially growing and is only once differentiable, the implicit Euler method converges strongly to the mild solution with rate exceeding $\frac{1}{2}$. More precisely, we prove that the fully discrete scheme achieves convergence rate $\mathcal{O}(h^{\beta} + h^{\beta+c_1-\frac{dc_1}{2}}|\ln(h)|^{\nu} + \Delta t^{\frac{\beta}{2}-\xi})$, where $\nu, \xi > 0$ are arbitrarily small real numbers and the parameter $c_1 \geq 0$ is such that $c_1 + 1$ is the order of the polynomial growth of the nonlinearity, see Assumption 2.3. In particular, for $\beta = 2$ we achieve a convergence rate $1-\xi$ in time. The key idea is that, instead of applying Taylor's formula of order 2, we only apply Taylor's formula of order 1. The challenge when applying only first order Taylor's formula is that the resulting stochastic integrals are no longer adapted to the filtration generated by the corresponding Wiener noise. Thus the resulting process is not a martingale, and one cannot directly apply the stochastic Fubini theorem, the Burkhölder-Davis-Gundy inequality and the Itô isometry. To handle the lack of global Lipschitz condition of the nonlinearity, we introduce appropriate auxiliaries processes $(\widetilde{X}^h(t))_t$ and $(\widetilde{X}^h_m)_m$ (see (28) and (52)) and analyze the errors $X(t)-\widetilde{X}^h(t)$, $\widetilde{X}^h(t)-X^h(t)$, $X^h(t_m)-\widetilde{X}^h_m$ and $\widetilde{X}^h_m - X^h_m$ separately (see the proofs of Theorems 3.1 and 4.1). Additional ingredients useful to handle the lack of global Lipschitz condition of the nonlinearity are the inverse estimate in Lemma 3.3 and the regularity estimates in Propositions 3.1 and 2.4. These inverse and regularity estimates lead to a rate of convergence in space depending on the spatial dimension and the growth of the nonlinearity. We remark that for $d=1,2$, we obtain a rate of convergence in space almost β and for $d=3$ we obtain a rate of convergence in space almost $\beta-\frac{c_1}{2}$ (see Theorem 4.1). If the nonlinearity satisfies the global Lipschitz condition (i.e., Assumption 2.3 is fulfilled with $c_1=0$), it follows from Theorem 4.1 that we recover the well-known convergence rates in the literature, that is, we achieve convergence rate $\mathcal{O}(h^\beta + h^\beta|\ln(h)|^\nu + \Delta t^{\frac{\beta}{2}-\xi})$.

The novelties in our result can be summarized in the following points:

- In contrast to many existing results, we consider more general nonlinearities and do not restrict ourself to cubic nonlinearity, see Assumption 2.3 and Remark 2.2.
- To achieve converge rate in time approximately 1, we assume the nonlinearity to be only once differentiable, while in the literature the requirement is the twice differentiability of the nonlinearity.

The rest of this paper is organized as follows: Section 2 deals with some preliminaries and regularity estimates of the mild solution. Section 3 deals with the finite element approximation and the error estimate in space. In Section 4, we investigate the error estimate of the fully discrete scheme. We end the paper in Section 5 with some numerical experiments illustrating the theoretical result.

2. Mathematical setting

2.1. **Notations and preliminaries.** Let $E:=\mathcal{C}(\overline{\Lambda},\mathbb{R})$ be the space of continuous functions on the closure of Λ, equipped with the norm $\|u\|_E := \sup_{x\in\overline{\Lambda}}|u(x)|$, $u\in E$. Let $(\mathcal{H},\langle.,.\rangle,\|.\|)$ and $(U,\langle.,.\rangle_U,\|.\|_U)$ be two separable Hilbert spaces. We denote by $L^p(\Omega,U)$, $p\geq 2$ the Banach space of all equivalence classes of p-integrable U-valued random variables. The norm in the Sobolev space $H^r(\Lambda)$, $r\geqslant 0$ is denoted by $\|.\|_r$. By $\mathcal{L}(U,\mathcal{H})$, we denote the space of bounded linear mappings from U to $\mathcal{H}$ endowed with the usual operator norm $\|.\|_{\mathcal{L}(U,\mathcal{H})}$. By $\mathcal{L}_2(U,\mathcal{H}) := HS(U,\mathcal{H})$ we denote the space of Hilbert-Schmidt operators from U to $\mathcal{H}$. We equip $\mathcal{L}_2(U,\mathcal{H})$ with the norm

$$\|l\|^2_{\mathcal{L}_2(U,\mathcal{H})} := \sum_{i=1}^{\infty} \|l\psi_i\|^2, \quad l \in \mathcal{L}_2(U,\mathcal{H}), \tag{3}$$

where $(\psi_i)_{i=1}^{\infty}$ is an orthonormal basis of U. We denote by $\mathcal{L}_1(U,\mathcal{H})$ the space of nuclear operators from U to $\mathcal{H}$. The trace of $l \in \mathcal{L}_1(U)$ is defined by

$$\mathrm{Tr}(l) := \sum_{i\in\mathbb{N}^d} \langle l\psi_i, \psi_i\rangle, \tag{4}$$

Note that definitions (3) and (4) are independent of the orthonormal basis of U. For simplicity, we write $\mathcal{L}(U,U) =: \mathcal{L}(U)$, $\mathcal{L}_2(U,U) =: \mathcal{L}_2(U)$, $\mathcal{L}_1(U,U) =: \mathcal{L}_1(U)$ and when $U = Q^{\frac{1}{2}}(\mathcal{H})$ we write $\mathcal{L}_2^0 := \mathcal{L}_2(Q^{\frac{1}{2}}(\mathcal{H}),\mathcal{H})$.

Proposition 2.1. (*cf.* [8]) *Let l, l_1, l_2 be linear operators in Hilbert spaces.*

(i) *If $l \in \mathcal{L}(U,\mathcal{H})$ and $l_1 \in \mathcal{L}_2(U)$, then $ll_1 \in \mathcal{L}_2(U,\mathcal{H})$ with*
$$\|ll_1\|_{\mathcal{L}_2(U,\mathcal{H})} \leq \|l\|_{\mathcal{L}(U,\mathcal{H})}\|l_1\|_{\mathcal{L}_2(U)}.$$

(ii) *If $l_1 \in \mathcal{L}_2(U,\mathcal{H})$ and $l_2 \in \mathcal{L}_2(\mathcal{H},U)$, then $l_1l_2 \in \mathcal{L}_1(\mathcal{H})$ with*
$$\|l_1l_2\|_{\mathcal{L}_1(\mathcal{H})} \leq \|l_1\|_{\mathcal{L}_2(U,\mathcal{H})}\|l_2\|_{\mathcal{L}_2(\mathcal{H},U)}.$$

In the rest of this paper, we take $\mathcal{H} = L^2(\Lambda)$.

From now we consider the linear operator A to be given by

$$Au = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(D_{ij}(x)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{d} q_i(x)\frac{\partial u}{\partial x_i}, \ \mathbf{D} = \left(D_{i,j}\right)_{1\leq i,j\leq d}, \ \mathbf{q} = \left(q_i\right)_{1\leq i\leq d},$$

with $D_{ij}, q_i \in L^{\infty}(\Lambda)$. We assume that there exists a constant $c_1 > 0$ such that

$$\sum_{i,j=1}^{d} D_{ij}(x)\xi_i\xi_j \geq c_1|\xi|^2, \quad \xi \in \mathbb{R}^d, \quad x \in \overline{\Lambda}.$$

As in [12, 13] we introduce two spaces $\mathbb{H}$ and V, such that $\mathbb{H} \subset V$; the two spaces depend on the boundary conditions and the domain of the operator A. For Dirichlet (or first-type) boundary conditions we take

$$V = \mathbb{H} = H_0^1(\Lambda) = \overline{\mathcal{C}_c^{\infty}(\Lambda)}^{H^1(\Lambda)}.$$

For Robin (third-type) boundary condition and Neumann (second-type) boundary condition, which is a special case of Robin boundary condition, we take $V = H^1(\Lambda)$

$$\mathbb{H} = \{v \in H^2(\Lambda) : \partial v/\partial \mathrm{v}_A + \alpha_0 v = 0, \text{ on } \partial\Lambda\}, \quad \alpha_0 \in \mathbb{R},$$

where $\partial v/\partial \mathtt{v}_A$ is the normal derivative of v and $\mathtt{v}_A$ is the exterior pointing normal $n = (n_i)$ to the boundary of A, given by

$$\partial v/\partial \mathtt{v}_A = \sum_{i,j=1}^{d} n_i(x) D_{ij}(x) \frac{\partial v}{\partial x_j}, \quad x \in \partial\Lambda.$$

Using Green's formula and the boundary conditions, we obtain the following corresponding bilinear form associated to A

$$a(u,v) = \int_\Lambda \left(\sum_{i,j=1}^{d} D_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{i=1}^{d} q_i \frac{\partial u}{\partial x_i} v \right) dx, \quad u, v \in V,$$

for Dirichlet and Neumann boundary conditions, and

$$a(u,v) = \int_\Lambda \left(\sum_{i,j=1}^{d} D_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{i=1}^{d} q_i \frac{\partial u}{\partial x_i} v \right) dx + \int_{\partial\Lambda} \alpha_0 uv dx, \quad u, v \in V,$$

for Robin boundary conditions. Using Gårding's inequality (see e.g., [39]), it holds that there exist two constants $c_0 \geq 0$ and $\lambda_0 > 0$ such that

$$a(v,v) \geq \lambda_0 \|v\|_1^2 - c_0 \|v\|^2, \quad v \in V.$$

By adding and substracting $c_0 X dt$ in both sides of (1), we obtain a new linear operator still denoted by A, and the corresponding bilinear form is also still denoted by a. Therefore, the following coercivity property holds

$$a(v,v) \geq \lambda_0 \|v\|_1^2, \quad v \in V. \tag{5}$$

Note that the expression of the nonlinear term F has changed as we included the term $c_0 X$ in a new nonlinear term that we still denote by F. The coercivity property (5) implies that A is sectorial in $L^2(\Lambda)$, i.e. there exist $C_1 > 0$, $\theta \in (\frac{1}{2}\pi, \pi)$ such that

$$\|(\lambda I - A)^{-1}\|_{L(L^2(\Lambda))} \leq \frac{C_1}{|\lambda|}, \qquad \lambda \in S_\theta,$$

where $S_\theta := \left\{\lambda \in \mathbb{C} : \lambda = \rho e^{i\phi},\ \rho > 0,\ 0 \leq |\phi| \leq \theta\right\}$ (see [16]). Then $-A$ is the infinitesimal generator of a bounded analytic semigroup $S(t) =: e^{-tA}$ on $L^2(\Lambda)$ such that

$$S(t) = e^{-tA} = \frac{1}{2\pi i} \int_{\gamma_A} e^{t\lambda} (\lambda I - A)^{-1} d\lambda, \quad t > 0,$$

where γ_A denotes a path that surrounds the spectrum of $-A$. The coercivity property (5) also implies that A is a positive operator and its fractional powers are well defined for any $\alpha > 0$, by

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} \mathrm{e}^{-tA} dt, \quad A^\alpha = (A^{-\alpha})^{-1},$$

where $\Gamma(\alpha)$ is the Gamma function (see e.g., [16]). Following [28, 13, 12], we characterize the domain of the operator $A^{\frac{r}{2}}$ denoted by $\mathcal{D}(A^{\frac{r}{2}})$, $r\in\{1,2\}$ with the following equivalence of norms

$$\mathcal{D}(A^{\frac{r}{2}})=\mathbb{H}\cap H^r(\Lambda),\qquad \text{(for Dirichlet boundary conditions)},$$
$$\mathcal{D}(A)=\mathbb{H},\quad \mathcal{D}(A^{\frac{1}{2}})=H^1(\Lambda),\quad \text{(for Robin boundary conditions)}.$$

Endowed with the norm $\Vert A^{\frac{r}{2}}.\Vert$, $\mathcal{D}\left(A^{\frac{r}{2}}\right)$ is a Banach space, see e.g., [16]. The following equivalence of norms holds $\Vert v\Vert_{H^1(\Lambda)}\equiv\Vert A^{\frac{r}{2}}v\Vert=:\Vert v\Vert_r$ for any $v\in\mathcal{D}(A^{\frac{r}{2}})$. The semigroup $S(t)$ satisfies the following properties (known as smoothing properties):

$$\Vert A^{\gamma}S(t)\Vert_{\mathcal{L}(\mathcal{H})}\leq Ct^{-\gamma},\quad \Vert A^{\gamma}S(t)A^{\sigma}\Vert_{\mathcal{L}(\mathcal{H})}\leq Ct^{-\gamma-\sigma}\quad t>0,\ \gamma,\sigma\geq 0, \tag{6}$$
$$\Vert A^{-\alpha}(\mathbf{I}-S(t))\Vert_{\mathcal{L}(\mathcal{H})}\leq Ct^{\alpha},\quad \Vert A^{\eta}(\mathbf{I}-S(t))A^{-\alpha}\Vert_{\mathcal{L}(\mathcal{H})}\leq Ct^{\alpha-\eta},\quad t\geq 0, \tag{7}$$

for $0\leq\eta\leq\alpha\leq 1$.

In addition, $A^{\gamma}S(t)=S(t)A^{\gamma}$ on $\mathcal{D}(A^{\gamma})$ for any $\gamma\geq 0$ (see e.g., [16, 11, 28]). The following Sobolev embedding holds:

$$\mathcal{D}(A^{\frac{\delta}{2}})\hookrightarrow\mathcal{C}(\overline{\Lambda},\mathbb{R}),\quad \delta>\frac{d}{2},\quad d\in\{1,2,3\}. \tag{8}$$

Remark 2.1. *Let A_1 and A_2 be respectively the self-adjoint and the non-self-adjoint parts of A. The following equivalence of norms hold (see e.g.,* [12, 28, 29]*):*

(i) $\Vert A^{\gamma}v\Vert\approx\Vert A_1^{\gamma}v\Vert$ *for any* $\gamma\in[-1,1]$ *and* $v\in\mathcal{D}(A^{\gamma})$*,*

(ii) $\Vert A^{\frac{\gamma}{2}}v\Vert\approx\Vert A_2^{\gamma}v\Vert$ *for any* $\gamma\in[-1,1]$ *and* $v\in\mathcal{D}(A^{\frac{\gamma}{2}})$*.*

To ensure the existence of the unique mild solution to (1) and for the purpose of the convergence analysis of the numerical solution, we make the following assumptions.

Assumption 2.1. *The initial data is such that $X_0\in L^{2p}(\Omega,\mathcal{D}(A^{\frac{\beta}{2}}))\cap L^{2p}(\Omega,\mathcal{C}(\overline{\Lambda},\mathbb{R}))$, for some $\beta\in\left(\max\left(1,\frac{d}{2}\right),2\right]$ and for all $p\in[1,\infty)$.*

Assumption 2.2. *The covariance operator $Q:\mathcal{H}\longrightarrow\mathcal{H}$ satisfies*

$$\Vert A^{\frac{\beta-1}{2}}Q^{\frac{1}{2}}\Vert_{\mathcal{L}_2(\mathcal{H})}<C,$$

for some constant $C>0$, where β is as in Assumption 2.1.

Assumption 2.3. *The nonlinearity $F:\mathcal{H}\longrightarrow\mathcal{H}$ is a Nemytskii-type operator, that is, there exists $\varphi:\Lambda\times\mathbb{R}\longrightarrow\mathbb{R}$ such that*

$$F\left(u\right)\left(x\right)=\varphi\left(x,u(x)\right)\quad \forall u\in\mathcal{H},\ x\in\Lambda. \tag{9}$$

In addition, the real-valued function φ satisfies:

(i) *For all* $x \in \Lambda$, $\varphi(x,.) \in C^1(\Lambda)$ *and there exists* $L_3 > 0$ *such that* $\left|\frac{\partial \varphi}{\partial u}(x,0)\right| \leq L_3$, *where* $\frac{\partial \varphi}{\partial u}$ *is the partial derivative of* φ *w.r.t. the second variable.*

(ii) *There exists a positive constant* L_0 *such that the following one-sided estimate holds:*

$$(y_1 - y_2)(\varphi(x,y_1) - \varphi(x,y_2)) \leq L_0|y_1 - y_2|^2 \quad \forall y_1, y_2 \in \mathbb{R},\ x \in \bar{\Lambda}.$$

(iii) *There exist constants* $L_1 \geq 0$ *and* $c_1 > 0$ *if* $d \in \{1,2\}$ *and* $c_1 \in (0, 2\beta)$ *if* $d = 3$, *such that the following polynomial growth estimate holds:*

$$|\varphi(x,y_1) - \varphi(x,y_2)| \leq L_1|y_1 - y_2|\left(1 + |y_1|^{c_1} + |y_2|^{c_1}\right) \quad \forall y_1, y_2 \in \mathbb{R},\ x \in \bar{\Lambda}.$$

(iv) *There exist positive constants* L_2 *and* c_2 *such that the following polynomial growth estimate of the derivative of* φ *holds:*

$$\left|\frac{\partial \varphi}{\partial u}(x,y_1) - \frac{\partial \varphi}{\partial u}(x,y_2)\right| \leq L_2|y_1 - y_2|\left(1 + |y_1|^{c_2} + |y_2|^{c_2}\right) \quad \forall y_1, y_2 \in \mathbb{R},\ x \in \bar{\Lambda}.$$

Remark 2.2. *The choice of the range of the parameter* c_1 *in Assumption 2.3 is justified in Remark 3.1 below. We observe that for* $d \in \{1,2\}$ *the nonlinearity is more general and is not restricted to Allen-Cahn equation. For* $d = 3$, *our assumption cover the case of Allen-Cahn equation (and even more) since* $\beta \in (0, 2\beta)$. *The restriction* $c_1 \in (0, 2\beta)$ *for* $d = 3$ *is only because one wishes the error estimates in Theorems 3.1 and 4.1 to converge to* 0 *as* $h \to 0$. *The proofs of the error estimates do not require any constraint on* $c_1 > 0$.

We derive in the following lemmas some useful properties of the nonlinearity F.

Lemma 2.1. *[One-sided Lipschitz] Under Assumption 2.3 (ii), there exists a positive constant* $C_F > 0$ *such that the nonlinearity* F *satisfies*

$$\langle u - v, F(u) - F(v)\rangle \leq C_F\|u - v\|^2 \quad \forall u, v \in \mathcal{H}.$$

Proof. The proof follows easily by using Assumption 2.3 (ii). □

Using Assumption 2.3 (iii), one can easily proves the following lemma.

Lemma 2.2. *[Polynomial growth estimates] Under Assumption 2.3 (iii) there exists a constant* $L \geq 0$ *such that the nonlinear function* F *satisfies:*

$$\begin{aligned}
&\|F(u)\| \leq L + L\|u\|\left(1 + \|u\|_E^{c_1}\right), \quad \|F(u)\|_E \leq L + L\|u\|_E^{c_1} \quad \forall u \in \mathcal{H} \cap E,\\
&\|F(u_1) - F(u_2)\| \leq L\|u_1 - u_2\|\left(1 + \|u_1\|_E^{c_1} + \|u_2\|_E^{c_1}\right) \quad \forall u_1, u_2 \in \mathcal{H} \cap E,\\
&\|F(u) - F(v)\|_E \leq L\|u - v\|_E\left(1 + \|u\|_E^{c_1} + \|v\|_E^{c_1}\right) \quad \forall u, v \in \mathcal{H} \cap E,
\end{aligned}$$

where we recall that $E = \mathcal{C}(\overline{\Lambda}, \mathbb{R})$.

Lemma 2.3. *Under Assumption 2.3 the nonlinear function F is Gâteaux differentiable and there exist $\eta \in \left(\frac{3}{4}, 1\right)$ and $L \geq 0$ such that*

$$\|A^{-\eta} F'(u)\|_{\mathcal{L}(\mathcal{H})} \leq L\|u\| \left(1 + \|u\|_E^{c_2}\right) + L \quad \forall u \in \mathcal{H} \cap E. \tag{10}$$

In addition, there exists a constant $L_3 \geq 0$ such that

$$\|A^{-\eta} \left(F'(u) - F'(v)\right) \|_{\mathcal{L}(\mathcal{H})} \leq L_3 \|u - v\| \left(1 + \|u\|_E^{c_2} + \|v\|_E^{c_2}\right) \quad \forall u, v \in \mathcal{H} \cap E. \tag{11}$$

Proof. From [14, Theorem 8], it follows that F is Gâteaux differentiable and its Gâteaux derivative is given by

$$\left(F'(u)(v)\right)(x) = \frac{\partial \varphi}{\partial u}(x, u(x)) v(x) \quad \forall x \in \Lambda,\ u, v \in \mathcal{H}.$$

Using the triangle inequality, Assumption 2.3 and Cauchy-Schwarz's inequality, we obtain

$$\begin{aligned}
\|F'(u)v\|_{L^1(\Lambda,\mathbb{R})} &= \int_\Lambda |F'(u)(v)(x)| dx = \int_\Lambda \left|\frac{\partial \varphi}{\partial u}(x, u(x)) v(x)\right| dx \\
&\leq \int_\Lambda \left|\left(\frac{\partial \varphi}{\partial u}(x, u(x)) - \frac{\partial \varphi}{\partial u}(x, 0)\right) v(x)\right| dx + \int_\Lambda \left|\frac{\partial \varphi}{\partial u}(x, 0) v(x)\right| dx \\
&\leq C \int_\Lambda |u(x)|(1 + |u(x)|^{c_2})|v(x)| dx + C \int_\Lambda |v(x)| dx \\
&\leq C \left(\int_\Lambda |u(x)|^2 (1 + |u(x)|^{2c_2}) dx\right)^{\frac{1}{2}} \|v\| + C\|v\| \\
&\leq C\|u\|(1 + \|u\|_E^{c_2})\|v\| + \|v\|.
\end{aligned}$$

Using Hölder's inequality and the embedding $\mathcal{D}(A^\delta) \hookrightarrow L^\infty(\Lambda, \mathbb{R})$ for $\delta > \frac{d}{4}$, we obtain

$$\begin{aligned}
\|A^{-\eta} F'(u)v\| &= \sup_{\|w\| \leq 1} \left|\langle A^{-\eta} F'(u)v, w\rangle\right| = \sup_{\|w\| \leq 1} \left|\langle F'(u)v, (A^*)^{-\eta} w\rangle\right| \\
&\leq \|F'(u)v\|_{L^1(\Lambda,\mathbb{R})} \sup_{\|w\| \leq 1} \|(A^*)^{-\eta} w\|_{L^\infty(\Lambda,\mathbb{R})} \\
&\leq C \left\{\|u\|(1 + \|u\|_E^{c_2})\|v\| + \|v\|\right\} \sup_{\|w\| \leq 1} \|A^\eta (A^*)^{-\eta} w\|.
\end{aligned}$$

Using [42, Lemma 3.1], it follows from the preceding estimate that

$$\|A^{-\eta} F'(u)v\| \leq C\|u\|(1 + \|u\|_E^{c_2})\|v\| + \|v\|.$$

Using the definition of the operator norm and the preceding estimate, it follows that

$$\|A^{-\eta} F'(u)\|_{\mathcal{L}(\mathcal{H})} = \sup_{\|v\| \leq 1} \|A^{-\eta} F'(u)v\| \leq C\|u\|(1 + \|u\|_E^{c_2}) + C.$$

This proves (10). The proof of (11) goes follows the same steps as those of (10). □

2.2. **Existence and regularity.** We introduce the following stochastic convolution

$$W_A(t) := \int_0^t e^{-(t-s)A} dW(s), \quad t \in [0,T].$$

Proposition 2.2. *Let Assumption 2.2 be fulfilled. Then for any $p \geq 2$, W_A satisfies*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|W_A(t)\|_E^p\right] \leq C < \infty.$$

Proof. Using the Sobolev embedding (8), the Doob's martingale inequality (cf. [35, Theorem 3.9]), the Burkhölder-Davis-Gundy inequality (cf. [35, Theorem 4.36]), the smoothing properties of the semigroup (6) and Assumption 2.2, we obtain

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]} \|W_A(t)\|_E^p\right] &\leq C\mathbb{E}\left[\sup_{t\in[0,T]} \|A^{\frac{d}{4}+\xi} W_A(t)\|^p\right] \leq C\mathbb{E}\left[\|A^{\frac{d}{4}+\xi} W_A(T)\|^p\right] \\
&\leq C\mathbb{E}\left[\left\|\int_0^T A^{\frac{d}{4}+\xi} e^{-(T-s)A} dW(s)\right\|^p\right] \\
&\leq C\left(\int_0^T \left\|A^{\frac{d}{4}+\xi} e^{-(T-s)A} Q^{\frac{1}{2}}\right\|^2_{\mathcal{L}_2(\mathcal{H})} ds\right)^{\frac{p}{2}} \\
&\leq C\left(\int_0^T \left\|A^{\frac{d}{4}+\xi} A^{\frac{1-\beta}{2}} e^{-(T-s)A} A^{\frac{\beta-1}{2}} Q^{\frac{1}{2}}\right\|^2_{\mathcal{L}_2(\mathcal{H})} ds\right)^{\frac{p}{2}} \\
&\leq C\left(\int_0^T \left\|A^{\frac{d+2-2\beta}{4}+\xi} e^{-(T-s)A}\right\|^2_{\mathcal{L}(\mathcal{H})} \left\|A^{\frac{\beta-1}{2}} Q^{\frac{1}{2}}\right\|^2_{\mathcal{L}_2(\mathcal{H})} ds\right)^{\frac{p}{2}} \\
&\leq C\left(\int_0^T (T-s)^{\min\left(0,-1+\beta-\frac{d}{2}-\frac{\xi}{2}\right)} ds\right)^{\frac{p}{2}} \leq C,
\end{aligned}$$

for any arbitrarily small $\xi > 0$, where at the last step we used the fact that $\beta > \frac{d+\xi}{2}$. □

Proposition 2.3. *Let Assumptions 2.1, 2.3 and 2.2 be fulfilled. Then the SPDE* (1) *has a unique mild solution X, satisfying $\mathbb{P}$-a.s.*

$$X(t) = S(t)X_0 + \int_0^t S(t-s)F(X(s))ds + \int_0^t S(t-s)dW(s), \quad t \in [0,T]. \tag{12}$$

Moreover, $X \in \mathcal{C}\left([0,T], \mathcal{C}(\overline{\Lambda}, \mathbb{R})\right)$ $\mathbb{P}$-a.s. and $X \in L^p\left(\Omega; \mathcal{C}\left([0,T], \mathcal{C}(\overline{\Lambda}, \mathbb{R})\right)\right)$ $\forall p \geq 2$.

Proof. The proof is an application of [9, Theorem 7.14] with $E = \mathcal{C}(\overline{\Lambda}, \mathbb{R})$. One can readily check that the requirements of [9, Theorem 7.14] are fulfilled. Indeed, from Section 2.1, $-A$ generates an analytic semigroup on $\mathcal{H} = L^2(\Lambda)$. From [9, Appendix A.5.2], we know that $-A$ generates a strongly continuous semigroup on $\mathcal{C}(\overline{\Lambda}, \mathbb{R})$. Since $d \leq 3$, it follows from [7, Remark 6.1.1 (2)] that [7, Hypothesis 6.1] is fulfilled. Note that from (5) it follows that

$-A$ is dissipative, see also [7, (6.1.2)]. The operator $-A+F$ is then dissipative. Hence [9, Hypothesis 7.7] is fulfilled. Finally applying [9, Theorem 7.14] proves the existence of the unique mild solution X to (1), with $X \in \mathcal{C}([0,T],\mathcal{C}(\bar{\Lambda},\mathbb{R}))$ $\mathbb{P}$-a.s. The fact that $X \in L^p\left(\Omega;\mathcal{C}\left([0,T],\mathcal{C}(\overline{\Lambda},\mathbb{R})\right)\right)$ follows therefore immediately. □

Lemma 2.4. *The following sharp stability estimates of the semigroup hold*[1]

(i) *Let $0 \leq \rho \leq 1$. Then the following estimate holds*

$$\int_{t_1}^{t_2} \|A^{\frac{\varrho}{2}} S(t_2-r)\|^2_{\mathcal{L}(\mathcal{H})} dr \leq C(t_2-t_1)^{1-\rho}, \quad 0 \leq t_1 \leq t_2 \leq T.$$

(ii) *Let $0 \leq \rho \leq 1$ and $u \in \mathcal{H}$. The following estimate holds*

$$\left\| A^{\rho} \int_{t_1}^{t_2} S(t_2-r) u dr \right\| \leq C(t_2-t_1)^{1-\rho}\|u\|, \quad 0 \leq t_1 \leq t_2 \leq T.$$

Proof. The proof of (i) for self-adjoint operator can be found in [26, Lemma 3.2 (iii)]. The proof in the case of non self-adjoint operator can be found in [33, Lemma 2.1, (16)]. The proof of (ii) can be found in [26, Lemma 3.2 (iv)]. □

Proposition 2.4. *[Regularity of the mild solution] For any $p \geq 1$, there exists a constant $C = C(p,T,X_0) > 0$ such that the following space regularity estimates hold*

$$\|X(t)\|_{L^{2p}(\Omega,\mathcal{C}(\overline{\Lambda},\mathbb{R}))} \leq C, \quad \|X(t)\|_{L^{2p}(\Omega,\mathcal{H})} \leq C, \quad \|A^{\frac{\beta}{2}} X(t)\|_{L^{2p}(\Omega,\mathcal{H})} \leq C. \tag{13}$$

In addition $\|F(X(t))\|_{L^{2p}(\Omega,\mathcal{H})} \leq C$ and for any $\gamma \in [0,1]$, the following estimate holds

$$\|X(t)-X(s)\|_{L^{2p}(\Omega,\dot{\mathcal{H}}^{\gamma})} \leq C(t-s)^{\min\left(\frac{1}{2},\frac{\beta-\gamma}{2}\right)}, \quad 0 \leq s \leq t \leq T. \tag{14}$$

Proof. The estimate $\|X(t)\|_{L^{2p}(\Omega,\mathcal{C}(\overline{\Lambda},\mathbb{R}))} \leq C$ follows from Proposition 2.3. Since Λ is bounded, using the embedding $\mathcal{C}(\overline{\Lambda},\mathbb{R}) \hookrightarrow \mathcal{H} = L^2(\Lambda)$ and the estimate $\|X(t)\|_{L^{2p}(\Omega,\mathcal{C}(\overline{\Lambda},\mathbb{R}))} \leq C$ (cf. Proposition 2.3), it follows that

$$\|X(t)\|_{L^{2p}(\Omega,\mathcal{H})} \leq C \quad \forall t \in [0,T]. \tag{15}$$

Using Lemma 2.2, the estimate $\|X(t)\|_{L^{2p}(\Omega,\mathcal{C}(\overline{\Lambda},\mathbb{R}))} \leq C$ and (15), we obtain

$$\|F(X(t))\|_{L^{2p}(\Omega,\mathcal{H})} \leq C\|X(t)\|_{L^{4p}(\Omega,\mathcal{H})}\left(1+\|X(t)\|^{c_1}_{L^{4pc_1}(\Omega,\mathcal{C}(\bar{\Lambda},\mathbb{R}))}\right) \leq C. \tag{16}$$

It remains now to prove $\|A^{\frac{\beta}{2}}X(t)\|_{L^{2p}(\Omega,\mathcal{H})} \leq C$ and (14). To this end, we use a circular argument, which consists of proving $\|A^{\frac{\beta}{2}}X(t)\|_{L^{2p}(\Omega,\mathcal{H})} \leq C$ for $\beta \in \left(\frac{d}{2},2\right)$, proving (14) and using the later to prove $\|AX(t)\|_{L^{2p}(\Omega,\mathcal{H})} \leq C$. We start with the proof of

[1] Lemma 2.4 still holds if A and S are replaced by their discrete versions A_h and S_h respectively; defined in Section 4.

$\|A^{\frac{\beta}{2}}X(t)\|_{L^{2p}(\Omega,\mathcal{H})} \leq C$ for $\beta \in \left(\frac{d}{2}, 2\right)$. From the mild solution (12), it follows that

$$\begin{aligned}
\|A^{\frac{\beta}{2}}X(t)\|_{L^{2p}(\Omega,\mathcal{H})} &\leq \|S(t)A^{\frac{\beta}{2}}X_0\|_{L^{2p}(\Omega,\mathcal{H})} + \left\|\int_0^t A^{\frac{\beta}{2}}S(t-s)F(X(s))ds\right\|_{L^{2p}(\Omega,\mathcal{H})}\\
&+ \left\|\int_0^t S(t-s)A^{\frac{\beta}{2}}dW(s)\right\|_{L^{2p}(\Omega,\mathcal{H})} =: \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3. \qquad (17)
\end{aligned}$$

Using Assumption 2.1, it follows that

$$\mathrm{I}_1 \leq \|S(t)\|_{\mathcal{L}(\mathcal{H})}\|A^{\frac{\beta}{2}}X_0\|_{L^{2p}(\Omega,\mathcal{H})} \leq C. \qquad (18)$$

Using (16) and the smoothing properties of the semigroup (cf. (6)), we obtain

$$\mathrm{I}_2 \leq C\int_0^t \|A^{\frac{\beta}{2}}S(t-s)\|_{\mathcal{L}(\mathcal{H})}\|F(X(s))\|_{L^{2p}(\Omega,\mathcal{H})}ds \leq C\int_0^t (t-s)^{-\frac{\beta}{2}}ds \leq C. \qquad (19)$$

Using the Burkhölder-Davis-Gundy inequality (cf. [35, Theorem 4.37]), Assumption 2.2 and Lemma 2.4 (i), it holds that

$$\begin{aligned}
\mathrm{I}_3 &\leq \left(\int_0^t \|S(t-s)A^{\frac{\beta}{2}}Q^{\frac{1}{2}}\|^2_{L^{2p}(\Omega,\mathcal{L}_2(\mathcal{H}))}ds\right)^{\frac{1}{2}}\\
&\leq \left(\int_0^t \|A^{\frac{1}{2}}S(t-s)\|^2_{\mathcal{L}(\mathcal{H})}\|A^{\frac{\beta-1}{2}}Q^{\frac{1}{2}}\|^2_{\mathcal{L}_2(\mathcal{H})}ds\right)^{\frac{1}{2}} \qquad (20)\\
&\leq C\left(\int_0^t \|A^{\frac{1}{2}}S(t-s)\|^2_{\mathcal{L}(\mathcal{H})}ds\right)^{\frac{1}{2}} \leq C.
\end{aligned}$$

Substituting (20), (19) and (18) into (17) yields $\|A^{\frac{\beta}{2}}X(t)\|_{L^{2p}(\Omega,\mathcal{H})} \leq C$ for $\beta \in \left(\frac{d}{2}, 2\right)$.

Let us now prove (14). We start by proving it for $\beta \in \left(\frac{d}{2}, 2\right)$. Using the mild representation (12) and the triangle inequality, it follows that

$$\begin{aligned}
\|X(t)-X(s)\|_{L^{2p}(\Omega,\dot{\mathcal{H}}^\gamma)} &= \|(S(t-s)-\mathbf{I})X(s)\|_{L^{2p}(\Omega,\dot{\mathcal{H}}^\gamma)}\\
&+ \left\|\int_s^t S(t-\sigma)F(X(\sigma))d\sigma\right\|_{L^{2p}(\Omega,\dot{\mathcal{H}}^\gamma)}\\
&+ \left\|\int_s^t S(t-\sigma)dW(\sigma)\right\|_{L^{2p}(\Omega,\dot{\mathcal{H}}^\gamma)} =: \mathrm{II}_1 + \mathrm{II}_2 + \mathrm{II}_3. \qquad (21)
\end{aligned}$$

Using the smoothing properties of the semigroup ((6)–(7)) and (13) for $\beta \in \left(\frac{d}{2}, 2\right)$, yields

$$\mathrm{II}_1 \leq \|A^{\frac{\gamma}{2}}\left(S(t-s)-\mathbf{I}\right)A^{-\frac{\beta}{2}}\|_{\mathcal{L}(\mathcal{H})}\|A^{\frac{\beta}{2}}X(s)\|_{L^{2p}(\Omega,\mathcal{H})} \leq C(t-s)^{\frac{\beta-\gamma}{2}}.$$

Using the smoothing properties of the semigroup (cf. (6)) and (16), it holds that

$$\mathrm{II}_2 \leq \int_s^t \|A^{\frac{\gamma}{2}}S(t-\sigma)F(X(\sigma))\|_{L^{2p}(\Omega,\mathcal{H})}d\sigma \leq C\int_s^t (t-s)^{-\frac{\gamma}{2}}ds \leq C(t-s)^{1-\frac{\gamma}{2}}. \qquad (22)$$

Using the Burkhölder-Davis-Gundy inequality (cf. [35, Theorem 4.37]), Assumption 2.2 and Lemma 2.4 (i), we obtain

$$\begin{aligned}
\mathrm{II}_3 &\leq \left(\int_s^t \|A^{\frac{\gamma}{2}}S(t-\sigma)Q^{\frac{1}{2}}\|^2_{L^{2p}(\Omega,\mathcal{L}_2(\mathcal{H}))}d\sigma\right)^{\frac{1}{2}}\\
&\leq \left(\int_s^t \|A^{\frac{1+\gamma-\beta}{2}}S(t-\sigma)\|^2_{\mathcal{L}(\mathcal{H})}\|A^{\frac{\beta-1}{2}}Q^{\frac{1}{2}}\|^2_{\mathcal{L}_2(\mathcal{H})}d\sigma\right)^{\frac{1}{2}} \leq C(t-s)^{\min\left(\frac{1}{2},\frac{\beta-\gamma}{2}\right)}. \qquad (23)
\end{aligned}$$

Substituting (23), (22) and (21) into (21) completes the proof of (14) for $\beta \in \left(\frac{d}{2}, 2\right)$.

Now we prove $\|AX(t)\|_{L^{2p}(\Omega,\mathcal{H})} \leq C$. From the mild representation (12), we have

$$\begin{aligned}
AX(t) &= S(t)AX_0 + \int_0^t AS(t-s)F(X(t))ds + \int_0^t AS(t-s)\left(F(X(s)-F(X(t))\right)ds\\
&\quad + \int_0^t AS(t-s)dW(s). \qquad (24)
\end{aligned}$$

Taking the norm in both sides of (24) and using Assumption 2.1, Lemma 2.4 (ii) and the smoothing properties of the semi-group (cf. (6)), yields

$$\begin{aligned}
\|AX(t)\|_{L^{2p}(\Omega,\mathcal{H})} &\leq \|S(t)AX_0\|_{L^{2p}(\Omega,\mathcal{H})} + \left\|\int_0^t AS(t-s)F(X(t))ds\right\|_{L^{2p}(\Omega,\mathcal{H})}\\
&\quad + \int_0^t \|AS(t-s)\|_{\mathcal{L}(\mathcal{H})}\|F(X(s))-F(X(t))\|_{L^{2p}(\Omega,\mathcal{H})}ds\\
&\quad + \left\|\int_0^t AS(t-s)dW(s)\right\|_{L^{2p}(\Omega,\mathcal{H})}\\
&\leq C\|AX_0\|_{L^{2p}(\Omega,\mathcal{H})} + C\|F(X(t))\|_{L^{2p}(\Omega,\mathcal{H})}\\
&\quad + C\int_0^t (t-s)^{-1}\|X(t)-X(s)\|_{L^{4p}(\Omega,\mathcal{H})}ds + \left\|\int_0^t AS(t-s)dW(s)\right\|_{L^{2p}(\Omega,\mathcal{H})}. \qquad (25)
\end{aligned}$$

Using the Burkhölder-Davis-Gundy inequality (cf. [35, Theorem 4.37]), Assumption 2.2 and Lemma 2.4 (i), we obtain

$$\begin{aligned}
\left\|\int_0^t AS(t-s)dW(s)\right\|_{L^{2p}(\Omega,\mathcal{H})} &\leq \left(\int_0^t \|AS(t-s)Q^{\frac{1}{2}}\|^2_{L^{2p}(\Omega,\mathcal{L}_2(\mathcal{H}))}ds\right)^{\frac{1}{2}}\\
&\leq \left(\int_0^t \|A^{\frac{1}{2}}S(t-s)\|^2_{\mathcal{L}(\mathcal{H})}\|A^{\frac{1}{2}}Q^{\frac{1}{2}}\|^2_{\mathcal{L}_2(\mathcal{H})}ds\right)^{\frac{1}{2}} \leq C. \qquad (26)
\end{aligned}$$

Substituting (26) into (25) and using (14) with $\beta = 2$ and $\gamma = 1$, yields

$$\|AX(t)\| \leq C + C\int_0^t (t-s)^{-1}(t-s)^{\frac{1}{2}}ds \leq C.$$

This completes the proof of the last estimate of (13) for $\beta = 2$. Since we have proved $\|AX(t)\|_{L^{2p}(\Omega,\mathcal{H})} \leq C$, the proof of (14) for $\beta = 2$ goes along the same lines as those in the case $\beta \in [0, 2)$. This ends the proof of the lemma. □

3. Finite element approximation

We now perform the space approximation of the SPDE (1). We start by splitting the domain Λ in finite triangles. Let $\mathcal{T}_h$ be a triangulation with maximal length h. Let $V_h \subset V$ be the space of continuous functions that are piecewise linear over the triangulation $\mathcal{T}_h$. We define the projection $P_h : L^2(\Lambda) \to V_h$ and the discrete operator $A_h : V_h \to V_h$, by

$$\langle P_h u, \chi\rangle = \langle u, \chi\rangle, \quad \chi \in V_h,\ u \in \mathcal{H}, \quad \text{and} \quad \langle A_h\phi, \chi\rangle = a(\phi, \chi), \quad \phi, \chi \in V_h.$$

The discrete operator $-A_h$ is also a generator of an analytic semigroup $S_h(t) =: e^{-tA_h}$ on $L^2(\Lambda)$ (cf. [28]). The semi-discrete version of (1) is: find $X^h(t) \in V_h$ such that

$$dX^h(t) + A_h X^h(t)dt = P_h F(X^h(t))dt + P_h dW(t),\ X^h(0) = P_h X_0,\ t \in (0, T]. \tag{27}$$

Note that $S_h(t)$ and $P_h F$ satisfy the same properties as $S(t)$ and F respectively, therefore as in Proposition 2.3, it is easy to check that the semi-discrete problem (27) has a unique mild solution $X^h(t)$ in $V_h \cap \mathcal{C}(\bar{\Lambda}, \mathbb{R})$, given by $X^h(0) = P_h X_0$ and for all $t \in (0, T]$

$$X^h(t) = S_h(t)X^h(0) + \int_0^t S_h(t-s)P_h F(X^h(s))ds + \int_0^t S_h(t-s)P_h dW(s) \quad \mathbb{P}\text{-a.s.}$$

Let us introduce the following semi-discrete auxiliary process

$$\widetilde{X}^h(t) := S_h(t)P_h X_0 + \int_0^t S_h(t-s)P_h F(X(s))ds + W_{A_h}(t), \quad t \in [0, T], \tag{28}$$

where the semi-discrete stochastic convolution $W_{A_h}(t)$ is given by

$$W_{A_h}(t) := \int_0^t S_h(t-s)P_h dW(s), \quad t \in [0, T]. \tag{29}$$

The proof of the following lemma can be found in [34, Lemma 11].

Lemma 3.1. *Under Assumption 2.2, there exists $C > 0$, independent of h such that*

$$\|A_h^{\frac{\beta-1}{2}} P_h Q^{\frac{1}{2}}\|_{\mathcal{L}_2(\mathcal{H})} < C.$$

We prove in Lemma 3.4 below some regularity estimates of the semi-discrete convolution $W_{A_h}(t)$. In particular we prove an estimate of $\|W_{A_h}(t)\|_{L^p(\Omega;L^\infty(\Lambda))}$. Note that the approach of Proposition 2.2 cannot be used here, since the estimate

$$\|A^\alpha A_h^{-\alpha} P_h\|_{\mathcal{L}(\mathcal{H})} \leq C, \quad \frac{1}{2} < \alpha \leq 1$$

is not true, because of $V_h \not\subseteq \mathcal{D}(A^\alpha)$. Therefore we need a difference appraoch. The approach used here is based on the inverse estimates in Lemmas 3.2 and 3.3 below. The proof of the inverse estimate in Lemma 3.2 below can be found in [1, Lemma 3.5].

Lemma 3.2. *Let $1 \leq r \leq \infty$, $1 \leq p \leq \infty$. Then it holds that*

$$\|v_h\|_{L^p(\Lambda)} \leq Ch^{\min\{0,d(r-p)/(pr)\}}\|v_h\|_{L^r(\Lambda)}, \quad v_h \in V_h.$$

The following lemma (which is a slight modification of [1, Remark 3.8]) will be useful.

Lemma 3.3. *For any $c > 0$ the following inverse estimate holds*

$$\|v_h\|_{L^\infty(\Lambda)} \leq Ch^{1-\frac{d}{2}}|\ln(h)|^{\frac{1}{c}}\|v_h\|_{W^{1,2}(\Lambda)} \quad \forall v_h \in V_h,$$

where $C > 0$ is independent of h and c.

Proof. The proof goes along the same lines as those of [1, Remark 3.8]. For the sake of completness we provide some details. Using Lemma 3.2 with $p = \infty$, it follows that

$$\|v_h\|_{L^\infty(\Lambda)} \leq Ch^{-\frac{d}{r}}\|v_h\|_{L^r(\Lambda)}, \quad \forall r \in [1, \infty],\ v_h \in V_h.$$

Using the Sobolev estimate $\|u\|_{L^r(\Lambda)} \leq Cr\|u\|_{W^{1,q}(\Lambda)}$ for $1 \leq q < d$ and $r = \frac{dq}{d-q}$, we obtain

$$\|v_h\|_{L^\infty(\Lambda)} \leq crh^{-\frac{d}{r}}\|v_h\|_{W^{1,q}(\Lambda)}, \quad v_h \in V_h.$$

Noting that for $r = \frac{dq}{d-q}$ we have $-\frac{d}{r} = 1 - \frac{d}{q}$. It follows from the preceding estimate that

$$\|v_h\|_{L^\infty(\Lambda)} \leq Crh^{1-\frac{d}{q}}\|v_h\|_{W^{1,q}(\Lambda)}, \quad v_h \in V_h. \tag{30}$$

Taking $q = d - |\ln(h)|^{-\frac{1}{c}}$, one can easily check that $1 \leq q < d$ and for h small enough we have $q \leq 2$. This implies that $1 - \frac{d}{q} \leq 1 - \frac{d}{2}$ and $W^{1,2}(\Lambda) \hookrightarrow W^{1,q}(\Lambda)$. One also proves that

$$r = \frac{dq}{d-q} = dq|\ln(h)|^{\frac{1}{c}} = (d^2 - d|\ln(h)|^{-\frac{1}{c}})|\ln(h)|^{\frac{1}{c}} \leq d^2|\ln(h)|^{\frac{1}{c}} \leq 9|\ln(h)|^{\frac{1}{c}}, \tag{31}$$

where at the last step we used the fact that $d \in \{1, 2, 3\}$.

Substituting (31) into (30) and using the embedding $W^{1,2}(\Lambda) \hookrightarrow W^{1,q}(\Lambda)$, it follows that

$$\|v_h\|_{L^\infty(\Lambda)} \leq Ch^{1-\frac{d}{2}}|\ln(h)|^{\frac{1}{c}}\|v_h\|_{W^{1,2}(\Lambda)}, \quad v_h \in V_h,$$

which ends the proof. □

In the next lemma, we provide regularity estimates of the stochastic convolution $W_{A_h}(t)$.

Lemma 3.4. *Let Assumptions 2.3 and 2.2 hold. Let $t \in [0, T]$, $p > 1$ and $c > 0$. Then*

$$\|W_{A_h}(t)\|_{L^{2p}(\Omega,\dot{\mathcal{H}}^1)} \leq C, \quad \|W_{A_h}(t)\|_{L^p(\Omega;L^\infty(\Lambda))} \leq Ch^{1-\frac{d}{2}}|\ln(h)|^{\frac{1}{c}},$$
$$\|W_{A_h}(t)\|_{L^{2p}(\Omega,\mathcal{H})} \leq C, \quad \|F(W_{A_h}(t))\|_{L^{2p}(\Omega,\mathcal{H})} \leq C + Ch^{c_1-\frac{dc_1}{2}}|\ln(h)|^{\frac{c_1}{c}},$$

where C is a positive constant independent of h, t and c and $W_{A_h}(t)$ is given by (29).

Proof. Note that the following equivalence of norms holds (see e.g., [28, (2.3)])

$$\|A_h^{\frac{1}{2}} v\| \approx \|v\|_1 \quad \forall v \in V_h. \tag{32}$$

Using (32), the BDG inequality (cf. [35, Theorem 4.37]), Lemmas 3.1 and 2.4 (i), we obtain

$$\begin{aligned}
\|W_{A_h}(t)\|_{L^{2p}(\Omega,\dot{\mathcal{H}}^1)} &\leq \|A_h^{\frac{1}{2}} W_{A_h}(t)\|_{L^{2p}(\Omega,\mathcal{H})} \leq \left(\int_0^t \|A_h^{\frac{1}{2}} S_h(t-s) P_h Q^{\frac{1}{2}}\|^2_{L^{2p}(\Omega,\mathcal{L}_2(\mathcal{H}))} ds\right)^{\frac{1}{2}} \\
&\leq \left(\int_0^t \|A_h^{\frac{2-\beta}{2}} S_h(t-s)\|^2_{\mathcal{L}(\mathcal{H})} \|A_h^{\frac{\beta-1}{2}} P_h Q^{\frac{1}{2}}\|^2_{\mathcal{L}_2(\mathcal{H})} ds\right)^{\frac{1}{2}} \\
&\leq C \left(\int_0^t \|A_h^{\frac{2-\beta}{2}} S_h(t-s)\|^2_{\mathcal{L}(\mathcal{H})} ds\right)^{\frac{1}{2}} \leq C \quad \forall t \in [0,T].
\end{aligned}$$

Using the embedding $\dot{\mathcal{H}}^1 \hookrightarrow \mathcal{H}$ and the preceding estimate, it follows that

$$\|W_{A_h}(t)\|_{L^{2p}(\Omega,\mathcal{H})} \leq C\|W_{A_h}(t)\|_{L^{2p}(\Omega,\dot{\mathcal{H}}^1)} \leq C \quad t \in [0,T].$$

Using Lemma 3.4 and the preceding estimate, we obtain for any $t \in [0,T]$

$$\|W_{A_h}(t)\|_{L^p(\Omega,L^\infty(\Lambda))} \leq Ch^{1-\frac{d}{2}} |\ln(h)|^{\frac{1}{c}} \|W_{A_h}(t)\|_{L^p(\Omega,L^2(\Lambda))} \leq Ch^{1-\frac{d}{2}} |\ln(h)|^{\frac{1}{c}}.$$

Using Lemma 2.2, Hölder's inequality and the preceding estimates, it follows that

$$\begin{aligned}
\|F(W_{A_h}(t))\|_{L^p(\Omega,\mathcal{H})} &\leq C + C\|W_{A_h}(t)\|_{L^p(\Omega,\mathcal{H})} + C\|W_{A_h}(t)\|_{L^{2p}(\Omega,\mathcal{H})} \|W_{A_h}(t)\|^{c_1}_{L^{2pc_1}(\Omega,L^\infty(\Lambda))} \\
&\leq C + Ch^{c_1 - \frac{dc_1}{2}} |\ln(h)|^{\frac{c_1}{c}} \leq C + Ch^{c_1 - \frac{dc_1}{2}} |\ln(h)|^{\frac{c_1}{c}} \quad \forall t \in [0,T].
\end{aligned}$$

□

The following proposition provides regularity of the auxiliary process $\widetilde{X}^h(t)$ (cf. (28)).

Proposition 3.1. *For any $p \geq 1$ and $c > 0$, there exists $C = C(p,T,X_0) > 0$ such that*

$$\|\widetilde{X}^h(t)\|_{L^{2p}(\Omega,\dot{\mathcal{H}}^1)} \leq C, \quad \|\widetilde{X}^h(t)\|_{L^{2p}(\Omega,L^\infty(\Lambda,\mathbb{R}))} \leq Ch^{1-\frac{d}{2}} |\ln(h)|^{\frac{1}{c}},$$

$$\|\widetilde{X}^h(t)\|_{L^{2p}(\Omega,\mathcal{H})} \leq C, \quad \|F(\widetilde{X}^h(t))\|_{L^{2p}(\Omega,\mathcal{H})} \leq C + Ch^{c_1 - \frac{dc_1}{2}} |\ln(h)|^{\frac{c_1}{c}}$$

for all $t \in [0,T]$. Moreover, for any $\gamma \in [0,1]$, the following regularity estimate holds

$$\|\widetilde{X}^h(t) - \widetilde{X}^h(s)\|_{L^{2p}(\Omega,\dot{\mathcal{H}}^\gamma)} \leq C|t-s|^{\min\left(\frac{1}{2}, \frac{\beta-\gamma}{2}\right)}, \quad t,s \in [0,T]. \tag{33}$$

Proof. Using the equivalence of norms (32) and the triangle inequality, it follows that

$$\begin{aligned}
\|\widetilde{X}^h(t)\|_{L^{2p}(\Omega,\dot{\mathcal{H}}^1)} &\leq \|A_h^{\frac{1}{2}} \widetilde{X}^h(t)\|_{L^{2p}(\Omega,\mathcal{H})} \\
&\leq \|A_h^{\frac{1}{2}} S_h(t) P_h X_0\|_{L^{2p}(\Omega,\mathcal{H})} + \int_0^t \|A_h^{\frac{1}{2}} S_h(t-s) F(X(s))\|_{L^{2p}(\Omega,\mathcal{H})} ds \qquad (34) \\
&\quad + \|A_h^{\frac{1}{2}} W_{A_h}(t)\|_{L^{2p}(\Omega,\mathcal{H})}.
\end{aligned}$$

Using [34, Lemma 1], Assumption 2.1 and the boundedness of $S_h(t)$ in $\mathcal{L}(\mathcal{H})$, we obtain

$$\|A_h^{\frac{1}{2}}S_h(t)P_hX_0\|_{L^{2p}(\Omega,\mathcal{H})}\leq \|S_h(t)A_h^{\frac{1}{2}}P_hX_0\|_{L^{2p}(\Omega,\mathcal{H})}\leq C\|S_h(t)\|_{\mathcal{L}(\mathcal{H})}\|A^{\frac{1}{2}}X_0\|_{L^{2p}(\Omega,\mathcal{H})}\leq C.$$

Using the smoothing property of the semigroup (6) and Proposition 2.4, we obtain

$$\int_0^t\|A_h^{\frac{1}{2}}S_h(t-s)F(X(s))\|_{L^{2p}(\Omega,\mathcal{H})}ds\leq C\int_0^t(t-s)^{-\frac{1}{2}}ds\leq C.$$

Using the equivalence of norms (32) and Lemma 3.1, it follows that

$$\|A_h^{\frac{1}{2}}W_{A_h}(t)\|_{L^{2p}(\Omega,\mathcal{H})}\leq C\|W_{A_h}(t)\|_{L^{2p}(\Omega,\dot{\mathcal{H}}^1)}\leq C.$$

Substituting the preceding estimates into (34) yields $\|\widetilde{X}^h(t)\|_{L^{2p}(\Omega,\dot{\mathcal{H}}^1)}\leq C$. Using the embedding $\dot{\mathcal{H}}^1\hookrightarrow\mathcal{H}$ and the preceding estimate, it follows that $\|\widetilde{X}^h(t)\|_{L^{2p}(\Omega,\mathcal{H})}\leq C$.

Using Lemma 3.4 and the preceding estimates, it follows that

$$\|\widetilde{X}^h(t)\|_{L^{2p}(\Omega,L^\infty(\Lambda,\mathbb{R}))}\leq Ch^{1-\frac{d}{2}}|\ln(h)|^{\frac{1}{c}}\|\widetilde{X}^h(t)\|_{L^{2p}(\Omega,\dot{\mathcal{H}}^1)}\leq Ch^{1-\frac{d}{2}}|\ln(h)|^{\frac{1}{c}}.$$

Using Lemma 2.2, Hölder's inequality and the preceding estimates, it follows that

$$\begin{aligned}\|F(\widetilde{X}^h(t))\|_{L^{2p}(\Omega,\mathcal{H})}&\leq C+C\|\widetilde{X}^h(t)\|_{L^{2p}(\Omega,\mathcal{H})}+C\|\widetilde{X}^h(t)\|_{L^{4p}(\Omega,\mathcal{H})}\|\widetilde{X}^h(t)\|^{c_1}_{L^{4pc_1}(\Omega,L^\infty(\Lambda))}\\&\leq C+Ch^{c_1-\frac{dc_1}{2}}|\ln(h)|^{\frac{c_1}{c}}.\end{aligned}$$

Using the mild representation and the triangle inequality, it follows that

$$\begin{aligned}\|\widetilde{X}^h(t)-\widetilde{X}^h(s)\|_{L^{2p}(\Omega,\dot{\mathcal{H}}^\gamma)}&\leq\left\|\int_s^tS_h(t-\sigma)P_hF(X(\sigma))d\sigma\right\|_{L^{2p}(\Omega,\dot{\mathcal{H}}^\gamma)}\\&+\left\|\int_s^tS(t-\sigma)dW(\sigma)\right\|_{L^{2p}(\Omega,\dot{\mathcal{H}}^\gamma)}=:\mathrm{III}_1+\mathrm{III}_2.\end{aligned}\tag{35}$$

Using the smoothing property of the semigroup (6) and the estimate (16), it holds that

$$\mathrm{III}_1\leq\int_s^t\|A^{\frac{\gamma}{2}}S_h(t-\sigma)P_hF(X(\sigma))\|_{L^{2p}(\Omega,\mathcal{H})}d\sigma\leq C\int_s^t(t-s)^{-\frac{\gamma}{2}}ds\leq C(t-s)^{1-\frac{\gamma}{2}}.\tag{36}$$

Using the Burkhölder-Davis-Gundy inequality (cf. [35, Theorem 4.37]), Lemma 3.1 and the smoothing property of the semigroup (6), yields

$$\begin{aligned}\mathrm{III}_2&\leq\left(\int_s^t\|A^{\frac{\gamma}{2}}S_h(t-\sigma)P_hQ^{\frac{1}{2}}\|^2_{L^{2p}(\Omega,\mathcal{L}_2(\mathcal{H}))}d\sigma\right)^{\frac{1}{2}}\\&\leq\left(\int_s^t\|S_h(t-\sigma)A_h^{\frac{1+\gamma-\beta}{2}}\|^2_{\mathcal{L}(\mathcal{H})}\|A_h^{\frac{\beta-1}{2}}P_hQ^{\frac{1}{2}}\|^2_{\mathcal{L}_2(\mathcal{H})}d\sigma\right)^{\frac{1}{2}}\leq C(t-s)^{\min\left(\frac{1}{2},\frac{\beta-\gamma}{2}\right)}.\end{aligned}\tag{37}$$

Substituting (37) and (36) into (35) completes the proof of (33). $\square$

The following lemma provides error estimate for the approximation of the semigroup.

Lemma 3.5.

(i) *For any* $r \in [0,2]$ *and* $\alpha \in [0,r]$, *it holds that*

$$\|\left(S(t)-S_h(t)P_h\right)v\| \leq Ch^r t^{-\frac{(r-\alpha)}{2}}\|v\|_\alpha, \quad v \in \mathcal{D}\left(A^{\frac{\alpha}{2}}\right), \quad t>0.$$

(ii) *For any* $0 \leq \gamma \leq 2$, *it holds that*

$$\left(\int_0^t \|\left(S(t)-S_h(t)P_h\right)v\|^2 ds\right)^{\frac{1}{2}} \leq Ch^\gamma \|v\|_{\gamma-1}, \quad v \in \mathcal{D}(A^{\frac{\gamma-1}{2}}), \quad t>0.$$

(iii) *Let* $0 \leq \rho \leq 1$. *Then it holds that*

$$\left\|\int_0^t \left(S(t)-S_h(t)P_h\right)vds\right\| \leq Ch^{2-\rho}\|v\|_{-\rho}, \quad v \in \mathcal{D}\left(A^{-\rho}\right), \quad t>0.$$

Proof. The proof of (i)-(ii) can be found in [42, Lemma 6.1]. The proof of (iii) can be found in [41, Lemma 3.2 (iv)]. □

Theorem 3.1. *[Space error] Let* $X(t)$ *and* $X^h(t)$ *be solution of* (1) *and* (27) *respectively. Let Assumptions 2.1, 2.3 and 2.2 be fulfilled. Then for any* $p \geq 1$, *it holds that*

$$\|X(t)-X^h(t)\|_{L^{2p}(\Omega,\mathcal{H})} \leq C\left(h^\beta + h^{\beta+c_1-\frac{dc_1}{2}}|\ln(h)|^\nu\right), \quad t \in [0,T],$$

where $\nu>0$ *is any arbitrarily positive real number, small enough.*

Proof. Using the triangle inequality, we split the error as follows

$$\|X(t)-X^h(t)\|_{L^{2p}(\Omega,\mathcal{H})} \leq \|X(t)-\widetilde{X}^h(t)\|_{L^{2p}(\Omega,\mathcal{H})} + \|\widetilde{X}^h(t)-X^h(t)\|_{L^{2p}(\Omega,\mathcal{H})}. \tag{38}$$

Subtracting (28) from (12) and using the triangle inequality, it holds that

$$\begin{aligned}\|X(t)-\widetilde{X}^h(t)\|_{L^{2p}(\Omega,\mathcal{H})} &\leq \|\left(S(t)-S_h(t)P_h\right)X_0\|_{L^{2p}(\Omega,\mathcal{H})}\\ &\quad+\left\|\int_0^t \left(S(t-s)-S_h(t-s)P_h\right)F(X(s))ds\right\|_{L^{2p}(\Omega,\mathcal{H})}\\ &\quad+\left\|\int_0^t \left(S(t-s)-S_h(t-s)P_h\right)dW(s)\right\|_{L^{2p}(\Omega,\mathcal{H})}\\ &=: \mathrm{IV}_1+\mathrm{IV}_2+\mathrm{IV}_3.\end{aligned} \tag{39}$$

Using Lemma 3.5 (i) with $r=\alpha=\beta$ and Assumption 2.1, yields

$$\mathrm{IV}_1 \leq Ch^\beta\|A^{\frac{\beta}{2}}X_0\|_{L^{2p}(\Omega,\mathcal{H})} \leq Ch^\beta. \tag{40}$$

Using the triangle inequality, we split IV_2 as follows

$$\begin{aligned}\mathrm{IV}_2 &\leq \left\|\int_0^t \left(S(t-s)-S_h(t-s)P_h\right)\left[F(X(s))-F(X(t))\right]ds\right\|_{L^{2p}(\Omega,\mathcal{H})}\\ &\quad+\left\|\int_0^t \left(S(t-s)-S_h(t-s)P_h\right)F(X(t))ds\right\|_{L^{2p}(\Omega,\mathcal{H})} =: \mathrm{IV}_{21}+\mathrm{IV}_{22}. \end{aligned}\tag{41}$$

Using Lemma 3.5 (i) with $r=\beta$ and $\gamma=0$, the triangle inequality, Lemma 2.2 and Proposition 2.4, we obtain

$$\begin{aligned}\mathrm{IV}_{21} &\leq Ch^\beta\int_0^t (t-s)^{-\frac{\beta}{2}}\|F(X(s))-F(X(t))\|_{L^{2p}(\Omega,\mathcal{H})}ds\\ &\leq Ch^\beta\int_0^t \Big\{(t-s)^{-\frac{\beta}{2}}\|X(t)-X(s)\|_{L^{4p}(\Omega,\mathcal{H})}\\ &\qquad\times\left(1+\|X(t)\|^{c_1}_{L^{4pc_1}(\Omega,E)}+\|X(s)\|^{c_1}_{L^{4pc_1}(\Omega,E)}\right)\Big\}ds\\ &\leq Ch^\beta\int_0^t (t-s)^{-\frac{\beta}{2}+\frac{1}{2}}ds \leq Ch^\beta. \end{aligned}\tag{42}$$

Using Lemma 3.5 (iii) with $\rho=0$ and estimate (16), yields

$$\mathrm{IV}_{22}\leq Ch^2\|F(X(t))\|_{L^{2p}(\Omega,\mathcal{H})}\leq Ch^2. \tag{43}$$

Substituting (43) and (42) into (41), we obtain

$$\mathrm{IV}_2\leq Ch^\beta. \tag{44}$$

Using the Burkhölder-Davis-Gundy inequality (cf. [35, Theorem 4.37]), Lemma 3.5 (ii) and Assumption 2.2, we obtain

$$\begin{aligned}\mathrm{IV}_3 &\leq \left(\int_0^t \|(S(t-s)-S_h(t-s)P_h)Q^{\frac{1}{2}}\|^2_{L^{2p}(\Omega,\mathcal{L}_2(\mathcal{H}))}ds\right)^{1/2}\\ &\leq Ch^\beta\|A^{\frac{\beta-1}{2}}Q^{\frac{1}{2}}\|_{\mathcal{L}_2(\mathcal{H})}\leq Ch^\beta. \end{aligned}\tag{45}$$

Substituting (45), (44) and (40) into (39), yields

$$\|X(t)-\widetilde{X}^h(t)\|_{L^{2p}(\Omega,\mathcal{H})}\leq Ch^\beta \quad \forall t\in[0,T]. \tag{46}$$

Let us introduce the following error representation $\widetilde{e}^h(t):=\widetilde{X}^h(t)-X^h(t)$, where $\widetilde{X}^h(t)$ is given by (28). Obviously $\widetilde{e}^h(t)$ is differentiable with repect to time and satisfies

$$\frac{d}{dt}\widetilde{e}^h(t)+A_h\widetilde{e}^h(t)=P_h\left(F(X(t))-F(X^h(t))\right), \quad t\in(0,T], \quad \widetilde{e}^h(0)=0. \tag{47}$$

Testing (47) with $\widetilde{e}^h(t)$, using Lemma 2.1 and Cauchy-Schwarz's inequality, yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{ds}\|\widetilde{e}^h(s)\|^2 + \left\langle A_h\widetilde{e}^h(s), \widetilde{e}^h(s)\right\rangle\\
&\quad= \left\langle F(\widetilde{X}^h(s)) - F(X^h(s)), \widetilde{e}^h(s)\right\rangle + \left\langle F(X(s)) - F(\widetilde{X}^h(s)), \widetilde{e}^h(s)\right\rangle\\
&\quad\leq C\|\widetilde{e}^h(s)\|^2 + C\|F(X(s)) - F(\widetilde{X}^h(s))\|\|\widetilde{e}^h(s)\|\\
&\quad\leq C\|\widetilde{e}^h(s)\|^2 + C\|F(X(s)) - F(\widetilde{X}^h(s))\|^2.
\end{aligned} \tag{48}$$

Using the coercivity estimate (5) and the fact that $\widetilde{e}^h(s) \in V_h$, yields

$$\lambda_0\|\widetilde{e}^h(s)\|_1^2 \leq a\left(\widetilde{e}^h(s), \widetilde{e}^h(s)\right) = \left\langle A_h\widetilde{e}^h(s), \widetilde{e}^h(s)\right\rangle.$$

Substituting the preceding estimate into (48), we obtain

$$\frac{1}{2}\frac{d}{ds}\|\widetilde{e}^h(s)\|^2 + \lambda_0\|\widetilde{e}^h(s)\|_1^2 \leq C\|\widetilde{e}^h(s)\|^2 + C\|F(X(s)) - F(\widetilde{X}^h(s))\|^2.$$

Integrating the preceding estimate over $[0, t]$, yields

$$\|\widetilde{e}^h(t)\|^2 \leq C\int_0^t \|\widetilde{e}^h(s)\|^2 ds + C\int_0^t \|F(X(s)) - F(\widetilde{X}^h(s))\|^2 ds. \tag{49}$$

Taking the $L^p(\Omega, \mathcal{H})$-norm in (49), using Hölder's inequality, Proposition 2.4, the estimate (46) and Proposition 3.1 with $c = \frac{\nu}{c_1}$ for an arbitrarily small $\nu > 0$, we obtain

$$\begin{aligned}
&\|\widetilde{e}^h(t)\|^2_{L^{2p}(\Omega,\mathcal{H})}\\
&\leq C\int_0^t \|\widetilde{e}^h(s)\|^2_{L^{2p}(\Omega,\mathcal{H})} ds + C\int_0^t \|F(X(s)) - F(\widetilde{X}^h(s))\|^2_{L^{2p}(\Omega,\mathcal{H})} ds\\
&\leq C\int_0^t \|\widetilde{e}^h(s)\|^2_{L^{2p}(\Omega,\mathcal{H})} ds\\
&\quad + C\int_0^t \|X(s) - \widetilde{X}^h(s)\|^2_{L^{8p}(\Omega,\mathcal{H})}\left(1 + \|X(s)\|^{2c_1}_{L^{8pc_1}(\Omega,E)} + \|\widetilde{X}^h(s)\|^{2c_1}_{L^{8pc_1}(\Omega,E)}\right) ds\\
&\leq C\int_0^t \|\widetilde{e}^h(s)\|^2_{L^{2p}(\Omega,\mathcal{H})} ds + Ch^{2\beta+2c_1-dc_1}|\ln(h)|^{2\nu}.
\end{aligned}$$

Applying Gronwall's lemma to the preceding estimate and taking the square root, yields

$$\|\widetilde{e}^h(t)\|_{L^{2p}(\Omega,\mathcal{H})} \leq Ch^{\beta+c_1-\frac{dc_1}{2}}|\ln(h)|^{\nu}.$$

Substituting the preceding estimate and (46) into (38) ends the proof. □

Below, we discuss the range of the parameter c_1 necessary for the semi-discrete approximation $X^h(t)$ converges to $X(t)$. This justifies the requirements on c_1 in Assumption 2.3.

Remark 3.1. *In order to have convergence to 0 of the upper bound of the error estimate in Theorem 3.1, one can require* $h^{\beta+c_1-\frac{dc_1}{2}-\nu}\to 0$ *as* $h\to 0$*, since* $\lim\limits_{h\to 0} h|\ln(h)|=0$*. It is therefore enough to require* $\beta+c_1-\frac{dc_1}{2}-\nu>0$*.*

(i) *For* $d\in\{1,2\}$*,* $\beta+c_1-\frac{dc_1}{2}-\nu>0$ *holds for all* $c_1>0$ *and* $0<\nu<1$*, since* $\beta\geq 1$*.*

(ii) *For* $d=3$*,* $\beta+c_1-\frac{dc_1}{2}-\nu>0$ *holds for all* $\nu>0$ *and* $0<c_1<2\beta-\nu$*. The restriction on* c_1 *in Assumption 2.3 for* $d=3$ *is only due to the fact one whishes the error estimate in Theorem 3.1 to converges to* 0 *as* $h\to 0$*.*

Remark 3.2. *We observe from Theorem 3.1 that the rate of convergence depends on the polynomial growth of the nonlinearity. For* $d=1,2$*, since* $c_1-\frac{d}{2}c_1\geq 0$*, the rate of convergence in space is* $\mathcal{O}\left(h^\beta+h^\beta|\ln(h)|^\nu\right)$*, which is independent of the growth* c_1 *and is almost* β*. For* $d=3$*, a reduction of the order of convergence occurs. In particular, we obtain convergence rate* $\mathcal{O}\left(h^{\beta-\frac{c_1}{2}}|\ln(h)|^\nu\right)$ *in space. In the case of Lipschitz nonlinearity (i.e., Assumption 2.3 with* $c_1=0$*) it follows from Theorem 3.1 that the rate of convergence in space is* $\mathcal{O}(h^\beta+h^\beta|\ln(h)|^\nu)$ *for* $d\in\{1,2,3\}$*, which is in agreement with existing results in the literature.*

4. Fully discrete scheme and main result

Let $t_m=m\Delta t\in[0,T]$, where $\Delta t=T/M$ and $M\in\mathbb{N}$. Applying the backward Euler method to (27) yields the following fully discrete scheme

$$\left\{\begin{array}{l} X_0^h=P_hX_0,\\ X_{m+1}^h=S_{h,\Delta t}X_m^h+\Delta t S_{h,\Delta t}P_hF(X_{m+1}^h)+S_{h,\Delta t}P_h\Delta W_m, \quad m=0,1,\cdots,M-1, \end{array}\right. \tag{50}$$

where ΔW_m and $S_{h,\Delta t}$ are given respectively by

$$\Delta W_m:=W(t_{m+1})-W(t_m) \quad \text{and} \quad S_{h,\Delta t}:=(\mathbf{I}+\Delta t A_h)^{-1}.$$

The fully discrete scheme (50) can be equivalently written as

$$\langle X_m^h,\varphi_h\rangle+\Delta t\langle A_hX_m^h,\varphi_h\rangle=\langle X_{m-1}^h,\varphi_h\rangle+\Delta t\langle P_hF(X_m^h),\varphi_h\rangle+\langle P_h\Delta W_{m-1},\varphi_h\rangle \tag{51}$$

for all $\varphi_h\in V_h$ and $m=1,\cdots,M$.

In the next lemma, we prove the solvability and the measurability of the scheme (51).

Lemma 4.1. *Let the step-size be such that* $\Delta t\leq\frac{1}{C_F}$ *(where* $C_F>0$ *is the constant in Lemma 2.1). Then the numerical scheme (51) has a unique solution* $\{X_h^m\}_{m=1}^M$ *in* V_h*. Furthermore, the* V_h*-valued random variable* X_h^m *is* $\mathcal{F}_{t_m}$*-measurable, for* $m=1,\cdots,M$*.*

Proof. We use an induction argument. We assume that for $X_0^h=P_hX_0\in L^p(\Omega,\mathcal{F}_0,\mathbb{P};\mathcal{H})$ there exist V_h-valued random variables $\{X_j^h\}_{j=1}^{m-1}$ that satisfy (51) and that X_j^h are $\mathcal{F}_{t_j}$-measurable for $j=1,\cdots,m-1$. We aim to prove the existence of an $\mathcal{F}_{t_m}$-measurable random variable X_m^h satisfying (51).

For each $\omega \in \Omega$ the scheme (51) defines a canonical mapping $\mathbf{h}_\omega : V_h \to V_h$ for which it holds $\mathbf{h}_\omega(X^h_m(\omega)) \equiv 0$. Consequently for each $U \in V_h$ we write

$$\mathbf{h}_\omega(U) := \frac{1}{\Delta t}\left(U - X^h_{m-1}(\omega)\right) + A_h U - P_h F(U) - \frac{1}{\Delta t}\Delta W_{m-1}(\omega).$$

Using Cauchy-Schwarz's inequality, it follows that

$$-\langle X^h_{m-1}(\omega), U\rangle \geq -\|X^h_{m-1}(\omega)\|\|U\| \text{ and } -\langle \Delta W_{m-1}(\omega), U\rangle \geq -\|\Delta W_{m-1}(\omega)\|\|U\|.$$

Using the coercivity estimate (5), Lemma 2.1 and the preceding estimates, we obtain

$$\langle \mathbf{h}_\omega(U), U\rangle \geq \frac{1}{\Delta t}\|U\|\left\{(1 - C_F\Delta t)\|U\| - \|X^h_{m-1}(\omega)\| - \|\Delta W_{m-1}(\omega)\|\right\}.$$

Let $\mathbf{R}_\omega$ be a real number such that

$$(1 - C_F\Delta t)\mathbf{R}_\omega - \|X^h_{m-1}(\omega)\| - \|\Delta W_{m-1}(\omega)\|.$$

Choosing $U \in V_h$ such that $\|U\| = \mathbf{R}_\omega$, it follows from above that $\langle \mathbf{h}_\omega(U), U\rangle \geq 0$. Consequently, applying [10, Lemma 3.1], it follows that for each $\omega \in \Omega$ there exists an V_h-valued random variable $X^h_m(\omega)$ satisfying (51).

To show the uniqueness, we consider $U, \widetilde{U} \in V_h$, such that $\mathbf{h}_\omega(U) \equiv \mathbf{h}_\omega(\widetilde{U}) \equiv 0$. Then

$$0 \equiv \mathbf{h}_\omega(U) - \mathbf{h}_\omega(\widetilde{U}) = \frac{1}{\Delta t}(U - \widetilde{U}) + A_h(U - \widetilde{U}) - (P_h F(U) - P_h F(\tilde{U})).$$

Testing the preceding estimate with $U - \widetilde{U}$, using (5) and Lemma 2.1, we obtain

$$0 = \langle \mathbf{h}_\omega(U) - \mathbf{h}_\omega(\widetilde{U}), U - \widetilde{U}\rangle \geq \frac{1}{\Delta t}(1 - C_F\Delta t)\|U - \widetilde{U}\|^2.$$

For $\Delta t \leq \frac{1}{C_F}$, it follows that $\|U - \widetilde{U}\|^2 \leq 0$, which yields the uniqueness.

Finally, the $\mathcal{F}_{t_m}$-measurability of X^h_m follows by applying [10, Lemma 3.2]. $\square$

In the rest of this paper, C is a positive constant independent of h, m, M and Δt; that may change from one place to another.

We state in the next theorem the convergence of the fully discrete scheme.

Theorem 4.1. *Let X be the mild solution of* (1) *and X^h_m be the numerical solution defined in* (50). *Let Assumptions 2.1, 2.3 and 2.2 be fulfilled. Let $\Delta t \leq \frac{1}{C_F}$ (where $C_F > 0$ is the constant in Lemma 2.1). Then for any $p \geq 1$ the following error estimate holds*

$$\|X(t_m) - X^h_m\|_{L^{2p}(\Omega,\mathcal{H})} \leq C\left(h^\beta + h^{\beta + c_1 - \frac{dc_1}{2}}|\ln(h)|^\nu + \Delta t^{\frac{\beta}{2} - \xi}\right), \quad m = 0, 1, \cdots, M.$$

where c_1 is as in Assumption 2.3, β is as in Assumption 2.1, $\nu, \xi > 0$ are arbitrarily small real numbers and the constant C is independent of m, M, h, Δt, ν and ξ.

Proof. As in the semi-discrete case, we introduce the following intermediate process

$$\widetilde{X}^h_m - \widetilde{X}^h_{m-1} + \Delta t A_h \widetilde{X}^h_m = \Delta t P_h F(X(t_m)) + P_h \Delta W_m, \quad \widetilde{X}^h_0 = P_h X_0. \tag{52}$$

Using the triangle inequality we split the error as follows

$$\|X(t_m) - X^h_m\|_{L^{2p}(\Omega,\mathcal{H})} \leq \|X(t_m) - \widetilde{X}^h_m\|_{L^{2p}(\Omega,\mathcal{H})} + \|\widetilde{X}^h_m - X^h_m\|_{L^{2p}(\Omega,\mathcal{H})}. \tag{53}$$

The numerical scheme (52) can be written in the following equivalent form

$$\widetilde{X}^h_m = S_{h,\Delta t}\widetilde{X}^h_{m-1} + \Delta t S_{h,\Delta t} P_h F(X(t_m)) + S_{h,\Delta t} P_h \Delta W_m.$$

Iterating the preceding numerical solution, yields

$$\widetilde{X}^h_m = S^m_{h,\Delta t} P_h X_0 + \Delta t \sum_{j=0}^{m-1} S^{m-j}_{h,\Delta t} P_h F\left(X(t_{j+1})\right) + W^m_{A_h}.$$

Iterating the numerical scheme (50), yields

$$X^h_m = S^m_{h,\Delta t} P_h X_0 + \Delta t \sum_{j=0}^{m-1} S^{m-j}_{h,\Delta t} P_h F(X^h_{j+1}) + W^m_{A_h}.$$

Subtracting the two preceding identities, taking the $L^{2p}(\Omega,\mathcal{H})$-norm, we obtain

$$\begin{aligned}
&\|X(t_m) - \widetilde{X}^h_m\|_{L^{2p}(\Omega,\mathcal{H})} \\
&\quad \leq \| \left(S(t_m) - S^m_{h,\Delta t} P_h\right) X_0\|_{L^{2p}(\Omega,\mathcal{H})} \\
&\qquad + \left\| \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \left[S(t_m - s)F(X(s)) - S^{m-j}_{h,\Delta t} P_h F(X(t_{j+1}))\right] ds \right\|_{L^{2p}(\Omega,\mathcal{H})} \\
&\qquad + \left\| \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \left(S(t_m - s) - S^{m-j}_{h,\Delta t} P_h\right) dW(s) \right\|_{L^{2p}(\Omega,\mathcal{H})} \\
&\quad =: \mathrm{J}_1 + \mathrm{J}_2 + \mathrm{J}_3.
\end{aligned} \tag{54}$$

Using Lemma 3.5 (i) with $r = \alpha = \beta$, [41, Lemma 3.3 (iv)] and Assumption 2.1, yields

$$\begin{aligned}
\mathrm{J}_1 &\leq \| \left(S(t_m) - S_h(t_m) P_h\right) X_0\|_{L^{2p}(\Omega,\mathcal{H})} + \| \left(S_h(t_m) - S^m_{h,\Delta t} P_h\right) X_0\|_{L^{2p}(\Omega,\mathcal{H})} \\
&\leq C\left(h^\beta + \Delta t^{\frac{\beta}{2}}\right).
\end{aligned} \tag{55}$$

In order to estimate J_2, we use the triangle inequality to decompose it as follows:

$$\begin{aligned}
\mathrm{J}_2 \leq &\left\| \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} S(t_m - s)\left(F(X(s)) - F(X(t_{j+1}))\right) \right\|_{L^{2p}(\Omega,\mathcal{H})} \\
&+ \left\| \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \left(S(t_m - s) - S^{m-j}_{h,\Delta t} P_h\right) F(X(t_{j+1})) ds \right\|_{L^{2p}(\Omega,\mathcal{H})} := \mathrm{J}_{21} + \mathrm{J}_{22}.
\end{aligned} \tag{56}$$

Let us start by estimating J_{22}. Using the triangle inequality, we split J_{22} as follows

$$\begin{aligned}\mathrm{J}_{22} &\leq \left\|\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\left(S(t_m-s)-S_h(t_m-s)P_h\right)F(X(t_{j+1}))ds\right\|_{L^{2p}(\Omega,\mathcal{H})}\\ &\quad+\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\|(S_h(t_m-s)-S_{h,\Delta t}^{m-j})P_hF(X(t_{j+1}))\|_{L^{2p}(\Omega,\mathcal{H})}ds\\ &=: \mathrm{J}_{22}^{(1)}+\mathrm{J}_{22}^{(2)}.\end{aligned} \tag{57}$$

Using [41, Lemma 3.3 (ii)], the estimates (16) and $\Delta t\sum_{j=1}^{m} t_j^{-1+\alpha}<C$, for $\alpha>0$, we obtain

$$\mathrm{J}_{22}^{(2)} \leq C\Delta t^{1-\xi}\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}} t_{m-j}^{-1+\xi}\|P_hF(X(t_{j+1}))\|_{L^{2p}(\Omega,\mathcal{H})}ds \leq C\Delta t^{2-\xi}\sum_{j=0}^{m-1} t_{m-j}^{-1+\xi}\leq C\Delta t^{1-\xi}. \tag{58}$$

Using the triangle inequality we split $\mathrm{J}_{22}^{(1)}$ as follows

$$\begin{aligned}\mathrm{J}_{22}^{(1)} &\leq \left\|\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\left(S(t_m-s)-S_h(t_m-s)P_h\right)F(X(t_m))ds\right\|_{L^{2p}(\Omega,\mathcal{H})}\\ &\quad+\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\|\left(S(t_m-s)-S_h(t_m-s)P_h\right)\left(F(X(t_{j+1}))-F(X(t_m))\right)\|_{L^{2p}(\Omega,\mathcal{H})}\,ds \\ &=: \mathrm{J}_{22}^{(11)}+\mathrm{J}_{22}^{(12)}.\end{aligned} \tag{59}$$

Using Lemma 3.5 (iii) with $\rho=0$ and the estimate (16), it follows that

$$\mathrm{J}_{22}^{(11)}=\left\|\int_0^{t_m}\left(S(t_m-s)-S_h(t_m-s)P_h\right)F(X(t_m))ds\right\|_{L^{2p}(\Omega,\mathcal{H})}\leq Ch^2. \tag{60}$$

Using Lemma 3.5 (i) with $r=2$, $\alpha=0$ and Proposition 2.4, it holds that

$$\begin{aligned}\mathrm{J}_{22}^{(12)} &\leq Ch^2\sum_{j=0}^{m-2}\int_{t_j}^{t_{j+1}}(t_m-s)^{-1}\|F(X(t_{j+1})-F(X(t_m))\|_{L^{2p}(\Omega,\mathcal{H})}ds\\ &\leq Ch^2\sum_{j=0}^{m-2}\int_{t_j}^{t_{j+1}}(t_m-s)^{-1}\|X(t_{j+1})-X(t_m)\|_{L^{4p}(\Omega,\mathcal{H})}ds\\ &\leq Ch^2\sum_{j=0}^{m-2}\int_{t_j}^{t_{j+1}}(t_m-t_{j+1})^{-1}(t_m-t_{j+1})^{\frac{\beta-1}{2}}ds\\ &\leq Ch^2\sum_{j=0}^{m-2}\int_{t_j}^{t_{j+1}}(t_m-t_{j+1})^{-1+\frac{\beta-1}{2}}ds\leq Ch^2.\end{aligned} \tag{61}$$

Substituting (61) and (60) into (59) yields

$$\mathrm{J}_{22}^{(1)} \leq Ch^2. \tag{62}$$

Substituting (62) and (58) into (57) yields

$$\mathrm{J}_{22} \leq C\left(h^2 + \Delta t^{1-\xi}\right). \tag{63}$$

To estimate J_{21}, we first use Taylor's formula in Banach space. This gives

$$\begin{aligned} &F(X(s)) - F(X(t_{j+1})) \\ &= \left(\int_0^1 F'\left(X(t_{j+1}) + \lambda\left(X(s) - X(t_{j+1})\right)\right) d\lambda\right)\left(X(s) - X(t_{j+1})\right). \end{aligned} \tag{64}$$

Let $s \in [t_j, t_{j+1}]$. The mild solution $X(t_{j+1})$ can be written as follows

$$X(t_{j+1}) = S(t_{j+1}-s)X(s) + \int_s^{t_{j+1}} S(t_{j+1}-\sigma)F(X(\sigma))d\sigma + \int_s^{t_{j+1}} S(t_{j+1}-\sigma)dW(\sigma). \tag{65}$$

Substituting (65) into (64) yields

$$\begin{aligned} F(X(s)) - F(X(t_{j+1})) = &-\mathcal{I}_{t_{j+1},s}\left(S(t_{j+1}-s) - \mathbf{I}\right)X(s) \\ &- \mathcal{I}_{t_{j+1},s}\int_s^{t_{j+1}} S(t_{j+1}-\sigma)F(X(\sigma))d\sigma \\ &- \mathcal{I}_{t_{j+1},s}\int_s^{t_{j+1}} S(t_{j+1}-\sigma)dW(\sigma), \end{aligned} \tag{66}$$

where the remainder $\mathcal{I}_{t_k,s}$ is given by

$$\mathcal{I}_{t_k,s} := \int_0^1 F'\left(X(t_k) + \lambda\left(X(s) - X(t_k)\right)\right) d\lambda, \quad k \in \{0, 1, \cdots, M-1\}. \tag{67}$$

Using Lemma 2.3 and Proposition 2.4 one can easily check that for any $p \geq 1$

$$\|\mathcal{I}_{t_k,s}\|_{L^{2p}(\Omega,\mathcal{H})} \leq C, \quad k \in \{0, \cdots, M-1\}, \quad s \in [t_k, t_{k+1}]. \tag{68}$$

Substituting (66) into the expression of J_{21} in (56) and using the triangle inequality, yields

$$\begin{aligned} \mathrm{J}_{21} \leq &\left\|\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}} S(t_m-s)\mathcal{I}_{t_{j+1},s}\left(S(t_{j+1}-s) - \mathbf{I}\right)X(s)\right\|_{L^{2p}(\Omega,\mathcal{H})} \\ &+ \left\|\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}} S(t_m-s)\mathcal{I}_{t_{j+1},s}\int_s^{t_{j+1}} S(t_{j+1}-\sigma)F(X(\sigma))d\sigma ds\right\|_{L^{2p}(\Omega,\mathcal{H})} \\ &+ \left\|\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}} S(t_m-s)\mathcal{I}_{t_{j+1},s}\int_s^{t_{j+1}} S(t_{j+1}-\sigma)dW(\sigma)ds\right\|_{L^{2p}(\Omega,\mathcal{H})} \\ =: &\ \mathrm{J}_{21}^{(1)} + \mathrm{J}_{21}^{(2)} + \mathrm{J}_{21}^{(3)}. \end{aligned} \tag{69}$$

Using the smoothing properties of the semigroup (cf. (6)), (68) and Proposition 2.4, yields

$$
\begin{aligned}
\mathrm{J}_{21}^{(1)} &\leq \sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}} \|S(t_m-s)\mathcal{I}_{t_{j+1},s}\|_{\mathcal{L}(\mathcal{H})}\|(S(t_{j+1}-s)-\mathbf{I})A^{-\frac{\beta}{2}}A^{\frac{\beta}{2}}X(s)\|_{L^{2p}(\Omega,\mathcal{H})}ds\\
&\leq C\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}} \|(S(t_{j+1}-s)-\mathbf{I})A^{-\frac{\beta}{2}}\|_{\mathcal{L}(\mathcal{H})}\|A^{\frac{\beta}{2}}X(s)\|_{L^{2p}(\Omega,\mathcal{H})}ds \qquad (70)\\
&\leq C\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}(t_{j+1}-s)^{\frac{\beta}{2}}ds \leq C\Delta t^{\frac{\beta}{2}}.
\end{aligned}
$$

Using the smoothing properties of the semi-group (cf. (6)), (68) and (16), we obtain

$$
\begin{aligned}
\mathrm{J}_{21}^{(2)} &\leq \sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\int_s^{t_{j+1}} \left\|S(t_m-s)\mathcal{I}_{t_{j+1},s}S(t_{j+1}-\sigma)F(X(\sigma))\right\|_{L^{2p}(\Omega,\mathcal{H})}d\sigma ds\\
&\leq C\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}(t_{j+1}-s)ds \leq C\Delta t. \qquad (71)
\end{aligned}
$$

Using the triangle inequality, we split $\mathrm{J}_{21}^{(3)}$ as follows

$$
\begin{aligned}
\mathrm{J}_{21}^{(3)} &\leq \left\|\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}} S(t_m-s)\mathcal{I}_{t_{j+1},t_{j+1}}\int_s^{t_{j+1}} S(t_{j+1}-\sigma)dW(\sigma)ds\right\|_{L^{2p}(\Omega,\mathcal{H})}\\
&\quad+\left\|\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}} S(t_m-s)\left(\mathcal{I}_{t_{j+1},s}-\mathcal{I}_{t_{j+1},t_{j+1}}\right)\int_s^{t_{j+1}} S(t_{j+1}-\sigma)dW(\sigma)ds\right\|_{L^{2p}(\Omega,\mathcal{H})}\\
&=: \mathrm{J}_{21}^{(31)}+\mathrm{J}_{21}^{(32)}. \qquad (72)
\end{aligned}
$$

Let us start with the estimate of $\mathrm{J}_{21}^{(32)}$. Using triangle and Hölder's inequalities, yields

$$
\begin{aligned}
&\|\mathrm{J}_{21}^{(32)}\|_{L^{2p}(\Omega,\mathcal{H})}\\
&\leq \sum_{j=1}^{m-1}\left\|\int_{t_j}^{t_{j+1}}\left(S(t_m-s)\left(\mathcal{I}_{t_{j+1},s}-\mathcal{I}_{t_{j+1},t_{j+1}}\right)\int_s^{t_{j+1}} S(t_{j+1}-r)dW(r)\right)ds\right\|_{L^{2p}(\Omega,\mathcal{H})}\\
&\leq \sum_{j=1}^{m-1}\int_{t_j}^{t_{j+1}}\left\|S(t_m-s)\left(\mathcal{I}_{t_{j+1},s}-\mathcal{I}_{t_{j+1},t_{j+1}}\right)\int_s^{t_{j+1}} S(t_{j+1}-r)dW(r)\right\|_{L^{2p}(\Omega,\mathcal{H})}ds\\
&\leq \sum_{j=1}^{m-1}\left(\int_{t_j}^{t_{j+1}}1^2ds\right)^{\frac{1}{2}}\left[\int_{t_j}^{t_{j+1}}\left\|S(t_m-s)\left(\mathcal{I}_{t_{j+1},s}-\mathcal{I}_{t_{j+1},t_{j+1}}\right)\int_s^{t_{j+1}} S(t_{j+1}-r)dW(r)\right\|^2_{L^{2p}(\Omega,\mathcal{H})}ds\right]^{\frac{1}{2}}\\
&\leq C\Delta t^{\frac{1}{2}}\sum_{j=1}^{m-1}\left[\int_{t_j}^{t_{j+1}}\left\|\int_s^{t_{j+1}} S(t_m-s)\left(\mathcal{I}_{t_{j+1},s}-\mathcal{I}_{j+1,j+1}\right)S(t_{j+1}-r)dW(r)\right\|^2_{L^{2p}(\Omega,\mathcal{H})}ds\right]^{\frac{1}{2}}.
\end{aligned}
$$

Using the Burkhölder-Davis-Gundy inequality (cf. [35, Theorem 4.37] or [24, Proposition 2.6]), we obtain from the preceding estimate that

$$
\begin{aligned}
&\|\mathrm{J}_{21}^{(32)}\|_{L^{2p}(\Omega,\mathcal{H})}\\
&\leq C\Delta t^{\frac{1}{2}}\sum_{j=1}^{m-1}\left[\int_{t_j}^{t_{j+1}}\int_{s}^{t_{j+1}}\left\|S(t_m-s)\left(\mathcal{I}_{t_{j+1},s}-\mathcal{I}_{t_{j+1},t_{j+1}}\right)S(t_{j+1}-r)\right\|^2_{L^{2p}(\Omega,\mathcal{L}_2^0)}drds\right]^{\frac{1}{2}}.
\end{aligned}
\tag{73}
$$

Using the smoothing properties of the semigroup (cf. (6)) and Assumption 2.2, we have

$$
\begin{aligned}
&\mathbb{E}\|S(t_m-s)\left(\mathcal{I}_{t_{j+1},s}-\mathcal{I}_{t_{j+1},t_{j+1}}\right)S(t_{j+1}-r)Q^{\frac{1}{2}}\|^{4p}_{\mathcal{L}_2(\mathcal{H})}\\
&\leq \|A^{\eta}S(t_m-s)\|^{4p}_{\mathcal{L}(H)}\mathbb{E}\|A^{-\eta}(\mathcal{I}_{t_{j+1},s}-\mathcal{I}_{t_{j+1},t_{j+1}})\|^2_{L^{2p}(\Omega,\mathcal{H})}\|S(t_{j+1}-r)A^{\frac{1-\beta}{2}}A^{\frac{\beta-1}{2}}Q^{\frac{1}{2}}\|^{4p}_{\mathcal{L}_2(\mathcal{H})}\\
&\leq \|A^{\eta}S(t_m-t_{m-j})\|^{4p}_{\mathcal{L}(\mathcal{H})}\|S(t_{m-j}-s)\|^{4p}_{\mathcal{L}(\mathcal{H})}\\
&\quad\times\mathbb{E}\|A^{-\eta}(\mathcal{I}_{t_{j+1},s}-\mathcal{I}_{t_{j+1},t_{j+1}})\|^{4p}_{\mathcal{L}(\mathcal{H})}\|A^{\frac{1-\beta}{2}}S(t_{j+1}-r)\|^{4p}_{\mathcal{L}(\mathcal{H})}\|A^{\frac{\beta-1}{2}}Q^{\frac{1}{2}}\|^{4p}_{\mathcal{L}_2(\mathcal{H})}\\
&\leq Ct_{m-j-1}^{-4p\eta}\mathbb{E}\|A^{-\eta}(\mathcal{I}_{t_{j+1},s}-\mathcal{I}_{t_{j+1},t_{j+1}})\|^{4p}_{\mathcal{L}(\mathcal{H})}.
\end{aligned}
\tag{74}
$$

From the definition of $\mathcal{I}_{t_{j+1},s}$ in (67), using Lemma 2.3 and Proposition 2.4 we estimate

$$
\begin{aligned}
&\|A^{-\eta}\left(\mathcal{I}_{t_{j+1},s}-\mathcal{I}_{t_{j+1},t_{j+1}}\right)\|_{\mathcal{L}(H)}\\
&\qquad\leq\int_0^1\left\|A^{-\eta}\left[F'\left(X(t_{j+1})+\lambda\left(X(s)-X(t_{j+1})\right)\right)-F'\left(X(t_{j+1})\right)\right]\right\|_{\mathcal{L}(H)}d\lambda\\
&\qquad\leq C\int_0^1\lambda\|X(s)-X(t_{j+1})\|\left(1+\|X(s)\|_E^{c_1}+\|X(t_{j+1})\|_E^{c_1}\right)d\lambda\\
&\qquad\leq C\|X(s)-X(t_{j+1})\|\left(1+\|X(s)\|_E^{c_1}+\|X(t_{j+1})\|_E^{c_1}\right).
\end{aligned}
\tag{75}
$$

Substituting (75) into (74) and using Proposition 2.4, we obtain

$$
\begin{aligned}
&\mathbb{E}\left\|S(t_m-s)\left(\mathcal{I}_{t_{j+1},s}-\mathcal{I}_{t_{j+1},t_{j+1}}\right)S(s-r)Q^{\frac{1}{2}}\right\|^{4p}_{\mathcal{L}_2(\mathcal{H})}\\
&\qquad\leq Ct_{m-j-1}^{-4p\eta}\|X(s)-X(t_{j+1})\|^{4p}_{L^{8p}(\Omega,\mathcal{H})}\\
&\qquad\quad\times\left(1+\|X(s)\|^{4pc_1}_{L^{8pc_1}(\Omega,E)}+\|X(t_{j+1})\|^{4pc_1}_{L^{8pc_1}(\Omega,E)}\right)\\
&\qquad\leq Ct_{m-j-1}^{-4p\eta}(t_{j+1}-s)^{2p}.
\end{aligned}
\tag{76}
$$

Substituting (76) into (73), yields

$$
\begin{aligned}
\|\mathrm{J}_{21}^{(32)}\|_{L^{2p}(\Omega,\mathcal{H})} &\leq C\Delta t + C\Delta t^{\frac{1}{2}}\sum_{j=0}^{m-2}\left[\int_{t_j}^{t_{j+1}}\int_s^{t_{j+1}} t_{m-j-1}^{-2\eta}(t_{j+1}-s)drds\right]^{\frac{1}{2}}\\
&\leq C\Delta t + C\Delta t^{\frac{1}{2}}\sum_{j=0}^{m-2}\left[\int_{t_j}^{t_{j+1}} t_{m-j-1}^{-2\eta}(t_{j+1}-s)^{\min(2\beta,2)}ds\right]^{\frac{1}{2}}\\
&\leq C\Delta t + C\Delta t\Delta t^{\frac{1}{2}}\sum_{j=0}^{m-2}\left[\int_{t_j}^{t_{j+1}} t_{m-j-1}^{-2\eta}ds\right]^{\frac{1}{2}}\\
&\leq C\Delta t + C\Delta t\sum_{j=0}^{m-2}\Delta t\, t_{m-j-1}^{-\eta} \leq C\Delta t.
\end{aligned}
\tag{77}
$$

Now let us estimate $J_{21}^{(31)}$. Using triangle inequality, we split $\mathrm{J}_{21}^{(31)}$ as follows

$$
\begin{aligned}
\mathrm{J}_{21}^{(31)} \leq &\left\|\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}} S(t_m-s)\left(\mathcal{I}_{t_{j+1},t_{j+1}}-\mathcal{I}_{t_j,t_j}\right)\int_s^{t_{j+1}} S(t_{j+1}-\sigma)dW(\sigma)ds\right\|_{L^{2p}(\Omega,\mathcal{H})}\\
&+\left\|\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}} S(t_m-s)\mathcal{I}_{t_j,t_j}\int_s^{t_{j+1}} S(t_{j+1}-\sigma)dW(\sigma)ds\right\|_{L^{2p}(\Omega,\mathcal{H})}\\
=:&\ \mathrm{J}_{21}^{(311)}+\mathrm{J}_{21}^{(312)}.
\end{aligned}
\tag{78}
$$

For a set $A\subseteq\mathbb{R}$, let χ_A be its characteristic function. We can rewrite $\mathrm{J}_{21}^{(311)}$ as follows

$$
\mathrm{J}_{21}^{(311)} = \left\|\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\int_{t_j}^{t_{j+1}} \chi_{[s,t_{j+1})}(\sigma)S(t_m-s)\mathcal{I}_{t_j,t_j}S(t_{j+1}-\sigma)dW(\sigma)ds\right\|_{L^{2p}(\Omega,\mathcal{H})}.
$$

Using the stochastic Fubini's Theorem (cf. [9, 36]) and the Burkhölder-Davis-Gundy inequality [35, Theorem 4.37], yields

$$
\begin{aligned}
\mathrm{J}_{21}^{(311)} &= \left\|\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\int_{t_j}^{t_{j+1}} \chi_{[s,t_{j+1})}(\sigma)S(t_m-s)\mathcal{I}_{t_j,t_j}S(t_{j+1}-\sigma)dsdW(\sigma)\right\|_{L^{2p}(\Omega,\mathcal{H})}\\
&\leq C\left(\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\left\|\int_{t_j}^{t_{j+1}} \chi_{[s,t_{j+1})}(\sigma)S(t_m-s)\mathcal{I}_{t_j,t_j}S(t_{j+1}-\sigma)Q^{\frac{1}{2}}ds\right\|^2_{L^{2p}(\Omega,\mathcal{L}_2(\mathcal{H}))}d\sigma\right)^{\frac{1}{2}}.
\end{aligned}
$$

Using Hölder's inequality, we estimate

$$
\begin{aligned}
&\mathrm{J}_{21}^{(311)}\\
&\leq C\left[\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\left(\int_{t_j}^{t_{j+1}}\left\|\chi_{[s,t_{j+1})}(\sigma)S(t_m-s)\mathcal{I}_{t_j,t_j}S(t_{j+1}-\sigma)Q^{\frac{1}{2}}\right\|_{L^{2p}(\Omega,\mathcal{L}_2(\mathcal{H}))}\right)^2 dsd\sigma\right]^{\frac{1}{2}}\\
&\leq C\left[\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\left(\int_{t_j}^{t_{j+1}}1^2ds\right)\right.\\
&\qquad\left.\times\left(\int_{t_j}^{t_{j+1}}\left\|\chi_{[s,t_{j+1})}(\sigma)S(t_m-s)\mathcal{I}_{t_j,t_j}S(t_{j+1}-\sigma)Q^{\frac{1}{2}}\right\|^2_{L^{2p}(\Omega,\mathcal{L}_2(\mathcal{H}))}ds\right)d\sigma\right]^{\frac{1}{2}}\\
&\leq C\Delta t^{\frac{1}{2}}\left[\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\left(\int_{t_j}^{t_{j+1}}\left\|\chi_{[s,t_{j+1})}(\sigma)S(t_m-s)\mathcal{I}_{t_j,t_j}S(t_{j+1}-\sigma)Q^{\frac{1}{2}}\right\|^2_{L^{2p}(\Omega,\mathcal{L}_2(\mathcal{H}))}ds\right)d\sigma\right]^{\frac{1}{2}}.
\end{aligned}
$$

Using Proposition 2.1, the estimate (68), Assumption 2.2 the smoothing properties of the semigroup (cf. (6)), we obtain

$$
\begin{aligned}
\mathrm{J}_{21}^{(311)}&\leq C\Delta t^{\frac{1}{2}}\left(\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\int_{t_j}^{t_{j+1}}\|\chi_{[s,t_{j+1})}(\sigma)S(t_m-s)\mathcal{I}_{t_j,t_j}S(t_{j+1}-\sigma)Q^{\frac{1}{2}}\|^2_{L^{2p}(\Omega,\mathcal{L}_2(\mathcal{H}))}dsd\sigma\right)^{\frac{1}{2}}\\
&\leq C\Delta t^{\frac{1}{2}}\left(\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\int_{t_j}^{t_{j+1}}\|A^{\frac{1-\beta}{2}}S(t_{j+1}-\sigma)\|^2_{\mathcal{L}(\mathcal{H})}\|A^{\frac{\beta-1}{2}}Q^{\frac{1}{2}}\|^2_{\mathcal{L}_2(\mathcal{H})}dsd\sigma\right)^{\frac{1}{2}} \qquad (79)\\
&\leq C\Delta t^{\frac{1}{2}}\left(\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}}\int_{t_j}^{t_{j+1}}d\sigma ds\right)^{\frac{1}{2}}\leq C\Delta t.
\end{aligned}
$$

Along the same lines as in the estimate of $\mathrm{J}_{21}^{(32)}$ (cf. (73)–(77)), one gets

$$\mathrm{J}_{21}^{(312)}\leq C\Delta t\leq C\Delta t^{\frac{\beta}{2}}. \qquad (80)$$

Substituting (79) and (80) into (78) yields

$$\mathrm{J}_{21}^{(31)}\leq C\Delta t^{\frac{\beta}{2}}. \qquad (81)$$

Substituting (77) and (81) into (72), we obtain

$$\mathrm{J}_{21}^{(3)}\leq C\Delta t. \qquad (82)$$

Substituting (82), (71) and (70) into (69), yields

$$\mathrm{J}_{21}\leq C\Delta t^{\frac{\beta}{2}-\xi}. \qquad (83)$$

Substituting (83) and (63) into (56), leads to

$$\mathrm{J}_2 \leq C\left(h^2 + \Delta t^{\frac{\beta}{2}-\xi}\right). \tag{84}$$

Using the Burkhölder-Davis-Gundy inequality (cf. [35, Theorem 4.37]), the triangle inequality and the elementary inequality $(a+b)^2 \leq 2a^2 + 2b^2$, $a, b \in \mathbb{R}$, we get

$$\begin{aligned}
\mathrm{J}_3 &\leq C\left(\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}} \|\left(S(t_m - s) - S_{h,\Delta t}^{m-j}P_h\right)Q^{\frac{1}{2}}\|^2_{L^{2p}(\Omega,\mathcal{L}_2(\mathcal{H}))}ds\right)^{\frac{1}{2}}\\
&\leq C\left(\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}} \|\left(S(t_m - s) - S_h(t_m - s)P_h\right)Q^{\frac{1}{2}}\|^2_{L^{2p}(\Omega,\mathcal{L}_2(\mathcal{H}))}ds\right)^{\frac{1}{2}} \qquad (85)\\
&\quad + C\left(\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}} \|\left(S_h(t_m - s) - S_{h,\Delta t}^{m-j}\right)P_hQ^{\frac{1}{2}}\|^2_{L^{2p}(\Omega,\mathcal{L}_2(\mathcal{H}))}ds\right)^{\frac{1}{2}} =: \mathrm{J}_{31} + \mathrm{J}_{32}.
\end{aligned}$$

Using Proposition 2.1, Lemma 3.5 (ii) and Assumption 2.2, it holds that

$$\begin{aligned}
\mathrm{J}_{31} &= C\left(\int_0^{t_m} \|\left(S(t_m - s) - S_h(t_m - s)P_h\right)Q^{\frac{1}{2}}\|^2_{\mathcal{L}_2(\mathcal{H})}ds\right)^{\frac{1}{2}}\\
&\leq Ch^{\beta}\|A^{\frac{\beta-1}{2}}Q^{\frac{1}{2}}\|_{\mathcal{L}_2(\mathcal{H})} \leq Ch^{\beta}. \qquad (86)
\end{aligned}$$

Using Proposition 2.1, the estimate (68), Lemma 3.1 and [41, Lemma 3.3 (iii)], yields

$$\begin{aligned}
\mathrm{J}_{32} &\leq C\left(\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}} \Delta t^{\beta}t_{m-j}^{-1}\|A_h^{\frac{\beta-1}{2}}P_hQ^{\frac{1}{2}}\|^2_{\mathcal{L}_2(\mathcal{H})}ds\right)^{\frac{1}{2}}\\
&\leq C\Delta t^{\frac{\beta}{2}-\xi}\left(\sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}} t_{m-j}^{-1+\xi}ds\right)^{\frac{1}{2}} \leq C\Delta t^{\frac{\beta}{2}-\xi}, \qquad (87)
\end{aligned}$$

for any arbitrarily small $\xi > 0$.

Substituting (87) and (86) into (85), yields

$$\mathrm{J}_3 \leq C\left(h^{\beta} + \Delta t^{\frac{\beta}{2}-\xi}\right). \tag{88}$$

Substituting (88), (84) and (55) into (54), we obtain

$$\|X(t_m) - \widetilde{X}^h(t_m)\|_{L^{2p}(\Omega,\mathcal{H})} \leq C\left(h^{\beta} + \Delta t^{\frac{\beta}{2}-\xi}\right). \tag{89}$$

It remains to estimate $\|\widetilde{X}^h(t_m) - X^h_m\|_{L^{2p}(\Omega,\mathcal{H})}$. Set $\widetilde{e}^h_m := \widetilde{X}^h(t_m) - X^h_m$. It is easy to check that $\widetilde{e}^h_m$ satisfies the following equation

$$\widetilde{e}^h_m - \widetilde{e}^h_{m-1} + \Delta t A_h\widetilde{e}^h_m = \Delta t P_h\left(F(X(t_m)) - F(X^h_m)\right), \quad \widetilde{e}^h_0 = 0,\ m = 1, \cdots, M.$$

Taking the inner product with $\widetilde{e}^h_m$ in the preceding identity, yields

$$\langle \widetilde{e}^h_m - \widetilde{e}^h_{m-1}, \widetilde{e}^h_m\rangle + \Delta t\langle A_h\widetilde{e}^h_m, \widetilde{e}^h_m\rangle = \Delta t\left\langle \left(F(X(t_m)) - F(X^h_m)\right), \widetilde{e}^h_m\right\rangle.$$

Using the identity $(a-b)a = \frac{1}{2}[a^2 - b^2 + (a-b)^2]$ and Lemma 2.1, it follows that

$$\begin{aligned}
&\frac{1}{2}\left(\|\widetilde{e}^h_m\|^2 - \|\widetilde{e}^h_{m-1}\|^2\right) + \Delta t\langle A_h\widetilde{e}^h_m, \widetilde{e}^h_m\rangle\\
&\leq \Delta t\left\langle \left(F(X(t_m)) - F(X^h_m)\right), \widetilde{e}^h_m\right\rangle\\
&= \Delta t\left\langle \left(F(X(t_m)) - F(\widetilde{X}^h(t_m))\right), \widetilde{e}^h_m\right\rangle + \Delta t\left\langle \left(F(\widetilde{X}^h(t_m)) - F(X^h_m)\right), \widetilde{e}^h_m\right\rangle \qquad (90)\\
&\leq \Delta t\left\langle \left(F(X(t_m)) - F(\widetilde{X}^h(t_m))\right), \widetilde{e}^h_m\right\rangle + C\Delta t\|\widetilde{e}^h_m\|^2.
\end{aligned}$$

Using the estimate (5) and Cauchy-Schwarz's inequality, it follows from (90) that

$$\begin{aligned}
&\frac{1}{2}\left(\|\widetilde{e}^h_m\|^2 - \|\widetilde{e}^h_{m-1}\|^2\right) + \lambda_0\Delta t\|\widetilde{e}^h_m\|^2_1\\
&\leq C\Delta t\|F(X(t_m)) - F(\widetilde{X}^h(t_m))\|^2 + C\Delta t\|\widetilde{e}^h_m\|^2.
\end{aligned}$$

Summing the preceding estimate and noting that $\widetilde{e}^0_h = 0$, we obtain

$$\|\widetilde{e}^h_m\|^2 \leq C\Delta t\sum_{j=1}^m \|\widetilde{e}^h_j\|^2 + C\Delta t\sum_{j=1}^m \|F(X(t_j)) - F(\widetilde{X}^h(t_j))\|^2.$$

Taking the $L^p(\Omega, \mathcal{H})$-norm in the preceding estimate and using Lemma 2.2, we get

$$\begin{aligned}
\|\widetilde{e}^h_m\|^2_{L^{2p}(\Omega,\mathcal{H})} &\leq C\Delta t\sum_{j=1}^m \|\widetilde{e}^h_j\|^2_{L^{2p}(\Omega,\mathcal{H})} + C\Delta t\sum_{j=0}^m \|F(X(t_j)) - F(\widetilde{X}^h(t_j))\|^2_{L^{2p}(\Omega,\mathcal{H})}\\
&\leq C\Delta t\sum_{j=1}^m \Big\{\|X(t_j) - \widetilde{X}^h(t_j)\|^2_{L^{8p}(\Omega,\mathcal{H})}\\
&\quad\times\left(1 + \|X(t_j)\|^{2c_1}_{L^{8pc_1}(\Omega,E)} + \|\widetilde{X}^h(t_j)\|^{2c_1}_{L^{8pc_1}(\Omega,E)}\right)\Big\} + C\Delta t\sum_{j=1}^m \|\widetilde{e}^h_j\|_{L^{2p}(\Omega,\mathcal{H})}.
\end{aligned}$$

Using (46), Lemma 2.2, Proposition 2.4 and Theorem 3.1 with $c = \frac{\nu}{c_1}$, it follows that

$$\begin{aligned}
\|\widetilde{e}^h_m\|^2_{L^{2p}(\Omega,\mathcal{H})} &\leq Ch^{2\beta}\Delta t\sum_{j=1}^m \left(\|\widetilde{X}^h(t_j)\|^{2c_1}_{L^{8pc_1}(\Omega,\mathcal{H})} + h^{2c_1-dc_1}|\ln(h)|^{2\nu}\right)\\
&\quad + C\Delta t\sum_{j=1}^m \|\widetilde{e}^h_j\|^2_{L^{2p}(\Omega,\mathcal{H})}\\
&\leq C\left(h^{2\beta} + h^{2\beta+2c_1-dc_1}|\ln(h)|^{2\nu}\right) + C\Delta t\sum_{j=1}^m \|\widetilde{e}^h_j\|^2_{L^{2p}(\Omega,\mathcal{H})}.
\end{aligned}$$

Applying the discrete Gronwall lemma to the preceding estimate, yields

$$\|\widetilde{e}_m^h\|_{L^{2p}(\Omega,\mathcal{H})} \leq C\left(h^\beta + h^{\beta+c_1-\frac{dc_1}{2}}|\ln(h)|^\nu\right). \tag{91}$$

Substituting (91) and (89) into (53) ends the proof. □

5. Numerical experiments

In this section, we provide some numerical experiments to illustrate our theoretical results. The reference solution or "the exact solution" used in the errors computation is the numerical solution with small step-size.

5.1. **Dirichlet boundary condition.** We consider the following two dimensional stochastic reactive dominated advection diffusion equation

$$dX(t) + \left[D\Delta X(t) - \nabla\cdot(\mathbf{q}X(t))\right]dt = \frac{1}{\varepsilon^2}f(X(t))dt + dW(t), \quad t\in(0,T]. \tag{92}$$

with homogeneous Dirichlet boundary conditions on $\Lambda = [0, L_1]\times[0, L_2]$, where $\varepsilon > 0$ is a small parameter. In the framework of (1), the linear operator A is the $L^2(\Lambda)$ realisation of the second-oder differential operator $\mathcal{A}u = D\Delta u - \nabla\cdot(\mathbf{q}u)$ and the nonlinear function is given by $F(u) = \frac{1}{\varepsilon^2}f(u)$. In the case $f(u) = G'(u) = -u^3 + u$, with $G(u) = -\frac{1}{4}(u^2-1)^2$, Eq. (92) is the well-known stochastic Allen-Cahn equation, which is a popular model for phase separation with ε being the intefracial parameter. The nonlinearity f in (92) ensures that asymptotically the solution remains within the physically meaningful range $-1\leq u\leq 1$ in the deterministic setting, see e.g., [32]. We assume the diffusion function D and the velocity field $\mathbf{q}$ to be constant. In the case $f(u) = -u^5 + u$, the associated function φ defined in (9) is given by $\varphi(x) = -x^5 + x$. In order to check that Assumption 2.3 is fulfilled, let us first recall the following identity

$$a^5 - b^5 = (a-b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4), \quad a, b\in\mathbb{R}.$$

We claim that the following estimate holds

$$\psi(a,b) := a^4 + a^3b + a^2b^2 + ab^3 + b^4 \geq 0, \quad a, b\in\mathbb{R}. \tag{93}$$

In fact, we distinguish two cases:

- If $a\geq b$, then it follows that
$$\psi(a,b) = a^4 + \underbrace{a^3b + a^2b^2}_{=a^2b(a+b)} + \underbrace{ab^3 + b^4}_{=b^3(a+b)} \geq a^4 + 2a^2b^2 + 2b^4 \geq 0.$$

- If $a\leq b$, then it follows that
$$\psi(a,b) = \underbrace{a^4 + a^3b}_{=a^3(a+b)} + \underbrace{a^2b^2 + ab^3}_{=ab^2(a+b)} + b^4 \geq 2a^4 + 2a^2b^2 + b^4 \geq 0.$$

Hence (93) holds. Using (93), that is, it follows that

$$(a-b)(\varphi(a)-\varphi(b))=(a-b)^2-(a-b)(a^5-b^5)=|a-b|^2-(a-b)^2\psi(a,b)\leq|a-b|^2.$$

Hence Assumption 2.3 is fulfilled.

For the numerical experiments, we consider the initial data to be $X_0(x,y)=-\tanh\left(\frac{d(x,y)}{\sqrt{2}\varepsilon}\right)$, where $d(x,y)=\max\{-d_1(x,y),-d_2(x,y)\}$ with

$$d_j(x,y)=\sqrt{(x-L_1/2)^2+(y-L_2/2)^2}-r_j,\quad j=1,2.$$

In the simulations we take $r_1=0.2$ and $r_2=0.55$. The eigenvalues and eigenfunctions of the Laplace operator $-\Delta$ on $H_0^1(\Lambda)$ are given by (see e.g., [27]):

$$\lambda_{i,j}=\pi^2\left[\left(\frac{i}{L_1}\right)^2+\left(\frac{j}{L_2}\right)^2\right]\text{ and }e_{i,j}(x,y)=\sin\left(\frac{i\pi x}{L_1}\right)\sin\left(\frac{j\pi y}{L_2}\right),\quad i,j=1,2,\cdots.$$

In the noise representation (2), we take

$$q_{i,j}=\left(i^2+j^2\right)^{-(\beta+\delta)},\ \delta>0. \tag{94}$$

Using Remark 2.1 (i), it follows that Assumption 2.2 is equivalent to

$$\|(-\Delta)^{\frac{\beta-1}{2}}Q^{\frac{1}{2}}\|_{\mathcal{L}_2(\mathcal{H})}<\infty. \tag{95}$$

One can easily prove that (95) is fulfilled (and hence Assumption 2.2 is fulfilled), since

$$\sum_{(i,j)\in\mathbb{N}^2}\lambda_{i,j}^{\beta-1}q_{i,j}\leq C\sum_{(i,j)\in\mathbb{N}^2}(i^2+j^2)^{-1+\delta}<\infty.$$

We truncate the noise (2) after 60 terms in the x-direction and 60 terms in the y-direction. The triangulation $\mathcal{T}_h$ is constructed from uniform Cartesian grids of sizes $\Delta x=L_1/50$ and $\Delta y=L_2/50$. In the simulations, we take $L_1=L_2=2$ and $\delta=0.001$.

We plot one path of the numerical solution in Figure 1. We observe in Figure 1(a) that the numerical solution remains in the mainingfull range $-1\leq u\leq 1$ for the case of stochastic Allen-Cahn equation with double-well potential (i.e., Eq. (92) with $f(u)=-u^3+u$).

In Figure 2 we plot the mean square error of the implicit Euler scheme. We used 50 sample paths and $\beta=2$. We observe that the rate of convergence is in agreement with the teoretical result in Theorem 4.1.

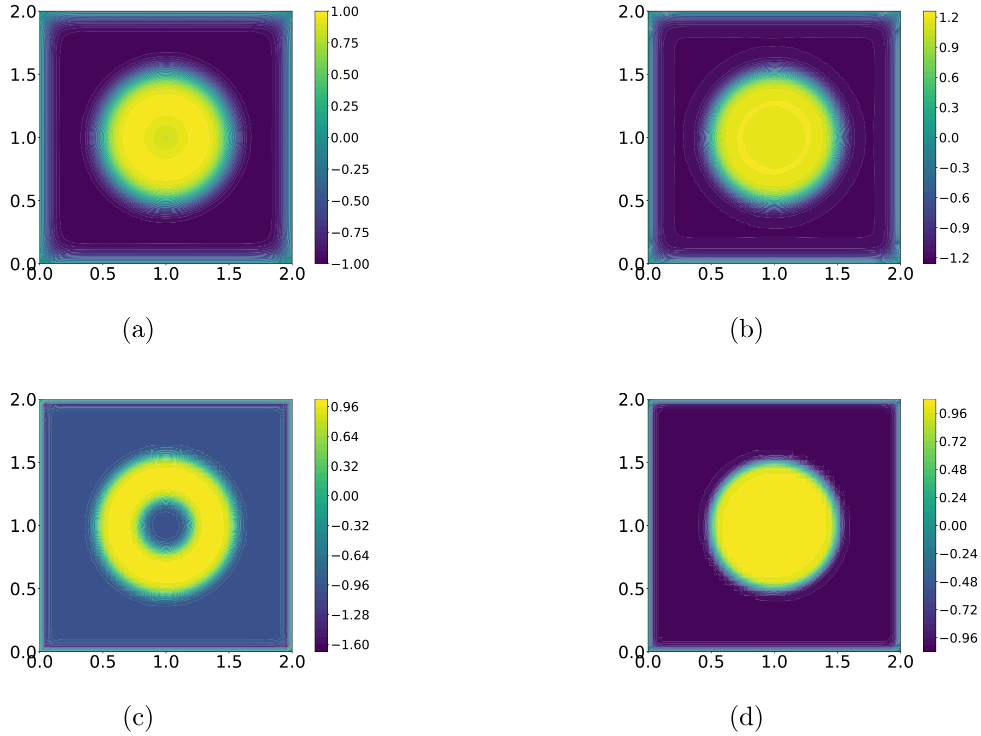

Figure 1. One path of the numerical solution of the SPDE (92) at time $T = 0.02$ with $D = \mathbf{q} = 1$, $\varepsilon = 1/32$, with setp-size $\Delta t = 1/1280$ for different nonlinearities. Graph (a): $f(u) = -u^3 + u$, Graph (b): $f(u) = -u^3 + u + \frac{u}{1+10|u|}$. Gaph (c): $f(u) = -u^5 + u$, Graph (d): $f(u) = -u^5 + u + \frac{u}{1+10|u|}$.

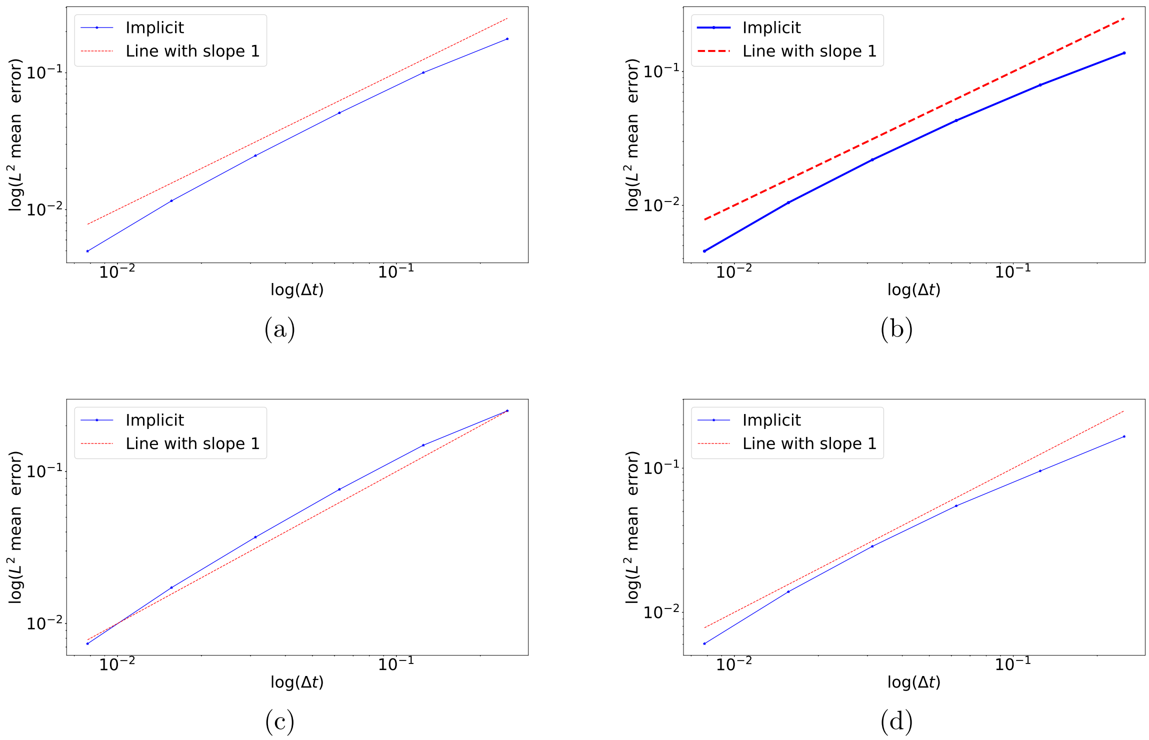

FIGURE 2. Mean square error of the implicit scheme for the SPDE (92) at time $T = 1$ with $\varepsilon = 1/2$, $D = 10^{-3}$, $\mathbf{q} = 10^2$, $\beta = 2$ for different nonlinearities. The total number of samples used is 50. The "exact solution" is taken to be the numerical one with small step-size $\Delta t = 1/512$. Graph (a): $f(u) = -u^3 + u$, the rate of convergence in time is 0.92. Graph (b): $f(u) = -u^3 + u + \frac{u}{1+|u|}$. The rate of convergence is 0.98. Gaph (c): $f(u) = -u^5 + u$, the rate of convergence in time is 1.06. Graph (d): $f(u) = -u^5 + u + \frac{u}{1+10|u|}$. The rate of convergence is 0.947.

5.2. **Mixed Neumann-Dirichlet boundary conditions.** We consider the two dimensional stochastic reactive dominated advection diffusion equation with constant diagonal diffusion function

$$dX(t) = \left[D\Delta X(t) - \nabla \cdot (\mathbf{q}X(t)) - (X^5(t) - X(t))\right] dt + dW(t), \tag{96}$$

with mixed Neumann-Dirichlet boundary conditions on $\Lambda = [0, L_1] \times [0, L_2]$. The Dirichlet boundary condition is $X = 1$ on $\Gamma = \{(x, y) : \ x = 0\}$ and we use the homogeneous Neumann boundary conditions elsewhere. Note that $\mathbf{q}$ is the Darcy velocity and is obtained as in [30]. The noise has the same eigenfunctions $\{e_i^{(1)} e_j^{(2)}\}_{i,j\geq 0}$ as the operator $-\Delta$ with homogeneous Neumann boundary conditions; where e_i are given by

$$e_0^{(l)} = \sqrt{\frac{1}{L_l}}, \quad \lambda_0^{(l)} = 0, \quad e_i^{(l)} = \sqrt{\frac{2}{L_l}} \cos(\lambda_i^{(l)} x), \quad \lambda_i^{(l)} = \frac{i\,\pi}{L_l},$$

where $l \in \{1, 2\}$ and $i = \{1, 2, 3, \cdots\}$, with the corresponding eigenvalues $(\lambda_{i,j})_{i,j}$ given by $\lambda_{i,j} = (\lambda_i^{(1)})^2 + (\lambda_j^{(2)})^2$. In the noise representation (2), we take $q_{i,j}$ as in (94).

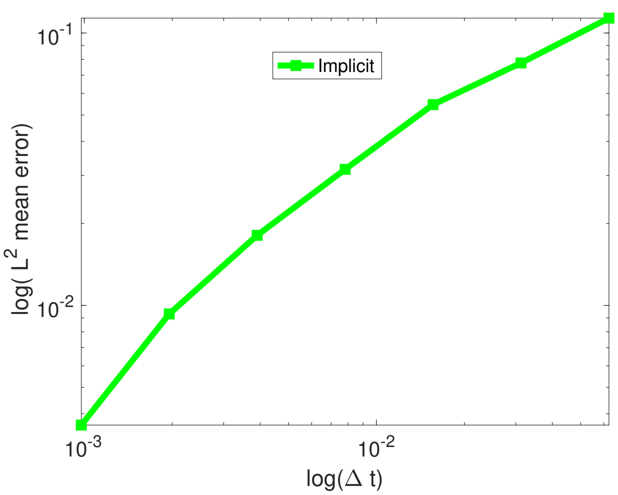

Figure 3. Convergence of the implicit scheme with $\beta = 2$ and $\delta = 0.001$ in (94) at the final time $T = 1$ for the SPDE (96). The order of convergence in time is 0.92. The total number of samples used is 50. Note that the "reference solution" for each sample is the numerical solution with the smaller time step $\Delta t = 1/2018$.

Figure 3 shows the convergence of the implicit scheme with $\beta = 2$ and $\delta = 0.001$ in (94) at the final time $T = 1$. The computational order of convergence in time is 0.92, which is close to the theoretical order in Theorem 4.1.

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